

An introduction to mathematical fluid mechanics: from classical theory to modern challenges

This PhD course provides a rigorous introduction to the mathematical analysis of the fundamental models for incompressible fluid flow.

Tailored for PhD students with strong analytic skills, the lectures bridge the gap between classical 20th-century foundations and the recent breakthroughs of the 2020s.

The first half of the course focuses on the Cauchy problem for smooth flows. We detail the local-in-time well-posedness of classical solutions in Sobolev spaces, utilizing harmonic analysis tools such as commutator estimates and the non-local structure of the pressure term. We then address the potential for singularity formation, the Beale-Kato-Majda criterion and the geometric role of vortex stretching in 3D.

The second half shifts to the framework of weak solutions addressing the global existence of Leray-Hopf solutions for the 3D Navier-Stokes equations and weak-strong uniqueness results. Special attention is given to the 2D Yudovich theory and an excursus on why this theory fails in 3D—specifically for axisymmetric flows with swirl and the formally analogous 2D Boussinesq equations.

The course concludes with an outlook to more recent developments depending on the audience's interest. Topics can include anomalous dissipation and the Onsager conjecture, recent results on the finite-time singularity formation for the 3D Euler equations (Elgindi, Chen-Hou) and the dynamics of vortex (filaments).

Due to time constraints, we focus on the key analytical strategies and provide a comprehensive roadmap for future research in nonlinear PDE and mathematical fluid dynamics. For further reading, the course mostly relies on the textbooks [1, 3, 4] and the review [2].

Prerequisites: Functional analysis and a basic course in PDEs. Knowledge in evolution equations is helpful but not required.

Date: The short course will run from May 19th to June 1th. It consists of six lectures and tutorials. Final examination consists of an oral exam.

References

- [1] J.-Y. Chemin. *Perfect incompressible fluids*. Transl. from the French by Isabelle Gallagher and Dragos Iftimie, volume 14 of *Oxf. Lect. Ser. Math. Appl.* Cl. Press, 1998.
- [2] T. D. Drivas and T. M. Elgindi. Singularity formation in the incompressible Euler equation in finite and infinite time. *EMS Surv. Math. Sci.*, 10(1):1–100, 2023.
- [3] A. J. Majda and A. L. Bertozzi. *Vorticity and incompressible flow*. Camb. Texts Appl. Math. Cambridge: Cambridge University Press, 2002.
- [4] C. Marchioro and M. Pulvirenti. *Mathematical theory of incompressible nonviscous fluids*, volume 96 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1994.