

Seminar of the Work Group
Nonlinear Partial Differential Equations
Institute for Analysis
WS 24/25

Speaker: Siliang Weng
October 22nd, 2024, 11:30 - 13:00
Seminar room: SR 3.069

Magnetic Quantization and Phase Space Methods

Siliang Weng,
Karlsruhe Institute of Technology (KIT)

Abstract

In this talk we introduce the framework of magnetic quantization, aiming to treat wave or Schrödinger equations in the presence of a strong magnetic field. Essentially this quantization gives a pseudo-differential operator theory that incorporate the magnetic field, allowing magnetic potentials that grows towards infinity. With this framework, we are able to adapt the phase space transform methods, which were designed originally for the usual wave or Schrödinger equations. Then such methods can be used to obtain well-posedness results for magnetic equations, in the setting of more general magnetic fields.

Seminar of the Work Group
Nonlinear Partial Differential Equations
Institute for Analysis
WS 24/25

Speaker: Felix Brandt
October 29th, 2024, 11:30 - 13:00
Seminar room: SR 3.061

Interaction of liquid crystals with a rigid body

Felix Brandt,
Technical University of Darmstadt

Abstract

This talk addresses the interaction problem of liquid crystals with a rigid body. The physical motivation for such problems is the presence of so-called liquid crystal colloids formed by dispersion of colloidal particles in the liquid crystal host medium, where a colloidal particle is viewed as a rigid body.

In the first part of the talk, we investigate the interaction problem involving a simplified Ericksen-Leslie model. We verify that the director condition $|d| = 1$ is preserved in the interaction problem. After transforming the moving boundary problem to a fixed domain, we establish the local strong well-posedness by showing maximal L^p - L^q -regularity of the linearized problem. Moreover, we prove global strong well-posedness close to constant equilibria, where we perform a splitting argument of the director into its mean value zero and average part to overcome the lack of invertibility.

The second part of the talk is dedicated to the study of the interaction problem of a general Beris-Edwards Q -tensor model. In contrast to Ericksen-Leslie models, which are vector models, Q -tensor models build on symmetric, traceless 3×3 -matrices Q to describe the biaxial alignment of molecules. In order to tackle the resulting quasilinear mixed-order problem with moving boundary, we first transform it to a fixed domain and then establish maximal L^p -regularity in an anisotropic ground space of the form $L^2 \times H^1$ by means of a “monolithic” approach. The proofs of the local strong well-posedness for large data and the global strong well-posedness for small data are completed by suitable nonlinear estimates.

The talk is based on joint work with Tim Binz, Matthias Hieber and Arnab Roy.

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Speaker: Ferdinand Eitler
November 5th, 2024, 11:30 - 13:00
Seminar room: SR 3.061

Gradient integrability for bounded BD-minimizers

Ferdinand Eitler,
University of Augsburg

Abstract

This talk focuses on regularity theory for bounded generalised minimisers of μ -elliptic linear growth functionals in the space BD of functions of bounded deformation. For this kind of variational problems the latter space is natural, but a priori the symmetric gradient exists only as a matrix valued Radon measure. For generalised minimisers which are locally bounded we establish a higher gradient integrability result for the full range of $\mu \in (1, 3]$. In order to use the available a priori bounds on the symmetric gradients, we must simultaneously allow for algebraic manipulations and keep track of the L^∞ -constraint. The potential non-uniqueness of generalised minimisers moreover requires these tasks to be compatible with a suitable application of Ekeland's variational principle. This is joint work with Lisa Beck (Augsburg) and Franz Gmeineder (Konstanz).

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Speaker: Louis Garénaux
November 12th, 2024, 11:30 - 13:00
Seminar room: SR 3.061

Existence of a monostable front for a KPP numerical scheme

Louis Garénaux,
Karlsruhe Institute of Technology (KIT)

Abstract

The KPP equation is a reaction-diffusion equation, that admits travelling wave solutions. In this presentation, we will study the corresponding natural numerical scheme, and will discuss how existence of a front for the space-continuous equation can lead to existence of a front for the space-discrete equation. This is a joint work with Hermen Jan Hupkes.

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Speaker: Wei Lian
November 26th, 2024, 11:30 - 13:00
Seminar room: SR 3.061

Transverse instability of line periodic waves to the KP-I equation

Wei Lian,
Lund University

Abstract

The passage from linear instability to nonlinear instability has been shown for 1D solitary waves under 2D perturbations. Although transverse instability of periodic waves to the KdV equation under the KP-I flow has been expected to be true from spectral instability for a long time, it has not been clear how to adapt the general instability theory for solitary waves to periodic waves until now. In this talk, we present how such an adaptation works with the aid of exponential trichotomies and multivariable Puiseux series.

Joint work with Erik Wahlén.

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Speaker: Emile Bukieda
November 19th, 2024, 11:30 - 13:00
Seminar room: SR 3.061

Nonlinear Stability at the Eckhaus Boundary in case of a Shortwave Destabilization

Emile Bukieda (Joint w. Björn de Rijk),
Karlsruhe Institute of Technology (KIT)

Abstract

We consider the complex Ginzburg-Landau equation in two spatial dimensions, which depends on two real parameters a and b and admits a family of periodic traveling stripe solutions parametrized by the wave number. We are interested in the nonlinear stability of these solutions against localized perturbations. It is known that, depending on a and b , these waves are either always unstable or there exists a critical value for the wave number, the so-called Eckhaus boundary, above which they are stable and below which they are unstable. If the wave number passes through the Eckhaus boundary from above a short- or longwave destabilization occurs. It has been shown in case of longwave destabilization that periodic traveling waves are still nonlinearly stable at the Eckhaus boundary. Here, we settle the other case and establish nonlinear diffusive stability at the Eckhaus boundary in case of a shortwave destabilization. Our proof relies on iterative L^1 - L^∞ estimates on the Duhamel formulation of the perturbation. To this end, we decompose the underlying semigroup which allows to separately handle the interactions of different critical modes in the nonlinear terms. A challenge is that standard L^1 - L^∞ estimates are insufficient to control the most critical interaction terms. Inspired by the space-time resonance method, developed for dispersive systems, we uncover a generic absence of space resonances to exploit oscillations in frequency yielding additional decay which allows to close a nonlinear argument.

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Speaker: Richard Nutt
December 10th, 2024, 11:30 - 13:00
Seminar room: SR 3.061

Exponential Decay of the Linear Maxwell System due to Interior Conductivity

Richard Nutt,
Karlsruhe Institute of Technology (KIT)

Abstract

We study the anisotropic, linear Maxwell system on a bounded domain with an L^∞ conductivity in a collar around the boundary and perfectly conductive boundary conditions. We show that solutions with divergence free initial values decay exponentially to 0. In our approach we split the solution via a Helmholtz decomposition and then show an observability estimate for the homogenous Maxwell problem.

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Speaker: Lukas Bengel
December 3rd, 2024, 11:30 - 13:00
Seminar room: SR 3.061

Soliton-based frequency combs in the Lugiato-Lefever equation

Lukas Bengel,
Karlsruhe Institute of Technology (KIT)

Abstract

Kerr frequency combs are optical signals consisting of a multitude of equally spaced excited modes in frequency space. They are generated by converting a continuous-wave pump laser within an optical microresonator. In its simplest form, the physics in the microresonator is modeled by the Lugiato-Lefever equation, a damped nonlinear Schrödinger equation with forcing. In this talk we demonstrate that the Lugiato-Lefever equation supports arbitrarily broad Kerr frequency combs by proving the existence and stability of periodic solutions consisting of any number of well-separated, strongly localized and highly nonlinear pulses.

The talk is based on a joint work with Björn de Rijk (KIT).

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Speaker: Rebekka Zimmermann
December 17th, 2024, 11:30 - 13:00
Seminar room: SR 3.061

Global regularity of 2D density patches for the incompressible inhomogeneous Navier-Stokes equations with largely varying viscosity coefficient

Rebekka Zimmermann,
Karlsruhe Institute of Technology (KIT)

Abstract

We consider the two-dimensional inhomogeneous incompressible Navier-Stokes equations with density-dependent viscosity coefficient. These equations describe the motion of a fluid with variable density and viscosity, which arises for example when mixing two immiscible fluids with different densities and viscosities. One question one might ask in this situation is whether the regularity of the interface between the two fluids is preserved over time. In this talk we prove that under suitable assumptions on the initial data this is indeed the case, even for largely varying, discontinuous viscosity coefficients.

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Speaker: Sebastian Ohrem
January 14th, 2024, 11:30 - 13:00
Seminar room: SR 3.061

Breather solutions for semilinear wave equations - Part 1

Sebastian Ohrem,
Karlsruhe Institute of Technology (KIT)

Abstract

We consider the semilinear wave equation

$$V(x)u_{tt} - u_{xx} = \Gamma(x)|u|^{p-1}u, \quad \text{for } (x, t) \in \mathbb{R} \times \mathbb{R},$$

with $p \in (1, \infty)$. For carefully chosen material coefficients V, Γ we show existence of time-periodic, real-valued, space-localized solutions, called breathers. We use variational methods.

In Part 1, we investigate a suitable domain for the energy functional

$$J(u) = \frac{1}{2} \int -V(x)u_t^2 + u_x^2 \, d(x, t) - \frac{1}{p+1} \int \Gamma(x)|u|^{p+1} \, d(x, t),$$

and its embedding properties, which are used for the variational argument.

This is joint work with Julia Henninger and Wolfgang Reichel, and part 2 is presented next week by Julia Henninger.

Seminar of the Work Group
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Speaker: Julia Henninger
January 21st, 2024, 11:30 - 13:00
Seminar room: SR 3.061

Breather solutions for semilinear wave equations - Part 2

Julia Henninger,
Karlsruhe Institute of Technology (KIT)

Abstract

We consider the semilinear wave equation

$$V(x)u_{tt} - u_{xx} = \Gamma(x)|u|^{p-1}u, \text{ for } (x, t) \in \mathbb{R} \times \mathbb{R},$$

with $p \in (1, \infty)$. For carefully chosen material coefficients V, Γ we show existence of time-periodic, real-valued, space-localized solutions, called breathers. We use variational methods. In Part 1, we developed a suitable domain for the energy functional and proved its embedding properties. In part 2 we show the existence of a critical point of our functional which leads to a solution of our equation. Further, we present suitable examples of V and Γ . This is joint work with Sebastian Ohrem and Wolfgang Reichel.

Seminar of the Work Group
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Speaker: Robert Wegner
February 4th, 2025, 11:30 - 13:00
Seminar room: SR 3.061

Gaussian White Noise

Robert Wegner,
Karlsruhe Institute of Technology (KIT)

Abstract

We give an overview over different definitions Gaussian White noise and their formal relations, ultimately concluding that it is characterized by an embedding $H \hookrightarrow X$ of a separable Hilbert space into a separable Banach space. An example is space-time white noise ξ , which is a key object in the theory of stochastic partial differential equations.

Gaussian White Noise

Robert Wegner

04.02.2025

References.

- “Gaussian Measures” by V. Bogachev.
- Lecture notes by A. Lussnardi, M. Miranda, D. Pallara for 19th ISEM:
dm.unife.it/it/ricerca-dmi/seminari/isem19/lectures

Notation.

- $\mathbb{R}_+ = [0, \infty)$.
- $(\Omega, \mathcal{G}, \mathbb{P})$ is a probability space.
- $(g_n)_{n \in \mathbb{N}}, (h_n)_{n \in \mathbb{N}}$ are independent Gaussian random variables (RV) with law $\mathcal{N}(0, 1)$.
- X is a separable Banach space. $\mathcal{B}(X)$ is the Borel- σ -algebra
- H is a separable Hilbert space and $(e_n)_{n \in \mathbb{N}}$ an ONB.

We have studied the following stochastic PDE:

$$\partial_t^2 u + \partial_t u + (1 - \Delta)u + u^3 = \sqrt{2} \langle \nabla \rangle^{-s} \xi \quad (t, x) \in \mathbb{R}_+ \times \mathbb{T}^2.$$

Conceptually, **(space-time) white noise** ξ is...

1... the **isonormal distribution** on $H = L^2_{t,x}$, i.e. the unique weak distribution $\langle \xi, - \rangle : H \longrightarrow L^2(\Omega, \mathbb{P})$ which fulfills

$$(\langle \xi, f \rangle)_{f \in H} \text{ are centered jointly Gaussian RV with } \mathbb{E}[\langle \xi, f \rangle \langle \xi, g \rangle] = \langle f, g \rangle_H. \quad (1)$$

2... the time derivative of a **Brownian motion** in each dimension:

$$\xi = \sum_{n=1}^{\infty} dB_t^n e_n \text{ where } (B_t^n)_{n \in \mathbb{N}} \text{ is a family of independent Brownian motions.}$$

3... the **isonormal series** on $L^2_{t,x}$: $\xi = \sum_{n=1}^{\infty} g_n e_n$.

4... the **isonormal cylinder set probability measure** on $L^2_{t,x}$.

5... a RV with **Gaussian law** μ on $C_t^{-\frac{1}{2}-\varepsilon} H_x^{-1-\varepsilon}$.

6... the separable Hilbert space $\ell^2(\mathbb{N})$.

1 $\langle \xi, - \rangle$ as a weak distribution

Definition 1. A **weak distribution** on a separable Hilbert space H is an equivalence class of linear operators $\langle \xi, - \rangle : H \longrightarrow \{RV \text{ on } (\Omega_\xi, \mathcal{F}_\xi, \mathbb{P}_\xi)\}$ under the relation

$$\langle \xi, - \rangle \sim \langle \xi', - \rangle \iff \text{Law}_{\mathbb{P}_\xi}(\langle \xi, x_1 \rangle, \dots, \langle \xi, x_d \rangle) = \text{Law}_{\mathbb{P}_{\xi'}}(\langle \xi', x_1 \rangle, \dots, \langle \xi', x_d \rangle) \quad \forall x_1, \dots, x_d \in H.$$

Lemma 2. If $\langle \xi, x_1 \rangle$ and $\langle \xi', x_1 \rangle$ fulfill (1) then $\langle \xi, x_1 \rangle \sim \langle \xi', x_1 \rangle$, hence the isonormal distribution is a weak distribution if it exists.

Proof. The RV $\langle \xi, f \rangle$ and $\langle \xi', f \rangle$ are Gaussian, hence their joint laws determined by the covariances. By (1) they agree. $\square$

2 ξ as a time derivative of Brownian motion

Definition 3. A family of RV $(B_t)_{t \in \mathbb{R}_+}$ is a **Brownian motion (BM)** if

- (i) $B_0 = 0$ a.s.
- (ii) $t \mapsto B_t$ is a.s. continuous.
- (iii) For any $0 \leq t_1 < \dots < t_n$ the increments $B_{t_{k+1}} - B_{t_k}$ are independent and have law $\mathcal{N}(0, t_{k+1} - t_k)$.

scaling limit of discrete random walk $\longrightarrow$ Brownian motion .

Definition 4 (Stochastic integral). Let $f \in C_c^1(\mathbb{R}_+)$.

$$\int_{\mathbb{R}_+} f(t) dB_t := - \int_{\mathbb{R}_+} f'(t) B_t dt .$$

- “Sum” of Gaussian RV = Gaussian RV
- B_t has a.s. unbounded variation so the oscillations of dB_t don't cancel out to 0.
- B_t has a.s. finite quadratic variation, which allows extending to $f \in L^2(\mathbb{R}_+)$ (Ito-integral).
- If $B_t = \$$ to buy/sell a stock and $f(t) = \$$ invested then $\int f dB_t = \$$ return.

Lemma 5. Let $(B_t^n)_{n \in \mathbb{N}}$ be independent BM and $(e_n)_{n \in \mathbb{N}}$ an ONB of H . Then $\langle \xi, - \rangle := \langle \sum_{n=1}^{\infty} dB_t^n e_n, - \rangle$ is space-time white noise.

Proof. for $f \in C_{c,t}^1 L_x^2$ we have

$$\begin{aligned} \langle \xi, f \rangle &= \left\langle \sum_{n=1}^{\infty} e_n dB_t^n, f \right\rangle_{t,x} = \sum_{n=1}^{\infty} \int_{\mathbb{R}_+} \langle f(t), e_n \rangle_x dB_t^n = \sum_{n=1}^{\infty} \int_{\mathbb{R}_+} \langle \partial_t f(t), e_n \rangle_x B_t^n dt \quad \text{is a centered Gaussian RV} \quad \checkmark \\ \mathbb{E} [\langle \xi, f \rangle \langle \xi, g \rangle] &= \sum_{n,k=1}^{\infty} \int_{\mathbb{R}_+} \langle \partial_t f(t), e_n \rangle_x \langle \partial_s g(s), e_k \rangle_x \mathbb{E} [B_t^n B_s^k] dt ds \\ &= \sum_{n=1}^{\infty} \int_{\mathbb{R}_+} \langle f(t), e_n \rangle_x \langle g(t), e_n \rangle_x dt = \langle f, g \rangle_{t,x} \quad \checkmark \end{aligned} \tag{2}$$

Extend to $f \in L_{t,x}^2 \dots$ $\square$

3 ξ as the isonormal series

Let H be a separable Hilbert space and consider the formal random series $\xi = \sum_{n=1}^{\infty} g_n e_n$.

Lemma 6. $f \mapsto \langle \xi, f \rangle$ is the isonormal distribution.

Proof. We check (1):

$$\begin{aligned}\langle \xi, f \rangle &= \sum_{n=1}^{\infty} g_n \langle f, e_n \rangle \quad \text{is a centered Gaussian RV} \quad \checkmark \\ \mathbb{E}[\langle \xi, f \rangle \langle \xi, g \rangle] &= \sum_{n,k=1}^{\infty} \mathbb{E}[g_n g_k] \langle f, e_n \rangle \langle g, e_k \rangle = \sum_{n=1}^{\infty} \langle f, e_n \rangle \langle g, e_n \rangle = \langle f, g \rangle \quad \checkmark\end{aligned}$$

□

Lemma 7. (i) *Almost surely, ξ diverges in H if H is infinite dimensional.*

(ii) *Almost surely, ξ converges in $H^s = \overline{H}^{\|\cdot\|_{H^s}}$ for $s < -\frac{1}{2}$ where $\|x\|_{H^s}^2 = \sum_{n=1}^{\infty} n^{2s} \langle x, e_n \rangle^2$.*

Proof. (i)

$$\mathbb{P}\left(\sum_{n=1}^{\infty} g_n e_n \text{ converges}\right) = \mathbb{P}\left(\sum_{n=1}^{\infty} g_n^2 < \infty\right) \leq \mathbb{P}\left(\frac{1}{N} \sum_{n=1}^N g_n^2 \xrightarrow{N \rightarrow \infty} 0\right).$$

But by the law of large numbers

$$\mathbb{P}\left(\frac{1}{N} \sum_{n=1}^N |g_n|^2 \xrightarrow{N \rightarrow \infty} \mathbb{E}[g_n^2] = 1\right) = 1.$$

(ii) Let $s < -\frac{1}{2}$. Then

$$\mathbb{E}\left[\underbrace{\sup_{N \in \mathbb{N}} \left\| \sum_{n=1}^N g_n e_n \right\|_{H^s}^2}_{< \infty \text{ a.s.}}\right] = \mathbb{E}\left[\sup_{N \in \mathbb{N}} \sum_{n=1}^N n^{2s} g_n^2\right] = \sup_{N \in \mathbb{N}} \sum_{n=1}^N n^{2s} \mathbb{E}[g_n^2] < \infty.$$

Therefore the series converges a.s.

□

4 Law(ξ) as a cylinder set probability measure

Let X be a separable Hilbert space. Let $\pi_K : H \rightarrow K \cong \mathbb{R}^d$ be a projection with finite dimensional range. Since $K \cong \mathbb{R}^d$ it has an obvious Borel- σ -algebra $\mathcal{B}(K)$. Define

$$\mathcal{F}_K^{\text{cy}1} = \{\pi_K^{-1}(A) : A \in \mathcal{B}(K) \cong \mathcal{B}(\mathbb{R}^d)\} \quad \mathcal{F}^{\text{cy}1} = \bigcup_{K \subset H, \dim(K) < \infty} \mathcal{F}_K^{\text{cy}1} \quad \mathcal{F} = \sigma(\mathcal{F}^{\text{cy}1}).$$

Then $\mathcal{F}_K^{\text{cy}1}$ and $\mathcal{F}$ are σ -algebras but $\mathcal{F}^{\text{cy}1}$ is only a set-algebra (closed under only **finite** unions).

Lemma 8. $\mathcal{F} = \mathcal{B}(H)$.

Definition 9. A *cylinder set probability measure (CSPM) on H* is probability pre-measure on $(H, \mathcal{F}^{\text{cy}1})$, i.e. an additive set function $\mu : \mathcal{F}^{\text{cy}1} \rightarrow \mathbb{R}_+ \cup \{\infty\}$ which is countable additive on $\mathcal{F}_K^{\text{cy}1}$ for any finite dimensional $K \subset H$, and fulfills $\mu(H) = 1$.

Lemma 10. Let μ be a CSPM and $K = \text{span}(x_1, \dots, x_d) \subset H$. Then

$$H^* \ni x_j^* = [y \mapsto \langle y, x_j \rangle] \text{ is a RV on } (H, \mathcal{F}_K^{\text{cy}1}, \mu) \quad \forall j.$$

By some abstract construction: There exists a large probability space $(\Omega, \mathcal{G}, \mathbb{P})$ and an embedding

$$\langle \zeta, - \rangle : H^* \longrightarrow \{RV \text{ on } (\Omega, \mathcal{G}, \mathbb{P})\}.$$

such that

$$\text{Law}_{(H, \mathcal{F}_K^{\text{cyl}}, \mu)}(x_1^*, \dots, x_d^*) = \text{Law}_{(\Omega, \mathcal{G}, \mathbb{P})}(\langle \zeta, x_1^* \rangle, \dots, \langle \zeta, x_d^* \rangle) \quad \forall K = \text{span}(x_1, \dots, x_d).$$

Hence we have a correspondence

$$\{\text{CSPMs } \mu\} \longleftrightarrow \{\text{weak distributions } \langle \zeta, - \rangle\}.$$

Definition 11. Let $K = \text{span}(x_1, \dots, x_d) \subset H$ be finite dimensional. We define a CSPM by

$$\mu(\pi_K^{-1}(A)) = \mu((x_1^*, \dots, x_d^*)^{-1}(A)) := \mathcal{N}(0, \langle x_i, x_j \rangle_{1 \leq i, j \leq d})(A).$$

for an arbitrary set $\pi_K^{-1}(A) \in \mathcal{F}_K^{\text{cyl}}$, $A \in \mathcal{B}(\mathbb{R}^d)$. Then

$$\text{Law}_{(H, \mathcal{F}_K^{\text{cyl}}, \mu)}(x_1^*, \dots, x_d^*) = \mathcal{N}(0, \langle x_i, x_j \rangle_{1 \leq i, j \leq d}) = \text{Law}(\langle \xi, x_1 \rangle \dots \langle \xi, x_d \rangle),$$

hence μ corresponds to ξ . We call μ the **isonormal CSPM**.

Lemma 12. Formally, $\mu = \text{Law}(\sum_{n=1}^{\infty} g_n e_n)$ for any ONB $(e_n)_{n \in \mathbb{N}}$. Therefore if H is infinite dimensional then μ does **not** extend to a measure on $(H, \mathcal{B}(H))$.

Theorem 13 (L. Gross 1967). Let μ be a CSPM on H and consider $j : H \hookrightarrow X$ where X is a separable Banach space.

- (i) $j_{\#}\mu$ is a CSPM on X .
- (ii) Iff $\|\cdot\|_X$ is a “**measurable seminorm**” on H , then $j_{\#}\mu$ extends to a measure $\bar{\mu}$ on $(X, \mathcal{B}(X))$.
- (iii) In this case $\sum_{n=1}^{\infty} g_n j(e_n)$ converges $\bar{\mu}$ -a.s. in X for any ONB $(e_n)_{n \in \mathbb{N}}$ of H .

Definition 14 (L. Gross 1967). An **Abstract Wiener space** is an emedding $H \hookrightarrow X$ where X is sufficiently large so that one of the following equivalent conditions holds:

- (i) The isonormal CSPM extends to a measure on μ on $(X, \mathcal{B}(X))$.
- (ii) The isonormal series $\xi = \sum_{n=1}^{\infty} g_n e_n$ converges in $L^2(\Omega; X)$.
- (iii) The isonormal distribution $\langle \xi, - \rangle$ can be extended to a RV in $L^2(\Omega; X^*)$.

Then μ is a **Gaussian** measure.

$L_{t,x}^2 \hookrightarrow C_{t \in \mathbb{R}_+}^{-\frac{1}{2}-\varepsilon} H_{x \in \mathbb{T}^2}^{-1-\varepsilon}$ yields space-time white noise.

5 ξ as a RV with Gaussian law μ

Example 15. Some Gaussian measures:

- (i) $\mathcal{N}(a, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\sigma^{-2}(x-a)^2} dx$, $a \in \mathbb{R}$, $\sigma^2 > 0$.
- (ii) $\sigma = 0$ degenerate case: δ_0 on $\mathbb{R}$
- (iii) $\mathcal{N}(a, \Sigma) = \frac{1}{\det(\sqrt{2\pi|\Sigma|})} e^{-\frac{1}{2}(x-a)^T \Sigma^{-1}(x-a)} d\mathcal{L}^n(x)$, $a \in \mathbb{R}^d$, $\Sigma \in \mathbb{R}^{d \times d}$ invertible.
- (iv) $\text{rank}(\Sigma) = k < d$ degenerate case: $\mathcal{L}^n$ replaced by $\mathcal{H}^k|_{\{a+\Sigma^{\frac{1}{2}}(\mathbb{R}^d)\}}$.
- (v) The **Wiener measure** μ on $(X, \mathcal{B}(X))$ where $X = L^2([0, 1])$ or $X = C_b([0, 1])$ such that for all $A \in \mathcal{B}(X)$, $\mu(A) = \mathbb{P}(B_t \in A)$.

(vi) There exists a Gaussian measure μ on $C_{t \in \mathbb{R}_+}^{-\frac{1}{2}-\varepsilon} H_{x \in \mathbb{T}^2}^{-1-\varepsilon}$ s.t.

$$\text{Law}_{x \sim \mu}(\langle x, f_1 \rangle_{t,x}, \dots, \langle x, f_d \rangle_{t,x}) = \text{Law}(\langle \xi, f_1 \rangle, \dots, \langle \xi, f_d \rangle) \quad \forall f_1, \dots, f_d \in L_{t,x}^2.$$

Let X be a separable Banach space.

Definition 16. A measure μ on $(X, \mathcal{B}(X))$ is **Gaussian** if these equivalent definitions hold:

- (i) For every $f \in X^*$ we have $f_{\#}\mu = \mathcal{N}(0, \sigma^2)$ for some $\sigma > 0$.
- (ii) The characteristic function

$$\widehat{\mu}(f) = \int_X e^{if(x)} d\mu(x) \quad \text{fulfills} \quad \widehat{\mu}(f) = e^{i\mathbb{E}_\mu[f] - \frac{1}{2} \text{Cov}_\mu(f, f)}$$

where

$$\begin{aligned} \mathbb{E}_\mu[f] &= \int_X f(x) d\mu(x) \\ \text{Cov}_\mu(f, g) &= \mathbb{E}_\mu[(f - \mathbb{E}_\mu[f])(g - \mathbb{E}_\mu[g])] = \int_X (f(x) - \mathbb{E}_\mu[f])(g(x) - \mathbb{E}_\mu[g]) d\mu(x). \end{aligned}$$

Lemma 17. Some facts:

- Fernique theorem: $\exists \alpha > 0$ such that $\int_X e^{\alpha \|x\|_X^2} d\mu(x) < \infty$.
- $\mathbb{E}_\mu \in (X^*)^*$ and $\text{Cov}_\mu \in (X^* \otimes X^*)^*$, i.e. they are bounded multilinear forms w.r.t. $\|\cdot\|_{X^*}$.
- $\exists a_\mu \in X$, the **mean** of μ , such that $\mathbb{E}_\mu[f] = f(a_\mu) \forall f \in X^*$.
Suffices: $\mathbb{E}_\mu \in (X^*)^*$ is weak*-continuous. Use uniform boundedness and dominated convergence.
Alternatively: $a_\mu = \mathbb{E}_\mu[\text{Id}_X]$ exists as a Bochner integral.
- The measure $\mu_0(A) = \mu(A - a_\mu)$ is **centered Gaussian**. For simplicity we **always** assume μ is centered.
- $X^* \subset L^2(X, \mu)$, so every $f \in X^*$ is a linear RV.
- $\langle f, g \rangle_{L^2(X, \mu)} = \int_X f(x)g(x) d\mu(x) = \text{Cov}_\mu(f, g)$.
- $\text{Cov}_\mu(-, g) \in L^2(X, \mu)^* \subset (X^*)^* \xleftarrow{\text{ev}_x(f)=f(x)} X$.

Question: When is $\text{Cov}_\mu(-, g) \in \text{ev}_X$, i.e. $\text{Cov}_\mu(f, g) = f(h)$ for some $h \in X$?

Definition 18. The **Reproducing kernel Hilbert space (RKHS)** X_μ^* of μ is the closure $\overline{X^*}^{\|\cdot\|_{L^2(X, \mu)}}$. It is a Hilbert space with inner product Cov_μ . Let $\tau : (X_\mu^*)^* \rightarrow X_\mu^*$ be the Riesz representation.

Definition 19. The **Cameron-martin space (CMS)** H of μ is the Hilbert space

$$H = \{h \in X : \|h\|_H < \infty\} \quad \|h\|_H = \|\text{ev}_h\|_{(X_\mu^*)^*}.$$

Lemma 20. (i) $\mathbb{E}_\mu \in (X_\mu^*)^*$ and $\text{Cov}_\mu \in (X_\mu^* \otimes X_\mu^*)^*$, i.e. they are bounded w.r.t. $\|\cdot\|_{L^2(X, \mu)}$.

- (ii) If $g \in X_\mu^*$ then $\tau^{-1}(g) = \text{Cov}_\mu(-, g) \in \text{ev}_X \subset (X^*)^*$
Again: weak*-continuity by uniform boundedness and dominated convergence.

(iii) We have isomorphisms

$$H \xrightarrow{\tau \circ \text{ev}} X_\mu^* \xrightarrow{\text{ev}^{-1} \circ \tau^{-1}} H.$$

(iv) If $h \in H$ and $g = \tau(\text{ev}_h) \in X_\mu^*$ are identified, then $\text{Cov}_\mu(f, g) = \text{Cov}_\mu(f, \tau(\text{ev}_h)) = \text{ev}_h(f) = f(h)$.

Proof. (iii)

$$\left\| \text{ev}_{\text{Cov}_\mu(-,g)}^{-1} \right\|_H = \|\text{Cov}_\mu(-,g)\|_{(X_\mu^*)^*} = \|g\|_{X_\mu^*}.$$

□

Example 21. Consider the Wiener measure μ on $X = L^2([0, 1])$.

Then

$$X^* = \left\{ \left[B \mapsto \int_0^1 f(t) B_t \, dt \right] : f \in L^2([0, 1]) \right\}.$$

What are $X_\mu^* = \overline{X^*}^{L^2(X, \mu)}$ and H ? Suppose $(f_n)_{n \in \mathbb{N}}$ is Cauchy in $L^2([0, 1])$. Write

$$Ih(t) = \int_0^t h(s) \, ds.$$

Then

$$0 \xleftarrow{n, m \rightarrow \infty} \mathbb{E} \left[\left(\int_0^1 (f_n - f_m) B_t \, dt \right)^2 \right] \stackrel{IBP}{=} \mathbb{E} \left[\left(\int_0^1 I(f_n - f_m) \, dB_t \right)^2 \right] \stackrel{(2)}{=} \int_0^1 (I(f_n - f_m))^2 \, dt.$$

Therefore $\exists G \in L^2([0, 1]) : If_n \rightarrow G$. Hence

$$X_\mu^* = \left\{ \left[B \mapsto \int_0^1 f(t) \, dB_t \right] : f \in L^2([0, 1]) \right\}, \quad \left\langle \left[B \mapsto \int_0^1 f(t) \, dB_t \right], \left[B \mapsto \int_0^1 h(t) \, dB_t \right] \right\rangle_{X_\mu^*} = \int_0^1 fh \, dt.$$

For $h \in L^2([0, 1])$ calculate

$$\left\langle \underbrace{\left[B \mapsto \int_0^1 f(t) \, dB_t \right]}_{\in X_\mu^*}, \underbrace{(\tau \circ \text{ev})(Ih)}_{\in X_\mu^* ?} \right\rangle_{X_\mu^*} = \text{ev}_{Ih} \left(\left[B \mapsto \int_0^1 f(t) \, dB_t \right] \right) = \int_0^1 f(t) \, d(Ih)(t) = \int_0^1 fh \, dt.$$

Therefore $(\tau \circ \text{ev})(Ih) = \left[B \mapsto \int_0^1 h(t) \, dB_t \right]$. Hence

$$H = \{Ih : h \in L^2([0, 1])\}, \quad \langle If, Ih \rangle_H = \int_0^1 fh \, dt.$$

Remark 22.

$$\begin{array}{ccc} \underbrace{H}_{\{\text{stochastic integrands}\}} & \hookrightarrow & \underbrace{(X, \mu)}_{\{\text{samples of Gaussian noise}\}} \rightsquigarrow \underbrace{X_\mu^*}_{\{\text{RV which are stochastic integrals}\}} \end{array}$$

$$\frac{dF(\cdots \in X_\mu^*)}{d(\cdots \in H)} = \text{Malliavin calculus} = \text{\$}\text{\$}\text{\$}.$$

Lemma 23 (CMS Facts). (i) $H \xrightarrow{\text{cpt.}} X$.

(ii) If H is infinite dimensional then $\mu(H) = 0$.

(iii) $H = \bigcap_{\substack{\mu(E)=1 \\ E \text{ Borel subspace}}} E$.

(iv) Cameron-Martin theorem: Let $h \in X$.

$$\mu(\cdot + h) \ll \mu \iff h \in H \text{ and then } \frac{d\mu(\cdot + h)}{d\mu}(x) = e^{\tau(\text{ev}_h)(x) - \frac{1}{2}\|h\|_H^2}.$$

(v) If X is a Hilbert space then $H = Q_\mu^{\frac{1}{2}}(X)$ where $\langle Q_\mu f, g \rangle = \text{Cov}_\mu(f, g)$.

(vi) Let $X \xrightarrow{j} Y$ and μ be Gaussian on X . The CMS H_μ of μ and $H_{j_\# \mu}$ of $j_\# \mu$ fulfill $H_\mu \xleftarrow{j} H_{j_\# \mu}$.

Theorem 24 (R.M. Dudley, J. Feldman, L. Le Cam 1971). Let $H \hookrightarrow X$ and μ be a measure on $(X, \mathcal{B}(X))$. The following are equivalent:

(i) $H \hookrightarrow X$ is an abstract Wiener space and μ the isonormal CSPM.

(ii) μ is Gaussian and H is the CMS of μ .

6 ξ as a Hilbert space

Remark 25. Fix a separable Hilbert space H . Various abstract Wiener spaces $H \hookrightarrow X$ are just “presentations” of the same “isonormal” noise on H :

$$\{\text{Gaussian measures on separable Banach spaces}\} \begin{array}{c} \xrightarrow{\text{Cameron-Martin space}} \\ \xleftarrow{\text{abstract Wiener spaces}} \end{array} \{\text{separable Hilbert spaces}\}.$$

Therefore there are really only $\mathbb{N} + 1$ types of Gaussian noise:

$$\mathbb{R}^1, \mathbb{R}^2, \mathbb{R}^3, \dots, \ell^2(\mathbb{N}).$$

Seminar of the Work Group
Nonlinear Partial Differential Equations
Institute for Analysis
WS 24/25

Speaker: Joannis Alexopoulos
February 11th, 2025, 12:15 - 13:00
Seminar room: SR 3.061

Nonlinear dynamics of periodic Lugiato-Lefever waves against nonlocalized perturbations

Joannis Alexopoulos,
Karlsruhe Institute of Technology (KIT)

Abstract

I give an overview of nonlinear stability results of diffusively spectrally stable periodic wave trains in the Lugiato-Lefever equation. Inspired by previous works, I add a further nonlinear stability result for a class of nonlocalized perturbations. This extends and unifies existing results in a natural manner. Furthermore, I discuss some insights of the proof.

Seminar of the Work Group
Nonlinear Partial Differential Equations
Institute for Analysis
WS 24/25

Speaker: Martino Caliaro
February 11th, 2025, 11:30 - 12:15
Seminar room: SR 3.061

On the Gross-Pitaevskii flow past a delta potential in one dimension

Martino Caliaro,
Gran Sasso Science Institute (GSSI)

Abstract

In this talk we investigate the flow of a one-dimensional Gross-Pitaevskii fluid past a repulsive delta potential. For subsonic velocities of the flow at infinity, this system admits a critical value of the potential strength, below which two steady solutions exist. At the critical value of the potential strength the two solutions merge, while above it they disappear in a saddle node bifurcation. Numerical simulations suggest that, in the subcritical regime, the first of these steady solutions is stable while the second is unstable. We employ the method of the Evans function to show the linear instability of the second solution, assuming that the potential strength is small enough. The talk is based on joint work with Paolo Antonelli (GSSI).