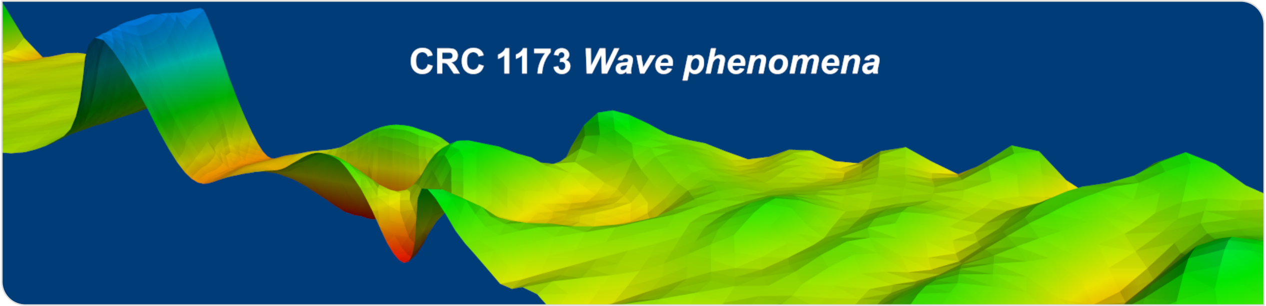
Nonlinear stability of periodic wave trains in the FitzHugh-Nagumo system against C^k_{ub}-perturbations

M.Sc. Joannis Alexopoulos | April 12th 2023



Inhaltsverzeichnis

Notation

- Given some set S and two functions $f, g : S \to \mathbb{R}$, we write

$$f \lesssim g :\Leftrightarrow \exists C > 0 : \forall x \in S : f(x) \leq Cg(x)$$

- Let $c > 0$ be a constant.

$$c \ll 1, \text{ i.e. } c \text{ shall be sufficiently small}$$

- We set

$$C^k_{\mathrm{ub}}(\mathbb{R}) := \{f \in C^k(\mathbb{R}) : \forall j = 1, ..., k : \partial_x^j f \text{ is uniformly continuous and bounded}\}$$

with norms $||u||_{C^k_{\mathrm{ub}}(\mathbb{R})} := ||u||_{W^{k,\infty}}$

A very short motivation

Localized vs. nonlocalized perturbations

Consider

$$\begin{cases} \partial_t u = \partial_x^2 u + u^4 \\ u(\cdot, 0) = u_0 \end{cases} \tag{1}$$

If $\boldsymbol{u_0} \in \boldsymbol{L^1(\mathbb{R})} \cap L^\infty(\mathbb{R})$, then:

- Origin is stable
- **Note**: $||e^{t\partial_x^2} u_0||_{L^\infty} \lesssim \frac{1}{(1+t)^{\frac{1}{2}}} ||u_0||_{L^1 \cap L^\infty}$

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If $\boldsymbol{u_0} \in \boldsymbol{L^\infty(\mathbb{R})}$, then:

- Origin is unstable
- for any $\alpha > 0$:
  $u(x, t) = -\frac{1}{(3(t-\alpha))^{\frac{1}{3}}}$ is a solution of (1) with
  $u_0 = (3\alpha)^{-\frac{1}{3}}$, **blowing up in finite time**

Another example

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Consider

$$\begin{cases} \partial_t u = \partial_x^2 u + \boldsymbol{(\partial_x u)^q} \\ u(\cdot, 0) = \boldsymbol{u_0} \in \boldsymbol{W^{1,\infty}(\mathbb{R})} \end{cases} \tag{2}$$

If $q > 2$, then:

- Origin is stable
- By iterative estimates on the Duhamel formulation of (2)

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Consider

$$\begin{cases} \partial_t u = \partial_x^2 u + \boldsymbol{(\partial_x u)^q} \\ u(\cdot, 0) = \boldsymbol{u_0 \in W^{1,\infty}}(\mathbb{R}) \end{cases} \tag{2}$$

If $q > 2$, then:

- Origin is stable
- By iterative estimates on the Duhamel formulation of (2)

If $\boldsymbol{q = 2}$, then:

- Origin is also stable
- Apply **Cole-Hopf transform**: $v = e^u - 1$ is a solution of $\partial_t v = \partial_x^2 v$

The FitzHugh-Nagumo system and goals

Equation of our interest: the FHN

The FitzHugh-Nagumo system (FHN) is given by

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \underbrace{\begin{pmatrix} \partial_x^2 u \\ 0 \end{pmatrix}}_{=:D\cdot(u,v)^T} + \underbrace{\begin{pmatrix} u(1-u)(u-a)-v \\ \varepsilon(u-\gamma v-a) \end{pmatrix}}_{=:F(u,v)} \tag{3}$$

with $x \in \mathbb{R}$, $t \geq 0$ and parameters $a \in (0, \frac{1}{2})$, $\gamma, \varepsilon > 0$.

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- Derived from the Hodgkin-Huxley model; playing an important role in neuroscience
- Prototype equation in the study of pattern formation, traveling waves and their stability

Periodic wave trains and derivation of the perturb. equation

$(u, v)^T(x, t) = \phi_0(x - ct)$ is a solution of (3)

Procedure:

- The profile $\phi_0 \in C^\infty(\mathbb{R}, \mathbb{R}^2)$ is T-periodic

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Procedure:

- The profile $\phi_0 \in C^\infty(\mathbb{R}, \mathbb{R}^2)$ is T-periodic
- Pass to co-moving frame $\zeta = x - ct$
- ϕ_0 is stationary solution
- Insert the perturbed solution

$$(u, v)^T(\zeta, t) - \phi_0(\zeta)$$

to derive the perturbation equation

The perturbation equation

The perturbation equation about ϕ_0 is given by

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \mathcal{L} \begin{pmatrix} u \\ v \end{pmatrix} + R(u, v) \tag{4}$$

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with linearization

$$\mathcal{L} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \partial_\zeta^2 u + c\partial_\zeta u \\ c\partial_\zeta v \end{pmatrix} + \begin{pmatrix} -a - 3(\phi_0)_1^2(\zeta) + 2(1-a)(\phi_0)_1(\zeta) & -1 \\ \varepsilon & -\varepsilon\gamma \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

and some remainder term R with $||R(u, v)||_{L^\infty} \lesssim ||(u, v)||_{L^\infty}^2$ as long as $||(u, v)||_{L^\infty} \leq 1$.

What we want to show

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- For mild solutions $(u, v)^T \in C([0, T), C^l_{ub}(\mathbb{R}))$ of (4), $T \in (0, \infty]$ with initial data $(u_0, v_0)^T \in C^k_{ub}(\mathbb{R})$ and $E_0 := ||(u_0, v_0)||_{W^{k,\infty}} \ll 1$, we aim to show an estimate

$$||(u, v)(t)||_{W^{l,\infty}} \lesssim (1 + t)^s ||(u_0, v_0)||_{W^{k,\infty}}, t \in [0, T), \tag{5}$$

with some suitable $s \leq 0$

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- Note that $(u,v)^T$ satisfies

$$\begin{pmatrix} u \\ v \end{pmatrix}(t) = e^{\mathcal{L}t}\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} + \int_0^t e^{\mathcal{L}(t-s)} R(u,v)(s)\, ds \quad \textbf{(Duhamel formulation)}$$

  and either blows up in finite time or exists globally

The iterative argument

- Choose a suitable template function

$$\eta(t) = \sup_{0\leq\tau\leq t} \left(||(u,v)(\tau)||_{W^{l,\infty}}(1+\tau)^s + \ldots\right)$$

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$$\eta(t) \lesssim E_0 + \eta^2(t), \tag{6}$$

implies $\eta(t) \lesssim E_0$

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$\rightsquigarrow$ **We obtain $T = \infty$ and (5)!**

$\rightsquigarrow$ **Task: show (6) for a proper choice of η!**

The linearization and estimates on the high-frequency part

Spectral assumptions

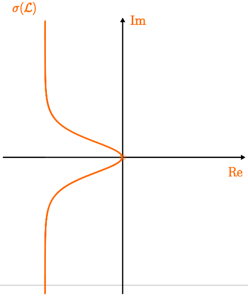
By Floquet-Bloch theory, we observe

$$\sigma(\mathcal{L}) = \bigcup_{\xi \in (-\frac{\pi}{T}, \frac{\pi}{T}]} \sigma(\mathcal{L}_\xi)$$

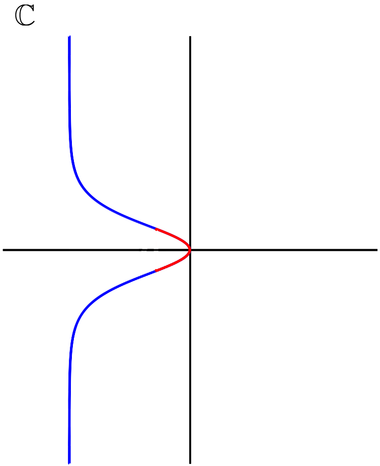
with $\mathcal{L}_\xi u = D(\partial_\zeta + i\xi)^2 u + c(\partial_\zeta + i\xi)u + F'(\phi_0)u$ posed on $L^2_{per}(0, T)$

and assume

1. $\sigma(\mathcal{L}) \subset \{\lambda \in \mathbb{C} : Re(\lambda) < 0\} \cup \{0\}$
2. $\exists \Theta > 0 : \forall \xi \in (-\frac{\pi}{T}, \frac{\pi}{T}] : \sup Re\, \sigma(\mathcal{L}_\xi) \leq -\Theta\xi^2$
3. 0 is a simple eigenvalue of $\mathcal{L}_0$ (with associated eigenfunction $\Phi_0 = \phi_0'$)

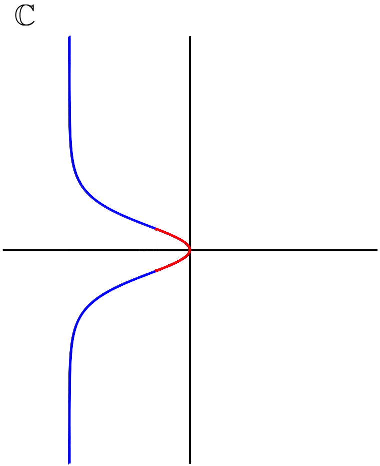


Qualitative picture of the spectrum



- High-frequency part

Qualitative picture of the spectrum



- High-frequency part
- Low-frequency part

$$\lambda_c(\xi) = ia\xi + d\xi^2 + O(|\xi|^3)$$

with $\mathcal{L}_\xi \Phi_\xi = \lambda_c(\xi)\Phi_\xi$ where

$$||\Phi_\xi - \phi_0' + i\xi F||_{H^m(0,T)} \lesssim |\xi|^2, |\xi| \ll 1$$

for some periodic $F \in C^\infty(\mathbb{R})$ and any $m \in \mathbb{N}_0$

Challenges on the linear level

The linearization $\mathcal{L}$ is non-elliptic, we consider it **on $\boldsymbol{C^k_{ub}(\mathbb{R})}$** and therefore...

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- ... standard Floquet-Bloch decomposition seems problematic to control the high-frequency part
- ... Gearhart-Prüss theorems are not available

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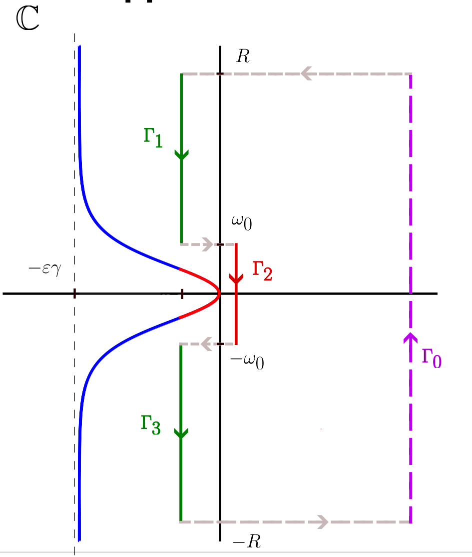
$\rightsquigarrow$ We employ a different approach than (usually) used in the literature!

Our approach towards linear estimates



- Let $g \in C^\infty(\mathbb{R}, \mathbb{R}^2)$ and write

$$-2\pi i \cdot e^{\mathcal{L}t} g = \lim_{R\to\infty} \int_{\Gamma_0} e^{\lambda t}(\mathcal{L} - \lambda)^{-1} g \, d\lambda$$

$$= -\lim_{R\to\infty} \int_{\Gamma_1 \cup \Gamma_3} e^{\lambda t}(\mathcal{L} - \lambda)^{-1} g \, d\lambda - \lim_{R\to\infty} \int_{\Gamma_2} e^{\lambda t}(\mathcal{L} - \lambda)^{-1} g \, d\lambda$$

$$=: S_e(t)g + S_c(t)g$$

- Lift $g \in C^\infty(\mathbb{R}, \mathbb{R}^2)$ to $g \in C^k_{ub}(\mathbb{R}, \mathbb{R}^2)$, $k \in \mathbb{N}_0$ properly!

Estimating the high-frequency part

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- We expect $||S_e(t)g||_{W^{k,\infty}} \lesssim e^{-\mu t}||g||_{W^{k,\infty}}$ for some $\mu > 0$

  $\rightsquigarrow$ Interpret the resolvent problem as Neumann-series

Estimating the high-frequency part

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  $\rightsquigarrow$ Interpret the resolvent problem as Neumann-series

- Exemplarily, one has to control

$$e^{\frac{-\varepsilon\gamma t}{2}} \int_{\mathbb{R}} g_2(y) G^{tr}(x-y) \int_{R \geq |\omega| > \omega_0} e^{i\frac{\omega}{c}(x-y+ct)}\, d\omega\, dy \tag{7}$$

as $R \to \infty$ with given integrable, bounded G^{tr}

Further decomposition and decay rates

Now, we arrive at the semigroup decompostion

$$S(t) = S_e(t) + (\phi_0' + F\partial_\zeta)\, S_p(t) + S_r(t)$$

with the estimates

Further decomposition and decay rates

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$$S(t) = S_e(t) + (\phi_0' + F\partial_\zeta)\, S_p(t) + S_r(t)$$

with the estimates

- $||S_e(t)g||_{W^{k,\infty}} \lesssim e^{-\mu t}||g||_{W^{k,\infty}}$
- $||S_r(t)g||_{L^\infty} \lesssim (1+t)^{-1}||g||_{L^\infty}$

and the remaining part

- $S_p(t)v(x) = \int\limits_{\mathbb{R}} \int\limits_{\Gamma_2} e^{\nu(\lambda)(x-y)+\lambda t}\tilde{\Psi}^*(y,\lambda)G(y)\, d\lambda\, dy$

The nonlinear, iterative estimates on the Duhamel formulation

Introduction of the phase modulation

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- We take translation invariance into account

We set $\tilde{w} = (u, v)$ and the inverse-modulation perturbation $w(\zeta, t) = \tilde{w}(\zeta - \psi(\zeta, t), t) - \phi_0(\zeta)$ satisfies

$$(\partial_t - \mathcal{L})(w + \phi_0' \psi) = \mathcal{N}(w, \psi, \partial_t \psi) + (\partial_t - \mathcal{L})(w \partial_\zeta \psi) \tag{8}$$

which suggest the choice:

$$\psi(t) := S_p(t) w_0 + \int_0^t S_p(t-s) \mathcal{N}(w(s), \psi(s), \partial_t \psi(s)) \, ds$$

Challenges on the nonlinear level

Control of ψ:

- Slow decay rates due to lack of localization
- $\mathcal{N}$ contains terms of the form $(\partial_\zeta \psi)^2$ as worst behaviour which cannot be controlled through iterative estimates on the Duhamel formula

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Control of *w*:

- Damping in the second component does not come from highest-order derivative
- High-frequency part is not smoothing $\rightarrow$ How to control derivatives?

  $\rightsquigarrow$ Regularity control of the quasilinear equation is difficult!

Controling ψ

- Recall: N contains only derivatives of ψ
- Application of generalized Cole-Hopf transform resolves this problem

Controling the derivatives of w

- The forward-modulated perturbation

$$\mathring{w}(\zeta, t) = \tilde{w}(\zeta, t) - \phi_0(\zeta + \psi(\zeta, t), t)$$

  satisfies a semilinear equation
- Show a damping estimate for $\mathring{w}$ and relate w and $\mathring{w}$; assuming that $|\partial_\zeta \psi| \ll 1$

Controling the derivatives of w

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$$\mathring{w}(\zeta,t) = \tilde{w}(\zeta,t) - \phi_0(\zeta + \psi(\zeta,t),t)$$

satisfies a semilinear equation
- Show a damping estimate for $\mathring{w}$ and relate w and $\mathring{w}$; assuming that $|\partial_\zeta \psi| \ll 1$
- We arrive at, for some $\kappa > 0$,

$$||\partial_\zeta^k w(t)||_{L^\infty} \lesssim e^{-\kappa t}||w_0||_{W^{k,\infty}} + \int_0^t e^{-\kappa(t-s)} \left(||(w(s))_1||_{L^\infty}^2 + ||(\partial_t\psi, \partial_\zeta\psi)||^2_{W^{k+2,\infty}\times W^{k+1,\infty}}\right) ds, k \geq 1$$

$\rightsquigarrow$ Bound derivatives of w !