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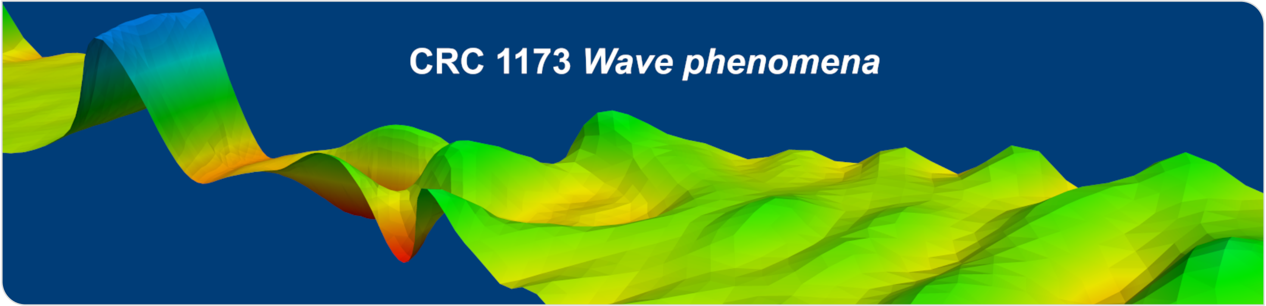
Wave phenomena

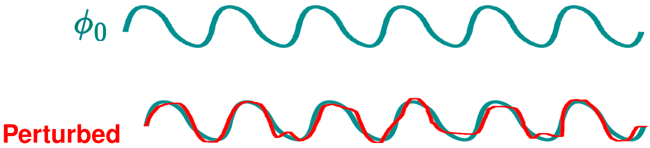
Nonlinear stability of periodic Lugiato-Lefever waves against $H^k_{\mathrm{per}} \oplus H^l$-perturbations

M.Sc. Joannis Alexopoulos | March 31st 2025



PhD project

- We develop nonlinear stability theory for periodic waves against nonlocalized perturbations



- Results for large class of pattern-forming systems (**fully nonlocalized**)
  - Reaction diffusion systems [R ’24]
  - FitzHugh-Nagumo system (AR ’24)
- Developed within a $C_{\mathrm{ub}}(\mathbb{R})$-framework

Towards extensions

There are **pattern-forming systems** obstructing an application of our $C_{\mathrm{ub}}(\mathbb{R})$-theory:

System of viscous conservation laws

- Multiple critical modes [JNRZ '14]

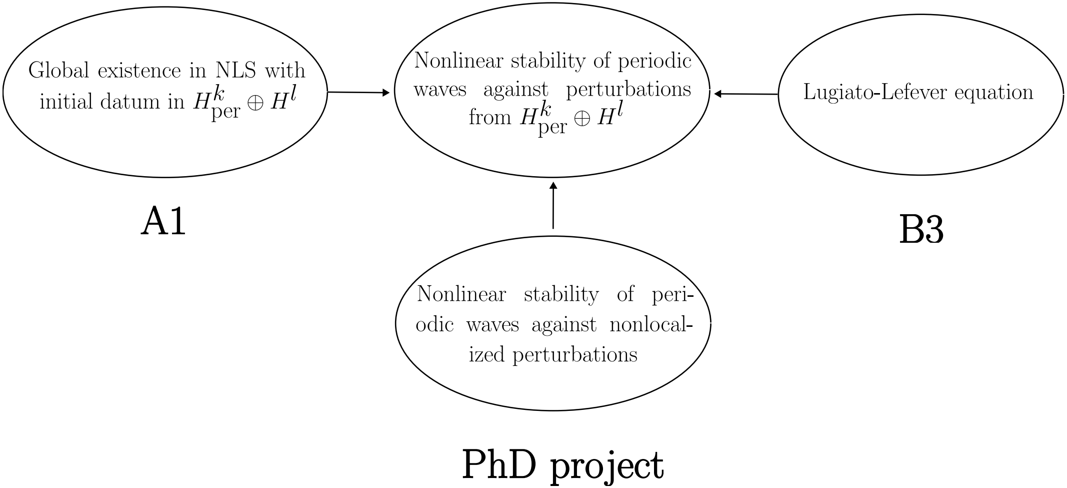
Lugiato-Lefever equation

- Local well-posedness fails in $C_{\mathrm{ub}}(\mathbb{R})$-spaces [BPSS '14]

$\rightsquigarrow$ **In this talk:** alternative class of **nonlocalized perturbations** from space $H^k_{\mathrm{per}}(0, T) \oplus H^l(\mathbb{R})$

$\rightsquigarrow$ Arise in applications due to combination of co-periodic and localized effects

Embedding into CRC



The Lugiato-Lefever equation

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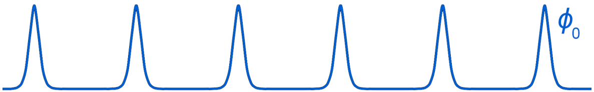
We study the **Lugiato-Lefever equation** on the extended real line

$$\partial_t u = -\beta i \partial_x^2 u - (1 + i\alpha) u + i|u|^2 u + F, \quad \beta \in \{-1, 1\}, \quad \alpha \in \mathbb{R}, \quad F > 0, \tag{1}$$

for $u : \mathbb{R} \times [0, \infty) \to \mathbb{C}$

Assumptions:

- There exists a smooth, nonconstant and T-periodic stationary solution $\phi_0 : \mathbb{R} \to \mathbb{C}$ of (1)



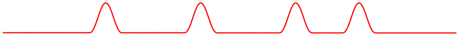
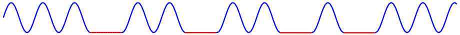
(thanks to Björn)

- ϕ_0 is **diffusively spectrally stable** (to be illustrated)

See [HD '18], (BR '25)

Short overview

Let $u(t)$ be a solution of (1) with initial datum u_0

	Small initial perturbation $u_0 - \phi_0$
[SS '18], L^2_{per}	
[HJPR '22], L^2	
(A '25)*, $L^2_{\text{per}} \oplus L^2$	

*: in preparation

Nonlinear stability:

There exists $C > 0$ such that:

If $\|u_0 - \phi_0\|$ is small, then

$$\|u(t) - \phi_0\| \leq C\|u_0 - \phi_0\|$$

for all $t \geq 0$

⇜ "Knocked out teeth" (**A1**)

Reformulation

Setting $\mathbf{u} = (\mathrm{Re}(u), \mathrm{Im}(u))^T : \mathbb{R} \to \mathbb{R}^2$, we transform (1) into the **real system**

$$\mathbf{u}_t = \mathcal{J}\left(\begin{pmatrix}-\beta & 0\\ 0 & -\beta\end{pmatrix}\mathbf{u}_{xx} - \begin{pmatrix}\alpha & 0\\ 0 & \alpha\end{pmatrix}\mathbf{u}\right) - \mathbf{u} + \mathcal{N}(\mathbf{u}) + \begin{pmatrix}F\\ 0\end{pmatrix} \tag{2}$$

where

$$\mathcal{J} = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \quad \mathcal{N}(\mathbf{u}) = |\mathbf{u}|^2\mathcal{J}\mathbf{u}$$

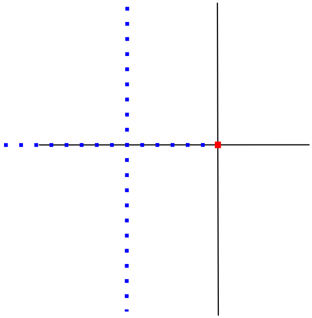
- $\phi = (\mathrm{Re}(\phi_0), \mathrm{Im}(\phi_0))^T : \mathbb{R} \to \mathbb{R}^2$ is T-periodic **stationary solution** of (2)
- **Linearization about** ϕ is given by

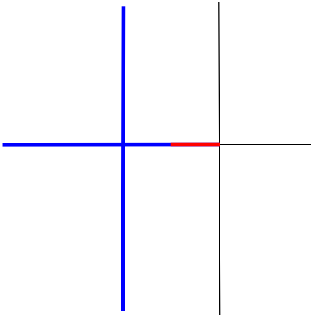
$$\mathcal{L}_0 = \mathcal{J}\begin{pmatrix}-\beta\partial_x^2 - \alpha + 3\phi_1^2 + \phi_2^2 & 2\phi_1\phi_2\\ 2\phi_1\phi_2 & -\beta\partial_x^2 - \alpha + \phi_1^2 + 3\phi_2^2\end{pmatrix} - \mathcal{I}$$

Diffusive spectral stablility

$\sigma_{L^2_{\mathrm{per}}(0,T)}(\mathcal{L}_0)$ $\sigma_{L^2}(\mathcal{L}_0)$

[HD ’18], (BR ’25):





⇝ How to come from spectral properties to nonlinear stability?

Main result

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Theorem (A '25)

Assume ϕ_0 is diffusively spectrally stable. Then, there exist constants $C, \varepsilon > 0$ such that for initial data $\mathbf{w}_0 \in H^6_{per}(0,T)$ and $\mathbf{v}_0 \in H^3(\mathbb{R})$ with

$$E_0 := \|\mathbf{w}_0 + \mathbf{v}_0\|_{H^6_{per}(0,T)\oplus H^3(\mathbb{R})} < \varepsilon$$

there exists a unique classical solution

$$\mathbf{u}(t) \in C([0,\infty); H^6_{per}(0,T) \oplus H^3(\mathbb{R})) \cap C^1([0,\infty); H^4_{per}(0,T) \oplus H^1(\mathbb{R}))$$

of (2) with initial condition $\mathbf{u}(0) = \phi + \mathbf{w}_0 + \mathbf{v}_0$ such that

$$\|\mathbf{u}(t) - \phi\|_{W^{2,\infty}} \leq CE_0, \quad \textit{for all } t \geq 0.$$

Unmodulated perturbation equations

Inspired by [KK '22]:

- Set $\mathbf{u}(t) = \mathbf{w}(t) + \mathbf{v}(t) + \phi$
- This gives the **coupled perturbation system**

$$\mathbf{w}_t = \mathcal{L}_0\mathbf{w} + \mathcal{R}_1(\mathbf{w})$$
$$\mathbf{w}(0) = \mathbf{w}_0 \in H^k_{\text{per}}(0, T),$$

$$\mathbf{v}_t = \mathcal{L}_0\mathbf{v} + \mathcal{R}_2(\mathbf{w}, \mathbf{v})$$
$$\mathbf{v}(0) = \mathbf{v}_0 \in H^l(\mathbb{R}),$$

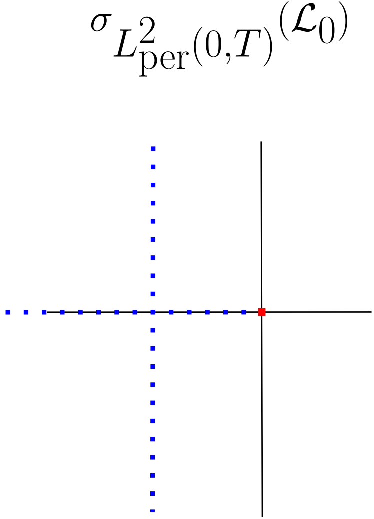
where $\mathcal{R}_2(\tilde{\mathbf{w}}, \mathbf{v}) = \mathcal{R}_{2,1}(\mathbf{w}, \mathbf{v}) + \mathcal{R}_{2,2}(\mathbf{w}, \mathbf{v})$, with **nonlinear bounds**

$$|\mathcal{R}_1(\mathbf{w})| \leq C|\mathbf{w}|^2$$

$$|\mathcal{R}_{2,1}(\mathbf{w}, \mathbf{v})| \leq C|\mathbf{v}|^2,$$
$$|\mathcal{R}_{2,2}(\mathbf{w}, \mathbf{v})| \leq C|\mathbf{v}||\mathbf{w}|$$

whenever $|\mathbf{v}|, |\mathbf{w}| \leq 1$

Linear decomposition in $L^2_{per}(0, T)$



Set

$$\tilde{S}_1(t)\mathbf{g} = (e^{\mathcal{L}_0 t} - \Pi)\mathbf{g},$$

where Π is the spectral projection of $\mathcal{L}_0$ onto its translational eigenspace $\mathrm{span}\{\phi'\}$,

and estimate

$$\|\tilde{S}_1(t)\mathbf{g}\|_{H^1_{\mathrm{per}}(0,T)} \leq C e^{-\delta_0 t} \|\mathbf{g}\|_{H^1_{\mathrm{per}}(0,T)}$$

See [SS '18]

Linear decomposition in $L^2(\mathbb{R})$

Split

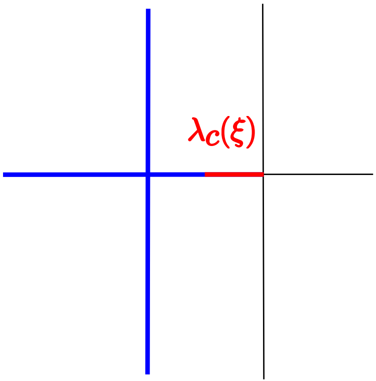
$$e^{\mathcal{L}_0 t}\mathbf{g} = \tilde{S}_2(t)\mathbf{g} + \phi' s_p(t)\mathbf{g}$$

- The principal part $s_p(t)$ decays as $e^{\partial_x^2 t}$
- The residual part $\tilde{S}_2(t)$ decays exponentially (but is not smoothing)

See [HJPR '22], [HJPR'24]

Notice: $\lambda_c(0) = 0, \lambda_c'(0) = 0$ and $\lambda_c''(0) < 0$

$\sigma_{L^2}(\mathcal{L}_0)$



Strategy

- First modulate **w** and then **v** with suitable modulation functions
- Exploit that the modulated perturbation of **w** obeys exponential decay measured in $H^k_{per}(0, T)$
- Bound the mixed nonlinearity in L^2 by $\|\tilde{\mathbf{w}}\|_{L^\infty}$ and $\|\mathbf{v}\|_{L^2}$
- Establish nonlinear damping estimate to control regularity

A toy example

Consider

$$\partial_t w = i\partial_x^2 w - w + w^2$$
$$\partial_t v = \partial_x^2 v + (\partial_x v)(w + \partial_x v)$$

with $w(0) = w_0 \in H^1_{\mathrm{per}}(0,1)$ and $v(0) = v_0 \in H^1(\mathbb{R})$

Strategy:

1. Show first that $\|w(t)\|_{H^1_{\mathrm{per}}(0,1)} \leq Ce^{-t}\|w_0\|_{H^1_{\mathrm{per}}(0,1)}$
2. Then $\|\partial_x v(t)\|_{L^2} \leq C(1+t)^{-\frac{1}{2}}\|v_0\|_{H^1}$
3. Conclude $\|\partial_x v(t)\|_{L^\infty} \leq C(1+t)^{-\frac{3}{4}}\|v_0\|_{H^1}$

Key observation for (2)

Set

$$\eta(t) = \sup_{0\leq s\leq t} (1+s)^{\frac{1}{2}} \|\partial_x v(s)\|_{L^2}.$$

Let $0 \leq s \leq t$. We have the **Duhamel formula**

$$\partial_x v(s) = \partial_x e^{\partial_x^2 s} v_0 + \int_0^t \partial_x e^{\partial_x^2 (t-s)} (\partial_x v(s))^2 \, ds + \int_0^t \partial_x e^{\partial_x^2 (t-s)} w(s) \partial_x v(s) \, ds$$

and estimate

$$\begin{aligned} \| \int_0^t \partial_x e^{\partial_x^2 (t-s)} w(s) \partial_x v(s) \, ds \|_{L^2} &\lesssim \int_0^t \|\partial_x e^{\partial_x^2 (t-s)}|_{L^2} \|w(s)\|_{L^\infty} \|\partial_x v(s)\|_{L^2} \, ds \\ &\lesssim \eta(t) \|w_0\|_{H^1_{\mathrm{per}}(0,1)} \int_0^t \frac{1}{(t-s)^{\frac{1}{2}}} e^{-s} (1+s)^{-\frac{1}{2}} \, ds \lesssim \eta(t)(1+t)^{-\frac{1}{2}} \|w_0\|_{H^1_{\mathrm{per}}(0,1)}. \end{aligned}$$

Outlook

- Apply scheme to system of viscous conservation laws [JNRZ '14]
- Related to **project A1**:

  Can we prove nonlinear stability against perturbations from the modulation space $M^m_{\infty,1}(\mathbb{R})$ for m large enough?

  - Notice: $C^\infty(\mathbb{R}) \subset M^m_{\infty,1}(\mathbb{R})$ while $C^\infty(\mathbb{R}) \not\subset H^k_{\text{per}}(0,T) \oplus H^l(\mathbb{R})$