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Hyperbolic relaxation systems: Chapman-Enskog

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Hyperbolic relaxation systems

We investigate **hyperbolic relaxation systems**

$$\begin{aligned} u_t &= f(u,v)_x, \\ v_t &= g(u,v)_x + q(u,v), \quad (x,t) \in \mathbb{R} \times [0,\infty), \end{aligned}$$

with $u, f \in \mathbb{R}^n$, $v, g, q \in \mathbb{R}^r$, and

- a smooth equilibrium manifold $\{u, v^*(u)\} \subset \mathbb{R}^n \times \mathbb{R}^r$, i.e., $q(u, v^*(u)) = 0$
- $\sigma(q_v(u, v^*(u))) \subset \{\lambda : \mathrm{Re}\lambda < 0\}$.

We note that for $f^*(u) = f(u, v^*(u))$, we obtain the **equilibrium system**

$$u_t = f^*(u)_x.$$

What to expect towards stability close to equilibria: **Chapman-Enskog-type expansion**.

Chapman-Enskog view I

We follow Liu 1987 and assume a **slightly simpler situation** (triangle form). Let $n = r = 1$, $(u_0, v_0) \in \mathbb{R}^2$ an equilibrium with

$$\frac{\partial(f,g)}{\partial(u,v)}(u_0,v_0) = \begin{pmatrix} f_u(u_0,v_0) & f_v(u_0,v_0) \\ 0 & g_v(u_0,v_0) \end{pmatrix} =: \begin{pmatrix} \lambda_1 & f_v(u_0,v_0) \\ 0 & \lambda_2 \end{pmatrix}$$

with $v_u^*(u_0) \neq 0$. We write $\lambda^*(u) = \partial_u f^*(u)$, assume **strict hyperbolicity**, i.e., $\lambda_2 > \lambda_1$.

First order expansion: $\partial_t u = \lambda^*(u_0) u_x$.

Second order expansion: We introduce $v_1 + v^*(u) = v$ and insert into the v-equation

$$\partial_t(v^*(u) + v_1) - \partial_x g(u, v^*(u) + v_1) = q(u,v) - q(u, v^*(u)).$$

Ignoring "faster decaying terms" and with Taylor, we have

$$\partial_t v^*(u) - \partial_x g(u, v^*(u)) = q_v(u, v^*(u)) v_1.$$

Chapman-Enskog view II

Remembering

$$\partial_t v^*(u) - \partial_x g(u, v^*(u)) = q_v(u, v^*(u))v_1,$$

we use equilbrium equation to arrive at

$$v_1 = \underbrace{q_v^*(u)^{-1}(\lambda^*(u)v_u^*(u) - g_u^*(u) - g_v^*(u)v_u^*(u))}_{=:\alpha(u)} u_x$$

using the notations $g^*(u) = g(u, v^*(u))$...

We insert this into the u-equation, that is,

$$\partial_t u - \partial_x f(u, v^*(u) + \alpha(u)u_x) = 0.$$

With Taylor, we find the **second order equation**

$$\partial_t u - \partial_x f^*(u) - \partial_x(f_v^*(u)\alpha(u)u_x) = 0.$$

Chapman-Enskog view III

Remembering

$$\partial_t u - \partial_x f^*(u) - \partial_x(f_v^*(u)\alpha(u)u_x) = 0, \quad \beta(u) := f_v^*(u)\alpha(u)$$

and observing

$$f_v^*(u) = v_u^*(u)^{-1}(\lambda^*(u) - f_u^*(u)),$$

we determine

$$\begin{aligned}\beta(u_0) &= v_u^*(u_0)^{-1}(\lambda^*(u_0) - \lambda_1)v_u^*(u_0)q_v^*(u_0)^{-1}(\lambda^*(u_0) - \lambda_2)\\ &= q_v^*(u_0)^{-1}(\lambda^*(u_0) - \lambda_1)(\lambda^*(u_0) - \lambda_2).\end{aligned}$$

We want $\beta(u_0) > 0$, therefore, our **stability condition** for (u_0, v_0) reads

$$\lambda_1 < \lambda^*(u_0) < \lambda_2.$$

Note: since $\lambda^*(u_0) = f_u(u_0, v_0) + v_u^*(u_0)f_v(u_0, v_0)$, the nontrivial coupling $f_v(u_0, v_0) \neq 0$ is necessary to guarantee the stability condition.

Excercise: nonlinear stability of equilibria and smooth patterns close to equilibria under this condition?

Example: inviscid St. Venant equations

The **inviscid St. Venant equations** are given by

$$\partial_t h + \partial_x q = 0,$$
$$\partial_t q + \partial_x \left(\frac{q^2}{h} + \frac{h^2}{2F^2} \right) = h - \frac{|q|q}{h^2}$$

and the equilibrium equation reads

$$\partial_t h = q^*(h), \quad q^*(h) = h^{\frac{3}{2}}.$$

Many things may be made explicitly, see Yang and Zumbrun 2020, Sukhtayev, Yang, and Zumbrun 2020. Regimes:

- $F < 2$: all equilibria satisfy the stability condition (exercise).
- $F > 2$ comes with more complexity.

Literature

[1] Tai-Ping Liu. “Hyperbolic conservation laws with relaxation”. In: *Comm. Math. Phys.* 108.1 (1987), pp. 153–175. ISSN: 0010-3616,1432-0916. URL: http://projecteuclid.org/euclid.cmp/1104116362.

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