Dynamics of periodic wave trains under large and fully nonlocalized modulations

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Introduction

Let $n \in \mathbb{N}$. We are interested in **reaction-diffusion systems**

$$\partial_t u = D\partial_x^2 u + f(u) \qquad (x,t) \in \mathbb{R} \times [0,\infty) \tag{1}$$

with $D \in \mathbb{R}^{n\times n}$ positive definite and smooth nonlinearity $f : \mathbb{R}^n \to \mathbb{R}^n$.

- Let $\phi_0 : \mathbb{R} \to \mathbb{R}^n$ be a 1-periodic, smooth profile such that

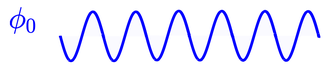
$$(x,t) \mapsto \phi_0(k_0 x - \omega_0 t)$$

is a periodic traveling-wave (wave train) solution of (1) with **wavenumber** $k_0 \in \mathbb{R}\setminus\{0\}$ and **wavespeed** $\frac{\omega_0}{k_0} \in \mathbb{R}$.

Switching to the co-moving frame $\zeta = k_0 x - \omega_0 t$ yields

$$\partial_t u = k_0^2 D\partial_\zeta^2 u + \omega_0 \partial_\zeta u + f(u) \qquad (\zeta,t) \in \mathbb{R}\times[0,\infty). \tag{2}$$

$\rightsquigarrow \phi_0$ is a stationary solution of (2).



Linearization and spectral assumptions

- Measuring the deviation $\check{v} = u - \phi_0$, we obtain the perturbation equation

$$\partial_t \check{v} = \mathcal{L}\check{v} + N(\check{v}) \text{ with } N(\check{v}) = f(\phi_0+\check{v}) - f(\phi_0) - f'(\phi_0)\check{v},$$

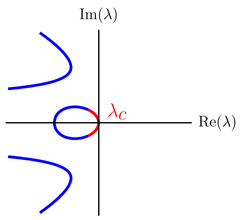
where $\mathcal{L}$ is the linearization of (2) about ϕ_0 posed on $C_{\text{ub}}(\mathbb{R})$ given by

$$\mathcal{L}\check{v} = k_0^2 D\partial_\zeta^2\check{v} + \omega_0\partial_\zeta\check{v} + f'(\phi_0)\check{v}.$$

- Let $\mathcal{L}_\xi = e^{-i\xi\cdot}\mathcal{L}e^{i\xi\cdot}$ be the Bloch operators posed on $L^2_{\text{per}}(0,1)$. It holds

$$\sigma(\mathcal{L}) = \bigcup_{\xi\in[-\pi,\pi)} \sigma(\mathcal{L}_\xi).$$

- Assume that ϕ_0 is **diffusively spectrally stable**, [9,10]. That is,



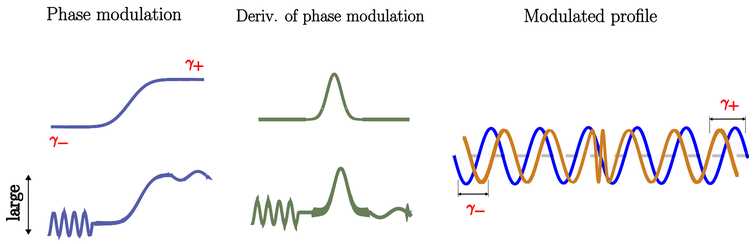
- 0 is a simple eigenvalue of $\mathcal{L}_0$ and $\ker(\mathcal{L}_0) = \text{span}\{\phi_0'\}$.
- The critical spectral curve obeys the expansion

$$\lambda_c(\xi) = ia\xi - d\xi^2 + O(|\xi|^3) \in \sigma(\mathcal{L}_\xi)$$

with $a \in \mathbb{R}$ and $d > 0$.
- $\sigma(\mathcal{L})\setminus\{0\}$ is confined to the left-half plane.

Our result

Initial setup:



- Let u be a solution of (2) with $u(0) = u_0$.
- Let $\gamma_0 : \mathbb{R} \to \mathbb{R}$ be a bounded initial modulation.
- Now, $\phi_0(\cdot + \gamma_0(\cdot))$ is a modulated wave train.

We write $E_0 := ||\underbrace{u_0 - \phi_0(\cdot+\gamma_0(\cdot))}_{=:v_0}||_{L^\infty} + ||\gamma_0'||_{L^\infty}$.

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Our result [1] (modulational stability): For small E_0, we find a phase modulation

$$\gamma : \mathbb{R}\times[0,\infty) \to \mathbb{R} \text{ with } \gamma(\cdot,0) = \gamma_0$$

such that

$$||u(\cdot,t) - \phi_0(\cdot+\gamma(\cdot,t))||_{L^\infty} \lesssim_\alpha \frac{E_0^\alpha}{(1+t)^{\frac{1}{2}-\alpha}}, \quad t \geq 0, \tag{3}$$

for every $\alpha \in [0, \frac{1}{6})$.

Furthermore, there exists a solution $\check{\gamma}(t)$ to

$$\partial_t\check{\gamma} = d\partial_\zeta^2\check{\gamma} + a\partial_\zeta\check{\gamma} + \nu(\partial_\zeta\check{\gamma})^2 \text{ for some } \nu \in \mathbb{R}, \tag{4}$$

with $\check{\gamma}(0) = \gamma_0$, such that

$$||u(\cdot,t) - \phi_0(\cdot+\check{\gamma}(\cdot,t))||_{L^\infty} \lesssim_\alpha E_0^{\frac{2}{3}\alpha}, \quad t \geq 0.$$

Heuristic: 'solutions stay close and converge to modulated wave train'.

- Our method **extends to general semilinear dissipative systems** using techniques from [2].
- Our result **lifts all localization requirements** on γ_0' and v_0 present in the literature [10,8,7].
- In particular, all previous results require $\gamma_0' \in L^1(\mathbb{R})$ and consequently $\gamma_0(x) \to \gamma_\pm$ as $x \to \pm\infty$.
- In case $\gamma_0(x) \to \gamma_\pm$ as $x \to \pm\infty$, we call $\gamma_d := \gamma_+ - \gamma_-$ the phase-offset and 'γ_- and γ_+ mix through the diffusion equation (4)'.

Semigroup decomposition

We decompose

$$e^{\mathcal{L}t} = (\phi_0' + \partial_k\phi(\cdot;k_0)\partial_\zeta)s_p(t) + S_r(t) + S_e(t),$$

where $\phi(\cdot;k)$ is the continuation of ϕ_0 w.r.t. the wavenumber k s.t. $\phi(\cdot;k_0) = \phi_0$, and

$$||S_r(t)g||_{L^\infty} \lesssim (1+t)^{-1}||g||_{L^\infty} \text{ and } ||S_e(t)g||_{L^\infty} \lesssim e^{-\mu t}||g||_{L^\infty}.$$

Given $0 \leq \alpha < \frac{1}{6}$, we establish the key estimate

$$||\partial_\zeta s_p(t)(\phi_0'\gamma_0)||_{L^\infty} \lesssim \min\left\{E_0, \frac{||\gamma_0||_{L^\infty}}{(1+t)^{\frac{1}{2}}}\right\} \lesssim_{\alpha,||\gamma_0||_{L^\infty}} \frac{E_0^\alpha}{(1+t)^{\frac{1}{2}-\alpha}}.$$

- **Rule of thumb**: trade derivatives on $s_p(t)$ for temporal decay.

Critical nonlinear contribution

Introduce the inverse-modulated perturbation, c.f. [4,5,8,9],

$$\widetilde{v}(\zeta,t) = u(\zeta+\gamma(\zeta,t),t) - \phi_0(\zeta).$$

- $\widetilde{v}$ and γ satisfy a **quasilinear equation** (in $\widetilde{v}$)

$$(\partial_t - \mathcal{L})(\widetilde{v} + \phi_0'\gamma) = \widetilde{N}(\widetilde{v},\widetilde{v}_\zeta,\widetilde{v}_{\zeta\zeta},\gamma_\zeta,\gamma_t)$$

and thus we choose, inspired by [8],

$$\gamma(t) = s_p(t)(\widetilde{v}(\cdot,0) + \phi_0'\gamma_0) + (1-\chi(t))(\gamma_0 - s_p(t)(\widetilde{v}(\cdot,0)+\phi_0'\gamma_0)) + \int_0^t s_p(t-s)\widetilde{N}(\widetilde{v},\widetilde{v}_\zeta,\widetilde{v}_{\zeta\zeta},\gamma_\zeta,\gamma_t)(s)\,ds$$

with a smooth cut-off function $\chi : \mathbb{R} \to [0,1]$ having support in $[1,\infty)$ such that $\chi \equiv 1$ on $(2,\infty)$, so that $\gamma(0) = \gamma_0$.

- Choice of γ yields a **perturbed viscous Hamilton-Jacobi equation**

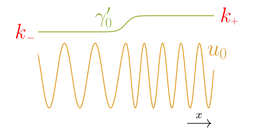
$$\partial_t\gamma = d\partial_\zeta^2\gamma + a\partial_\zeta\gamma + \nu(\partial_\zeta\gamma)^2 + h.o.t \text{ with some } \nu \in \mathbb{R}.$$

- Note: γ appears only as derivatives w.r.t. ζ or t in the nonlinearities!

Outlook: wave number modulations

- We consider initial data of the form

$$u_0(x) = \phi(k_0 x + \gamma_0(x); k_0 + \gamma_0'(x)) + v_0,$$
$$\lim_{x\to\pm\infty}\gamma_0'(x) = k_\pm, \; ||\gamma_0'||_{L^\infty}, ||v_0||_{L^\infty} \ll 1.$$



- **Aim**: show similar estimate as (3) replacing $\phi_0(\cdot+\gamma(\cdot,t))$ by $\phi(\cdot+\gamma(\cdot,t);k_0+\gamma'(\cdot,t))$.
- For the real Ginzburg-Landau equation resolved in [3,6].

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*PhD-Thesis