Nonlinear stability of periodic waves in the FHN system in a pure L^∞-framework

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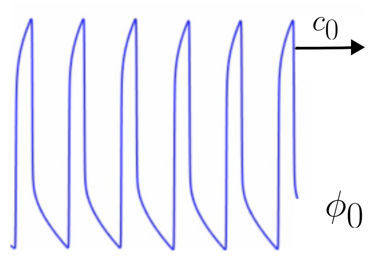
Introduction

We consider the FitzHugh-Nagumo system (FHN)

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} u_{xx} \\ 0 \end{pmatrix} + \underbrace{\begin{pmatrix} u(1-u)(u-\mu) - v \\ \varepsilon(u - \delta v - \mu) \end{pmatrix}}_{=:F(u,v)} \qquad (1)$$

with $x \in \mathbb{R}, t \geq 0$ and parameters $\mu \in \mathbb{R}$ and $\delta, \varepsilon > 0$.

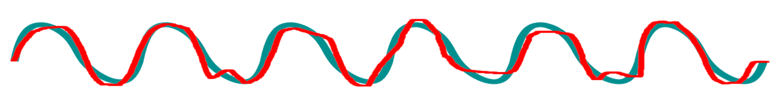
- Simplification of the Hodgkin-Huxley model for nerve propagation [9].
- Paradigm model for pattern formation:



- (1) admits a family of wave trains $\phi_0(x - c_0 t)$ with $c_0 \neq 0$ and $\phi_0 : \mathbb{R} \to \mathbb{R}^2$ smooth and T-**periodic**, [2,11].

Aim:

Show nonlinear stability of ϕ_0 against C_{ub}-perturbations. That is, lift any localization or periodicity requirement on perturbations.



$\rightsquigarrow$ We go beyond earlier nonlinear stability analyses against C_{ub}-perturbations relying on parabolic smoothing properties, [3,6].

- Switch to co-moving frame $\zeta = x - c_0 t$:

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = D\begin{pmatrix} u_{\zeta\zeta} \\ v_{\zeta\zeta} \end{pmatrix} + c_0 \begin{pmatrix} u_\zeta \\ v_\zeta \end{pmatrix} + F(u,v), \qquad D := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad (2)$$

$\rightsquigarrow$ ϕ_0 is a stationary solution of (2).

- Let $(u,v)^T$ be a solution of (2). Measuring the deviation $w := (u,v)^T - \phi_0$, we obtain the perturbation equation

$$w_t = \mathcal{L}w + N(w) \qquad (3)$$

where $\mathcal{L}$ is the linearization about ϕ_0 posed on $C_{\text{ub}}(\mathbb{R}) \times C_{\text{ub}}(\mathbb{R})$ with dense domain $C^2_{\text{ub}}(\mathbb{R}) \times C^1_{\text{ub}}(\mathbb{R})$ given by

$$\mathcal{L}w = Dw_{\zeta\zeta} + c_0 w_\zeta + F'(\phi_0)w.$$

Main result:

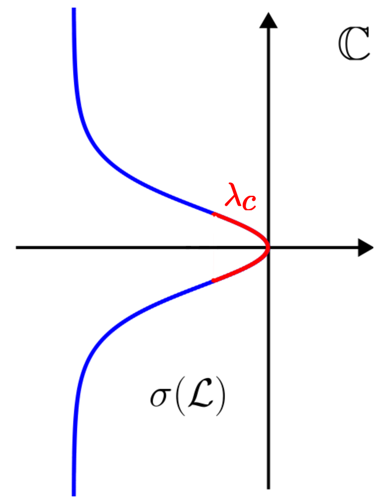
$\exists \epsilon_0, C_0 > 0 : \forall w_0 \in C^3_{\text{ub}}(\mathbb{R}) \times C^2_{\text{ub}}(\mathbb{R})$ with $E_0 := ||w_0||_{W^{3,\infty} \times W^{2,\infty}} < \epsilon_0$
the solution w of (3) with $w(0) = w_0$ is global and satisfies
$\forall t \geq 0 : ||w(t)||_{W^{2,\infty} \times W^{1,\infty}} \leq C_0 E_0$

Spectral Preliminaries

- Let $\mathcal{L}_\xi = e^{-i\xi\cdot}\mathcal{L}e^{i\xi\cdot}$ be the Bloch operators posed on $L^2_{per}(0,T)$. It holds

$$\sigma(\mathcal{L}) = \bigcup_{\xi \in [-\frac{\pi}{T}, \frac{\pi}{T})} \sigma(\mathcal{L}_\xi).$$

- Assume that ϕ_0 is **diffusively spectrally stable**, [4]. That is,



- 0 is a simple eigenvalue of $\mathcal{L}_0$ and $\ker(\mathcal{L}_0) = \text{span}\{\phi_0'\}$.
- The critical spectral curve obeys the expansion:

$$\lambda_c(\xi) = ia\xi - d\xi^2 + O(|\xi|^3) \in \sigma(\mathcal{L}_\xi)$$

with $a \neq 0$ and $d > 0$.

Linear Estimates

Approach: Inverse Laplace transform and resolvent analysis, [1,8].

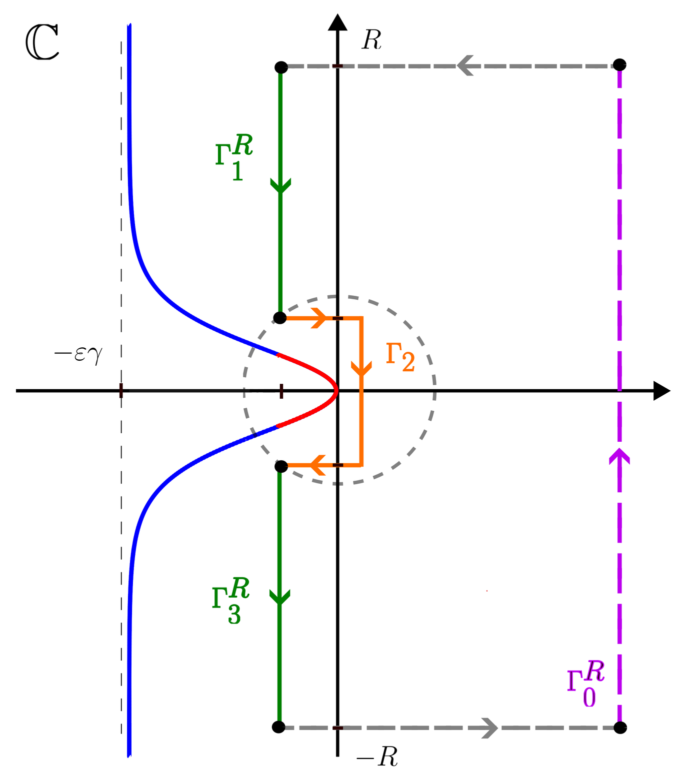
For $f \in C^\infty(\mathbb{R})$ and for $t > 1$,

$$e^{\mathcal{L}t} f = \lim_{R\to\infty} \frac{1}{2\pi i} \int_{\Gamma_0^R} e^{\lambda t}(\lambda - \mathcal{L})^{-1} f \, d\lambda$$
$$= -(S_e(t) + S_c(t))f$$

where (S_e: high-, S_c: low-frequency part)

$$S_e(t)f = \lim_{R\to\infty} \frac{1}{2\pi i} \int_{\Gamma_1^R \cup \Gamma_3^R} e^{\lambda t}(\lambda - \mathcal{L})^{-1} f \, d\lambda,$$

$$S_c(t)f = \frac{1}{2\pi i} \int_{\Gamma_2} e^{\lambda t}(\lambda - \mathcal{L})^{-1} f \, d\lambda.$$

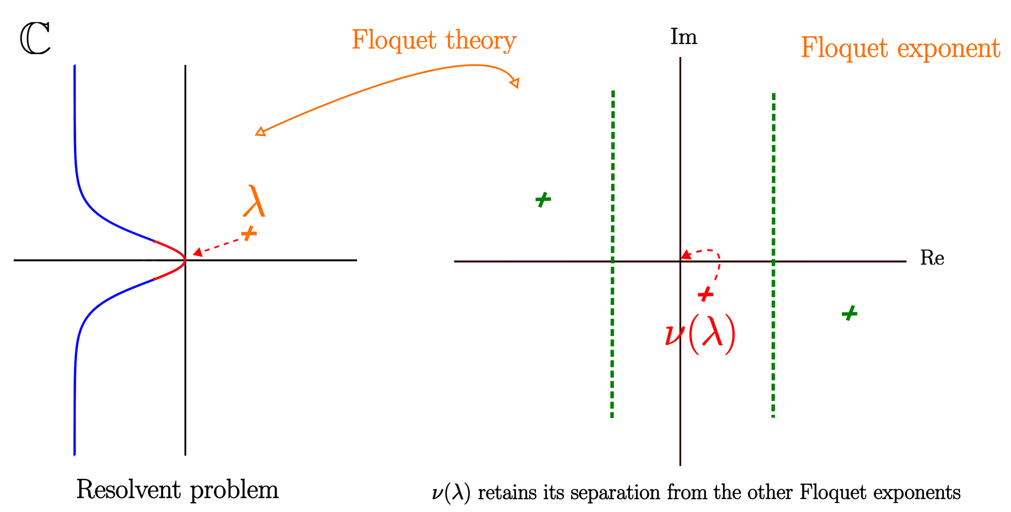


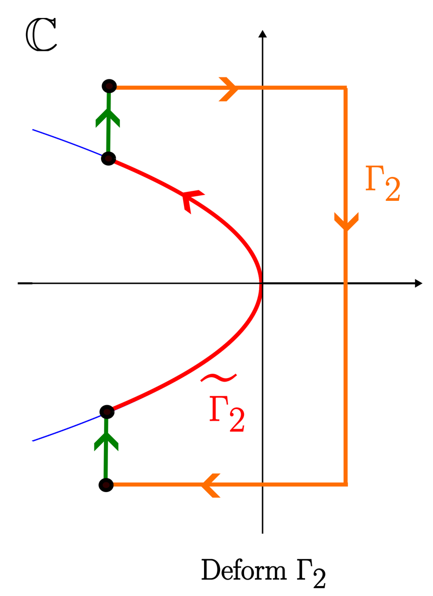
1st Challenge: $\mathcal{L}$ is not sectorial $\rightsquigarrow$ How to control S_e?

Our solution:
- Neumann series expansion of resolvent for $|\text{Im}(\lambda)| \gg 1$.
- Critical terms can, via complex inversion formula, be identified as **convolutions of simpler C_0-semigroups**.
- Obtain $||S_e(t)||_{L^\infty \to L^\infty} \leq C_1 e^{-\alpha t}$ for some $\alpha, C_1 > 0$.

2nd Challenge: How to establish decay on S_c?

Our solution: Relate the inverse Laplace representation to the **Bloch representation** on $\widetilde{\Gamma}_2$, via:





Through the Bloch representation of $S_c(t)$, we can use knowledge from [3]:

- We further decompose

$$S_c(t) = (\phi_0' + \partial_k\phi(\cdot;1)\partial_\zeta)\, S_p(t) + O_{L^\infty \to L^\infty}((1+t)^{-1}),$$

where $\phi(\cdot,k)$ is the continuation of $\phi(\cdot,1) = \phi_0$ and ∂_k denotes the derivative w.r.t. the wavenumber k.

- For the critical part we have, with $\widetilde{\Phi}_0$ as the adjoint eigenfunction,

$$S_p(t) = e^{(d\partial_\zeta^2 + a\partial_\zeta)t}\widetilde{\Phi}_0^* + O_{L^\infty \to L^\infty}((1+t)^{-\frac{1}{2}}).$$

Nonlinear Iterative Estimates

Introduce the inverse-modulated perturbation, [3,7],

$$\widetilde{w}(\zeta,t) = (u,v)^T(\zeta + \gamma(\zeta,t),t) - \phi_0(\zeta).$$

- $\widetilde{w}$ and γ satisfy a **quasilinear equation** (in $\widetilde{w}$)

$$(\partial_t - \mathcal{L})(\widetilde{w} + \phi_0'\gamma) = \widetilde{N}(\widetilde{w}, \widetilde{w}_\zeta, \widetilde{w}_{\zeta\zeta}, \gamma_\zeta, \gamma_t) \qquad (4)$$

and thus we choose

$$\gamma(t) = S_p(t)w_0 + \int_0^t S_p(t-s)\widetilde{N}(\widetilde{w}, \widetilde{w}_\zeta, \widetilde{w}_{\zeta\zeta}, \gamma_\zeta, \gamma_t)(s)\,ds.$$

- Choice of γ yields a **perturbed viscous Hamilton-Jacobi equation**

$$\partial_t\gamma = d\partial_\zeta^2\gamma + a\partial_\zeta\gamma + \kappa\gamma_\zeta^2 + h.o.t \text{ for some } \kappa \in \mathbb{R}. \qquad (5)$$

- Note: γ appears only as derivatives w.r.t. ζ or t in the nonlinearities!

1st Challenge: How to control regularity in (4)?

Our solution: Using uniformly local Sobolev spaces [5,10] and forward modulation [12], we find the nonlinear damping estimate

$$||\widetilde{w}(t)||^2_{W^{2,\infty}\times W^{1,\infty}} \leq C_2\Big[e^{-\theta t}E_0^2 + \int_0^t \frac{||\widetilde{w}(s)||^2_{L^\infty} + ||(\gamma_\zeta,\gamma_t)(s)||^2_{W^{4,\infty}\times W^{3,\infty}}}{e^{\theta(t-s)}}\,ds\Big]$$

for some $\theta, C_2 > 0$.

2nd Challenge: Slowest decaying nonlinear term γ_ζ^2 cannot be controlled through iterative estimates on the Duhamel formula of (5).

Our solution: As in [3], we apply the **Cole-Hopf transform** $z = e^{\frac{\kappa}{d}\gamma} - 1$ to (5) which eliminates the critical nonlinear term γ_ζ^2.

Finally, we close a **nonlinear argument** with

$$||\widetilde{w}(t)||_{W^{2,\infty}\times W^{1,\infty}}, ||(\gamma_\zeta,\gamma_t)(t)||_{W^{4,\infty}\times W^{3,\infty}} \sim O((1+t)^{-\frac{1}{2}}).$$

Outlook

Our scheme can be applied to show nonlinear stability results of periodic waves against non-localized perturbations for other dissipative systems such as the **Lugiato-Lefever equation** or the **Taylor-Couette flow**.

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