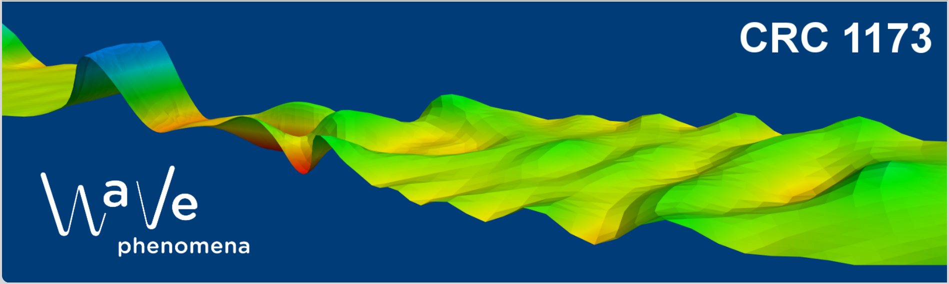
Karlsruhe Institute of Technology

Real-valued time-periodic localized standing waves for a class of nonlinear wave equations

Andreas Hirsch, Simon Kohler and Wolfgang Reichel

6th GAMM Workshop on Analysis of PDEs, Stuttgart • September 20, 2018

Institute for Analysis



The problem

Find $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ solving

(semi) $$V(x)u_{tt} - u_{xx} = \Gamma(x)|u|^{p-1}u$$

(quasi) $$V(x)u_{tt} - u_{xx} = \Gamma(x)(u_t^3)_t$$

such that

- u is T-periodic, real valued
- $\lim_{|x|\to\infty} u(x,t) = 0$

for $p > 1$ & suitable conditions on $V, \Gamma : \mathbb{R} \to \mathbb{R}$

Outline:

(A) history of semilinear problem
(B) results for semilinear wave equation
(C) physical background of quasilinear problem
(D) results for quasilinear wave equation

History of the problem

1973 Ablowitz, Kaup, Newell, Segur: Sine-Gordon breather

$$u_{tt} - u_{xx} + \sin u = 0$$

$$u(x,t) = 4\arctan\left(\frac{m\sin(\omega t)}{\omega\cosh(mx)}\right), m^2 + \omega^2 = 1$$

History of the problem

1973 Ablowitz, Kaup, Newell, Segur: Sine-Gordon breather

$$u_{tt} - u_{xx} + \sin u = 0$$

$$u(x,t) = 4\arctan\left(\frac{m\sin(\omega t)}{\omega\cosh(mx)}\right), m^2 + \omega^2 = 1$$

1993 Denzler, 1994 Birnir, McKean, Weinstein: non-persistence

$$u_{tt} - u_{xx} + f(u) = 0, \qquad f(0) = 0, f'(0) = 1$$

has no breather-solution except for $f(u) = \sin u$.

History of the problem

1973 Ablowitz, Kaup, Newell, Segur: Sine-Gordon breather

$$u_{tt} - u_{xx} + \sin u = 0$$

$$u(x,t) = 4 \arctan\left(\frac{m \sin(\omega t)}{\omega \cosh(mx)}\right), m^2 + \omega^2 = 1$$

1993 Denzler, 1994 Birnir, McKean, Weinstein: non-persistence

$$u_{tt} - u_{xx} + f(u) = 0, \qquad f(0) = 0, f'(0) = 1$$

has no breather-solution except for $f(u) = \sin u$.

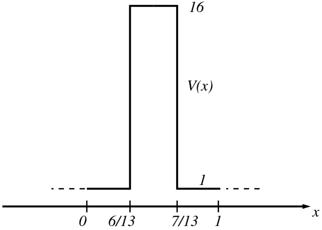
1994 McKay, Aubry:
replace u_{xx} by $u_{n+1} - 2u_n + u_{n-1} \Rightarrow$ breathers are back

History of the problem

2011: Blank, Chirilus-Bruckner, Lescarret, Schneider

$$V(x)u_{tt} - u_{xx} + q(x)u = \pm u^3$$

V 1-periodic, $q(x) = (q_0 - \epsilon^2)V(x)$, $q_0 \approx 3.7703$



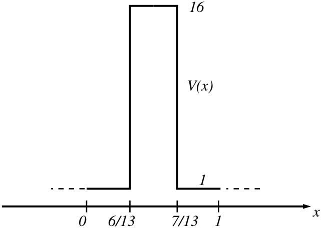
$\exists$ breather for $0 < \epsilon < \epsilon_0$
$u(x,t) = O(\epsilon)$
$T = \frac{32}{13}$

History of the problem

2011: Blank, Chirilus-Bruckner, Lescarret, Schneider

$$V(x)u_{tt} - u_{xx} + q(x)u = \pm u^3$$

V 1-periodic, $q(x) = (q_0 - \epsilon^2)V(x)$, $q_0 \approx 3.7703$



$\exists$ breather for $0 < \epsilon < \epsilon_0$

$u(x,t) = O(\epsilon)$

$T = \frac{32}{13}$

2017: M.Plum, W.R., JEPE. Existence of breather $U : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$

$$V(x)U_{tt} + \nabla \times \nabla \times U + q(x)U = \pm\Gamma(x)|U|^{p-1}U \text{ in } \mathbb{R}^3 \times \mathbb{R}$$

- $1 < p < \infty$, V, q, Γ radially symmetric, positive
- $\sup \frac{q}{\Gamma} < \infty$ & smooth. condition on $\frac{q}{V}$ at $|x| = r = 0$ and at $r = \infty$

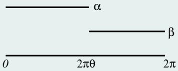
Our result for the semilinear case

(semi) $$V(x)u_{tt} - u_{xx} = \pm|u|^{p-1}u$$

Theorem 1 (A. Hirsch & W.R., 2018)

Breathers exist in the following three cases (V is positive, 2π-periodic):

(1) $V(x) = \alpha + \beta\delta^{per}(x), \beta > 32\alpha,$ $\quad 1 < p < p^* = 2$

(2) $V(x) =$ 

$, 0 < \theta \ll \frac{1}{2}, \frac{\alpha}{\beta} = \frac{(1-\theta)^2}{\theta^2}$ $\quad 1 < p < p^* = 3$

(3) $\exists$ *suitable* $V \in H^r_{per}(\mathbb{R})$ *near* $V_0 \equiv 1$, $r \in [1, 3/2)$, $\quad 1 < p < p^* = \frac{7-2r}{1+2r}$

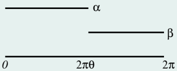
Our result for the semilinear case

(semi) $$V(x)u_{tt} - u_{xx} = \pm|u|^{p-1}u$$

Theorem 1 (A. Hirsch & W.R., 2018)

Breathers exist in the following three cases (V is positive, 2π-periodic):

(1) $V(x) = \alpha + \beta\delta^{per}(x), \beta > 32\alpha,$ $\quad 1 < p < p^* = 2$

(2) $V(x) =$ 

$, 0 < \theta \ll \frac{1}{2}, \frac{\alpha}{\beta} = \frac{(1-\theta)^2}{\theta^2}$ $\quad 1 < p < p^* = 3$

(3) $\exists$ *suitable* $V \in H^r_{per}(\mathbb{R})$ *near* $V_0 \equiv 1$, $r \in [1, 3/2)$, $\quad 1 < p < p^* = \frac{7-2r}{1+2r}$

frequencies $\omega =$

(1) $(4\alpha)^{-1}$,

(2) $(4\theta\sqrt{\alpha})^{-1}$

(3) $\pi\left(\int_0^{2\pi} \sqrt{V(x)}\,dx\right)^{-1}$

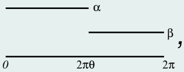
Our result for the semilinear case

(semi) $$V(x)u_{tt} - u_{xx} = \pm|u|^{p-1}u$$

Theorem 1 (A. Hirsch & W.R., 2018)

Breathers exist in the following three cases (V is positive, 2π-periodic):

(1) $V(x) = \alpha + \beta\delta^{per}(x), \beta > 32\alpha,$ $\quad 1 < p < p^* = 2$

(2) $V(x) =$ 

$, 0 < \theta \ll \frac{1}{2}, \frac{\alpha}{\beta} = \frac{(1-\theta)^2}{\theta^2}$ $\quad 1 < p < p^* = 3$

(3) $\exists$ *suitable* $V \in H^r_{per}(\mathbb{R})$ *near* $V_0 \equiv 1$, $r \in [1, 3/2)$, $\quad 1 < p < p^* = \frac{7-2r}{1+2r}$

Approach: $u(x,t) = \sum_{k \text{ odd}} u_k(x)e^{ik\omega t}$, $\bar{u}_k = u_{-k}$

$$V(x)\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} = \bigoplus_{k \text{ odd}} L_k \text{ with } L_k = -\frac{d^2}{dx^2} - k^2\omega^2 V(x)$$

$$\operatorname{dist}(0, \sigma(L_k)) \geq \text{const.} \begin{cases} |k| & \text{case (1), (2)} \\ |k|^{\frac{3}{2}-r} & \text{case (3)} \end{cases}$$

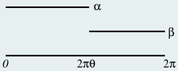
Our result for the semilinear case

(semi) $$V(x)u_{tt} - u_{xx} = \pm|u|^{p-1}u$$

Theorem 1 (A. Hirsch & W.R., 2018)

Breathers exist in the following three cases (V is positive, 2π-periodic):

(1) $V(x) = \alpha + \beta\delta^{per}(x), \beta > 32\alpha,$ $\quad 1 < p < p^* = 2$

(2) $V(x) =$ 

$, 0 < \theta \ll \frac{1}{2}, \frac{\alpha}{\beta} = \frac{(1-\theta)^2}{\theta^2}$ $\quad 1 < p < p^* = 3$

(3) $\exists$ *suitable* $V \in H^r_{per}(\mathbb{R})$ *near* $V_0 \equiv 1, r \in [1, 3/2),$ $\quad 1 < p < p^* = \frac{7-2r}{1+2r}$

construction of $V \in H^r_{per}(\mathbb{R})$: real-analytic inverse spectral theory
M. Chirilus-Bruckner & C.E. Wayne, JMAA 2015.

Variational method

space-time domain $D = \mathbb{R} \times (0, T)$, $T = 2\pi/\omega$:

$$J(u) = \int_D u_x^2 - V(x) u_t^2 \, d(x,t) \mp \frac{2}{p+1} \int_D |u|^{p+1} \, d(x,t)$$

Variational method

space-time domain $D = \mathbb{R} \times (0, T)$, $T = 2\pi/\omega$:

$$J(u) = \int_D u_x^2 - V(x)u_t^2 \, d(x,t) \mp \frac{2}{p+1} \int_D |u|^{p+1} \, d(x,t)$$

$\hat{u} = (u_k)_{k\ odd}$, $S\hat{u}(x,t) = u(x,t) = \sum_{k\ odd} u_k(x) e^{ik\omega t}$

$$J(\hat{u}) = \sum_{k\ odd} b_{L_k}(u_k, u_k) \mp \frac{2}{p+1} \int_D |S\hat{u}|^{p+1} \, d(x,t)$$

Variational method

space-time domain $D = \mathbb{R} \times (0, T)$, $T = 2\pi/\omega$:

$$J(u) = \int_D u_x^2 - V(x) u_t^2 \, d(x,t) \mp \frac{2}{p+1} \int_D |u|^{p+1} \, d(x,t)$$

$\hat{u} = (u_k)_{k\ odd}$, $S\hat{u}(x,t) = u(x,t) = \sum_{k\ odd} u_k(x) e^{ik\omega t}$

$$J(\hat{u}) = \sum_{k\ odd} b_{L_k}(u_k, u_k) \mp \frac{2}{p+1} \int_D |S\hat{u}|^{p+1} \, d(x,t)$$

for $\hat{u}$ in Hilbert -space

$$H = \Big\{ \hat{u} : \sum_{k\ odd} b_{|L_k|}(u_k, u_k) < \infty, u_k \in \mathrm{dom}(b_{|L_k|}) = \mathrm{dom}(b_{L_k}) = H^1(\mathbb{R}), \\ \bar{u}_k = u_{-k} \Big\}$$

Variational method – continued

$$J(\hat{u}) = \sum_{k \text{ odd}} b_{L_k}(u_k, u_k) \mp \frac{2}{p+1} \int_D |S\hat{u}|^{p+1} \, d(x,t)$$

$$H = \Big\{ \hat{u} : \sum_{k \text{ odd}} b_{|L_k|}(u_k, u_k) < \infty, u_k \in \mathrm{dom}(b_{|L_k|}) = \mathrm{dom}(b_{L_k}) = H^1(\mathbb{R}),$$
$$\bar{u}_k = u_{-k} \Big\}$$

- J is indefinite, $0 \notin \sigma(V(x)\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}) = \overline{\bigcup_{k \text{ odd}} \sigma(L_k)}$
- locally compact embedding $S : H \to L^q(D), 2 \le q < p^* + 1$
- concentration compactness

$\Rightarrow \exists$ critical point $\hat{=}$ real-valued breather $u(x,t) = \sum_{k \text{ odd}} u_k(x) e^{ik\omega t}$

Extensions & further properties

generalize

$$V(x)u_{tt} - u_{xx} = \pm|u|^{p-1}u$$

to

$$V(x)u_{tt} - u_{xx} + q(x)u = f(x,u)$$

with $q(x) = \tau V(x)$, $\tau \in (-\tau_0, \tau_0)$ and

Extensions & further properties

generalize

$$V(x)u_{tt} - u_{xx} = \pm|u|^{p-1}u$$

to

$$V(x)u_{tt} - u_{xx} + q(x)u = f(x,u)$$

with $q(x) = \tau V(x)$, $\tau \in (-\tau_0, \tau_0)$ and

(H1) f continuous, 2π-periodic in x, $|f(x,s)| \le c(1+|s|^p)$, $1 < p < p^*$

(H2) $f(x,s) = o(s)$ as $s \to 0$ uniformly in $x \in \mathbb{R}$

(H3) $f(x,s)$ odd in $s \in \mathbb{R}$, $s \mapsto f(x,s)/|s| \nearrow\!\!\nearrow_s$ on $(-\infty,0)$ and $(0,\infty)$

(H4) $\frac{F(x,s)}{s^2} \to \infty$ as $s \to \infty$ uniformly in $x \in \mathbb{R}$, $F(x,s) := \int_0^s f(x,t)\,dt$

inspired by: Szulkin, Weth, The method of Nehari manfold, 2010.

Extensions & further properties: regularity

$u \in L^{p+1}(D)$, $1 < p < p^*$

$$u = \sum_{k \text{ odd}} u_k(x) e^{ik\omega t} \in H^\alpha(0,T;H^1(\mathbb{R})) \cap H^\beta(0,T;L^2(\mathbb{R}))$$

i.e.

$$\sum_{k \text{ odd}} |k|^{2\alpha} \|u_k'\|_{L^2}^2 + |k|^{2\beta} \|u_k\|_{L^2}^2 < \infty$$

with

$$2\beta \begin{cases} = 1 & \text{case 1)} \\ = 1 & \text{case 2)} \\ = \frac{3}{2} - r & \text{case 3)} \end{cases} \qquad 2\alpha \begin{cases} = -3 & \text{case 1)} \\ < -1 & \text{case 2)} \\ = -\frac{1}{2} - r & \text{case 3)} \end{cases}$$

Concept of weak solution

$\phi \in C_c^\infty(\mathbb{T} \times \mathbb{R})$

$$\int_D V(x) u \phi_{tt} \, d(x,t) - \int_D u \phi_{xx} \, d(x,t) = \int_D \Gamma(x) |u|^{p-1} u \phi \, d(x,t)$$

For Case 1): $V(x) = \alpha + \beta \delta_{per}$

$$„ \int_D \delta_{per}(x) u \phi_{tt} \, d(x,t) \text{“} = \sum_{n \in \mathbb{Z}} \int_0^T u(2\pi n, t) \phi_{tt}(2\pi n, t) \, dt$$

A. Hirsch, W.R.: Real-valued, time-periodic localized weak solutions for a semilinear wave equation with periodic potentials, ArXiv 2018

Concept of weak solution

$\phi \in C_c^\infty(\mathbb{T} \times \mathbb{R})$

$$\int_D V(x) u \phi_{tt} \, d(x,t) - \int_D u \phi_{xx} \, d(x,t) = \int_D \Gamma(x) |u|^{p-1} u \phi \, d(x,t)$$

For Case 1): $V(x) = \alpha + \beta \delta_{per}$

$$„\int_D \delta_{per}(x) u \phi_{tt} \, d(x,t)“ = \sum_{n \in \mathbb{Z}} \int_0^T u(2\pi n, t) \phi_{tt}(2\pi n, t) \, dt$$

A. Hirsch, W.R.: Real-valued, time-periodic localized weak solutions for a semilinear wave equation with periodic potentials, ArXiv 2018

— end of part A and B —

The problem

Find $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ solving

$$\text{(semi)} \qquad V(x)u_{tt} - u_{xx} = \Gamma(x)|u|^{p-1}u$$

$$\text{(quasi)} \qquad V(x)u_{tt} - u_{xx} = \Gamma(x)(u_t^3)_t$$

such that

- u is T-periodic, real valued
- $\lim_{|x|\to\infty} u(x,t) = 0$

for $p > 1$ & suitable conditions on $V, \Gamma : \mathbb{R} \to \mathbb{R}$

Outline:

(A) history of semilinear problem
(B) results for semilinear wave equation
(C) physical background of quasilinear problem
(D) results for quasilinear wave equation

The nonlinear Maxwell model

$$\nabla \times \mathbf{E} + \partial_t \mathbf{B} = 0, \qquad \nabla \cdot \mathbf{D} = 0,$$

$$\nabla \times \mathbf{H} - \partial_t \mathbf{D} = 0, \qquad \nabla \cdot \mathbf{B} = 0.$$

Material laws:

$$\mathbf{B} = \mu_0 \mathbf{H}, \quad \mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}(x, \mathbf{E}) = \epsilon_0 (1 + \chi_1(x) + \chi_3(x)|\mathbf{E}|^2)\mathbf{E}$$

$\mu_0 \epsilon_0 = 1/c^2$ ($c =$ speed of light in vacuum)

$\hookrightarrow$
$$\nabla \times \nabla \times \mathbf{E} + \mu_0 \partial_t^2 \mathbf{D} = 0$$

The nonlinear Maxwell model

$$\nabla \times \mathbf{E} + \partial_t \mathbf{B} = 0, \qquad \nabla \cdot \mathbf{D} = 0,$$

$$\nabla \times \mathbf{H} - \partial_t \mathbf{D} = 0, \qquad \nabla \cdot \mathbf{B} = 0.$$

Material laws:

$$\mathbf{B} = \mu_0 \mathbf{H}, \quad \mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}(x, \mathbf{E}) = \epsilon_0 (1 + \chi_1(x) + \chi_3(x)|\mathbf{E}|^2)\mathbf{E}$$

$\mu_0 \epsilon_0 = 1/c^2$ (c = speed of light in vacuum)

$$\hookrightarrow \qquad \nabla \times \nabla \times \mathbf{E} + \mu_0 \partial_t^2 \mathbf{D} = 0$$

Quasilinear wave-equation for $\mathbf{E}$:

$$\hookrightarrow \qquad \nabla \times \nabla \times \mathbf{E} + \mu_0 \partial_t^2 \Big(\epsilon_0 (1 + \chi_1(x))\mathbf{E} + \epsilon_0 \chi_3(x) |\mathbf{E}|^2 \mathbf{E} \Big) = 0$$

The nonlinear Maxwell model

$$\nabla \times \mathbf{E} + \partial_t \mathbf{B} = 0, \qquad \nabla \cdot \mathbf{D} = 0,$$
$$\nabla \times \mathbf{H} - \partial_t \mathbf{D} = 0, \qquad \nabla \cdot \mathbf{B} = 0.$$

Material laws:

$$\mathbf{B} = \mu_0 \mathbf{H}, \quad \mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}(x, \mathbf{E}) = \epsilon_0 (1 + \chi_1(x) + \chi_3(x)|\mathbf{E}|^2)\mathbf{E}$$

$\mu_0 \epsilon_0 = 1/c^2$ (c = speed of light in vacuum)

$$\hookrightarrow \qquad \nabla \times \nabla \times \mathbf{E} + \mu_0 \partial_t^2 \mathbf{D} = 0$$

Quasilinear wave-equation for **E**:

$$\hookrightarrow \qquad \nabla \times \nabla \times \mathbf{E} + \mu_0 \partial_t^2 \Big(\epsilon_0 (1 + \chi_1(x))\mathbf{E} + \epsilon_0 \chi_3(x) |\mathbf{E}|^2 \mathbf{E} \Big) = 0$$

For $\mathbf{E} = (0, 0, v(x_1, t))$:

$$\hookrightarrow \qquad -v_{x_1 x_1} + \mu_0 \epsilon_0 (1 + \chi_1(x_1)) v_{tt} = -\mu_0 \epsilon_0 \chi_3(x_1)(v^3)_{tt}$$

For $v(x, t) = u_t(x, t)$

$$\hookrightarrow \qquad -u_{xx} + V(x) u_{tt} = \Gamma(x)(u_t^3)_t$$

The quasilinear case – concept of solution

Joint work with Simon Kohler

$$(quasi) \qquad \begin{cases} V(x)u_{tt} - u_{xx} &= \Gamma(x)(u_t^3)_t \text{ in } \mathbb{R}\times\mathbb{R} \\ u(x,t) &\to 0 \text{ as } |x|\to\infty \\ u(x,t+T) &= u(x,t) \end{cases}$$

An extreme example: $\Gamma(x) = \pm\delta_0(x)$.

The quasilinear case – concept of solution

Joint work with Simon Kohler

$$(quasi) \qquad \begin{cases} V(x)u_{tt} - u_{xx} &= \Gamma(x)(u_t^3)_t \text{ in } \mathbb{R}\times\mathbb{R} \\ u(x,t) &\to 0 \text{ as } |x|\to\infty \\ u(x,t+T) &= u(x,t) \end{cases}$$

An extreme example: $\Gamma(x) = \pm\delta_0(x)$.
For $\phi \in C_c^\infty(\mathbb{T}\times\mathbb{R})$

$$\int_D -V(x)u_t\phi_t \, d(x,t) + \int_D u_x\phi_x \, d(x,t) = \mp \int_0^T u_t(0,t)^3\phi_t(0,t)\, dt$$

The quasilinear case – concept of solution

Joint work with Simon Kohler

$$(quasi) \qquad \begin{cases} V(x)u_{tt} - u_{xx} &= \Gamma(x)(u_t^3)_t \text{ in } \mathbb{R}\times\mathbb{R} \\ u(x,t) &\to 0 \text{ as } |x|\to\infty \\ u(x,t+T) &= u(x,t) \end{cases}$$

An extreme example: $\Gamma(x) = \pm\delta_0(x)$.
For $\phi \in C_c^\infty(\mathbb{T}\times\mathbb{R})$

$$\int_D -V(x)u_t\phi_t \, d(x,t) + \int_D u_x\phi_x \, d(x,t) = \mp \int_0^T u_t(0,t)^3\phi_t(0,t)\, dt$$

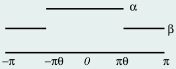
If $V(x)$ and $u(x,t)$ are even in x then

$$(nN) \qquad \begin{cases} V(x)u_{tt} - u_{xx} &= 0 \text{ in } (0,\infty)\times\mathbb{R}, \\ -2u_x(0,t) &= \pm(u_t(0,t)^3)_t, \\ u(x,t) &\to 0 \text{ as } x\to\infty \\ u(x,t+T) &= u(x,t) \end{cases}$$

The quasilinear case – existence of solution

$$(nN) \qquad \begin{cases} V(x)u_{tt} - u_{xx} = 0 \text{ in } (0,\infty)\times\mathbb{R}, \\ -2u_x(0,t) = \pm(u_t(0,t)^3)_t, \end{cases}$$

Theorem 2 (S. Kohler & W.R., 2018)

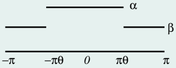
Breathers exist if $V(x) =$ 

$, 0 < \theta < 1, \theta \neq \frac{1}{2}, \frac{\alpha}{\beta} = \frac{(1-\theta)^2}{\theta^2}$

The quasilinear case – existence of solution

$$(nN) \qquad \begin{cases} V(x)u_{tt} - u_{xx} &= 0 \text{ in } (0,\infty)\times\mathbb{R}, \\ -2u_x(0,t) &= \pm(u_t(0,t)^3)_t, \end{cases}$$

Theorem 2 (S. Kohler & W.R., 2018)

Breathers exist if $V(x) =$ 

$, 0 < \theta < 1, \theta \neq \frac{1}{2}, \frac{\alpha}{\beta} = \frac{(1-\theta)^2}{\theta^2}$

Approach: $u(x,t) = \sum_{k \text{ odd}} u_k(x) e^{ik\omega t}$, $\bar{u}_k = u_{-k}$

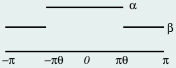
$$V(x)\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} = \bigoplus_{k \text{ odd}} L_k \text{ with } L_k = -\frac{d^2}{dx^2} - k^2\omega^2 V(x)$$

$$\operatorname{dist}(0, \sigma(L_k)) \geq \text{const.}\, |k|$$

The quasilinear case – existence of solution

$$(nN) \qquad \begin{cases} V(x)u_{tt} - u_{xx} &= 0 \text{ in } (0,\infty)\times\mathbb{R}, \\ -2u_x(0,t) &= \pm(u_t(0,t)^3)_t, \end{cases}$$

Theorem 2 (S. Kohler & W.R., 2018)

Breathers exist if $V(x) =$ 

$, 0 < \theta < 1, \theta \neq \frac{1}{2}, \frac{\alpha}{\beta} = \frac{(1-\theta)^2}{\theta^2}$

Approach: $u(x,t) = \sum_{k \text{ odd}} u_k(x) e^{ik\omega t}$, $\bar{u}_k = u_{-k}$

$$V(x)\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} = \bigoplus_{k \text{ odd}} L_k \text{ with } L_k = -\frac{d^2}{dx^2} - k^2\omega^2 V(x)$$

$$\operatorname{dist}(0, \sigma(L_k)) \geq \text{const.} |k|$$

more precise: $u(x,t) = \sum_{k \text{ odd}} \frac{\alpha_k}{k} \phi_k(x) e^{ik\omega t}$, $\bar{\alpha}_k = -\alpha_{-k}$

$\phi_k =$ Bloch-mode, $L_k\phi_k = 0$ on $(0,\infty)$, exp. decaying at $+\infty$, $\phi_k(0) = 1$

The quasilinear case – variational method

$D = \mathbb{R} \times (0, T)$, $T = 2\pi/\omega$, $u(x,t) = \sum_{k\ odd} \frac{\alpha_k}{k} \phi_k(x) e^{ik\omega t}$

$$\begin{aligned}
J(u) &= \int_D u_x^2 - V(x) u_t^2 \, d(x,t) \mp \frac{1}{2} \int_0^T |u_t(0,t)|^4 \, dt \\
&= 2 \int_0^T u(0,t) u_x(0,t) \, dt \mp \frac{1}{2} \int_0^T |u_t(0,t)|^4 \, dt \\
&= 2 \sum_{k\ odd} \frac{\phi_k'(0)}{k^2} |\alpha_k|^2 \mp \frac{\omega^4}{2} \|\hat{\alpha} * \hat{\alpha}\|_{l^2}^2
\end{aligned}$$

$$J(\hat{\alpha}) := \omega^4 \|\hat{\alpha} * \hat{\alpha}\|_{l^2}^2 \mp 4 \sum_{k\ odd} \frac{\phi_k'(0)}{k^2} |\alpha_k|^2$$

The quasilinear case – variational method

$D = \mathbb{R} \times (0, T)$, $T = 2\pi/\omega$, $u(x,t) = \sum_{k\ odd} \frac{\alpha_k}{k} \phi_k(x) e^{ik\omega t}$

$$
\begin{aligned}
J(u) = & \int_D u_x^2 - V(x) u_t^2 \, d(x,t) \mp \frac{1}{2} \int_0^T |u_t(0,t)|^4 \, dt \\
= & 2 \int_0^T u(0,t) u_x(0,t) \, dt \mp \frac{1}{2} \int_0^T |u_t(0,t)|^4 \, dt \\
= & 2 \sum_{k\ odd} \frac{\phi_k'(0)}{k^2} |\alpha_k|^2 \mp \frac{\omega^4}{2} \|\hat{\alpha} * \hat{\alpha}\|_{l^2}^2
\end{aligned}
$$

$$
J(\hat{\alpha}) := \omega^4 \|\hat{\alpha} * \hat{\alpha}\|_{l^2}^2 \mp 4 \sum_{k\ odd} \frac{\phi_k'(0)}{k^2} |\alpha_k|^2
$$

note: $\phi_k'(0) = \text{const.}\, k (-1)^{\frac{k-1}{2}}$

The quasilinear case – variational method

$D = \mathbb{R} \times (0, T)$, $T = 2\pi/\omega$, $u(x,t) = \sum_{k \text{ odd}} \frac{\alpha_k}{k} \phi_k(x) e^{ik\omega t}$

$$J(u) = \int_D u_x^2 - V(x) u_t^2 \, d(x,t) \mp \frac{1}{2} \int_0^T |u_t(0,t)|^4 \, dt$$
$$= 2 \int_0^T u(0,t) u_x(0,t) \, dt \mp \frac{1}{2} \int_0^T |u_t(0,t)|^4 \, dt$$
$$= 2 \sum_{k \text{ odd}} \frac{\phi_k'(0)}{k^2} |\alpha_k|^2 \mp \frac{\omega^4}{2} \|\hat{\alpha} * \hat{\alpha}\|_{l^2}^2$$
$$J(\hat{\alpha}) := \omega^4 \|\hat{\alpha} * \hat{\alpha}\|_{l^2}^2 \mp 4 \sum_{k \text{ odd}} \frac{\phi_k'(0)}{k^2} |\alpha_k|^2$$

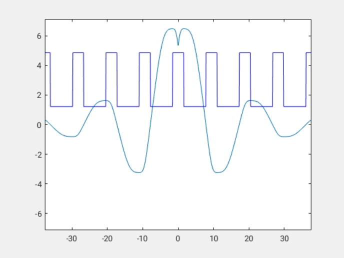
note: $\phi_k'(0) = \text{const.}\, k (-1)^{\frac{k-1}{2}}$
J well-defined, coercive, weakly-lsc on

$$X = \{\hat{\alpha} : \hat{\alpha} * \hat{\alpha} \in l^2, \bar{\alpha}_k = -\alpha_{-k}\} \hookrightarrow l^2$$

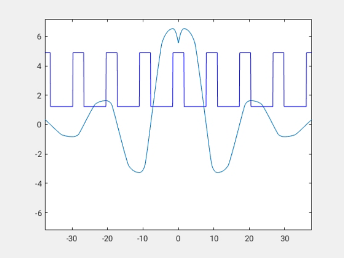
$\Rightarrow \exists$ minimizer $\hat{=}$ real-valued breather $u(x,t) = \sum_{k \text{ odd}} \frac{\alpha_k}{k} \phi_k(x) e^{ik\omega t}$

Numerical experiments

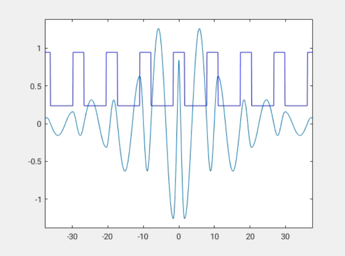
$u(x,t) = \sum_{k=-K}^{K} \frac{\alpha_k}{k} \phi_k(x) e^{ik\omega t}$. Top: $\gamma = 1$, Bottom: $\gamma = -1$



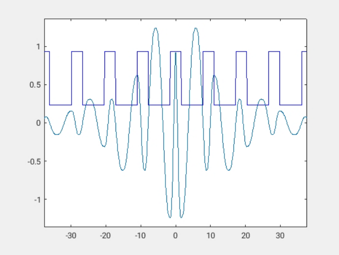
$K = 5$



$K = 81$



$K = 5$



$K = 81$

Regularity

- $u \in H^1(D)$
- $|u(x,t)| \leq \text{const.}\, e^{-\rho|x|}$
- $u \in H^{1+\nu}(0,T;L^2(\mathbb{R})) \cap H^\nu(0,T;H^1(\mathbb{R}))$, $\nu < 1/4$

Recall concept of solution: for $\phi \in C_c^\infty(\mathbb{T}\times\mathbb{R})$

$$\int_D -V(x)u_t\phi_t\, d(x,t) + \int_D u_x\phi_x\, d(x,t) = \mp \int_0^T u_t(0,t)^3\phi_t(0,t)\, dt$$

Regularity

- $u \in H^1(D)$
- $|u(x,t)| \leq \text{const.}\, e^{-\rho|x|}$
- $u \in H^{1+\nu}(0,T;L^2(\mathbb{R})) \cap H^\nu(0,T;H^1(\mathbb{R}))$, $\nu < 1/4$

Recall concept of solution: for $\phi \in C_c^\infty(\mathbb{T} \times \mathbb{R})$

$$\int_D -V(x)u_t\phi_t \, d(x,t) + \int_D u_x\phi_x \, d(x,t) = \mp \int_0^T u_t(0,t)^3\phi_t(0,t)\, dt$$

— end of part C and D —

The problem

Find $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ solving

$$\text{(semi)} \qquad V(x)u_{tt} - u_{xx} = \Gamma(x)|u|^{p-1}u$$

$$\text{(quasi)} \qquad V(x)u_{tt} - u_{xx} = \Gamma(x)(u_t^3)_t$$

such that

- u is T-periodic, real valued
- $\lim_{|x|\to\infty} u(x,t) = 0$

for $p > 1$ & suitable conditions on $V, \Gamma : \mathbb{R} \to \mathbb{R}$

Outline:

(A) history of semilinear problem
(B) results for semilinear wave equation
(C) physical background of quasilinear problem
(D) results for quasilinear wave equation