

EXISTENCE AND MULTIPLICITY PROOFS FOR SEMILINEAR ELLIPTIC BOUNDARY VALUE PROBLEMS BY COMPUTER ASSISTANCE

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ABSTRACT. Many boundary value problems for semilinear elliptic partial differential equations allow very stable numerical computations of approximate solutions, but are still lacking analytical existence proofs. In the present article, we propose a method which exploits the knowledge of a “good” numerical approximate solution, in order to provide a rigorous proof of existence of an exact solution close to the approximate one. This goal is achieved by a fixed-point argument which takes all numerical errors into account, and thus gives a mathematical proof which is not “worse” than any purely analytical one. The method is used to prove existence and multiplicity statements for some specific examples, including cases where purely analytical methods had not been successful.

Keywords: elliptic boundary value problem, semilinear, computer-assisted proof, existence, enclosures, error bounds, multiplicity.

MSC: 35J25, 35J65, 65N15

1. Introduction

Semilinear elliptic differential equations of the form

$$-\Delta u(x) + f(x, u(x)) = 0 \quad (x \in \Omega) \quad (1)$$

(with $\Omega \subset \mathbb{R}^n$ denoting some given domain, and $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ some given nonlinearity), together with boundary conditions, e. g. of Dirichlet type

$$u(x) = 0 \quad (x \in \partial\Omega), \quad (2)$$

have been (and still are) extensively studied in the differential equations literature. Such *semilinear boundary value problems* have a lot of applications e. g. in Mathematical Physics, and often serve as model problems for more complex mathematical situations, and last but not least, they form a very exciting and challenging object for purely mathematical investigations. Starting perhaps with Picard’s successive iterations at the end of the 19th century, various analytical methods and techniques have been (and are being) developed to study *existence* and *multiplicity* of solutions to problem (1), (2), such as variational methods (including mountain pass methods), index and degree theory, monotonicity methods, fixed-point methods, and more; see e. g. [2]-[6], [12]-[14], [18]-[21], [25, 26, 30, 31, 33, 34, 36], [39]-[41], [43, 44, 55, 61], and the references therein.

In this article, we want to report on a supplement to these purely analytical methods by a *computer-assisted* approach, which in the recent years has turned out to be successful with various examples where purely analytical methods have failed. In spite of many numerical calculations involved, the existence and multiplicity proofs given by our method are completely rigorous and not “worse” than any other proof. One might ask if (systematic

or accidental) hardware errors could spoil the correctness of a computer-assisted proof, but the probability of the permanent occurrence of such errors can be made very small by use of different hardware platforms and by repeating the computations many times. Of course, some uncertainty concerning the correctness of the hardware actions or of the program codes remains, but is this uncertainty really larger than the uncertainty attached to a complex “theoretical” proof?

Recently, various mathematical problems have been solved by computer-assisted proofs, among them the Kepler conjecture, the existence of chaos, the existence of the Lorenz attractor, the famous four-colour problem, and more.

In many cases, computer-assisted proofs have the remarkable advantage (compared with a “theoretical” proof) of providing accurate *quantitative* information. Coming back to our approach concerning problem (1), (2), such quantitative information is given in form of tight and explicit *bounds* for the solution.

We start with an *approximate* solution ω to (1), (2), which can be obtained by any numerical method which gives approximations in the function space needed (to be specified later). In this first step, there is no need for any mathematical rigor, and the field is open for the whole rich variety of modern numerics.

Next, we use a Newton-Cantorovich-type argument to prove the existence of a solution to (1), (2) in some “close” and “explicit” neighborhood of ω . For this purpose, we consider the boundary value problem for the *error* $v = u - \omega$ and rewrite it as a fixed-point equation

$$v \in X, \quad v = Tv \tag{3}$$

in a Banach space X , which we treat by some fixed-point theorem. More precisely, we aim at Schauder’s fixed-point theorem if compactness is available (which essentially requires the domain Ω in (1) to be bounded), or at Banach’s fixed-point theorem (if we are ready to accept an additional contraction condition; see (17) below). The existence of a solution v^* of (3) in some suitable set $V \subset X$ then follows from the fixed-point theorem, provided that

$$TV \subset V. \tag{4}$$

Consequently, $u^* := \omega + v^*$ is a solution of (1), (2) (which gives the desired existence result), and the statement “ $u^* \in \omega + V$ ” (implied by $v^* \in V$) gives the desired bounds, or enclosures, for u^* .

So the crucial condition to be verified, for some suitable set V , is (4). Restricting ourselves to *norm balls* V (centered at the origin), we find that (4) results in an inequality involving the radius of V , and various other terms generated by the “data” of our problem (1), and by the numerical approximation ω . All these terms are *computable*, either directly or via additional computer-assisted means (like the eigenvalue bounds discussed briefly in Section 3.3). In these computations (in contrast to the computation of ω mentioned above), all possible numerical errors have to be taken into account, in order to be able to check the aforementioned inequality (implying (4)) with mathematical rigor. For example, remainder term bounds need to be computed when quadrature formulas are applied, and interval arithmetic [35, 57] is needed to take rounding errors into account.

Computer-assisted means for obtaining enclosures for solutions to elliptic partial differential equations have been proposed by Collatz [16, 17] already more than 50 years ago. He used maximum-principle-type arguments to obtain two-sided bounds for the error function $u - \omega$, with ω denoting a numerical C^2 -approximation. Schröder [58]–[60], Walter [62] and

others generalized these ideas, which resulted in the method of *differential inequalities*. It was successfully applied to many examples with first or second order ordinary differential equations, or with second order elliptic or parabolic differential equations. However, there are drawbacks of differential inequalities methods concerning the size of the class of problems (1), (2) to which they can be applied: At least for obtaining “tight” solution enclosures, all eigenvalues of the linearization L of (1), (2) at ω need to be *positive*, which excludes many interesting situations. Furthermore, differential inequalities techniques are essentially restricted to first- and second-order problems (with the exception of some fourth-order problems which can be handled as second-order systems). In contrast, the enclosure method proposed in this article requires the eigenvalues of the linearization L to be *non-zero* only (which is checked by *eigenvalue enclosures*), and at least in principle it can be used for elliptic problems of any (even) order; see also the remarks at the end of Section 2.

An existence and enclosure method similar to ours has been developed by Nakao and his group [46]–[48]. This approach avoids the computation of eigenvalue enclosures for L , which constitutes a significant advantage in some cases. Instead, a finite-dimensional *projection* of L is used, and treated by well-established means of verifying numerical linear algebra. However, also the (infinite-dimensional) *projection error* needs to be bounded in a suitable way, which is well possible for “simple” domains Ω , but problematic e.g. for unbounded domains.

Another more recent approach is based on the Conley index and the numerical verification of corresponding topological conditions; it is suited for proving the existence of stationary solutions for certain classes of problems, as well as for detecting global dynamics (see e.g. [22, 29]).

For *ordinary* differential equation problems (possibly originating from a partial differential equation after symmetry reductions), many existence and enclosure methods can be found in the literature, which we will not address in this article.

2. Abstract formulation

It turns out to be useful to explain the basics of our computer-assisted approach first for the following abstract problem:

$$\text{Find } u \in X \text{ satisfying } \mathcal{F}(u) = 0, \quad (5)$$

with $(X, \langle \cdot, \cdot \rangle_X)$ and $(Y, \langle \cdot, \cdot \rangle_Y)$ denoting two real Hilbert spaces, and $\mathcal{F} : X \rightarrow Y$ some Fréchet differentiable mapping.

Let $\omega \in X$ denote some *approximate* solution to (5) (computed e.g. by numerical means), and

$$L := \mathcal{F}'(\omega) : X \rightarrow Y \quad (6)$$

the Fréchet derivative of $\mathcal{F}$ at ω , i.e. $L \in \mathcal{B}(X, Y)$ (the Banach space of all bounded linear operators from X to Y), and

$$\lim_{\substack{h \in X \setminus \{0\} \\ h \rightarrow 0}} \frac{1}{\|h\|_X} \|\mathcal{F}(\omega + h) - \mathcal{F}(\omega) - L[h]\|_Y = 0.$$

Suppose that we know constants δ and K , and a non-decreasing function $g : [0, \infty) \rightarrow [0, \infty)$ such that

$$\|\mathcal{F}(\omega)\|_Y \leq \delta, \quad (7)$$

i.e. δ bounds the *defect* (residual) of the approximate solution ω to (5),

$$\|u\|_X \leq K \|L[u]\|_Y \text{ for all } u \in X, \quad (8)$$

i.e. K bounds the *inverse* of the linearization L ,

$$\|\mathcal{F}'(\omega + u) - \mathcal{F}'(\omega)\|_{\mathcal{B}(X,Y)} \leq g(\|u\|_X) \text{ for all } u \in X, \quad (9)$$

i.e. g majorizes the modulus of continuity of $\mathcal{F}'$ at ω , and

$$g(t) \rightarrow 0 \text{ as } t \rightarrow 0 \quad (10)$$

(which in particular requires $\mathcal{F}'$ to be continuous at ω).

The concrete computation of such δ , K , and g is the main challenge in our approach, with particular emphasis on K . We will however not address these questions in this section, i.e. on the abstract level, but postpone them to the more specific case of the boundary value problem (1), (2), to be treated in the following sections. For now, we assume that (7) - (10) hold true.

In order to obtain a suitable fixed-point formulation (3) for our problem (5), we will need that the operator L is *onto*. (Note that L is one-to-one by (8).) For this purpose, we propose two alternative ways, both suited for the later treatment of problem (1), (2).

1) “*The compact case*”. Suppose that $\mathcal{F}$ admits a splitting

$$\mathcal{F} = L_0 + \mathcal{G} \quad (11)$$

with a *bijective* linear operator $L_0 \in \mathcal{B}(X, Y)$ and a *compact* and Fréchet differentiable operator $\mathcal{G} : X \rightarrow Y$ with compact Fréchet derivative $\mathcal{G}'(\omega)$.

Noting that $L_0^{-1} \in \mathcal{B}(Y, X)$ by the Open Mapping Theorem, we find that the linear operator

$$L_0^{-1}\mathcal{G}'(\omega) : X \rightarrow X \text{ is compact.}$$

Moreover, since $L = L_0 + \mathcal{G}'(\omega)$ by (11), we have the equivalence

$$L[u] = r \Leftrightarrow u + (L_0^{-1}\mathcal{G}'(\omega)) [u] = L_0^{-1}[r] \quad (12)$$

for every $u \in X, r \in Y$. Fredholm’s Alternative Theorem for compact linear operators tells us that the equation on the right of (12) has a unique solution $u \in X$ for every $r \in Y$, provided that the homogeneous equation ($r = 0$) admits only the trivial solution $u = 0$. By the equivalence (12), the same is true for the equation $L[u] = r$. Since the homogeneous equation $L[u] = 0$ indeed admits only the trivial solution by (8), L is therefore onto.

2) “*The dual and symmetric case*”. Suppose that $Y = X'$, the (topological) *dual* of X , i.e. the space of all bounded linear functionals $l : X \rightarrow \mathbb{R}$. $X' (= \mathcal{B}(X, \mathbb{R}))$ is a Banach space endowed with the usual operator sup-norm. Indeed, this norm is generated by an inner product (which therefore makes X' a Hilbert space) as explained in the following: Consider the linear mapping $\Phi : X \rightarrow X'$ given by

$$(\Phi[u])[v] := \langle u, v \rangle_X \quad (u, v \in X). \quad (13)$$

For all $u \in X$,

$$\|\Phi[u]\|_{X'} = \sup_{v \in X \setminus \{0\}} \frac{|(\Phi[u])[v]|}{\|v\|_X} = \sup_{v \in X \setminus \{0\}} \frac{|\langle u, v \rangle_X|}{\|v\|_X} = \|u\|_X,$$

i.e. Φ is an *isometry* (and hence one-to-one).

Furthermore, Φ is *onto* by Riesz’ representation theorem for bounded linear functionals on

a Hilbert space: Given any $r \in X'$, some (unique) $u \in X$ exists such that $r[v] = \langle u, v \rangle_X$ for all $v \in X$, i.e. $\Phi[u] = r$ by (13). Φ is therefore called the *canonical isometric isomorphism* between X and X' . It immediately gives an inner product on X' by

$$\langle r, s \rangle_{X'} := \langle \Phi^{-1}[r], \Phi^{-1}[s] \rangle_X \quad (r, s \in X'), \quad (14)$$

and the norm generated by this inner product is the “old” norm $\|\cdot\|_{X'}$, because Φ is isometric.

In theoretical functional analysis, the Hilbert spaces X and X' are often identified via the isometric isomorphism Φ , i.e. they are not distinguished, which however we will *not* do because this might lead to confusion when X is a Sobolev function space, as it will be later.

To ensure that $L : X \rightarrow Y = X'$ is onto, we make the additional assumption that $\Phi^{-1}L : X \rightarrow X$ is *symmetric* with respect to $\langle \cdot, \cdot \rangle_X$, which by (13) amounts to the relation

$$(L[u])[v] = (L[v])[u] \text{ for all } u, v \in X. \quad (15)$$

This implies the denseness of the range $(\Phi^{-1}L)(X) \subset X$: Given any u in its orthogonal complement, we have, for all $v \in X$,

$$0 = \langle u, (\Phi^{-1}L)[v] \rangle_X = \langle (\Phi^{-1}L)[u], v \rangle_X,$$

and hence $(\Phi^{-1}L)[u] = 0$, which implies $L[u] = 0$ and thus $u = 0$ by (8).

Therefore, since Φ is isometric, the range $L(X) \subset X'$ is *dense*. For proving that L is onto, we are therefore left to show that $L(X) \subset X'$ is *closed*. For this purpose, let $(L[u_n])_{n \in \mathbb{N}}$ denote some sequence in $L(X)$ converging to some $r \in X'$. Then (8) shows that $(u_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in X . With $u \in X$ denoting its limit, the boundedness of L implies $L[u_n] \rightarrow L[u]$ ($n \rightarrow \infty$). Thus, $r = L[u] \in L(X)$, which proves closedness of $L(X)$.

We are now able to formulate and prove our main theorem, which is similar to the Newton-Cantorovich-Theorem:

Theorem 1: *Let δ, K, g satisfy conditions (7) - (10). Suppose that some $\alpha > 0$ exists such that*

$$\delta \leq \frac{\alpha}{K} - G(\alpha), \quad (16)$$

where $G(t) := \int_0^t g(s)ds$. Moreover, suppose that

1) “the compact case” is at hand,

or

2) “the dual and symmetric case” is at hand, and the additional condition

$$Kg(\alpha) < 1 \quad (17)$$

holds true.

Then, there exists a solution $u \in X$ of the equation $\mathcal{F}(u) = 0$ satisfying

$$\|u - \omega\|_X \leq \alpha. \quad (18)$$

Remark 1: a) Due to (10), $G(t) = \int_0^t g(s)ds$ is *superlinearly* small as $t \rightarrow 0$. Therefore, the crucial condition (16) is indeed satisfied for some “small” α if K is “moderate” (i.e. not too large) and δ is sufficiently small, which means according to (7) that the approximate solution ω to problem (5) must be computed with *sufficient accuracy*, and (16)

tells us *how* accurate the computation has to be. This meets the general philosophy of computer-assisted proofs: The “hard work” of the proof is left to the computer!

b) For proving Theorem 1, we will use the (abstract) *Green’s operator* L^{-1} to re-formulate problem (5) as a fixed-point equation, and apply some fixed-point theorem. If the space X were finite-dimensional, *Brouwer’s* Fixed-Point Theorem would be most suitable for this purpose. In the application to differential equation problems like (1), (2), however, X has to be infinite-dimensional, whence Brouwer’s Theorem is not applicable. We have two choices: i) Either we can use the generalization of Brouwer’s Theorem to infinite-dimensional spaces, i. e. *Schauder’s* Fixed-Point-Theorem, which explicitly requires additional compactness properties (holding automatically in the finite-dimensional case). In our application to (1), (2) discussed later, this compactness is given by compact embeddings of Sobolev function spaces, provided that the domain Ω is bounded (or at least has finite measure). Since we want to include unbounded domains in our consideration, too, we make also use of the second option: ii) We can use *Banach’s* Fixed-Point Theorem. No compactness is needed then, but instead an additional contraction condition (which is condition (17)) is required. Due to (10), this condition is however not too critical if α (computed according to (16)) is “small”.

Proof of Theorem 1. We rewrite problem (5) as

$$L[u - \omega] = -\mathcal{F}(\omega) - \{\mathcal{F}(u) - \mathcal{F}(\omega) - L[u - \omega]\},$$

which due to the bijectivity of L amounts to the equivalent fixed-point equation

$$v \in X, \quad v = -L^{-1}[\mathcal{F}(\omega) + \{\mathcal{F}(\omega + v) - \mathcal{F}(\omega) - L[v]\}] =: T(v) \quad (19)$$

for the error $v = u - \omega$.

Now we are going to show the following properties of the fixed-point operator $T : X \rightarrow X$:

- i) $T(V) \subset V$ for the closed, bounded, non-empty, and convex norm ball

$$V := \{v \in X : \|v\|_X \leq \alpha\},$$

- ii) T is continuous and compact (in case 1)) or contractive on V (in case 2)), respectively.

Then, Schauder’s Fixed-Point Theorem (in case 1)) or Banach’s Fixed-Point Theorem (in case 2)), respectively, gives a solution $v^* \in V$ of the fixed-point equation (19), whence by construction $u^* := \omega + v^*$ is a solution of $\mathcal{F}(u) = 0$ satisfying (18).

For proving i) and ii), we first note that for every differentiable function $f : [0, 1] \rightarrow Y$, the real-valued function $\|f\|_Y$ is differentiable almost everywhere on $[0, 1]$, and $(d/dt)\|f\|_Y \leq \|f'\|_Y$ a.e. on $[0, 1]$. Hence, for every $v, \tilde{v} \in X$,

$$\begin{aligned} & \|\mathcal{F}(\omega + v) - \mathcal{F}(\omega + \tilde{v}) - L[v - \tilde{v}]\|_Y \\ &= \int_0^1 \frac{d}{dt} \|\mathcal{F}(\omega + (1-t)\tilde{v} + tv) - \mathcal{F}(\omega + \tilde{v}) - tL[v - \tilde{v}]\|_Y dt \\ &\leq \int_0^1 \|\{\mathcal{F}'(\omega + (1-t)\tilde{v} + tv) - L\}[v - \tilde{v}]\|_Y dt \\ &\leq \int_0^1 \|\mathcal{F}'(\omega + (1-t)\tilde{v} + tv) - L\|_{\mathcal{B}(X,Y)} dt \cdot \|v - \tilde{v}\|_X \\ &\leq \int_0^1 g(\|(1-t)\tilde{v} + tv\|_X) dt \cdot \|v - \tilde{v}\|_X, \end{aligned} \quad (20)$$

using (6) and (9) in the last step. Choosing $\tilde{v} = 0$ in (20) we obtain, for each $v \in X$,

$$\|\mathcal{F}(\omega + v) - \mathcal{F}(\omega) - L[v]\|_Y \leq \int_0^1 g(t\|v\|_X) dt \cdot \|v\|_X = \int_0^{\|v\|_X} g(s) ds = G(\|v\|_X). \quad (21)$$

Furthermore, (20) and the fact that g is non-decreasing imply, for all $v, \tilde{v} \in V$,

$$\begin{aligned} \|\mathcal{F}(\omega + v) - \mathcal{F}(\omega + \tilde{v}) - L[v - \tilde{v}]\|_Y &\leq \int_0^1 g((1-t)\|\tilde{v}\|_X + t\|v\|_X) dt \cdot \|v - \tilde{v}\|_X \\ &\leq g(\alpha)\|v - \tilde{v}\|_X. \end{aligned} \quad (22)$$

To prove i), let $v \in V$, i.e. $\|v\|_X \leq \alpha$. Now (19), (8), (7), (21), and (16) imply

$$\begin{aligned} \|T(v)\|_X &\leq K\|\mathcal{F}(\omega) + \{\mathcal{F}(\omega + v) - \mathcal{F}(\omega) - L[v]\}\|_Y \\ &\leq K(\delta + G(\|v\|_X)) \leq K(\delta + G(\alpha)) \leq \alpha, \end{aligned}$$

which gives $T(v) \in V$. Thus, $T(V) \subset V$.

For proving ii), suppose first that “the compact case” is at hand. So (11), which in particular gives $L = L_0 + \mathcal{G}'(\omega)$, and (19) imply

$$T(v) = -L^{-1}[\mathcal{F}(\omega) + \{\mathcal{G}(\omega + v) - \mathcal{G}(\omega) - \mathcal{G}'(\omega)[v]\}] \text{ for all } v \in X,$$

whence continuity and compactness of T follow from continuity and compactness of $\mathcal{G}$ and $\mathcal{G}'(\omega)$, and the boundedness of L^{-1} ensured by (8).

If the “dual and symmetric case” is at hand, (19), (8), and (22) imply, for $v, \tilde{v} \in V$,

$$\begin{aligned} \|T(v) - T(\tilde{v})\|_X &= \|L^{-1}\{\mathcal{F}(\omega + v) - \mathcal{F}(\omega + \tilde{v}) - L[v - \tilde{v}]\}\|_X \\ &\leq K\|\mathcal{F}(\omega + v) - \mathcal{F}(\omega + \tilde{v}) - L[v - \tilde{v}]\|_Y \leq Kg(\alpha)\|v - \tilde{v}\|_X, \end{aligned}$$

whence (17) shows that T is contractive on V . This completes the proof of Theorem 1. $\square$

In the following two sections, we will apply the abstract approach developed in this section to the elliptic boundary value problem (1), (2). This can be done in (essentially two) different ways, i.e. by different choices of the Hilbert spaces X and Y , resulting in different general assumptions (e.g. smoothness conditions) to be made for the “data” of the problem and the numerical approximation ω , and different conditions (7) - (9), (16), (17), as well as different “results”, i.e. existence statements and error bounds (18).

At this point, we want to report briefly on some other applications of our abstract setting which we cannot discuss in more detail in this article.

For *parameter-dependent problems* (where $\mathcal{F}$ in (5), or f in (1), depends on an additional parameter λ), one is often interested in *branches* $(u_\lambda)_{\lambda \in I}$ of solutions. By additional perturbation techniques, our method can indeed be generalized to computer-assisted proofs for such solution branches, as long as the parameter-interval I defining the branch is compact [51]. Such branches may however contain *turning points* (where a branch “returns” at some value λ^*) or *bifurcation points* (where several -usually two- branches cross each other). Near such points, the operator L defined in (6) is “almost” singular, i.e. (8) holds only with a very large K , or not at all, which makes our approach break down. However, there are means to overcome these problems:

In case of (simple) turning points, the well-known method of augmenting the given equation by a *bordering* equation can also be used here; the “new” operator $\mathcal{F}$ in (5) contains the “old” one *and* the bordering functional, and the “new” operator L is regular near the turning point if the bordering equation has been chosen appropriately [50].

In case of (simple) symmetry-breaking bifurcations, we can, in a first step, include the symmetry in the spaces X and Y , which excludes the symmetry-breaking branch and regularizes the problem, whence an existence and enclosure result for the symmetric branch can be obtained. In a second step, we exclude the symmetric branch by some transformation (similar to the Lyapunov-Schmidt reduction), and defining a corresponding new operator $\mathcal{F}$ we can perform our method to obtain an existence and enclosure result also for the symmetry-breaking branch [52].

Non-selfadjoint eigenvalue problems have been treated in [38], again using bordering equation techniques normalizing the unknown eigenfunction. So $\mathcal{F}$ now acts on pairs (u, λ) , and is defined via the eigenvalue equation and the (scalar) normalizing equation. In this way, we were able to give the first known instability proof of the Orr-Sommerfeld equation with Blasius profile, which is a fourth-order ODE eigenvalue problem on $[0, \infty)$.

Also (other) *higher order* problems are covered by our abstract setting. In [11], we could prove the existence of 36 travelling wave solutions of a fourth-order nonlinear beam equation on the real line. Biharmonic problems (with $\Delta\Delta u$ as leading term) are presently investigated by B. Fazekas; see also [23].

3. Strong solutions

Now we study the elliptic boundary value problem (1), (2) under the additional assumptions that f and $\partial f/\partial y$ are continuous on $\bar{\Omega} \times \mathbb{R}$, and that the domain $\Omega \subset \mathbb{R}^n$ (with $n \leq 3$) is *bounded* with Lipschitz boundary, and H^2 -*regular* (i.e., for each $r \in L^2(\Omega)$, the Poisson problem $-\Delta u = r$ in Ω , $u = 0$ on $\partial\Omega$ has a unique solution $u \in H^2(\Omega) \cap H_0^1(\Omega)$).

Here and in the following, $L^2(\Omega)$ denotes the Hilbert space of all (equivalence classes of) square-integrable Lebesgue-measurable functions on Ω , endowed with the inner product

$$\langle u, v \rangle_{L^2} := \int_{\Omega} uv \, dx,$$

and $H^k(\Omega)$ is the Sobolev space of all functions $u \in L^2(\Omega)$ with weak derivatives up to order k in $L^2(\Omega)$. $H^k(\Omega)$ is a Hilbert space with the inner product

$$\langle u, v \rangle_{H^k} := \sum_{\substack{\alpha \in \mathbb{N}_0^n \\ |\alpha| \leq k}} \langle D^\alpha u, D^\alpha v \rangle_{L^2},$$

and it can also be characterized as the completion of $C^\infty(\bar{\Omega})$ with respect to $\langle \cdot, \cdot \rangle_{H^k}$. If we replace here $C^\infty(\bar{\Omega})$ by $C_0^\infty(\Omega)$ (with the subscript 0 indicating compact support in Ω), we obtain, by completion, the Sobolev space $H_0^k(\Omega)$, which incorporates the vanishing of all derivatives up to order $k-1$ on $\partial\Omega$ in a weak sense.

We note that *piecewise* C^k -smooth functions u (e.g. form functions of Finite Element methods) belong to $H^k(\Omega)$ if and only if they are (globally) in $C^{k-1}(\bar{\Omega})$.

Our assumption that Ω is H^2 -regular is satisfied e.g. for C^2 - (or $C^{1,1}$ -) *smoothly* bounded domains (see e.g. [27]), and also for *convex* polygonal domains $\Omega \subset \mathbb{R}^2$ [28]; it is *not* satisfied e.g. for domains with re-entrant corners, like the L -shaped domain $(-1, 1)^2 \setminus [0, 1)^2$.

Under the assumptions made, we can choose the spaces

$$X := H^2(\Omega) \cap H_0^1(\Omega), \quad Y := L^2(\Omega), \quad (23)$$

and the operators

$$\mathcal{F} := L_0 + \mathcal{G}, \quad L_0[u] := -\Delta u, \quad \mathcal{G}(u) := f(\cdot, u), \quad (24)$$

whence indeed our problem (1), (2) amounts to the abstract problem (5). Moreover, $L_0 : X \rightarrow Y$ is bijective by the assumed unique solvability of the Poisson problem, and clearly bounded, i.e. in $\mathcal{B}(X, Y)$. Finally, $\mathcal{G} : X \rightarrow Y$ is Fréchet differentiable with derivative given by

$$\mathcal{G}'(u)[v] = \frac{\partial f}{\partial y}(\cdot, u)v, \quad (25)$$

which follows from the fact that $\mathcal{G}$ has this derivative as an operator from $C(\bar{\Omega})$ (endowed with the maximum norm $\|\cdot\|_\infty$) into itself, and that the embeddings $H^2(\Omega) \hookrightarrow C(\bar{\Omega})$ and $C(\bar{\Omega}) \hookrightarrow L^2(\Omega)$ are bounded. In fact, $H^2(\Omega) \hookrightarrow C(\bar{\Omega})$ is even a *compact* embedding by the famous Sobolev-Kondrachev-Rellich Embedding Theorem [1] (and since $n \leq 3$), which shows that $\mathcal{G}$ and $\mathcal{G}'(u)$ (for any $u \in X$) are compact. Thus, “the compact case” (see (11)) is at hand.

For the application of Theorem 1, we are therefore left to comment on the computation of constants δ and K , and a function g which satisfy (7) - (10) (in the setting (23), (24)). But first, some comments on the computation of the approximate solution ω should be made.

3.1. Computation of ω . Since ω is required to be in $X = H^2(\Omega) \cap H_0^1(\Omega)$, it has to satisfy the boundary condition *exactly* (in the sense of being in $H_0^1(\Omega)$), and it needs to have weak derivatives in $L^2(\Omega)$ up to order 2. If Finite Elements shall be used, this implies the need for C^1 -elements (i.e. *globally* C^1 -smooth Finite Element basis functions), which is a drawback at least on a technical level. (In the alternative approach proposed in the next section, this drawback is avoided.) If $\Omega = (0, a) \times (0, b)$ is a *rectangle*, there are however many alternatives to Finite Elements, for example polynomial or trigonometric polynomial basis functions. E.g. in the latter case, ω is put up in the form

$$\omega(x_1, x_2) = \sum_{i=1}^N \sum_{j=1}^M \alpha_{ij} \sin\left(i\pi \frac{x_1}{a}\right) \sin\left(j\pi \frac{x_2}{b}\right), \quad (26)$$

with coefficients α_{ij} to be determined by some numerical procedure. Such a procedure usually consists of a Newton iteration, together with e.g. a Ritz-Galerkin or a collocation method, and some linear algebraic system solver, which possibly incorporates multigrid methods. To *start* the Newton iteration, a rough initial approximation is needed, which can e.g. be obtained by path-following methods, or by use of the numerical mountain pass algorithm proposed in [15].

An important remark is that, no matter how ω is put up or which numerical method is used, there is *no* need for any *rigorous* (i.e. error free) computation at this stage, i.e. the whole variety of numerical methods is at hand.

3.2. Defect bound δ . Computing some δ satisfying (7) means, due to (23) and (24), computing an upper bound for (the square root of)

$$\int_{\Omega} [-\Delta\omega + f(\cdot, \omega)]^2 dx \quad (27)$$

(which should be “small” if ω is a “good” approximate solution). In some cases this integral can be calculated in closed form, by hand or by computer algebra routines, for example if f is polynomial and ω is piecewise polynomial (as it is if Finite Element methods have been used to compute it), or if $f(x, \cdot)$ is polynomial and both $f(\cdot, y)$ and ω are trigonometric polynomial (compare (26)). The resulting formulas have to be evaluated *rigorously*, to obtain a true upper bound for the integral in (27). For this purpose, *interval arithmetic* [35, 57] must be used in this evaluation, in order to take rounding errors into account.

If closed form integration is impossible, a *quadrature formula* should be applied, possibly piecewise, to the integral in (27), again with evaluation in interval arithmetic. To obtain a true upper bound for the integral, we need in addition a *remainder term bound* for the quadrature formula, which usually requires rough $\|\cdot\|_{\infty}$ -bounds for some higher derivatives of the integrand. Such rough bounds can be obtained e.g. by subdividing Ω into (many) small boxes, and performing interval evaluations of the needed higher derivatives over each of these boxes (which gives true supersets of the function value ranges over each of the boxes, and thus, by union, over Ω).

3.3. Bound K for L^{-1} . The next task is the computation of a constant K satisfying (8), which due to (23) - (25) means

$$\|u\|_{H^2} \leq K \|L[u]\|_{L^2} \quad \text{for all } u \in H^2(\Omega) \cap H_0^1(\Omega), \quad (28)$$

where $L : H^2(\Omega) \cap H_0^1(\Omega) \rightarrow L^2(\Omega)$ is given by

$$L[u] = -\Delta u + cu, \quad c(x) := \frac{\partial f}{\partial y}(x, \omega(x)) \quad (x \in \bar{\Omega}). \quad (29)$$

The first (and most crucial) step towards (28) is the computation of a constant K_0 such that

$$\|u\|_{L^2} \leq K_0 \|L[u]\|_{L^2} \quad \text{for all } u \in H^2(\Omega) \cap H_0^1(\Omega). \quad (30)$$

Choosing some constant lower bound $\underline{c}$ for c on $\bar{\Omega}$, and using the compact embedding $H^2(\Omega) \hookrightarrow L^2(\Omega)$, we find by standard means that $(L - \underline{c})^{-1} : L^2(\Omega) \rightarrow L^2(\Omega)$ is compact, symmetric, and positive definite, and hence has a $\langle \cdot, \cdot \rangle_{L^2}$ -orthonormal and complete system $(\varphi_k)_{k \in \mathbb{N}}$ of eigenfunctions $\varphi_k \in H^2(\Omega) \cap H_0^1(\Omega)$, with associated sequence $(\mu_k)_{k \in \mathbb{N}}$ of (positive) eigenvalues converging monotonically to 0. Consequently, $L[\varphi_k] = \lambda_k \varphi_k$ for $k \in \mathbb{N}$, with $\lambda_k = \mu_k^{-1} + \underline{c}$ converging monotonically to $+\infty$. Series expansion yields, for every $u \in H^2(\Omega) \cap H_0^1(\Omega)$,

$$\begin{aligned} \|L[u]\|_{L^2}^2 &= \sum_{k=1}^{\infty} \langle L[u], \varphi_k \rangle_{L^2}^2 = \sum_{k=1}^{\infty} \langle u, L[\varphi_k] \rangle_{L^2}^2 = \sum_{k=1}^{\infty} \lambda_k^2 \langle u, \varphi_k \rangle_{L^2}^2 \\ &\geq \left(\min_{j \in \mathbb{N}} \lambda_j^2 \right) \sum_{k=1}^{\infty} \langle u, \varphi_k \rangle_{L^2}^2 = \left(\min_{j \in \mathbb{N}} \lambda_j^2 \right) \|u\|_{L^2}^2, \end{aligned}$$

which shows that (30) holds if (and only if) $\lambda_j \neq 0$ for all $j \in \mathbb{N}$, and

$$K_0 \geq \left(\min_{j \in \mathbb{N}} |\lambda_j| \right)^{-1}. \quad (31)$$

Thus, bounds for the eigenvalue(s) of L neighboring 0 are needed to compute K_0 . Such *eigenvalue bounds* can be obtained by computer-assisted means of their own. For example, *upper* bounds to $\lambda_1, \dots, \lambda_N$ (with $N \in \mathbb{N}$ given) are easily and efficiently computed by the *Rayleigh-Ritz* method [56]:

Let $\tilde{\varphi}_1, \dots, \tilde{\varphi}_N \in H^2(\Omega) \cap H_0^1(\Omega)$ denote linearly independent trial functions, for example approximate eigenfunctions obtained by numerical means, and form the matrices

$$A_1 := (\langle L[\tilde{\varphi}_i], \tilde{\varphi}_j \rangle_{L^2})_{i,j=1,\dots,N}, \quad A_0 := (\langle \tilde{\varphi}_i, \tilde{\varphi}_j \rangle_{L^2})_{i,j=1,\dots,N}.$$

Then, with $\Lambda_1 \leq \dots \leq \Lambda_N$ denoting the eigenvalues of the matrix eigenvalue problem

$$A_1 x = \Lambda A_0 x$$

(which can be enclosed by means of verifying numerical linear algebra; see [8]), the Rayleigh-Ritz method gives

$$\lambda_i \leq \Lambda_i \text{ for } i = 1, \dots, N.$$

However, for computing K_0 via (31), also *lower* eigenvalue bounds are needed, which constitute a more complicated task than upper bounds. The most accurate method for this purpose has been proposed by Lehmann [42], and improved by Goerisch concerning its range of applicability [9]. Its numerical core is again (as in the Rayleigh-Ritz method) a matrix eigenvalue problem, but the accompanying analysis is more involved. In most cases, the Lehmann-Goerisch method must be combined with a *homotopy* method connecting the given eigenvalue problem to a simple “base” problem with known eigenvalues. A detailed description of these methods would be beyond the scope of this article. Instead, we refer to [53] for more details.

Once a constant K_0 satisfying (30) is known, the desired constant K (satisfying (28)) can relatively easily be calculated by explicit *a priori estimates*: With $\underline{c}$ denoting a constant lower bound for c , we obtain by partial integration, for each $u \in H^2(\Omega) \cap H_0^1(\Omega)$,

$$\|u\|_{L^2} \|L[u]\|_{L^2} \geq \langle u, L[u] \rangle_{L^2} = \int_{\Omega} (|\nabla u|^2 + cu^2) dx \geq \|\nabla u\|_{L^2}^2 + \underline{c} \|u\|_{L^2}^2,$$

which implies, together with (30), that

$$\|\nabla u\|_{L^2} \leq K_1 \|L[u]\|_{L^2}, \text{ where } K_1 := \begin{cases} \sqrt{K_0(1 - \underline{c}K_0)} & \text{if } \underline{c}K_0 \leq \frac{1}{2} \\ \frac{1}{2\sqrt{\underline{c}}} & \text{otherwise} \end{cases}. \quad (32)$$

To complete the H^2 -bound required in (28), we need to estimate the L^2 -norm of the (Frobenius matrix norm of the) Hessian matrix u_{xx} of $u \in H^2(\Omega) \cap H_0^1(\Omega)$. If Ω is convex (as we shall assume now), we have

$$\|u_{xx}\|_{L^2} \leq \|\Delta u\|_{L^2} \text{ for all } u \in H^2(\Omega) \cap H_0^1(\Omega) \quad (33)$$

(see e.g. [28, 37]); for the non-convex case, we refer to [28, 49]. Now, with $\bar{c}$ denoting an additional upper bound for c , we choose $\mu := \max\{0, \frac{1}{2}(\bar{c} - \underline{c})\}$, and calculate

$$\|\Delta u\|_{L^2} \leq \|-\Delta u + \mu u\|_{L^2} \leq \|L[u]\|_{L^2} + \|\mu - c\|_{\infty} \|u\|_{L^2}.$$

Using that $\|\mu - c\|_{\infty} = \max\{-\underline{c}, \frac{1}{2}(\bar{c} - \underline{c})\}$, and combining with (30), we obtain

$$\|\Delta u\|_{L^2} \leq K_2 \|L[u]\|_{L^2}, \text{ where } K_2 := 1 + K_0 \max\{-\underline{c}, \frac{1}{2}(\bar{c} - \underline{c})\}. \quad (34)$$

Now, (30), (32), (34) give (28) as follows. For quantitative purposes, we use the modified inner product

$$\langle u, v \rangle_{H^2} := \gamma_0 \langle u, v \rangle_{L^2} + \gamma_1 \langle \nabla u, \nabla v \rangle_{L^2} + \gamma_2 \langle \Delta u, \Delta v \rangle_{L^2} \quad (35)$$

(with positive weights $\gamma_0, \gamma_1, \gamma_2$) on X , which due to (33) (and to the obvious reverse inequality $\|\Delta u\|_{L^2} \leq \sqrt{n} \|u_{xx}\|_{L^2}$) is equivalent to the canonical one. Then, (28) obviously holds for

$$K := \sqrt{\gamma_0 K_0^2 + \gamma_1 K_1^2 + \gamma_2 K_2^2}, \quad (36)$$

with K_0, K_1, K_2 from (30), (32), (34).

3.4. Local Lipschitz bound g for $\mathcal{F}'$. By (23), (24), and (25), condition (9) reads

$$\left\| \left[\frac{\partial f}{\partial y}(\cdot, \omega + u) - \frac{\partial f}{\partial y}(\cdot, \omega) \right] v \right\|_{L^2} \leq g(\|u\|_{H^2}) \|v\|_{H^2} \quad \text{for all } u, v \in H^2(\Omega) \cap H_0^1(\Omega). \quad (37)$$

We start with a monotonically non-decreasing function $\tilde{g} : [0, \infty) \rightarrow [0, \infty)$ satisfying

$$\left| \frac{\partial f}{\partial y}(x, \omega(x) + y) - \frac{\partial f}{\partial y}(x, \omega(x)) \right| \leq \tilde{g}(|y|) \quad \text{for all } x \in \Omega, \ y \in \mathbb{R}, \quad (38)$$

and $\tilde{g}(t) \rightarrow 0$ as $t \rightarrow 0+$. In practice, such a function $\tilde{g}$ can usually be calculated by hand, if a bound for $\|\omega\|_\infty$ is available, which in turn can be computed by interval evaluations of ω over small boxes (as described at the end of Subsection 3.2).

Using $\tilde{g}$, the left-hand side of (37) can be bounded by

$$\tilde{g}(\|u\|_\infty) \|v\|_{L^2}, \quad (39)$$

whence we are left to estimate both the norms $\|\cdot\|_{L^2}$ and $\|\cdot\|_\infty$ by $\|\cdot\|_{H^2}$. With ρ^* denoting the smallest eigenvalue of

$$-\Delta u = \rho u, \quad u \in H^2(\Omega) \cap H_0^1(\Omega),$$

we obtain by eigenfunction expansion that

$$\|\nabla u\|_{L^2}^2 = \langle u, -\Delta u \rangle_{L^2} \geq \rho^* \|u\|_{L^2}^2, \quad \|\Delta u\|_{L^2}^2 \geq (\rho^*)^2 \|u\|_{L^2}^2,$$

and thus, by (35),

$$\|u\|_{L^2} \leq [\gamma_0 + \gamma_1 \rho^* + \gamma_2 (\rho^*)^2]^{-\frac{1}{2}} \|u\|_{H^2} \quad \text{for all } u \in H^2(\Omega) \cap H_0^1(\Omega). \quad (40)$$

Furthermore, in [49, Corollary 1], we calculate constants C_0, C_1, C_2 , which depend on Ω in a rather simple way allowing explicit computation, such that

$$\|u\|_\infty \leq C_0 \|u\|_{L^2} + C_1 \|\nabla u\|_{L^2} + C_2 \|u_{xx}\|_{L^2} \quad \text{for all } u \in H^2(\Omega) \cap H_0^1(\Omega),$$

whence by (33) and (35) we obtain

$$\|u\|_\infty \leq [\gamma_0^{-1} C_0^2 + \gamma_1^{-1} C_1^2 + \gamma_2^{-1} C_2^2]^{\frac{1}{2}} \|u\|_{H^2} \quad \text{for all } u \in H^2(\Omega) \cap H_0^1(\Omega). \quad (41)$$

Using (40) and (41) in (39), we find that (37) (and (10)) hold for

$$g(t) := [\gamma_0 + \gamma_1 \rho^* + \gamma_2 (\rho^*)^2]^{-\frac{1}{2}} \tilde{g} \left([\gamma_0^{-1} C_0^2 + \gamma_1^{-1} C_1^2 + \gamma_2^{-1} C_2^2]^{\frac{1}{2}} t \right). \quad (42)$$

Remark 2: Via (36) and (42), the parameters $\gamma_0, \gamma_1, \gamma_2$ enter the crucial inequality (16). One can choose these parameters in order to minimize the error bound α (under some normalization condition on $(\gamma_0, \gamma_1, \gamma_2)$, e.g. $\gamma_0 + \gamma_1 + \gamma_2 = 1$), or to maximize $\max\{\alpha/K - G(\alpha) : \alpha \geq 0\}$ (to allow a larger defect bound δ in (16)). Of course, this optimization need only be carried out *approximately*.

3.5. A numerical example. Consider the problem

$$\Delta u + u^2 = s \cdot \sin(\pi x_1) \sin(\pi x_2) \quad (x = (x_1, x_2) \in \Omega := (0, 1)^2), \quad u = 0 \text{ on } \partial\Omega. \quad (43)$$

The results reported here have been established in [10] in joint work with P. J. McKenna and B. Breuer.

It had been conjectured in the PDE community since the 1980's that problem (43) has at least 4 solutions for $s > 0$ sufficiently large.

For $s = 800$, we were able to compute 4 essentially different approximate solutions by the numerical mountain pass algorithm developed in [15], where “essentially different” means that none of them is an elementary symmetry transform of another one. Using finite Fourier series of the form (26), and a Newton iteration in combination with a collocation method, we improved the accuracy of the mountain pass solutions, resulting in highly accurate approximations $\omega_1, \dots, \omega_4$ of the form (26).

We applied our computer-assisted enclosure method to each of these four approximations, and were successful in verifying the corresponding four inequalities (16), with four error bounds $\alpha_1, \dots, \alpha_4$. Therefore, Theorem 1 guarantees the existence of four solutions $u_1, \dots, u_4 \in H^2(\Omega) \cap H_0^1(\Omega)$ of problem (43) such that

$$\|u_i - \omega_i\|_{H^2} \leq \alpha_i \quad (i = 1, \dots, 4).$$

Using the embedding inequality (41), we obtain in addition

$$\|u_i - \omega_i\|_{\infty} \leq \beta_i \quad (i = 1, \dots, 4) \quad (44)$$

for $\beta_i := [\gamma_0^{-1}C_0^2 + \gamma_1^{-1}C_1^2 + \gamma_2^{-1}C_2^2]^{\frac{1}{2}}\alpha_i$. Finally, it is easy to check on the basis of the numerical data that

$$\|S\omega_i - \omega_j\|_{\infty} > \beta_i + \beta_j \quad (i, j = 1, \dots, 4, \quad i \neq j)$$

for each elementary (rotation or reflection) symmetry transformation S of the square Ω , whence (44) shows that $Su_i \neq u_j$ ($i, j = 1, \dots, 4, \quad i \neq j$) for each of these S , i.e. that $u_1, \dots, u_4$ are indeed *essentially different*.

The following Figure 1 shows plots of $\omega_1, \dots, \omega_4$ (we might say as well: of $u_1, \dots, u_4$, since the error bounds β_i are much smaller than the “optical accuracy” of the figure). The first two solutions are fully symmetric (with respect to reflection at the axes $x_1 = \frac{1}{2}, x_2 = \frac{1}{2}, x_1 = x_2, x_1 = 1 - x_2$), while the third is symmetric only with respect to $x_2 = \frac{1}{2}$, and the fourth only with respect to $x_1 = x_2$.

Table 1 shows the defect bounds δ (see (7), (27)), the constants K satisfying (8) (or (28)), and the $\|\cdot\|_{\infty}$ -error bounds β (see (44)) for the four solutions.

We wish to remark that, two years after publication of our result, Dancer and Yan [21] gave a more general analytical proof (which we believe was stimulated by our result); they even proved that the number of solutions of problem (43) becomes unbounded as $s \rightarrow \infty$.

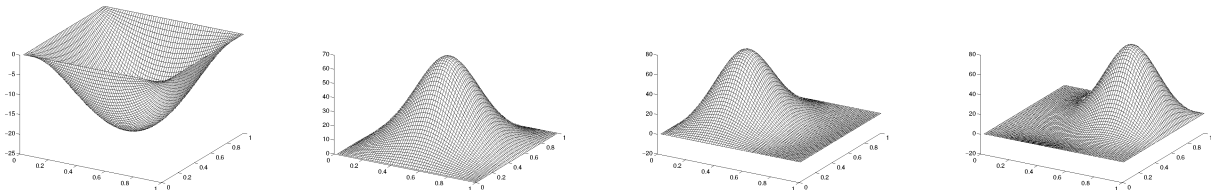


Figure 1: Four solutions to problem (43), $s = 800$.

approximate solution	defect bound δ	K (see (28))	error bound β
ω_1	0.0023	0.2531	$5.8222 \cdot 10^{-4}$
ω_2	0.0041	4.9267	0.0228
ω_3	0.0059	2.8847	0.0180
ω_4	0.0151	3.1436	0.0581

Table 1: Enclosure results for problem (43).

4. Weak solutions

We will now investigate problem (1), (2) under weaker assumptions on the domain $\Omega \subset \mathbb{R}^n$ and on the numerical approximation method, but stronger assumptions on the nonlinearity f , compared with the “strong solutions” approach described in the previous section. Ω is now allowed to be any (bounded or unbounded) domain with Lipschitz boundary. We choose the spaces

$$X := H_0^1(\Omega), \quad Y := H^{-1}(\Omega) \quad (45)$$

for our abstract setting, where $H^{-1}(\Omega) := (H_0^1(\Omega))'$ denotes the topological dual space of $H_0^1(\Omega)$, i.e. the space of all bounded linear functionals on $H_0^1(\Omega)$. We endow $H_0^1(\Omega)$ with the inner product

$$\langle u, v \rangle_{H_0^1} := \langle \nabla u, \nabla v \rangle_{L^2} + \sigma \langle u, v \rangle_{L^2} \quad (46)$$

(with some parameter $\sigma > 0$ to be chosen later), and $H^{-1}(\Omega)$ with the “dual” inner product given by (14), with Φ from (13).

To interpret our problem (1), (2) in these spaces, we first need to define Δu (for $u \in H_0^1(\Omega)$), or more general, $\operatorname{div} \rho$ (for $\rho \in L^2(\Omega)^n$), as an element of $H^{-1}(\Omega)$. This definition simply imitates partial integration: The functional $\operatorname{div} \rho : H_0^1(\Omega) \rightarrow \mathbb{R}$ is given by

$$(\operatorname{div} \rho)[\varphi] := - \int_{\Omega} \rho \cdot \nabla \varphi \, dx \quad \text{for all } \varphi \in H_0^1(\Omega), \quad (47)$$

implying in particular that $|(\operatorname{div} \rho)[\varphi]| \leq \|\rho\|_{L^2} \|\nabla \varphi\|_{L^2} \leq \|\rho\|_{L^2} \|\varphi\|_{H_0^1}$, whence $\operatorname{div} \rho$ is indeed a *bounded* linear functional, and

$$\|\operatorname{div} \rho\|_{H^{-1}} \leq \|\rho\|_{L^2}. \quad (48)$$

Using this definition of $\Delta u (= \operatorname{div}(\nabla u))$, it is easy to check that the canonical isometric isomorphism $\Phi : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$ defined in (13) is now given by (note (46))

$$\Phi[u] = -\Delta u + \sigma u \quad (u \in H_0^1(\Omega)), \quad (49)$$

where $\sigma u \in H_0^1(\Omega)$ is interpreted as an element of $H^{-1}(\Omega)$ as explained in the following.

Next, we give a meaning to a *function* being an element of $H^{-1}(\Omega)$, in order to define $f(\cdot, u)$ in (1) (and σu in (49)) in $H^{-1}(\Omega)$. For this purpose, let $\mathcal{L}$ denote the linear space consisting of all (equivalence classes of) Lebesgue-measurable functions $w : \Omega \rightarrow \mathbb{R}$ such that

$$\sup \left\{ \frac{1}{\|\varphi\|_{H_0^1}} \int_{\Omega} |w \varphi| \, dx : \varphi \in H_0^1(\Omega) \setminus \{0\} \right\} < \infty. \quad (50)$$

For each $w \in \mathcal{L}$, we can define an associated linear functional $\ell_w : H_0^1(\Omega) \rightarrow \mathbb{R}$ by

$$\ell_w[\varphi] := \int_{\Omega} w\varphi dx \quad \text{for all } \varphi \in H_0^1(\Omega).$$

ℓ_w is bounded due to (50) and hence in $H^{-1}(\Omega)$. Identifying $w \in \mathcal{L}$ with its associated functional $\ell_w \in H^{-1}(\Omega)$, we obtain

$$\mathcal{L} \subset H^{-1}(\Omega), \quad (51)$$

and $\|w\|_{H^{-1}}$ is less than or equal to the left-hand side of (50), for every $w \in \mathcal{L}$.

To get a better impression of the functions contained in $\mathcal{L}$, we recall that Sobolev's Embedding Theorem [1, Theorem 5.4] gives $H_0^1(\Omega) \subset L^p(\Omega)$, with bounded embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$ (i.e. there exists some constant $C_p > 0$ such that $\|u\|_{L^p} \leq C_p \|u\|_{H_0^1}$ for all $u \in H_0^1(\Omega)$), for each

$$p \in [2, \infty) \text{ if } n = 2, \quad p \in \left[2, \frac{2n}{n-2}\right] \text{ if } n \geq 3. \quad (52)$$

Here, $L^p(\Omega)$ denotes the Banach space of all (equivalence classes of) Lebesgue-measurable functions $u : \Omega \rightarrow \mathbb{R}$ with finite norm

$$\|u\|_{L^p} := \left[\int_{\Omega} |u|^p dx \right]^{\frac{1}{p}}. \quad (53)$$

With p in the range (52), and p' denoting its dual number (i.e. $p^{-1} + (p')^{-1} = 1$), we obtain by Hölder's Inequality, combined with the above embedding, that for all $w \in L^{p'}(\Omega)$

$$\int_{\Omega} |w\varphi| dx \leq \|w\|_{L^{p'}} \|\varphi\|_{L^p} \leq C_p \|w\|_{L^{p'}} \|\varphi\|_{H_0^1},$$

implying $w \in \mathcal{L}$, and $\|w\|_{H^{-1}} \leq C_p \|w\|_{L^{p'}}$. Consequently,

$$L^{p'}(\Omega) \subset \mathcal{L}, \quad (54)$$

and (note (51)) the embedding $L^{p'}(\Omega) \hookrightarrow H^{-1}(\Omega)$ is bounded, with the same embedding constant C_p as in the “dual” embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$. Note that the range (52) for p amounts to the range

$$p' \in (1, 2] \text{ if } n = 2, \quad p' \in \left[\frac{2n}{n+2}, 2\right] \text{ if } n \geq 3 \quad (55)$$

for the dual number p' .

By (54), the linear span of the union of all $L^{p'}(\Omega)$, taken over p' in the range (55), is a subspace of $\mathcal{L}$, and this subspace is in fact all of $\mathcal{L}$ which we need (and can access) in practical applications.

Coming back to our problem (1), (2), we now simply require that

$$f(\cdot, u) \in \mathcal{L} \quad \text{for all } u \in H_0^1(\Omega), \quad (56)$$

in order to define the term $f(\cdot, u)$ as an element of $H^{-1}(\Omega)$.

Our abstract setting requires furthermore that

$$\mathcal{F} : \begin{cases} H_0^1(\Omega) & \rightarrow & H^{-1}(\Omega) \\ u & \mapsto & -\Delta u + f(\cdot, u) \end{cases} \quad (57)$$

is Fréchet-differentiable. Since $\Delta : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$ is linear and bounded by (48), this amounts to the Fréchet-differentiability of

$$\mathcal{G} : \begin{cases} H_0^1(\Omega) & \rightarrow H^{-1}(\Omega) \\ u & \mapsto f(\cdot, u) \end{cases}. \quad (58)$$

For this purpose, we require (as in the previous section) that $\partial f/\partial y$ is continuous on $\bar{\Omega} \times \mathbb{R}$. But in contrast to the “strong solutions” setting, this is not sufficient here; the main reason is that $H_0^1(\Omega)$ does not embed into $C(\bar{\Omega})$. We need additional *growth restrictions* on $f(x, y)$ or $(\partial f/\partial y)(x, y)$ as $|y| \rightarrow \infty$.

An important (but not the only) admissible class consists of those functions f which satisfy

$$f(\cdot, 0) \in \mathcal{L}, \quad (59)$$

$$\frac{\partial f}{\partial y}(\cdot, 0) \text{ is a bounded function on } \Omega, \quad (60)$$

$$\left| \frac{\partial f}{\partial y}(x, y) - \frac{\partial f}{\partial y}(x, 0) \right| \leq c_1 |y|^{r_1} + c_2 |y|^{r_2} \quad (x \in \Omega, y \in \mathbb{R}), \quad (61)$$

with non-negative constants c_1, c_2 , and with

$$0 < r_1 \leq r_2 < \infty \quad \text{if } n = 2, \quad 0 < r_1 \leq r_2 \leq \frac{4}{n-2} \quad \text{if } n \geq 3. \quad (62)$$

(A “small” r_1 will make condition (61) weak near $y = 0$, and a “large” r_2 will make it weak for $|y| \rightarrow \infty$.)

Lemma 1: *Let f satisfy (59) - (61), besides the continuity of $\partial f/\partial y$. Then $\mathcal{G}$ given by (58) is well-defined and Fréchet-differentiable, with derivative $\mathcal{G}'(u) \in \mathcal{B}(H_0^1(\Omega), H^{-1}(\Omega))$ (for $u \in H_0^1(\Omega)$) given by*

$$(\mathcal{G}'(u)[v])[\varphi] = \int_{\Omega} \frac{\partial f}{\partial y}(\cdot, u) v \varphi dx \quad (v, \varphi \in H_0^1(\Omega)). \quad (63)$$

The proof of Lemma 1 is rather technical, and therefore omitted here.

According to (47) and (63), we have

$$(\mathcal{F}'(u)[\varphi])[\psi] = \int_{\Omega} \left[\nabla \varphi \cdot \nabla \psi + \frac{\partial f}{\partial y}(\cdot, u) \varphi \psi \right] dx = (\mathcal{F}'(u)[\psi])[\varphi] \quad (u, \varphi, \psi \in H_0^1(\Omega)) \quad (64)$$

for the operator $\mathcal{F}$ defined in (57), which in particular implies condition (15) (for *any* $\omega \in H_0^1(\Omega)$; note (6)), in the setting (45), (57). Thus, the “dual and symmetric case” (see Section 2) is at hand.

Remark 3: If the domain Ω is *bounded*, several simplifications and extensions are possible:

- a) The range $\sigma > 0$ for the parameter in (46) can be extended to $\sigma \geq 0$.
- b) Condition (61) can be simplified to

$$\left| \frac{\partial f}{\partial y}(x, y) \right| \leq \tilde{c}_1 + \tilde{c}_2 |y|^r \quad (x \in \Omega, y \in \mathbb{R}) \quad (65)$$

for some r in the range (62). Condition (60) is satisfied automatically and can therefore be omitted.

c) In the case $n = 2$, the power-growth condition (61) (or (65)) is too restrictive (for bounded domains). Instead, *exponential* growth can be allowed, based on the Trudinger-Moser inequality [45, Theorem 1 and the first part of its proof] which states that

$$\frac{1}{\text{meas}(\Omega)} \int_{\Omega} \exp \left[\left(\frac{u(x)}{c\|u\|_{H_0^1}} \right)^2 \right] dx \leq 1 + \frac{1}{4\pi c^2 - 1} \quad (u \in H_0^1(\Omega)) \quad (66)$$

for each $c > (4\pi)^{-\frac{1}{2}}$. In [54], we showed that e.g. in the case $f(x, y) = -\lambda e^y$, the Fréchet differentiability (and other properties) of the mapping $\mathcal{G}$ defined in (58) can easily be derived from (66); see also the second example in Subsection 4.5.

Again, we comment now on the computation of an approximate solution ω , and of the terms δ, K , and g satisfying (7) - (10), needed for the application of Theorem 1, here in the setting (45), (57).

4.1. Computation of ω . By (45), ω needs to be in $X = H_0^1(\Omega)$ only (and no longer in $H^2(\Omega)$, as in the “strong solutions” approach of the previous Section). In the Finite Element context, this increases the class of allowed elements significantly; for example, the “usual” linear (or quadratic) triangular elements can be used. In case of an unbounded domain Ω , we are furthermore allowed to use approximations ω of the form

$$\omega = \left\{ \begin{array}{ll} \omega_0 & \text{on } \Omega_0 \\ 0 & \text{on } \Omega \setminus \Omega_0 \end{array} \right\}, \quad (67)$$

with $\Omega_0 \subset \Omega$ denoting some bounded subdomain (the “computational” domain), and $\omega_0 \in H_0^1(\Omega_0)$ some approximate solution of the differential equation (1) on Ω_0 , subject to Dirichlet boundary conditions on $\partial\Omega_0$.

We pose the additional condition of ω being *bounded*, which on one hand is satisfied anyway for all practical numerical schemes, and on the other hand turns out to be very useful in the following.

4.2. Defect bound δ . By (45) and (57), condition (7) for the defect bound δ now amounts to

$$\| -\Delta\omega + f(\cdot, \omega) \|_{H^{-1}} \leq \delta, \quad (68)$$

which is a slightly more complicated task than computing an upper bound for an integral (as it was needed in Section 3). The best general way seems to be the following. First we compute an additional approximation $\rho \in H(\text{div}, \Omega)$ to $\nabla\omega$. (Here, $H(\text{div}, \Omega)$ denotes the space of all vector-valued functions $\tau \in L^2(\Omega)^n$ with weak derivative $\text{div } \tau$ in $L^2(\Omega)$. Hence, obviously $H(\text{div}, \Omega) \supset H^1(\Omega)^n$.) ρ can be computed e.g. by interpolation (or some more general projection) of $\nabla\omega$ in $H(\text{div}, \Omega)$, or in $H^1(\Omega)^n$. It should be noted that ρ comes “for free” as a part of the approximation, if *mixed* Finite Elements are used to compute ω .

Furthermore, according to the arguments before and after (54), applied with $p = p' = 2$,

$$\|w\|_{H^{-1}} \leq C_2 \|w\|_{L^2} \quad \text{for all } w \in L^2(\Omega). \quad (69)$$

For *explicit* calculation of C_2 , we refer to the appendix. By (48) and (69),

$$\begin{aligned} \| -\Delta\omega + f(\cdot, \omega) \|_{H^{-1}} &\leq \| \text{div}(-\nabla\omega + \rho) \|_{H^{-1}} + \| -\text{div}\rho + f(\cdot, \omega) \|_{H^{-1}} \\ &\leq \| \nabla\omega - \rho \|_{L^2} + C_2 \| -\text{div}\rho + f(\cdot, \omega) \|_{L^2}, \end{aligned} \quad (70)$$

which reduces the computation of a defect bound δ (satisfying (68)) to computing bounds for two *integrals*, i.e. we are back to the situation discussed in Subsection 3.2 already.

There is an alternative way to compute δ if ω is of the form (67), with $\omega_0 \in H^2(\Omega_0) \cap H_0^1(\Omega_0)$, and with Ω_0 having a Lipschitz boundary. This situation can arise e.g. if Ω is the whole of $\mathbb{R}^n$, and the “computational” domain Ω_0 is chosen as a “large” rectangle, whence ω_0 can be put up e.g. in the form (26).

Using partial integration on Ω_0 , we obtain now

$$\begin{aligned} \|\cdot - \Delta\omega + f(\cdot, \omega)\|_{H^{-1}} &\leq \\ &\leq C_2 \left[\|\cdot - \Delta\omega_0 + f(\cdot, \omega_0)\|_{L^2(\Omega_0)}^2 + \|f(\cdot, 0)\|_{L^2(\Omega \setminus \Omega_0)}^2 \right]^{\frac{1}{2}} + C_{tr} \left\| \frac{\partial\omega_0}{\partial\nu_0} \right\|_{L^2(\partial\Omega_0)}, \end{aligned} \quad (71)$$

with C_{tr} denoting a constant for the trace embedding $H^1(\Omega_0) \hookrightarrow L^2(\partial\Omega_0)$, the explicit computation of which will be addressed in the appendix, and $\partial\omega_0/\partial\nu_0$ the normal derivative on $\partial\Omega_0$.

4.3. Bound K for L^{-1} . According to (45), condition (8) now reads

$$\|u\|_{H_0^1} \leq K \|L[u]\|_{H^{-1}} \text{ for all } u \in H_0^1(\Omega), \quad (72)$$

with L , defined in (6), now given by (note (57), (58))

$$L = -\Delta + \mathcal{G}'(\omega) : H_0^1(\Omega) \rightarrow H^{-1}(\Omega).$$

Under the growth conditions (59) - (62), Lemma 1 (or (63)) shows that, more concretely,

$$(L[\varphi])[\psi] = \int_{\Omega} \left[\nabla\varphi \cdot \nabla\psi + \frac{\partial f}{\partial y}(\cdot, \omega)\varphi\psi \right] dx \quad (\varphi, \psi \in H_0^1(\Omega)); \quad (73)$$

the same formula holds true also in the exponential case mentioned in Remark 3c). So we will assume from now on that L is given by (73).

Making use of the isomorphism $\Phi : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$ given by (13) or (49), we obtain

$$\|L[u]\|_{H^{-1}} = \|\Phi^{-1}L[u]\|_{H_0^1} \quad (u \in H_0^1(\Omega)).$$

Since moreover $\Phi^{-1}L$ is $\langle \cdot, \cdot \rangle_{H_0^1}$ -symmetric by (73) and (15), and defined on the whole Hilbert space $H_0^1(\Omega)$, and hence *selfadjoint*, we find that (72) holds for any

$$K \geq [\min \{|\lambda| : \lambda \text{ is in the spectrum of } \Phi^{-1}L\}]^{-1}, \quad (74)$$

provided that the min is positive (which is clearly an unavoidable condition for $\Phi^{-1}L$ being invertible with bounded inverse). Thus, in order to compute K , we need bounds for

- i) the essential spectrum of $\Phi^{-1}L$ (i.e. accumulation points of the spectrum, and eigenvalues of infinite multiplicity),
- ii) isolated eigenvalues of $\Phi^{-1}L$ of finite multiplicity, more precisely those neighboring 0.

ad i) If Ω is unbounded, we suppose again that ω is given in the form (67), with some bounded Lipschitz domain $\Omega_0 \subset \Omega$. If Ω is bounded, we may assume the same, simply choosing $\Omega_0 := \Omega$ (and $\omega_0 := \omega$).

Now define $L_0 : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$ by (73), but with $(\partial f/\partial y)(x, \omega(x))$ replaced by $(\partial f/\partial y)(x, 0)$. Using the Sobolev/Kondratiev/Rellich Embedding Theorem [1], implying the compactness of the embedding $H^1(\Omega_0) \hookrightarrow L^2(\Omega_0)$, we find that $\Phi^{-1}L - \Phi^{-1}L_0 :$

$H_0^1(\Omega) \rightarrow H_0^1(\Omega)$ is compact. Therefore, the perturbation result given in [32, IV, Theorem 5.35] shows that the essential spectra of $\Phi^{-1}L$ and $\Phi^{-1}L_0$ coincide. Thus, being left with the computation of bounds for the essential spectrum of $\Phi^{-1}L_0$, we can use e.g. Fourier transform methods if $\Omega = \mathbb{R}^n$ and $(\partial f/\partial y)(\cdot, 0)$ is constant, or Floquet theory if $(\partial f/\partial y)(\cdot, 0)$ is periodic. Alternatively, if

$$\frac{\partial f}{\partial y}(x, 0) \geq c_0 > -\rho^* \quad (x \in \Omega), \quad (75)$$

with $\rho^* \in [0, \infty)$ denoting the minimal point of the spectrum of $-\Delta$ on $H_0^1(\Omega)$, we obtain by straightforward estimates of the Rayleigh quotient that the (full) spectrum of $\Phi^{-1}L_0$, and thus in particular the essential spectrum, is bounded from below by $\min\{1, (c_0 + \rho^*)/(\sigma + \rho^*)\}$.

ad ii) For computing bounds to eigenvalues of $\Phi^{-1}L$, we choose the parameter σ in the H_0^1 -product (46) such that

$$\sigma > \frac{\partial f}{\partial y}(x, \omega(x)) \quad (x \in \Omega); \quad (76)$$

thus, we have to assume that the right-hand side of (76) is bounded above. Furthermore, we assume that the infimum s_0 of the essential spectrum of $\Phi^{-1}L$ is *positive*, which is true e.g. if (75) holds. As a particular consequence of (76) (and (49)) we obtain that $s_0 \leq 1$ and all eigenvalues of $\Phi^{-1}L$ are less than 1, and that, via the transformation $\kappa = 1/(1-\lambda)$, the eigenvalue problem $\Phi^{-1}L[u] = \lambda u$ is equivalent to

$$-\Delta u + \sigma u = \kappa \left(\sigma - \frac{\partial f}{\partial y}(\cdot, \omega) \right) u \quad (77)$$

(to be understood as an equation in $H^{-1}(\Omega)$), which is furthermore equivalent to the eigenvalue problem for the selfadjoint operator $R := (I_{H_0^1(\Omega)} - \Phi^{-1}L)^{-1}$. Thus, *defining* the essential spectrum of problem (77) to be the one of R , we find that it is bounded from below by $1/(1-s_0)$ if $s_0 < 1$, and is empty if $s_0 = 1$. In particular, its infimum is larger than 1, since $s_0 > 0$ by assumption.

Therefore, the computer-assisted eigenvalue enclosure methods mentioned in Subsection 3.3 (which are applicable to eigenvalues *below* the essential spectrum; see [63]) can be used to enclose the eigenvalue(s) of problem (77) neighboring 1 (if they exist), whence by the transformation $\kappa = 1/(1-\lambda)$ we obtain enclosures for the eigenvalue(s) of $\Phi^{-1}L$ neighboring 0 (if they exist). Taking also s_0 into account, we can now easily compute the desired constant K via (74). (Note that $K = s_0^{-1}$ can be chosen if no eigenvalues below the essential spectrum exist.)

4.4. Local Lipschitz bound g for $\mathcal{F}'$. In the setting (45), (57), condition (9) now reads

$$\left| \int_{\Omega} \left[\frac{\partial f}{\partial y}(x, \omega(x) + u(x)) - \frac{\partial f}{\partial y}(x, \omega(x)) \right] v(x) \varphi(x) dx \right| \leq g(\|u\|_{H_0^1}) \|v\|_{H_0^1} \|\varphi\|_{H_0^1} \quad (78)$$

for all $u, v, \varphi \in H_0^1(\Omega)$. Here, we have assumed that the Fréchet derivative of $\mathcal{G}$ (defined in (58)) is given by (63), which is true e.g. under the growth conditions (59)–(62), but also in the exponential case (with $n = 2$ and Ω bounded) mentioned in Remark 3c). We will now concentrate on the case where (59)–(62) hold true. For the exponential case, we

refer to [54] and to the second example in Subsection 4.5.

As in the strong solutions approach treated in Section 3, we start with a monotonically non-decreasing function $\tilde{g} : [0, \infty) \rightarrow [0, \infty)$ satisfying

$$\left| \frac{\partial f}{\partial y}(x, \omega(x) + y) - \frac{\partial f}{\partial y}(x, \omega(x)) \right| \leq \tilde{g}(|y|) \quad \text{for all } x \in \Omega, y \in \mathbb{R}, \quad (79)$$

and $\tilde{g}(t) \rightarrow 0$ as $t \rightarrow 0+$, but now we require in addition that $\tilde{g}(t^{1/r})$ is a *concave* function of t . Here, $r := r_2$ is the (larger) exponent in (61).

In practice, $\tilde{g}$ can often be put up in the form

$$\tilde{g}(t) = \sum_{j=1}^N a_j t^{\mu_j} \quad (0 \leq t < \infty),$$

where $a_1, \dots, a_N > 0$ and $\mu_1, \dots, \mu_N \in (0, r]$ are arranged in order to satisfy (79).

Now defining $\psi(t) := \tilde{g}(t^{1/r})$, the left-hand side of (78) can be bounded by (note (79))

$$\int_{\Omega} \tilde{g}(|u(x)|) |v(x)\varphi(x)| dx = \int_{\Omega} \psi(|u(x)|^r) |v(x)\varphi(x)| dx. \quad (80)$$

Without loss of generality we may assume that $v\varphi$ does not vanish identically (almost everywhere) on Ω (otherwise, (78) is trivial because the left-hand side is zero). Since $v\varphi \in L^1(\Omega)$ and hence $|v(x)\varphi(x)| dx$ induces a *finite* measure, and since ψ is concave, Jensen's Inequality [7] shows that

$$\frac{\int_{\Omega} \psi(|u(x)|^r) |v(x)\varphi(x)| dx}{\int_{\Omega} |v(x)\varphi(x)| dx} \leq \psi \left(\frac{\int_{\Omega} |u(x)|^r |v(x)\varphi(x)| dx}{\int_{\Omega} |v(x)\varphi(x)| dx} \right). \quad (81)$$

Furthermore, for $\lambda \in (0, 1]$ and $t \in [0, \infty)$, $\psi(\lambda t) = \psi(\lambda t + (1-\lambda)0) \geq \lambda\psi(t) + (1-\lambda)\psi(0) = \lambda\psi(t)$, i.e. $\psi(t) \leq \lambda^{-1}\psi(\lambda t)$. By Cauchy-Schwarz and the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$,

$$\lambda := \frac{\int_{\Omega} |v(x)\varphi(x)| dx}{C_2^2 \|v\|_{H_0^1} \|\varphi\|_{H_0^1}} \in (0, 1],$$

whence the right-hand side of (81) is bounded by

$$\frac{C_2^2 \|v\|_{H_0^1} \|\varphi\|_{H_0^1}}{\int_{\Omega} |v(x)\varphi(x)| dx} \cdot \psi \left(\frac{\int_{\Omega} |u(x)|^r |v(x)\varphi(x)| dx}{C_2^2 \|v\|_{H_0^1} \|\varphi\|_{H_0^1}} \right). \quad (82)$$

According to (62), we can find some

$$q \in (1, \infty) \text{ if } n = 2, \quad q \in \left[\frac{n}{2}, \infty \right) \text{ if } n \geq 3, \quad (83)$$

such that qr is in the range (52). Since (83) implies that also $p := 2q/(q-1)$ is in the range (52), both the embeddings $H_0^1(\Omega) \hookrightarrow L^{qr}(\Omega)$ and $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$ are bounded. Furthermore, $q^{-1} + p^{-1} + p^{-1} = 1$, whence the generalized Hölder Inequality gives

$$\int_{\Omega} |u(x)|^r |v(x)\varphi(x)| dx \leq \|u\|_{L^{qr}}^r \|v\|_{L^p} \|\varphi\|_{L^p} \leq C_{qr}^r C_p^2 \|u\|_{H_0^1}^r \|v\|_{H_0^1} \|\varphi\|_{H_0^1}.$$

Using this estimate in (82), and combining it with (81) and (80), we find that the left-hand side of (78) is bounded by

$$C_2^2 \|v\|_{H_0^1} \|\varphi\|_{H_0^1} \cdot \psi \left(C_{qr}^r (C_p/C_2)^2 \|u\|_{H_0^1}^r \right).$$

Since $\psi(t) = \tilde{g}(t^{\frac{1}{r}})$, (78) therefore holds for

$$g(t) := C_2^2 \cdot \tilde{g} \left(C_{qr} (C_p/C_2)^{\frac{2}{r}} t \right) \quad (0 \leq t < \infty), \quad (84)$$

which also satisfies (10) and is non-decreasing.

4.5. Examples. In our first example, we consider the problem of finding nontrivial solutions to

$$-\Delta u + V(x)u - u^2 = 0 \quad \text{on } \Omega := \mathbb{R}^2, \quad (85)$$

where $V(x) = A + B \sin(\pi(x_1 + x_2)) \sin(\pi(x_1 - x_2))$, with real parameters A and B . The results presented here have been obtained in joint work with B. Breuer and P. J. McKenna.

We are interested only in solutions which are symmetric with respect to reflection about both coordinate axes. Thus, we include these symmetries into all function spaces used, and into the numerical approximation spaces.

We treated the particular case $A = 6$, $B = 2$. On a “computational” domain $\Omega_0 := (-\ell, \ell) \times (-\ell, \ell)$, we computed an approximation $\omega_0 \in H^2(\Omega_0) \cap H_0^1(\Omega_0)$ of the differential equation in (85), with Dirichlet boundary conditions on $\partial\Omega_0$, in a finite Fourier series form like (26) (with $N = M = 80$). For finding ω_0 , we started with a nontrivial approximate solution for Emden’s equation (which is (85) with $A = B = 0$) on Ω_0 , and performed a path following Newton method, deforming (A, B) from $(0, 0)$ into $(6, 2)$.

In the single Newton steps, we used a collocation method with equidistant collocation points. By increasing the sidelength of Ω_0 in an additional path following, we found that the approximation ω_0 remains “stable”, with rapidly decreasing normal derivative $\partial\omega_0/\partial\nu_0$ (on $\partial\Omega_0$), as ℓ increases; this gives rise to some hope that a “good” approximation ω for problem (85) is obtained in the form (67). For $\ell = 8$, $\|\partial\omega_0/\partial\nu_0\|_{L^2(\partial\Omega_0)}$ turned out to be small enough compared with $\|-\Delta\omega_0 + V\omega_0 - \omega_0^2\|_{L^2(\Omega_0)}$, and we computed a defect bound δ (satisfying (68)) via (71) as

$$\delta = 0.7102 \cdot 10^{-2}; \quad (86)$$

note that, by the results mentioned in the appendix, $C_2 = \sigma^{-\frac{1}{2}}$, and $C_{tr} = \sigma^{-\frac{1}{2}} [\ell^{-1} + \sqrt{\ell^{-2} + 2\sigma}]^{\frac{1}{2}}$. Moreover, (76) requires $\sigma > A + B = 8$ (since ω turns out to be non-negative). Choosing $\sigma := 9$, we obtain $C_2 \leq 0.3334$ and $C_{tr} \leq 0.6968$.

Since condition (75) holds for $c_0 = A - B = 4$ (and $\rho^* = 0$), the arguments following (75) give the lower bound $s_0 := 4/9 \geq 0.4444$ for the essential spectrum of $\Phi^{-1}L$, and hence the lower bound $1/(1 - s_0) = 1.8$ for the essential spectrum of problem (77).

By the eigenvalue enclosure methods mentioned in Subsection 3.3, we were able to compute the bounds

$$\kappa_1 \leq 0.5293, \quad \kappa_2 \geq 1.1769$$

for the first two eigenvalues of problem (77), which by (74) leads to the constant

$$K = 6.653 \quad (87)$$

satisfying (72).

For computing g satisfying (9) or (78), we first note that (79) holds for

$$\tilde{g}(t) := 2t,$$

and (61) for $r_1 = r_2 = 1$, whence the additional concavity condition is satisfied. Choosing $q := 2$ we obtain $qr = 2$ and $p = 4$ in the arguments following (83), whence (84) gives

$$g(t) = 2C_2C_4^2t = \frac{1}{9}t \quad (88)$$

since $2C_2C_4^2 = \sigma^{-1}$ by Lemma 2a) in the appendix.

Using (86) - (88), we find that (16) and (17) hold for $\alpha = 0.04811$, whence Theorem 1 implies the existence of a solution $u^* \in H_0^1(\mathbb{R}^2)$ to problem (85) such that

$$\|u^* - \omega\|_{H_0^1} \leq 0.04811. \quad (89)$$

It is easy to check on the basis of the numerical data that $\|\omega\|_{H_0^1} > 0.04811$, whence (89) shows in particular that u^* is non-trivial.

We wish to remark that it would be of great interest to achieve such results also for cases where $0 < A < B$ in the potential V , because V is then no longer non-negative, which excludes an important class of purely analytical approaches to prove existence of a non-trivial solution. So far, we were not successful with such cases due to difficulties in the homotopy method which has to be used for our computer-assisted eigenvalue enclosures (see the brief remarks in Subsection 3.3); note that these difficulties occur on a rather “technical” level. We were however able to compute an (apparently) “good” approximation ω , e.g. in the case $A = 6$, $B = 26$.

The following Figure 2 shows plots of ω for the successful case $A = 6$, $B = 2$, and for the non-successful case $A = 6$, $B = 26$.

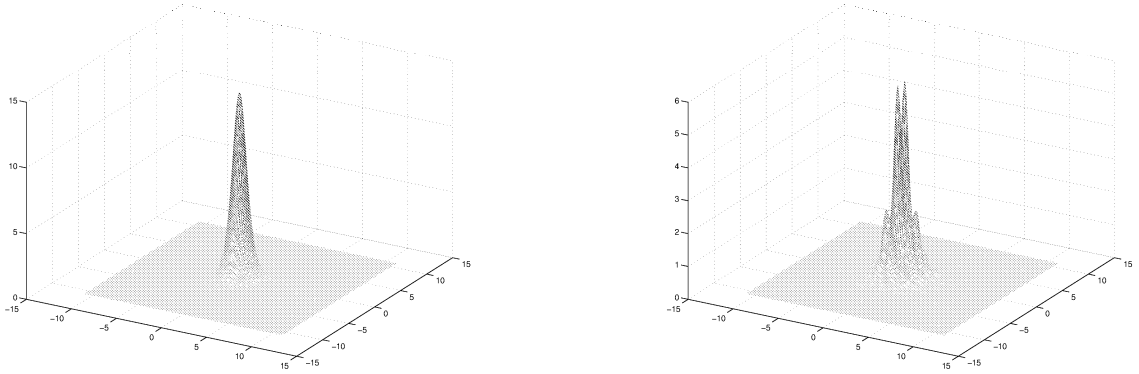


Figure 2: Example (85); $A = 6$, $B = 2$ (left) and $A = 6$, $B = 26$ (right).

In our second example, we consider the *Gelfand* equation

$$-\Delta u = \lambda e^u \quad \text{on } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \quad (90)$$

depending on a real parameter λ . We are interested in parameter values $\lambda \geq 0$ only; negative values of λ are less important. The results reported on here are joint work with C. Wiemers and published in [54].

It is known that, on “simple” domains Ω like the unit square or the unit ball, problem (90) has a “nose”-shaped branch (λ, u) of solutions, starting in $(\lambda = 0, u \equiv 0)$, going up to some maximal value of λ where the branch has a turning point, and then returning to

$\lambda = 0$ but with $\|u\|_\infty$ tending to ∞ as $\lambda \rightarrow 0$. Moreover, there are no other solutions (on these “simple” domains).

Here (and in [54]) we are concerned with a special *non-convex* domain $\Omega \subset \mathbb{R}^2$ plotted in Figure 3. (For an exact quantitative definition of Ω , see [54].) Ω is symmetric with respect to the x_1 -axis but not quite symmetric with respect to the x_2 -axis; it is a bit shorter on the left-hand side than on the right. Starting at $(\lambda = 0, u \equiv 0)$, and performing numerical branch following, we obtained the usual “nose”-shaped branch (of approximate solutions) plotted in Figure 4; the plot consists in fact of an interpolation of many grid points.

Obviously, the approximations develop substantial unsymmetries along the branch. In order to find new (approximate) solutions, we *reflected* such an unsymmetric approximation about the x_2 -axis, re-arranged the boundary values (which is necessary but easily possible due to the slight unsymmetry of Ω), and re-started the Newton iteration. Fortunately, it “converged” to a new approximation, and by branch following we could detect a new branch of approximate solutions plotted (together with the “old” one) in Figure 5; in order to obtain a nicely visible separation of the two branches, we introduced the difference $d(u)$ between the two peak values of each approximation as a third dimension in the bifurcation diagram.

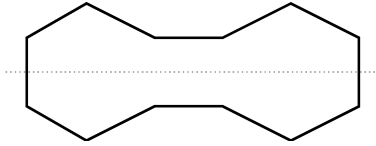


Figure 3: Domain Ω for example (90).

In order to *prove* the existence of a new solution branch, we performed the computer-assisted method described above for the selected value $\lambda = 15/32$. Here, our “new” approximation ω was computed with 65536 quadratic triangular finite elements, corresponding to 132225 unknowns.

For calculating a defect bound δ (satisfying (68)), we used essentially (up to some technical refinements) the estimate (70), where the approximation $\rho \in H(\text{div}, \Omega)$ to $\nabla \omega$ was computed by linear Raviart-Thomas elements. The result is

$$\delta = 0.8979 \cdot 10^{-2}. \quad (91)$$

Since $(\partial f / \partial y)(x, y) = -\lambda e^y < 0$ here, condition (76) is satisfied for $\sigma = 0$; indeed, this choice is allowed because Ω is bounded (see Remark 3a)). We computed eigenvalue bounds for problem (77) by the Rayleigh-Ritz and the Lehmann-Goerisch method, exploiting symmetry properties, with the final result that (72) holds for

$$K = 3.126; \quad (92)$$

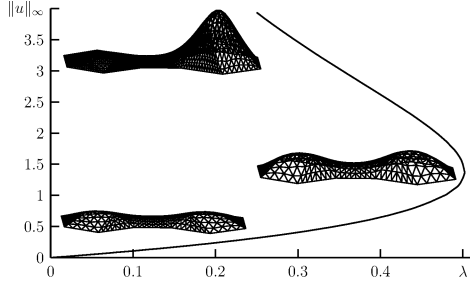


Figure 4: Main branch of (approximate) solutions for problem (90).

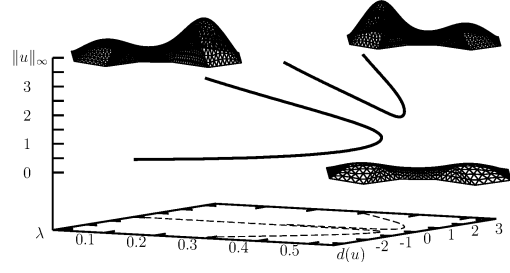


Figure 5: Main and new branch for problem (90).

note that problem (77) has no essential spectrum here since Ω is bounded.

For proving that $\mathcal{G}$ defined in (58) is Fréchet differentiable and for computing a function g satisfying (9) or (78), we make essential use of the Trudinger-Moser inequality (66) (note that Lemma 1 does not apply here due to the exponential nonlinearity). For each $u \in H_0^1(\Omega) \setminus \{0\}$,

$$4|u(x)| = 2 \cdot 2\|u\|_{H_0^1} \cdot \frac{|u(x)|}{\|u\|_{H_0^1}} \leq 4\|u\|_{H_0^1}^2 + \left(\frac{|u(x)|}{\|u\|_{H_0^1}} \right)^2,$$

whence (66) (with $c := 1$) gives, since $[4\pi/(4\pi - 1)]^{1/4} \leq 1.03$,

$$\|\exp(|u|)\|_{L^4} \leq 1.03 \operatorname{meas}(\Omega)^{\frac{1}{4}} \exp(\|u\|_{H_0^1}^2). \quad (93)$$

For all $u_0, u, v, \varphi \in H_0^1(\Omega)$, the generalized Hölder Inequality and (93) imply

$$\begin{aligned} \int_{\Omega} |e^{u_0+u} - e^{u_0}| |v| |\varphi| dx &\leq \int_{\Omega} e^{u_0} e^{|u|} |u| |v| |\varphi| dx \leq \|e^{u_0}\|_{L^4} \|e^{|u|}\|_{L^4} \|u\|_{L^6} \|v\|_{L^6} \|\varphi\|_{L^6} \\ &\leq \|e^{u_0}\|_{L^4} \cdot 1.03 \operatorname{meas}(\Omega)^{\frac{1}{4}} \exp\left(\|u\|_{H_0^1}^2\right) C_6^3 \|u\|_{H_0^1} \|v\|_{H_0^1} \|\varphi\|_{H_0^1}. \end{aligned} \quad (94)$$

By an argument similar to the abstract estimate (20), (21), we obtain the desired Fréchet differentiability from (94). Furthermore, for $u_0 := \omega$, (94) shows that (78) holds for

$$g(t) = \gamma t e^{t^2}, \quad \text{where } \gamma := \|\lambda e^{\omega}\|_{L^4} \cdot 1.03 \operatorname{meas}(\Omega)^{\frac{1}{4}} C_6^3, \quad (95)$$

and thus $G(t) = \int_0^t g(s) ds = \frac{1}{2} \gamma (\exp(t^2) - 1) \leq \frac{1}{2} \gamma t^2 \exp(t^2)$. From the numerical data, Lemma 2 (appendix), and the result $\rho^* \geq 1.4399$ (obtained by eigenvalue bounds), we obtain that $\gamma \leq 5.62$. Together with (91), (92), (95), we obtain that (16) and (17) hold for $\alpha := 0.05066$, whence Theorem 1 gives the existence of a solution $u^* \in H_0^1(\Omega)$ of problem (90) (with $\lambda = 15/32$) such that

$$\|u^* - \omega\|_{H_0^1} \leq 0.05066. \quad (96)$$

(It should be remarked that we could do without condition (17) being satisfied, since Ω is bounded and hence we could use compactness properties, and Schauder's instead of Banach's Fixed Point Theorem.)

In the same way, we also obtained existence results with H_0^1 -error bounds for two solutions of (90) on the “old” (nose-shaped) branch, again for $\lambda = 15/32$. From the numerical data, and all three error bounds, we can easily deduce that the three solutions

are pairwise different, whence u^* established above lies on a new independent solution branch; the Implicit Function Theorem (plus some perturbation type argument showing that $-\Delta - \lambda e^{u^*} : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$ is one-to-one and onto) shows that indeed a solution *branch* through $(\lambda = 15/32, u^*)$ exists.

5. Appendix: Embedding constants

At various points in this paper, an explicit norm bound for the embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$, i.e. a constant C_p such that

$$\|u\|_{L^p} \leq C_p \|u\|_{H_0^1} \quad \text{for all } u \in H_0^1(\Omega), \quad (97)$$

is needed, for p in the range (52), and with $\|\cdot\|_{H_0^1}$ and $\|\cdot\|_{L^p}$ defined in (46) and (53), respectively. Here, we are not aiming at the optimal constants, but at “good” constants which are *easy to compute*.

Lemma 2: *Let $\rho^* \in [0, \infty)$ denote the minimal point of the spectrum of $-\Delta$ on $H_0^1(\Omega)$. a) Let $n = 2$ and $p \in [2, \infty)$. With ν denoting the largest integer $\leq p/2$, (97) holds for*

$$C_p = \left(\frac{1}{2}\right)^{\frac{1}{2} + \frac{2\nu-3}{p}} \left[\frac{p}{2} \left(\frac{p}{2} - 1\right) \cdots \left(\frac{p}{2} - \nu + 2\right)\right]^{\frac{2}{p}} \frac{1}{(\rho^* + \frac{p}{2}\sigma)^{\frac{1}{p}}} \quad (98)$$

(where the bracket-term is put equal to 1 if $\nu = 1$).

b) Let $n \geq 3$ and $p \in [2, \frac{2n}{n-2}]$. With $s := n \left(\frac{1}{p} - \frac{1}{2} + \frac{1}{n}\right) \in [0, 1]$, (97) holds for

$$C_p = \left(\frac{n-1}{\sqrt{n}(n-2)}\right)^{1-s} \left(\frac{s}{s\rho^* + \sigma}\right)^{\frac{s}{2}} \quad (99)$$

(where the second factor is put equal to 1 if $s = 0$).

Proof. ad a) Since $C_0^\infty(\Omega)$ is dense in $H_0^1(\Omega)$, it suffices to prove (97) for $u \in C_0^\infty(\Omega)$. By zero extension outside Ω , we may regard u as a function in $C_0^\infty(\mathbb{R}^2)$.

For all $(x_1, x_2) \in \mathbb{R}^2$,

$$|u(x_1, x_2)|^{\frac{p}{2}} = \frac{p}{2} \int_{-\infty}^{x_1} |u(t, x_2)|^{\frac{p}{2}-1} \operatorname{sgn}(u(t, x_2)) \frac{\partial u}{\partial x_1}(t, x_2) dt \leq \frac{p}{2} \int_{-\infty}^{x_1} |u(t, x_2)|^{\frac{p}{2}-1} \left| \frac{\partial u}{\partial x_1}(t, x_2) \right| dt$$

and analogously,

$$|u(x_1, x_2)|^{\frac{p}{2}} \leq \frac{p}{2} \int_{x_1}^{\infty} |u(t, x_2)|^{\frac{p}{2}-1} \left| \frac{\partial u}{\partial x_1}(t, x_2) \right| dt.$$

Adding these two inequalities gives

$$|u(x_1, x_2)|^{\frac{p}{2}} \leq \frac{p}{4} \int_{-\infty}^{\infty} |u(t, x_2)|^{\frac{p}{2}-1} \left| \frac{\partial u}{\partial x_1}(t, x_2) \right| dt.$$

An analogous inequality is obtained by integration over x_2 instead of x_1 . Multiplication of these two inequalities yields

$$|u(x_1, x_2)|^p \leq \frac{p^2}{16} \left(\int_{-\infty}^{\infty} |u(t, x_2)|^{\frac{p}{2}-1} \left| \frac{\partial u}{\partial x_1}(t, x_2) \right| dt \right) \left(\int_{-\infty}^{\infty} |u(x_1, t)|^{\frac{p}{2}-1} \left| \frac{\partial u}{\partial x_2}(x_1, t) \right| dt \right).$$

Note that, on the right-hand side, the first factor depends only on x_2 , and the second only on x_1 . Thus, integrating this inequality over $\mathbb{R}^2$ we obtain, using Cauchy-Schwarz,

$$\int_{\mathbb{R}^2} |u|^p dx \leq \frac{p^2}{32} \left(\int_{\mathbb{R}^2} |u|^{p-2} dx \right) \left(\int_{\mathbb{R}^2} |\nabla u|^2 dx \right). \quad (100)$$

By iteration of this inequality,

$$\int_{\mathbb{R}^2} |u|^p dx \leq \frac{p^2}{32} \frac{(p-2)^2}{32} \dots \frac{(p-2\nu+4)^2}{32} \left(\int_{\mathbb{R}^2} |u|^{p-2\nu+2} dx \right) \left(\int_{\mathbb{R}^2} |\nabla u|^2 dx \right)^{\nu-1}. \quad (101)$$

Let $q := p - 2\nu + 2$. By the choice of ν , we have $2 \leq q < 4$. Thus, Hölder's Inequality gives the following simple interpolation inequality:

$$\int_{\mathbb{R}^2} |u|^q dx = \int_{\mathbb{R}^2} |u|^{2q-4} |u|^{4-q} dx \leq \left(\int_{\mathbb{R}^2} u^4 dx \right)^{\frac{q}{2}-1} \left(\int_{\mathbb{R}^2} u^2 dx \right)^{2-\frac{q}{2}}. \quad (102)$$

Using (100) with 4 in place of p , inserting the result into (102), and further inserting into (101) gives, since $(q/2) - 1 = (p/2) - \nu$,

$$\int_{\mathbb{R}^2} |u|^p dx \leq \frac{p^2}{32} \frac{(p-2)^2}{32} \dots \frac{(p-2\nu+4)^2}{32} \left(\frac{1}{2} \right)^{\frac{p}{2}-\nu} \left(\int_{\mathbb{R}^2} u^2 dx \right) \left(\int_{\mathbb{R}^2} |\nabla u|^2 dx \right)^{\frac{p}{2}-1}. \quad (103)$$

Moreover,

$$\begin{aligned} & \left(\int_{\mathbb{R}^2} u^2 dx \right)^{\frac{2}{p}} \left(\int_{\mathbb{R}^2} |\nabla u|^2 dx \right)^{1-\frac{2}{p}} \\ & \leq \left(\frac{1}{\rho^* + \frac{p}{2}\sigma} \left[\int_{\mathbb{R}^2} |\nabla u|^2 dx + \frac{p}{2}\sigma \int_{\mathbb{R}^2} u^2 dx \right] \right)^{\frac{2}{p}} \left(\int_{\mathbb{R}^2} |\nabla u|^2 dx \right)^{1-\frac{2}{p}} \\ & \leq \frac{1}{(\rho^* + \frac{p}{2}\sigma)^{\frac{2}{p}}} \left\{ \frac{2}{p} \left[\int_{\mathbb{R}^2} |\nabla u|^2 dx + \frac{p}{2}\sigma \int_{\mathbb{R}^2} u^2 dx \right] + \left(1 - \frac{2}{p} \right) \int_{\mathbb{R}^2} |\nabla u|^2 dx \right\} \\ & = \frac{1}{(\rho^* + \frac{p}{2}\sigma)^{\frac{2}{p}}} \|u\|_{H_0^1}^2. \end{aligned} \quad (104)$$

Using this inequality in (103), and moreover calculating

$$\frac{p^2}{32} \frac{(p-2)^2}{32} \dots \frac{(p-2\nu+4)^2}{32} \left(\frac{1}{2} \right)^{\frac{p}{2}-\nu} = \left[\frac{p}{2} \left(\frac{p}{2} - 1 \right) \dots \left(\frac{p}{2} - \nu + 2 \right) \right]^2 \cdot \left(\frac{1}{2} \right)^{\frac{p}{2}+2\nu-3},$$

we obtain the assertion.

ad b) In [24, proof of Theorem 9.2, (9.10)], it is shown that, again for $u \in C_0^\infty(\mathbb{R}^n)$,

$$\|u\|_{L^{\frac{2n}{n-2}}} \leq \frac{n-1}{n-2} \prod_{i=1}^n \left\| \frac{\partial u}{\partial x_i} \right\|_{L^2}^{\frac{1}{n}}.$$

Thus, by the arithmetic-geometric mean inequality,

$$\|u\|_{L^{\frac{2n}{n-2}}} \leq \frac{n-1}{\sqrt{n}(n-2)} \|\nabla u\|_{L^2}, \quad (105)$$

which implies the result (even with $\sigma = 0$ in (46)) if $p = 2n/(n-2)$. Now let $p \in [2, 2n/(n-2))$, whence $s = n\left(\frac{1}{p} - \frac{1}{2} + \frac{1}{n}\right) \in (0, 1]$. Again, we use the interpolation inequality (note that $\frac{n-2}{2n}p(1-s) + \frac{1}{2}ps = 1$)

$$\int_{\mathbb{R}^n} |u|^p dx = \int_{\mathbb{R}^n} |u|^{p(1-s)} |u|^{ps} dx \leq \left(\int_{\mathbb{R}^n} |u|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{2n}p(1-s)} \left(\int_{\mathbb{R}^n} u^2 dx \right)^{\frac{1}{2}ps},$$

whence, by (105),

$$\|u\|_{L^p} \leq \left(\frac{n-1}{\sqrt{n}(n-2)} \right)^{1-s} \left(\int_{\mathbb{R}^n} |\nabla u|^2 dx \right)^{\frac{1}{2}(1-s)} \left(\int_{\mathbb{R}^n} u^2 dx \right)^{\frac{1}{2}s}. \quad (106)$$

Moreover, by arguments similar to (104)

$$\left(\int_{\mathbb{R}^n} u^2 dx \right)^s \left(\int_{\mathbb{R}^n} |\nabla u|^2 dx \right)^{1-s} \leq \frac{1}{(\rho^* + \frac{\sigma}{s})^s} \|u\|_{H_0^1}^2.$$

Inserting into (106) gives the assertion. $\square$

Remark 4: The embedding constants given in Lemma 2 depend on the minimum ρ^* of the spectrum of $-\Delta$ on $H_0^1(\Omega)$. If no information on ρ^* is available, one may simply use the lower bound 0 for ρ^* . If Ω contains balls of arbitrarily large radius, ρ^* is 0. In these cases the parameter σ in (46) must of course be chosen positive.

In many cases, however, positive lower bounds for ρ^* can easily be computed, since ρ^* depends in an antitone way on the domain Ω . If e.g. Ω is contained in a rectangle $(a_1, b_1) \times \cdots \times (a_n, b_n)$, where $a_i = -\infty$ and $b_i = \infty$ are admitted, then $\rho^* \geq \pi^2 \sum_{i=1}^n (b_i - a_i)^{-2}$.

If $\Omega \subset \mathbb{R}^2$ has finite measure, another simple lower bound for ρ^* is obtained by using (100) for $p := 2$, implying that the Rayleigh quotient for $-\Delta$, and hence ρ^* , is $\geq 8/\text{meas}(\Omega)$.

More accurate lower bounds for ρ^* can be computed by the eigenvalue enclosure methods mentioned in Subsection 3.3.

In Subsection 4.2, a *trace embedding* constant C_{tr} satisfying

$$\|u|_{\partial\Omega}\|_{L^2(\partial\Omega)} \leq C_{tr} \|u\|_{H^1(\Omega)} \quad (u \in H^1(\Omega)) \quad (107)$$

is required, with Ω denoting a bounded Lipschitz domain. Here, the norm $\|\cdot\|_{H^1}$ is given by (the square root of) the right-hand side of (46). Clearly, $\sigma > 0$ must be required now, since otherwise (107) would be violated for constant functions u . Again, we are not

aiming at the optimal constant, but at a “good” and easily computable one.

Lemma 3: *Let $\rho : \bar{\Omega} \rightarrow \mathbb{R}^n$ be continuous, with bounded weak first derivatives, such that*

$$\rho \cdot \nu \geq 1 \text{ on } \partial\Omega, \quad (108)$$

where $\nu : \partial\Omega \rightarrow \mathbb{R}^n$ denotes the outer unit normal field (which exists almost everywhere on $\partial\Omega$). Then, with $\|\rho\|_\infty := \left\| \sqrt{\sum_{i=1}^n \rho_i^2} \right\|_\infty$, (107) holds for

$$C_{tr} = \left[\frac{1}{\sigma} \left(\frac{1}{2} \|\operatorname{div} \rho\|_\infty + \sqrt{\frac{1}{4} \|\operatorname{div} \rho\|_\infty^2 + \sigma \|\rho\|_\infty^2} \right) \right]^{\frac{1}{2}}.$$

Proof: We have to show (107) for $u \in C^1(\bar{\Omega})$. By (108) and Gauß’ Divergence Theorem,

$$\begin{aligned} \int_{\partial\Omega} u^2 dS &\leq \int_{\partial\Omega} (u^2 \rho) \cdot \nu dS = \int_{\Omega} \operatorname{div}(u^2 \rho) dx = \int_{\Omega} (\operatorname{div} \rho) u^2 dx + 2 \int_{\Omega} u (\nabla u) \cdot \rho dx \\ &\leq \|\operatorname{div} \rho\|_\infty \|u\|_{L^2(\Omega)}^2 + 2 \|\rho\|_\infty \|u\|_{L^2(\Omega)} \|\nabla u\|_{L^2(\Omega)} \\ &\leq \|\operatorname{div} \rho\|_\infty \|u\|_{L^2(\Omega)}^2 + \|\rho\|_\infty \left(\lambda \|u\|_{L^2(\Omega)}^2 + \frac{1}{\lambda} \|\nabla u\|_{L^2(\Omega)}^2 \right) \\ &= \frac{\|\rho\|_\infty}{\lambda} \left[\|\nabla u\|_{L^2(\Omega)}^2 + \left(\frac{\lambda}{\|\rho\|_\infty} \|\operatorname{div} \rho\|_\infty + \lambda^2 \right) \|u\|_{L^2(\Omega)}^2 \right] \end{aligned}$$

for arbitrary $\lambda > 0$. Choosing $\lambda := \|\rho\|_\infty^{-1} \left[-\frac{1}{2} \|\operatorname{div} \rho\|_\infty + \sqrt{\frac{1}{4} \|\operatorname{div} \rho\|_\infty^2 + \sigma \|\rho\|_\infty^2} \right]$ gives the assertion. $\square$

If for example Ω is a bounded rectangle $(-\ell_1, \ell_1) \times \cdots \times (-\ell_n, \ell_n)$, we can choose $\rho(x) := (x_1/\ell_1, \dots, x_n/\ell_n)$, satisfying (108). Lemma 3 therefore yields

$$C_{tr} = \left[\frac{1}{\sigma} \left(\frac{1}{2} \sum_{i=1}^n \frac{1}{\ell_i} + \sqrt{\frac{1}{4} \left(\sum_{i=1}^n \frac{1}{\ell_i} \right)^2 + n\sigma} \right) \right]^{\frac{1}{2}}.$$

If Ω is a ball with radius R , centered at 0, we choose $\rho(x) := R^{-1}x$, which satisfies (108), whence Lemma 3 gives

$$C_{tr} = \left[\frac{1}{\sigma} \left(\frac{n}{2R} + \sqrt{\frac{n^2}{4R^2} + \sigma} \right) \right]^{\frac{1}{2}}.$$

Note that the shear *existence* of a vector field ρ with the required properties is ensured by the Lipschitz continuity of $\partial\Omega$ (see [28, Lemma 1.5.1.9]).

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