AN OPIAL-TYPE INEQUALITY WITH AN INTEGRAL BOUNDARY CONDITION

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Abstract. We determine the best constant K and extremals of the Opial-type inequality $\int_a^b |yy'|\,dx \le K(b-a)\int_a^b |y'|^2\,dx$ where y is required to satisfy the boundary condition $\int_a^b y\,dx = 0$. The techniques employed differ from either those used recently by Denzler to solve this problem or originally to prove the classical inequality; but they also yield a new proof of that inequality.

1. Introduction

In 1962 C. Olech [11] gave a simplified proof of the following inequality originally due in a slightly less general form to Zdzisław Opial [1][12].

Theorem A. *If y is a real absolutely continuous function on the interval $[a,b]$, $-\infty < a < b < \infty$ and $y(a) = y(b) = 0$, $\int_a^b (y')^2dx < \infty$, then the best constant K of the inequality*

$$\int_a^b |yy'|\,dx \le K(b-a)\int_a^b (y')^2\,dx \tag{1.1}$$

is $1/4$. Equality holds in (1.1) if and only if

$$y(s) = \begin{cases} c(s-a) & \text{if } a \le s \le \frac{a+b}{2}, \\ c(b-s) & \text{if } \frac{a+b}{2} < s \le b \end{cases}$$

where c is an arbitrary constant.

Embedded in Olech's proof is the half-interval form of Opial's inequality discovered also by Beesack [2] which is satisfied by those y vanishing only at a.

Theorem B. *If y is a real absolutely continuous on the interval $[a,b]$, $-\infty < a < b < \infty$, and $y(a) = 0$, $\int_a^b (y')^2dx < \infty$, then*

$$\int_a^b |yy'|\,dx \le \frac{b-a}{2}\int_a^b (y')^2\,dx. \tag{1.2}$$

Equality holds in (1.2) if and only if $y = c(s-a)$ for some constant c.

Since their discovery both (1.1) and (1.2) have attracted enormous interest. At least six proofs are known and a very large number of generalizations have been given. For a survey of the literature on Opial-type inequalities see the books of Agarwal and Pang [1] and Mitrinović, Pečarić, and Fink [10]. For an important recent

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[1]Opial required that y' be continuous and $y > 0$. Also he did not characterize the extremals.

result showing that Opial inequalities are equivalent to those of Hardy-type (without, however, preserving the constants) see [13]. Besides having intrinsic interest Opial-type inequalities have proved essential in developing disconjugacy/stability criteria for differential equations, obtaining sufficient conditions for the positivity of eigenvalues, bounds on the spacing of zeros of a solution, or improving other inequalities, such as the Lyapunov inequalitiy, see e.g. [4], [6], or [7].

We note that the existence of an inequality of the form (1.1) or (1.2) is quite easy to prove. As already noticed in [12] if we apply the Cauchy-Schwarz and the one-dimensional Poincaré inequality we find that

$$\begin{aligned}\int_a^b |yy'|\,dx &\le \left(\int_a^b y^2\,dx\right)^{1/2}\left(\int_a^b (y')^2\,dx\right)^{1/2}\\ &\le \frac{b-a}{\pi}\int_a^b (y')^2\,dx. \qquad (1.3)\end{aligned}$$

The nontrivial part of (1.1) is the determination of the least value of K and the characterization of the extremals, and in all applications this knowledge has been essential. In order to get a feeling for subtleties involved in Opial's inequality we sketch an argument for (1.1) which is close to Olech's.

Outline of a proof. Let y be an absolutely continuous function such that $y(a) = y(b) = 0$ and $\int_a^b (y')^2\,dx < \infty$ and let $p \in (a, b)$ satisfy

$$\int_a^p |y'|\,dx = \int_p^b |y'|\,dx.$$

Define

$$Y(x) = \begin{cases} \int_a^x |y'|\,dx & \text{if } x \in [a, p],\\ \int_x^b |y'|\,dx & \text{if } x \in (p, b].\end{cases}$$

Evidently Y is absolutely continuous, $|y| \le |Y|$, and $|y'| = |Y'|$ so that an extremal of (1.1) (if any) will be found among the class of appropriate functions y which are nondecreasing on $(0, p]$, nonincreasing on $(p, 1]$, and such that $y(p) = 1$. ¿From this it follows that the extremal is a linear spline with a unique knot at p and by varying p we find that the least value of K is $1/4$. By a variation of the above argument (see [5]) one can show that

$$\int_a^b |yy'|\,dx \le \frac{b-a}{4}\int_a^b (y')^2\,dx,$$
$$\text{if } y(a) + y(b) = 0.$$

Most of the other generalizations of Opial's inequality familiar to us involve boundary conditions similar to those of (1.1) or (1.2). [2]

In this paper we will consider the inequality

$$\int_a^b |yy'|\,dx \le K(b-a)\int_a^b (y')^2\,dx, \qquad (1.4)$$

$$\int_a^b y\,dx = 0 \qquad (1.5)$$

[2]However for a discussion of Opial-type inequalities satisfying the nonhomogenous conditions $y(a) = c$, $y(b) = d$, $c, d \neq 0$ see [5].

where y again is absolutely continous and $\int_a^b (y')^2\,dx < \infty$. As in the case of (1.1) it is not difficult to show that $K < \infty$ in (1.4) exists. The argument (1.3) using a form of the Wirtinger inequality [10, p. 67] shows that an upper bound for K in (1.4), (1.5) is also $1/\pi$. If we set $y = x - (a+b)/2$ a calculation shows that a lower bound on K is $1/4$.

We will prove the following result which was conjectured by one of the authors in 2001 and presented as an open problem in the meeting "General Inequalities 8" at Noszvaj, Hungary, in September 2002.

Theorem 1. *The best value of K in* (1.4), (1.5) *is also* $1/4$ *and all extremals are of the form $y_c(x) = c(x - (a+b)/2)$ for any constant c.*

If one assumes that there is a unique extremal for (1.4), (1.5) then it is not hard to show that Theorem 1 is true (see [3]). In an earlier version of the present paper we showed that the mere *existence* of an extremal implies Theorem 1, but could not prove its actual existence.

While (1.4), (1.5) is simple in form it is much harder to handle than (1.1). As in the previous case the main difficulty is caused by the absolute value signs on the left side, but the technique we used to prove (1.1) no longer seems applicable since it is hard to construct a piecewise monotone function y with the properties of Y while preserving the condition $\int_a^b y\,dx = 0$. We will be forced therefore to use a much more complicated technique based on transformation of variables and variational ideas. The proof will be given in Section 3. In Section 2 we show that an extremal of (1.4), (1.5) exists in a restricted function set where y' is required to have an arbitrary but finite number N of sign changes. This limited existence result turns out to be sufficient preparation for a complete proof of the Theorem which will be given in in Section 3. In Section 4 it will be shown how the argument for (1.4), (1.5) will also work to give yet another proof (number 7?) of both Theorem A and Theorem B.

We should point out that our work is now the second proof of Theorem 1. In 2003 Jochen Denzler [8] found a constructive proof of Theorem 1 on entirely different lines from our variational method. His basic idea was to sequentially modify an admissible function y using various rearrangements and normalizations so as to decrease $\int_0^1 (y')^2\,dx - 4\int_0^1 |yy'|\,dx$ either by decreasing $\int_0^1 (y')^2\,dx$ while leaving the second term fixed or by increasing $\int_0^1 |yy'|\,dx$ while leaving the first term fixed.

However since Denzler's approach and ours are so different, we feel justified in giving another proof of this problem especially since the methodology underlying our variational approach might be helpful in dealing with other inequalities, e.g., a generalization of Theorem 1 to higher dimensions. In fact, it is likely that additional proofs will be found just as in the case of the original Opial inequality. In particular since both Denzler's proof and ours are much more complicated than any proof of Theorem A, a third simpler proof would be desirable.

We close this section with a few remarks on notation. We denote the Lebesgue space of (equivalence classes) of real square integrable functions by $L^2(a,b)$ and the class of absolutely continuous real functions on $[a,b]$ by $AC[a,b]$. We set

$$H^1(a,b) := \{y \in AC[a,b] : y' \in L^2(a,b)\},$$

endowed with its Hilbert space norm $[\|y\|^2_{L^2(a,b)} + \|y'\|^2_{L^2(a,b)}]^{1/2}$. The class of admissible functions for which the inequalities (1.1), (1.2), or (1.4) are defined is called $\mathcal{D}$. It is the subspace of $H^1(a,b)$ satisfying the appropriate boundary condition, e.g. in the case of Theorem 1

$$\mathcal{D} := \left\{ y \in H^1(a,b) : \int_a^b y\,dx = 0 \right\}.$$

Finally, $\mu(S)$ denotes the Lebesgue measure of a measurable set S.

2. The existence of an extremal on a restricted function set

An essential argument in our proof of Theorem 1 will be the Euler equation for some suitably chosen variational problem based on transformation of the independent variable. So we (seem to) need the *existence* of an extremal maximizing

$$\mathcal{Q}(y) := \frac{1}{b-a} \cdot \frac{\int_a^b |yy'|\,dx}{\int_a^b (y')^2\,dx}$$

on $\mathcal{D}$ a priori. As mentioned above, we have not been able to find such an a priori existence proof. It turns out, however, that a priori existence of an extremal on a restricted set involving artificial compactness (instead of the full set $\mathcal{D}$) is sufficient for our argument.

Let $N \in \mathbb{N}$ be fixed. We say that a function $f \in L^2(a,b)$ has *at most N sign changes on* $[a,b]$ if $t_1, \ldots, t_N \in [a,b]$ exist such that $a =: t_0 \le t_1 \le t_2 \le \cdots \le t_N \le t_{N+1} := b$ and, for $j = 1, \ldots, N+1$,

$$f \ge 0 \text{ a.e. on } [t_{j-1}, t_j] \text{ or } f \le 0 \text{ a.e. on } [t_{j-1}, t_j]$$

(which is trivially satisfied if $t_{j-1} = t_j$).

We define the restricted function set (still for N fixed)

$$\mathcal{D}_N := \{ y \in \mathcal{D} : y' \text{ has at most } N \text{ sign changes on } [a,b] \}. \tag{2.1}$$

Proposition 1. *There exists $y \in \mathcal{D}_N \setminus \{0\}$ which maximizes $\mathcal{Q}$ on $\mathcal{D}_N$.*

Proof. Let

$$K_N := \sup_{y \in \mathcal{D}_N \setminus \{0\}} \mathcal{Q}(y),$$

and choose some sequence (y_n) in $\mathcal{D}_N$ such that

$$\int_a^b (y_n')^2\,dx = 1 \quad (n \in \mathbb{N}), \quad \int_a^b |y_n y_n'|dx \to (b-a)K_N \quad (n \to \infty). \tag{2.2}$$

Since, for each n, y_n' has at most N sign changes, there exist $a =: t_0^{(n)} \le t_1^{(n)} \cdots \le t_N^{(n)} \le t_{N+1}^{(n)} := b$ such that, for $j = 1, \ldots, N+1$,

$$y_n' \ge 0 \text{ a.e. on } \left[t_{j-1}^{(n)}, t_j^{(n)}\right] \text{ or } y_n' \le 0 \text{ a.e. on } \left[t_{j-1}^{(n)}, t_j^{(n)}\right]. \tag{2.3}$$

Since $\int_a^b y_n\,dx = 0$ and therefore y_n has at least one zero, (2.2) implies that (y_n) is bounded in $H^1(a,b)$. Hence, a subsequence (denoted again by (y_n)) can be chosen such that

$$y_n \rightharpoonup y \text{ (weakly) in } H^1(a,b) \tag{2.4}$$

for some $y \in H^1(a,b)$, and in addition,

$$t_j^{(n)} \to t_j \quad (j = 1,\dots,N, n \to \infty) \tag{2.5}$$

where $a =: t_0 \le t_1 \le t_2 \le \dots \le t_N \le t_{N+1} := b$ (2.4) implies, by Sobolev-Kondrachev-Rellich's Embedding Theorem,

$$y_n \to y \text{ uniformly on } [a,b] \tag{2.6}$$

and

$$y_n' \rightharpoonup y' \text{ (weakly) in } L^2(a,b). \tag{2.7}$$

The zero integral condition for y_n and (2.6) imply that y has zero integral, i.e., $y \in \mathcal{D}$. To prove $y \in \mathcal{D}_N$ we show that, for each $j = 1,\dots,N+1$,

$$y' \ge 0 \text{ a.e. on } \big[t_{j-1},t_j\big] \text{ or } y' \le 0 \text{ a.e. on } \big[t_{j-1},t_j\big]. \tag{2.8}$$

Assuming the contrary we obtain, for some $j \in \{1,\dots,N+1\}$, subsets $U^+, U^- \subset [t_{j-1},t_j]$ which both have positive measure, such that $y' > 0$ on U^+ and $y' < 0$ on U^-. Possibly after reducing U^+ and U^- (but still keeping their measures positive), we may assume that $U^+, U^- \subset [t_{j-1}+\delta, t_j-\delta]$ for some $\delta > 0$, whence (2.3) and (2.5) imply

$$y_n' \ge 0 \text{ a.e. on } U^+ \cup U^- \text{ or } y_n' \le 0 \text{ a.e. on } U^+ \cup U^- \tag{2.9}$$

for n sufficiently large. On the other hand, with χ_+ denoting the characteristic function of U^+, (2.7) implies

$$\int\limits_{U^+} y_n'\,dx = \int\limits_a^b y_n'\chi_+\,dx \longrightarrow \int\limits_a^b y'\chi_+\,dx = \int\limits_{U^+} y'\,dx > 0,$$

and analogously, $\int\limits_{U^-} y_n'\,dx \to \int\limits_{U^-} y'\,dx < 0$. This contradicts (2.9), and thus proves (2.8). In particular, $y \in \mathcal{D}_N$.

Furthermore, by (2.3),

$$\begin{aligned}
\int_a^b |y_n y_n'|\,dx &= \sum_{j=1}^{N+1} \left| \int_{t_{j-1}^{(n)}}^{t_j^{(n)}} |y_n|\, y_n'\,dx \right| \\
&= \frac{1}{2} \sum_{j=1}^{N+1} \left| (y_n\,|y_n|)(t_j^{(n)}) - (y_n\,|y_n|)(t_{j-1}^{(n)}) \right| \\
&\le (N+1)\big\| y_n\,|y_n| - y\,|y| \big\|_\infty + \frac{1}{2} \sum_{j=1}^{N+1} \left| (y\,|y|)(t_j^{(n)}) - (y\,|y|)(t_{j-1}^{(n)}) \right| \\
&= (N+1)\big\| y_n\,|y_n| - y\,|y| \big\|_\infty + \sum_{j=1}^{N+1} \left| \int_{t_{j-1}^{(n)}}^{t_j^{(n)}} |y|\, y'\,dx \right| \\
&\le (N+1)\big\| y_n\,|y_n| - y\,|y|\, \big\|_\infty + \int_a^b |yy'|dx\,,
\end{aligned}$$

whence (2.6) (implying $y_n|y_n| \to y|y|$ uniformly) and (2.2) give

$$\int_a^b |yy'|\,dx \ge (b-a)K_N\ .$$

In particular, $y \neq 0$. Furthermore, $\int_a^b (y')^2 dx \le 1$ by (2.2) and (2.7). Thus, $\mathcal{Q}(y) \ge K_N$. But also $\mathcal{Q}(y) \le K_N$ since $y \in \mathcal{D}_N$. Hence y is the maximizer we are looking for. □

Remark 1. The proof shows that the statement of the proposition remains true when the zero integral condition is replaced by any set of conditions

$$\phi_i[y] = 0 \qquad \text{for } i \in I, \tag{2.10}$$

with some index set I, and with $\phi_i (i \in I)$ denoting some bounded linear functionals on $H^1(a,b)$ which are such that $\mathcal{D} := \{y \in H^1(a,b) : y \text{ satisfies (2.10)}\}$ and $\mathcal{D}_N$ defined by (2.1) (using this $\mathcal{D}$) contain nonzero elements, and such that the Poincaré inequality $\|y\|_{L^2(a,b)} \le C\|y'\|_{L^2(a,b)}$ is true for $y \in \mathcal{D}_N$.

Choosing e.g. $\phi_1[y] = y(a)$ and $\phi_2[y] = y(b)$, or just $\phi_1[y] = y(a)$, one obtains the a priori statement of Proposition 1 for the situations underlying Theorem A or Theorem B, respectively.

3. Proof of Theorem 1

For fixed $N \in \mathbb{N}$, we define

$$\text{either (i) } \widetilde{\mathcal{D}} := \mathcal{D}_N, \text{ or (ii) } \widetilde{\mathcal{D}} := \mathcal{D}, \tag{3.1}$$

and

$$K := \sup_{y \in \widetilde{\mathcal{D}} \setminus \{0\}} \mathcal{Q}(y). \tag{3.2}$$

We will prove for both alternatives in (3.1):

$$\begin{aligned}&\text{If } y \in \widetilde{\mathcal{D}}\backslash\{0\} \text{ is a maximizer of } \mathcal{Q} \text{ on } \widetilde{\mathcal{D}}, \text{ i.e. } \mathcal{Q}(y) = K,\\ &\text{then } K = \tfrac{1}{4} \text{ and } y(x) = c(x - \tfrac{a+b}{2}) \text{ for some } c \in \mathbb{R}.\end{aligned} \tag{3.3}$$

The proof of Theorem 1 is then easy: Using the alternative (i) in (3.1), and Proposition 1, we obtain from (3.3) that $K = K_N = 1/4$. This holds for every $N \in \mathbb{N}$. Moreover, $\cup_{N=0}^{\infty}\mathcal{D}_N$ is dense in $\mathcal{D}$ with respect to the $H^1(a,b)$-norm, since for any given $y \in \mathcal{D}$, the density of $C[a,b]$ in $L^2(a,b)$ in combination with Weierstrass' Approximation Theorem gives a sequence (Q_n) of polynomials converging to y' in $L^2(a,b)$, whence defining

$$P_n(x) := \int_a^x Q_n(s)ds - \frac{1}{b-a}\int_a^b \left(\int_a^t Q_n(s)ds\right) dt \qquad (x \in [a,b],\ n \in \mathbb{N}) \ ,$$

and noting that $\int_a^b y(x)dx = 0$ implies

$$y(x) = \int_a^x y'(s)ds - \frac{1}{b-a}\int_a^b \left(\int_a^t y'(s)ds\right) dt \qquad (x \in [a,b]) \ ,$$

we obtain $P_n \to y$ in $H^1(a,b)$, and $P_n \in \bigcup_{N=0}^{\infty}\mathcal{D}_N$ for each $n \in \mathbb{N}$.

This density result proves that $\sup\{\mathcal{Q}(y) : y \in \mathcal{D}\backslash\{0\}\}$ is also 1/4, and thus the first part of Theorem 1. The second part about the form of extremals follows immediately from (3.3) when using the alternative (ii) in (3.1).

To prove (3.3) we proceed in a series of Lemmas. We restrict ourselves to the case $a = 0$, $b = 1$; the general case then follows by a change of variables.

So let $y \in \widetilde{\mathcal{D}}\backslash\{0\}$ be a maximizer of $\mathcal{Q}$ on $\widetilde{\mathcal{D}}$. We normalize y by

$$\int_0^1 |y'|^2\, dx = 1, \tag{3.4}$$

so that (3.2) gives

$$\frac{1}{K}\int_0^1 |yy'|\, dx = 1. \tag{3.5}$$

Let

$$M_+ := \{x \in [0,1] : y(x) > 0\}, \quad M_- := \{x \in [0,1] : y(x) < 0\}.$$

Since y has zero integral, we have

$$\int_{M_+} y\, dx = -\int_{M_-} y\, dx > 0. \tag{3.6}$$

With no loss of generality (possibly after replacing y by $-y$) we may assume that

$$\mu(M_-) \leq \mu(M_+), \tag{3.7}$$

whence in particular

$$0 < \mu(M_-) \le \frac{1}{2}. \tag{3.8}$$

Lemma 1. *Let*

$$\begin{aligned} \gamma &:= \frac{1}{\int\limits_{M_+} y\,dx} \left[\frac{1}{K} \int\limits_{M_+} |yy'|\,dx - \int\limits_{M_+} |y'|^2\,dx \right] \\ &\equiv \frac{1}{\int\limits_{M_-} y\,dx} \left[\frac{1}{K} \int\limits_{M_-} |yy'|\,dx - \int\limits_{M_-} |y'|^2\,dx \right] \end{aligned} \tag{3.9}$$

where the last equality follows from (3.4)*,* (3.5)*,* (3.6)*. Then,*

$$|y'|^2 = 1 + 2\gamma y \quad \text{a.e. on } [0,1]. \tag{3.10}$$

Proof. Let $\varphi \in C[0,1]$ be fixed, and choose $\varepsilon_0 > 0$ such that, for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$,

$$\min_{[0,1]}(1+\varepsilon\varphi) > 0, \qquad \int\limits_{M_+} (1+\varepsilon\varphi) y\,dx > 0, \qquad \int\limits_{M_-} (1+\varepsilon\varphi) y\,dx < 0$$

(note (3.6)). Define for $x \in [0,1]$

$$\Phi_\varepsilon(x) := \frac{\int_0^x (1+\varepsilon\varphi)dt}{\int_0^1 (1+\varepsilon\varphi)\,dt}, \qquad \Psi_\varepsilon := \Phi_\varepsilon^{-1},$$

which both are increasing C^1-mappings of $[0,1]$ onto itself. Set

$$y_\varepsilon(x) := \begin{cases} \lambda_\varepsilon^+ y(\Psi_\varepsilon(x)) & \text{if } x \in \Phi_\varepsilon(M_+), \\ \lambda_\varepsilon^- y(\Psi_\varepsilon(x)) & \text{if } x \in \Phi_\varepsilon(M_-), \\ 0 & \text{otherwise,} \end{cases} \tag{3.11}$$

where

$$\lambda_\varepsilon^+ := \frac{1}{\int\limits_{M_+} (1+\varepsilon\varphi) y\,dx}, \quad \lambda_\varepsilon^- := -\frac{1}{\int\limits_{M_-} (1+\varepsilon\varphi) y\,dx}. \tag{3.12}$$

Since y is continuous and thus vanishes on $\partial M_+ \cap (0,1)$ and on $\partial M_- \cap (0,1)$, y_ε vanishes on $\partial\Phi_\varepsilon(M_+) \cap (0,1)$ and on $\partial\Phi_\varepsilon(M_-) \cap (0,1)$, and is therefore in $H^1(0,1)$. Furthermore, $M_+ \cap (0,1)$ and $M_- \cap (0,1)$ are both disjoint unions of open intervals, whence we can use the transformation $t = \Psi_\varepsilon(x)$ to obtain

$$\begin{aligned}\int_0^1 y_\varepsilon(x)\,dx &= \lambda_\varepsilon^+ \int_{\Phi_\varepsilon(M_+)} y(\Psi_\varepsilon(x))\,dx + \lambda_\varepsilon^- \int_{\Phi_\varepsilon(M_-)} y(\Psi_\varepsilon(x))\,dx \\ &= \lambda_\varepsilon^+ \int_{M_+} y(t)\Phi_\varepsilon'(t)\,dt + \lambda_\varepsilon^- \int_{M_-} y(t)\Phi_\varepsilon'(t)\,dt \\ &= \frac{1}{\int_0^1 (1+\varepsilon\varphi)\,dt}\left[\lambda_\varepsilon^+ \int_{M_+} (1+\varepsilon\varphi)y\,dt + \lambda_\varepsilon^- \int_{M_-} (1+\varepsilon\varphi)y\,dt\right] = 0\end{aligned}$$

by (3.12), whence $y_\varepsilon \in \mathcal{D}$, i.e. $y_\varepsilon \in \widetilde{\mathcal{D}}$ in case of the alternative (ii) in (3.1). In case of the alternative (i), there exist $a =: t_0 \le t_1 \le \cdots \le t_N \le t_{N+1} := b$ such that $y' \ge 0$ a.e. on $[t_{j-1}, t_j]$ or $y' \le 0$ a.e. on $[t_{j-1}, t_j]$, for each $j = 1, \ldots, N+1$. Hence (3.11) and the positivity of λ_ε^+, λ_ε^- and Ψ_ε' show that, for $j = 1, \ldots, N+1$, $y_\varepsilon' \ge 0$ a.e. on $[\Phi_\varepsilon(t_{j-1}), \Phi_\varepsilon(t_j)]$ or $y_\varepsilon' \le 0$ a.e. on $[\Phi_\varepsilon(t_{j-1}), \Phi_\varepsilon(t_j)]$. This gives $y_\varepsilon \in \mathcal{D}_N$, i.e. $y_\varepsilon \in \widetilde{\mathcal{D}}$ also in this case.

Therefore, since y maximizes $\mathcal{Q}$ on $\widetilde{\mathcal{D}}$, $\mathcal{Q}[y_\varepsilon] \le \mathcal{Q}[y]$ for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$. Moreover, $\lambda_0^+ = \lambda_0^-$ by (3.6) and (3.12), implying that $y_0 = \lambda_0^+ y$ and hence $\mathcal{Q}[y_0] = \mathcal{Q}[y]$. Consequently,

$$\frac{d}{d\varepsilon}\mathcal{Q}[y_\varepsilon]\mid_{\varepsilon=0} = 0, \tag{3.13}$$

provided that the derivative exists, which however will follow from the calculations below. The next step is to compute $\mathcal{Q}[y_\varepsilon]$ and $\frac{d}{d\varepsilon}\mathcal{Q}[y_\varepsilon]\mid_{\varepsilon=0}$. By (3.11),

$$\begin{aligned}\int_0^1 |y_\varepsilon y_\varepsilon'|\,dx &= (\lambda_\varepsilon^+)^2 \int_{\Phi_\varepsilon(M_+)} |y(\Psi_\varepsilon(x))y'(\Psi_\varepsilon(x))|\;\Psi_\varepsilon'(x)\,dx \\ &\quad + (\lambda_\varepsilon^-)^2 \int_{\Phi_\varepsilon(M_-)} |y(\Psi_\varepsilon(x))y'(\Psi_\varepsilon(x))|\;\Psi_\varepsilon'(x)\,dx \\ &= (\lambda_\varepsilon^+)^2 \int_{M_+} |y(t)y'(t)|\,dt + (\lambda_\varepsilon^-)^2 \int_{M_-} |y(t)y'(t)|\,dt \\ &=: f(\varepsilon). \end{aligned} \tag{3.14}$$

Moreover,

$$\begin{aligned}
\int_0^1 (y_\varepsilon')^2\,dx &= (\lambda_\varepsilon^+)^2 \int_{\Phi_\varepsilon(M_+)} y'(\Psi_\varepsilon(x))^2 \Psi_\varepsilon'(x)^2\,dx \\
&\qquad + (\lambda_\varepsilon^-)^2 \int_{\Phi_\varepsilon(M_-)} y'(\Psi_\varepsilon(x))^2 \Psi_\varepsilon'(x)^2\,dx \\
&= (\lambda_\varepsilon^+)^2 \int_{M_+} y'(t)^2 \frac{1}{\Phi_\varepsilon'(t)}\,dt + (\lambda_\varepsilon^-)^2 \int_{M_-} y'(t)^2 \frac{1}{\Phi_\varepsilon'(t)}\,dt \\
&= \int_0^1 (1+\varepsilon\varphi)\,dt \left[(\lambda_\varepsilon^+)^2 \int_{M_+} \frac{|y'|^2}{1+\varepsilon\varphi}\,dt + (\lambda_\varepsilon^-)^2 \int_{M_-} \frac{|y'|^2}{1+\varepsilon\varphi}\,dt \right] \\
&=: g(\varepsilon). \qquad (3.15)
\end{aligned}$$

By (3.12),

$$\frac{d\lambda_\varepsilon^+}{d\varepsilon} = -(\lambda_\varepsilon^+)^2 \int_{M_+} \varphi y\,dx, \quad \frac{d\lambda_\varepsilon^-}{d\varepsilon} = (\lambda_\varepsilon^-)^2 \int_{M_-} \varphi y\,dx,$$

whence (3.14) and (3.15) (and the fact that $\lambda_0^+ = \lambda_0^- =: \lambda_0$) give

$$\begin{aligned}
f'(0) &= -2(\lambda_0^+)^3 \int_{M_+} |yy'|\,dx \int_{M_+} \varphi y\,dx + 2(\lambda_0^-)^3 \int_{M_-} |yy'|\,dx \int_{M_-} \varphi y\,dx \\
&= 2\lambda_0^3 \left[-\int_{M_+} |yy'|\,dx \int_{M_+} \varphi y\,dx + \int_{M_-} |yy'|\,dx \int_{M_-} \varphi y\,dx \right],
\end{aligned}$$

and

$$\begin{aligned}
g'(0) &= \int_0^1 \varphi\,dx \left[(\lambda_0^+)^2 \int_{M_+} |y'|^2\,dx + (\lambda_0^-)^2 \int_{M_-} |y'|^2\,dx \right] \\
&\quad - 2(\lambda_0^+)^3 \int_{M_+} |y'|^2\,dx \int_{M_+} \varphi y\,dx + 2(\lambda_0^-)^3 \int_{M_-} |y'|^2\,dx \int_{M_-} \varphi y\,dx \\
&\quad - (\lambda_0^+)^2 \int_{M_+} \varphi |y'|^2\,dx - (\lambda_0^-)^2 \int_{M_-} \varphi |y'|^2\,dx \\
&= \lambda_0^2 \left[\int_0^1 \varphi\,dx - \int_0^1 \varphi |y'|^2\,dx \right] \\
&\quad + 2\lambda_0^3 \left[-\int_{M_+} |y'|^2\,dx \int_{M_+} \varphi y\,dx + \int_{M_-} |y'|^2\,dx \int_{M_-} \varphi y\,dx \right]
\end{aligned}$$

(note (3.4)). Since $\mathcal{Q}[y_\varepsilon] = f(\varepsilon)/g(\varepsilon)$ and $f(0)/g(0) = K$, condition (3.13) amounts to $g'(0) - \frac{1}{K}f'(0) = 0$, i.e.,

$$\int_0^1 \varphi[1-|y'|^2]\,dx + 2\lambda_0 \left\{ \left[\frac{1}{K}\int_{M_+} |yy'|\,dx - \int_{M_+} |y'|^2\,dx \right] \int_{M_+} \varphi y\,dx \right.$$
$$\left. - \left[\frac{1}{K}\int_{M_-} |yy'|\,dx - \int_{M_-} |y'|^2\,dx \right] \int_{M_-} \varphi y\,dx \right\} = 0. \tag{3.16}$$

Using (3.12), (3.6), and (3.9), we find that (3.16) is equivalent to

$$\int_0^1 \varphi[1-|y'|^2 + 2\gamma y]\,dx = 0$$

which implies (3.10) since $\varphi \in C[0,1]$ is arbitrary. $\square$

Lemma 2. *Let f be any function in $H^1(0,1)$ such that $f > 0$ on M_+, $f < 0$ on M_-, and $|f'| \le 1$ a.e. on $[0,1]$. Then,*

$$\int_{M_+} f\,dx \le \frac{1}{2}\mu(M_+)^2, \tag{3.17}$$

with equality holding if and only if M_+ is an interval and $f(x) = |x-\xi|$ on M_+, with ξ denoting one of the endpoints of M_+. Correspondingly,

$$-\int_{M_-} f\,dx \le \frac{1}{2}\mu(M_-)^2,$$

with equality holding if and only if M_- is an interval and $f(x) = -|x-\eta|$ on M_-, with η denoting one of the endpoints of M_-.

Proof. Since y is continuous, $M_+ \cap (0,1)$ is the disjoint union of finitely or countably many open intervals I_i. For each fixed i, the function f vanishes at least at one endpoint ξ_i of I_i, since f is continuous and at least one endpoint of I_i does not belong to M_+. Thus, since $|f'| \le 1$ a.e. on $[0,1]$,

$$f(x) \le |x-\xi_i| \text{ on } I_i, \tag{3.18}$$

which implies

$$\int_{I_i} f\,dx \le \int_{I_i} |x-\xi_i|\,dx = \frac{1}{2}\mu(I_i)^2 \le \frac{1}{2}\mu(M_+)\cdot\mu(I_i), \tag{3.19}$$

with equality holding (everywhere in this chain) if and only if equality holds in (3.18), and $\mu(I_i) = \mu(M_+)$. Summation over i in (3.19) yields (3.17), with equality holding if and only if there is just *one* interval I_i and equality holds in (3.18). The corresponding statement on M_- follows analogously. $\square$

Lemma 3. *$\gamma \geq 0$, and*

$$\Gamma := \gamma \left| \int\limits_{M_-} y\,dx \right| \leq \min\left\{\frac{\mu}{2}, 1-2\mu\right\} \tag{3.20}$$

where $\mu := \mu(M_-)$ (cf. (3.8)).

Proof. We use again (as in the proof of Lemma 2) the subdivision of $M_+ \cap (0,1)$ and of $M_- \cap (0,1)$ into disjoint open intervals $\{I_i^+\}$ and $\{I_j^-\}$, respectively, and the fact that, for each i and j, y vanishes at least at one endpoint of I_i^+ and I_j^-, respectively.

Using the half interval inequality (1.2) in Theorem B, we obtain for each i that

$$\int\limits_{I_i^+} |yy'|\,dx \leq \frac{1}{2}\mu(I_i^+) \int\limits_{I_i^+} |y'|^2\,dx \leq \frac{1}{2}\mu(M_+) \int\limits_{I_i^+} |y'|^2\,dx,$$

whence summation over i gives

$$\int\limits_{M_+} |yy'|\,dx \leq \frac{1}{2}\mu(M_+) \int\limits_{M_+} |y'|^2\,dx. \tag{3.21}$$

Since $K \geq \frac{1}{4}$ (which follows from (3.2) and $\mathcal{Q}[\widetilde{y}] = \frac{1}{4}$ for $\widetilde{y}(x) = x - \frac{1}{2}$), (3.21) further implies that

$$\frac{1}{K} \int\limits_{M_+} |yy'|\,dx \leq 2\mu(M_+) \int\limits_{M_+} |y'|^2\,dx \leq 2(1-\mu) \int\limits_{M_+} |y'|^2\,dx.$$

Analogously,

$$\frac{1}{K} \int\limits_{M_-} |yy'|\,dx \leq 2\mu(M_-) \int\limits_{M_-} |y'|^2\,dx = 2\mu \int\limits_{M_-} |y'|^2\,dx.$$

By (3.9), we therefore obtain

$$\gamma \int\limits_{M_+} y\,dx \leq (1-2\mu) \int\limits_{M_+} |y'|^2\,dx \leq 1-2\mu \tag{3.22}$$

(cf. (3.8)), as well as

$$\gamma \int\limits_{M_-} y\,dx \leq (2\mu - 1) \int\limits_{M_-} |y'|^2\,dx \leq 0$$

by (3.8), whence (3.6) proves $\gamma \geq 0$. Moreover (3.10) implies

$$\gamma \int\limits_{M_-} y\,dx = \frac{1}{2}\left[\int\limits_{M_-} |y'|^2\,dx - \mu(M_-)\right] \geq -\frac{\mu}{2},$$

which together with (3.6) and (3.22) gives (3.20). □

Now we want to "solve" the differential equation (3.10). Formally (i.e. without regarding zeroes of $1+2\gamma y$), (3.10) yields $\left|\frac{1}{\gamma}\left(\sqrt{1+2\gamma y}\right)'\right| = 1$ if $\gamma \neq 0$, i.e. $\frac{1}{\gamma}\sqrt{1+2\gamma y} = z(+\text{const})$ for some function z such that $|z'| \equiv 1$. A "natural" specification for the constant is $\frac{1}{\gamma}$, since then

$$z = \frac{1}{\gamma}\left[\sqrt{1+2\gamma y} - 1\right]$$

gives $z \equiv y$ in the limit $\gamma \to 0$, whence $|z'| \equiv 1$ in every case.
For a rigorous proof, we have to circumvent differentiability problems (and possible multiple solutions of initial value problems) when $1+2\gamma y$ has zeroes. For this purpose, we "regularize" the above function z by a parameter $\varepsilon > 0$, which leads to the definition (3.23) below.

Lemma 4. $\mu(M_+) = \mu(M_-) = \frac{1}{2}, \gamma = 0$, *and* $K = \frac{1}{4}$.

Proof. Let $\varepsilon > 0$, and define $z \in H^1(0,1)$ by

$$z := \begin{cases} \frac{1}{\gamma}\left[\sqrt{1+\varepsilon+2\gamma y} - \sqrt{1+\varepsilon}\right] & \text{if } \gamma \neq 0, \\ \frac{y}{\sqrt{1+\varepsilon}} & \text{if } \gamma = 0. \end{cases} \tag{3.23}$$

(Note that $1+2\gamma y \geq 0$ by (3.10)). Therefore, in both cases, $\sqrt{1+\varepsilon+2\gamma y} = \sqrt{1+\varepsilon} + \gamma z$, implying

$$\gamma z \geq -\sqrt{1+\varepsilon} \text{ on } [0,1] \tag{3.24}$$

and

$$y = z\left(\sqrt{1+\varepsilon} + \frac{1}{2}\gamma z\right). \tag{3.25}$$

(3.24) shows in particular that $\sqrt{1+\varepsilon}+\frac{1}{2}\gamma z > 0$ on $[0,1]$, whence (3.25) gives $z > 0$ on M_+ and $z < 0$ on M_-. Moreover, (3.23) and (3.10) imply

$$|z'| = \frac{|y'|}{\sqrt{1+\varepsilon+2\gamma y}} = \sqrt{\frac{1+2\gamma y}{1+\varepsilon+2\gamma y}} \leq 1 \text{ a.e. on } [0,1], \tag{3.26}$$

so that by Lemma 2

$$2\left|\int\limits_{M_-} z\,dx\right| \leq \mu(M_-)^2 = \mu^2. \tag{3.27}$$

Furthermore using (3.25),

$$\int\limits_{M_-} yz\,dx - \frac{1}{\mu}\int\limits_{M_-} y\,dx \int\limits_{M_-} z\,dx = \frac{1}{2\mu}\iint\limits_{M_-\times M_-}[y(x)-y(\widetilde{x})][z(x)-z(\widetilde{x})]\,dx\,d\widetilde{x}$$

$$= \frac{1}{2\mu}\iint\limits_{M_-\times M_-}[z(x)-z(\widetilde{x})]^2\left[\sqrt{1+\varepsilon}+\frac{1}{2}\gamma(z(x)+z(\widetilde{x}))\right]dx\,d\widetilde{x}$$

which is non-negative by (3.24). Thus,

$$\int\limits_{M_-} yz\,dx \geq \frac{1}{\mu}\left|\int\limits_{M_-} y\,dx\right|\left|\int\limits_{M_-} z\,dx\right|. \tag{3.28}$$

In addition by (3.25) and Lemma 3, $y \geq \sqrt{1+\varepsilon}\ z$ on $[0,1]$, whence $|y| \leq \sqrt{1+\varepsilon}\,|z|$ on M_-, and thus,

$$\int\limits_{M_-} z^2\,dx \geq \frac{1}{\sqrt{1+\varepsilon}} \int\limits_{M_-} yz\,dx. \tag{3.29}$$

Moreover, with $x_0 \in M_+$ chosen such that $z(x_0) = \max\{z(x) : x \in M_+\} =: z_{\max}$, (3.26) implies

$$z(x) \geq z_{\max} - |x - x_0| \text{ for } x \in [0,1]. \tag{3.30}$$

Since z has at least one zero, (3.30) shows that at least one of the two intervals $(x_0 - z_{\max}, x_0)$ and $(x_0, x_0 + z_{\max})$ is completely contained in M_+. Thus, with I denoting this interval, (3.30) gives

$$\int\limits_{M_+} z(x)\,dx \geq \int\limits_{I} z(x)\,dx \geq \int\limits_{I} [z_{\max} - |x - x_0|]\,dx = \frac{1}{2} z_{\max}^2. \tag{3.31}$$

Finally, since $\gamma \geq 0$ (by Lemma 3) and $\int\limits_0^1 y\,dx = 0$, (3.25) implies that $\int\limits_0^1 z\,dx \leq 0$, i.e., $\int\limits_{M_+} z\,dx \leq \left|\int\limits_{M_-} z\,dx\right|$, whence (3.31) gives

$$z_{\max} \leq \left[2 \left|\int\limits_{M_-} z\,dx\right|\right]^{\frac{1}{2}}. \tag{3.32}$$

Now, using (3.5), (3.25), (3.24), (3.26), and (3.6) we obtain

$$\begin{aligned}
K &= \int\limits_0^1 |yy'|\,dx = \int\limits_0^1 |y|\,|z'|\left(\sqrt{1+\varepsilon} + \gamma z\right)\,dx \\
&\leq \sqrt{1+\varepsilon} \int\limits_0^1 |y|\,dx + \gamma \int\limits_0^1 |y| z\,dx \\
&= -2\sqrt{1+\varepsilon} \int\limits_{M_-} y\,dx + \gamma \left[- \int\limits_{M_-} yz\,dx + \int\limits_{M_+} yz\,dx\right] \\
&= -2(1+\varepsilon) \int\limits_{M_-} z\,dx + \gamma \left[-\sqrt{1+\varepsilon} \int\limits_{M_-} z^2\,dx - \int\limits_{M_-} yz\,dx + \int\limits_{M_+} yz\,dx\right].
\end{aligned}$$

Thus, going on with estimating according to (3.29), (3.28), (3.32), and using $\gamma \geq 0$ and Γ defined in Lemma 3, we obtain

$$\begin{aligned}
K &\leq -2(1+\varepsilon)\int\limits_{M_-} z\,dx + \gamma\left[-2\int\limits_{M_-} yz\,dx + z_{\max}\int\limits_{M_+} y\,dx\right] \\
&\leq -2(1+\varepsilon)\int\limits_{M_-} z\,dx + \gamma\left[-\frac{2}{\mu}\left|\int\limits_{M_-} y\,dx\right|\left|\int\limits_{M_-} z\,dx\right| + \left(2\left|\int\limits_{M_-} z\,dx\right|\right)^{\frac{1}{2}}\left|\int\limits_{M_-} y\,dx\right|\right] \\
&= (1+\varepsilon-\frac{1}{\mu}\Gamma)\cdot 2\left|\int\limits_{M_-} z\,dx\right| + \Gamma\left[2\left|\int\limits_{M_-} z\,dx\right|\right]^{\frac{1}{2}}.
\end{aligned}$$

Since $1+\varepsilon-\frac{1}{\mu}\Gamma \geq 0$ due to (3.20), we go on using (3.27) to obtain

$$K \leq (1+\varepsilon-\frac{1}{\mu}\Gamma)\mu^2 + \Gamma\mu = (1+\varepsilon)\mu^2.$$

This holds for every $\varepsilon > 0$, whence $K \leq \mu^2$. On the other hand, $K \geq \frac{1}{4}$. So by (3.8) we obtain

$$K = \frac{1}{4}, \text{ and } \mu = \mu(M_-) = \frac{1}{2}.$$

Now (3.7) shows that also $\mu(M_+) = \frac{1}{2}$, and finally (3.20) gives $\Gamma = 0$, and hence $\gamma = 0$. □

It is now easy to complete the proof of (3.3). Lemma 4 and (3.10) give

$$|y'| = 1 \text{ a.e. on } [0,1], \tag{3.33}$$

so Lemma 2 provides

$$\int\limits_{M_+} y\,dx \leq \frac{1}{8}, \qquad -\int\limits_{M_-} y\,dx \leq \frac{1}{8}. \tag{3.34}$$

Moreover, by (3.5), (3.33) and Lemma 4,

$$\frac{1}{4} = K = \int\limits_0^1 |y|\,dx = \int\limits_{M_+} y\,dx - \int\limits_{M_-} y\,dx.$$

So equality must hold in both inequalities in (3.34). Therefore, Lemma 2 implies that both M_+ and M_- must be intervals (of length $\frac{1}{2}$), i.e.,

$$M_+ \cap (0,1) = \left(0, \tfrac{1}{2}\right), \quad M_- \cap (0,1) = \left(\tfrac{1}{2}, 1\right) \tag{3.35}$$

or vice versa, and that

$$y(x) = x \text{ or } \tfrac{1}{2} - x \text{ on } \left(0, \tfrac{1}{2}\right), \quad y(x) = x - 1 \text{ or } \tfrac{1}{2} - x \text{ on } \left(\tfrac{1}{2}, 1\right)$$

in the case (3.35), and

$$y(x) = -x \text{ or } x - \tfrac{1}{2} \text{ on } \left(0, \tfrac{1}{2}\right), \quad y(x) = 1 - x \text{ or } x - \tfrac{1}{2} \text{ on } \left(\tfrac{1}{2}, 1\right)$$

in the opposite case. Since y is continuous, we obtain $y(x) = x - \frac{1}{2}$ or $\frac{1}{2} - x$ on $[0,1]$.

4. A new proof of Theorems A and B

The following is by no means the easiest or shortest proof of Opial's inequality or its half interval version[3]. However it is a new argument and we include it since the technique may be useful in the proof of other inequalities. Again, it suffices to consider the case $a = 0$, $b = 1$.
Using Remark 1 after Proposition 1, it is sufficient to prove a statement corresponding to (3.3), with the obvious changes concerning the value of K and the form of extremals. So, with $\widetilde{\mathcal{D}}$ defined correspondingly, let $y \in \widetilde{\mathcal{D}}\backslash\{0\}$ be a maximizer of

$$\mathcal{Q}[z] := \frac{\int_0^1 |zz'|\,dx}{\int_0^1 (z')^2\,dx}$$

on $\widetilde{\mathcal{D}}$, normalized by (3.4), (3.5), with K given by (3.2). Let $\varphi \in C[0,1]$ be fixed, and choose $\varepsilon_0 > 0$ such that, for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$,

$$\min_{[0,1]}(1 + \varepsilon\varphi) > 0,$$

and define again

$$\Phi_\varepsilon(x) := \frac{\int_0^x (1+\varepsilon\varphi)dt}{\int_0^1 (1+\varepsilon\varphi)dt} \text{ on } [0,1], \quad \Psi_\varepsilon := \Phi_\varepsilon^{-1},$$

and this time

$$y_\varepsilon(x) := y(\Psi_\varepsilon(x)) \text{ on } [0,1].$$

Since $\Psi_\varepsilon(0) = 0$ and $\Psi_\varepsilon(1) = 1$, y_ε lies in $\widetilde{\mathcal{D}}$ in both Theorem A and B (by the same argument as given in the proof of Lemma 1), whence

$$\frac{d}{d\varepsilon}\mathcal{Q}[y_\varepsilon]\,|_{\varepsilon=0} = 0. \tag{4.1}$$

Also because

$$\begin{aligned}
\int_0^1 |y_\varepsilon y_\varepsilon'|\,dx &= \int_0^1 |y(\Psi_\varepsilon(x))y'(\Psi_\varepsilon(x))|\,\Psi_\varepsilon'(x)\,dx = \int_0^1 |yy'|\,dt, \\
\int_0^1 (y_\varepsilon')^2\,dx &= \int_0^1 y'(\Psi_\varepsilon(x))^2 \Psi_\varepsilon'(x)^2\,dx = \int_0^1 y'(t)^2 \frac{1}{\Phi_\varepsilon'(t)}\,dt \\
&= \int_0^1 (1+\varepsilon\varphi)dt \int_0^1 \frac{(y')^2}{1+\varepsilon\varphi}\,dt,
\end{aligned}$$

[3]This distinction probably belongs to Mallows whose proof of Theorem B takes two lines. The proofs of Olech and Levinson are also short. See [1, pp. 5–9].

(4.1) amounts to

$$0 = \frac{d}{d\varepsilon}\left[\int_0^1 (1+\varepsilon\varphi)dt \int_0^1 \frac{(y')^2}{1+\varepsilon\varphi}\,dt\right]\Bigg|_{\varepsilon=0} = \int_0^1 \varphi\,dt - \int_0^1 \varphi(y')^2\,dt$$

(note (3.4)). Since $\varphi \in C[0,1]$ is arbitrary, we obtain

$$|y'| = 1 \text{ a.e. on } [0,1], \tag{4.2}$$

and (3.5) gives

$$K = \int_0^1 |y|\,dx. \tag{4.3}$$

Now to distinguish between the two theorems we wish to prove, we set $\widetilde{y} := \min\{x, 1-x\}$ for Theorem A and $\widetilde{y} := x$ for Theorem B. Property (4.2) and the condition $y(0) = y(1) = 0$ for Theorem A or $y(0) = 0$ for Theorem B imply that $|y| \le \widetilde{y}$ on $[0,1]$, whence

$$\int_0^1 |y|\,dx \le \int_0^1 \widetilde{y}\,dx = \mathcal{Q}[\widetilde{y}] \le K. \tag{4.4}$$

Now (4.3) shows that equality must hold everywhere in the inequality chain (4.4), so that $K = \mathcal{Q}[\widetilde{y}] = \frac{1}{2}$ in Theorem B ($= \frac{1}{4}$ in Theorem A), and $|y| = \widetilde{y}$ on $[0,1]$, whence by continuity $y = \widetilde{y}$ or $y = -\widetilde{y}$. $\square$

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