

A computer-assisted existence and multiplicity proof for travelling waves in a nonlinearly supported beam

B. Breuer¹, J. Horák², P. J. McKenna³, M. Plum¹

Abstract: For a nonlinear beam equation with exponential nonlinearity, we prove existence of at least 36 travelling wave solutions for the specific wave speed $c = 1.3$. This complements the result in [30] stating that for almost all $c \in (0, \sqrt{2})$ there exists at least one solution. Our proof makes heavy use of computer assistance: Starting from numerical approximations, we use a fixed point argument to prove existence of solutions “close to” the computed approximations.

Keywords: Travelling waves, existence, multiplicity, computer-assisted proof.

1 Introduction

Inspired by an old report of the existence of travelling waves on the Golden Gate Bridge in San Francisco in 1938, [2], the study of travelling waves in nonlinearly supported beams was begun in [16]. The first type of nonlinearity that was studied was a piecewise linear one, reflecting the fact that when cables loose tension, they do not resist compression.

The first result, in [16], was partly numerical. The equation

$$u_{tt} + u_{xxxx} + u^+ = 1 \tag{1}$$

was studied on $\mathbb{R}^1$ and solutions of the form $1 + y(x - ct)$ were found by reducing the partial differential equation (1) to the ordinary differential equation on the real line

$$y^{iv} + c^2 y'' + (1 + y)^+ = 1$$

and then solving explicitly the two linear equations $y^{iv} + c^2 y'' + y = 0$ where $y \geq -1$ and $y^{iv} + c^2 y'' = 1$ where $y \leq -1$. Solutions of both equations were constructed which matched at the boundary $y = -1$ and which tended to zero exponentially as $|x| \rightarrow \infty$ by showing that solutions corresponded to zeroes of a certain transcendental function. These zeroes were then found numerically for $c \in [c_1, c_2]$ where c_1 and c_2 were certain constants satisfying $0 < c_1 < c_2 < \sqrt{2}$.

Later, in [9], two important developments took place. First, a rigorous proof of the existence of solutions of (1) was given for *all* $c \in (0, \sqrt{2})$ via the mountain pass theorem and the method of concentrated compactness. Second, the investigation of interaction properties of this new class of waves was begun.

As explained in [9], the piecewise nonlinearity was not very suitable for highly accurate numerical investigations of the initial value problem for (1), although some

¹Mathematisches Institut I, Universität Karlsruhe, 76218 Karlsruhe, Germany

²Mathematisches Institut, Universität zu Köln, Weyertal 86-90, 50931 Köln, Germany

³Department of Mathematics, University of Connecticut, Storrs, CT 06269-3009, U.S.A

numerical experiments were promising. Therefore the authors in [9] decided to change the nonlinearity in (1) to a smoother one with many of the same characteristics. The model they chose was to replace the piecewise $(1 + y)^+ - 1$ by $f(y) = e^y - 1$. Like the piecewise version, this tended to -1 as $y \rightarrow -\infty$, $f(0) = 0$ and $f'(0) = 1$. This substitution led to the discovery of a large class of travelling wave solutions with extraordinary interaction, stability, and fission properties which remain unexplained to date, [9, 14]. Initially, these solutions were calculated by the mountain pass algorithm, [10], although later it became clear that shooting methods were a faster and more efficient substitute, [7].

However, the substitution of the new nonlinearity, while making for beautiful numerical results, introduced a new problem; the existence of homoclinic solutions of

$$y^{iv} + c^2 y'' + e^y - 1 = 0 \quad (2)$$

was not proven.

Numerical evidence suggested that there were many solutions of equation (2), with many different shapes. Some were stable, some unstable. Their minima appeared to go to $-\infty$ as $c \rightarrow 0$. As $c \rightarrow \sqrt{2}$, they appear to go to zero, but this becomes difficult to compute since they begin to resemble a sinusoidal function, and are supported on larger and larger sets.

Until recently, there has been little progress on the proof of existence of solutions of equation (2). In this journal, Smets and van den Berg [30] showed that for *almost all* c in the interval $(0, \sqrt{2})$, there exists at least one solution.

In this paper, we go in a different direction to prove existence of many homoclinic solutions of equation (2) for one fixed c , also assuming that $c \in (0, \sqrt{2})$. We first calculate approximate solutions numerically. The next step is to verify that there are true solutions of (2) close to each of the approximate solutions. This is done by a fixed point argument applied to the differential equation for the error function. The general idea for this verification is presented in Section 3.

While we recognize that this result is not ideal, it is in some sense a complement to that of [30], emphasizing that we expect eventually to establish existence for all c , and presumably with larger multiplicity. The plan of this paper is as follows: following some notational preliminaries in section 2, we outline the existence and enclosure theorem which is at the heart of this paper. (This is the result that allows us to "capture" true solutions in the neighborhood of approximate solutions.) Showing how the essential constant needed for this theorem is obtained computationally is the subject of section 4. Although not strictly necessary for the purpose of proving existence of the solutions, we have also calculated the Morse indices of the solutions we have verified. By this, we mean the number of negative eigenvalues of the linearization about each solution. (In calculating this index, we restrict ourselves to the space of symmetric solutions about the origin, since otherwise the solutions would be degenerate with an eigenvalue of zero, by translation invariance.)

Section 6 is devoted to the way we found 40 approximate solutions, first rather crudely, via shooting, and then how we refined about them so as to allow us, for 36 of them, to capture the true solutions via the enclosure method. Section 7 describes the verified computation of some more constants needed for the method. In section

8, we summarize our verified results on the existence of 36 solutions of equation (2) with $c = 1.3$. This leads eventually to our final completely verified result:

Theorem 1. *For $c = 1.3$, equation (2) has at least 36 solutions. Their Morse indices are given in Table 1.*

In reading this section, it is important to distinguish between the *verified* results and the approximations. When we speak of the existence of the 36 solutions and their Morse indices, it is clear that these have been proved. However, when we speak of the "branches" of solutions obtained by the continuation methods, this has not been verified and has been included more as an aid to the reader's intuitive understanding of the probable big picture of the solution set supported by "reasonable" computational evidence.

Finally, in section 9, we mention some open problems and directions for future research.

The computer-assisted method used in this paper has been applied successfully to several other problems already (see e.g. [24, 25, 26, 28]). Also other research groups have developed methods for proving results in the field of boundary value problems for ordinary and partial differential equations by computer assistance. M. T. Nakao and his co-workers use a splitting of the problem into a finite-dimensional part and an infinite-dimensional "remainder"; the former is treated directly by (verified) numerical methods, the latter is captured by projection error bounds (see e.g. [19, 20, 21]). Another more recent approach is based on the Conley index and the numerical verification of corresponding topological conditions; it is suited for proving the existence of stationary solutions for certain classes of problems, as well as for detecting global dynamics (see e.g. [11, 13]).

It is also worth remarking that a result of three of the authors (which is similar to the one presented here), [5], on a long-standing open question in elliptic partial differential equations stimulated major progress and a solution of "most" of that open problem. Certainly, we hope that a similar result occurs for this problem.

2 Basic notation and inequalities

Here we formulate and prove some basic inequalities which we will need for our computer-assisted existence proofs. The fourth order problem (2) will be formulated weakly in the space $H_S^2(\mathbb{R}) := \{u \in H^2(\mathbb{R}) : u(x) = u(-x) \text{ for all } x \in \mathbb{R}\}$, which we endow with the inner product

$$\langle u, v \rangle_{H^2} := \langle u'', v'' \rangle_{L^2} + \sigma \langle u, v \rangle_{L^2}, \quad (3)$$

where $\langle \cdot, \cdot \rangle_{L^2}$ denotes the usual inner product in $L^2(\mathbb{R})$, and $\sigma > 0$ is some constant to be specified later. Indeed, the inequality $\|u'\|_{L^2}^2 \leq \|u\|_{L^2} \|u''\|_{L^2}$ ensures that $\langle \cdot, \cdot \rangle_{H^2}$ is equivalent to the "usual" H^2 -inner product in $H_S^2(\mathbb{R})$; it is however better suited for our quantitative purposes.

Our weak formulation of problem (2) reads: Find $u \in H_S^2(\mathbb{R})$ such that

$$\int_{\mathbb{R}} [u'' \varphi'' - c^2 u' \varphi' + (e^u - 1) \varphi] dx = 0 \quad \text{for all } \varphi \in H_S^2(\mathbb{R}). \quad (4)$$

Note that it amounts to the same if we pose the equation in (4) for all $\varphi \in H^2(\mathbb{R})$, since the antisymmetric part of any $\varphi \in H^2(\mathbb{R})$ satisfies the equation automatically (as long as $u \in H_S^2(\mathbb{R})$).

Besides $H_S^2(\mathbb{R})$, we will need its (topological) dual space $H_S^{-2}(\mathbb{R})$, endowed with the canonical dual norm $\|\cdot\|_{H^{-2}}$. Functions $u \in L_S^2(\mathbb{R}) := \{v \in L^2(\mathbb{R}) : v(x) = v(-x) \text{ for almost all } x \in \mathbb{R}\}$ are identified with elements in $H_S^{-2}(\mathbb{R})$ by

$$u[\varphi] := \int_{\mathbb{R}} u\varphi \, dx \quad \text{for all } \varphi \in H_S^2(\mathbb{R}),$$

and their second derivative $u'' \in H_S^{-2}(\mathbb{R})$ is defined by

$$u''[\varphi] := \int_{\mathbb{R}} u\varphi'' \, dx \quad \text{for all } \varphi \in H_S^2(\mathbb{R}).$$

Riesz' Representation Lemma for bounded linear functionals shows that the mapping

$$\Phi : \left\{ \begin{array}{l} H_S^2(\mathbb{R}) \rightarrow H_S^{-2}(\mathbb{R}) \\ u \mapsto u^{iv} + \sigma u \end{array} \right\} \quad (\text{i.e. } (\Phi u)[\varphi] := \langle u, \varphi \rangle_{H^2} \text{ for } u, \varphi \in H_S^2(\mathbb{R})) \quad (5)$$

is an isometric isomorphism.

Lemma 1. *The following inequalities hold true:*

- a) $\|u\|_{L^2} \leq \frac{1}{\sqrt{\sigma}} \|u\|_{H^2}$ for $u \in H_S^2(\mathbb{R})$,
- b) $\|u\|_{H^{-2}} \leq \frac{1}{\sqrt{\sigma}} \|u\|_{L^2}$ for $u \in L_S^2(\mathbb{R})$,
- c) $\|u\|_{\infty} \leq \widehat{C} \|u\|_{H^2}$ for $u \in H_S^2(\mathbb{R})$, where $\widehat{C} := \frac{1}{2} \left(\frac{3}{\sigma}\right)^{3/8}$,
- d) $e^u - 1 \in L_S^2(\mathbb{R})$ for $u \in H_S^2(\mathbb{R})$, and

$$\|e^u - 1 - u\|_{L^2} \leq \frac{\widehat{C}}{2\sqrt{\sigma}} \exp(\widehat{C} \|u\|_{H^2}) \|u\|_{H^2}^2,$$
- e) $\|(e^u - 1 - u) - (e^v - 1 - v)\|_{L^2} \leq \frac{\widehat{C}}{\sqrt{\sigma}} \exp(\widehat{C} \max\{\|u\|_{H^2}, \|v\|_{H^2}\}) \cdot \max\{\|u\|_{H^2}, \|v\|_{H^2}\} \cdot \|u - v\|_{H^2}$ for $u, v \in H_S^2(\mathbb{R})$.

Proof: a) is trivial. b) follows from a) by the usual dual estimate

$$\begin{aligned} \|u\|_{H^{-2}} &= \sup \left\{ \frac{1}{\|\varphi\|_{H^2}} \left| \int_{\mathbb{R}} u\varphi \, dx \right| : \varphi \in H_S^2(\mathbb{R}), \varphi \neq 0 \right\} \\ &\leq \sup \left\{ \frac{\|u\|_{L^2} \|\varphi\|_{L^2}}{\|\varphi\|_{H^2}} : \varphi \in H_S^2(\mathbb{R}), \varphi \neq 0 \right\} \leq \frac{1}{\sqrt{\sigma}} \|u\|_{L^2}. \end{aligned}$$

For proving c), we first note that $H^2(\mathbb{R})$ embeds continuously into the space of bounded continuous functions on $\mathbb{R}$ (endowed with the sup-norm $\|\cdot\|_{\infty}$) by Sobolev's

Embedding Theorem; what we have to prove is the validity of the asserted embedding constant $\widehat{C}$. For each $x \in \mathbb{R}$, we have

$$u(x)^2 = 2 \int_{-\infty}^x uu' dt \leq 2 \int_{-\infty}^x |uu'| dt, \quad u(x)^2 = -2 \int_x^{\infty} uu' dt \leq 2 \int_x^{\infty} |uu'| dt,$$

whence by addition we obtain

$$u(x)^2 \leq \int_{\mathbb{R}} |uu'| dt \leq \|u\|_{L^2} \|u'\|_{L^2}. \quad (6)$$

Moreover, $\|u'\|_{L^2}^2 \leq \|u\|_{L^2} \|u''\|_{L^2}$ by partial integration, which together with (6) implies

$$\begin{aligned} \|u\|_{\infty}^2 &\leq \|u\|_{L^2}^{3/2} \|u''\|_{L^2}^{1/2} = \left[\left(\frac{\sigma}{3} \right)^{\frac{3}{16}} \|u\|_{L^2}^{3/2} \right] \cdot \left[\left(\frac{3}{\sigma} \right)^{\frac{3}{16}} \|u''\|_{L^2}^{1/2} \right] \\ &\leq \frac{\left[\left(\frac{\sigma}{3} \right)^{\frac{3}{16}} \|u\|_{L^2}^{3/2} \right]^{\frac{4}{3}}}{\frac{4}{3}} + \frac{\left[\left(\frac{3}{\sigma} \right)^{\frac{3}{16}} \|u''\|_{L^2}^{1/2} \right]^4}{4} \\ &= \frac{3}{4} \left(\frac{\sigma}{3} \right)^{\frac{1}{4}} \|u\|_{L^2}^2 + \frac{1}{4} \left(\frac{3}{\sigma} \right)^{\frac{3}{4}} \|u''\|_{L^2}^2 \\ &= \frac{1}{4} \left(\frac{3}{\sigma} \right)^{\frac{3}{4}} \{ \sigma \|u\|_{L^2}^2 + \|u''\|_{L^2}^2 \} = \widehat{C}^2 \|u\|_{H^2}^2, \end{aligned}$$

which proves c).

To prove d) we note that, by Taylor's Theorem,

$$|e^{u(x)} - 1 - u(x)| = \frac{1}{2} e^{\theta u(x)} |u(x)|^2 \leq \frac{1}{2} e^{\|u\|_{\infty}} \|u\|_{\infty} |u(x)|$$

for each $x \in \mathbb{R}$ and some $\theta = \theta(x) \in [0, 1]$, whence

$$\|e^u - 1 - u\|_{L^2} \leq \frac{1}{2} e^{\|u\|_{\infty}} \|u\|_{\infty} \|u\|_{L^2},$$

so that a) and c) give the assertion.

For proving e), we use Taylor's Theorem again to obtain

$$\begin{aligned} |(e^{u(x)} - 1 - u(x)) - (e^{v(x)} - 1 - v(x))| &= |[e^{v(x)+\theta(u(x)-v(x))} - 1] (u(x) - v(x))| \\ &\leq [\exp(\max\{\|u\|_{\infty}, \|v\|_{\infty}\}) - 1] |u(x) - v(x)| \\ &\leq \exp(\max\{\|u\|_{\infty}, \|v\|_{\infty}\}) \cdot \max\{\|u\|_{\infty}, \|v\|_{\infty}\} \cdot |u(x) - v(x)|, \end{aligned}$$

whence again a) and c) prove the assertion. $\square$

Remark 1. The proof shows that Lemma 1 remains true with $H_S^2(\mathbb{R})$ and $L_S^2(\mathbb{R})$ replaced by the full spaces $H^2(\mathbb{R})$ and $L^2(\mathbb{R})$.

3 The Existence and Enclosure Theorem

In this section we will present the theorem forming the basis of our computational existence and multiplicity proof for problem (2) (resp. (4)). Besides existence of a solution $u^* \in H_S^2(\mathbb{R})$, the theorem yields a *bound* for u^* of the form

$$\|u^* - \omega\|_{H^2} \leq \alpha, \quad (7)$$

with $\omega \in H_S^2(\mathbb{R})$ denoting an approximate solution computed by numerical means, and with $\alpha > 0$ denoting a “small” constant provided by the theorem. By Lemma 1a) and c), we obtain

$$\|u^* - \omega\|_{L^2} \leq \frac{1}{\sqrt{\sigma}} \alpha, \quad \|u^* - \omega\|_{\infty} \leq \hat{C} \alpha \quad (8)$$

as consequences of (7).

Thus, with $\omega_1, \dots, \omega_k \in H_S^2(\mathbb{R})$ denoting approximations such that, with $\alpha_1, \dots, \alpha_k$ denoting the error bounds given by the theorem,

$$\begin{aligned} \|\omega_i - \omega_j\|_{H^2} > \alpha_i + \alpha_j \quad \text{or} \quad \|\omega_i - \omega_j\|_{L^2} > \frac{1}{\sqrt{\sigma}} (\alpha_i + \alpha_j) \\ \text{or} \quad \|\omega_i - \omega_j\|_{\infty} > \hat{C} (\alpha_i + \alpha_j) \end{aligned} \quad (9)$$

for $i, j = 1, \dots, k$, $i \neq j$, our method yields the existence of k different solutions $u_1^*, \dots, u_k^* \in H_S^2(\mathbb{R})$ and thus, the desired multiplicity result. Note that (9) can be checked rather directly from the numerical data.

So let $\omega \in H_S^2(\mathbb{R})$ denote an approximate solution to problem (4) obtained by the numerical methods described in Section 6. We need the following two quantities:

- (i) a bound $\delta \geq 0$ for the *defect* (residual) of ω :

$$\|\omega^{iv} + c^2 \omega'' + e^\omega - 1\|_{H^{-2}} \leq \delta, \quad (10)$$

the computation of which will be described in Section 7,

- (ii) a constant $K \geq 0$ such that

$$\|u\|_{H^2} \leq K \|Lu\|_{H^{-2}} \quad \text{for all } u \in H_S^2(\mathbb{R}), \quad (11)$$

with $L : H_S^2(\mathbb{R}) \rightarrow H_S^{-2}(\mathbb{R})$ denoting the linearization of (2) at ω :

$$Lu := u^{iv} + c^2 u'' + e^\omega u, \quad \text{i.e.} \quad (Lu)[\varphi] = \int_{\mathbb{R}} (u'' \varphi'' - c^2 u' \varphi' + e^\omega u \varphi) dx. \quad (12)$$

Clearly, K satisfying (11) is a bound for the inverse operator L^{-1} . The calculation of K , which is the most involved part of our method and needs computer-assisted methods of its own, will be described in Section 4.

Theorem 2. Suppose that some $\alpha \geq 0$ exists such that

$$\delta \leq \frac{\alpha}{K} - \alpha^2 \frac{\hat{C}}{2\sigma} \exp(\bar{\omega} + \hat{C}\alpha), \quad (13)$$

where $\bar{\omega} := \sup_{x \in \mathbb{R}} \omega(x)$, and

$$\alpha K \frac{\hat{C}}{\sigma} \exp(\bar{\omega} + \hat{C}\alpha) < 1. \quad (14)$$

Then, there exists a solution $u^* \in H_S^2(\mathbb{R})$ of problem (2) resp. (4) satisfying (7).

Proof: The first step is to prove that

$$L : H_S^2(\mathbb{R}) \rightarrow H_S^{-2}(\mathbb{R}) \text{ is one-to-one and onto.} \quad (15)$$

Indeed, L is one-to-one by (11). Moreover, L is clearly bounded, and defined on the whole Hilbert space $H_S^2(\mathbb{R})$; hence L is closed. Therefore, L^{-1} is closed, and moreover bounded by (11). Thus, the domain of L^{-1} , i.e. the range of L , is closed. For proving (15) we are therefore left to show that the range of L is dense. Since Φ defined in (5) is an isometric isomorphism, this density requirement is equivalent to the density of $\{\Phi^{-1}L\varphi : \varphi \in H_S^2(\mathbb{R})\}$ in $H_S^2(\mathbb{R})$. So let $v \in H_S^2(\mathbb{R})$ be in its orthogonal complement, i.e.

$$\langle v, \Phi^{-1}L\varphi \rangle_{H^2} = 0 \text{ for all } \varphi \in H_S^2(\mathbb{R}).$$

By (5), this implies

$$0 = (L\varphi)[v] = \int_{\mathbb{R}} (\varphi''v'' - c^2\varphi'v' + e^\omega\varphi v) dx = (Lv)[\varphi]$$

for all $\varphi \in H_S^2(\mathbb{R})$, i.e. $Lv = 0$. By (11), this yields $v = 0$ and thus, proves (15).

Via the transformation $v = u - \omega$ problem (2) is therefore equivalent to

$$v = -L^{-1} [e^\omega(e^v - 1 - v) + (\omega^{iv} + c^2\omega'' + e^\omega - 1)] =: Tv, \quad (16)$$

which amounts to a fixed-point equation for $T : H_S^2(\mathbb{R}) \rightarrow H_S^2(\mathbb{R})$. Let

$$\mathcal{D} := \{v \in H_S^2(\mathbb{R}) : \|v\|_{H^2} \leq \alpha\},$$

with α satisfying (13) and (14). Using (16), (11), Lemma 1b) and d), (10), and (13), we obtain for $v \in \mathcal{D}$:

$$\begin{aligned} \|Tv\|_{H^2} &\leq K [\|e^\omega(e^v - 1 - v)\|_{H^{-2}} + \|\omega^{iv} + c^2\omega'' + e^\omega - 1\|_{H^{-2}}] \\ &\leq K \left[\frac{1}{\sqrt{\sigma}} e^{\bar{\omega}} \frac{\hat{C}}{2\sqrt{\sigma}} \exp(\hat{C}\|v\|_{H^2}) \|v\|_{H^2}^2 + \delta \right] \\ &\leq K \left[\frac{\hat{C}}{2\sigma} \exp(\bar{\omega} + \hat{C}\alpha) \alpha^2 + \delta \right] \leq \alpha, \end{aligned}$$

i.e. $T\mathcal{D} \subset \mathcal{D}$. Moreover, by (16), (11), Lemma 1b) and e), we obtain for $v, \tilde{v} \in \mathcal{D}$:

$$\begin{aligned} \|Tv - T\tilde{v}\|_{H^2} &\leq K \|e^\omega[(e^v - 1 - v) - (e^{\tilde{v}} - 1 - \tilde{v})]\|_{H^{-2}} \\ &\leq K \frac{1}{\sqrt{\sigma}} e^{\bar{\omega}} \frac{\hat{C}}{\sqrt{\sigma}} \exp(\hat{C}\alpha)\alpha \|v - \tilde{v}\|_{H^2}, \end{aligned}$$

whence by (14) T is a contraction on $\mathcal{D}$. Thus, Banach's Fixed-Point Theorem yields a fixed point $v^* \in \mathcal{D}$ of T . By (16), $u^* := \omega + v^*$ is therefore a solution of problem (2) resp. (4) satisfying $\|u^* - \omega\|_{H^2} \leq \alpha$. $\square$

Remark 2. a) Suppose that (11) holds for some “moderate” K . Then, the crucial conditions (13) and (14) are obviously satisfied for some “small” α provided that δ is sufficiently small, which means according to (10) that the approximate solution ω has to be computed with sufficient accuracy! So the “hard work” of the proof is left to the computer, a fact which describes the general idea of computer-assisted proofs.

b) For practically computing a constant α satisfying (13) and (14), we first solve (13) (with “ $\leq$ ” replaced by “ $=$ ”) approximately by a Newton iteration, starting from $\alpha^{(0)} = 0$. The result of the iteration is then slightly enlarged, e.g. multiplied by 1.01, which gives a candidate for the constant α we are looking for. The two inequalities (13) and (14) are then checked using interval arithmetic [12, 17]. The computation of (an upper bound for) the number $\bar{\omega}$ needed for this check will be mentioned at the beginning of Section 7.

Remark 3. If we had formulated problem (2) in the full space $H^2(\mathbb{R})$ instead of $H_S^2(\mathbb{R})$, it would have been impossible to compute a constant K satisfying (11) (with $H_S^2(\mathbb{R})$ replaced by $H^2(\mathbb{R})$), since L is (at least close to being) not one-to-one on $H^2(\mathbb{R})$. In fact, supposing that a solution u^* of problem (2) exists and defining $\tilde{L}$ as L by (12), but with the exact solution u^* in place of the approximation ω , we obtain $\tilde{L}v = 0$ for $v := (u^*)'$, as is readily obtained by differentiating equation (2) (after proving some higher regularity of u^*). This reflects the fact that together with a solution u^* , a whole continuous family of solutions is obtained simply by translation of u^* .

Using however the space $H_S^2(\mathbb{R})$ of symmetric functions, translations of a solution u^* are no longer in the space, as well as the derivative $(u^*)'$, whence $\tilde{L}$ (and L) may well be one-to-one on $H_S^2(\mathbb{R})$. Indeed, our numerical results based on the methods developed in the next section prove that (11) holds true (with “moderate” K) for the examples under consideration.

Another (minor) advantage of using the space $H_S^2(\mathbb{R})$ is the fact that the symmetry reduces the numerical effort.

4 Computation of K

In this section, we describe how a constant K satisfying (11) can be computed explicitly, as needed for Theorem 2. We will use analytical as well as additional computer-assisted arguments.

With $\Phi : H_S^2(\mathbb{R}) \rightarrow H_S^{-2}(\mathbb{R})$ denoting the isometric isomorphism introduced in (5), we note that

$$\|Lu\|_{H^{-2}} = \|\Phi^{-1}Lu\|_{H^2} \quad \text{for } u \in H_S^2(\mathbb{R}), \quad (17)$$

and that, by (12),

$$\langle \Phi^{-1}Lu, v \rangle_{H^2} = (Lu)[v] = \int_{\mathbb{R}} (u''v'' - c^2u'v' + e^\omega uv) dx \quad (18)$$

for $u, v \in H_S^2(\mathbb{R})$, which in particular implies that $\Phi^{-1}L$ is $\langle \cdot, \cdot \rangle_{H^2}$ -symmetric. Since $\Phi^{-1}L$ is moreover defined on the whole of $H_S^2(\mathbb{R})$, it is therefore selfadjoint (and bounded). Thus, using (17) and the spectral decomposition of $\Phi^{-1}L$, we see that (11) holds if and only if

$$\gamma := \min\{|\lambda| : \lambda \text{ is in the spectrum of } \Phi^{-1}L\} > 0, \quad (19)$$

and that in the affirmative case one can choose any

$$K \geq \frac{1}{\gamma}. \quad (20)$$

Thus, we have to compute a positive lower bound for γ (proving simultaneously that (19) holds true). The first step is to calculate the *essential* spectrum σ_{ess} of $\Phi^{-1}L$ (defined as the set of all accumulation points of the spectrum, i.e. the spectrum except isolated eigenvalues; note that eigenvalues of infinite multiplicity cannot occur for our ODE problem). For technical simplification, we will now assume that

$$\text{the approximate solution } \omega \text{ has compact support,} \quad (21)$$

in coincidence with our numerical schemes described in Section 6. Recall also that $c \in (0, \sqrt{2})$ throughout the paper.

Lemma 2.

$$\sigma_{\text{ess}} = \left[\frac{1}{2} \left(1 + \frac{1}{\sigma} \right) - \sqrt{\frac{1}{4} \left(1 - \frac{1}{\sigma} \right)^2 + \frac{c^4}{4\sigma}}, \max \left\{ 1, \frac{1}{\sigma} \right\} \right].$$

Proof: First we show that $\Phi^{-1}L$ is a compact perturbation of $\Phi^{-1}L_0$, where the constant coefficient operator $L_0 : H_S^2(\mathbb{R}) \rightarrow H_S^{-2}(\mathbb{R})$ is given by

$$L_0u := u^{iv} + c^2u'' + u. \quad (22)$$

Indeed, due to (21) there exists $R_0 > 0$ such that $e^\omega - 1 = 0$ outside $[-R_0, R_0]$. Now let (u_n) denote a bounded sequence in $H_S^2(\mathbb{R})$. By Sobolev-Kondrachev-Rellich's Embedding Theorem, there exists a subsequence (u_{n_k}) which converges in $L_S^2(-R_0, R_0)$. Thus, $((L - L_0)u_{n_k}) = ((e^\omega - 1)u_{n_k})$ converges in $L_S^2(\mathbb{R})$, and hence in $H_S^{-2}(\mathbb{R})$. Therefore, $(\Phi^{-1}(L - L_0)u_{n_k})$ converges in $H_S^2(\mathbb{R})$, whence $\Phi^{-1}(L - L_0)$ is compact.

Since the essential spectrum is invariant under relative compact perturbations [15, Chapter IV, Theorem 5.35], the essential spectra of $\Phi^{-1}L$ and $\Phi^{-1}L_0$ coincide. So we are left to show that the essential spectrum σ_{ess}^0 of $\Phi^{-1}L_0$ equals I , the interval on the right-hand side of the asserted equality. An essential tool is the polynomial family

$$p_\lambda(s) := (1 - \lambda)s^4 - c^2s^2 + 1 - \lambda\sigma \quad (s \in \mathbb{R}, \lambda \in \mathbb{R}). \quad (23)$$

An elementary calculation shows that, for all $\lambda \in \mathbb{R} \setminus \{1\}$,

$$p_\lambda \text{ has real zeroes} \Leftrightarrow \lambda \in I. \quad (24)$$

To prove that $\sigma_{\text{ess}}^0 \subset I$, let $\lambda \in \mathbb{R} \setminus I$. We show that λ is in the resolvent set of $\Phi^{-1}L_0$, i.e. that, for each $r \in H_S^2(\mathbb{R})$, there exists a unique $u \in H_S^2(\mathbb{R})$ satisfying $(\Phi^{-1}L_0 - \lambda)u = r$.

By (5) and (22), this equation reads

$$(1 - \lambda)u^{iv} + c^2u'' + (1 - \lambda\sigma)u = r^{iv} + \sigma r,$$

and with $\mathcal{F}$ denoting the Fourier transformation, it is equivalent to

$$p_\lambda(s)\mathcal{F}[u](s) = (s^4 + \sigma)\mathcal{F}[r](s) \quad (s \in \mathbb{R}). \quad (25)$$

Since $\lambda \in \mathbb{R} \setminus I$ and thus p_λ has order 4 and no real zeroes by (24), the rational function $q(s) := (s^4 + \sigma)/p_\lambda(s)$ is bounded on $\mathbb{R}$, whence indeed $u := \mathcal{F}^{-1}[q\mathcal{F}[r]]$ solves (25), and $s^2|\mathcal{F}[u](s)| \leq \text{const} \cdot s^2|\mathcal{F}[r](s)|$ for $s \in \mathbb{R}$, implying $u \in H^2(\mathbb{R})$. Moreover, u is symmetric since $\mathcal{F}$ preserves symmetry and r and q are symmetric. Thus, $u \in H_S^2(\mathbb{R})$. Finally, u is the *unique* solution of (25) since p_λ has no zeroes and thus $r = 0$ implies $u = 0$.

Conversely, let $\lambda \in I$. First we exclude the case $\lambda = 1$. Then, (24) shows that p_λ has at least one real zero s_0 . We choose a function $\theta \in C^\infty(\mathbb{R})$ satisfying $\theta = 1$ on $(-\infty, 0]$, $\theta = 0$ on $[1, \infty)$, and

$$u_n(x) := \cos(s_0x)\theta(x - n)\theta(-x - n) \quad (x \in \mathbb{R}, n \in \mathbb{N}).$$

Clearly $u_n \in C^\infty(\mathbb{R})$ with compact support in $[-n - 1, n + 1]$, and $u_n(x) = \cos(s_0x)$ for $x \in [-n, n]$. Thus, for $x \in [-n, n]$,

$$L_0u_n - \lambda\Phi u_n = [s_0^4 - c^2s_0^2 + 1 - \lambda(s_0^4 + \sigma)] \cos(s_0x) = p_\lambda(s_0) \cos(s_0x) = 0.$$

Consequently,

$$\|L_0u_n - \lambda\Phi u_n\|_{H^{-2}}^2 \leq \frac{1}{\sigma} \|L_0u_n - \lambda\Phi u_n\|_{L^2}^2 = \frac{2}{\sigma} \int_n^{n+1} |L_0u_n - \lambda\Phi u_n|^2 dx$$

is bounded as $n \rightarrow \infty$, whence

$$\|(\Phi^{-1}L_0 - \lambda)u_n\|_{H^2} \text{ is bounded as } n \rightarrow \infty. \quad (26)$$

Moreover,

$$\|u_n\|_{H^2}^2 \geq \sigma \|u_n\|_{L^2}^2 \geq \sigma \int_{-n}^n \cos(s_0 x)^2 dx$$

tends to infinity as $n \rightarrow \infty$, which together with (26) yields that λ is in the spectrum of $\Phi^{-1}L_0$. This implies $\lambda \in \sigma_{\text{ess}}^0$ since $\Phi^{-1}L_0$ has no eigenvalues, which follows from the fact that all solutions of the constant coefficient equation $L_0 u - \lambda \Phi u = 0$ are linear combinations of fundamental solutions of the form $x^k e^{\rho x}$ (with $k \in \mathbb{N}_0$, $\rho \in \mathbb{C}$), none of which (except 0) is in $H_S^2(\mathbb{R})$.

We are left to prove that $\lambda = 1$ is in σ_{ess}^0 . Again, since $\Phi^{-1}L_0$ has no eigenvalues, it suffices to show that $\lambda = 1$ is in the spectrum of $\Phi^{-1}L_0$. Indeed, if 1 were in the resolvent set, the equation $\Phi^{-1}L_0 u - u = r$, i.e.

$$c^2 u'' + (1 - \sigma)u = r^{iv} + \sigma r,$$

would have a unique solution $u \in H_S^2(\mathbb{R})$ for each $r \in H_S^2(\mathbb{R})$, implying $r^{iv} \in L_S^2(\mathbb{R})$ for each $r \in H_S^2(\mathbb{R})$, which is obviously false. $\square$

Remark 4. The general assumption $c^2 < 2$ ensures that $\min \sigma_{\text{ess}} = \frac{1}{2} \left(1 + \frac{1}{\sigma}\right) - \sqrt{\frac{1}{4} \left(1 - \frac{1}{\sigma}\right)^2 + \frac{c^4}{4\sigma}}$ is *positive*.

Since besides σ_{ess} only isolated eigenvalues of $\Phi^{-1}L$ contribute to its spectrum, we are left to compute a positive lower bound for

$$\gamma_0 := \min\{|\lambda| : \lambda \text{ is isolated eigenvalue of } \Phi^{-1}L\}. \quad (27)$$

Since, by Lemma 2, γ defined in (19) is given by

$$\gamma = \min \left\{ \gamma_0, \frac{1}{2} \left(1 + \frac{1}{\sigma}\right) - \sqrt{\frac{1}{4} \left(1 - \frac{1}{\sigma}\right)^2 + \frac{c^4}{4\sigma}} \right\}, \quad (28)$$

we then have the desired positive lower bound for γ .

For computing a lower bound for γ_0 , we need *eigenvalue bounds*. We will describe their computation by additional computer-assisted means in the following subsection.

Mainly for numerical reasons, it turns out to be advantageous to transform the eigenvalue problem $\Phi^{-1}Lu = \lambda u$ as follows. We restrict the possible choices for σ by requiring

$$\sigma \geq e^{\bar{\omega}} \quad (\text{with } \bar{\omega} := \sup_{x \in \mathbb{R}} \omega(x)). \quad (29)$$

Using (5), (12), (29) we find that, for $u \in H_S^2(\mathbb{R}) \setminus \{0\}$,

$$\langle u - \Phi^{-1}Lu, u \rangle_{H^2} = (\Phi u - Lu)[u] = \int_{\mathbb{R}} [c^2 (u')^2 + (\sigma - e^{\omega})u^2] dx > 0, \quad (30)$$

i.e. $I_{H_S^2(\mathbb{R})} - \Phi^{-1}L$ is positive, and hence one-to-one. The selfadjointness of $\Phi^{-1}L$ implies the selfadjointness of

$$R := \left(I_{H_S^2(\mathbb{R})} - \Phi^{-1}L \right)^{-1} : D(R) \subset H_S^2(\mathbb{R}) \rightarrow H_S^2(\mathbb{R}). \quad (31)$$

Noting that, by (30), all eigenvalues of $\Phi^{-1}L$ are less than 1, we immediately deduce from (31) that

$$\lambda \text{ is eigenvalue of } \Phi^{-1}L \Leftrightarrow \frac{1}{1-\lambda} \text{ is eigenvalue of } R. \quad (32)$$

The spectral mapping theorem [15, III. Theorem 6.15] gives a corresponding relation also for the complete (and thus, also for the essential) spectra: The essential spectrum Σ_{ess} of R is characterized by

$$\Sigma_{\text{ess}} \cup \{\infty\} = \left\{ \frac{1}{1-\lambda} : \lambda \in \sigma_{\text{ess}} \right\}.$$

Using Lemma 2 (and the fact that $\sigma \geq 1$ due to (29)), we therefore obtain

$$\Sigma_{\text{ess}} = \left[\left\{ \frac{1}{2} \left(1 - \frac{1}{\sigma} \right) + \sqrt{\frac{1}{4} \left(1 - \frac{1}{\sigma} \right)^2 + \frac{c^4}{4\sigma}} \right\}^{-1}, \infty \right). \quad (33)$$

Note that $\min \Sigma_{\text{ess}} > 1$ since $c^2 < 2$.

Moreover, using (31), (5), (12) we find that, for $\kappa \in \mathbb{R}$,

$$\begin{aligned} u \in D(R), \quad Ru = \kappa u &\Leftrightarrow u \in H_S^2(\mathbb{R}), \quad u = \kappa \left(I_{H_S^2(\mathbb{R})} - \Phi^{-1}L \right) u \\ &\Leftrightarrow u \in H_S^2(\mathbb{R}), \quad \Phi u = \kappa(\Phi u - Lu) \\ &\Leftrightarrow u \in H_S^2(\mathbb{R}), \quad u^{iv} + \sigma u = \kappa(-c^2 u'' + (\sigma - e^\omega)u) \\ &\Leftrightarrow u \in H_S^2(\mathbb{R}), \quad \langle u, \varphi \rangle_{H^2} = \kappa N(u, \varphi) \text{ for all } \varphi \in H_S^2(\mathbb{R}), \end{aligned} \quad (34)$$

where

$$N(u, \varphi) := \int_{\mathbb{R}} (c^2 u' \varphi' + (\sigma - e^\omega) u \varphi) dx. \quad (35)$$

The methods presented in the next subsection are used, in our numerical examples, to obtain eigenvalue bounds for problem (34), (35), and thus for R . The relation (32) then gives the desired eigenvalue bounds for $\Phi^{-1}L$.

Note that for computing a positive lower bound for γ_0 (defined in (27)), we need bounds for the eigenvalue(s) of $\Phi^{-1}L$ which are next to 0. Thus, (32) shows that we need bounds for the eigenvalue(s) of problem (34), (35) which are next to 1 (and therefore in particular are below the essential spectrum Σ_{ess} of R resp. problem (34), (35)).

It turns out that the best way to solve this task is to compute bounds for the n *smallest* eigenvalues $\kappa_1, \dots, \kappa_n$ of problem (34), (35), together with a lower bound $\rho \leq \kappa_{n+1}$ (or $\rho \leq \inf \Sigma_{\text{ess}}$ if no $(n+1)$ -st eigenvalue exists). Here, n has to be large enough to give the desired separating distance between 1 and the eigenvalues of (34), (35) with guarantee, which is e.g. the case if $\kappa_{n-1} < 1 < \kappa_n$ or if $\kappa_n < 1 < \rho$ (and the enclosing intervals are accurate enough). Note that this includes the possibility of choosing $n = 0$, in which case we have to compute $\rho \leq \kappa_1$ (or $\rho \leq \inf \Sigma_{\text{ess}}$ if no eigenvalue exists) and need $1 < \rho$.

4.1 Variational eigenvalue bounds

Here, we will describe variational methods for computing two-sided bounds for the n *smallest* eigenvalues (with suitable $n \in \mathbb{N}_0$), and a lower bound ρ for the $(n+1)$ -st eigenvalue, if it exists, of problem (34), (35), or of the more general eigenvalue problem

$$\langle u, \varphi \rangle = \kappa N(u, \varphi) \text{ for all } \varphi \in H, \quad (36)$$

with $(H, \langle \cdot, \cdot \rangle)$ denoting a separable complex Hilbert space, and N a bounded, positive and Hermitian sesquilinear form on H . Under these hypotheses, problem (36) is equivalent to an eigenvalue problem for a selfadjoint operator in H (which is R in our concrete case (34), (35)). In this sense the usual spectral terms are well defined for problem (36). In particular, let $\sigma_0 \in \mathbb{R} \cup \{+\infty\}$ denote the infimum of the essential spectrum of problem (36), and suppose that $\sigma_0 > 0$.

Poincaré's min-max principle [31, Chapter 2] is the basis of obtaining *upper* eigenvalue bounds via the well-known

Rayleigh-Ritz method (see, e.g. [29, Theorem 40.1 and Remarks 40.1, 40.2, 39.10]) Let $n \in \mathbb{N}$ and $v_1, \dots, v_n \in H$ be linearly independent trial functions. Define the matrices

$$A_0 := (\langle v_i, v_j \rangle)_{i,j=1,\dots,n}, \quad A_1 := (N(v_i, v_j))_{i,j=1,\dots,n} \quad (37)$$

and let $\hat{\kappa}_1 \leq \hat{\kappa}_2 \leq \dots \leq \hat{\kappa}_n$ denote the eigenvalues of

$$A_0 x = \hat{\kappa} A_1 x. \quad (38)$$

Then, if $\hat{\kappa}_n < \sigma_0$, there are at least n eigenvalues of (36) below σ_0 , and the n smallest of these (counted by multiplicity), ordered by magnitude and denoted by $\kappa_1, \dots, \kappa_n$, satisfy

$$\kappa_j \leq \hat{\kappa}_j \quad (j = 1, \dots, n). \quad (39)$$

Since the matrix eigenvalues $\hat{\kappa}_1, \dots, \hat{\kappa}_n$ can be enclosed by more direct methods combining numerical linear algebra ideas with interval analysis (see [3]), the Rayleigh-Ritz method provides a rather direct access to upper eigenvalue bounds.

For computing *lower* bounds we use the method given by the following theorem. In its original form, it is due to Lehmann [18], and later it has been considerably improved by Goerisch (see e.g. [4]) in its range of applicability. The following version (admitting essential spectrum) can be extracted from [32, Theorem 2.4].

Theorem 3. Let $(X, b(\cdot, \cdot))$ denote a complex Hilbert space and $T : H \rightarrow X$ an isometric linear operator, i.e., $b(T\varphi, T\psi) = \langle \varphi, \psi \rangle$ for all $\varphi, \psi \in H$. Let $v_1, \dots, v_n \in H$ be linearly independent. Let $w_1, \dots, w_n \in X$ satisfy

$$b(T\varphi, w_j) = N(\varphi, v_j) \text{ for all } \varphi \in H. \quad (40)$$

In addition to A_0 and A_1 in (37), define the matrix

$$A_2 := (b(w_i, w_j))_{i,j=1,\dots,n}. \quad (41)$$

Let some $\rho \in (0, \sigma_0]$ be chosen such that there are at most finitely many eigenvalues of (36) below ρ , and such that

$$\left[v \in \text{span}\{v_1, \dots, v_n\} \text{ and } \langle v, \varphi \rangle = \rho N(v, \varphi) \text{ for all } \varphi \in H \right] \Rightarrow v = 0. \quad (42)$$

Let $\tau_1 \leq \dots \leq \tau_k < 0$ denote the negative eigenvalues (counted by multiplicity) of

$$(A_0 - \rho A_1)x = \tau(A_0 - 2\rho A_1 + \rho^2 A_2)x \quad (43)$$

(here, the matrix on the right-hand side is positive definite). Then, there are at least k eigenvalues of problem (36) below ρ , and the k largest of these (counted by multiplicity), denoted by $\kappa_k^\rho \leq \kappa_{k-1}^\rho \leq \dots \leq \kappa_1^\rho (< \rho)$, satisfy

$$\kappa_j^\rho \geq \rho - \frac{\rho}{1 - \tau_j} \quad (j = 1, \dots, k). \quad (44)$$

Proof: As mentioned above, Theorem 3 follows essentially from [32, Theorem 2.4]. Only for proving that problem (36) has at least k eigenvalues below ρ , we need a little additional argument. Indeed, since (43) has k negative eigenvalues and the matrix on its right-hand side is positive definite, the matrix $A_0 - \rho A_1$ has at least k negative eigenvalues. Thus, problem (38) has at least k eigenvalues below ρ , whence the assertion follows from the Rayleigh-Ritz method. $\square$

To obtain two-sided eigenvalue bounds for problem (36), together with the guarantee that no eigenvalue has been missed between the enclosing intervals, we first compute *approximate* eigenelements $v_1, \dots, v_n \in H$ by numerical means, which we use as trial function for the Rayleigh-Ritz method and for Theorem 3. Here, n is chosen such that $\hat{\kappa}_n$, the largest eigenvalue of problem (38), turns out to be less than σ_0 ; if this is false even for $n = 1$ (which never happens in our examples), we try $n = 0$ in the following, putting formally $\hat{\kappa}_0 := -\infty$. If however $n \geq 1$, the Rayleigh-Ritz method gives at least n (smallest) eigenvalues $\kappa_1, \dots, \kappa_n$ of (36) below σ_0 , which satisfy (39). If furthermore we choose ρ such that

$$\hat{\kappa}_n < \rho \leq \sigma_0 \quad (45)$$

(which in particular implies condition (42)), and such that there are at most finitely many eigenvalues of (36) below ρ , the matrix on the left-hand side of problem (43) is clearly negative definite, whence problem (43) has $k = n$ negative eigenvalues $\tau_1 \leq \dots \leq \tau_n < 0$, and thus Theorem 3 gives the lower bounds (44) for the n largest

eigenvalues of (36) below ρ . We know that these are the n *smallest* eigenvalues $\kappa_1, \dots, \kappa_n$ provided that

$$\rho \leq \kappa_{n+1} \quad (46)$$

if an $(n+1)$ -st eigenvalue $\kappa_{n+1} < \sigma_0$ exists, or if $\rho \leq \sigma_0$ (as already stated in (45)) if such an eigenvalue does not exist. Then, (44) yields

$$\kappa_j \geq \rho - \frac{\rho}{1 - \tau_{n+1-j}} \quad (j = 1, \dots, n), \quad (47)$$

which together with (39) gives the desired two-sided bounds for the n smallest eigenvalues of (36).

The task of deciding whether κ_{n+1} exists, and (in the affirmative case) of computing a lower bound for it which allows to choose ρ satisfying (45) and (46), is not easy. In particular, note that (46) requires a lower eigenvalue bound as an *assumption* for Theorem 3, while Theorem 3 is just *aiming* at lower eigenvalue bounds. However, a rather *rough* lower bound for κ_{n+1} is sufficient in (46) (to produce very *precise* lower bounds for $\kappa_1, \dots, \kappa_n$ by Theorem 3). Such rough lower eigenvalue bounds can often be obtained by a homotopy method, as explained in the next subsection.

In the course of this homotopy, we will make strong use of the following corollary which is readily obtained from Theorem 3 by choosing $n = 1$ (and noting that (42) then follows from condition (48) below).

Corollary 1. *Let X, b, T as in Theorem 3. Let $v \in H$, $v \neq 0$, and $w \in X$ such that (40) holds (with w, v instead of w_j, v_j). Moreover, let $\rho \in (0, \sigma_0]$ be chosen such that there are at most finitely many eigenvalues of (36) below ρ , and*

$$\frac{\langle v, v \rangle}{N(v, v)} < \rho. \quad (48)$$

Then, there is an eigenvalue κ of problem (36) satisfying

$$\frac{\rho N(v, v) - \langle v, v \rangle}{\rho b(w, w) - N(v, v)} \leq \kappa < \rho. \quad (49)$$

We close this subsection by commenting how we choose the terms X, b, T (and $w_1, \dots, w_n \in X$) needed for Theorem 3, for our concrete problem (34), (35) (and for the problems (60) below, with obvious changes to be made in the following). Let

$$\begin{aligned} X &:= L_S^2(\mathbb{R}) \times L_S^2(\mathbb{R}), \quad T\varphi := \begin{pmatrix} \varphi'' \\ \varphi \end{pmatrix} \text{ for } \varphi \in H_S^2(\mathbb{R}), \\ b \left(\begin{pmatrix} w^{(1)} \\ w^{(2)} \end{pmatrix}, \begin{pmatrix} \tilde{w}^{(1)} \\ \tilde{w}^{(2)} \end{pmatrix} \right) &:= \langle w^{(1)}, \tilde{w}^{(1)} \rangle_{L^2} + \sigma \langle w^{(2)}, \tilde{w}^{(2)} \rangle_{L^2}. \end{aligned} \quad (50)$$

Clearly, $T : H_S^2(\mathbb{R}) \rightarrow (X, b(\cdot, \cdot))$ is isometric. Condition (40) now requires that, for $j = 1, \dots, n$,

$$\langle \varphi'', w_j^{(1)} \rangle_{L^2} + \sigma \langle \varphi, w_j^{(2)} \rangle_{L^2} = \langle \varphi, -c^2 v_j'' + (\sigma - e^\omega) v_j \rangle_{L^2} \quad (51)$$

for all $\varphi \in H_S^2(\mathbb{R})$. So we need to choose

$$w_j^{(1)} \in H_S^2(\mathbb{R}), \quad (52)$$

and (51) is then equivalent to prescribing

$$w_j^{(2)} := \frac{1}{\sigma} \left[-(w_j^{(1)})'' - c^2 v_j'' + (\sigma - e^\omega) v_j \right]. \quad (53)$$

In principle, *every* choice for $w_j^{(1)}$ satisfying (52) is possible, but a more detailed analysis of the proof of Theorem 3 shows that “good” eigenvalue bounds can be expected (only) if one chooses

$$w_j \approx T w_{j,L}, \quad (54)$$

where the “Lehmann choice” $w_{j,L} \in H_S^2(\mathbb{R})$ is the solution of

$$\langle \varphi, w_{j,L} \rangle_{H^2} = N(\varphi, v_j) \text{ for all } \varphi \in H_S^2(\mathbb{R}). \quad (55)$$

(The fact that (55) is usually not solvable in closed form is the reason why Goerisch’s extension via X, b, T is so helpful!) If we now assume again that $v_1, \dots, v_n$ are approximate eigenfunctions to problem (34), with corresponding approximate eigenvalues $\tilde{\kappa}_1, \dots, \tilde{\kappa}_n$, we see that (55) gives $w_{j,L} \approx \tilde{\kappa}_j^{-1} v_j$, whence (54) requires

$$w_j^{(1)} \approx \frac{1}{\tilde{\kappa}_j} v_j''. \quad (56)$$

(Note that the second approximate equation $w_j^{(2)} \approx \tilde{\kappa}_j^{-1} v_j$ in (54) cannot be used because $w_j^{(2)}$ is fixed by (53)). In general, $w_j^{(1)}$ cannot be chosen *equal* to $\tilde{\kappa}_j^{-1} v_j''$ due to condition (52). So one needs to *approximate* $\tilde{\kappa}_j^{-1} v_j''$ in $H_S^2(\mathbb{R})$ to obtain $w_j^{(1)}$, which in our practical examples we do by interpolating $\tilde{\kappa}_j^{-1} v_j''$ in the finite dimensional subspace of $H_S^2(\mathbb{R})$ which we use for numerical approximations (see Section 6).

4.2 A homotopy method

In this subsection, we will describe a method for computing a constant ρ satisfying (45), (46), as needed for our eigenvalue enclosures. For this purpose, we use a homotopy method which connects our given problem (36) to a “base problem” with some knowledge on its spectrum. In fact, the version of the homotopy method described here is new (on a general level); compared with the versions e.g. in [22, 23, 27], it needs much less computational effort.

Suppose that a bounded, positive, Hermitian sesquilinear form N_0 on $(H, \langle \cdot, \cdot \rangle)$ is at hand such that

$$N_0(u, u) \geq N(u, u) \text{ for all } u \in H, \quad (57)$$

and moreover, some $\rho_0 \in \mathbb{R}$ and some $n_0 \in \mathbb{N}_0$ are known such that the “base problem”

$$\langle u, \varphi \rangle = \kappa^{(0)} N_0(u, \varphi) \text{ for all } \varphi \in H \quad (58)$$

has *precisely* n_0 eigenvalues $\kappa_1^{(0)} \leq \dots \leq \kappa_{n_0}^{(0)}$ (counted by multiplicity) in $(0, \rho_0)$, and such that $\rho_0 \leq \sigma_0^{(0)}$ (with $\sigma_0^{(0)}$ denoting the infimum of the essential spectrum of (58)). For simplicity of presentation, we assume moreover that the infima of the essential spectra of problems (36) and (58) coincide, i.e. $\sigma_0^{(0)} = \sigma_0$. We define

$$N_s(u, v) := (1 - s)N_0(u, v) + sN(u, v) \text{ for } u, v \in H, \ s \in [0, 1], \quad (59)$$

and consider the family of eigenvalue problems

$$\langle u, \varphi \rangle = \kappa^{(s)} N_s(u, \varphi) \text{ for all } \varphi \in H. \quad (60)$$

By (57) and (59), $N_s(u, u)$ is non-increasing in s , for each fixed $u \in H$. Thus, Poincaré's min-max principle shows, since $\sigma_0^{(0)} = \sigma_0$, that the infimum of the essential spectrum of (60) equals σ_0 for all $s \in [0, 1]$, and that, with $\kappa_1^{(s)} \leq \kappa_2^{(s)} \leq \dots$ denoting the eigenvalues of (60) below σ_0 , we have for $0 \leq s \leq t \leq 1$:

$$\kappa_j^{(s)} \leq \kappa_j^{(t)} \text{ for all } j \text{ such that } \kappa_j^{(t)} \text{ exists (implying that } \kappa_j^{(s)} \text{ exists)}. \quad (61)$$

To start the homotopy (in the case $n_0 \geq 1$) we suppose that the gap between $\kappa_{n_0}^{(0)}$ and ρ_0 is not too small. For some $s_1 > 0$, we compute approximate eigenpairs $(\tilde{\kappa}_j^{(s_1)}, \tilde{u}_j^{(s_1)})$ ($j = 1, \dots, n_0$) of problem (60), with $\tilde{\kappa}_1^{(s_1)}, \dots, \tilde{\kappa}_{n_0}^{(s_1)}$ ordered by magnitude. Indeed, if s_1 is not too large, we may expect that (and easily check if) we find the “full” number n_0 of approximate eigenpairs such that, in addition, the Rayleigh quotient formed with $\tilde{u}_{n_0}^{(s_1)}$ (which approximately equals $\tilde{\kappa}_{n_0}^{(s_1)}$) satisfies

$$\frac{\langle \tilde{u}_{n_0}^{(s_1)}, \tilde{u}_{n_0}^{(s_1)} \rangle}{N_{s_1}(\tilde{u}_{n_0}^{(s_1)}, \tilde{u}_{n_0}^{(s_1)})} < \rho_0. \quad (62)$$

Corollary 1, applied to problem (60) with $s = s_1$ and with $v := \tilde{u}_{n_0}^{(s_1)}$, therefore yields the existence of an eigenvalue $\kappa^{(s_1)}$ of that problem in the interval given by (49), the lower bound of which we now denote by ρ_1 , i.e. we obtain

$$\rho_1 \leq \kappa^{(s_1)} < \rho_0. \quad (63)$$

Furthermore, since problem (58) has precisely n_0 eigenvalues in $(0, \rho_0)$, property (61) shows that problem (60) (with $s = s_1$) has *at most* n_0 eigenvalues in $(0, \rho_0)$, which together with (63) implies:

$$\text{problem (60), with } s = s_1, \text{ has at most } n_0 - 1 \text{ eigenvalues in } (0, \rho_1). \quad (64)$$

Let s_1 be chosen “almost” maximal with property (62), so that the inequality in (62) is “almost” an equality (or that $s_1 = 1$, in which case the argumentation further below completes the homotopy). The structure of ρ_1 (given as the lower bound in (49)), and the choice of our test functions then show that also ρ_1 is “not far” below ρ_0 . Consequently, if $\kappa_{n_0-1}^{(s_1)}$ and $\kappa_{n_0}^{(s_1)}$ exist and are “well separated” (as can be guessed on the basis of the approximations $\tilde{\kappa}_{n_0-1}^{(s_1)}$ and $\tilde{\kappa}_{n_0}^{(s_1)}$), we expect that the (only)

eigenvalue in (63) is $\kappa_{n_0}^{(s_1)}$, and thus, that problem (60) (with $s = s_1$) has *precisely* $n_0 - 1$ eigenvalues in $(0, \rho_1)$. (We *could* check if this expectation is true, using the “hard” statement (64) and a Rayleigh-Ritz computation, but this is not necessary. We simply continue on the basis of this expectation, and the final Rayleigh-Ritz computation at the *end* of the homotopy will either prove it a posteriori, or show that the homotopy was not successful. Of course, we will not use this expectation for proving further intermediate “hard” statements like (66) below.)

In the second homotopy step (taking place if $n_0 \geq 2$ and $s_1 < 1$), we repeat the above procedure with s_1 in place of 0, $n_0 - 1$ in place of n_0 , and ρ_1 in place of ρ_0 : For some $s_2 > s_1$ (to be chosen “almost” maximal) we compute approximate eigenpairs $(\tilde{\kappa}_j^{(s_2)}, \tilde{u}_j^{(s_2)})$ ($j = 1, \dots, n_0 - 1$) of problem (60) such that

$$\frac{\langle \tilde{u}_{n_0-1}^{(s_2)}, \tilde{u}_{n_0-1}^{(s_2)} \rangle}{N_{s_2}(\tilde{u}_{n_0-1}^{(s_2)}, \tilde{u}_{n_0-1}^{(s_2)})} < \rho_1, \quad (65)$$

whence Corollary 1 gives an eigenvalue $\kappa^{(s_2)}$ in the interval $[\rho_2, \rho_1)$, with ρ_2 denoting the lower bound now occurring in (49). Furthermore, (64) and (61) show that problem (60) (with $s = s_2$) has at most $n_0 - 1$ eigenvalues in $(0, \rho_1)$, whence altogether

$$\text{problem (60), with } s = s_2, \text{ has at most } n_0 - 2 \text{ eigenvalues in } (0, \rho_2). \quad (66)$$

As before, we see that, if $\kappa_{n_0-2}^{(s_2)}$ and $\kappa_{n_0-1}^{(s_2)}$ are “well separated”, we may expect that (60) (with $s = s_2$) has *precisely* $n_0 - 2$ eigenvalues in $(0, \rho_2)$.

We go on with this algorithm until, for some $r \in \mathbb{N}_0$, either $s_r = 1$ and $r \leq n_0$, or $s_r < 1$ and $r = n_0$ (in which case the homotopy cannot be continued beyond s_r); here, we formally put $s_0 := 0$. In both cases, we obtain in analogy to (64), (66), that problem (60), with $s = s_r$, has at most $n_0 - r$ eigenvalues in $(0, \rho_r)$. Using (61) in the case $s_r < 1$, we obtain the same statement with s_r replaced by 1. Thus, in both cases,

$$\text{problem (36) has at most } n_0 - r \text{ eigenvalues in } (0, \rho_r),$$

which gives (46) for $\rho := \rho_r$ and $n := n_0 - r$.

Finally, if $n \geq 1$, we perform a Rayleigh-Ritz computation for problem (36) and check if condition (45) is satisfied (as it *will* be if our “expectations” mentioned before are correct, including the well-separatedness of $\kappa_{n_0}^{(0)}$ and ρ_0 , $\kappa_{n_0-1}^{(s_1)}$ and $\kappa_{n_0}^{(s_1)}$, $\kappa_{n_0-2}^{(s_2)}$ and $\kappa_{n_0-1}^{(s_2)}$ etc., and if the numerics are sufficiently accurate).

If this check is successful, i.e. (45) and (46) are satisfied, we can compute the desired two-sided bounds for the n smallest eigenvalues of problem (36) as described after Theorem 3.

Remark 5. Note that the additional numerical effort needed for this version of the homotopy method consists mainly of *approximate* computations (for problem (60) with $s = s_1, s_2, \dots, s_r$) which are comparatively cheap, and a *rigorous* check only of the simple conditions (62), (65) etc. In particular, there is no need for the rigorous (interval analytic) solution of larger matrix eigenvalue problems during the homotopy, as it was the case in earlier versions [22, 23, 27].

Remark 6. In all our numerical examples, the assumptions that $\kappa_{n_0}^{(0)}$ and ρ_0 , $\kappa_{n_0-1}^{(s_1)}$ and $\kappa_{n_0}^{(s_1)}$, $\kappa_{n_0-2}^{(s_2)}$ and $\kappa_{n_0-1}^{(s_2)}$ etc., are “well separated”, which we made in the above description, are satisfied; the smallest distance occurring in this context in the examples is about 0.001, which is still enough separation.

If in other applications this assumption happens to be violated, we need to apply, in the course of the homotopy, Theorem 3 with n “locally” equal to the number of clustered eigenvalues, instead of $n = 1$ (i.e. Corollary 1). Then, at each of the points $s_1, s_2, \dots$, the whole cluster occurring there is “dropped” (and the number of eigenvalues to continue with is reduced correspondingly), instead of the single eigenvalue $\kappa_{n_0}^{(s_1)}$ resp. $\kappa_{n_0-1}^{(s_2)}$ etc.

We are left to describe the choice of the form N_0 , defining the base problem (58), for our concrete problem (34), (35). We choose points $0 < \xi_1 < \dots < \xi_k$ such that $\text{supp}(\omega) \subset [-\xi_k, \xi_k]$ (compare assumption (21)), and compute values $\omega_1, \dots, \omega_k \in \mathbb{R}$ such that (with $\xi_0 := 0$)

$$\omega(x) \geq \omega_j \text{ for } x \in [\xi_{j-1}, \xi_j], \quad j = 1, \dots, k. \quad (67)$$

We define the step-function $\underline{\omega} \in L_S^2(\mathbb{R})$ by

$$\underline{\omega}(x) := \begin{cases} \omega_j & \text{for } x \in [\xi_{j-1}, \xi_j), \quad j = 1, \dots, k \\ 0 & \text{for } x \in [\xi_k, \infty) \end{cases} \quad (68)$$

and $\underline{\omega}(x) := \omega(-x)$ for $x < 0$. The form N_0 is now defined as N in (35), but with $\underline{\omega}$ in place of ω . Since $\omega \geq \underline{\omega}$ on $\mathbb{R}$, condition (57) is satisfied. So we are left to compute (enclosures for) *all* eigenvalues $\kappa^{(0)}$ of the base problem (58), i.e. of

$$u \in H_S^2(\mathbb{R}), \quad u^{iv} + \sigma u = \kappa^{(0)}(-c^2 u'' + (\sigma - e^{\underline{\omega}})u) \quad (69)$$

in $(0, \rho_0)$, where now we choose ρ_0 in $(1, \Sigma_{\text{ess}}]$, Σ_{ess} given by (33), large enough to justify the expectation that at the end of the homotopy there is still enough information about the spectral points of (34),(35) neighboring 1, see Figure 2.

Note that $\underline{\omega}$ has compact support, so that the proof of Lemma 2 and the arguments following it work as well with $\underline{\omega}$ in place of ω , i.e. the essential spectrum of the base problem (69) is also given by (33).

First we calculate a rough lower bound for all eigenvalues of problem (69): Condition (29) gives $0 \leq e^{\underline{\omega}} \leq e^{\bar{\omega}} \leq \sigma$, whence $\|\sigma - e^{\underline{\omega}}\|_{\infty} \leq \sigma$. Thus, with $(\kappa^{(0)}, u)$ denoting an eigenpair of (69), we find (with Φ given by (5))

$$\begin{aligned} \|u\|_{H^2} &= \|\Phi u\|_{H^{-2}} = \kappa^{(0)} \| -c^2 u'' + (\sigma - e^{\underline{\omega}})u \|_{H^{-2}} \\ &\leq \frac{\kappa^{(0)}}{\sqrt{\sigma}} \| -c^2 u'' + (\sigma - e^{\underline{\omega}})u \|_{L^2} \leq \frac{\kappa^{(0)}}{\sqrt{\sigma}} [c^2 \|u''\|_{L^2} + \sigma \|u\|_{L^2}] \\ &\leq \frac{\kappa^{(0)}}{\sqrt{\sigma}} \sqrt{c^4 + \sigma} \|u\|_{H^2} \end{aligned}$$

and thus, $\kappa^{(0)} \geq \left(\frac{c^4}{\sigma} + 1\right)^{-\frac{1}{2}}$.

So we have to compute (enclosures for) *all* eigenvalues of problem (69) in the interval

$$I := \left[\left(\frac{c^4}{\sigma} + 1 \right)^{-\frac{1}{2}}, \left\{ \frac{1}{2} \left(1 - \frac{1}{\sigma} \right) + \sqrt{\frac{1}{4} \left(1 - \frac{1}{\sigma} \right)^2 + \frac{c^4}{4\sigma}} \right\}^{-1} \right).$$

On each of the intervals (ξ_{j-1}, ξ_j) , $j = 1, \dots, k$, and on (ξ_k, ∞) , problem (69) has constant coefficients by (68), whence a $(\kappa^{(0)})$ -dependent fundamental system of the differential equation in (69) can be put up in closed form on each of these $k + 1$ intervals. On (ξ_k, ∞) , precisely two of the four fundamental solutions happen to decay (exponentially) at ∞ , for each $\kappa^{(0)} \in I$. On $(0, \xi_1)$, precisely two of the four fundamental solutions have vanishing first and third derivative at 0, for each $\kappa^{(0)} \in I$.

So each eigenfunction u of problem (69), which is automatically in $H_S^4(\mathbb{R})$ by equation (69) itself, and satisfies $u'(0) = u'''(0) = 0$ by symmetry, is a linear combination of two $(\kappa^{(0)})$ -dependent fundamental solutions on $(0, \xi_1)$ and on (ξ_k, ∞) , respectively, and of four $(\kappa^{(0)})$ -dependent fundamental solutions on each of the remaining intervals (ξ_{j-1}, ξ_j) ($j = 2, \dots, k$). The H^4 -smoothness requires four matching conditions at each of the points $\xi_1, \dots, \xi_k$. This generates a $(4k \times 4k)$ -determinant $D(\kappa^{(0)})$, with $\kappa^{(0)}$ -dependent but explicitly known entries, which has to vanish in order to give an eigenvalue $\kappa^{(0)}$.

To find all zeroes of D in the interval I (including the information that these zeroes are simple), we use interval analytical means, in particular, interval bisection and the interval Newton algorithm; for details, see [1, 12]. These methods give the precise number of eigenvalues of problem (69) in I , which in our examples varies between 11 and 23, with a choice of k , the number of breakpoints, between 5 and 12.

5 Morse index

The eigenvalue enclosures for problem (34), (35), which are needed for our computer-assisted existence proof, also give a rather direct access to the Morse index of the enclosed solutions.

Without the imposition of symmetry on our solution space, all our solutions would be degenerate since they are translation invariant. By removing the translation invariance, we arrive at a linearized eigenvalue problem in which there are a finite number of negative eigenvalues. Our initial hope in doing this calculation was that this might throw some light on stability properties of the waves. So far, this has not been successful.

This is in marked contrast with parabolic problems like the Swift-Hohenberg equation, which are first-order in time, where the Morse index directly determines the stability of steady states. (Of course, the general methods of this paper apply equally well to stationary solutions of such parabolic equations.)

For computing the Morse index, we have to find the number of negative eigenvalues of the problem

$$u \in H_S^4(\mathbb{R}), \quad \hat{L}u = \lambda u, \tag{70}$$

with $\hat{L} : H_S^4(\mathbb{R}) \subset L_S^2(\mathbb{R}) \rightarrow L_S^2(\mathbb{R})$ defined by

$$\hat{L}u := u^{iv} + c^2 u'' + e^{u^*} u, \quad (71)$$

where $u^* \in H_S^2(\mathbb{R})$ denotes the solution under consideration. Let $\tilde{L} : H_S^2(\mathbb{R}) \rightarrow H_S^{-2}(\mathbb{R})$ be defined by the same differential expression as $\hat{L}$.

Lemma 3. $\hat{L}$ and $\Phi^{-1}\tilde{L}$ have the same finite number of negative eigenvalues.

Proof: First we note that

$$\sigma_{\text{ess}}(\hat{L}) \subset [1 - \frac{c^4}{4}, \infty), \quad (72)$$

which can be seen as follows. $\hat{L}$ is a relatively compact perturbation of $\hat{L}_0$, defined as $\hat{L}$ but with 1 in place of e^{u^*} , because for each sequence (u_n) in $H_S^4(\mathbb{R})$ such that (u_n) and $(\hat{L}_0 u_n)$ are bounded in $L_S^2(\mathbb{R})$, $\|u_n''\|_{L^2}^2 = \langle \hat{L}_0 u_n, u_n \rangle_{L^2} - \langle c^2 u_n'' + u_n, u_n \rangle_{L^2} \leq C_1 + C_2 \|u_n''\|_{L^2}$, and so (u_n) is bounded in $H_S^2(\mathbb{R})$, whence for each compact interval $[-N, N]$ ($N \in \mathbb{N}$) a subsequence converges in $L_S^2(-N, N)$, and so a diagonal-type subsequence (u_{n_k}) converges in $L_S^2(-N, N)$ for every N , which implies convergence of $((e^{u^*} - 1) u_{n_k})$ in $L_S^2(\mathbb{R})$ since $u^* \rightarrow 0$ as $x \rightarrow \pm\infty$. Consequently, $\hat{L}$ and $\hat{L}_0$ have the same essential spectrum. The characteristic polynomial $p_\lambda(s) = s^4 - c^2 s^2 + 1 - \lambda$ of $\hat{L}_0 - \lambda$ (arising by Fourier transformation) has no real zeroes (i.e. λ belongs to the resolvent set of $\hat{L}_0$; compare the proof of Lemma 2) if $\lambda < 1 - \frac{c^4}{4}$, which implies (72).

By similar arguments (see also the proof of Lemma 2), we obtain $\sigma_{\text{ess}}(\Phi^{-1}\tilde{L})$ to be the interval given in Lemma 2. In particular, our general assumption $c < \sqrt{2}$ implies that the minima of the essential spectra of $\hat{L}$ and of $\Phi^{-1}\tilde{L}$ are both positive.

Consequently, $\hat{L}$ and $\Phi^{-1}\tilde{L}$ both have at most finitely many negative eigenvalues $\hat{\lambda}_1 \leq \dots \leq \hat{\lambda}_{\hat{m}}$ and $\tilde{\lambda}_1 \leq \dots \leq \tilde{\lambda}_{\tilde{m}}$, respectively, with corresponding eigenfunctions $\hat{u}_1, \dots, \hat{u}_{\hat{m}} \in H_S^4(\mathbb{R})$ and $\tilde{u}_1, \dots, \tilde{u}_{\tilde{m}} \in H_S^2(\mathbb{R})$. The eigenvalue equation itself shows that a fortiori $\hat{u}_1, \dots, \hat{u}_{\hat{m}} \in H_S^5(\mathbb{R})$. Moreover, for each $u \in H_S^4(\mathbb{R})$,

$$\langle \Phi^{-1}\tilde{L}u, u \rangle_{H^2} = (\tilde{L}u)[u] = \langle \hat{L}u, u \rangle_{L^2},$$

and thus, if $\hat{m} \geq 1$,

$$\begin{aligned} \tilde{\mu}_{\hat{m}} &:= \min_{\substack{U \subset H_S^2(\mathbb{R}) \text{ subspace} \\ \dim U = \hat{m}}} \max_{u \in U \setminus \{0\}} \frac{\langle \Phi^{-1}\tilde{L}u, u \rangle_{H^2}}{\langle u, u \rangle_{H^2}} \\ &\leq \max_{u \in \text{span}\{\hat{u}_1, \dots, \hat{u}_{\hat{m}}\} \setminus \{0\}} \frac{\langle \hat{L}u, u \rangle_{L^2}}{\langle u, u \rangle_{H^2}} \\ &< 0, \end{aligned}$$

whence Poincaré's min-max principle and the positivity of $\min \sigma_{\text{ess}}(\Phi^{-1}\tilde{L})$ show that $\tilde{\mu}_{\hat{m}} = \tilde{\lambda}_{\hat{m}}$, and in particular $\tilde{m} \geq \hat{m}$. Trivially, this also holds if $\hat{m} = 0$.

The reverse inequality $\hat{m} \geq \tilde{m}$ follows analogously. $\square$

So we are left to compute the number of negative eigenvalues of $\Phi^{-1}\tilde{L}$. For this purpose, let $\omega \in H_S^2(\mathbb{R})$ denote the approximate solution from which u^* was obtained via our computer-assisted proof, with error bound α (satisfying (7), (8)) obtained from Theorem 2. Moreover, let L be given by (12), i.e. as $\tilde{L}$, but with e^ω in place of e^{u^*} . Let $\sigma_0 := \min \sigma_{\text{ess}}(\Phi^{-1}L) = \min \sigma_{\text{ess}}(\Phi^{-1}\tilde{L})$, where the latter equality holds since $\Phi^{-1}L$ and $\Phi^{-1}\tilde{L}$ are compact perturbations of each other. σ_0 is known by Lemma 2. Finally, with $\hat{C}$ given in Lemma 1c) and $\bar{\omega}$ defined in Theorem 2, let

$$\eta := \frac{1}{\sigma} e^{\bar{\omega}} \left(e^{\hat{C}\alpha} - 1 \right). \quad (73)$$

Note that η is “small” when the error bound α is “small”.

Lemma 4. *Let $\lambda_1 \leq \dots \leq \lambda_m$ denote the m smallest eigenvalues of $\Phi^{-1}L$ (counted by multiplicity), and suppose that $\lambda_m + \eta < \sigma_0$. Then, $\Phi^{-1}\tilde{L}$ has at least m eigenvalues below σ_0 , the m smallest of which, denoted by $\tilde{\lambda}_1 \leq \dots \leq \tilde{\lambda}_m$ (and counted by multiplicity) satisfy*

$$\lambda_i - \eta \leq \tilde{\lambda}_i \leq \lambda_i + \eta \quad (i = 1, \dots, m). \quad (74)$$

If there is no $(m+1)$ -st eigenvalue of $\Phi^{-1}L$ below σ_0 , then there is no $(m+1)$ -st eigenvalue of $\Phi^{-1}\tilde{L}$ which in addition is below $\sigma_0 - \eta$.

Proof: Using (8) and (73) we obtain

$$\|e^{u^*} - e^\omega\|_\infty = \|e^\omega (e^{u^* - \omega} - 1)\|_\infty \leq e^{\bar{\omega}} (e^{\|u^* - \omega\|_\infty} - 1) \leq \sigma\eta. \quad (75)$$

Moreover, for all $u \in H_S^2(\mathbb{R})$,

$$\langle \Phi^{-1}Lu, u \rangle_{H^2} = (Lu)[u] = \int_{\mathbb{R}} [(u'')^2 - c^2(u')^2 + e^\omega u^2] dx,$$

and correspondingly for $\Phi^{-1}\tilde{L}$. Together with (75) (and (3)) this gives

$$\frac{\langle \Phi^{-1}Lu, u \rangle_{H^2}}{\langle u, u \rangle_{H^2}} - \eta \leq \frac{\langle \Phi^{-1}\tilde{L}u, u \rangle_{H^2}}{\langle u, u \rangle_{H^2}} \leq \frac{\langle \Phi^{-1}Lu, u \rangle_{H^2}}{\langle u, u \rangle_{H^2}} + \eta \quad (76)$$

for all $u \in H_S^2(\mathbb{R})$, whence (74) follows from Poincaré’s min-max principle.

Supposing now that $\Phi^{-1}\tilde{L}$ has an $(m+1)$ -st eigenvalue which is below $\sigma_0 - \eta$, we obtain from the *first* inequality in (76), using the min-max principle again, that $\Phi^{-1}L$ has an $(m+1)$ -st eigenvalue below σ_0 . This completes the proof. $\square$

Corollary 2. *Let $\lambda_1, \dots, \lambda_m$ as in Lemma 4, and suppose that $\lambda_m + \eta < 0$, and that $\eta < \sigma_0$. Suppose moreover that either*

- i) there exists an $(m+1)$ -st eigenvalue λ_{m+1} of $\Phi^{-1}L$ (such that $\lambda_1, \dots, \lambda_{m+1}$ are the $m+1$ smallest ones), and $\lambda_{m+1} - \eta > 0$, $\lambda_{m+1} + \eta < \sigma_0$,*

or

ii) there is no $(m + 1)$ -st eigenvalue of $\Phi^{-1}L$ below σ_0 .

Then, $\Phi^{-1}\tilde{L}$ has precisely m negative eigenvalues.

Proof: The assertion is an immediate consequence of Lemma 4 (applied with $m + 1$ in place of m in case i)). $\square$

By the bounds computed for the eigenvalues of problem (34), (35) in the course of our computer-assisted existence proof, we have a direct access to bounds for the eigenvalues $\lambda_1, \dots, \lambda_m, \lambda_{m+1}$ of $\Phi^{-1}L$ needed for Corollary 2, by the equivalences (32) and (34). So if η defined in (73) is sufficiently small (as indeed it turns out to be in our concrete examples), Corollary 2 and Lemma 3 together yield the precise number of negative eigenvalues of problem (70), i.e. the Morse index of u^* .

6 Computation of approximations

In this section, we give a brief description of the numerical methods we used to compute approximate solutions $\omega \in H_S^2(\mathbb{R})$ for the given problem (2) or (4), and approximate eigenpairs for problem (34), (35) or (60), (59), respectively. Finding “many” approximate solutions for problem (2) (and avoiding the trivial solution) is not easy. We use a shooting method introduced in [8] for this purpose, which we describe in Subsection 6.1. To obtain defect bounds δ (see (10)) which are small enough to satisfy (13) and (14) (compare Remark 2a)), we need to improve the accuracy of the shooting approximations. Starting from these, we use a Newton iteration, and a spectral collocation method (in a trigonometric approximation space) for the linear subproblems, to obtain highly accurate approximations; see Subsection 6.2. To compute the approximate eigenpairs needed, we use an approximate Rayleigh-Ritz procedure, described in Subsection 6.3.

6.1 The shooting method

As in [7], in order to compute starting approximations for problem (2), we apply a shooting method introduced in [8]. Let us first rewrite the fourth order equation as a first order system. We denote $z = (z_0, z_1, z_2, z_3)^T = (y, y', y'', y''')^T$ and write

$$z' = (z_1, z_2, z_3, -c^2 z_2 - e^{z_0} + 1)^T =: f(z) . \quad (77)$$

We seek solutions with an even symmetry, therefore we complement (77) by the boundary conditions

$$z(-\infty) = 0 , \quad z_1(0) = z_3(0) = 0 , \quad (78)$$

and consider (77) on $(-\infty, 0]$. For $c^2 \in (0, 2)$ the Jacobi matrix $Df(z)|_{z=0}$ has exactly two eigenvalues with positive real parts (hence (78) represents four boundary conditions). Since they are complex conjugate, there exist real unit vectors v_1, v_2 such that $v_1 \pm iv_2$ are the corresponding eigenvectors.

Let $\varepsilon > 0$ be a small fixed number. After omitting terms of higher order in ε the boundary condition at infinity can be approximated by $z(-R) = \varepsilon v$, where $R > 0$ is some unknown number and $v \in \text{span}\{v_1, v_2\}$ is a unit vector. System (77) will be solved numerically on the finite interval $(-R, 0)$ with this new boundary condition. Since R is an unknown parameter, a transformation of variables $\xi = (R + x)/R$, $w(\xi) = z(x)$ is performed in order to work on the unit interval $(0, 1)$. We obtain a new problem

$$\begin{aligned} w'_0 &= R w_1, \quad w'_1 = R w_2, \quad w'_2 = R w_3, \quad w'_3 = R(-c^2 w_2 - e^{w_0} + 1), \\ &\text{subject to } w(0) = \varepsilon(v_1 \cos \eta + v_2 \sin \eta), \end{aligned} \quad (79)$$

where η chooses a unit vector in $\text{span}\{v_1, v_2\}$. Hence for ε fixed and a choice of parameters R and η , (79) represents an initial value problem that can be solved using some standard numerical integration technique. Let us denote $w(\xi; R, \eta)$ the solution of this initial value problem given by R and η . Our goal is to find R and η such that the solution also satisfies the two remaining boundary conditions $w_1(1; R, \eta) = w_3(1; R, \eta) = 0$.

Newton's method will be used to find the shooting parameters R and η . Denote

$$\theta = \begin{pmatrix} R \\ \eta \end{pmatrix}, \quad \Phi(\theta) = \begin{pmatrix} w_1(1; R, \eta) \\ w_3(1; R, \eta) \end{pmatrix}.$$

Given an initial guess $\theta^{(0)}$, we generate a sequence of improved guesses

$$\theta^{(n+1)} = \theta^{(n)} - [D\Phi(\theta^{(n)})]^{-1} \Phi(\theta^{(n)}), \quad (80)$$

such that $\Phi(\theta^{(n)}) \rightarrow 0$ as $n \rightarrow \infty$. The derivatives

$$D\Phi(\theta^{(n)}) = \left(\begin{array}{cc} \frac{\partial w_1}{\partial R} & \frac{\partial w_1}{\partial \eta} \\ \frac{\partial w_3}{\partial R} & \frac{\partial w_3}{\partial \eta} \end{array} \right)_{\xi=1; R=R^{(n)}, \eta=\eta^{(n)}}$$

can be obtained by appending additional equations to system (79). Define

$$w_{j+4} = \frac{\partial w_j}{\partial R}, \quad w_{j+8} = \frac{\partial w_j}{\partial \eta} \quad j \in \{0, 1, 2, 3\}.$$

After differentiating these new variables with respect to ξ we get

$$\begin{aligned} w'_4 &= R w_5 + w_1, \quad w'_5 = R w_6 + w_2, \quad w'_6 = R w_7 + w_3, \\ w'_7 &= R(-c^2 w_6 - e^{w_0} w_4) - c^2 w_2 - e^{w_0} + 1, \\ w'_8 &= R w_9, \quad w'_9 = R w_{10}, \quad w'_{10} = R w_{11}, \quad w'_{11} = R(-c^2 w_{10} - e^{w_0} w_8) \\ &\text{subject to } w_{j+4}(0) = 0, \quad w_{j+8}(0) = \varepsilon(-v_{1,j} \sin \eta + v_{2,j} \cos \eta), \quad j \in \{0, 1, 2, 3\}, \end{aligned} \quad (81)$$

where the new initial conditions were obtained by taking a derivative of the initial conditions in (79) with respect to R and η .

For given shooting parameters R and η systems (79) and (81) can be integrated from $\xi = 0$ up to $\xi = 1$. Then the Newton step (80) can be evaluated.

6.2 The spectral Newton-collocation method

In this section we are describing how we improve the accuracy of the relatively rough approximations obtained by the shooting method. We look for high accuracy approximations in the space

$$V_{R,M} := \text{span} \{ \varphi_{R,k} : k \in \{1, \dots, M\} \}$$

for fixed (“large”) $R > 0$ and $M \in \mathbb{N}$, where, for $k = 1, \dots, M$,

$$\varphi_{R,k}(x) := \begin{cases} \sin\left(\pi \frac{x+R}{2R}\right) \sin\left((2k-1)\pi \frac{x+R}{2R}\right) & \text{for } |x| \leq R \\ 0 & \text{for } |x| > R \end{cases}. \quad (82)$$

Clearly $V_{R,M} \subset H_S^2(\mathbb{R})$, since $\varphi_{R,k}(\pm R) = \varphi'_{R,k}(\pm R) = 0$ and $\varphi_{R,k}(x) = \varphi_{R,k}(-x)$ for all $x \in \mathbb{R}$, $k \in \{1, \dots, M\}$. The first step is to subject the (discrete) shooting approximations to an interpolation process to obtain approximations ω_0 in $V_{R,M}$, which are then used to start a Newton iteration ($n = 0, 1, 2, \dots$):

(i) Find $v_n \in V_{R,M}$ such that

$$v_n^{iv} + c^2 v_n'' + e^{\omega_n} v_n \approx -(\omega_n^{iv} + c^2 \omega_n'' + e^{\omega_n} - 1), \quad (83)$$

(ii) $\omega_{n+1} := \omega_n + v_n$,

until “convergence” is achieved (in the sense that some suitable norm of v_n is below some tolerance).

The approximate solution of the linear subproblems in (83) is carried out by *collocation* at equidistant collocation points $\xi_j := \left(\frac{j}{M} - 1\right)R$ ($j = 1, \dots, M$), i.e. $v_n \in V_{R,M}$ is required to satisfy the linear differential equation in (83) at these collocation points; note that for symmetry reasons no additional collocation points in $(0, R)$ are needed. Rewriting $\varphi_{R,k}$ on $[-R, 0]$ as a difference of two cos-functions, we see that these collocation conditions lead to the linear algebraic system

$$[C(D^2 - c^2 D) - \tilde{C}(\tilde{D} - c^2 \tilde{D}) + W_n(C - \tilde{C})]\hat{v}_n = R_n, \quad (84)$$

with $M \times M$ matrices $C, \tilde{C}, D, \tilde{D}, W_n$, and $R_n \in \mathbb{R}^M$ defined by

$$\begin{aligned} C_{jk} &:= \cos\left((k-1)j\frac{\pi}{M}\right), & \tilde{C}_{jk} &:= \cos\left(kj\frac{\pi}{M}\right), \\ D &:= \text{diag}\left((k-1)^2\frac{\pi^2}{R^2}\right), & \tilde{D} &:= \text{diag}\left(k^2\frac{\pi^2}{R^2}\right), \\ W_n &:= \text{diag}\left(e^{\omega_n(\xi_j)}\right), & R_{n,j} &:= -2\left[\omega_n^{iv} + c^2 \omega_n'' + e^{\omega_n} - 1\right](\xi_j), \end{aligned} \quad (85)$$

which all are very simple to compute.

Since in our 36 successful examples it turns out that M between 1200 and 2500 is sufficient, problem (84) can be solved directly by Gaussian elimination. The solution vector $\hat{v}_n \in \mathbb{R}^M$ defines the approximate solution v_n of (83) by

$$v_n := \sum_{k=1}^M \hat{v}_{n,k} \varphi_{R,k}. \quad (86)$$

6.3 Computation of approximate eigenpairs

In principle, we could use the (simple) collocation method described above also for the eigenvalue problems (34), (35) or (60), (59), respectively. This would however destroy the symmetry of the arising matrices. For this reason, we prefer to use a Rayleigh-Ritz procedure for this task. First we give a description for problem (34), (35). We choose some $\bar{R} > R$ (e.g. $\bar{R} = 2R$), with R denoting the value used in (82) to compute the approximation ω , and calculate (approximations to) the integrals ($j, k = 1, \dots, M$)

$$\begin{aligned} A_{jk} &:= \int_{-\bar{R}}^{\bar{R}} \left[\varphi''_{\bar{R},j} \varphi''_{\bar{R},k} + \sigma \varphi_{\bar{R},j} \varphi_{\bar{R},k} \right] dx, \\ B_{jk} &:= \int_{-\bar{R}}^{\bar{R}} \left[c^2 \varphi'_{\bar{R},j} \varphi'_{\bar{R},k} + (\sigma - e^\omega) \varphi_{\bar{R},j} \varphi_{\bar{R},k} \right] dx. \end{aligned} \quad (87)$$

For this purpose, we compute an approximation $p \in V_{\bar{R},M}$ to $e^\omega - 1$ by interpolation, and replace e^ω by $1 + p$ in (87). Now, all integrals in (87) can be calculated in closed form.

The matrix eigenvalue problem

$$Ax = \kappa Bx, \quad (88)$$

with $A = (A_{jk})$, $B = (B_{jk})$, is the Rayleigh-Ritz approximation to problem (34), (35) in $V_{\bar{R},M}$. We solve (88) (approximately) by standard numerical methods. The results give approximate eigenpairs to (34), (35), using a formula analogous to (86) for the eigenfunctions.

For problem (60), (59), with N_0 defined as described after (68), we form the additional matrix $\underline{B} = (\underline{B}_{jk})$ given by

$$\underline{B}_{jk} := \int_{-\bar{R}}^{\bar{R}} \left[c^2 \varphi'_{\bar{R},j} \varphi'_{\bar{R},k} + (\sigma - e^{\underline{\omega}}) \varphi_{\bar{R},j} \varphi_{\bar{R},k} \right] dx, \quad (89)$$

which is possible in closed form since $\underline{\omega}$ (defined in (68)) is piecewise constant. The matrix eigenvalue problem

$$Ax = \kappa^{(s)} [(1-s)\underline{B} + sB]x \quad (90)$$

is the Rayleigh-Ritz approximation to problem (60), (59), and is again solved (approximately) by standard methods.

7 Some verified computations

In several places in the previous sections, the *rigorous* computation of certain terms is required, in order to preserve the proof character of our method. For many of these

terms, it suffices to evaluate them simply in *interval arithmetic* [12, 17] instead of standard arithmetic, e.g. the constants in Lemma 1, the right hand side of (13) and the left-hand side of (14), the infimum of σ_{ess} (Lemma 2) and of Σ_{ess} (33), etc. Some other terms need additional effort, e.g. the constant $\bar{\omega}$ needed in Theorem 2 and in (29), (73), and the constants ω_j in (67) (which we obtain by grid point evaluation of ω and rough bounds for ω' , all in interval arithmetic), and the matrix eigenvalues $\hat{\kappa}_j$ of problem (38) and τ_j of problem (43). For the verified *solution* of these eigenvalue problems, we use the methods in [3] providing enclosures for eigenvalues of symmetric interval matrix eigenvalue problems. For *putting up* the matrices for these problems (see (37), (41)), as well as for evaluating the left-hand sides of (48), (49), (62), (65) etc., we need verified (interval) evaluations of expressions of the form

$$\langle v, \tilde{v} \rangle_{H^2}, \quad N(v, \tilde{v}), \quad N_0(v, \tilde{v}), \quad b(w, \tilde{w}), \quad (91)$$

with given (approximate eigenfunctions) $v, \tilde{v} \in V_{\bar{R}, M}$, w and $\tilde{w}$ of the form (52), (53), and with N, N_0 , and b defined by (35), the lines after (68), and (50), respectively. Again, after approximating $e^\omega - 1$ by some $p \in V_{\bar{R}, M}$ (e.g. by interpolation), and replacing e^ω by $1 + p$ in $N(v, \tilde{v})$ and in $b(w, \tilde{w})$, all integrals in (91) can be computed in closed form, using interval evaluations of sin and cos. The error in $N(v, \tilde{v})$ caused by the approximation is less than or equal to

$$\|e^\omega - 1 - p\|_\infty \|v\|_{L^2} \|\tilde{v}\|_{L^2}$$

and can therefore be bounded by standard techniques (involving higher derivatives of e^ω and p , to be bounded by interval arithmetical tools). Analogously, the approximation error in $b(w, \tilde{w})$ is bounded. Taking these error bounds into account we obtain the desired enclosures for the terms in (91).

Finally, we comment on the verified computation of a defect bound δ (see (10)). In view of Remark 2a), it is worth putting some effort into this computation in order to obtain a “small” bound δ . Again we start with a (high accuracy) approximation $p \in V_{\bar{R}, M}$ to $e^\omega - 1$, and a bound δ_2 for the error:

$$\|e^\omega - 1 - p\|_{L^2} \leq \delta_2. \quad (92)$$

Let $v := \Phi^{-1}[\omega^{iv} + c^2\omega'' + p] \in H_S^2(\mathbb{R})$, i.e., v is the solution in $H_S^2(\mathbb{R})$ of

$$v^{iv} + \sigma v = \omega^{iv} + c^2\omega'' + p \quad (\text{in } H_S^{-2}(\mathbb{R})). \quad (93)$$

Since ω and p are in $V_{\bar{R}, M}$, a special (symmetric) solution $\hat{v}$ of (93) on $[-R, R]$ can easily be calculated. Using fundamental systems on $(-\infty, -R]$, $[-R, R]$, and $[R, \infty)$, respectively, we find that the general *piecewise* solution of (93) which is symmetric and decays at $\pm\infty$, is given by

$$v(x) = \begin{cases} e^{\alpha x} [A \cos(\alpha x) + B \sin(\alpha x)] & (-\infty < x < -R), \\ \hat{v}(x) + C \cosh(\alpha x) \cos(\alpha x) + D \sinh(\alpha x) \sin(\alpha x) & (-R < x < R), \\ e^{-\alpha x} [A \cos(\alpha x) - B \sin(\alpha x)] & (R < x < \infty), \end{cases}$$

with $\alpha := \sigma^{\frac{1}{4}}/\sqrt{2}$, and free constants A, B, C, D . Since (93) is required to hold in $H_S^{-2}(\mathbb{R})$, we find four matching conditions at R :

$$\begin{aligned} v(R-0) - v(R+0) &= v'(R-0) - v'(R+0) = 0, \\ v''(R-0) - v''(R+0) &= \omega''(R-0), \quad v'''(R-0) - v'''(R+0) = \omega'''(R-0), \end{aligned}$$

which determine A, B, C, D uniquely. The corresponding matching conditions at $-R$ are then automatically satisfied by symmetry. Consequently, v is known in closed form, and (an upper bound for) $\|v\|_{H^2}$ can be computed.

Now the desired defect bound can easily be obtained, using (93), Lemma 1b), and (92), by

$$\begin{aligned} \|\omega^{iv} + c^2\omega'' + e^\omega - 1\|_{H^{-2}} &\leq \|\omega^{iv} + c^2\omega'' + p\|_{H^{-2}} + \|e^\omega - 1 - p\|_{H^{-2}} \\ &\leq \|v^{iv} + \sigma v\|_{H^{-2}} + \frac{1}{\sqrt{\sigma}}\|e^\omega - 1 - p\|_{L^2} \leq \|v\|_{H^2} + \frac{1}{\sqrt{\sigma}}\delta_2 =: \delta. \end{aligned} \quad (94)$$

Note that, in realistic applications, the approximation error bound δ_2 is much smaller than the essential term $\|v\|_{H^2}$, whence the defect bound δ in (94) is close to the true defect norm.

8 Numerical results

In this section, we report on the numerical results obtained, which finally prove the desired existence and multiplicity result for problem (2) with the specific choice

$$c = 1.3. \quad (95)$$

A large number of numerical solutions was found using the shooting method, as Fig. 1 shows. The graphs show profiles of travelling waves with the fixed value of speed $c = 1.3$. The shooting method provided even more numerical solutions. For example, there seems to be a family of solutions with large amplitudes. Here we included only the simplest two of these, number 12 (lower and upper branch) in the Figure 1. The 40 shooting approximations plotted in Fig. 1 were investigated in detail according to our method.

Starting from these shooting approximations, we applied the Newton-collocation method with R between 70 and 110 and M between 1200 and 2500. In all 40 cases, the Newton iteration “converged” within about 6 steps, with a tolerance of 10^{-7} , to highly accurate approximations ω .

Applying the methods described in Subsections 4.1 and 4.2, we computed enclosures for the lowest eigenvalues of problem (34), (35), providing a separation between 1 and the spectrum of (34), (35). Figure 2 illustrates the course of the homotopy algorithm leading to these eigenvalue enclosures for one particular of the 40 approximations.

So we were able to compute the constants K satisfying (11), via (20), (28), (27), (32), (34).

The results are displayed in Table 1, as well as the defect bounds δ (see (10)) computed according to Section 7, and the error bounds α provided by Theorem 2;

the crucial conditions (13), (14) are satisfied in 36 of the 40 cases. In the remaining 4 cases, the constant K is too large, and no α satisfying (13), (14) could be found for the values of δ obtained within our approximation quality.

Let us discuss briefly why the solution-pairs 9 and 11 fail. Consider an approximate solution ω symmetric about $x = 0$. One way to produce a new approximate solution is to superimpose $\omega(x + L)$ and $\omega(x - L)$. This gives a symmetric function, and if L is large, one that we can imagine as a plausible approximate solution. As we vary L continuously, we get a one-parameter family of approximate solutions. This at least suggests that if we linearize about this approximate solution, at least one eigenvalue would be very close to 0, and verification would fail. This is what we think is going on in solution-pairs 9 and 11, where two symmetric copies of solution pair 1 are “glued” together. This is one possible explanation why the verification failed for these pairs.

Of course, another possibility is that our calculations were not accurate enough. Using higher M (e.g., $M = 3500$) does not help much because then rounding errors prevent smaller defect bounds δ . So we would need, in addition, an arithmetic with longer mantissa to make δ substantially smaller. We did however not carry this out, and are content with 36 verified solutions.

The last column of Table 1 shows the Morse index of the respective solution u^* obtained according to Section 5.

It should be remarked that the defect bounds δ obtained in the 36 cases are not directly comparable because different approximation accuracies (i.e. different M and R in the approximation space $V_{R,M}$) have been used in order to satisfy conditions (13), (14). E.g. for solution 6 (upper branch) we used $M = 2500$ to manage the large constant K .

Finally, it is easy to check that condition (9) holds true. This completes the desired existence and multiplicity result, i.e. the proof of Theorem 1.

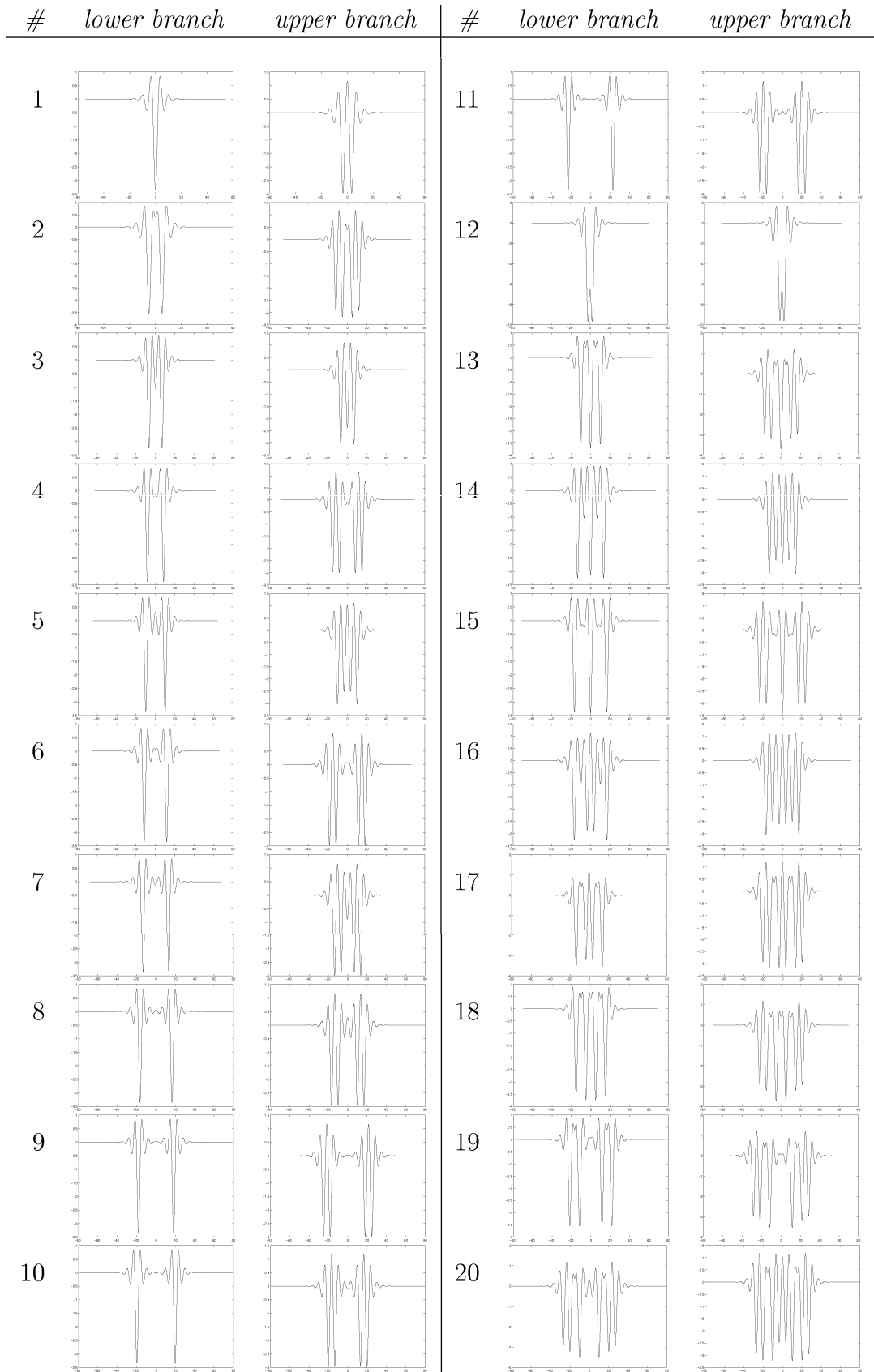


Figure 1: Numerical solutions for $c = 1.3$ and the number of the corresponding curve in the continuation diagram in Fig. 3.

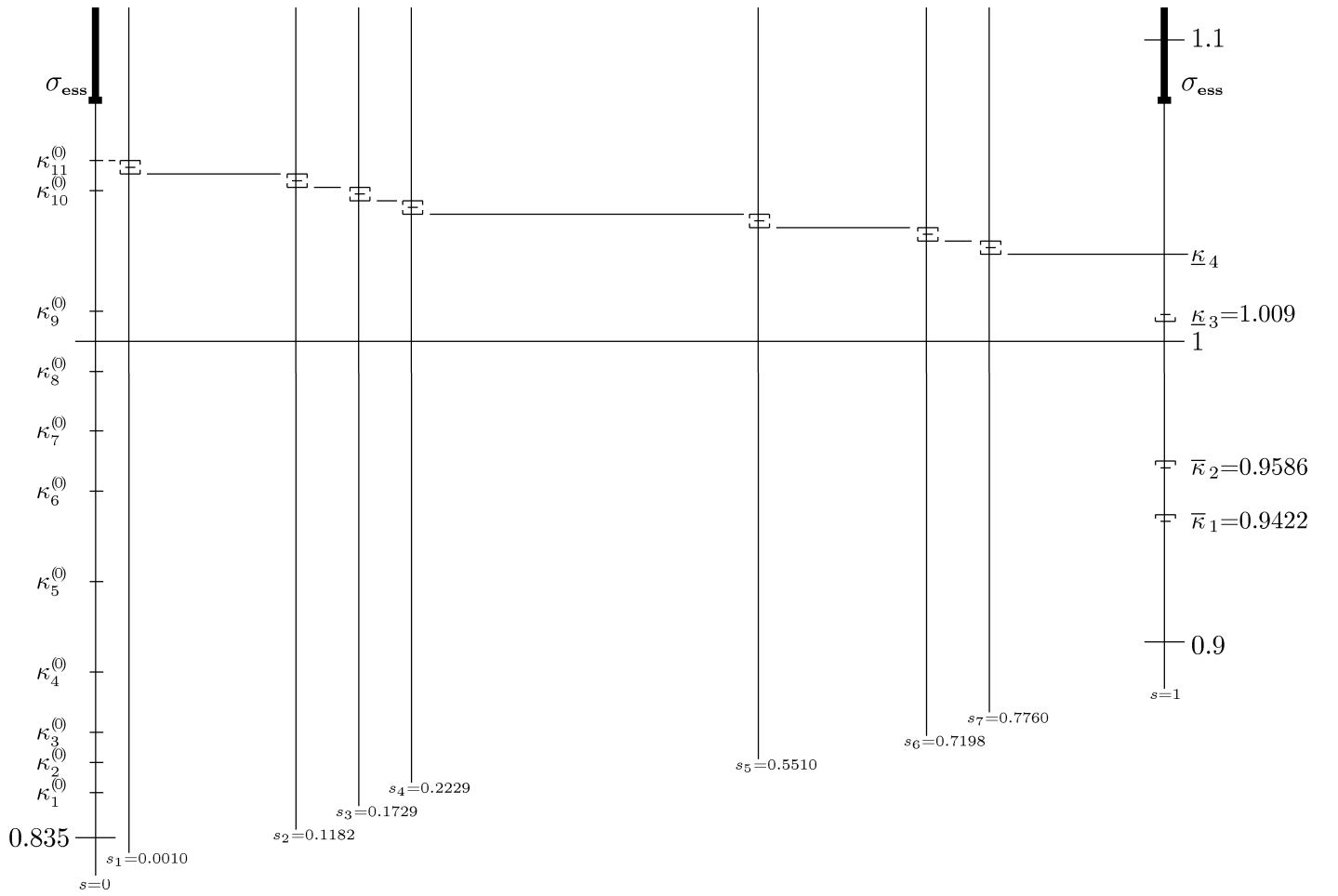


Figure 2: Course of homotopy algorithm for solution number 5 (upper branch).

	lower branch				upper branch			
Solution	K	δ	α	Morse Index	K	δ	α	Morse Index
1	1.51e+01	5.36e-08	8.05e-07	1	2.48e+01	4.21e-08	1.05e-06	1
2	6.52e+01	4.56e-08	2.97e-06	2	1.27e+02	4.40e-08	5.59e-06	3
3	1.22e+02	2.06e-08	2.50e-06	1	6.21e+01	4.62e-08	2.87e-06	2
4	3.61e+02	4.87e-08	1.76e-05	2	8.55e+02	4.41e-08	3.80e-05	3
5	8.06e+02	5.32e-08	4.33e-05	1	1.09e+02	4.02e-08	4.37e-06	2
6	2.11e+03	5.18e-08	1.18e-04	2	5.24e+03	6.53e-11	3.42e-07	3
7	5.11e+03	4.70e-08	4.33e-05	1	3.48e+02	4.33e-08	1.51e-05	2
8	3.19e+04	1.13e-10	3.72e-06	1	2.07e+03	1.62e-10	3.34e-07	2
9	7.87e+04	1.57e-10	—	-	1.99e+05	5.37e-10	—	-
10	3.19e+04	1.57e-10	5.21e-06	1	1.30e+04	2.62e-10	3.44e-06	2
11	1.87e+06	7.69e-11	—	-	8.12e+04	3.08e-10	—	-
12	9.20e+01	5.18e-08	4.77e-06	2	1.14e+02	2.65e-08	3.02e-06	3
13	1.20e+02	4.69e-08	5.62e-06	3	2.35e+02	4.40e-08	1.04e-05	4
14	2.65e+02	2.03e-08	5.35e-06	2	1.65e+02	4.47e-08	7.35e-06	3
15	7.00e+02	5.25e-08	3.71e-05	3	1.56e+03	1.67e-08	2.61e-05	4
16	3.80e+02	4.85e-08	1.85e-05	2	2.32e+02	4.62e-08	1.07e-05	3
17	1.45e+02	4.97e-08	7.16e-06	3	2.23e+02	1.65e-08	3.65e-06	4
18	1.97e+02	2.11e-08	4.16e-06	4	3.70e+02	1.73e-08	6.38e-06	5
19	4.12e+03	5.50e-08	4.16e-06	4	6.81e+03	3.34e-09	2.37e-05	5
20	2.43e+03	7.36e-08	2.02e-04	4	2.17e+02	6.38e-10	3.34e-07	5

Table 1: Verified upper bounds for the crucial constants K , α , δ . The 40 approximations (36 of which are verified) are ordered as in Figure 1.

9 Concluding remarks and open questions

In addition to the shooting computations for $c = 1.3$, a numerical continuation was performed. Figure 3 shows parts of curves in the plane given by c on the horizontal axis and the H^2 -norm $\|u\|_{H^2} = (\|u\|_{L^2}^2 + \|u''\|_{L^2}^2)^{1/2}$ on the vertical axis. As in [7], we observe that the two branches marked by 1 in Fig. 3 appear to bifurcate from $u \equiv 0$ at $c = \sqrt{2}$ (this is difficult to compute). The rest of the curves consist of two branches (lower and upper) that persist up to a certain value of $c < \sqrt{2}$. To prevent the figure from being too cluttered, only a small part of the curves close to the respective turning point is shown. It was numerically observed, however, that for decreasing values of c the branches seem to exist and to become unbounded as $c \rightarrow 0$.

We are not claiming that we have completely solved the problem of describing the solution set of equation (2) for all values of c in $(0, \sqrt{2})$. We now know that (2) has at least one solution for *almost all* c in $(0, \sqrt{2})$. We also know that for $c = 1.3$, there are at least 36 solutions. Many questions remain.

- The most obvious question is whether at least one solution exists for *all* c in $(0, \sqrt{2})$.

- Next, of course, can one prove that there is more than one solution for all c in $(0, \sqrt{2})$? It appears that near $c = \sqrt{2}$ there are exactly two solutions but it is very difficult to compute as the solutions tend to spread out and resemble sinusoidal functions.

We wish to remark that at least in principle (i.e. up to problems of numerical accuracy or computing time), an extension of our method (see [26]) is able to prove existence of solution *branches* $(u_c)_{c \in [c_1, c_2]}$, with $[c_1, c_2]$ denoting a compact subinterval of $(0, \sqrt{2})$. In this way, large parts of the bifurcation diagram in Figure 3 could possibly be verified. We have not tried this yet, mainly for reasons of numerical effort.

- One suspects that the branches which we have shown in Figure 3 continue all the way to $c = 0$, perhaps with more branches accumulating in the process. Can one prove that as $c \rightarrow 0$, the number of solutions goes to infinity?
- The most vexing question of all, barely mentioned in this paper, is the question of the stability, fusion, and interaction (soliton) properties of these travelling waves. For more on these properties, the reader is referred to [7, 14].
- Is there any connection between the Morse index of a solution and its other properties, such as for example, its shape?
- For the piecewise linear model that preceded equation (2), transversality arguments are used in [6] to generate solutions by essentially patching together a primary solution and a translate by a large distance and showing there is a true solution close to this approximate solution. Also with the exponential nonlinearity of this paper, there is a temptation to produce many approximate solutions by gluing together widely separated single solutions. As mentioned earlier, we suspect that any time we try to produce true solutions by applying our method to this candidate to obtain a verified solution, we will find an eigenvalue very close to 0, which we expect to cause the enclosure method to fail. So for the exponential, as opposed to the piecewise linear model, some new techniques appear to be needed for this type of gluing to work.

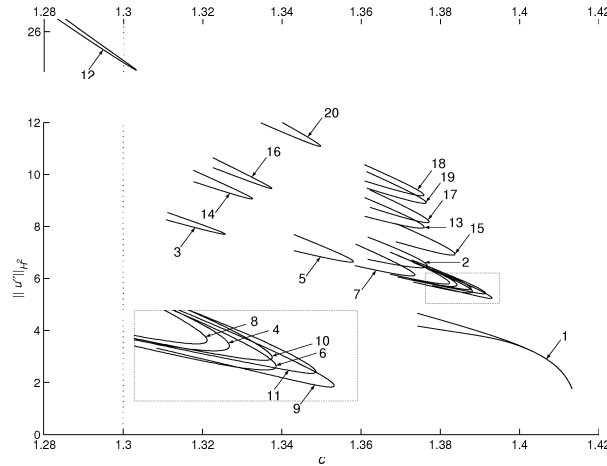


Figure 3: Continuation of numerical solutions of Fig. 1. This figure shows portions of the solution curves as c is varied above 1.3. Presumably, these branches continue as $c \rightarrow 0$. What this paper proves is that at $c = 1.3$, 36 solutions exist.

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