Absorption evolution families and exponential stability of non–autonomous diffusion equations

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Abstract. For a positive, strongly continuous evolution family $\mathcal{U}$ on $L^p(\Omega)$ and a positive, measurable, time dependent potential $V(\cdot)$ we construct a corresponding *absorption evolution family* $\mathcal{U}_V$ by a procedure introduced by J. Voigt, [24, 25], in the autonomous case. We give sufficient conditions on $\mathcal{U}$ and V such that $\mathcal{U}_V$ is strongly continuous and satisfies variation of constants formulas. For an evolution family $\mathcal{U}$ on $L^1(\mathbb{R}^N)$ satisfying upper and lower Gaussian estimates exponential stability of $\mathcal{U}_V$ is characterized by a condition on the size of V extending recent results by W. Arendt and C.J.K. Batty, [2, 3, 6], in the autonomous case and by D. Daners, M. Hieber, P. Koch Medina, and S. Merino, [8, 12], in the time periodic case. An application to a second order parabolic equation with real coefficients and singular potential is given.

1 Introduction

In the present paper we prove the existence of mild solutions u for a non–autonomous diffusion equation with absorption on $X = L^p(\mathbb{R}^N)$, $1 \le p < \infty$, given by

$$\begin{aligned} \tfrac{d}{dt}\, u(t) &= \textstyle\sum_{k,l=1}^N D_k\left(a_{kl}(t) D_l\, u(t)\right) - V(t)u(t), \\ u(s) &= x, \qquad t \ge s \ge 0. \end{aligned} \tag{1.1}$$

Here we assume $a_{kl} \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^N, \mathbb{R})$, uniform ellipticity of the differential operator and that the potential $0 \le V \in L^1_{loc}(\mathbb{R}_+, L^p_{loc}(\mathbb{R}^N))$ is growing at infinity less than $e^{\alpha|\xi|^2}$, $\alpha > 0$. Further, for $p = 1$ and a large class of potentials we characterize exponential stability of the solutions of (1.1) by a condition on the size of V. All this is part of Section 5 and follows from our investigations of the following more general situation.

Consider $X = L^p(\Omega)$ for a σ–finite measure space (Ω, μ) and $1 \le p < \infty$. For $I \in \{[a,b], [a,\infty), \mathbb{R}\}$ set $D = D_I := \{(t,s) \in I^2 : t \ge s\}$. Let $\mathcal{U} = (U(t,s))_{(t,s)\in D} \subseteq \mathcal{L}(X)$ be a strongly continuous evolution family of positive

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operators on X (see Section 2 for this notion). At first, in Section 2, it is shown for potentials $V \in L^\infty(I \times \Omega)$ that there is a unique strongly continuous evolution family $\mathcal{U}_V = (U_V(t,s))_{(t,s)\in D}$ satisfying the *variation of constants formula*

$$U_V(t,s)x = U(t,s)x - \int_s^t U(t,\tau)V(\tau)U_V(\tau,s)x\,d\tau \tag{1.2}$$

for $(t,s) \in D$ and $x \in X$. Following a procedure introduced by J. Voigt, [24, 25], in the autonomous case, we then construct the so-called *absorption evolution family* $\mathcal{U}_V$ for arbitrary measurable $0 \le V : I\times\Omega \to \mathbb{R}$ in Section 3. The operators $U_V(t,s)$ are defined as the strong limit of the operators $U_n(t,s)$ determined by (1.2) where V is replaced by the truncated potential $V \wedge n\mathbb{1}$. We prove that $\mathcal{U}_V$ is strongly continuous and that (1.2) holds if V satisfies the integrability condition mentioned above and $\mathcal{U}$ satisfies an *upper Gaussian estimate*, that is, $0 \le U(t,s) \le Me^{a(t-s)\Delta}$ for $(t,s) \in D$ and constants $M, a > 0$. Here, Ω is an open subset of $\mathbb{R}^N$, $(e^{t\Delta})_{t\ge0}$ is the semigroup generated by the Laplacian on $L^p(\mathbb{R}^N)$, and $L^p(\Omega)$ is embedded in $L^p(\mathbb{R}^N)$ by extending functions by 0. The *lower Gaussian estimate* is defined analogously.

In Section 4 we investigate for $\Omega = \mathbb{R}^N$ the asymptotic behaviour of $\mathcal{U}_V$. If $U(t,s) = e^{(t-s)\Delta}$ and $0 \le V(t) \equiv V_0 \in L^1(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$, then exponential stability of $\mathcal{U}_V$ was characterized by W. Arendt and C.J.K. Batty, [3], by a condition on the size of V_0. (See [2] and [6] for further results in the autonomous case.) For $p = 1$ this characterization was extended to the periodic case by D. Daners, M. Hieber, P. Koch Medina, and S. Merino, [8], [12]. We generalize these results to positive evolution families satisfying Gaussian estimates.

Finally, in the last section we apply our results to the diffusion equation (1.1). Here, $\mathcal{U}$ is given by the weak fundamental solution of (1.1) with $V = 0$. Due to (1.2) we can interpret $t \mapsto U_V(t,s)x$ as a *mild solution* of (1.1). We point out that for measurable coefficients and singular potentials we cannot expect to obtain a differentiable solution of (1.1).

2 Bounded perturbations

In this section we study bounded perturbations of evolution families on arbitrary Banach spaces X. Let $I \in \{[a,b], [a,\infty), \mathbb{R}\}$ and $D = D_I := \{(t,s) \in I^2 : t \ge s\}$. A family $\mathcal{U} = (U(t,s))_{(t,s)\in D}$ of bounded operators on X is called an *evolution family* if $U(s,s) = Id$ and $U(t,s) = U(t,r)U(r,s)$ for $t \ge r \ge s$. We say that $\mathcal{U}$ is *strongly continuous* if the mapping $D \ni (t,s) \mapsto U(t,s)$ is strongly continuous. The *exponential growth bound* $\omega(\mathcal{U})$ of $\mathcal{U}$ is defined by

$$\omega(\mathcal{U}) := \inf\{w \in \mathbb{R} : \text{there is } M_w \ge 1 \text{ with } \|U(t,s)\| \le M_w\, e^{w(t-s)} \text{ for } (t,s) \in D\}.$$

The evolution family $\mathcal{U}$ is called *exponentially bounded* if $\omega(\mathcal{U}) < \infty$, and $\mathcal{U}$ is *exponentially stable* if $\omega(\mathcal{U}) < 0$.

In the sequel we make use of the following "truncation–extension" procedure. Let $\mathcal{U}$ be a strongly continuous evolution family with index set D_I. Set $I_n = [a_n, b_n] := I \cap [-n, n]$ for $n \in \mathbb{N}$. Then we define

$$U_n(t,s) := \begin{cases} Id, & t \geq s \geq b_n \ \text{ or } \ a_n \geq t \geq s, \\ U(\min\{t, b_n\}, \max\{s, a_n\}), & \text{otherwise,} \end{cases} \tag{2.1}$$

cf. [16]. Clearly, $\mathcal{U}_n$ is a strongly continuous evolution family with index set $D_{\mathbb{R}}$. Moreover, due to the principle of uniform boundedness we obtain $\sup_{(t,s)\in D_{\mathbb{R}}} \|U_n(t,s)\| < \infty$.

For an exponentially bounded, strongly continuous evolution family $\mathcal{U}$ with index set $D_{\mathbb{R}}$ we define an operator $T(t)$ on the Bochner–Lebesgue space $E = L^1(\mathbb{R}, X)$ by setting

$$T(t)f := U(\cdot, \cdot - t)f(\cdot - t), \qquad f \in E, \ t \geq 0.$$

One can easily see that $\mathcal{T} = (T(t))_{t\geq 0}$ is a $\mathcal{C}_0$–semigroup on E, cf. [22]. We call $\mathcal{T}$ the *evolution semigroup* associated with $\mathcal{U}$. Its generator is denoted by $(G, D(G))$. For further information concerning this approach we refer to [14, 15, 16, 17, 18, 19, 21, 22, 23] and the references therein.

Denote by $L^\infty_{loc}(I, \mathcal{L}_s(X))$ the space of a.e. defined, strongly measurable, operator–valued functions $B(\cdot)$ (i.e., $B(\cdot)x$ is strongly measurable for all $x \in X$) such that $B(\cdot)$ is essentially bounded on compact intervals in $I \subseteq \mathbb{R}$. Further, let $L^\infty(I, \mathcal{L}_s(X))$ be the subspace of a.e. defined, essentially bounded functions $B(\cdot)$. We stress that the operators $B(t)$ are defined for $t \in I \setminus N$, where the null set N only depends on $B(\cdot)$. Observe that $B(\cdot) \in L^\infty(I, \mathcal{L}_s(X))$ induces a bounded multiplication operator $\mathcal{B}$ on E by setting $\mathcal{B}f := B(\cdot)f(\cdot)$, $f \in E$, and $\|\mathcal{B}\| \leq \|B(\cdot)\|_\infty$. With χ_A we denote the characteristic function of a set A.

We now obtain a result on bounded perturbations of evolution families by means of standard perturbation theory for $\mathcal{C}_0$–semigroups applied to the evolution semigroup $\mathcal{T}$ on E. Perturbations by strongly continuous $B(\cdot)$ were studied in [15] and [17] with the same approach. Other versions of Theorem 2.1 based on the Dyson–Phillips expansion are contained in [7, Ch. 2,9].

Theorem 2.1. *Let $\mathcal{U} = (U(t,s))_{(t,s)\in D}$ be a strongly continuous evolution family on a Banach space X. Let $B(\cdot) \in L^\infty_{loc}(I, \mathcal{L}_s(X))$. Then there is a strongly continuous evolution family $\mathcal{U}_B = (U_B(t,s))_{(t,s)\in D}$ on X satisfying*

$$U_B(t,s)x = U(t,s)x + \int_s^t U(t,\tau)B(\tau)U_B(\tau,s)x\,d\tau, \tag{2.2}$$

$$U_B(t,s)x = U(t,s)x + \int_s^t U_B(t,\tau)B(\tau)U(\tau,s)x\,d\tau \tag{2.3}$$

for $x \in X$ and $(t,s) \in D$. Moreover, if $\mathcal{V} = (V(t,s))_{(t,s)\in D}$ is a family of bounded operators which is strongly continuous with respect to t and satisfies (2.2), then $\mathcal{V} = \mathcal{U}_B$.

Proof. Set $I_n := I \cap [-n,n]$ and let $B_n(\cdot) := \chi_{I_n} B(\cdot)$, $n \in \mathbb{N}$. We consider the bounded, strongly continuous evolution family $\mathcal{U}_n$ with index set $D_{\mathbb{R}}$ as defined in (2.1). Let $\mathcal{T}_n$ be the associated evolution semigroup on $E = L^1(\mathbb{R}, X)$ with generator $(G_n, D(G_n))$. As observed above, $B_n(\cdot)$ induces a bounded multiplication operator $\mathcal{B}_n$ on E. Thus by [20], Thm. 3.1.1, Prop. 3.1.2, the perturbed operator $G_{n,B} := G_n + \mathcal{B}_n$ with domain $D(G_{n,B}) = D(G_n)$ generates a $\mathcal{C}_0$–semigroup $\mathcal{T}_{n,B}$ on E satisfying

$$T_{n,B}(t)f = T_n(t)f + \int_0^t T_n(t-\tau)\mathcal{B}_n T_{n,B}(\tau)f\,d\tau, \tag{2.4}$$

$$T_{n,B}(t)f = T_n(t)f + \int_0^t T_{n,B}(t-\tau)\mathcal{B}_n T_n(\tau)f\,d\tau \tag{2.5}$$

for $f \in E$ and $t \geq 0$. From [21, Thm. 3.4] it follows that $\mathcal{T}_{n,B}$ is an evolution semigroup induced by a strongly continuous evolution family $\mathcal{U}_{n,B}$ on X. Using [18, Thm. 4.2] we derive from (2.4) and (2.5)

$$U_{n,B}(t,s)x = U(t,s)x + \int_s^t U(t,\tau)B(\tau)U_{n,B}(\tau,s)x\,d\tau, \tag{2.6}$$

$$U_{n,B}(t,s)x = U(t,s)x + \int_s^t U_{n,B}(t,\tau)B(\tau)U(\tau,s)x\,d\tau, \tag{2.7}$$

respectively, for $x \in X$ and $(t,s) \in D$ with $t,s \in [-n,n]$. A standard application of Gronwall's inequality shows that $U_B(t,s) := U_{n,B}(t,s)$ for $(t,s) \in D_I$ with $t,s \in [-n,n]$ is a (well–defined) strongly continuous evolution family $\mathcal{U}_B$ which satisfies the assertions. $\square$

We now apply the above result to the following situation. Consider $X = L^p(\Omega)$ for a σ–finite measure space (Ω, μ) and $1 \leq p < \infty$. Assume that the evolution family $\mathcal{U}$ on X is *positive*, that is, $U(t,s)x \geq 0$ for $(t,s) \in D$ and $0 \leq x \in X$. Let $V \in L^\infty(I \times \Omega)$. Then the function $V(t,\cdot) : \xi \mapsto V(t,\xi)$ is measurable and bounded for a.e. $t \in I$. So V induces bounded multiplication operators $V(t)$ on X with $\|V(t)\| \leq \|V\|_\infty$ for a.e. $t \in I$.

Lemma 2.2. *Let $V \in L^\infty(I \times \Omega)$ and $X = L^p(\Omega)$, $1 \leq p < \infty$. Then $V(\cdot) \in L^\infty(I, \mathcal{L}_s(X))$.*

Proof. By the preceeding observations we only have to show that $t \mapsto V(t)x$ is strongly measurable for $x \in X$. There exist uniformly bounded, simple functions

V_n converging to V pointwise a.e. as $n \to \infty$. Hence, for a.e. $t \in I$ we have $V_n(t,\cdot) \to V(t,\cdot)$ pointwise a.e.. By the dominated convergence theorem $V_n(t)x$ converges to $V(t)x$ in X as $n \to \infty$ for a.e. $t \in I$ and each $x \in X$. Moreover, it is easily seen that the mapping $t \mapsto V_n(t)x$ is strongly measurable. This yields the assertion. □

Proposition 2.3. *Let $\mathcal{U} = (U(t,s))_{(t,s)\in D}$ be a strongly continuous evolution family on $X = L^p(\Omega)$, $1 \le p < \infty$. Let $V \in L^\infty(I \times \Omega)$. Then the following assertions hold:*

(a) There is a strongly continuous evolution family $\mathcal{U}_V = (U_V(t,s))_{(t,s)\in D}$ on X satisfying

$$U_V(t,s)x = U(t,s)x - \int_s^t U(t,\tau)V(\tau)U_V(\tau,s)x\,d\tau, \tag{2.8}$$

$$U_V(t,s)x = U(t,s)x - \int_s^t U_V(t,\tau)V(\tau)U(\tau,s)x\,d\tau \tag{2.9}$$

for $x \in X$ and $(t,s) \in D$. Moreover, $\mathcal{U}_V$ is uniquely determined by (2.8).

(b) If $V_n \in L^\infty(I\times\Omega)$ with $\|V_n\|_\infty \le C$ and $V_n \to V$ pointwise a.e., then $U_{V_n}(t,s)$ converges strongly to $U_V(t,s)$ as $n \to \infty$ for each $(t,s) \in D$.

(c) If $\mathcal{U}$ is positive and $0 \le \tilde{V} \le V$ for $\tilde{V} \in L^\infty(I\times\Omega)$, then

$$0 \le e^{-(t-s)\|V\|_\infty} U(t,s) \le U_V(t,s) \le U_{\tilde{V}}(t,s) \le U(t,s) \tag{2.10}$$

for $(t,s) \in D$.

Proof. Assertion (a) follows from Lemma 2.2 and Theorem 2.1. Let V_n be as in (b). As in the proof of Lemma 2.2 one shows that for a.e. $t \in I$ the multiplication operators $V_n(t)$ converge strongly to $V(t)$ as $n \to \infty$. Fix $(t,s) \in D$ and $x \in X$. Then (2.8) yields

$$\begin{aligned} U_V(t,s)x - U_{V_n}(t,s)x &= \int_s^t U(t,\tau)(V_n(\tau) - V(\tau))U_V(\tau,s)x\,d\tau \\ &\quad - \int_s^t U(t,\tau)V_n(\tau)(U_V(\tau,s)x - U_{V_n}(\tau,s)x)\,d\tau. \end{aligned}$$

The first integral tends to 0 as $n \to \infty$ due to Lebesgue's theorem. Hence, for each $\varepsilon > 0$ there is an $N_\varepsilon \in \mathbb{N}$ such that

$$\|U_V(t,s)x - U_{V_n}(t,s)x\| \le \varepsilon + CM \int_s^t \|U_V(\tau,s)x - U_{V_n}(\tau,s)x\|\,d\tau$$

for $n \geq N_\varepsilon$, where $M := \sup_{t\geq\tau\geq s} \|U(t,\tau)\|$. So Gronwall's inequality implies (b).

We consider the bounded, strongly continuous evolution family $\mathcal{U}_n$ with index set $D_{\mathbb{R}}$ as defined in (2.1). Due to (a) there exists a perturbed evolution family $(\mathcal{U}_n)_V$. Let $\mathcal{T}_n$, resp. $(\mathcal{T}_n)_V$, be the associated evolution semigroup on $E = L^1(\mathbb{R}, X)$. Since V induces a bounded operator on E, the Trotter product formula, [20, Cor. 3.5.5], yields

$$(T_n)_V(t)f = \lim_{m\to\infty} \left(T_n(\tfrac{t}{m})e^{-\frac{t}{m}V}\right)^m f$$

for $f \in E$. For $0 \leq \tilde{V} \leq V$ and positive $\mathcal{U}$ we derive

$$0 \leq e^{-t\|V\|_\infty} T_n(t) \leq (T_n)_V(t) \leq (T_n)_{\tilde{V}}(t) \leq T_n(t). \tag{2.11}$$

Observe that $U_n(t,s) = U(t,s)$ and $(U_n)_V(t,s) = U_V(t,s)$ for $(t,s) \in D$ with $t,s \in [-n,n]$. Thus due to the strong continuity of $\mathcal{U}$, $\mathcal{U}_V$, $\mathcal{U}_{\tilde{V}}$, assertion (c) follows from (2.11). □

3 Absorption evolution families

We now consider a positive, strongly continuous evolution family $\mathcal{U} = (U(t,s))_{(t,s)\in D}$ on $X = L^p(\Omega)$, $1 \leq p < \infty$, and a positive, measurable function $V : I \times \Omega \to \mathbb{R}$. Set $V^{(n)} := n\mathbb{1} \wedge V$, $n \in \mathbb{N}$, where $\mathbb{1}$ denotes the constant function 1. Then $0 \leq V^{(n)} \in L^\infty(I \times \Omega)$ and $V^{(n)} \uparrow V$. Let $\mathcal{U}_n$ be the evolution family corresponding to $\mathcal{U}$ and $V^{(n)}$ as obtained in Proposition 2.3. By (2.10) we have $0 \leq U_n(t,s)x \leq U_m(t,s)x \leq U(t,s)x$ for $n \geq m$, $0 \leq x \in X$, and $(t,s) \in D$. Therefore we can define a bounded operator $U_V(t,s)$ on X by setting $U_V(t,s)x := \lim_{n\to\infty} U_n(t,s)x$, $(t,s) \in D$. We first collect several properties of the family $\mathcal{U}_V = (U_V(t,s))_{(t,s)\in D}$.

Lemma 3.1. *Let $\mathcal{U}$ be a positive, strongly continuous evolution family on $X = L^p(\Omega)$, $1 \leq p < \infty$, and let $V, \tilde{V} : I \times \Omega \to \mathbb{R}$ be measurable such that $0 \leq \tilde{V} \leq V$. Define $\mathcal{U}_V$ on X as above. Then the following assertions hold:*

(a) $0 \leq U_V(t,s) \leq U_{\tilde{V}}(t,s) \leq U(t,s)$ for $(t,s) \in D$.

(b) $\mathcal{U}_V$ is a strongly measurable evolution family on X, i.e., the mapping $(t,s) \mapsto U_V(t,s)x$ is strongly measurable for each $x \in X$.

(c) If $V_n : I \times \Omega \to \mathbb{R}$ is measurable such that $0 \leq V_n \leq V$ and $V_n \to V$ pointwise a.e. as $n \to \infty$, then $U_{V_n}(t,s)$ converges strongly to $U_V(t,s)$ for $(t,s) \in D$.

Proof. Assertion (a) is clear from the definition and (2.10). Since $\mathcal{U}_n$ is a strongly continuous evolution family, assertion (b) follows from the strong convergence of $U_n(t,s)$ to $U_V(t,s)$ as $n \to \infty$. Consider V_n as in (c). Let $V^{(m)} = m\mathbb{1} \wedge V$ and $V_n^{(m)} = m\mathbb{1} \wedge V_n$ for $n, m \in \mathbb{N}$. We have $0 \le V_n^{(m)} \le V^{(m)} \in L^\infty(I \times \Omega)$ and $V_n^{(m)} \to V^{(m)}$ pointwise a.e. as $n \to \infty$. Hence, by Proposition 2.3(b) the operator $U_{V_n^{(m)}}(t,s)$ converges strongly to $U_m(t,s)$ as $n \to \infty$ for $(t,s) \in D$. On the other hand, $V_n^{(m)} \le V_n$ yields $U_{V_n}(t,s) \le U_{V_n^{(m)}}(t,s)$ for $n, m \in \mathbb{N}$ and $(t,s) \in D$. Thus

$$U_V(t,s)x \le \liminf_{n\to\infty} U_{V_n}(t,s)x \le \limsup_{n\to\infty} U_{V_n}(t,s)x \le \lim_{n\to\infty} U_{V_n^{(m)}}(t,s)x = U_m(t,s)x$$

for $0 \le x \in X$, $(t,s) \in D$, and $m \in \mathbb{N}$. Now (c) follows by letting $m \to \infty$. $\square$

As a consequence, the evolution family $\mathcal{U}_V$ depends only on $\mathcal{U}$ and V and not on a particular approximating sequence (V_n) (satisfying $0 \le V_n \le V$). The following definition generalizes Voigt's notion of an *absorption semigroup* (if $V \ge 0$), see [24], [25].

Definition 3.2. *Let $0 \le V : I \times \Omega \to \mathbb{R}$ be measurable and let $\mathcal{U} = (U(t,s))_{(t,s)\in D}$ be a positive, strongly continuous evolution family on $X = L^p(\Omega)$, $1 \le p < \infty$. Then the evolution family $\mathcal{U}_V$ on X as defined above is called the* absorption evolution family *corresponding to $\mathcal{U}$ and V.*

We stress that in general $\mathcal{U}_V$ is not strongly continuous and there may exist $(t,s) \in D$ such that (2.8) fails. For instance, consider $X = \mathbb{C}$, $I = \mathbb{R}_+$, $U(t,s) \equiv 1$, and $V(t) = \frac{1}{t}$, yielding $U_V(t,s) = \exp(-\int_s^t V(\tau)\, d\tau)$, $t \ge s \ge 0$, see also [25, Ex. 4.1]. However, in a more specific situation we can show strong continuity and variation of constants formulas for $\mathcal{U}_V$.

For the next result we recall that a subspace Y of $X = L^p(\Omega)$ is called an *ideal* in X if $x \in X$, $0 \le y \in Y$, and $|x| \le y$ imply $x \in Y$. Further, we refer to [9, pp.52] for the definition of the Pettis–integral. If $V : I \times \Omega \to \mathbb{C}$ is measurable we denote by $(V(t), D(V(t)))$, $t \in I$, the induced multiplication operators on X with maximal domain.

Theorem 3.3. *Let $\mathcal{U} = (U(t,s))_{(t,s)\in D}$ be a positive, strongly continuous evolution family on $X = L^p(\Omega)$, $1 \le p < \infty$, and let $V : I \times \Omega \to \mathbb{R}$ be positive and measurable. Consider a dense ideal Y in X.*

(a) Assume that for $s \in I$ and a.e. $t \in I$ with $t > s$ we have $U(t,s)Y \subseteq D(V(t))$, and

$$\text{for } y \in Y \text{ and } [c,d] \subseteq D \text{ there is } \psi \in L^1[c,d] \text{ with } \|V(\cdot)U(\cdot,s)y\|_p \le \psi \tag{3.1}$$

on $[s,d]$ for all $s \in [c,d]$. Then, $U_V(t,s)Y \subseteq D(V(t))$ for $s \in I$ and a.e. $I \ni t > s$ and $\mathcal{U}_V$ satisfies (2.8) and (2.9) for $(t,s) \in D$ and $x \in Y$. In addition, suppose that the mapping $s \mapsto V(t)U(t,s)y$ is continuous for $y \in Y$, a.e. $t \in I$, and $s \in (-\infty, t) \cap I$. Then $\mathcal{U}_V$ is strongly continuous.

(b) *Assume $U(t,s)X \subseteq D(V(t))$ for $s \in I$ and a.e. $t \in I$ with $t > s$ and that (3.1) holds. Then $\mathcal{U}_V$ is strongly contiuous. Moreover, $U_V(t,s)X \subseteq D(V(t))$ for $s \in I$ and a.e. $I \ni t > s$, and (2.8) and (2.9) hold for $(t,s) \in D$ and $x \in X$, where the integrals are Pettis-integrals if $p > 1$.*

Proof. *(a):* From $0 \le U_V(t,s) \le U(t,s)$ it follows that $U_V(t,s)Y \subseteq D(V(t))$ for $s \in I$ and a.e. $I \ni t > s$. Consider the absorption evolution family $\mathcal{U}_n$ corresponding to $V^{(n)} = n\mathbb{I} \wedge V$, $n \in \mathbb{N}$. Using the dominated convergence theorem, we derive

$$\lim_{n\to\infty} V^{(n)}(t)U_n(t,s)y = V(t)U_V(t,s)y \tag{3.2}$$

for $y \in Y$, $s \in I$, and a.e. $I \ni t > s$. Further, by Proposition 2.3 we have

$$U_n(t,s)y = U(t,s)y - \int_s^t U(t,\tau)V^{(n)}(\tau)U_n(\tau,s)y\,d\tau, \tag{3.3}$$

$$U_n(t,s)y = U(t,s)y - \int_s^t U_n(t,\tau)V^{(n)}(\tau)U(\tau,s)y\,d\tau. \tag{3.4}$$

For $y \in Y$ and $(t,s) \in D$ we set

$$\begin{aligned} f_n(\tau) &:= U(t,\tau)V^{(n)}(\tau)U_n(\tau,s)y, & f(\tau) &:= U(t,\tau)V(\tau)U_V(\tau,s)y,\\ g_n(\tau) &:= U_n(t,\tau)V^{(n)}(\tau)U(\tau,s)y, & g(\tau) &:= U_V(t,\tau)V(\tau)U(\tau,s)y \end{aligned}$$

for a.e. $\tau \in (s,t]$ and $n \in \mathbb{N}$. Then (3.1) yields $\|f_n(\cdot)\|_p$, $\|g_n(\cdot)\|_p \le \varphi$ for a function $\varphi \in L^1[s,t]$. Further, by (3.2) we obtain $f_n(\tau) \to f(\tau)$ and $g_n(\tau) \to g(\tau)$ in X for a.e. $\tau \in (s,t]$. Consequently, for $y \in Y$ equations (2.8) and (2.9) follow from (3.3) and (3.4), respectively, due to Lebesgue's theorem.

To prove strong continuity of $\mathcal{U}_V$ it suffices to show that $D \ni (t,s) \mapsto U_V(t,s)y \in X$ is continuous for $y \in Y$. By (3.1) and the strong continuity of $\mathcal{U}$ we see that the mapping

$$t \mapsto \int_s^t U(t,\tau)V(\tau)U_V(\tau,s)y\,d\tau \in X, \qquad t \ge s,$$

is continuous for each $s \in I$. Thus (2.8) implies that $t \mapsto U_V(t,s)$ is strongly continuous for $t \ge s$. Because of (2.9) it remains to show the continuity of the mapping

$$\Phi : (t,s) \mapsto \int_s^t U_V(t,\tau)V(\tau)U(\tau,s)y\,d\tau \in X.$$

By (3.1) this is clear at points (t_0, t_0). So let $t > s$. By the assumption and the strong continuity of $U_V(\cdot, s)$, the mapping $D \ni (t,s) \mapsto U_V(t,\tau)V(\tau)U(\tau,s)y$ is continuous for a.e. $\tau \in (s,t)$. Then we can derive the continuity of Φ by using (3.1).

(b): From the assumption we infer that $U_V(t,s)X \subseteq D(V(t))$ for $s \in I$ and a.e. $I \ni t > s$. Thus, $V(t)U(t,s)$ and $V(t)U_V(t,s)$ are everywhere defined and hence bounded for $s \in I$ and a.e. $I \ni t > s$. Since $V(t)U(t,s) = V(t)U(t,r)U(r,s)$ for $t > r > s$ this implies strong continuity of $s \mapsto V(t)U(t,s)$ and, due to (a), the first assertion in (b) follows. Further, for $0 \le x \in X$ there is an increasing sequence $(y_n) \subseteq Y_+$ converging to x. By (a) we have

$$U(t,s)y_n - U_V(t,s)y_n = \int_s^t U(t,\tau)V(\tau)U_V(\tau,s)y_n \, d\tau =: \int_s^t h_n(\tau)\, d\tau, \quad (3.5)$$

$$U(t,s)y_n - U_V(t,s)y_n = \int_s^t U_V(t,\tau)V(\tau)U(\tau,s)y_n \, d\tau =: \int_s^t k_n(\tau)\, d\tau \quad (3.6)$$

for $(t,s) \in D$. Then $h_n(\tau)$ and $k_n(\tau)$ converge in X monotonically to $h(\tau) := U(t,\tau)V(\tau)U_V(\tau,s)x$ and $k(\tau) := U_V(t,\tau)V(\tau)U(\tau,s)x$, respectively, for a.e. $\tau \in (s,t]$. In particular, h and k are strongly measurable. Moreover, let $0 \le x' \in X'$. The positive functions $\langle h_n(\cdot), x'\rangle$ and $\langle k_n(\cdot), x'\rangle$ converge monotonically to $\langle h(\cdot), x'\rangle$ and $\langle k(\cdot), x'\rangle$ as $n \to \infty$, respectively. On the other hand,

$$\int_s^t h_n(\tau)\, d\tau \to U(t,s)x - U_V(t,s)x \quad \text{and} \quad \int_s^t k_n(\tau)\, d\tau \to U(t,s)x - U_V(t,s)x$$

in X by (3.5) and (3.6), respectively. By the monotone convergence theorem we obtain that $\langle h(\cdot), x'\rangle$ and $\langle k(\cdot), x'\rangle$ are integrable and

$$\langle U(t,s)x - U_V(t,s)x, x'\rangle = \int_s^t \langle h(\tau), x'\rangle \, d\tau = \int_s^t \langle k(\tau), x'\rangle \, d\tau.$$

If $p = 1$ we can choose $x' = 1\!\!1$. Since $\|z\|_1 = \langle z, 1\!\!1\rangle$ for each $0 \le z \in L^1(\Omega)$, the functions h and k are Bochner integrable, and (2.8) and (2.9) hold for each $x \in L^1(\Omega)$ with Bochner integrals. For $1 < p < \infty$ the functions h and k are Pettis integrable, and (2.8) and (2.9) hold for each $x \in L^p(\Omega)$, $1 < p < \infty$, where the integrals are Pettis integrals. □

Corollary 3.4. *Let $\mathcal{U}$ be a positive, strongly continuous evolution family on $X = L^p(\Omega)$, $1 \le p < \infty$. Let $0 \le V \in L^1_{loc}(I, L^\infty(\Omega))$. Then the conclusions of Theorem 3.3 hold for $Y = X$.*

In order to obtain further sufficient conditions for (3.1) we now introduce the concept of a Gaussian estimate. Let Ω be an open subset of $\mathbb{R}^N$. We identify

$X = L^p(\Omega)$ with a subspace of $L^p(\mathbb{R}^N)$ by extending functions by 0. Consider the semigroup $(e^{t\Delta})_{t\geq 0}$ generated by the Laplacian on $L^p(\mathbb{R}^N)$, $1 \leq p \leq \infty$. It is known that $e^{t\Delta}x = K_t * x$, where $K_t(\xi) = (4\pi t)^{-N/2}\exp(-\frac{|\xi|^2}{4t})$, $t > 0$, $\xi \in \mathbb{R}^N$, is the Gaussian kernel. A positive evolution family $\mathcal{U} = (U(t,s))_{(t,s)\in D}$ on X is said to satisfy an *upper Gaussian estimate* if there are constants $M, a > 0$ such that

$$0 \leq U(t,s)x \leq M\, e^{a(t-s)\Delta}x$$

for $0 \leq x \in L^p(\Omega)$. In Section 5 we consider examples for evolution families satisfying an upper Gaussian estimate. We now easily derive the following result.

Corollary 3.5. *Let $\Omega \subseteq \mathbb{R}^N$ be open and let $X = L^p(\Omega)$, $1 \leq p < \infty$. Let $\mathcal{U}$ be a positive, strongly continuous evolution family on X satisfying an upper Gaussian estimate. Let $V = V_1 + V_2 + V_3$, where $0 \leq V_1 \in L^1_{loc}(I, X)$, $0 \leq V_2 \in L^1_{loc}(I, L^\infty(\Omega))$, $0 \leq V_3 \in L^\infty(I \times \Omega)$. Then the conclusions of Theorem 3.3 hold for $Y = X \cap L^\infty(\Omega)$.*

Proof. Observe that the Gaussian estimate and Young's inequality yield $|U(t,s)x| \leq MK_{a(t-s)} * |x| \in Y \subseteq D(V(t))$ for $t > s$ and $x \in X$. In particular, $\|U(t,s)y\|_\infty \leq M\|y\|_\infty$ and

$$\|V(t)U(t,s)y\|_p \leq M(\|V_1(t)\|_p\,\|y\|_\infty + \|V_2(t)\|_\infty\,\|y\|_p + \|V_3\|_\infty\,\|y\|_p)$$

for $y \in Y = X \cap L^\infty(\Omega)$ and $(t,s) \in D$. □

The assumptions of part (a) and (b) of Theorem 3.3 can be checked for a much larger class of potentials V. For simplicity we concentrate on the case $\Omega = \mathbb{R}^N$. By λ we denote the Lebesgue measure on $\mathbb{R}^N$. We set $\varepsilon_\alpha(\xi) := e^{-\alpha|\xi|^2}$.

Corollary 3.6. *Let $X = L^p(\mathbb{R}^N)$, $1 \leq p < \infty$. Let $\mathcal{U}$ be a positive, strongly continuous evolution family on X satisfying an upper Gaussian estimate. Let $0 \leq V \in L^1_{loc}(I, L^p(\mathbb{R}^N, \varepsilon_\alpha\lambda))$ for all $\alpha > 0$. Then the conclusions of Theorem 3.3 (a) and (b) hold for $Y = \{y \in L^\infty(\mathbb{R}^N) : y$ has compact support$\}$.*

Proof. Fix $y \in Y$ and $d > 0$. Consider $t, s \in I$ with $d \geq t - s > 0$. Set $K := \operatorname{supp} y$. Choose a ball B of radius r such that $\operatorname{dist}(\xi, K) \geq 1$ for $\xi \notin B$. Further, observe that the Gaussian kernel K_τ satisfies

$$K_{a\tau}(\xi) \leq 2^{\frac{N}{2}}\, e^{-\frac{1}{8a\tau}}\, K_{2a\tau}(\xi) \tag{3.7}$$

for $a > 0$, $\tau > 0$, and $|\xi| \geq 1$. Then, by using the Gaussian estimate and (3.7) we compute

$\|V(t)U(t,s)y\|_p^p$

$$
\begin{aligned}
&\leq\ M^p \int_{\mathbb{R}^N} V(t,\xi)^p \left(\int_K K_{a(t-s)}(\xi-\eta)|y(\eta)|\,d\eta\right)^p d\xi \\
&\leq\ M^p \,\|y\|_\infty^p \left(\int_B V(t,\xi)^p\,d\xi + \int_{B^c} V(t,\xi)^p\, 2^{\frac{pN}{2}} e^{-\frac{p}{8a(t-s)}} \left(\int_K K_{2a(t-s)}(\xi-\eta)\,d\eta\right)^p d\xi\right) \\
&\leq\ M^p \,\|y\|_\infty^p \left(\|V(t)\|_{L^p(B)}^p + \lambda(K)^p\,(4a\pi(t-s))^{-\frac{pN}{2}}\, e^{-\frac{p}{8a(t-s)}} \int_{B^c} V(t,\xi)^p\, e^{-\frac{p\,\mathrm{dist}(\xi,K)^2}{8a(t-s)}}\,d\xi\right) \\
&\leq\ C_1 \left(\|V(t)\|_{L^p(B)}^p + \int_{B^c} V(t,\xi)^p\, e^{-\frac{p\,(|\xi|-r)^2}{8ad}}\,d\xi\right) \\
&\leq\ C_2 \left(\|V(t)\|_{L^p(B)}^p + \int_{B^c} V(t,\xi)^p\, e^{-\alpha|\xi|^2}\,d\xi\right) \\
&\leq\ C_3\,\|V(t)\|_{L^p(\mathbb{R}^N,\,\varepsilon_\alpha\lambda)}^p
\end{aligned}
$$

for a.e. t and constants $C_k, \alpha > 0$ depending on y and d. This establishes (3.1).

Next, notice that for each sequence $(s_n) \subseteq I$ with $s_n < t$ and $s_n \to s$ there is a subsequence (s_{n_k}) such that $V(t)U(t,s_{n_k})y$ converges to $V(t)U(t,s)y$ pointwise a.e.. Hence, by Lebesgue's theorem, for showing the continuity of $s \mapsto V(t)U(t,s)y$ it suffices to find $z \in X$ such that

$$
z \in D(V(t)) \ \text{ for a.e. } \ t \in I \quad \text{and} \quad |U(t,r)y| \leq z \ \text{ for } \ 0 \leq t-r \leq 2d. \tag{3.8}
$$

Clearly, by the Gaussian estimate we have $|U(t,r)y|\,(\xi) \leq M\|y\|_\infty$ for $\xi \in \mathbb{R}^N$. Moreover, for $\xi \notin B$ we infer from (3.7) that

$$
\begin{aligned}
|U(t,r)y|\,(\xi) \ &\leq\ M\,2^{\frac{N}{2}}\,\|y\|_\infty\, e^{-\frac{1}{8a(t-r)}} \int_K K_{2a(t-r)}(\xi-\eta)\,d\eta \\
&\leq\ M\,2^{\frac{N}{2}}\,\|y\|_\infty\,(8a\pi(t-r))^{-\frac{N}{2}}\, e^{-\frac{1}{8a(t-r)}} \int_K e^{-\frac{|\xi-\eta|^2}{16d}}\,d\eta \\
&\leq\ C\,\|y\|_\infty\,\lambda(K)\, e^{-\frac{(|\xi|-r)^2}{16d}} \ =: \varphi(\xi)
\end{aligned}
$$

for some constant $C > 0$. Setting $z(\xi) := M\|y\|_\infty$ for $\xi \in B$ and $z(\xi) := \varphi(\xi)$ for $\xi \notin B$ yields (3.8) for a.e. $t \in I$. □

4 Exponential stability

In this section we assume that I is unbounded, $\Omega = \mathbb{R}^N$, and $\mathcal{U}$ is positive. If $\mathcal{U}$ satisfies an upper Gaussian estimate, then the absorption evolution family $\mathcal{U}_V$ is uniformly bounded since $0 \leq U_V(t,s) \leq U(t,s)$. So the question arises under which conditions on V the evolution family $\mathcal{U}_V$ is exponentially stable. For this we introduce the following notion. Let $B(\xi,r) := \{\eta \in \mathbb{R}^N : |\xi-\eta| \leq r\}$.

Definition 4.1. *Let $0 \le V \in L^1_{loc}(I \times \mathbb{R}^N)$ and $t > 0$. We say that V* belongs to the class $\mathcal{E}_t$ *if there are constants $c, r > 0$ such that*

$$\int_s^{s+t} \int_{B(\xi,r)} V(\tau,\eta)\, d\eta\, d\tau \ge c \qquad \text{for all} \quad s \in I,\ \xi \in \mathbb{R}^N. \tag{4.1}$$

This definition generalizes a concept introduced by W. Arendt and C.J.K. Batty for time independent potentials and by D. Daners and P. Koch–Medina for time periodic potentials; see [3, Prop. 1.4] and [8].

We begin with some preparations. We first give a sufficient condition for (4.1).

Lemma 4.2. *Let $0 \le V \in L^1_{loc}(I \times \mathbb{R}^N)$. Let $c, r, t > 0$ and $m \ge 0$ be constants such that*

$$\int_s^{s+t} \int_{B(\xi,r)} V(\tau,\eta)\, d\eta\, d\tau \ge c$$

for $s \in I$ and $\xi \in \mathbb{R}^N \setminus A_s$, where $A_s \subseteq \mathbb{R}^N$ is measurable and $\lambda(A_s) \le m$. Then $V \in \mathcal{E}_t$.

Proof. Without loss of generality let $m > 0$. Set $r_1 := r + (2m)^{\frac{1}{N}} \lambda(B(0,1))^{-\frac{1}{N}}$. Fix $\xi \in \mathbb{R}^N$ and $s \in I$. Then there is a vector $\xi(s) \in \mathbb{R}^N \setminus A_s$ with $|\xi - \xi(s)| \le r_1 - r$. (Otherwise, $B(\xi, r_1 - r) \subseteq A_s$, and hence $m \ge \lambda(A_s) \ge 2m$, which is a contradiction.) Thus

$$\int_s^{s+t} \int_{B(\xi,r_1)} V(\tau,\eta)\, d\eta\, d\tau \ge \int_s^{s+t} \int_{B(\xi(s),r)} V(\tau,\eta)\, d\eta\, d\tau \ge c.$$

So (4.1) is satisfied with constants $t, c, r_1 > 0$. □

The following characterization of exponential stability of an evolution family can easily be shown by using that $\|T\| = \|T'\mathbb{1}\|_\infty$ for $0 \le T \in \mathcal{L}(L^1(\Omega))$, where $\mathbb{1}$ denotes the constant function 1.

Lemma 4.3. *Let $\mathcal{U}$ be a positive, exponentially bounded evolution family on $L^1(\Omega)$. Then $\mathcal{U}$ is exponentially stable if and only if there are constants $t_0 > 0$ and $\delta < 1$ such that $U(s + t_0, s)'\mathbb{1} \le \delta \mathbb{1}$ for all $s \in I$. In this case, $\delta > 0$ can be chosen arbitrarily small.*

A positive evolution family $\mathcal{U}$ on $L^p(\mathbb{R}^N)$ is said to satisfy a *lower Gaussian estimate* if there are constants $\tilde{M}, \tilde{a} > 0$ such that

$$U(t,s) \ge \tilde{M}\, e^{\tilde{a}(t-s)\Delta}$$

for $(t,s) \in D$. For $p = 1$ this implies $U(t,s)'\mathbb{1} \ge \tilde{M}\mathbb{1}$. Moreover, $\mathcal{U}$ is called *submarkovian* if $p = 1$ and $U(t,s)'\mathbb{1} \le \mathbb{1}$ for $(t,s) \in D$. Further, we use the space

$L^1_{loc,u}(I\times \mathbb{R}^N)$ of uniformly locally integrable functions f endowed with the norm

$$\|f\|_{loc,u} := \sup_{s\in I,\xi\in\mathbb{R}^N} \int_s^{s+1}\int_{B(\xi,1)} |f(\tau,\eta)|\,d\eta\,d\tau.$$

Notice that a different choice of constants in the integrals, say $t,r>0$ instead of 1, yields an equivalent norm.

In Theorem 4.4 and 4.6 we present for $X=L^1(\mathbb{R}^N)$ necessary and sufficient conditions on $\mathcal{U}$ and V such that the absorption evolution family $\mathcal{U}_V$ is exponentially stable. The case $X=L^p(\mathbb{R}^N)$ is considered afterwards.

Theorem 4.4. *Let $\mathcal{U}$ be a submarkovian, strongly continuous evolution family on $X=L^1(\mathbb{R}^N)$ satisfying a lower Gaussian estimate. Let $0\le V\in \overline{L^\infty(I\times\mathbb{R}^N)_+}^{L^1_{loc,u}}$. If $V\in\mathcal{E}_t$ for some $t>0$, then $\mathcal{U}_V$ is exponentially stable.*

Proof. (i) Let $V\in\mathcal{E}_t$ satisfy the above assumptions, where $t>0$ is fixed. Then there are functions $0\le\tilde{V}_n\in L^\infty(I\times\mathbb{R}^N)$ converging to V in $L^1_{loc,u}(I\times\mathbb{R}^N)$. Set $V_n:=\tilde{V}_n\wedge V$. Choose $c,r>0$ as in (4.1). Then

$$\begin{aligned}\int_s^{s+t}\int_{B(\xi,r)} V_n(\tau,\eta)\,d\eta\,d\tau &\ge c-\int_s^{s+t}\int_{B(\xi,r)}(V(\tau,\eta)-V_n(\tau,\eta))\,d\eta\,d\tau\\ &\ge c-C\|V-\tilde{V}_n\|_{loc,u}\,,\end{aligned}$$

where C only depends on t and r. Hence, $V_n\in\mathcal{E}_t$ for n large enough. We also have $0\le U_V(t,s)\le U_{V_n}(t,s)$ by Lemma 3.1. As a consequence, it suffices to consider $0\le V\in L^\infty(I\times\mathbb{R}^N)$.

(ii) Now let $0\le V\in L^\infty(I\times\mathbb{R}^N)$. Choose $t_0>t$. By assumption, $\mathbb{1}\ge U(t,s)'\mathbb{1}\ge\tilde{M}\mathbb{1}$ for $(t,s)\in D$. So we derive from (2.9), (2.10), and the lower Gaussian estimate that

$$\begin{aligned}\langle x,U_V(s+t_0,s)'\mathbb{1}\rangle &= \langle x,U(s+t_0,s)'\mathbb{1}\rangle-\int_s^{s+t_0}\langle x,U(\tau,s)'V(\tau)U_V(s+t_0,\tau)'\mathbb{1}\rangle\,d\tau\\ &\le \langle x,\mathbb{1}\rangle-\tilde{M}e^{-t_0\|V\|_\infty}\int_s^{s+t_0}\langle x,U(\tau,s)'V(\tau)\mathbb{1}\rangle\,d\tau\\ &\le \langle x,\mathbb{1}\rangle-\tilde{M}^2e^{-t_0\|V\|_\infty}\int_s^{s+t_0}\langle x,e^{\tilde{a}(\tau-s)\Delta}V(\tau)\mathbb{1}\rangle\,d\tau\end{aligned}$$

for $s\in I$ and $0\le x\in X$. Also by Fubini's theorem,

$$\begin{aligned}\int_s^{s+t_0}\langle x,e^{\tilde{a}(\tau-s)\Delta}V(\tau)\mathbb{1}\rangle\,d\tau &= \int_s^{s+t_0}\int_{\mathbb{R}^N}\Big(x(\eta)\int_{\mathbb{R}^N}K_{\tilde{a}(\tau-s)}(\xi-\eta)V(\tau,\xi)\,d\xi\Big)\,d\eta\,d\tau\\ &= \Big\langle x,\int_s^{s+t_0}\int_{\mathbb{R}^N}K_{\tilde{a}(\tau-s)}(\xi-\cdot)V(\tau,\xi)\,d\xi\,d\tau\Big\rangle\,.\end{aligned}$$

Set $\varepsilon := \tilde{a}(t_0 - t) > 0$ and choose $r > 0$ as in (4.1). Then (4.1) yields

$$\begin{aligned}\int_s^{s+t_0} \int_{\mathbb{R}^N} K_{\tilde{a}(\tau-s)}(\xi-\eta)V(\tau,\xi)\,d\xi\,d\tau &\geq \int_{s+t_0-t}^{s+t_0} \int_{B(\eta,r)} K_{\tilde{a}(\tau-s)}(\xi-\eta)V(\tau,\xi)\,d\xi\,d\tau \\ &\geq \frac{c}{(4\pi\tilde{a}t_0)^{N/2}}\, e^{-\frac{r^2}{4\varepsilon}} =: ck_{\varepsilon,r}\end{aligned}$$

for $s \in I$ and a.e. $\eta \in \mathbb{R}^N$. Altogether we obtain

$$\langle x, U_V(s+t_0,s)'\mathbb{1}\rangle \leq (1 - ck_{\varepsilon,r}\tilde{M}^2 e^{-t_0\|V\|_\infty})\langle x, \mathbb{1}\rangle \tag{4.2}$$

for all $0 \leq x \in L^1(\mathbb{R}^N)$. Hence, $U_V(s+t_0,s)'\mathbb{1} \leq \delta\mathbb{1}$ for $\delta := 1 - ck_{\varepsilon,r}\tilde{M}^2 e^{-t_0\|V\|_\infty} < 1$ and $s \in I$. Thus the assertion follows from Lemma 4.3. □

Remark 4.5 *(a) Theorem 4.4 is false if we only require $0 \leq V \in \mathcal{E}_t \cap L^1_{loc,u}(I \times \mathbb{R}^N)$. An example is given in [3], p. 1019, where $V(t) \equiv V_0$ and $U(t,s) = e^{(t-s)\Delta}$.*

(b) Let $V = V_1 + V_2 + V_3 + V_4$, where $0 \leq V_1 \in L^1(I, L^1(\mathbb{R}^N))$, $0 \leq V_2 \in L^1(I, L^\infty(\mathbb{R}^N))$, $0 \leq V_3 \in L^\infty(I \times \mathbb{R}^N)$, and $0 \leq V_4 \in \{f \in L^\infty(I, L^1(\mathbb{R}^N)) : f(\tau,\cdot) = f(\tau+T,\cdot)$ for some $T > 0\}$. Consider $V_k^{(n)} = n\mathbb{1} \wedge V_k \in L^\infty(I \times \mathbb{R}^N)$ for $k = 1,2,3,4$. Then $0 \leq V_n := \sum_{k=1}^4 V_k^{(n)}$ converges to V in $L^1_{loc,u}(I \times \mathbb{R}^N)$.

Proof of (b). It suffices to show $V_k^{(n)} \to V_k$ in $L^1_{loc,u}$ for $k = 1,2,3,4$. This is clear for V_1 and V_3. Further, for $s \in I$ and $\xi \in \mathbb{R}^N$

$$\int_s^{s+1} \int_{B(\xi,1)} (V_2(\tau,\eta) - V_2^{(n)}(\tau,\eta))\,d\eta\,d\tau \leq \lambda(B(0,1)) \int_I \|V_2(\tau,\cdot) - V_2^{(n)}(\tau,\cdot)\|_\infty\,d\tau.$$

For a.e. τ the integrand on the right–hand side equals 0 for n large enough. So by the dominated convergence theorem the right–hand side converges to 0 as $n \to \infty$. Finally, for $s \in I$ and $\xi \in \mathbb{R}^N$

$$\int_s^{s+T} \int_{B(\xi,1)} (V_4(\tau,\eta) - V_4^{(n)}(\tau,\eta))\,d\eta\,d\tau \leq \int_0^{2T} \|V_4(\tau,\cdot) - V_4^{(n)}(\tau,\cdot)\|_1\,d\tau.$$

Since for a.e. τ we have $V_4^{(n)}(\tau,\eta) \to V_4(\tau,\eta)$ for a.e. $\eta \in \mathbb{R}^N$, the integrand on the right–hand side tends to 0 as $n \to \infty$ for a.e. τ. By Lebesgue's theorem the claim follows. □

Theorem 4.6. *Let $\mathcal{U}$ be a positive, strongly continuous evolution family on $X = L^1(\mathbb{R}^N)$ satisfying an upper Gaussian estimate. Assume that there is a constant $d > 0$ such that $U(t,s)'\mathbb{1} \geq d\mathbb{1}$, $(t,s) \in D$. Let $V = V_1 + V_2 + V_3$, where $0 \leq V_1 \in L^1_{loc,u}(I, L^1(\mathbb{R}^N))$, $0 \leq V_2 \in L^1(I, L^\infty(\mathbb{R}^N))$, and $0 \leq V_3 \in L^\infty(I \times \mathbb{R}^N)$. Then exponential stability of $\mathcal{U}_V$ implies $V \in \mathcal{E}_t$ for some $t > 0$.*

Proof. By Lemma 4.3 there are constants $t > 0$, $0 < \delta < d$ such that $U_V(s+t,s)'\mathbb{1} \leq \delta\mathbb{1}$ for $s \in I$. Fix $s \in I$ and $0 \leq x \in X \cap L^\infty(\mathbb{R}^N)$. By Corollary 3.5 equation (2.9) holds. Then the assumptions yield

$$\begin{aligned}
\delta\langle x, \mathbb{1}\rangle &\geq \langle x, U_V(s+t,s)'\mathbb{1}\rangle \\
&= \langle x, U(s+t,s)'\mathbb{1}\rangle - \int_s^{s+t} \langle V(\tau)U(\tau,s)x, U_V(t+s,\tau)'\mathbb{1}\rangle \, d\tau \\
&\geq d\langle x, \mathbb{1}\rangle - M^2 \int_s^{s+t} \langle V(\tau)e^{a(\tau-s)\Delta}x, \mathbb{1}\rangle \, d\tau. \qquad (4.3)
\end{aligned}$$

Since $0 \leq x \in L^1 \cap L^\infty$, Fubini's theorem implies

$$\begin{aligned}
&\int_s^{s+t} \langle V(\tau)e^{a(\tau-s)\Delta}x, \mathbb{1}\rangle \, d\tau \\
&= \int_s^{s+t} \int_{\mathbb{R}^N} V(\tau,\xi) \int_{\mathbb{R}^N} K_{a(\tau-s)}(\xi-\eta)x(\eta) \, d\eta \, d\xi \, d\tau \\
&= \int_{\mathbb{R}^N} x(\eta) \left(\int_s^{s+t} \int_{\mathbb{R}^N} K_{a(\tau-s)}(\xi-\eta)V(\tau,\xi) \, d\xi \, d\tau \right) d\eta. \qquad (4.4)
\end{aligned}$$

From (4.3) and (4.4) we infer

$$\int_s^{s+t} \int_{\mathbb{R}^N} K_{a(\tau-s)}(\xi-\eta)V(\tau,\xi) \, d\xi \, d\tau \geq \frac{d-\delta}{M^2} \qquad (4.5)$$

for $s \in I$ and a.e. $\eta \in \mathbb{R}^N$. On the other hand we have

$$\left\| \int_s^{s+t} \int_{\mathbb{R}^N} K_{a(\tau-s)}(\xi-\cdot)V_1(\tau,\xi) \, d\xi \, d\tau \right\|_1 \leq \int_s^{s+t} \|V_1(\tau)\|_1 \, d\tau \leq \tilde{C}\|V_1\|_{L^1_{loc,u}} =: C,$$

where C does not depend on s. Thus the measure of the set

$$A_s := \left\{ \eta \in \mathbb{R}^N : \int_s^{s+t} \int_{\mathbb{R}^N} K_{a(\tau-s)}(\xi-\eta)V_1(\tau,\xi) \, d\xi \, d\tau \geq \frac{d-\delta}{2M^2} \right\}$$

is bounded by $m := \frac{2M^2C}{d-\delta}$ for all $s \in I$. We set $V_\infty := V_2 + V_3$. Then (4.5) implies

$$\int_s^{s+t} \int_{\mathbb{R}^N} K_{a(\tau-s)}(\xi-\eta)V_\infty(\tau,\xi) \, d\xi \, d\tau \geq \frac{d-\delta}{2M^2}$$

for $s \in I$ and $\eta \in \mathbb{R}^N \setminus A_s$. Moreover,

$$\int_s^{s+\varepsilon} \int_{\mathbb{R}^N} K_{a(\tau-s)}(\xi-\eta)V_\infty(\tau,\xi) \, d\xi \, d\tau \leq \int_s^{s+\varepsilon} (\|V_2(\tau)\|_\infty + \|V_3\|_\infty) \, d\tau$$

for $\varepsilon > 0$. By assumption the integral on the right–hand side tends to 0 as $\varepsilon \to 0$ uniformly in $s \in I$. So we can fix $\varepsilon \in (0, t)$ such that

$$\int_{s+\varepsilon}^{s+t} \int_{\mathbb{R}^N} K_{a(\tau-s)}(\xi - \eta) V_\infty(\tau, \xi)\, d\xi\, d\tau \geq \frac{d-\delta}{4M^2}$$

for $s \in I$ and $\eta \in \mathbb{R}^N \setminus A_s$. We further compute

$$\begin{aligned}
\frac{d-\delta}{4M^2} &\leq \int_{s+\varepsilon}^{s+t} \int_{B(\eta,r)} K_{a(\tau-s)}(\xi-\eta) V_\infty(\tau,\xi)\, d\xi\, d\tau \\
&\quad + \int_{s+\varepsilon}^{s+t} \int_{B(\eta,r)^c} K_{a(\tau-s)}(\xi-\eta) V_\infty(\tau,\xi)\, d\xi\, d\tau \\
&\leq \frac{1}{(4\pi a\varepsilon)^{N/2}} \Bigg(\int_{s+\varepsilon}^{s+t} \int_{B(\eta,r)} V_\infty(\tau,\xi)\, d\xi\, d\tau \\
&\quad + \int_{s+\varepsilon}^{s+t} \int_{B(0,r)^c} e^{-\frac{1}{4at}|\xi|^2}\, V_\infty(\tau, \eta-\xi)\, d\xi\, d\tau \Bigg) \\
&\leq \frac{1}{(4\pi a\varepsilon)^{N/2}} \Bigg(\int_{s}^{s+t} \int_{B(\eta,r)} V_\infty(\tau,\xi)\, d\xi\, d\tau \\
&\quad + \int_{s}^{s+t} (\|V_2(\tau)\|_\infty + \|V_3\|_\infty)\, d\tau \int_{B(0,r)^c} e^{-\frac{1}{4at}|\xi|^2}\, d\xi \Bigg)
\end{aligned}$$

for $s \in I$, $r > 0$, and a.e. $\eta \in \mathbb{R}^N \setminus A_s$. Thus we can fix a small constant $c > 0$ and choose $r > 0$ such that

$$\int_{s}^{s+t} \int_{B(\xi,r)} V_\infty(\tau,\eta)\, d\eta\, d\tau \geq c$$

for each $s \in I$ and a.e. $\xi \in \mathbb{R}^N \setminus A_s$, where $\lambda(A_s) \leq m$. Therefore Lemma 4.2 yields $V_\infty \in \mathcal{E}_t$, and hence $V \in \mathcal{E}_t$. □

Notice that potentials V as considered in Remark 4.5(b) satisfy the assumptions of Theorem 4.4 and 4.6. As a consequence, the next result follows immediately.

Corollary 4.7. *Let $V \geq 0$ be as in Remark 4.5(b). Let $\mathcal{U}$ be a submarkovian, strongly continuous evolution family on $L^1(\mathbb{R}^N)$ satisfying an upper and a lower Gaussian estimate. Then $\mathcal{U}_V$ is exponentially stable if and only if $V \in \mathcal{E}_t$ for some $t > 0$.*

The case $X = L^p(\mathbb{R}^N)$ is treated by means of the Riesz–Thorin interpolation theorem and the following lemma whose proof relies on an argument due to W. Arendt, cf. [1, p.1160].

Lemma 4.8. *Let $\mathcal{U} = (U(t,s))_{(t,s)\in D}$ be a positive, strongly continuous evolution family on $L^p(\Omega)$, $1 \le p < \infty$, where $\Omega \subseteq \mathbb{R}^N$ is open. Assume that $\mathcal{U}$ satisfies an upper Gaussian estimate. Then for each $1 \le r < \infty$ there exists a unique strongly continuous evolution family $\mathcal{U}_r$ on $L^r(\Omega)$ which coincides with $\mathcal{U}$ on $L^p(\Omega) \cap L^r(\Omega)$.*

Proof. Due to the upper Gaussian estimate the restriction of $U(t,s)$, $(t,s) \in D$, to $L^p(\Omega) \cap L^r(\Omega)$ has a unique bounded extension $U_r(t,s)$ to $L^r(\Omega)$, $1 \le r < \infty$. Clearly, $\mathcal{U}_r$ is a bounded evolution family. For proving strong continuity of $\mathcal{U}_r$ it is enough to consider $x \in L^p(\Omega) \cap L^r(\Omega)$. So let $D \ni (t_n, s_n) \to (t,s) \in D$ and set $y_n := U(t_n, s_n)x$ and $z_n := M\, K_{a(t_n - s_n)} * |x|$. Observe that it suffices to show that every subsequence of (y_n) has a subsequence which converges in L^r to $y := U(t,s)x$. Since $y_n \to y$ in L^p we see by passing to a subsequence that $y_n \to y$ pointwise a.e.. Taking another subsequence we obtain that $\|z_n - z_{n-1}\|_r \le 2^{-n}$. Let $w := \sum_{n\ge 2} |z_n - z_{n-1}| + |z_1|$. Then $w \in L^r(\mathbb{R}^N)$. Further, $|y_n| \le z_n \le w$. Hence, Lebesgue's theorem implies $y_n \to y$ in L^r. □

Corollary 4.9. *Assume $0 \le V \in \overline{L^\infty(I \times \mathbb{R}^N)_+}^{L^1_{loc,u}}$. Let $\mathcal{U} = (U(t,s))_{(t,s)\in D}$ be a strongly continuous evolution family on $L^p(\mathbb{R}^N)$, $1 < p < \infty$, satisfying an upper and a lower Gaussian estimate. Assume that the induced evolution family $\mathcal{U}_1$ on $L^1(\mathbb{R}^N)$ is submarkovian. If $V \in \mathcal{E}_t$ for some $t > 0$, then the absorption evolution family $\mathcal{U}_V$ on $L^p(\mathbb{R}^N)$ is exponentially stable.*

Proof. By Lemma 4.8 $\mathcal{U}$ induces a strongly continuous evolution family $\mathcal{U}_r$ on $L^r(\mathbb{R}^N)$, $1 \le r < \infty$, satisfying an upper and a lower Gaussian estimate. By Lemma 3.1 there exists a bounded absorption evolution family $\mathcal{U}_{r,V}$ on $L^r(\mathbb{R}^N)$. Choose $r \in (p, \infty)$. By interpolation of $U_{1,V}(t,s)$ and $U_{r,V}(t,s)$ we obtain operators $W(t,s) : L^p(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$, $(t,s) \in D$, which are exponentially stable due to Theorem 4.4. It remains to show that $W(t,s) = U_{p,V}(t,s)$ for $(t,s) \in D$. As in the proof of Theorem 4.4, it suffices to consider $V \in L^\infty(I \times \mathbb{R}^N)$. Then the evolution family $\mathcal{U}_{r,V}$, $1 \le r < \infty$, is strongly continuous and satisfies (2.8) by Proposition 2.3. By interpolation, $\mathcal{W}$ is strongly continuous and satisfies (2.8), too. Consequently, Proposition 2.3 implies $U_{p,V}(t,s) = W(t,s)$. □

Remark 4.10 *(a) The method of the proofs of Theorem 4.4 and 4.6 also works for $X = BUC(\mathbb{R}^N)$ and potentials V as considered in Theorem 4.4 and 4.6 provided that analogues of the results of Section 2 and 3 can be established, see [23, §4.3].*

(b) In the case of $U(t,s) = e^{(t-s)\Delta}$, $X = L^p(\mathbb{R}^N)$, and time independent potentials $V(t) \equiv V_0$ the above characterization was shown by W. Arendt and

C.J.K. Batty for $0 \leq V_0 \in L^1(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$ *in [3, Thm. 1.2]. For arbitrary measurable* $V_0 \geq 0$ *similar results of probabilistic nature can be found in [2], [6]. The characterization was extended by D. Daners and P. Koch Medina, [8, Thm. 5.1], to the time periodic case for* $X = BUC(\mathbb{R}^N)$ *or* $L^1(\mathbb{R}^N)$. *In [12] a generalization to certain periodic evolution families satisfying Gaussian estimates was given. We have used several ideas from [8].*

(c) *If* $U(t,s) = e^{(t-s)\Delta}$ *and* $V(t) \equiv V_0 \in L^1 + L^\infty$, *then in Corollary 4.9 the converse also holds, see [2]. We do not know if this is true in the non–autonomous situation, cf. [8], Remark 5.2.*

5 Applications

In this section we study the initial value problem

$$\begin{aligned} D_t\, u(t,\xi) &= \textstyle\sum_{k,l=1}^N D_k\,(a_{kl}(t,\xi) D_l\, u(t,\xi)) - V(t,\xi)u(t,\xi), \\ u(s,\xi) &= x(\xi), \qquad t \geq s \geq 0,\ \xi \in \mathbb{R}^N, \end{aligned} \tag{5.1}$$

where $x \in L^\infty(\mathbb{R}^N)$ and $D_t = \partial/\partial t$, $D_k = \partial/\partial \xi_k$ in the sense of distributions. We further assume $a_{kl} \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^N, \mathbb{R})$ for $k,l = 1,\cdots,N$ and $\sum_{k,l=1}^N a_{kl}(t,\xi)\eta_k\eta_l \geq \mu\,|\eta|^2$ for $\eta,\xi \in \mathbb{R}^N$, $t \geq 0$, and a constant $\mu > 0$. For $V \equiv 0$ it is known that there is a unique bounded weak solution u of (5.1) (see e.g. [10] for a definition) which is given by

$$u(t,\xi) = \int_{\mathbb{R}^N} k(t,s,\xi,\eta)x(\eta)\,d\eta, \quad \xi \in \Omega,\ t > s, \tag{5.2}$$

where $k : Q := \{(t,s,\xi,\eta) \in \mathbb{R}_+^2 \times \mathbb{R}^{2N} : t > s\} \to \mathbb{R}$ is measurable. The kernel k is called *weak fundamental solution.* There are constants $M, a, \tilde{M}, \tilde{a} > 0$ such that

$$\tilde{M}\, K_{\tilde{a}(t-s)}(\xi - \eta) \leq k(t,s,\xi,\eta) \leq M\, K_{a(t-s)}(\xi - \eta) \tag{5.3}$$

for $t > s$ and a.e. $\xi,\eta \in \mathbb{R}^N$, where K_t, $t > 0$, is the Gaussian kernel. Moreover, k is Hölder continuous uniformly on compact subsets of Q, and we have

$$k(t,s,\xi,\eta) = \int_\Omega k(t,r,\xi,\zeta)k(r,s,\zeta,\eta)\,d\zeta \quad \text{and} \tag{5.4}$$

$$\int_{\mathbb{R}^N} k(t,s,\xi,\eta)\,d\eta = 1 \tag{5.5}$$

on Q. These properties were established by S.D. Eidel'man and F.O. Porper in [10, Ch. 5]. (For further results see the paper [5] by D.G. Aronson, the monographs [11], [13], and the references therein.) We now define

$$U(t,s)x := \int_\Omega k(t,s,\cdot,\eta)x(\eta)\,d\eta$$

for $t > s \geq 0$ and $x \in X = L^p(\mathbb{R}^N)$, $1 \leq p < \infty$. Further, we set $U(s,s) := Id$ for $s \geq 0$. Using the above mentioned properties of k we can check the assumptions on the evolution family $\mathcal{U}$ we have made in the preceding sections. For $X = BUC(\mathbb{R}^N)$ a similar result can be found in [12].

Lemma 5.1. *The above defined family $\mathcal{U} = (U(t,s))_{t \geq s \geq 0}$ is a positive, strongly continuous evolution family on $X = L^p(\mathbb{R}^N)$, $1 \leq p < \infty$. Moreover, $\mathcal{U}$ satisfies an upper and a lower Gaussian estimate and is submarkovian on $L^1(\mathbb{R}^N)$.*

Proof. Since k is measurable and satisfies (5.3), $U(t,s)$ is a bounded operator on X. By (5.4) and (5.3) $\mathcal{U}$ is an evolution family satisfying an upper and a lower Gaussian estimate. Observe that the weak fundamental solution of the adjoint problem, cf. [5, p.621], is given by $\tilde{k}(t,s,\xi,\eta) = k(s,t,\eta,\xi)$ for $0 \leq t < s$ and $\xi, \eta \in \mathbb{R}^N$, [5, Thm. 10]. Moreover, $\tilde{k}$ satisfies (5.5). Hence, $\mathcal{U}$ is (sub-)markovian on $L^1(\mathbb{R}^N)$.

It remains to prove strong continuity of $\mathcal{U}$. Since $\mathcal{U}$ is uniformly bounded, it suffices to consider $x \in C_c(\mathbb{R}^N)$. So fix $x \in C_c(\mathbb{R}^N)$ and let $(t',s') \to (t,s)$ in $D_{\mathbb{R}_+}$. We may assume that $0 \leq t-s, t'-s' \leq c$ for a constant c. As in the proof of Corollary 3.6, due to the upper Gaussian estimate there exists $y \in X$ such that $|U(t',s')x| \leq y$ for $0 \leq t'-s' \leq c$. Consequently, by the dominated convergence theorem, we have to show that

$$U(t',s')x\,(\xi) \to U(t,s)x\,(\xi) \qquad \text{for } \ \xi \in \mathbb{R}^N. \tag{5.6}$$

So let $\xi \in \mathbb{R}^N$. At first, consider $t > s$. We can fix constants $d, \tilde{c} > 0$ such that $\tilde{c} \leq t-s, t'-s' \leq c$ and $|\xi| < d$. Since k is Hölder continuous uniformly in $Q_1 := \{(t',s',\zeta,\eta) \in Q : \tilde{c} \leq t-s, t'-s' \leq c,\ |\zeta| \leq d,\ \eta \in \operatorname{supp} x\}$, there are constants $C \geq 0$ and $0 < \alpha \leq 1$ depending on Q_1 such that

$$\begin{aligned} |U(t,s)x\,(\xi) - U(t',s')x\,(\xi)| &\leq \|x\|_\infty \int_{\operatorname{supp} x} |k(t,s,\xi,\eta) - k(t',s',\xi,\eta)|\, d\eta \\ &\leq C\|x\|_\infty\, \lambda(\operatorname{supp} x)\, (|t-t'|^\alpha + |s-s'|^\alpha). \end{aligned} \tag{5.7}$$

In the case $(t',s') \to (s,s)$ we may assume $t' > s'$. For a given $\varepsilon > 0$, choose $\delta > 0$ such that $|\xi - \eta| \leq \delta$ implies $|x(\xi) - x(\eta)| \leq \varepsilon$. Observe that the Gaussian kernel K_t satisfies

$$K_{a(t'-s')}(\xi - \eta) \leq 2^{\frac{N}{2}}\, e^{-\frac{\delta^2}{8a(t'-s')}}\, K_{2a(t'-s')}(\xi - \eta)$$

for $t' > s'$ and $|\xi - \eta| \geq \delta > 0$. Then by (5.5) and (5.3) we obtain that

$$|U(t',s')x(\xi) - x(\xi)| \;=\; \left| \int_{\mathbb{R}^N} k(t',s',\xi,\eta)(x(\eta) - x(\xi))\, d\eta \right|$$

$$
\begin{aligned}
&\le M \int_{\mathbb{R}^N} K_{a(t'-s')}(\xi-\eta)|x(\eta)-x(\xi)|\,d\eta \\
&\le \varepsilon M + 2M\|x\|_\infty \int_{|\xi-\eta|\ge\delta} K_{a(t'-s')}(\xi-\eta)\,d\eta \\
&\le \varepsilon M + 2M\|x\|_\infty\, 2^{\frac{N}{2}}\, e^{-\frac{\delta^2}{8a(t'-s')}} \int_{\mathbb{R}^N} K_{2a(t'-s')}(\xi-\eta)\,d\eta \\
&= \varepsilon M + 2M\|x\|_\infty\, 2^{\frac{N}{2}}\, e^{-\frac{\delta^2}{8a(t'-s')}}. \qquad (5.8)
\end{aligned}
$$

Now (5.7) and (5.8) yield (5.6). □

A measurable function $u : [s,\infty)\times\mathbb{R}^N \to \mathbb{C}$ is called a *mild solution* of (5.1) if

$$
u(t,\xi) = \int_{\mathbb{R}^N} k(t,\tau,\xi,\eta)x(\eta)\,d\eta - \int_s^t \int_{\mathbb{R}^N} k(t,\tau,\xi,\eta)V(\tau,\eta)u(\tau,\eta)\,d\eta\,d\tau \qquad (5.9)
$$

holds for a.e. (t,ξ). Thanks to Lemma 5.1 we can derive from Corollary 3.6 the existence of a mild solution of (5.1). Moreover, by Corollary 4.7 and 4.9 we find conditions for the exponential stability of such a solution. Recall that $\varepsilon_\alpha(\xi) = e^{-\alpha|\xi|^2}$.

Proposition 5.2. *Let $\mathcal{U}$ be the evolution family on $X = L^p(\mathbb{R}^N)$, $1 \le p < \infty$, as considered in Lemma 5.1. Further, consider a potential $0 \le V \in \bigcap_{\alpha>0} L^1_{loc}(\mathbb{R}_+, L^p(\mathbb{R}^N, \varepsilon_\alpha\lambda))$. Then there is a strongly continuous absorption evolution family $\mathcal{U}_V$ satisfying (2.8) and (2.9) for $x \in L^\infty(\mathbb{R}^N)$ with compact support. For such x, there exists a mild solution u of (5.1) with $u(t,\cdot) = U_V(t,s)x$ for a.e. t.*
If, in addition, $0 \le V \in \overline{L^\infty(\mathbb{R}_+\times\mathbb{R}^N)_+}^{L^1_{loc,u}} \cap \mathcal{E}_t$ for some $t>0$, then $\mathcal{U}_V$ (and hence u) is exponentially stable on X.
If $p=1$ and $V \ge 0$ is given as in Remark 4.5(b), then $\mathcal{U}_V$ (and hence u) is exponentially stable on $L^1(\mathbb{R}^N)$ if and only if $V \in \mathcal{E}_t$ for some $t>0$.

Proof. Due to [4, Lemma 2.2] there is a measurable function $u : [s,\infty)\times\mathbb{R}^N \to \mathbb{C}$ such that $u(t,\cdot) = U_V(t,s)x$. By using [4, Lemma 2.2] we see that (2.8) implies (5.9). □

We note that in [5, Thm. 3, 10] the existence of a unique bounded weak solution of (5.1) on a bounded interval I was shown under a stronger integrability assumption on V. Moreover, this weak solution coincides a.e. with the mild solution obtained above, see [5, Thm. 10].

Finally, we point out that it is possible to derive similar results for uniformly elliptic operators of second order with smooth real coefficients on bounded domains Ω with smooth boundary and Dirichlet boundary conditions. In fact, if one defines $\mathcal{U}$ on $L^p(\Omega)$ as in (5.2) (by replacing the fundamental solution by the Green's function, cf. e.g. [11], [13]) then it can be shown as in Lemma 5.1 that $\mathcal{U}$ is strongly continuous and satisfies an upper Gaussian estimate, see [23, Lemma 3.4].

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