EVOLUTION SEMIGROUPS, TRANSLATION ALGEBRAS, AND EXPONENTIAL DICHOTOMY OF COCYCLES

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Abstract. We study the exponential dichotomy of an exponentially bounded, strongly continuous cocycle over a continuous flow on a locally compact metric space Θ acting on a Banach space X. Our main tool is the associated evolution semigroup on $C_0(\Theta; X)$. We prove that the cocycle has exponential dichotomy if and only if the evolution semigroup is hyperbolic if and only if the imaginary axis is contained in the resolvent set of the generator of the evolution semigroup. To show the latter equivalence, we establish the spectral mapping/annular hull theorem for the evolution semigroup. In addition, dichotomy is characterized in terms of the hyperbolicity of a family of weighted shift operators defined on $c_0(\mathbb{Z}; X)$. Here we develop Banach algebra techniques and study weighted translation algebras that contain the evolution operators. These results imply that dichotomy persists under small perturbations of the cocycle and of the underlying compact metric space. Also, exponential dichotomy follows from pointwise discrete dichotomies with uniform constants. Finally, we extend to our situation the classical Perron theorem which says that dichotomy is equivalent to the existence and uniqueness of bounded, continuous, mild solutions to the inhomogeneous equation.

1. Introduction

Exponential dichotomy of cocycles and nonautonomous Cauchy problems is a classical and well-studied subject, see [13, 14, 17, 18, 30, 35, 42, 43] and the literature cited therein. Recently, important contributions were made in the infinite dimensional case, see [10, 11, 12, 28, 44], and in the applications, [46, 47]. In this paper we continue the investigation, begun in [21, 25], of the dichotomy of cocycles by means of the so-called *evolution semigroup.*

Let $\{\varphi^t\}_{t\in\mathbb{R}}$ be a continuous flow on a locally compact metric space Θ and let $\{\Phi^t\}_{t\in\mathbb{R}_+}$ be an exponentially bounded, strongly continuous cocycle over $\{\varphi^t\}$ with values in the set $\mathcal{L}(X)$ of bounded linear operators on a Banach space X. Cocycles arise, for instance, as solution operators for variational equations

$$\dot{u}(t) = A(\varphi^t\theta)u(t)$$

with possibly unbounded operators $A(\theta)$, see e.g. [21, 42, 44]. On the space $\mathcal{F} = C_0(\Theta; X)$ of continuous functions $f : \Theta \to X$ vanishing at infinity endowed with the sup-norm, we

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define the *evolution semigroup* $\{E^t\}_{t\in\mathbb{R}_+}$ by

$$(E^t f)(\theta) = \Phi^t(\varphi^{-t}\theta) f(\varphi^{-t}\theta) \quad \text{for} \ \ \theta \in \Theta, \ t \geq 0, \ f \in \mathcal{F}. \tag{1.1}$$

The semigroup $\{E^t\}$ is strongly continuous on $C_0(\Theta; X)$. Its generator is denoted by Γ.

We briefly review the history of the subject before we outline the results and methods presented in the current paper. Starting with the pioneering paper [31] by J. Mather, spectral properties of evolution semigroups have been used to characterize exponential dichotomy of the underlying cocycle. In fact, for the finite dimensional case $\dim X < \infty$ and compact Θ, the following *Dichotomy Theorem* is contained in [31]:

$\{\Phi^t\}$ has exponential dichotomy on Θ if and only if $\{E^t\}$ is hyperbolic;

that is, $\sigma(E^1) \cap \mathbb{T} = \emptyset$ for the spectrum $\sigma(E^1)$ and the unit circle $\mathbb{T}$. Moreover, in the finite dimensional, compact setting it was proved in [7, 8, 9, 19] that the semigroup $\{E^t\}$ satisfies the *spectral mapping theorem*

$$\sigma(E^t)\backslash\{0\} = \exp(t\sigma(\Gamma)) \qquad \text{for} \ \ t \geq 0 \tag{1.2}$$

provided that the flow $\{\varphi^t\}$ is *aperiodic* (i.e., $\mathrm{Int}\{\theta \in \Theta : \varphi^T\theta = \theta \text{ for some } T > 0\} = \emptyset$), and the *annular hull theorem*

$$(\sigma(E^t)\backslash\{0\}) \cdot \mathbb{T} = \exp(t\sigma(\Gamma)) \cdot \mathbb{T} \qquad \text{for} \ \ t \geq 0 \tag{1.3}$$

with no restrictions on the flow $\{\varphi^t\}$. We point out that the spectral mapping theorem *does not hold* without the assumption of the aperiodicity of the flow, see [9, Ex. 2.3]. As a consequence, the dichotomy of the cocycle is equivalent to the spectral condition $\sigma(\Gamma) \cap i\mathbb{R} = \emptyset$. An essential step in the proof of these results was a proposition from [29] known as the *Mañe Lemma*, see also [7, 8, 9]. It says that 1 belongs to the approximate point spectrum $\sigma_{ap}(E^1)$ if and only if there exists a point $\theta_0 \in \Theta$, called the *Mañe point*, such that $\sup\{|\Phi^t(\theta_0)x_0| : t \in \mathbb{R}\}$ is finite for a nonzero vector x_0. We stress that, in general, the Mañe Lemma does not hold in the infinite dimensional or/and locally compact setting due to the noncompactness of the unit sphere in X or/and Θ and to the fact that $\{\Phi^t\}$ is defined only for $t \geq 0$.

The operator $E^1 =: E$ can be expressed as a weighted translation operator $E = aV$ on $\mathcal{F}$, where $a(\theta) = \Phi^1(\varphi^{-1}\theta)$ and $(Vf)(\theta) = f(\varphi^{-1}\theta)$. An important C^*-algebra technique for the study of weighted translation operators E on the space $L^2(\Theta; X)$ was developed in [1, 2]. In particular, the hyperbolicity of the operator E was related to that of a family of weighted shift operators $\pi_\theta(E)$ acting on a space of X-valued sequences $\bar{x} = (x_n)_{n\in\mathbb{Z}}$ by the rule

$$\pi_\theta(E) : (x_n)_{n\in\mathbb{Z}} \mapsto (\Phi^1(\varphi^{n-1}\theta)x_{n-1})_{n\in\mathbb{Z}} = \mathrm{diag}((a(\varphi^n\theta))_{n\in\mathbb{Z}}\, S\, (x_n)_{n\in\mathbb{Z}} \tag{1.4}$$

for $\theta \in \Theta$, where $S(x_n)_{n\in\mathbb{Z}} = (x_{n-1})_{n\in\mathbb{Z}}$.

For compact Θ and infinite dimensional X, the evolution semigroup (1.1) was considered in [25] for eventually norm-continuous in θ cocycles, cf. Section 3.3, and in [40] for strongly continuous cocycles with invertible values. The proof of the Dichotomy Theorem from [40] requires just a straightforward modification to work in our setting.

The special case of the translation flow $\varphi^t\theta = \theta + t$ on $\Theta = \mathbb{R}$ is well understood by now, see [4, 5, 20, 22, 34, 36, 37, 39]. Here, the cocycle is given by $\Phi^t(\theta) = U(\theta + t, \theta)$ for a strongly continuous evolution family $\{U(t,s)\}_{t\geq s}$ on a Banach space X which can be thought as solution operator for the evolution equation $\dot{u}(t) = A(t)u(t)$ with unbounded operators $A(t)$. We refer to [45] for applications of the spectral theory of evolution semigroups to parabolic evolution equations.

In [21] and [24] the general locally compact and infinite dimensional setting was investigated under the additional and restrictive assumption that the underlying flow $\{\varphi^t\}$ is aperiodic. In particular, the spectral mapping theorem (1.2) was established in [21] for $\mathcal{F} = C_0(\Theta; X)$ and in [24] for $\mathcal{F} = L^p(\Theta; X)$. Also, the following Discrete Dichotomy Theorem was proved in [21]: The semigroup $\{E^t\}$ is hyperbolic on $C_0(\Theta; X)$ if and only if $\sigma(\pi_\theta(E)) \cap \mathbb{T} = \emptyset$ for each $\theta \in \Theta$ on the space $c_0(\mathbb{Z}; X)$ of X-valued sequences vanishing at infinity and

$$\sup_{z\in\mathbb{T},\theta\in\Theta} \|[z - \pi_\theta(E)]^{-1}\|_{\mathcal{L}(c_0(\mathbb{Z};X))} < \infty.$$

Via the Dichotomy Theorem, the Discrete Dichotomy Theorem relates "global" dichotomy of $\{\Phi^t\}$ on Θ and "pointwise" dichotomies at $\theta \in \Theta$, cf. [10, 18].

The proof of the Discrete Dichotomy Theorem in [21] required a further development of the algebraic techniques from [2, 20, 22, 25], see also [16]. Namely, the weighted translation operator E was "immersed" in the algebra $\mathfrak{B}$ of operators $b \in \mathcal{L}(C_0(\Theta; X))$ of the form

$$b = \sum_{k=-\infty}^{\infty} a_k V^k \quad \text{with} \quad \|b\|_1 := \sum_{k=-\infty}^{\infty} \|a_k\|_{\mathcal{L}(C_0(\Theta;X))} < \infty,$$

where a_k are the multiplication operators induced by strongly continuous, bounded functions $a_k : \Theta \to \mathcal{L}(X)$. The map $\pi_\theta : E \mapsto \pi_\theta(E)$ was extended to a representation of $\mathfrak{B}$ in $\mathcal{L}(c_0(\mathbb{Z}; X))$. We stress that without the aperiodicity assumption on $\{\varphi^t\}$ the algebra $\mathfrak{B}$ cannot even be defined. Thus, a generalization to arbitrary flows requires a radical modification of the method.

In the present paper, we consider evolution semigroups $\{E^t\}$ and cocycles $\{\Phi^t\}$ without any additional assumptions on $\{\varphi^t\}$. We recall necessary background material and prove the Dichotomy Theorem for strongly continuous cocycles in Section 2. The persistence of dichotomy under small perturbations of the cocycle follows immediately from this result, cf. [10].

A modification of the Banach algebra techniques mentioned above is developed in Section 3. Here, we replace the algebra $\mathfrak{B}$ by a certain convolution algebra $\hat{\mathfrak{B}}$. This allows to prove the Discrete Dichotomy Theorem. In addition, for $\theta \in \Theta$, we introduce a family of strongly continuous semigroups $\{\Pi_\theta^t\}_{t\in\mathbb{R}_+}$ on $C_0(\mathbb{R}; X)$ by setting

$$(\Pi_\theta^t h)(s) = \Phi^t(\varphi^{s-t}\theta)h(s-t) \quad \text{for} \;\; s \in \mathbb{R},\; t \geq 0,\; h \in C_0(\mathbb{R}; X).$$

Each of the semigroups $\{\Pi_\theta^t\}$ is an evolution semigroup in the sense of [20, 37]. If $\Phi^t(\theta)$ is the solution operator for the variational equation $\dot{u}(t) = A(\varphi^t\theta)u(t)$, then the generator Γ_θ of $\{\Pi_\theta^t\}$ is the closure of the operator $h \mapsto \left[-\frac{d}{ds} + A(\varphi^\cdot\theta)\right] h$, see [21]. We give an analogue of the Discrete Dichotomy Theorem for $\{\Pi_\theta^t\}$. As an application, we derive for eventually norm-continuous in θ cocycles an infinite dimensional generalization of the Sacker-Sell Perturbation Theorem [43, Thm. 6] on the semicontinuity of the Sacker-Sell dynamical spectrum as a function of the underlying compact set Θ.

In Section 4, we prove the spectral mapping theorem for aperiodic flows and the annular hull theorem provided that the function of prime periods is strictly separated from zero. (In fact, we assume somewhat weaker properties of the flow.) Note that the last assumption is needed for the Annular Hull Theorem 1.3 if $\dim X = \infty$. Counterexamples are furnished by the identical flow on a singleton and the cocycle given by a semigroup violating the annular hull theorem, see [33, 34] for such semigroups. We stress that the annular hull theorem does not hold even for the semigroup related to a first-order perturbation of the two-dimensional wave equation, see [41]. In the proof of the annular hull and spectral mapping theorems for the evolution semigroups we give an explicit construction of approximate eigenfunctions for Γ supported in a tube along certain trajectories of the flow. This localization method can be traced as far as to [31]. The idea of our construction goes back to the finite-dimensional case [7, 8, 9], but uses essentially different infinite dimensional methods of [20, 37]. The choice of the trajectories is related to the existence of *Mañe sequences* $\{\theta_m\} \subset \Theta$ which we define in this paper as an infinite dimensional generalization of the Mañe points mentioned above. It is easy to see that $\sigma_{ap}(E) \cap \mathbb{T} \neq \emptyset$ implies the existence of a Mañe sequence. As a generalization of the Mañe Lemma, we show that the existence of a Mañe sequence implies $\sigma_{ap}(E) \cap \mathbb{T} \neq \emptyset$ under an additional technical condition related to the residual spectrum.

Finally, we characterize in Section 5 the dichotomy of $\{\Phi^t\}$ by the existence and uniqueness of bounded, continuous solutions to the mild inhomogeneous equation

$$u(\varphi^t\theta) = \Phi^t(\theta)u(\theta) + \int_0^t \Phi^{t-\tau}(\varphi^\tau\theta)g(\varphi^t\theta)d\theta, \quad t \geq 0,\ \theta \in \Theta,$$

in $C_0(\Theta; X)$ or $C_b(\Theta; X)$. In $C_0(\Theta; X)$, this equation is just a reformulation of $\Gamma u = -g$. Our result gives an affirmative answer to a question remained unsolved in [23] and can be regarded as a far-reaching generalization of a classical theorem by O. Perron, see [13, 14, 23, 27, 32] for further comments and related results.

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Notation. We denote: $\mathbb{T}$ – unit circle in $\mathbb{C}$; $\mathbb{D}$ – open unit disc in $\mathbb{C}$; X – Banach space with norm $|\cdot|$; $\mathcal{L}(X)$ – bounded linear operators on X; $\mathcal{L}_s(X)$ – bounded linear operators on X with the strong topology; I – the identity operator; $\rho(A)$, $\sigma(A)$, $\sigma_{ap}(A)$, $\sigma_r(A)$ – resolvent set, spectrum, approximate point spectrum, residual spectrum of A.

2. The Dichotomy Theorem

We briefly introduce exponential dichotomy of cocycles (see the classical sources [17, 18, 42, 43] and [10, 44]) and evolution semigroups (see [21, 25, 40]), and prove the equivalence of the exponential dichotomy of a cocycle and the hyperbolicity of the corresponding evolution semigroup.

2.1. **Exponential dichotomy.** Let $\{\varphi^t\}_{t\in\mathbb{R}}$ be a *continuous flow* on a locally compact metric space Θ; that is, the map $\mathbb{R} \times \Theta \ni (t, \theta) \mapsto \varphi^t\theta \in \Theta$ is continuous and $\varphi^{t+\tau}\theta = \varphi^t(\varphi^\tau\theta)$, $\varphi^0\theta = \theta$ for all $t, \tau \in \mathbb{R}$ and $\theta \in \Theta$. Let $\{\Phi^t\}_{t\geq 0}$ be an *exponentially bounded, strongly continuous cocycle* over $\{\varphi^t\}_{t\in\mathbb{R}}$. This means that the map $\mathbb{R}_+ \times \Theta \ni (t, \theta) \mapsto \Phi^t(\theta) \in \mathcal{L}(X)$ is strongly continuous, $\|\Phi^t(\theta)\| \leq Ne^{\omega t}$ for $t \geq 0$, $\theta \in \Theta$, and some constants N and ω, and the following cocycle property holds:

$$\Phi^{t+\tau}(\theta) = \Phi^t(\varphi^\tau\theta)\Phi^\tau(\theta) \quad \text{and} \quad \Phi^0(\theta) = I \quad \text{for} \quad t, \tau \geq 0 \text{ and } \theta \in \Theta. \tag{2.1}$$

We will also consider cocycles for *discrete time* $t \in \mathbb{N}$ which are defined analogously. Cocycles $\{\Phi^t\}_{t\geq 0}$ are in one-to-one correspondence with *continuous linear skew-product flows* (LSPF) $\{\hat{\varphi}^t\}_{t\geq 0}$ defined by

$$\hat{\varphi}^t : \Theta \times X \to \Theta \times X : (\theta, x) \mapsto (\varphi^t\theta, \Phi^t(\theta)x) \quad \text{for} \quad t \geq 0.$$

Therefore, all assertions below concerning cocycles can be reformulated in terms of the corresponding linear skew-product flows.

Definition 2.1. *The cocycle* $\{\Phi^t\}_{t\geq 0}$ *has* exponential dichotomy *on* Θ *if there exists a bounded, strongly continuous, projection valued function* $P : \Theta \to \mathcal{L}_s(X)$ *satisfying*

(a) $P(\varphi^t\theta)\Phi^t(\theta) = \Phi^t(\theta)P(\theta)$,

(b) $\Phi^t_Q(\theta)$ *is invertible as an operator from* $\operatorname{Im} Q(\theta)$ *to* $\operatorname{Im} Q(\varphi^t\theta)$ *(with inverse* $\Phi^{-t}_Q(\varphi^t\theta)$*),*

(c) *there exist constants* $\beta > 0$ *and* $M = M(\beta) \geq 0$ *such that*

$$\|\Phi^t_P(\theta)\| \leq Me^{-\beta t} \quad \text{and} \quad \|[\Phi^t_Q(\theta)]^{-1}\| \leq Me^{-\beta t}$$

for all $\theta \in \Theta$ *and* $t \geq 0$. *The function* $P(\cdot)$ *is called* dichotomy projection *and the constants* β, M *are the* dichotomy constants.

Here and below, $\Phi^t_P(\theta)$ and $\Phi^t_Q(\theta)$ denote the restrictions $\Phi^t(\theta)P(\theta) : \operatorname{Im} P(\theta) \to \operatorname{Im} P(\varphi^t\theta)$ and $\Phi^t(\theta)Q(\theta) : \operatorname{Im} Q(\theta) \to \operatorname{Im} Q(\varphi^t\theta)$, respectively, and we set $Q = I - P$ for a projection P. Notice that $\Phi^t(\theta)Q(\theta)$ is strongly continuous on $\mathbb{R} \times \Theta$ and satisfies (2.1) with I replaced by $Q(\theta)$. Further, we define dichotomy along single orbits.

Definition 2.2. *The cocycle* $\{\Phi^t\}_{t\in\mathbb{K}_+}$ *has* exponential dichotomy (over $\mathbb{K} = \mathbb{R}$ or $\mathbb{Z}$) at a point $\theta_0 \in \Theta$ *if there are uniformly bounded projections* $P_\tau \in \mathcal{L}(X)$ *for* $\tau \in \mathbb{K}$ *which depend strongly continuous on* τ *if* $\mathbb{K} = \mathbb{R}$ *and satisfy*

(a) $P_{\tau+t}\Phi^t(\varphi^\tau\theta_0) = \Phi^t(\varphi^\tau\theta_0)P_\tau$,

(b) $\Phi^t_Q(\varphi^\tau\theta_0)$ *is invertible from* $\operatorname{Im} Q_\tau$ *to* $\operatorname{Im} Q_{\tau+t}$,

(c) *there exist constants* $\beta = \beta(\theta_0) > 0$ *and* $M = M(\beta) \geq 0$ *such that*

$$\|\Phi^t_P(\varphi^\tau\theta_0)\| \leq Me^{-\beta t} \quad \text{and} \quad \|[\Phi^t_Q(\varphi^\tau\theta_0)]^{-1}\| \leq Me^{-\beta t}$$

for $t, \tau \in \mathbb{K}$ *and* $t \geq 0$.

Clearly, dichotomy on Θ implies dichotomy at θ_0. If $\operatorname{Im} Q(\theta)$ and $\operatorname{Im} Q(\varphi^t\theta)$ are finite dimensional, then (b) and the second estimate in (c) follow from

$$|\Phi^t(\theta)x| \geq \tfrac{1}{M}\, e^{\beta t}|x| \quad \text{for} \quad x \in \operatorname{Im} Q(\theta),\ t \geq 0.$$

If each operator $\Phi^t(\theta)$ is invertible, exponential dichotomy at θ_0 is equivalent to the existence of a projection P such that

$$\|\Phi^t(\theta_0)P[\Phi^\tau(\theta_0)]^{-1}\| \leq Me^{-\beta(t-\tau)}, \quad t \geq \tau,$$
$$\|\Phi^t(\theta_0)Q[\Phi^\tau(\theta_0)]^{-1}\| \leq Me^{-\beta(\tau-t)}, \quad t \leq \tau,$$

cf. [40]. Then one has $P_\tau = \Phi^\tau(\theta)P_0[\Phi^\tau(\theta)]^{-1}$. This is the classical definition of exponential dichotomy, see [13, 14, 42]. We also recall the analogous concept for a single operator.

Definition 2.3. *A operator* $E \in \mathcal{L}(X)$ *is called* hyperbolic *if* $\sigma(E) \cap \mathbb{T} = \emptyset$.

Observe that the powers $\{E^n\}_{n\in\mathbb{N}}$ or a semigroup $\{E^t\}_{t\geq 0}$ have exponential dichotomy if and only if E or E^{t_0} for some $t_0 > 0$ are hyperbolic, respectively. In this case, the dichotomy projection is given by the *Riesz projection*

$$P = \frac{1}{2\pi i}\int_{\mathbb{T}} (\lambda I - E)^{-1}\, d\lambda.$$

We denote by $E^t_P = E^tP$ and $E^t_Q = E^tQ$ the restriction to the stable and the unstable subspace, $\operatorname{Im} P$ and $\operatorname{Im} Q$, respectively; and analogously for the generator of the semigroup, cf. [33, A-III].

For an exponentially bounded, strongly continuous cocycle $\{\Phi^t\}$ and $\lambda \in \mathbb{C}$, we define the rescaled cocycle $\Phi^t_\lambda(\theta) = e^{-\lambda t}\Phi^t(\theta)$ and the rescaled linear skew-product flow

$$\hat{\varphi}^t_\lambda : (\theta, x) \mapsto (\varphi^t\theta, e^{-\lambda t}\Phi^t(\theta)x) \qquad \text{for } \theta \in \Theta,\ x \in X,\ t \geq 0.$$

Note that the cocycle Φ^t_λ corresponds to the equation $\dot{u}(t) = [A(\varphi^t\theta) - \lambda]u(t)$.

Definition 2.4. *The* dynamical (or Sacker-Sell) spectrum Σ *is defined by*

$$\Sigma = \{\lambda \in \mathbb{R} | \quad \{\Phi^t_\lambda\}_{t\geq 0} \quad \textit{does not have exponential dichotomy}\}.$$

2.2. **Evolution semigroups.** By $\mathcal{F} = C_0(\Theta; X)$ we denote the space of continuous functions $f : \Theta \to X$ vanishing at infinity endowed with the sup-norm. For a cocycle $\{\Phi^t\}_{t\geq 0}$, we introduce on $C_0(\Theta; X)$ the *evolution semigroup* $\{E^t\}_{t\geq 0}$ by setting

$$(E^tf)(\theta) = \Phi^t(\varphi^{-t}\theta)f(\varphi^{-t}\theta) \qquad \text{for } \theta \in \Theta,\ t \geq 0,\ f \in C_0(\Theta; X).$$

One checks the following simple fact, cf. [21, Thm. 2.1], [40, Prop. 3].

Proposition 2.5. *$\{E^t\}_{t\geq 0}$ is a strongly continuous semigroup on $C_0(\Theta; X)$ if and only if $\{\Phi^t\}_{t\geq 0}$ is an exponentially bounded, strongly continuous cocycle.*

The following result is due to R. Rau, [40, Lemma 7], see [2, 20, 21, 25, 36] for earlier and different versions.

Theorem 2.6 (Spectral Projection Theorem)**.** *Let $\{\Phi^t\}_{t\geq 0}$ be an exponentially bounded, strongly continuous cocycle over a continuous flow $\{\varphi^t\}_{t\in\mathbb{R}}$ on Θ. If the induced evolution semigroup $\{E^t\}$ is hyperbolic on $C_0(\Theta; X)$, then there exists a bounded, strongly continuous, projection valued function $P : \Theta \to \mathcal{L}_s(X)$ such that the Riesz projection $\mathcal{P}$ for E^t corresponding to $\sigma(E^t) \cap \mathbb{D}$ is given by the formula $(\mathcal{P}f)(\theta) = P(\theta)f(\theta)$ for $\theta \in \Theta$ and $f \in C_0(\Theta; X)$.*

The idea of the proof is to check, see Lemma 3.2 in [20], that $\mathcal{P}$ commutes with the multiplication operators induced by continuous, bounded functions $\chi : \Theta \to \mathbb{R}$. This allows to define $P(\theta_0)x = (\mathcal{P}f)(\theta_0)$ for $\theta_0 \in \Theta$ and $x \in X$, where $f \in C_0(\Theta; X)$ is an arbitrary function such that $f(\theta_0) = x$.

We now come to the first main result; our proof is a modification of [40, Thm. 10,12].

Theorem 2.7 (Dichotomy Theorem)**.** *Let $\{\Phi^t\}_{t\geq 0}$ be an exponentially bounded, strongly continuous cocycle over a continuous flow $\{\varphi^t\}_{t\geq 0}$ on a locally compact metric space Θ, and let $\{E^t\}_{t\geq 0}$ be the induced evolution semigroup on $C_0(\Theta; X)$. Then the cocycle has exponential dichotomy on Θ if and only if the evolution semigroup is hyperbolic. In this case, the Riesz projection $\mathcal{P}$ corresponding to the spectral set $\sigma(E^1)\cap\mathbb{D}$ and the dichotomy* *projection $P : \Theta \to \mathcal{L}_s(X)$ of $\{\Phi^t\}$ are related by $(\mathcal{P}f)(\theta) = P(\theta)f(\theta)$ for $\theta \in \Theta$ and $f \in C_0(\Theta; X)$.*

Proof. (1) Assume that $\{\Phi^t\}$ has exponential dichotomy with projections $P(\theta)$. Define $\mathcal{P} = P(\cdot)$ on $C_0(\Theta; X)$. Then $\mathcal{P}E^t = E^t\mathcal{P}$ by (a) in Definition 2.1. The inequality $\|E^t_P\|_{\mathcal{L}(C_0(\Theta;X))} \le Me^{-\beta t}$ follows from the first inequality in (c). Also, E^t_Q on $\operatorname{Im}\mathcal{Q}$ has the inverse $(R^t f)(\theta) = [\Phi^t_Q(\theta)]^{-1} f(\varphi^t\theta)$. Now, (c) yields $\|[E^t_Q]^{-1}\|_{\mathcal{L}(C_0(\Theta;X))} \le Me^{-\beta t}$ for $t \ge 0$. This means that $\{E^t\}$ is hyperbolic.

(2) Assume that $\{E^t\}$ is hyperbolic with Riesz projection $\mathcal{P}$. The Spectral Projection Theorem 2.6 shows that $(\mathcal{P}f)(\theta) = P(\theta)f(\theta)$ for some bounded, strongly continuous, projection valued function $P : \Theta \to \mathcal{L}_s(X)$. Since $\mathcal{P}$ is the Riesz projection for E^1, E^t_Q is invertible on $\operatorname{Im}\mathcal{Q}$ and there exist constants $\beta > 0$ and $M > 0$ such that

$$\|E^t_P\|_{\mathcal{L}(C_0(\Theta;X))} \le Me^{-\beta t}, \quad t \ge 0, \tag{2.2}$$

$$\|E^t_Q\mathcal{Q}f\|_\infty \ge \tfrac{1}{M}\, e^{\beta t}\|\mathcal{Q}f\|_\infty, \quad t \ge 0. \tag{2.3}$$

Clearly, (a) in Definition 2.1 holds. Let $\theta \in \Theta$ and $x \in X$. Take a function $f \in C_0(\Theta; X)$ such that $f(\theta) = x$ and $\|f\|_\infty = |x|$. Using (2.2), we obtain

$$|\Phi^t_P(\theta)x| \le \|\Phi^t_P(\cdot)f(\cdot)\|_\infty = \|E^t_P f\|_\infty \le Me^{-\beta t}\|f\|_\infty = Me^{-\beta t}|x|$$

which gives the first inequality in (c). To prove (b), fix $t \ge 0$, $\theta \in \Theta$, and $x \in \operatorname{Im} Q(\theta)$. Recall that for any $f \in \operatorname{Im}\mathcal{Q}$ we have $\|E^t_Q f\|_\infty = \sup_{\theta\in\Theta}|\Phi^t_Q(\theta)f(\theta)|$. For any $\epsilon > 0$ choose $f \in \operatorname{Im}\mathcal{Q}$ such that $f(\theta) = x$ and $\|E^t_Q f\|_\infty \le |\Phi^t(\theta)x| + \epsilon$. This function can be chosen as $f = \chi x$ where χ is a continuous bump-function with $\chi(\theta) = 1$ and a sufficiently small support. Then (2.3) implies

$$\tfrac{1}{M}e^{\beta t}|x| \le \tfrac{1}{M}e^{\beta t}\|f\|_\infty \le \|E^t_Q f\|_\infty \le |\Phi^t(\theta)x| + \varepsilon,$$

and hence

$$|\Phi^t_Q(\theta)x| \ge \tfrac{1}{M}e^{\beta t}|x|. \tag{2.4}$$

In particular, the operator $\Phi^t_Q(\theta) : \operatorname{Im} Q(\theta) \to \operatorname{Im} Q(\varphi^t\theta)$ is injective. To see that it is surjective, take $y \in \operatorname{Im} Q(\varphi^t\theta)$. For a continuous, compactly supported function $\alpha : \Theta \mapsto [0, 1]$ with $\alpha(\varphi^t\theta) = 1$, set $f = \alpha(\cdot)Q(\cdot)y$. Then $f \in \operatorname{Im}\mathcal{Q}$. Since E^t_Q is invertible on $\operatorname{Im}\mathcal{Q}$, there exists $g \in \operatorname{Im}\mathcal{Q}$ such that $E^t\mathcal{Q}g = f$ For $x := g(\theta) \in \operatorname{Im} Q(\theta_0)$, this gives

$$\Phi^t_Q(\theta)x = \Phi^t_Q(\theta)g(\theta) = f(\varphi^t\theta) = \alpha(\varphi^t\theta)Q(\varphi^t\theta)y = y.$$

Thus, (b) in Definition 2.1 holds. Together with (2.4) this establishes the second inequality in (c) of this definition. $\square$

Corollary 2.8. *The dichotomy projection is uniquely determined by Definition 2.1(a)–(c).*

Theorem 2.7 combined with the argument used in part (1) of its proof shows that the existence of a dichotomy over $\mathbb{R}$ and $\mathbb{Z}$ for the cocycle $\{\Phi^t\}$ are equivalent:

Corollary 2.9. *Given a cocycle $\{\Phi^t\}_{t\ge0}$ the following assertions are equivalent:*

(1) *$\{\Phi^t\}_{t\ge0}$ has an exponential dichotomy over $\mathbb{R}$;*
(2) *the discrete time cocycle $\{\Phi^t\}_{t\in\mathbb{N}}$ has dichotomy.*

The Dichotomy Theorem 2.7 relates our approach with the Sacker-Sell spectral theory, [43], for infinite dimensional cocycles or linear skew-product flows. In fact, by rescaling one obtains

$$\Sigma = \log|\sigma(E^1) \setminus \{0\}|,$$

cf. [25, 40]. One can also derive robustness of dichotomy from Theorem 2.7, see also [10, 12, 13, 14, 18, 21, 22, 30, 45]. We present the following result in this direction.

Corollary 2.10. *Assume that the exponentially bounded, strongly continuous cocycle $\{\Phi^t\}_{t\geq 0}$ has exponential dichotomy. There exists $\varepsilon > 0$ such that every exponentially bounded, strongly continuous cocycle $\{\Psi^t\}_{t\geq 0}$ with*

$$\sup_{\theta\in\Theta} \|\Phi^t(\theta) - \Psi^t(\theta)\|_{\mathcal{L}(X)} < \varepsilon$$

for some $t > 0$ also has exponential dichotomy.

Proof. Let $\{E^t\}$ and $\{F^t\}$ denote the evolution semigroups induced by $\{\Phi^t\}$ and $\{\Psi^t\}$, respectively. Assume that

$$\sup_{\theta\in\Theta} \|\Phi^t(\theta) - \Psi^t(\theta)\|_{\mathcal{L}(X)} < \inf_{\lambda\in\mathbb{T}} \|(\lambda - E^t)^{-1}\|^{-1} =: \varepsilon.$$

Then, using $\lambda - F^t = [1 - (F^t - E^t)(\lambda - E^t)^{-1}]\,(\lambda - E^t)$ and

$$\|E^t - F^t\|_{\mathcal{L}(C_0(\Theta;X))} = \sup_{\theta\in\Theta} \|\Phi^t(\theta) - \Psi^t(\theta)\|_{\mathcal{L}(X)},$$

we derive that F^t, and hence $\{\Psi^t\}$, has exponential dichotomy. □

3. Global and Pointwise Dichotomies

We now want to characterize the (global) dichotomy of $\{\Phi^t\}$ on Θ by the (local) dichotomy at each $\theta_0 \in \Theta$ with uniform dichotomy constants. To that purpose, we develop Banach algebra techniques which allow one to describe the dichotomy of $\{\Phi^t\}$ (or, equivalently, the hyperbolicity of $\{E^t\}$) in terms of the hyperbolicity of the discrete operators $\pi_\theta(E)$ on $c_0(\mathbb{Z}; X)$ defined in (1.4).

3.1. **Weighted translation algebras.** For a continuous flow $\{\varphi^t\}_{t\in\mathbb{R}}$ on a locally compact metric space Θ, let us define the translation group $(V^t f)(\theta) = f(\varphi^{-t}\theta)$ for $t \in \mathbb{R}$ on the space $C_0(\Theta; X)$. Set $a_t(\theta) = \Phi^t(\varphi^{-t}\theta)$ for $t \geq 0$, $\theta \in \Theta$, and an exponentially bounded, strongly continuous cocycle $\{\Phi^t\}_{t\geq 0}$ over $\{\varphi^t\}_{t\in\mathbb{R}}$. We abbreviate $V = V^1$, $E = E^1$, $\varphi = \varphi^1$, $\Phi = \Phi^1$.

We denote by $\tilde{\mathfrak{A}} = C_b(\Theta; \mathcal{L}_s(X))$ the algebra of bounded, strongly continuous, operator valued functions $a = a(\cdot)$ on Θ with the sup-norm $\|a\|_{\tilde{\mathfrak{A}}} = \|a\|_\infty = \sup_{\theta\in\Theta} \|a(\theta)\|_{\mathcal{L}(X)}$ and point-wise multiplication. Clearly, $\tilde{\mathfrak{A}}$ can be isometrically identified with the subalgebra $\mathfrak{A}$ of $\mathcal{L}(C_0(\Theta; X))$ which contains the multiplication operators (denoted simply by a) given by $(af)(\theta) = a(\theta)f(\theta)$ for $\theta \in \Theta$ and $f \in C_0(\Theta; X)$.

Let $\hat{\mathfrak{B}}$ denote the set of all sequences $(a_k)_{k\in\mathbb{Z}}$ with entries $a_k \in \tilde{\mathfrak{A}}$ satisfying $(\|a_k\|_{\tilde{\mathfrak{A}}})_{k\in\mathbb{Z}} \in \ell^1(\mathbb{Z})$. Equipped with the norm

$$\|(a_k)_{k\in\mathbb{Z}}\|_1 := \sum_{k=-\infty}^{\infty} \|a_k\|_{\tilde{\mathfrak{A}}} = \sum_{k=-\infty}^{\infty} \sup_{\theta\in\Theta} \|a_k(\theta)\|_{\mathcal{L}(X)}\,,$$

$\hat{\mathfrak{B}}$ is a Banach space. For $\hat{b}' = (a'_n)_{n\in\mathbb{Z}}$ and $\hat{b}'' = (a''_n)_{n\in\mathbb{Z}}$ in $\hat{\mathfrak{B}}$, we define $\hat{b} := \hat{b}' * \hat{b}''$ by $\hat{b} = (a_n)_{n\in\mathbb{Z}}$ and $a_n(\theta) = \sum_{k\in\mathbb{Z}} a'_k(\theta)a''_{n-k}(\varphi^{-k}\theta)$ for $\theta \in \Theta$ and $n \in \mathbb{Z}$. Notice that

$$\|a_n(\theta)\|_{\mathcal{L}(X)} \le \|\hat{b}'\|_1 \, \|\hat{b}''\|_1.$$

The function $\theta \mapsto a'_k(\theta)a''_{n-k}(\varphi^{-k}\theta)x$ is continuous for $x \in X$ and $k, n \in \mathbb{Z}$. We have

$$\begin{aligned} |[a_n(\theta_1) - a_n(\theta_2)]x| \le & \sum_{|k|\le N} |\,[a'_k(\theta_1)a''_{n-k}(\varphi^{-k}\theta_1) - a'_k(\theta_2)a''_{n-k}(\varphi^{-k}\theta_2)]x\,| \\ & + 2\,|x|\,\|\hat{b}''\|_1 \sum_{|k|>N} \|a'_k\|_\infty\,. \end{aligned}$$

for $\theta_1, \theta_2 \in \Theta$ and $N \in \mathbb{N}$. Thus $a_n \in \tilde{\mathfrak{A}}$. We further estimate

$$\|\hat{b}\|_1 \le \sum\nolimits_n \sum\nolimits_k \|a'_k\|_\infty \, \|a''_{n-k} \circ \varphi^{-k}\|_\infty = \sum\nolimits_k \sum\nolimits_n \|a'_k\|_\infty \, \|a''_{n-k}\|_\infty \le \|\hat{b}'\|_1 \, \|\hat{b}''\|_1.$$

Therefore, $(\hat{\mathfrak{B}}, *, \|\cdot\|_1)$ is a Banach algebra with the unit element $\hat{e} = e \otimes \delta_{n0}$. Here and below, $e(\theta) = I_X$ for $\theta \in \Theta$ and δ_{nk} is the Kronecker delta. In other words, $\hat{e} = (e_n)$ where $e_0 = I$ and $e_n = 0$ for $n \ne 0$.

To relate the algebra $\hat{\mathfrak{B}}$ with $\mathcal{L}(C_0(\Theta; X)))$, we introduce the (algebra) homomorphism

$$\rho : \hat{\mathfrak{B}} \to \mathcal{L}(C_0(\Theta; X))) : \hat{b} = (a_k)_{k\in\mathbb{Z}} \mapsto \sum_{k=-\infty}^{\infty} a_k V^k. \tag{3.1}$$

Clearly, $\rho(\hat{e}) = I$ and $\|\rho(\hat{b})\|_{\mathcal{L}(C_0(\Theta;X))} \le \|\hat{b}\|_1$. We stress that, in general, ρ is not injective. As an example, let $\varphi^1\theta = \theta$ for all $\theta \in \Theta$. Then, $V = I$ and each $a \in \mathfrak{A} \subset \mathcal{L}(C_0(\theta; X))$ can be represented as

$$a = \tfrac{1}{2}\,a + \tfrac{1}{2}\,aV = \tfrac{1}{3}\,a + \tfrac{2}{3}\,aV \quad \text{etc.}$$

However, ρ is injective provided that φ is aperiodic, i.e., the set of aperiodic points for φ is dense in Θ. Assuming, for a moment, the aperiodicity of φ, let $\mathfrak{B}$ be the subspace of $\mathcal{L}(C_0(\Theta; X))$ containing the operators of the form

$$b = \sum_{k=-\infty}^{\infty} a_k V^k \quad \text{where} \quad a_k \in \mathfrak{A} \quad \text{and} \quad \|b\|_1 := \sum_{k=-\infty}^{\infty} \|a_k\|_{\mathfrak{A}} < \infty. \tag{3.2}$$

For aperiodic φ, it was proved in Prop. 2.14 and 2.15 of [21] that the representation $b \in \mathfrak{B}$ as in (3.2) is unique and that $(\mathfrak{B}, \|\cdot\|_1)$ is a Banach algebra (with respect to the composition of operators). Thus, $\rho : \hat{\mathfrak{B}} \to \mathfrak{B}$ is an isomorphism if φ is aperiodic.

Going back to a not necessarily aperiodic flow φ, let $\mathfrak{D} \subset \mathcal{L}(c_0(\mathbb{Z}; X))$ be the algebra of all bounded diagonal operators $d = \mathrm{diag}(d^{(n)})_{n\in\mathbb{Z}}$. Define

$$\mathfrak{C} := \{D = \sum_{k=-\infty}^{\infty} d_k S^k \in \mathcal{L}(c_0(\mathbb{Z}; X)) : d_k \in \mathfrak{D},\ \|D\|_{\mathfrak{C}} := \sum_{k=-\infty}^{\infty} \|d_k\|_{\mathcal{L}(c_0(\mathbb{Z};X))} < \infty\},$$

where $S : (x_n)_{n\in\mathbb{Z}} \mapsto (x_{n-1})_{n\in\mathbb{Z}}$ is the shift operator on $c_0(\mathbb{Z}; X)$. The representation of $D \in \mathfrak{C}$ as $D = \sum d_k S^k$ is unique since, for $m_0, n_0 \in \mathbb{Z}$ and $x \in X$ with $|x| = 1$, one has

$$\begin{aligned} \|D\|_{\mathcal{L}(c_0(\mathbb{Z};X))} \ge \|D(x \otimes \delta_{n,n_0})\|_{c_0(\mathbb{Z};X)} &= \Big\| \sum\nolimits_k \mathrm{diag}(d_k^{(n)})_{n\in\mathbb{Z}} \, (x \otimes \delta_{n-k,n_0})_{n\in\mathbb{Z}} \Big\|_{c_0(\mathbb{Z};X)} \\ &\ge |\sum\nolimits_k d_k^{(m_0)} x \, \delta_{m_0-k,n_0}| = |d_{m_0-n_0}^{(m_0)} x|. \end{aligned}$$

We further need, for each $\theta \in \Theta$, the map

$$\hat{\pi}_\theta : \hat{\mathfrak{B}} \to \mathcal{L}(c_0(\mathbb{Z}; X)) : \hat{b} = (a_k)_{n\in\mathbb{Z}} \mapsto \sum_{k=-\infty}^{\infty} \operatorname{diag}[a_k(\varphi^n\theta)]_{n\in\mathbb{Z}}\, S^k. \tag{3.3}$$

Observe that $\hat{\pi}_\theta : \hat{\mathfrak{B}} \to \mathfrak{C}$ and

$$\begin{aligned} \|\hat{\pi}_\theta(\hat{b})\|_{\mathfrak{C}} &= \sum_{k=-\infty}^{\infty} \|\operatorname{diag}(a_k(\varphi^n\theta))_{n\in\mathbb{Z}}\|_{\mathcal{L}(c_0(\mathbb{Z};X))} \\ &= \sum_{k=-\infty}^{\infty} \sup_{n\in\mathbb{Z}} \sup_{\theta\in\Theta} \|a_k(\varphi^n\theta)\|_{\mathcal{L}(X)} = \sum_{k=-\infty}^{\infty} \|a_k\|_\infty = \|\hat{b}\|_1 . \end{aligned}$$

Also, we compute for $\hat{b} = \hat{b}' * \hat{b}''$

$$\begin{aligned} \hat{\pi}_\theta(\hat{b}) &= \sum_{k=-\infty}^{\infty} \operatorname{diag}\left(\sum_{l=-\infty}^{\infty} a'_l(\varphi^n\theta) a''_{k-l}(\varphi^{n-l}\theta) \right)_{n\in\mathbb{Z}} S^k \\ &= \sum_{l=-\infty}^{\infty} \left[\operatorname{diag}(a'_l(\varphi^n\theta))_{n\in\mathbb{Z}}\, S^l S^{-l} \sum_{k=-\infty}^{\infty} \operatorname{diag}(a''_{k-l}(\varphi^{n-l}\theta))_{n\in\mathbb{Z}}\, S^k \right] \\ &= \sum_{l=-\infty}^{\infty} \left[\operatorname{diag}(a'_l(\varphi^n\theta))_{n\in\mathbb{Z}}\, S^l \sum_{k=-\infty}^{\infty} \operatorname{diag}(a''_{k-l}(\varphi^n\theta))_{n\in\mathbb{Z}}\, S^{k-l} \right] \\ &= \hat{\pi}_\theta(\hat{b}') \cdot \hat{\pi}_\theta(\hat{b}''). \end{aligned}$$

As a result, $\hat{\pi}_\theta$ is a continuous homomorphism. Its importance relies on the following fact.

Proposition 3.1. $\bigcap_{\theta\in\Theta} \ker \hat{\pi}_\theta = \{0\}$.

Proof. Assume that $\hat{\pi}_\theta(\hat{b}) = 0$ for all $\theta \in \Theta$ and some $\hat{b} = (a_n)_{n\in\mathbb{Z}} \in \hat{\mathfrak{B}}$. This means that

$$0 = \hat{\pi}_\theta(\hat{b}) = \sum\nolimits_k \operatorname{diag}(a_k(\varphi^n\theta))_{n\in\mathbb{Z}}\, S^k \in \mathfrak{C}$$

for $\theta \in \Theta$. Since the representation $D = \sum_k \operatorname{diag}(d_k^{(n)}) S^k$ for $D \in \mathfrak{C}$ is unique, we conclude that $a_k(\varphi^n\theta) = 0$ for each $k, n \in \mathbb{Z}$ and $\theta \in \Theta$. Thus, $\hat{b} = 0$ in $\hat{\mathfrak{B}}$. $\square$

For $a \in \mathfrak{A}$, we define the operator $b = I - aV \in \mathcal{L}(C_0(\Theta; X))$ and the sequence $\hat{b} = (a_k)_{k\in\mathbb{Z}} \in \hat{\mathfrak{B}}$ by $a_0 = e$, $a_1 = -a$, and $a_k = 0$ otherwise. Then $\rho(\hat{b}) = I - aV = b$ by (3.1). Also, given $b = I - aV$, we introduce for each $\theta \in \Theta$ the operator $\pi_\theta(b) \in \mathcal{L}(c_0(\mathbb{Z}; X))$ by the rule

$$\pi_\theta(b) = \pi_\theta(I - aV) := I - \operatorname{diag}(a(\varphi^n\theta))_{n\in\mathbb{Z}} S. \tag{3.4}$$

To relate these concepts to the evolution semigroup $\{E^t\}$ on $C_0(\Theta; X)$ induced by the cocycle $\{\Phi^t\}$, we remark that the operator $I - E^1$ is of the form $b = I - E = I - aV$ with $a(\theta) = \Phi(\varphi^{-1}\theta)$ for $\theta \in \Theta$. We can now give a sufficient condition for the invertibility of $I - E$ in terms of the operators $\pi_\theta(b)$.

Theorem 3.2. *Let $b = I - aV$ for $a \in \mathfrak{A}$. Assume that $\pi_\theta(b)$ is invertible in $\mathcal{L}(c_0(\mathbb{Z}; X))$ for each $\theta \in \Theta$ and that there exists a constant $B > 0$ such that*

$$\|[\pi_\theta(b)]^{-1}\|_{\mathcal{L}(c_0(\mathbb{Z};X))} \le B \quad \text{for all} \quad \theta \in \Theta. \tag{3.5}$$

Then b^{-1} exists in $\mathcal{L}(C_0(\Theta; X))$.

Proof. Let $\hat{b} = (a_k)_{k\in\mathbb{Z}} \in \hat{\mathfrak{B}}$ be given by $a_0 = I$, $a_1 = -a$, and $a_k = 0$ for $k \neq 0, 1$. The definitions (3.3) and (3.4) yield

$$\hat{\pi}_\theta(\hat{b}) = \pi_\theta(b) \qquad \text{for} \;\; \theta \in \Theta. \tag{3.6}$$

Also, $\rho(\hat{b}) = b$ by (3.1). Setting $d_0(\theta) = I$, $d_1(\theta) = \operatorname{diag}(a(\varphi^n\theta))_{n\in\mathbb{Z}}$, and $d_k(\theta) = 0$ for $k \neq 1, 2$ and $\theta \in \Theta$, we see that

$$\hat{\pi}_\theta(\hat{b}) = \sum\nolimits_k d_k(\theta) S^k \in \mathfrak{C}.$$

Next, we show that the inverse of $\hat{\pi}_\theta(\hat{b})$ also belongs to $\mathfrak{C}$. The proof of this fact is close to that of Lemma 1.6 in [22].

Proposition 3.3. *Under the assumptions of Theorem 3.2, the operator $[\hat{\pi}_\theta(\hat{b})]^{-1} = [\pi_\theta(b)]^{-1}$ belongs to $\mathfrak{C}$ for each $\theta \in \Theta$; that is,*

$$[\hat{\pi}_\theta(\hat{b})]^{-1} = \sum_{k=-\infty}^{\infty} C_k(\theta) S^k \quad \textit{where} \quad C_k(\theta) = \operatorname{diag}(C_k^{(n)}(\theta))_{n\in\mathbb{Z}} \quad \textit{satisfies} \tag{3.7}$$

$$\sum_{k=-\infty}^{\infty} \sup_{\theta\in\Theta} \|C_k(\theta)\|_{\mathcal{L}(c_0(\mathbb{Z};X))} < \infty. \tag{3.8}$$

Proof of Proposition 3.3. Set $D_\theta = \hat{\pi}_\theta(\hat{b}) = I - \operatorname{diag}(a(\varphi^n\theta))_{n\in\mathbb{Z}} S$. Assumption (3.5) and identity (3.6) yield

$$\sup_{\theta\in\Theta} \|D_\theta^{-1}\| \leq B. \tag{3.9}$$

For $\gamma > 0$, we define the operator

$$D_\theta(\gamma) = I - \operatorname{diag}(e^{\gamma(|n|-|n-1|)} a(\varphi^n\theta))_{n\in\mathbb{Z}}\, S$$

on $c_0(\mathbb{Z}; X)$. Note that $D_\theta(\gamma) = J_\gamma^{-1} D_\theta J_\gamma$ for $J_\gamma = \operatorname{diag}(e^{-\gamma|n|})_{n\in\mathbb{Z}}$. Clearly,

$$\|D_\theta - D_\theta(\gamma)\|_{\mathcal{L}(c_0(\mathbb{Z};X))} \leq \max\{|1 - e^\gamma|, |1 - e^{-\gamma}|\}\; \|a\|_{\mathfrak{A}} \to 0$$

as $\gamma \to 0$. Choose γ_0 such that

$$q := B \max\{|1 - e^{\pm\gamma_0}|\}\; \|a\|_{\mathfrak{A}} < 1.$$

Then $\|D_\theta - D_\theta(\gamma)\| \cdot \|D_\theta^{-1}\| \leq q$ for all $\theta \in \Theta$ and $0 \leq \gamma \leq \gamma_0$. Since D_θ is invertible in $\mathcal{L}(c_0(\mathbb{Z}; X))$, we conclude that $D_\theta(\gamma)$ is invertible and $\|[D_\theta(\gamma)]^{-1}\| \leq R := \frac{B}{1-q}$ for all $0 \leq \gamma \leq \gamma_0$ and $\theta \in \Theta$.

Consider D_θ^{-1} as an operator matrix $D_\theta^{-1} = [C_{kj}(\theta)]_{k,j\in\mathbb{Z}}$, where $C_{kj}(\theta) \in \mathcal{L}(X)$ are defined for $x \in X$ by

$$C_{kj}(\theta)x := (D_\theta^{-1}(\bar{x}_j))_k \quad \text{for} \;\; \bar{x}_j = (\delta_{jn}x)_{n\in\mathbb{Z}}.$$

From $D_\theta(\gamma)^{-1} = J_\gamma^{-1} D_\theta^{-1} J_\gamma$ we derive $D_\theta(\gamma)^{-1} = [e^{\gamma(|k|-|j|)} C_{kj}(\theta)]_{k,j\in\mathbb{Z}}$. Now,

$$R \geq \|D_\theta(\gamma)^{-1}\|_{\mathcal{L}(c_0(\mathbb{Z};X))} \geq e^{\gamma(|k|-|j|)} \|C_{kj}(\theta)\|_{\mathcal{L}(X)}$$

for $k, j \in \mathbb{Z}$ yields

$$\|C_{k,0}(\theta)\|_{\mathcal{L}(X)} \leq R\, e^{-\gamma|k|} \quad \text{for} \quad \theta \in \Theta \;\; \text{and} \;\; k \in \mathbb{Z}.$$

Observe that, for every $j, k, m \in \mathbb{Z}$, one has

$$D_{\varphi^m\theta} = S^{-m} D_\theta S^m \quad \text{and} \quad C_{kj}(\varphi^m\theta) = C_{k+m,j+m}(\theta). \tag{3.10}$$

In particular, from $C_{k+m,m}(\theta) = C_{k,0}(\varphi^m\theta)$ it follows that

$$\|C_{k+m,m}(\theta)\|_{\mathcal{L}(X)} \le Re^{-\gamma|k|} \quad \text{for} \quad k,m \in \mathbb{Z} \quad \text{and} \quad \theta \in \Theta. \tag{3.11}$$

So we can write D_θ^{-1} as

$$D_\theta^{-1} = \sum\nolimits_k C_k(\theta)S^k = \sum\nolimits_k \operatorname{diag}(C_k^{(n)}(\theta))_{n\in\mathbb{Z}}S^k$$

for $C_k^{(n)}(\theta) = C_{n,n-k}(\theta)$, and we estimate

$$\sum_{k\in\mathbb{Z}} \sup_{\theta\in\Theta} \|C_k(\theta)\|_{\mathcal{L}(c_0(\mathbb{Z};X))} = \sum_{k\in\mathbb{Z}} \sup_{\theta\in\Theta} \sup_{n\in\mathbb{Z}} \|C_{n,n-k}(\theta)\|_{\mathcal{L}(X)} \le \sum_{k\in\mathbb{Z}} Re^{-\gamma|k|} < \infty. \qquad \square$$

To conclude the proof of Theorem 3.2 we need the following fact.

Claim 1. $\theta \mapsto C_k^{(0)}(\theta)$, see (3.7), is a continuous, bounded function from Θ to $\mathcal{L}_s(X)$ with

$$\sum_{k\in\mathbb{Z}} \sup_{\theta\in\Theta} \|C_k^{(0)}(\theta)\|_{\mathcal{L}(X)} < \infty.$$

We finish the proof of the theorem and show Claim 1 later. First, notice that (3.10) implies

$$C_k^{(n)}(\theta) = C_{n,n-k}(\theta) = C_{0,-k}(\varphi^n\theta) = C_k^{(0)}(\varphi^n\theta) \tag{3.12}$$

for $k,n \in \mathbb{Z}$ and $\theta \in \Theta$. The sequence $\hat{d} = (C_k^{(0)})_{k\in\mathbb{Z}}$ belongs to $\hat{\mathfrak{B}}$ by Claim 1. Thus, we can apply to $\hat{d}$ the homomorphism $\rho : \hat{\mathfrak{B}} \to \mathcal{L}(C_0(\Theta;X))$. Let $d = \rho(\hat{d})$. We want to show that $db = bd = I$.

Consider the sequence $\hat{r} = \hat{d} * \hat{b} - \hat{e} \in \hat{\mathfrak{B}}$. Fix $\theta \in \Theta$ and apply $\hat{\pi}_\theta : \hat{\mathfrak{B}} \to \mathfrak{C}$. Using (3.3), (3.12), and Proposition 3.3, we compute

$$\begin{aligned}
\hat{\pi}_\theta(\hat{r}) = \hat{\pi}_\theta(\hat{d}) \cdot \hat{\pi}_\theta(\hat{b}) - I &= \Big[\sum\nolimits_k \operatorname{diag}(C_k^{(0)}(\varphi^n\theta))_{n\in\mathbb{Z}}\, S^k\Big] \cdot \hat{\pi}_\theta(\hat{b}) - I \\
&= \Big[\sum\nolimits_k \operatorname{diag}(C_k^{(n)}(\theta))_{n\in\mathbb{Z}}\, S^k\Big] \cdot \hat{\pi}_\theta(\hat{b}) - I \\
&= \Big[\sum\nolimits_k C_k(\theta)S^k\Big] \cdot \hat{\pi}_\theta(\hat{b}) - I = 0.
\end{aligned}$$

Since $\theta \in \Theta$ was arbitrary, Proposition 3.1 shows that $\hat{r} = 0$ in $\hat{\mathfrak{B}}$. As a consequence,

$$0 = \rho(\hat{r}) = \rho(\hat{d} * \hat{b} - \hat{e}) = \rho(\hat{d}) \cdot \rho(\hat{b}) - I = d \cdot b - I.$$

Similarly, one obtains $0 = bd - I$, and the theorem is proved.

Proof of Claim 1. Let $k \in \mathbb{Z}$, $x \in X$, and $\theta_0 \in \Theta$. Define $\bar{x} = (x_n) \in c_0(\mathbb{Z};X)$ by $x_n = x$ if $n = -k$ and $x_n = 0$ if $n \neq -k$. Then, for every $\theta \in \Theta$,

$$\begin{aligned}
\Big|[C_k^{(0)}(\theta) - C_k^{(0)}(\theta_0)]x\Big| &= \Bigg|\sum_{l=-\infty}^{\infty} \Big[C_l^{(0)}(\theta) - C_l^{(0)}(\theta_0)\Big]\, x_{-l}\Bigg| \\
&\le \sup_{n\in\mathbb{Z}} \Big|\sum\nolimits_l \Big[C_l^{(n)}(\theta) - C_l^{(n)}(\theta_0)\Big]\, x_{n-l}\Big| \\
&= \Big\|\Big(\sum\nolimits_l \operatorname{diag}\Big(C_l^{(n)}(\theta) - C_l^{(n)}(\theta_0)\Big)_{n\in\mathbb{Z}}\, S^l\Big)\,\bar{x}\Big\|_{c_0(\mathbb{Z};X)} \\
&= \Big\|\Big([\hat{\pi}_\theta(\hat{b})]^{-1} - [\hat{\pi}_{\theta_0}(\hat{b})]^{-1}\Big)\,\bar{x}\Big\|_{c_0(\mathbb{Z};X)} \\
&\le \Big\|[\hat{\pi}_\theta(\hat{b})]^{-1}\Big\|_{\mathcal{L}(c_0(\mathbb{Z};X))} \cdot \Big\|[\hat{\pi}_{\theta_0}(\hat{b}) - \hat{\pi}_\theta(\hat{b})]\bar{y}\Big\|_{c_0(\mathbb{Z};X)}\,,
\end{aligned} \tag{3.13}$$

where $\bar{y} := [\hat{\pi}_{\theta_0}(\hat{b})]^{-1}\bar{x} \in c_0(\mathbb{Z}; X)$. It is easy to see that

$$\Theta \ni \theta \mapsto \pi_\theta(b) \in \mathcal{L}_s(c_0(\mathbb{Z}; X))$$

is (strongly!) continuous. So (3.13) and (3.9) imply the strong continuity of $\theta \mapsto C_k^{(0)}(\theta) \in \mathcal{L}_s(X)$. ¿From (3.12) and (3.8) we derive

$$\sum_{k\in\mathbb{Z}} \sup_{\theta\in\Theta} \|C_k^{(0)}(\theta)\|_{\mathcal{L}(X)} = \sum_{k\in\mathbb{Z}} \sup_{\theta\in\Theta} \sup_{n\in\mathbb{Z}} \|C_k^{(0)}(\varphi^n\theta)\|_{\mathcal{L}(X)} = \sum_{k\in\mathbb{Z}} \sup_{\theta\in\Theta} \|C_k(\theta)\|_{\mathcal{L}(c_0(\mathbb{Z};X))} < \infty,$$

and Claim 1 is proved. □

Remark 3.4. Assume that φ is aperiodic. Then, $\rho : \hat{\mathfrak{B}} \to \mathfrak{B}$ is an isomorphism and $\pi_\theta = \hat{\pi}_\theta \circ \rho^{-1}$. The proof of Theorem 3.2 shows that in this case b^{-1} belongs to $\mathfrak{B}$, cf. [21, Lemma 4.4]. ♢

In the course of the proof of Theorem 3.2 we showed that for $b = I - aV$ the element $\hat{b}$ is invertible in $\hat{\mathfrak{B}}$ under condition (3.5). Its inverse is given by $\hat{d} = (C_k^{(0)})_{k\in\mathbb{Z}}$. Later, dealing with eventually norm-continuous in θ cocycles, we will need the following refinement of this fact. Let $\tilde{\mathfrak{A}}_{\text{norm}}$ be the closed subalgebra $C_b(\Theta; \mathcal{L}(X))$ of $\tilde{\mathfrak{A}}$ of all *norm* continuous, bounded, operator valued functions $a : \Theta \to \mathcal{L}(X)$. Let $\hat{\mathfrak{B}}_{\text{norm}}$ denote the corresponding closed subalgebra of $\hat{\mathfrak{B}}$, that is, $\hat{\mathfrak{B}}_{\text{norm}} = \{(a_k)_{k\in\mathbb{Z}} \in \hat{\mathfrak{B}} : a_k \in \tilde{\mathfrak{A}}_{\text{norm}}\}$.

Corollary 3.5. *Assume that $a \in C_b(\Theta; \mathcal{L}(X))$ is norm continuous. If the conditions of Theorem 3.2 hold, then $\hat{d} = \hat{b}^{-1}$ belongs to $\hat{\mathfrak{B}}_{\text{norm}}$.*

Proof. We have to check that $\theta \mapsto C_k^{(0)}(\theta) \in \mathcal{L}(X)$ is norm continuous (cf. Claim 1 in the proof of Theorem 3.2). Fix $k \in \mathbb{Z}$. Recall the estimate (3.13)

$$|[C_k^{(0)}(\theta) - C_k^{(0)}(\theta_0)]x| \le B\, \|[\hat{\pi}_{\theta_0}(\hat{b}) - \hat{\pi}_\theta(\hat{b})]\bar{y}\|_{c_0(\mathbb{Z};X)},$$

for $x \in X$ and the constant B given in (3.5), and

$$\bar{y} = [\hat{\pi}_{\theta_0}(\hat{b})]^{-1}\bar{x} = \sum_{l=-\infty}^{\infty} \operatorname{diag}(C_l^{(n)}(\theta_0))_{n\in\mathbb{Z}}\, S^l\bar{x} = (C_{n-k}^{(n)}(\theta_0)x)_{n\in\mathbb{Z}} = (C_{n,k}(\theta_0)x)_{n\in\mathbb{Z}},$$

where $\bar{x} = \delta_{n,-k} \otimes x$ and we have used (3.12). The inequality (3.11) yields $\|C_{n,k}(\theta_0)\| \le Re^{-\gamma|n-k|}$ for $n \in \mathbb{Z}$. Fix $\varepsilon > 0$ and take $N = N_\varepsilon$ such that $Re^{-\gamma|n-k|} < \varepsilon$ for $|n| \ge N$. Using the continuity of $a : \Theta \to \mathcal{L}(X)$, choose $\delta > 0$ such that

$$\|a(\varphi^n\theta) - a(\varphi^n\theta_0)\|_{\mathcal{L}(X)} < \varepsilon \quad \text{provided} \quad |n| < N \;\text{ and }\; d(\theta, \theta_0) < \delta.$$

For $d(\theta, \theta_0) < \delta$, we now conclude that

$$\begin{aligned}
&\|[\hat{\pi}_{\theta_0}(\hat{b}) - \hat{\pi}_\theta(\hat{b})]\bar{y}\|_{c_0(\mathbb{Z};X)} \\
&\quad \le \max\{\sup_{|n|<N} |[a(\varphi^n\theta) - a(\varphi^n\theta_0)]C_{n,k}(\theta_0)x|,\, \sup_{|n|\ge N} |[a(\varphi^n\theta) - a(\varphi^n\theta_0)]C_{n,k}(\theta_0)x|\} \\
&\quad \le \varepsilon \max\{R,\, 2\sup_{\theta\in\Theta} \|a(\theta)\|_{\mathcal{L}(X)}\}\, |x|.
\end{aligned}$$

□

3.2. **Discrete Dichotomy Theorem.** The algebraic method developed above now enables us to prove the main result of this section. This result connects global dichotomy on Θ as defined in Definition 2.1, dichotomy at points $\theta \in \Theta$ as defined in Definition 2.2, the hyperbolicity of the operator given by $(Ef)(\theta) = \Phi(\varphi^{-1}\theta)f(\varphi^{-1}\theta)$ on $C_0(\Theta; X)$, and the hyperbolicity of the operators $\pi_\theta(E) = \operatorname{diag}(\Phi(\varphi^{n-1}\theta))_{n\in\mathbb{Z}}\, S$ on $c_0(\mathbb{Z}; X)$.

We start with some elementary facts. First, observe that

$$\pi_\theta(E)^k = \pi_\theta(E^k) = \operatorname{diag}(\Phi^k(\varphi^{n-k}\theta))_{n\in\mathbb{Z}}\, S^k \tag{3.14}$$

for $k \in \mathbb{N}$ and $\theta \in \Theta$. We set $\|T\|_{\bullet,Y} = \inf\{\|Ty\| : \|y\| = 1\}$ for a bounded operator T on a Banach space Y. Observe that the operator $\pi_\theta(E)$ is also defined on $\ell^\infty(\mathbb{Z}, X)$, the space of bounded sequences endowed with the sup-norm.

Proposition 3.6. *Let $z \in \mathbb{T}$ and $\theta \in \Theta$. We have $\|I - \pi_\theta(E)\|_\bullet = \|zI - \pi_\theta(E)\|_\bullet$ on $c_0(\mathbb{Z}; X)$ or $\ell^\infty(\mathbb{Z}, X)$. Further, the spectrum $\sigma(\pi_\theta(E))$ in $c_0(\mathbb{Z}; X)$ or $\ell^\infty(\mathbb{Z}, X)$ is invariant with respect to rotations centered at the origin, and if $\sigma(\pi_\theta(E)) \cap \mathbb{T} = \emptyset$, then $\|(I - \pi_\theta(E))^{-1}\| = \|(zI - \pi_\theta(E))^{-1}\|$.*

Proof. Define $L_\xi = \operatorname{diag}(e^{-in\xi})_{n\in\mathbb{Z}}$ on $c_0(\mathbb{Z}; X)$ or $\ell^\infty(\mathbb{Z}, X)$ for $\xi \in \mathbb{R}$. Then L_ξ is an invertible isometry on both spaces. So the identity

$$L_\xi\, [zI - \pi_\theta(E)]\, L_\xi^{-1} = zI - e^{-i\xi} \operatorname{diag}(\Phi(\varphi^{n-1}\theta))_{n\in\mathbb{Z}} S = e^{-i\xi}(ze^{i\xi} I - \pi_\theta(E))$$

implies the result. $\square$

Recall that due to Corollary 2.9 the dichotomies of a cocycle for continuous and discrete times are equivalent. Thus, we can work with discrete time.

Theorem 3.7 (Discrete Dichotomy Theorem)**.** *Let $\varphi : \Theta \to \Theta$ be a homeomorphism on a locally compact metric space Θ and let $\{\Phi^t\}_{t\in\mathbb{N}}$ with $\Phi^1 = \Phi \in C_b(\Theta; \mathcal{L}_s(X))$ be a discrete time cocycle over φ. The following assertions are equivalent.*

(a) *$\{\Phi^t\}$ has exponential dichotomy on Θ over $\mathbb{Z}$.*
(b) *$\{\Phi^t\}$ has exponential dichotomy over $\mathbb{Z}$ at each point $\theta \in \Theta$ with dichotomy constants $\beta(\theta) \geq \beta > 0$ and $M(\beta(\theta)) \leq M$.*
(c) *$\pi_\theta(E)$ is hyperbolic on $c_0(\mathbb{Z}; X)$ for each $\theta \in \Theta$ and*

$$\sup_{z\in\mathbb{T}} \sup_{\theta\in\Theta} \|[zI - \pi_\theta(E)]^{-1}\|_{\mathcal{L}(c_0(\mathbb{Z};X))} < \infty.$$

(d) *E is hyperbolic on $C_0(\Theta; X)$.*

If (a)-(d) hold, let $P(\cdot)$ be the dichotomy projection for $\{\Phi^t\}$ on Θ, let $P_n, n \in \mathbb{Z}$, be the family of the dichotomy projections for $\{\Phi^t\}$ at $\theta \in \Theta$, let $p_\theta \in \mathcal{L}(c_0(\mathbb{Z}; X)), \theta \in \Theta$, be the Riesz projection for $\pi_\theta(E)$ on $c_0(\mathbb{Z}; X)$ that corresponds to $\sigma(\pi_\theta(E)) \cap \mathbb{D}$, and let $\mathcal{P}$ be the Riesz projection for E on $C_0(\Theta; X)$ that corresponds to $\sigma(E) \cap \mathbb{D}$. Then,

$$\mathcal{P} = P(\cdot), \qquad P_n = P(\varphi^n\theta), \quad \textit{and} \quad p_\theta = \operatorname{diag}(P_n)_{n\in\mathbb{Z}}.$$

Proof. (a) $\Rightarrow$ (b). Let $\{\Phi^t\}_{t\geq 0}$ have exponential dichotomy for discrete time with projections $P(\cdot) \in C_b(\Theta; \mathcal{L}_s(X))$ and constants $\beta, M > 0$, see Definition 2.1. For $\theta \in \Theta$, the projections $P_n := P(\varphi^n\theta)$, $n \in \mathbb{Z}$, clearly satisfy Definition 2.2 with constants $\beta(\theta) = \beta$ and $M(\beta(\theta)) = M$.

(b) $\Rightarrow$ (c) Fix $\theta \in \Theta$. Let $P_n, n \in \mathbb{Z}$, be the family of dichotomy projections for $\{\Phi^t\}$ at θ as in Definition 2.2. Define $p_\theta = \operatorname{diag}(P_n)_{n\in\mathbb{Z}}$ on $c_0(\mathbb{Z}; X)$. By (a) in Definition 2.2, we have $p_\theta \pi_\theta(E) = \pi_\theta(E) p_\theta$. The identity (3.14) and Definition 2.2(b) imply that

$$(\pi_\theta(E)p_\theta)^k = \operatorname{diag}(\Phi^k(\varphi^{n-k}\theta)P_{n-k})_{n\in\mathbb{Z}}\, S^k \quad \text{and}$$
$$(\pi_\theta(E)q_\theta)^{-k} = \operatorname{diag}([\Phi^k_Q(\varphi^n\theta)Q_n]^{-1})_{n\in\mathbb{Z}}\, S^{-k} = \operatorname{diag}(\Phi^{-k}_Q(\varphi^{n+k}\theta)Q_{n+k})_{n\in\mathbb{Z}}\, S^{-k},$$

for $k \in \mathbb{N}$. Thus, the estimates in Definition 2.2(c) give

$$\|(\pi_\theta(E)p_\theta)^k\|_{\mathcal{L}(c_0(\mathbb{Z};X))} \le M\, e^{-\beta k} \quad \text{and} \quad \|(\pi_\theta(E)q_\theta)^{-k}\|_{\mathcal{L}(c_0(\mathbb{Z};X))} \le M\, e^{-\beta k}.$$

As a result, $\pi_\theta(E)$ is hyperbolic and $(z,\theta) \mapsto \|[zI - \pi_\theta(E)]^{-1}\|_{\mathcal{L}(c_0(\mathbb{Z};X))}$ is bounded.

(c) $\Rightarrow$ (d). For every $z \in \mathbb{T}$, apply Theorem 3.2 to $b = I - aV$, where $a(\theta) = z^{-1}\Phi(\varphi^{-1}\theta)$ and $(Vf)(\theta) = f(\varphi^{-1}\theta)$ for $\theta \in \Theta$.

(d) $\Rightarrow$ (a) and $\mathcal{P} = P(\cdot)$ can be shown exactly as Theorem 2.7. $\square$

One can improve the equivalence (b)$\Leftrightarrow$(c) as follows.

Corollary 3.8. *The cocycle $\{\Phi^t\}_{t\in\mathbb{N}}$ has exponential dichotomy at $\theta \in \Theta$ over $\mathbb{Z}$ if and only if $\pi_\theta(E)$ is hyperbolic on $c_0(\mathbb{Z}; X)$.*

Proof. Necessity was proved, in fact, in Theorem 3.7 (b)$\Rightarrow$(c). So let $\pi_\theta(E)$ be hyperbolic. As in [20, Lemma 3.2], we see that $p_\theta\chi = \chi p_\theta$ for every $\chi = (\chi_n) \in \ell^\infty(\mathbb{Z})$. By arguments similar to those used in the proof of [40, Lemma 7], one obtains that $p_{\theta_0} = \operatorname{diag}(P_n)_{n\in\mathbb{Z}}$ for a family of uniformly bounded projections $P_n \in \mathcal{L}(X)$, $n \in \mathbb{Z}$. From $p_\theta\pi_\theta(E) = \pi_\theta(E)p_\theta$ we derive (a) in Definition 2.2. Further, $\Phi^k(\varphi^{n-k}\theta) : \operatorname{Im} Q_{n-k} \to \operatorname{Im} Q_n$ is surjective due to (3.14) and the surjectivity of $\pi_\theta(E)^k$ on $\operatorname{Im} q_\theta$. Finally, the estimates in (c) follow as in the proof of Theorem 2.7. $\square$

In addition, for the case $\mathbb{K} = \mathbb{R}$, we can characterize pointwise dichotomy of the cocycle by means of *evolution semigroups along trajectories*, $\{\Pi^t_\theta\}_{t\ge 0}$ on $C_0(\mathbb{R}; X)$, which are defined by

$$(\Pi^t_\theta h)(s) = \Phi^t(\varphi^{s-t}\theta)h(s-t), \quad s \in \mathbb{R},\ t \ge 0,\ h \in C_0(\mathbb{R}; X),$$

for each $\theta \in \Theta$, cf. Theorem 5.9 of [21]. Since $\{\Phi^t\}$ is an exponentially bounded, strongly continuous cocycle, the operators $U_\theta(s,\tau) := \Phi^{s-\tau}(\varphi^\tau\theta)$, $s \ge \tau$, yield an exponentially bounded, strongly continuous evolution family, see [21, p.110]. In particular, $(\Pi^t_\theta h)(s) = U_\theta(s, s-t)h(s-t)$ and $\{\Pi^t_\theta\}$ is an evolution semigroup on $C_0(\mathbb{R}; X)$, see [20, 37] for the definitions. By the results in [20, 36, 39], the evolution family $\{U_\theta(s,\tau)\}_{s\ge\tau}$ has exponential dichotomy on $\mathbb{R}$ if and only if $\{\Pi^t_\theta\}_{t\ge 0}$ is hyperbolic if and only if Γ_θ is invertible. But dichotomy of $\{U_\theta(s,\tau)\}_{s\ge\tau}$ means exactly the dichotomy of $\{\Phi^t\}_{t\ge 0}$ at θ over $\mathbb{R}$. The next result now follows from the Dichotomy Theorem 2.7 and the techniques used in the proof of Theorem 5.9 of [21].

Theorem 3.9. *The following assertions are equivalent for an exponentially bounded, strongly continuous cocycle $\{\Phi^t\}_{t\ge 0}$ over a continuous flow $\{\varphi^t\}_{t\in\mathbb{R}}$ on Θ.*

(a) *$\{\Phi^t\}_{t\ge 0}$ has exponential dichotomy on Θ.*

(b) *$\sigma(\Pi^1_\theta) \cap \mathbb{T} = \emptyset$ for each $\theta \in \Theta$ and*

$$\sup_{z\in\mathbb{T}} \sup_{\theta\in\Theta} \|[z - \Pi^1_\theta]^{-1}\|_{\mathcal{L}(C_0(\mathbb{R};X))} < \infty.$$

(c) $\sigma(\Gamma_\theta) \cap i\mathbb{R} = \emptyset$ *for each* $\theta \in \Theta$ *and*

$$\sup_{\xi\in\mathbb{R}} \sup_{\theta\in\Theta} \|[i\xi - \Gamma_\theta]^{-1}\|_{\mathcal{L}(C_0(\mathbb{R};X))} < \infty.$$

(d) $\{\Phi^t\}_{t\geq 0}$ *has exponential dichotomy over* $\mathbb{R}$ *at each point* $\theta \in \Theta$ *with dichotomy constants* $\beta(\theta) \geq \beta > 0$ *and* $M(\beta(\theta)) \leq M$.

Moreover, the Riesz projection $\mathcal{P}$ *for the operator* E*, the Riesz projection* p_θ *for the operator* Π_θ^1*, the dichotomy projection* $P(\cdot)$ *for the dichotomy of* $\{\Phi^t\}$ *on* Θ*, and the dichotomy projections* $\{P_\tau^\theta\}_{\tau\in\mathbb{R}}$ *for the dichotomy of* $\{\Phi^t\}$ *at a point* $\theta \in \Theta$ *are related as follows:*

$$\mathcal{P} = P(\cdot), \quad P_\tau^\theta = P(\varphi^\tau\theta), \quad (p_\theta h)(\tau) = P_\tau^\theta h(\tau),\ h \in C_0(\mathbb{R};X).$$

3.3. **Perturbation theorem.** In this subsection we give an infinite-dimensional generalization of the Sacker-Sell Perturbation Theorem (see Theorem 6 in [43]) which establishes the semicontinuity of the dynamical spectrum as a function of Θ. To this end, we assume that Θ is a compact metric space and that the cocycle $\{\Phi^t\}$ is *eventually norm continuous* in θ. This means (cf. [33, p. 38]) that for some $t_0 > 0$ the map $\Theta \ni \theta \mapsto \Phi^{t_0}(\theta) \in \mathcal{L}(X)$ is continuous in operator norm. By rescaling the time, we can assume that $t_0 = 1$. Also, we work with discrete time $t \in \mathbb{N}$. We start with a preliminary fact.

Proposition 3.10. *Assume that* $\Theta \ni \theta \mapsto \Phi^1(\theta) \in \mathcal{L}(X)$ *is norm continuous. If* $\sigma(E)\cap\mathbb{T} = \emptyset$*, then* $\{\Phi^n\}_{n\in\mathbb{N}}$ *has exponential dichotomy with a norm continuous dichotomy projection* $P(\cdot)$.

Proof. Assume that E is hyperbolic. By the Discrete Dichotomy Theorem 3.7 we know that the Riesz projection p_θ for $\pi_\theta(E) = \pi_\theta(aV)$ is given by $p_\theta = \mathrm{diag}(P(\varphi^n\theta))_{n\in\mathbb{Z}}$. Notice that $a = \Phi^1 \circ \varphi^{-1} \in \tilde{\mathfrak{A}}_{\mathrm{norm}}$, see the notations before Corollary 3.5. Define $\hat{a} = (a_n)_{n\in\mathbb{Z}} \in \hat{\mathfrak{B}}_{\mathrm{norm}}$ by $a_1 = a$ and $a_n = 0$ otherwise. Let $\hat{b}_z = \hat{e} - z^{-1}\hat{a}$ for $z \in \mathbb{T}$. Recall that $\hat{\pi}_\theta(\hat{b}_z) = \pi_\theta(I - z^{-1}aV) = I - z^{-1}\pi_\theta(E)$ by (3.6). The Discrete Dichotomy Theorem 3.7 yields

$$\|(I - z^{-1}\pi_\theta(E))^{-1}\|_{\mathcal{L}(c_0(\mathbb{Z};X))} \leq C \quad \text{for all} \quad \theta \in \Theta \text{ and } z \in \mathbb{T}.$$

Applying Corollary 3.5 to $b_z = I - z^{-1}aV$, we see that for each $z \in \mathbb{T}$ the element $z\hat{e} - \hat{a} \in \hat{\mathfrak{B}}_{\mathrm{norm}}$ is invertible in $\hat{\mathfrak{B}}_{\mathrm{norm}}$. Define the idempotent $\hat{p}$ in the algebra $\hat{\mathfrak{B}}_{\mathrm{norm}}$ by

$$\hat{p} = \frac{1}{2\pi i}\int_{\mathbb{T}} (z\hat{e} - \hat{a})^{-1} dz.$$

Since $\hat{\pi}_\theta : \hat{\mathfrak{B}} \to \mathfrak{C}$ is a continuous homomorphism for all $\theta \in \Theta$, we obtain

$$\hat{\pi}_\theta(\hat{p}) = \frac{1}{2\pi i}\int_{\mathbb{T}} \hat{\pi}_\theta([z\hat{e} - \hat{a}]^{-1})\, dz = \frac{1}{2\pi i}\int_{\mathbb{T}} [zI - \pi_\theta(aV)]^{-1}\, dz = p_\theta$$

for every $\theta \in \Theta$. On the order hand, let $\hat{p} = (P_k)_{k\in\mathbb{Z}}$ for $P_k \in \tilde{\mathfrak{A}}_{\mathrm{norm}}$. Then we have

$$0 = \hat{\pi}_\theta(\hat{p}) - p_\theta = \pi_\theta(P_0) - p_\theta + \sum_{k\neq 0} \pi_\theta(P_k) S^k$$

by the definition (3.3) of $\hat{\pi}_\theta$ and $\pi_\theta(P_k) = \mathrm{diag}[P_k(\varphi^n\theta)]_n$. Since the representation of $\hat{\pi}_\theta(\hat{p}) - p_\theta \in \mathfrak{C}$ as a series in powers of S is unique, we conclude that $\pi_\theta(P_0) = p_\theta$ and $\pi_\theta(P_k) = 0$ for $k \neq 0$ and each $\theta \in \Theta$. So Proposition 3.1 yields $P_k = 0$ for $k \neq 0$, and hence $P = P_0 \in \hat{\mathfrak{A}}_{\mathrm{norm}}$ (use the sequence $(\delta_{n0}P_k)_n \in \hat{\mathfrak{B}}$). □

Let $\mathcal{K}$ be the set of all closed, φ-invariant subsets of Θ. For fixed $\Theta_0 \in \mathcal{K}$ and the metric $d(\cdot,\cdot)$ on Θ, define

$$\mathrm{dist}_{\Theta_0}(\Theta_1) = \sup_{\theta_1\in\Theta_1}\inf_{\theta_0\in\Theta_0} d(\theta_0,\theta_1) \quad \text{for} \;\; \Theta_1 \in \mathcal{K}.$$

Further, denote by $\Sigma(\Theta_1) = \Sigma(\{\hat{\varphi}^n|\Theta_1 \times X\})$ the Sacker-Sell dynamical spectrum, see Definition 2.4, for the cocycle $\{\Phi^n\}_{n\in\mathbb{N}}$ restricted on Θ_1, that is, for $\{\Phi^n\}_{n\in\mathbb{N}} = \{\Phi^n(\theta) : n \in \mathbb{N},\, \theta \in \Theta_1\}$, or for the linear skew-product flow $\{\hat{\varphi}^n\}$ restricted to $\Theta_1 \times X$. The next result implies the upper semicontinuity of the function $\Theta_1 \mapsto \Sigma(\Theta_1)$ on $\mathcal{K}$.

Theorem 3.11 (Perturbation Theorem)**.** *Let $\{\Phi^t\}_{t\in\mathbb{Z}_+}$ with $\Phi^1 = \Phi \in C_b(\Theta;\mathcal{L}(X))$ be a discrete time cocycle over a homeomorphism φ on a compact metric space Θ. Assume that $\{\Phi^t\}_{t\in\mathbb{N}}$ has exponential dichotomy on the set $\Theta_0 \in \mathcal{K}$ in the sense of Definition 2.1. There exists $\delta_* > 0$ such that the cocycle $\{\Phi^t\}_{t\in\mathbb{N}}$ has exponential dichotomy on each $\Theta_1 \in \mathcal{K}$ with the property*

$$dist_{\Theta_0}(\Theta_1) \le \delta \quad \textit{for some} \;\; \delta \in (0,\delta_*). \tag{3.15}$$

The proof of the theorem uses nothing but elementary calculations with Neumann series and the Discrete Dichotomy Theorem. Let us sketch our argument. For $\varepsilon > 0$, a sequence $\bar{\theta} = \{\theta_n\}_{n\in\mathbb{Z}} \subset \Theta$ is called *ε-pseudo-orbit* for φ if $d(\varphi\theta_n, \theta_{n+1}) \le \varepsilon$ for all $n \in \mathbb{Z}$. As an analogue to the operators $\pi_\theta(E)$, we define the operator $T_{\bar{\theta}} : (x_n)_{n\in\mathbb{Z}} \mapsto (\Phi(\theta_{n-1})x_{n-1})_{n\in\mathbb{Z}}$ on $c_0(\mathbb{Z};X)$ for every ε-pseudo-orbit $\bar{\theta} = \{\theta_n\}_{n\in\mathbb{Z}} \subset \Theta_0$.

Assuming that $\{\Phi^t\}$ has exponential dichotomy on Θ_0, we will show that for sufficiently small ε the operators $I - T_{\bar{\theta}}$ are invertible on $c_0(\mathbb{Z}, X)$ with uniformly bounded norms $\|(I - T_{\bar{\theta}})^{-1}\|$ for all ε-pseudo-orbits $\bar{\theta} \subset \Theta_0$. Fix $\theta \in \Theta_1$ and choose an ε-pseudo-orbit $\bar{\theta} = \{\theta_n\} \subset \Theta_0$ such that $d(\theta_n, \varphi^n\theta)$ is small for each $n \in \mathbb{Z}$. We use the continuity of $\Phi(\cdot)$ to show that for this choice of $\{\theta_n\}$ the norm $\|\pi_\theta(E) - T_{\bar{\theta}}\|_{\mathcal{L}(c_0(\mathbb{Z};X))}$ is small for sufficiently small $\varepsilon > 0$. Since $I - T_{\bar{\theta}}$ is invertible for $\bar{\theta} \subset \Theta_0$, we conclude that $I - \pi_\theta(E)$ is invertible and $\|(I - \pi_\theta(E))^{-1}\| \le C$ for each $\theta \in \Theta_1$. Hence, $\{\Phi^t\}$ has exponential dichotomy on Θ_1 by the Discrete Dichotomy Theorem.

Proof. Assume that $\{\Phi^t\}$ has exponential dichotomy on $\Theta_0 \in \mathcal{K}$ with projection $P(\cdot)$ and constants M, β. Set $c = \max_{\theta\in\Theta_0}\{\|P(\theta)\|, 1\}$. Choose $\gamma > 0$ such that

$$\gamma \le \tfrac{1}{16c}. \tag{3.16}$$

Fix $N \in \mathbb{N}$ such that $Me^{-\beta N} \le \gamma$. Set $\psi = \varphi^N$ and $\Psi(\theta) = \Phi^N(\theta)$. We derive for all $\theta \in \Theta_0$ that

$$\Psi(\theta)P(\theta) = P(\psi\theta)\Psi(\theta), \tag{3.17}$$

$$\Psi_Q(\theta) : \operatorname{Im} Q(\theta) \to \operatorname{Im} Q(\psi\theta) \text{ is invertible}, \tag{3.18}$$

$$\|\Psi_P(\theta)\| \le \gamma, \tag{3.19}$$

$$\|\Psi_Q^{-1}(\theta)\| \le \gamma. \tag{3.20}$$

Recall that $P(\cdot)$ and $\Psi(\cdot)$ are norm-continuous and Θ_0 and Θ are compact. Fix $\varepsilon_* > 0$ such that for every $\varepsilon \in (0, \varepsilon_*)$ the following inequalities hold:

$$\sup_{d(\theta,\eta)\le\varepsilon,\,\theta,\eta\in\Theta} \|\Psi(\theta) - \Psi(\eta)\| \le 3/16, \tag{3.21}$$

$$\sup_{d(\theta,\eta)\le\varepsilon,\,\theta,\eta\in\Theta_0} \|P(\theta) - P(\eta)\| \le \tfrac{1}{2(1+2c)}, \tag{3.22}$$

$$c\,\sup_{d(\theta,\eta)\le\varepsilon,\,\theta,\eta\in\Theta_0} \|P(\theta) - P(\eta)\| \cdot \max_{\theta\in\Theta} \|\Psi(\theta)\| \le 3/8. \tag{3.23}$$

For a sequence $\bar{\theta} = \{\theta_n\}_{n\in\mathbb{Z}} \subset \Theta_0$, we define by

$$T_{\bar{\theta}} : (x_n) \mapsto (\Psi(\theta_{n-1})x_{n-1}),\;\; P_{\bar{\theta}} = \text{ diag }\{P(\theta_n)\},\;\; Q_{\bar{\theta}} = I - P_{\bar{\theta}},\;\; \hat{p}_{\bar{\theta}} = \text{ diag }\{P(\psi\theta_{n-1})\}$$

operators on $c_0(\mathbb{Z}; X)$. Then (3.17) implies

$$T_{\bar{\theta}}P_{\bar{\theta}} = \hat{p}_{\bar{\theta}}T_{\bar{\theta}} \quad \text{and} \quad T_{\bar{\theta}}Q_{\bar{\theta}} = (I - \hat{p}_{\bar{\theta}})T_{\bar{\theta}}. \tag{3.24}$$

The following lemma shows that exponential dichotomy, (3.17)–(3.20), implies the invertibility of $I - T_{\bar{\theta}}$ for every ε-pseudo-orbit $\bar{\theta}$ for ψ in Θ_0, where ε is given by (3.21)–(3.23).

Lemma 3.12. *Let $\bar{\theta} = \{\theta_n\} \subset \Theta_0$ be an ε-pseudo-orbit for ψ, where ε satisfies* (3.21)–(3.23)*. Then $I - T_{\bar{\theta}}$ is invertible on $c_0(\mathbb{Z}, X)$ and*

$$\|(I - T_{\bar{\theta}})^{-1}\| \le \tfrac{8}{3}. \tag{3.25}$$

We prove the lemma later and finish the proof of Theorem 3.11 first. Fix ε satisfying (3.21)–(3.23). Fix $\delta^* \le \varepsilon/4$ such that $d(\psi\theta, \psi\eta) \le \varepsilon/2$ as soon as $d(\theta, \eta) < 2\delta^*$. Let $\delta < \delta^*$ and let $\Theta_1 \in \mathcal{K}$ satisfy (3.15). For $\theta \in \Theta_1$, define the operator $T_\theta := \pi_\theta(E^N) : (x_n) \mapsto (\Psi(\psi^{n-1}\theta)x_{n-1})$.

For a given $\theta \in \Theta_1$ and each $n \in \mathbb{Z}$, we use (3.15) to find $\theta_n \in \Theta_0$ such that $d(\theta_n, \psi^n\theta) < 2\delta$. The sequence $\bar{\theta} = \{\theta_n\} \subset \Theta_0$ is an ε-pseudo-orbit for ψ because of

$$d(\theta_n, \psi\theta_{n-1}) \;\le d(\theta_n, \psi^n\theta) + d(\psi(\psi^{n-1}\theta),\; \psi(\theta_{n-1})) \le 2\delta + \varepsilon/2 \le \varepsilon.$$

Thus, Lemma 3.12 can be applied to $I - T_{\bar{\theta}}$ yielding $\|(I - T_{\bar{\theta}})^{-1}\| \le 8/3$. Also,

$$\|T_\theta - T_{\bar{\theta}}\| = \sup_{n\in\mathbb{Z}} \|\Psi(\psi^n\theta) - \Psi(\theta_n)\| \le \sup_{d(\vartheta,\eta)\le\varepsilon} \|\Psi(\vartheta) - \Psi(\eta)\| \le 3/16 \tag{3.26}$$

by (3.21). Then the operator

$$I - T_\theta = I - T_{\bar{\theta}} - (T_\theta - T_{\bar{\theta}}) = (I - T_{\bar{\theta}})\left[I - (I - T_{\bar{\theta}})^{-1}(T_\theta - T_{\bar{\theta}})\right]$$

is invertible by (3.25) and (3.26). Moreover, the usual estimate of the Neumann series implies $\|(I - T_\theta)^{-1}\| \le 16/3$.

As a consequence of the Discrete Dichotomy Theorem, E^N is hyperbolic on $C_0(\Theta_1; X)$. Hence, E is hyperbolic on $C_0(\Theta_1; X)$, and $\{\Phi^n\}$ has exponential dichotomy on Θ_1 by the Discrete Dichotomy Theorem. ☐

Proof of Lemma 3.12. In the decomposition $c_0(\mathbb{Z}, X) = \operatorname{Im} P_{\bar{\theta}} \oplus \operatorname{Im} Q_{\bar{\theta}}$, we represent $T_{\bar{\theta}}$ as $T_{\bar{\theta}} = A + B$ by

$$A = \begin{bmatrix} P_{\bar{\theta}}T_{\bar{\theta}}P_{\bar{\theta}} & 0 \\ 0 & Q_{\bar{\theta}}T_{\bar{\theta}}Q_{\bar{\theta}} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & P_{\bar{\theta}}T_{\bar{\theta}}Q_{\bar{\theta}} \\ Q_{\bar{\theta}}T_{\bar{\theta}}P_{\bar{\theta}} & 0 \end{bmatrix}. \tag{3.27}$$

We prove the lemma in four steps.

Step 1. We show that

$$\max\{\|P_{\bar\theta}T_{\bar\theta}P_{\bar\theta}\|,\ \|\,[Q_{\bar\theta}T_{\bar\theta}Q_{\bar\theta}]^{-1}\,\|\}\le 1/8. \tag{3.28}$$

First, (3.16) and (3.19) imply

$$\|P_{\bar\theta}T_{\bar\theta}P_{\bar\theta}\|=\sup_{n\in\mathbb{Z}}\|P(\theta_n)\Psi(\theta_{n-1})P(\theta_{n-1})\|\le c\gamma\le 1/16\,.$$

Further, by (3.24) one has

$$Q_{\bar\theta}T_{\bar\theta}Q_{\bar\theta}=Q_{\bar\theta}(I-\hat p_{\bar\theta})\cdot(I-\hat p_{\bar\theta})T_{\bar\theta}Q_{\bar\theta}. \tag{3.29}$$

The operator $(I-\hat p_{\bar\theta})T_{\bar\theta}Q_{\bar\theta}:\ \mathrm{Im}Q_{\bar\theta}\to\ \mathrm{Im}\ (I-\hat p_{\bar\theta})$ is invertible by (3.18) and (3.20), and

$$\|\,[(I-\hat p_{\bar\theta})T_{\bar\theta}Q_{\bar\theta}]^{-1}\,\|=\sup_{n\in\mathbb{Z}}\|\Psi_Q^{-1}(\theta_n)\|\le\gamma. \tag{3.30}$$

To prove that $Q_{\bar\theta}(I-\hat p_{\bar\theta}):\ \mathrm{Im}\ (I-\hat p_{\bar\theta})\to\mathrm{Im}\,Q_{\bar\theta}$ is invertible, we consider the operator

$$R:=I-(P_{\bar\theta}-Q_{\bar\theta})(P_{\bar\theta}-\hat p_{\bar\theta})=Q_{\bar\theta}(I-\hat p_{\bar\theta})+P_{\bar\theta}\hat p_{\bar\theta}$$

on $c_0(\mathbb{Z},X)$ and its decomposition

$$R:\ \mathrm{Im}\ \hat p_{\bar\theta}\oplus\ \mathrm{Im}\ (I-\hat p_{\bar\theta})\to\mathrm{Im}\,P_{\bar\theta}\oplus\mathrm{Im}\,Q_{\bar\theta}\,;$$

$$R=\begin{bmatrix}P_{\bar\theta}\\ Q_{\bar\theta}\end{bmatrix}R\ [\hat p_{\bar\theta},I-\hat p_{\bar\theta}]=\begin{bmatrix}P_{\bar\theta}\hat p_{\bar\theta} & 0\\ 0 & Q_{\bar\theta}(I-\hat p_{\bar\theta})\end{bmatrix}.$$

Since $\bar\theta=\{\theta_n\}$ is an ε-pseudo-orbit for ψ, the inequality (3.22) yields

$$\|(P_{\bar\theta}-Q_{\bar\theta})(P_{\bar\theta}-\hat p_{\bar\theta})\|\le(1+2c)\|P_{\bar\theta}-\hat p_{\bar\theta}\|=(1+2c)\sup_{n\in\mathbb{Z}}\|P(\theta_n)-P(\psi\theta_{n-1})\|\le 1/2.$$

Therefore, R is invertible and, hence, $Q_{\bar\theta}(I-\hat p_{\bar\theta})$ is invertible with $\|[Q_{\bar\theta}(I-\hat p_{\bar\theta})]^{-1}\|\le\|R^{-1}\|\le 2$. This estimate together with (3.29)–(3.30) and (3.16) gives $\|\,[Q_{\bar\theta}T_{\bar\theta}Q_{\bar\theta}]^{-1}\,\|\le 2\gamma\le 1/8$.

Step 2. We show that $I-A$ is invertible, and

$$\|(I-A)^{-1}\|\le 4/3\,. \tag{3.31}$$

The inequality (3.28) implies that the operators

$$P_{\bar\theta}-P_{\bar\theta}T_{\bar\theta}P_{\bar\theta}\quad\text{and}\quad Q_{\bar\theta}-Q_{\bar\theta}T_{\bar\theta}Q_{\bar\theta}=-Q_{\bar\theta}T_{\bar\theta}Q_{\bar\theta}\left(Q_{\bar\theta}-(Q_{\bar\theta}T_{\bar\theta}Q_{\bar\theta})^{-1}\right)$$

are invertible on $\mathrm{Im}\,P_{\bar\theta}$ and $\mathrm{Im}\,Q_{\bar\theta}$, respectively. Then

$$(I-A)^{-1}=(P_{\bar\theta}-P_{\bar\theta}T_{\bar\theta}P_{\bar\theta})^{-1}\oplus(Q_{\bar\theta}-Q_{\bar\theta}T_{\bar\theta}Q_{\bar\theta})^{-1}=\sum_{k=0}^{\infty}(P_{\bar\theta}T_{\bar\theta}P_{\bar\theta})^k\oplus-\sum_{k=1}^{\infty}(Q_{\bar\theta}T_{\bar\theta}Q_{\bar\theta})^{-k}.$$

As a result, $\|(I-A)^{-1}\|\le 4/3$.

Step 3. We estimate $\|B\|=\max\{\|P_{\bar\theta}T_{\bar\theta}Q_{\bar\theta}\|,\ \|Q_{\bar\theta}T_{\bar\theta}P_{\bar\theta}\|\}$. By (3.24),

$$P_{\bar\theta}T_{\bar\theta}Q_{\bar\theta}=P_{\bar\theta}(P_{\bar\theta}-\hat p_{\bar\theta})T_{\bar\theta}\quad\text{and}\quad Q_{\bar\theta}T_{\bar\theta}P_{\bar\theta}=(\hat p_{\bar\theta}-P_{\bar\theta})\hat p_{\bar\theta}T_{\bar\theta}.$$

We also have $\|P_{\bar\theta}\|\le c$, $\|\hat p_{\bar\theta}\|\le c$, $\|\hat p_{\bar\theta}-P_{\bar\theta}\|\le\sup_{d(\theta,\eta)\le\varepsilon}\|P(\theta)-P(\eta)\|$, $\|T_{\bar\theta}\|\le\sup_{\theta\in\Theta}\|\Psi(\theta)\|$. Thus, (3.23) yields

$$\|B\|\le c\sup_{d(\theta,\eta)\le\varepsilon}\|P(\theta)-P(\eta)\|\cdot\sup_{\theta\in\Theta}\|\Psi(\theta)\|\le 3/8\,. \tag{3.32}$$

Step 4. We use (3.31) and (3.32) to finish the proof. Indeed, $\|(I-A)^{-1}B\| \le \frac{1}{2}$ and

$$\|(I-T_{\bar\theta})^{-1}\| = \|[(I-A)(I-(I-A)^{-1}B]^{-1}\| \\ \le \|(I-A)^{-1}\| \, \|[I-(I-A)^{-1}B]^{-1}\| \le 8/3. \qquad \square$$

We conclude this section with a typical application of Perturbation Theorem 3.11. This theorem proves the roughness of dichotomy with respect to the perturbations which are small in the metric on $C(\mathbb{R};\mathcal{L}(X))$ (with the topology of uniform convergence on compacta) versus the norm on $C_b(\mathbb{R};\mathcal{L}(X))$, compare [43, §5].

Consider a variational equation

$$\dot u(t) = [A + B(\varphi^t\theta)]u(t), \tag{3.33}$$

where A is a generator of a strongly continuous semigroup on X and B is a bounded function such that

$$\Theta \ni \theta \mapsto B(\varphi^{(\cdot)}\theta) \in C(\mathbb{R};\mathcal{L}(X)) \quad \text{is continuous.} \tag{3.34}$$

This means that

$$\sup_{t\in K} \|B(\varphi^t\theta_1) - B(\varphi^t\theta_2)\|_{\mathcal{L}(X)} \to 0 \quad \text{as} \quad \theta_1 \to \theta_2$$

for each compact set $K \subset \mathbb{R}$. Assume that (3.33) has exponential dichotomy at a point $\theta_0 \in \Theta$. Perturbation Theorem 3.11 implies that there exists an $\varepsilon > 0$ such that (3.33) has dichotomy for each $\theta \in \Theta$ with the property that the orbits $\mathcal{O}(\theta)$ and $\mathcal{O}(\theta_0)$ satisfy $dist_{\overline{\mathcal{O}(\theta_0)}}(\overline{\mathcal{O}(\theta)}) < \varepsilon$. Indeed, the cocycle $\{\Phi^t(\theta)\}$ for (3.33) satisfies

$$\Phi^t(\theta)x = e^{tA}x + \int_0^t e^{(t-\tau)A}B(\varphi^\tau\theta)\Phi^\tau(\theta)x\, d\tau, \quad x \in X,$$

see [10, Thm. 5.1]. Using Grownwall inequality, boundedness of B, and condition (3.34), one checks that $\Theta \ni \theta \mapsto \Phi^1(\theta) \in \mathcal{L}(X)$ is norm continuous.

4. Annular Hull and Spectral Mapping Theorems

We now want to characterize the exponential dichotomy of the cocycle $\{\Phi^t\}_{t\ge0}$ by the spectrum of the generator Γ of the corresponding evolution semigroup $\{E^t\}_{t\ge0}$ on the space $\mathcal{F} = C_0(\Theta;X)$. To that purpose, we prove the spectral mapping and annular hull theorems for $\{E^t\}_{t\ge0}$. Assuming $\sigma_{ap}(E) \cap \mathbb{T} \ne \emptyset$, we give an explicit construction of approximate eigenfunctions for Γ. These approximate eigenfunctions are supported in flow boxes selected to contain *Mañe sequences* for $\{\Phi^n\}_{n\in\mathbb{Z}_+}$. Throughout, $\{\Phi^t\}_{t\ge0}$ is an exponentially bounded, strongly continuous cocycle over a continuous flow $\{\varphi^t\}_{t\in\mathbb{R}}$ on a locally compact metric space Θ.

Definition 4.1. *A sequence $\{\theta_m\}_{m=1}^\infty \subset \Theta$ is called* Mañe sequence *if there exist constants C, $c > 0$ and vectors $x_m \in X$ such that*

$$|\Phi^m(\theta_m)x_m| \ge c \quad \textit{and} \tag{4.1}$$

$$|\Phi^k(\theta_m)x_m| \le C \quad \textit{for} \quad k = 0,1,\dots,2m. \tag{4.2}$$

We first discuss this concept, see also Corollary 4.12 and 4.13. Assume, for a moment, that $\{\Phi^n\}_{n\in\mathbb{Z}}$ is a cocycle with *invertible* values and that θ_0 is a Mañe point, i.e., there exists a nonzero vector x_0 such that $C := \sup\{|\Phi^n(\theta_0)x_0| : n \in \mathbb{Z}\} < \infty$, see the Introduction. We claim that $\theta_m = \varphi^{-m}\theta_0$ form a Mañe sequence with vectors $x_m = \Phi^{-m}(\theta_0)x_0$. Indeed,

$$\begin{aligned}|\Phi^m(\theta_m)x_m| &= |\Phi^m(\varphi^{-m}\theta_0)\Phi^{-m}(\theta_0)x_0| = |x_0| =: c > 0 \quad \text{and}\\ |\Phi^k(\theta_m)x_m| &= |\Phi^k(\varphi^{-m}\theta_0)\Phi^{-m}(\theta_0)x_0| = |\Phi^{k-m}(\theta_0)x_0| \le C\end{aligned}$$

for $k \in \mathbb{Z}$. Thus, Definition 4.1 extends the notion of a Mañe point to the non-invertible case.

Going back to the situation of a cocycle with possibly noninvertible values, we remark the following.

Proposition 4.2. *If $\{\Phi^t\}_{t\in\mathbb{N}}$ has exponential dichotomy on Θ, then there are no Mañe sequences in Θ.*

Proof. Suppose, in the contrary, that $\{\theta_m\}_1^\infty$ is a Mañe sequence with the corresponding vectors x_m. Decompose $x_m = x_m^s + x_m^u$, where $x_m^s \in \operatorname{Im} P(\theta_m)$ and $x_m^u \in \operatorname{Im} Q(\theta_m)$. Let $y_m = \Phi^m(\theta_m)x_m$, $y_m^s = \Phi^m(\theta_m)x_m^s$, and $y_m^u = \Phi^m(\theta_m)x_m^u$. Using dichotomy and (4.2) with $k = 2m$ and $k = m$, we estimate

$$\begin{aligned}|y_m^u| &\le Me^{-\beta m}|\Phi_Q^m(\varphi^m\theta_m)y_m^u| \le Me^{-\beta m}\{|\Phi^{2m}(\theta_m)x_m| + |\Phi_P^m(\varphi^m\theta_m)y_m^s|\}\\ &\le Me^{-\beta m}\{C + Me^{-\beta m}(|y_m| + |y_m^u|)\} \le Me^{-\beta m}\{C + Me^{-\beta m}(C + |y_m^u|)\}.\end{aligned}$$

Thus, for sufficiently large m, we have

$$|y_m^u| \le Me^{-\beta m}\,\frac{C + MCe^{-\beta m}}{1 - M^2e^{-2\beta m}} \;\to\; 0 \quad \text{as} \quad m \to \infty.$$

Together with (4.1) and (4.2) with $k = m$, this implies $|y_m^s| \in [c/2,\, 2C]$ for large m. Since Φ^t has exponential dichotomy, we conclude

$$\frac{c}{2} \le |y_m^s| = |\Phi^m(\theta_m)x_m^s| \le Me^{-\beta m}|x_m^s| \quad \text{and} \quad 2C \ge |y_m^u| = |\Phi^m(\theta_m)x_m^u| \ge M^{-1}e^{\beta m}|x_m^u|.$$

Hence, $|x_m^s| \to \infty$ and $|x_m^u| \to 0$ as $m \to \infty$. This contradicts (4.2) with $k = 0$. $\square$

We now give sufficient conditions for the existence of a Mañe sequence, cf. [8, 29], using the following refinement of Theorem 3.7 (c)⇒(d).

Lemma 4.3. *Assume there exists a constant $c > 0$ such that*

$$\|I - \pi_\theta(E)\|_{\bullet,\ell^\infty(\mathbb{Z};X)} \ge c \quad \textit{for all} \;\; \theta \in \Theta.$$

Then we have $\|zI - E\|_{\bullet,\mathcal{F}} \ge c$ for all $z \in \mathbb{T}$.

Proof. Let $f \in \mathcal{F}$ and $z \in \mathbb{T}$. Notice that for each $\theta \in \Theta$ the sequence $\bar{x}_\theta := (f(\varphi^k\theta))_{k\in\mathbb{Z}}$ is bounded. The assumption and Proposition 3.6 yield

$$\begin{aligned}\|zf - Ef\|_{\mathcal{F}} &= \sup_{\theta\in\Theta}|zf(\theta) - \Phi^1(\varphi^{-1}\theta)f(\varphi^{-1}\theta)| = \sup_{\theta\in\Theta}\sup_{k\in\mathbb{Z}}|zf(\varphi^k\theta) - \Phi^1(\varphi^{k-1}\theta)f(\varphi^{k-1}\theta)|\\ &= \sup_{\theta\in\Theta}\|[zI - \pi_\theta(E)]\bar{x}_\theta\|_{\ell^\infty(\mathbb{Z};X)} \ge c\,\sup_{\theta\in\Theta}\|\bar{x}_\theta\|_{\ell^\infty(\mathbb{Z};X)} = c\,\|f\|_{\mathcal{F}}.\end{aligned} \qquad \square$$

Proposition 4.4. *A Mañe sequence* $\{\theta_m\}_1^\infty$ *exists provided one of the following conditions holds.*

(a) $\inf_{\theta\in\Theta} \|I - \pi_\theta(E)\|_{\bullet,\ell^\infty} = 0$.
(b) $\sigma_{ap}(E) \cap \mathbb{T} \neq \emptyset$.

Proof. In view of Lemma 4.3, it suffices to consider (a). Since $\{\Phi^t\}$ is exponentially bounded, $N := \sup_\theta \|\pi_\theta(E)\|$ is finite. Fix $m \in \mathbb{N}$ and let $\varepsilon := (4\sum_{k=0}^{2m} N^k)^{-1}$. By the assumption, there exists $\sigma_m \in \Theta$ and $\bar{y}^{(m)} = (y_l^{(m)})_{l\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z}; X)$ such that $\|\bar{y}^{(m)}\|_\infty = 1$ and $\|[I - \pi_{\sigma_m}(E)]\bar{y}^{(m)}\|_\infty < \varepsilon$. Hence,

$$\tfrac{1}{2} < \|[\pi_{\sigma_m}(E)]^k \bar{y}^{(m)}\|_{\ell^\infty(\mathbb{Z};X)} < 2 \quad \text{for} \quad k = 0, 1, \ldots 2m.$$

For a fixed $k \in \{0, 1, \ldots, 2m\}$, we have

$$\|[\pi_{\sigma_m}(E)]^k \bar{y}^{(m)}\|_{l^\infty(\mathbb{Z};X)} = \sup_{l\in\mathbb{Z}} |\Phi^k(\varphi^{l-k}\sigma_m) y_{l-k}^{(m)}| = \sup_{l\in\mathbb{Z}} |\Phi^k(\varphi^{l-m}\sigma_m) y_{l-m}^{(m)}|.$$

Select l such that $\frac{1}{2} \le |\Phi^m(\varphi^{l-m}\sigma_m) y_{l-m}^{(m)}|$. Note that $|\Phi^k(\varphi^{l-m}\sigma_m) y_{l-m}^{(m)}| \le 2$ holds for $k = 0, 1, \ldots 2m$. Therefore, $\theta_m := \varphi^{l-m}\sigma_m$ is a Mañe sequence with the vectors $x_m := y_{l-m}^{(m)}$. $\square$

In the construction of the approximate eigenfunctions we will use some basic facts concerning the period function and flow boxes, or tubes.

A point $\theta \in \Theta$ is called *periodic* with respect to the flow $\{\varphi^t\}_{t\in\mathbb{R}}$ if $\varphi^T\theta = \theta$ for some $T > 0$, and *aperiodic* if there is no such T. The *prime period function* $p : \Theta \to \mathbb{R} \cup \{\infty\}$ is defined as $p(\theta) = \inf\{T > 0 : \varphi^T\theta = \theta\}$ for periodic points θ and $p(\theta) = \infty$ for aperiodic θ. Further, set

$$\begin{aligned} \mathcal{B}(\Theta) &= \{\theta \in \Theta : p \text{ is bounded in a neighborhood of } \theta\} \quad \text{and} \\ \mathcal{BC}(\Theta) &= \{\theta \in \mathcal{B}(\Theta) : p \text{ is continuous at } \theta\}. \end{aligned}$$

Also, we denote by $\mathcal{O}(\theta) = \{\varphi^t\theta : t \in \mathbb{R}\}$ the orbit through $\theta \in \Theta$. The following result is taken from [6, Thm. IV.2.11].

Lemma 4.5. *Let* $\{\varphi^t\}_{t\in\mathbb{R}}$ *be a continuous flow on a locally compact, metric space* Θ *with prime period function* p. *Suppose that* $\theta_0 \in \Theta$ *and* τ *are such that* $0 < \tau < p(\theta_0)/4$. *Then there exists an open set* $\mathcal{U}$ *and a set* Σ *such that* $\theta_0 \in \Sigma \subset \mathcal{U}$, *the closure* $\overline{\mathcal{U}}$ *is compact, and for each* $\theta \in \mathcal{U}$ *there exists a unique number* $t = t(\theta)$ *in* $(-\tau, \tau)$ *such that the point* $\sigma := \varphi^{-t}\theta$ *belongs to* Σ. *Moreover, the map* $\theta \mapsto t(\theta)$ *is continuous on* $\mathcal{U}$.

The set $\mathcal{U}$ as in Lemma 4.5 is called *flow-box of length* τ with *cross-section* Σ at $\theta_0 \in \Theta$.

Lemma 4.6. *For a continuous flow* $\{\varphi^t\}_{t\in\mathbb{R}}$ *on a locally compact metric space* Θ *with prime period function* p, *we have*

(a) *p is lower semicontinuous, in particular, the set* $\{\theta : p(\theta) \le d\}$ *is closed for each* $d \ge 0$;
(b) *the points of continuity of* p *are dense in* Θ;
(c) *for* $\theta_0 \in \mathcal{BC}(\Theta)$ *with* $p(\theta_0) > 0$, *there exists a relatively compact, open set* $\mathcal{U}$ *and a set* $\Sigma \ni \theta_0$ *such that* $\mathcal{O}(\theta_0) \subset \mathcal{U} \subset \mathcal{BC}(\Theta)$, *for* $\theta \in \mathcal{U}$ *there is unique* $t = t(\theta) \in [0, p(\theta))$ *with* $\varphi^{-t}\theta \in \Sigma$, *and* $\mathcal{U} \ni \theta \mapsto t(\theta)$ *is continuous.*

Proof. Assertion (a) is shown in [3, p.314]. The second assertion follows from (a) and [15, pp.87]. To show (c), we use Lemma 4.5 to find a relatively compact, open flow box $\mathcal{U}_0$ at θ_0 of length $\tau \in (0, p(\theta_0)/4)$ with cross section Σ_0 such that $\theta \mapsto t(\theta)$ is continuous, the function p takes values in $(3p(\theta_0)/4, 5p(\theta_0)/4)$ on $\mathcal{U}_0$, and

$$\mathcal{O}(\theta_0) \cap \Sigma_0 = \{\theta_0\}. \tag{4.3}$$

Let Σ_1 be a relatively open subset of Σ_0 containing θ_0 such that $\overline{\Sigma}_1 \subset \mathcal{U}_0$. We claim that $\mathcal{O}(\theta_0)$ has an open neighborhood $\mathcal{U} \subset \mathcal{U}_0$ satisfying

$$\text{for all } \theta \in \mathcal{U} \text{ the set } \mathcal{O}(\theta) \cap \Sigma_1 \text{ contains exactly one element.} \tag{4.4}$$

Suppose that no such neighborhood exists. Then there were points θ_n converging to a point $\varphi^t\theta_0$ for some $0 \le t < p(\theta_0)$ such that $\mathcal{O}(\theta_n) \cap \Sigma_1$ is empty or contains more than one element. But, since $\varphi^{-t}\theta_n \to \theta_0$, we obtain $\varphi^{-t}\theta_n \in \mathcal{U}_0$ and, hence, points $\sigma_n \in \mathcal{O}(\theta_n) \cap \Sigma_0$ for large n. Moreover, $t(\varphi^{-t}\theta_n) \to t(\theta_0) = 0$ so that $\sigma_n \to \theta_0$. In particular, $\sigma_n \in \Sigma_1$ for large n. Then there must exist $\sigma_n' = \varphi^{s_n}\sigma_n \in \Sigma_1$ for some s_n with

$$0 < \tau \le s_n \le p(\sigma_n) - \tau < p(\theta_0) - \frac{\tau}{2}$$

and n sufficiently large, where we have used the continuity of p at θ_0. Taking a subsequence, we may assume that $s_n \to s$ so that

$$\sigma_n' \to \varphi^s\theta_0 \in \overline{\Sigma}_1 \subset \mathcal{U}_0\,.$$

Again by the continuity of $t(\cdot)$, we derive $\varphi^s\theta_0 \in \Sigma_0$. Due to (4.3), this yields $\varphi^s\theta_0 = \theta_0$. But this contradicts $0 < s < p(\theta_0)$, and so there exists a neighborhood $\mathcal{U}$ of $\mathcal{O}(\theta_0)$ satisfying (4.4).

It remains to show the continuity of $p(\cdot)$ and $t(\cdot)$ on $\mathcal{U}$. So let $\theta_n \to \theta$ in $\mathcal{U}$. Suppose that $p(\theta_n)$ does not converge to $p(\theta)$. But, $p(\theta_n) \to p_0$ for a subsequence, and so $\theta_n = \varphi^{p(\theta_n)}\theta_n \to \varphi^{p_0}\theta$. Hence, $p_0 = kp(\theta)$ for $k \neq 1$. This is impossible since $p(\theta), p_0 \in [3p(\theta_0)/4, 5p(\theta_0)/4]$. Further, $\varphi^r\theta_n \to \varphi^r\theta$ in $\mathcal{U}_0$ for some r and sufficiently large n. This yields the continuity of $\theta \mapsto t(\theta)$ on $\mathcal{U}$. ☐

Recall that Γ denotes the infinitesimal generator of the evolution semigroup $\{E^t\}_{t\ge 0}$ on $\mathcal{F} = C_0(\Theta; X)$ defined by formula (1.1). Our next goal is to construct the approximate eigenfunctions for Γ on $\mathcal{F}$ for a given Mañe sequence. We first treat the case that the Mañe sequence consists of aperiodic points.

Lemma 4.7. *Assume $\{\theta_m\}_1^\infty$ is a Mañe sequence such that $\theta_m \notin \mathcal{B}(\Theta)$ or $p(\theta_m) \to \infty$. Then, for each $\xi \in \mathbb{R}$, there exist functions $f_n \in \mathcal{D}(\Gamma)$ with $\|f_n\|_\infty = 1$ such that $\|(i\xi - \Gamma)f_n\|_\infty \to 0$ as $n \to \infty$.*

Proof. Without loss of generality, we assume that $p(\theta_m) \to \infty$ as $m \to \infty$. Denote $n = \min\{m, \left[\frac{1}{5}p(\theta_m)\right]\}$, where $[\,\cdot\,]$ is the integer part. Note that $n > 0$ for sufficiently large m. Set $\sigma_0 = \varphi^m(\theta_m)$ and $x = \Phi^{m-n}(\theta_m)x_m$. By the assumption there are constants $C, c > 0$ such that $c \le |\Phi^m(\theta_m)x_m|$ and $|\Phi^k(\theta_m)x_m| \le C$ for $k = 0, 1, \ldots, 2m$. This implies that

$$0 < c \le |\Phi^n(\varphi^{-n}\sigma_0)x| \quad \text{and} \quad |\Phi^k(\varphi^{-n}\sigma_0)x| \le C \quad \text{for} \quad k = 0, 1, \ldots, 2n. \tag{4.5}$$

Using Lemma 4.5, we find an open flow box $\mathcal{U}$ of length n at σ_0 with cross-section $\Sigma \ni \sigma_0$ such that

$$|\Phi^t(\varphi^{-n}\sigma)x - \Phi^t(\varphi^{-n}\sigma_0)x| \le 1 \quad \text{for } \ 0 \le t \le 2n \text{ and } \sigma \in \Sigma. \tag{4.6}$$

For fixed n, choose "bump"-functions $\beta \in C(\mathcal{U})$ with $0 \le \beta \le 1$, $\beta(\sigma_0) = 1$, and compact support and $\gamma \in C^1(\mathbb{R})$ with $0 \le \gamma \le 1$, $\operatorname{supp}\gamma \subset (0, 2n)$, $\gamma(n) = 1$, and $\|\gamma'\|_\infty \le 2/n$. For $\xi \in \mathbb{R}$, define

$$f_n(\theta) = e^{-i\xi t}\beta(\sigma)\gamma(t)\Phi^t(\varphi^{-n}\sigma)x \quad \text{if } \ \theta = \varphi^{t-n}\sigma \in \mathcal{U} \ \text{ with } \ t \in (0, 2n), \ \sigma \in \Sigma, \tag{4.7}$$

and $f_n(\theta) = 0$ if $\theta \notin \mathcal{U}$. Clearly, $f_n \in \mathcal{F}$ and $\|f_n\|_\infty \ge |f_n(\sigma_0)| \ge c > 0$ by (4.5). For $h > 0$ so small that $\gamma = 0$ on $[2n - h, 2n]$, one has

$$(E^h f_n)(\theta) = e^{-i\xi(t-h)}\beta(\sigma)\gamma(t-h)\Phi^t(\varphi^{-n}\sigma)x$$

for $\theta = \varphi^{t-n}\sigma \in \mathcal{U}$ and $(E^h f_n)(\theta) = 0$ for $\theta \notin \mathcal{U}$. Thus, $f_n \in \mathcal{D}(\Gamma)$ and

$$(\Gamma f_n)(\theta) - i\xi f_n(\theta) = -\gamma'(t)\beta(\sigma)e^{-i\xi t}\Phi^t(\varphi^{-n}\sigma)x \tag{4.8}$$

for $\theta = \varphi^{t-n}\sigma$ with $t \in (0, 2n)$ and $\sigma \in \Sigma$ and $(\Gamma f_n)(\theta) = 0$ for $\theta \notin \mathcal{U}$. Let $N = \sup\{\|\Phi^\tau(\theta)\| : \tau \in [0,1], \theta \in \Theta\}$. Then (4.6) and (4.5) imply

$$\|\Gamma f_n - i\xi f_n\|_\infty \le \frac{2N}{n} \max_{k=0,\dots,2n} |\Phi^k(\varphi^{-n}\sigma)x| \le \frac{2N(C+1)}{n}. \qquad \square$$

If the Mañe sequence consists of periodic points, we will make use of the following fact.

Lemma 4.8. *Assume there is a sequence $\{\theta_m\} \subset \mathcal{BC}(\Theta)$ with $0 < d_0 \le p(\theta_m) \le d_1$ and $x_m \in X$ such that $|x_m| = 1$ and $|[z - \Phi^{p(\theta_m)}(\theta_m)]x_m| \to 0$ as $m \to \infty$ for some $z \in \mathbb{T}$. Then there exist $\xi \in \mathbb{R}$ and $f_n \in \mathcal{D}(\Gamma)$ such that $\|f_n\|_\infty = 1$ and $\|(i\xi - \Gamma)f_n\|_\infty \to 0$ as $n \to \infty$.*

Proof. We may assume that $z = e^{i\eta} = e^{i\frac{\eta}{p(\theta_n)}p(\theta_n)}$ and $p(\theta_n) \to p_0 \in [d_0, d_1]$. Set $\xi = \eta/p_0$. By Proposition 4.6, there exist open tubular neighborhoods $\mathcal{U}_n$ of θ_n with sections $\Sigma_n \ni \theta_n$ such that on $\mathcal{U}_n$ the function p is continuous and takes values in $[d_0/2, 2d_1]$ and

$$|x_n - e^{-i\xi p(\sigma)}\Phi^{p(\sigma)}(\sigma)x_n| \to 0 \quad \text{for } \sigma \in \Sigma_n \ \text{ as } n \to \infty. \tag{4.9}$$

Take a continuous "bump"-function $\beta : \mathcal{U}_n \to [0, 1]$ with $\beta(\theta_n) = 1$ and compact support. Choose a function $\alpha \in C^1[0, 1]$ satisfying $0 \le \alpha \le 1$, $\alpha = 0$ on $[0, 1/3]$, $\alpha = 1$ on $[2/3, 1]$, and $\|\alpha'\|_\infty \le 4$. Define

$$f_n(\theta) = e^{-i\xi t}\beta(\sigma)\Phi^t(\sigma)[\alpha(t/p(\sigma))x_n + (1 - \alpha(t/p(\sigma)))e^{-i\xi p(\sigma)}\Phi^{p(\sigma)}(\sigma)x_n] \tag{4.10}$$

for $\theta = \varphi^t\sigma \in \mathcal{U}_n$, where $\sigma \in \Sigma_n$ and $0 \le t < p(\sigma)$, and $f_n(\theta) = 0$ for $\theta \notin \mathcal{U}_n$. Clearly, $f_n \in \mathcal{F}$ with

$$\|f_n\|_\infty \ge |f_n(\theta_n)| = |\Phi^{p(\theta_n)}(\theta_n)x_n| \ge 1/2$$

for large n. For $0 < h < d_0/6$, we compute

$$(E^h f_n)(\theta) = \begin{cases} e^{-i\xi(t-h)}\beta(\sigma)\Phi^t(\sigma)[\alpha(\frac{t-h}{p(\sigma)})x_n + (1 - \alpha(\frac{t-h}{p(\sigma)}))e^{-i\xi p(\sigma)}\Phi^{p(\sigma)}(\sigma)x_n], & t \ge h, \\ e^{-i\xi(t-h)}\beta(\sigma)\Phi^t(\sigma)e^{-i\xi p(\sigma)}\Phi^{p(\sigma)}(\sigma)x_n, & t < h, \end{cases}$$

for $\theta = \varphi^t\sigma \in \mathcal{U}_n$ and $(E^h f_n)(\theta) = 0$ otherwise. (Write $\varphi^{-h}\theta = \varphi^{p(\theta)+t-h}\sigma$ if $\theta \in \mathcal{U}_n$ and $t - h < 0$.) Thus, $f_n \in \mathcal{D}(\Gamma)$ and

$$\Gamma f_n(\theta) - i\xi f_n(\theta) = -e^{-i\xi t}\beta(\sigma)\frac{1}{p(\sigma)}\alpha'(\tfrac{t}{p(\sigma)})\Phi^t(\sigma)[x_n - e^{-i\xi p(\sigma)}\Phi^{p(\sigma)}(\sigma)x_n] \qquad (4.11)$$

for $\theta = \varphi^t\sigma \in \mathcal{U}_n$ and zero otherwise. Let $N = \sup\{\|\Phi^\tau(\theta)\| : \theta \in \Theta,\ 0 \le \tau \le 2d_1\}$. Finally we obtain

$$\|(\Gamma - i\xi)f_n\|_\infty \le \frac{8N}{d_0}\,|x_n - e^{-i\xi p(\sigma)}\Phi^{p(\sigma)}(\sigma)x_n| \;\to\; 0$$

as $n \to \infty$ thanks to (4.9). □

We are now in the position to prove the annular hull theorem and the spectral mapping theorem for the evolution semigroup

$$(E^t f)(\theta) = \Phi^t(\varphi^{-t}\theta)f(\varphi^{-t}\theta) \quad \text{on} \;\; \mathcal{F} = C_0(\Theta; X)$$

with generator Γ. The spectral mapping theorem says that

$$\sigma(E^t)\backslash\{0\} = \exp t\sigma(\Gamma) \quad \text{for} \;\; t \ge 0. \qquad (4.12)$$

In general, this result does not hold if φ^t is not aperiodic, see e.g. [9, Ex. 2.3]. The annular hull theorem states that

$$\exp t\sigma(\Gamma) \subseteq \sigma(E^t)\backslash\{0\} \subseteq \mathcal{H}(\exp t\sigma(\Gamma)) \quad \text{for} \;\; t \ge 0,$$

where $\mathcal{H}(\cdot)$ is the annular hull of the set $(\cdot)$, that is, the union of all circles centered at the origin that intersect the set $(\cdot)$. In other words, the annular hull theorem can be written as

$$\mathbb{T} \cdot (\sigma(E^t)\backslash\{0\}) = \mathbb{T} \cdot \exp t\sigma(\Gamma) \quad \text{for} \;\; t \ge 0. \qquad (4.13)$$

Observe that the annular hull theorem can be violated if there are fixed points. Indeed, for the one-point set $\Theta = \{\theta\}$ and identical flow $\varphi^t(\theta) \equiv \theta$ define a cocycle $\Phi^t(\theta) = e^{tA}$ for a strongly continuous semigroup $\{e^{tA}\}$ on X which does not satisfy the annular hull theorem (see [33, A-III] for examples). Clearly, in this case (4.13) does not hold for the corresponding evolution semigroup.

Theorem 4.9 (Spectral Mapping/Annular Hull Theorem). *Let $\{E^t\}_{t\ge 0}$ be the evolution semigroup on $\mathcal{F} = C_0(\Theta; X)$ with generator Γ induced by an exponentially bounded, strongly continuous cocycle $\{\Phi^t\}_{t\in\mathbb{R}_+}$ over a continuous flow $\{\varphi^t\}_{t\in\mathbb{R}}$ on a locally compact metric space Θ, and let p be the function of prime periods for $\{\varphi^t\}$.*

- (a) *If $\mathcal{B}(\Theta) = \emptyset$, then $\sigma(\Gamma)$ is invariant with respect to vertical translations, $\sigma(E^t)$ is invariant with respect to rotations centered at zero, and the spectral mapping theorem* (4.12) *holds.*
- (b) *If $p(\theta) \ge d_0 > 0$ for all $\theta \in \mathcal{B}(\Theta)$, then the annular hull theorem* (4.13) *holds.*

Proof. By virtue of the spectral inclusion theorem and the spectral mapping theorem for the residual spectrum, see e.g. [33, A-III.6], and a standard rescaling, (4.13) holds provided that

$$\sigma_{ap}(E) \cap \mathbb{T} \ne \emptyset \;\; \text{implies} \;\; \sigma(\Gamma) \cap i\mathbb{R} \ne \emptyset$$

for $E = E^1$. So let $z \in \sigma_{ap}(E) \cap \mathbb{T}$. By Proposition 4.4, there exists a Mañe sequence $\{\theta_m\}$ in Θ. We have to consider two cases:

Aperiodic Case: there is a subsequence with $\theta_m \notin \mathcal{B}(\Theta)$ or $p(\theta_m) \to \infty$ as $m \to \infty$;
Periodic Case: $\theta_m \in \mathcal{B}(\Theta)$ and $p(\theta_m) < d_1$ for a constant d_1.

In the aperiodic case, Lemma 4.7 shows that $i\mathbb{R} \subseteq \sigma_{ap}(\Gamma)$ which implies (4.12). The other assertions in (a) then follow from the spectral inclusion theorem [33, A-III.6.2] and $\partial\sigma(\Gamma) \subset \sigma_{ap}(\Gamma)$.

In the periodic case, using Lemma 4.6, we may and will assume that $\theta_m \in \Omega := \{\theta \in \mathcal{B}C(\Theta) : d_0 \le p(\theta) \le d_1\}$. The set Ω is a φ^t-invariant and locally compact metric space by Lemma 4.6. On Ω, we define a new flow $\{\psi^t\}_{t\in\mathbb{R}}$ by $\psi^t(\theta) = \varphi^{tp(\theta)}(\theta)$ and a new cocycle $\{\Psi^t\}_{t\in\mathbb{R}_+}$ over $\{\psi^t\}$ by $\Psi^t(\theta) = \Phi^{tp(\theta)}(\theta)$. Recall that $\theta \mapsto p(\theta)$ is continuous on $\mathcal{B}C(\Theta)$. Clearly, $\{\psi^t\}$ is continuous and $\{\Psi^t\}$ is strongly continuous and exponentially bounded on Ω. So there exists the induced evolution semigroup $\{E^t_\Psi\}$ on $C_0(\Omega; X)$. Let $E_\Psi = E^1_\Psi$. We stress that $(E_\Psi f)(\theta) = \Phi^{p(\theta)}(\theta)f(\theta)$ for $f \in C_0(\Omega; X)$ and $\theta \in \Omega$. The following fact is stated as a separate lemma for later use.

Lemma 4.10. *Let Ω be as defined above. Assume that $\sigma(\Phi^{p(\theta)}(\theta)) \cap \mathbb{T} = \emptyset$ for each $\theta \in \Omega$ and*

$$\sup_{\theta\in\Omega} \|[zI - \Phi^{p(\theta)}(\theta)]^{-1}\|_{\mathcal{L}(X)} < \infty \quad \text{for all } \ z \in \mathbb{T}.$$

Then $\{\Phi^t\}$ has exponential dichotomy on Ω.

Proof of Lemma 4.10. The assumptions imply $\sigma(E_\Psi) \cap \mathbb{T} = \emptyset$, where

$$(zI - E_\Psi)^{-1} f(\theta) = [zI - \Phi^{p(\theta)}(\theta)]^{-1} f(\theta)$$

for $\theta \in \Omega$, $z \in \mathbb{T}$, and $f \in C_0(\Omega; X)$. An application of the Dichotomy Theorem 2.7 to $\{E_\Psi\}$ shows that the cocycle $\{\Psi^t\}$ has exponential dichotomy on Ω with the dichotomy projection $P(\cdot)$ and the dichotomy constants $\beta, M > 0$. Therefore, the cocycle $\{\Phi^t\}$ has exponential dichotomy on Ω with $P(\cdot)$, β/d_1, and M. □

Since Ω contains a Mañe sequence, the cocycle $\{\Phi^t\}$ does not have exponential dichotomy on Ω due to Proposition 4.2. Therefore the assumptions in Lemma 4.10 cannot be satisfied. This means that either

(1) $\|zI - \Phi^{p(\theta_n)}(\theta_n)\|_{\bullet,X} \to 0$ for some $z \in \mathbb{T}$ and $\theta_n \in \Omega$ or
(2) there exist $z \in \mathbb{T}$, $\theta_0 \in \Omega, y \in X$, and $\delta > 0$ such that $|zx - \Phi^{p(\theta_0)}(\theta_0)x - y| \ge \delta$ for all $x \in X$.

In case (1), Lemma 4.8 implies $\sigma_{ap}(\Gamma) \cap i\mathbb{R} \neq \emptyset$. In case (2), take a function $g \in C_0(\Theta; X)$ such that $g(\theta_0) = y$. For all $f \in C_0(\Theta; X)$, we have

$$\|zf - E^{p(\theta_0)} f - g\|_\infty \ge |zf(\theta_0) - \Phi^{p(\theta_0)} f(\theta_0) - y| \ge \delta;$$

that is, $z \in \sigma_r(E^{p(\theta_0)}) \cap \mathbb{T}$. Hence, $\sigma_r(\Gamma) \cap i\mathbb{R} \neq \emptyset$ by the spectral mapping theorem for the residual spectrum [33, A-III.6.3]. Altogether, we have established (b) in Theorem 4.9. □

Combining the above result with the Dichotomy Theorem 2.7, we obtain the following.

Corollary 4.11. *Let $\{\Phi^t\}_{t\in\mathbb{R}_+}$ be an exponentially bounded, strongly continuous cocycle $\{\Phi^t\}_{t\in\mathbb{R}_+}$ over a continuous flow $\{\varphi^t\}_{t\in\mathbb{R}}$ on a locally compact metric space Θ, let $p(\theta) \geq d_0 > 0$ for all $\theta \in \mathcal{B}(\Theta)$, and let Γ be the generator of the induced evolution semigroup on $C_0(\Theta; X)$. Then the cocycle has exponential dichotomy on Θ if and only if $i\mathbb{R} \subset \rho(\Gamma)$. If $\mathcal{B}(\Theta) = \emptyset$, then these conditions are equivalent to $\rho(\Gamma) \cap i\mathbb{R} \neq \emptyset$.*

We have seen in Proposition 4.4 that the condition $\sigma_{\rm ap}(E)\cap\mathbb{T} \neq \emptyset$ implies the existence of a Mañe sequence. We can now give two results in the opposite direction.

Corollary 4.12. *Assume that $\sigma_r(E) \cap \mathbb{T} = \emptyset$. Then, $\sigma_{ap}(E) \cap \mathbb{T} \neq \emptyset$ if and only if there exists a Mañe sequence $\{\theta_m\}_1^\infty$.*

Proof. If $\sigma_{ap}(E) \cap \mathbb{T} \neq \emptyset$, then a Maǹe sequence exists by Proposition 4.4. If a Mañe sequence exists, then $\sigma(E) \cap \mathbb{T} \neq \emptyset$ by Proposition 4.2 and Dichotomy Theorem 2.7. So the assumption, $\sigma_r(E) \cap \mathbb{T} = \emptyset$, implies the result. $\square$

Corollary 4.13. *Assume that $\{\theta_m\}_1^\infty$ is a Mañe sequence.*

(a) *If the aperiodic case holds for $\{\theta_m\}_1^\infty$, that is, there is a subsequence such that $\theta_m \notin \mathcal{B}(\Theta)$ or $p(\theta_m) \to \infty$ as $m \to \infty$, then $\mathbb{T} \subset \sigma_{ap}(E)$.*

(b) *If the periodic case holds for $\{\theta_m\}_1^\infty$, that is, $\theta_m \in \Omega = \{\theta \in \mathcal{B}C(\theta) : d_0 \leq p(\theta) \leq d_1\}$ for some constant d_1, and if $\sigma_r(\Phi^{p(\theta)}(\theta)) \cap \mathbb{T} = \emptyset$ for all $\theta \in \Omega$, then $\sigma_{ap}(E) \cap \mathbb{T} \neq \emptyset$.*

Proof. In the aperiodic case (a), one has $i\mathbb{R} \subset \sigma_{ap}(\Gamma)$ for the generator Γ of the evolution semigroup $\{E^t\}$ on $C_0(\Theta; X)$ by Lemma 4.7. So the spectral inclusion theorem yields $\mathbb{T} \subset \sigma_{ap}(E)$.

In the periodic case (b), the cocycle $\{\Phi^t\}$ does not have exponential dichotomy on Ω by Proposition 4.2. Therefore, the assumptions of Lemma 4.10 do not hold. Since we have assumed $\sigma_r(\Phi^{p(\sigma)}(\sigma)) \cap \mathbb{T} = \emptyset$ for all $\theta \in \Omega$, we conclude that $\|zI - \Phi^{p(\theta_n)}(\theta_n)\|_{\bullet,X} \to 0$ for some $z \in \mathbb{T}$ and $\theta_n \in \Omega$. Lemma 4.8 implies $i\xi \in \sigma_{ap}(\Gamma)$ for some $\xi \in \mathbb{R}$, and hence $\sigma_{ap}(E) \cap \mathbb{T} \neq \emptyset$ by the spectral inclusion theorem. $\square$

5. Dichotomy and Mild Solutions

In the sequel, we relate dichotomy of the cocycle $\{\Phi^t\}$ or the linear skew-product flow $\{\hat{\varphi}^t\}$ on $\Theta \times X$ to the existence and uniqueness of the solution u for the mild form of the inhomogeneous variational equation $\dot{u}(\varphi^t\theta) = (A(\varphi^t\theta) - \lambda)u(\varphi^t\theta) + g(\varphi^t\theta)$. Recall that $\Phi^t_\lambda(\theta) = e^{-\lambda t}\Phi^t(\theta)$ for $\lambda \in \mathbb{C}$, $t \geq 0$, and $\theta \in \Theta$. Consider the mild inhomogeneous equation

$$u(\varphi^t\theta) = \Phi^t_\lambda(\theta)u(\theta) + \int_0^t \Phi^{t-\tau}_\lambda(\varphi^\tau\theta)g(\varphi^\tau\theta)\,d\tau, \quad t \geq 0,\ \theta \in \Theta, \tag{5.1}$$

on $C_0(\Theta; X)$ or $C_b(\Theta; X)$, the space of bounded, continuous functions $f : \Theta \to X$ endowed with the sup-norm.

Definition 5.1. *Let $\mathcal{F} \in \{C_0(\Theta; X), C_b(\Theta; X)\}$ and $\lambda \in \mathbb{C}$. We say that* condition $(M_\lambda, \mathcal{F})$ holds *if for each $g \in \mathcal{F}$ there exists a unique $u \in \mathcal{F}$ satisfying* (5.1).

If the condition $(M_\lambda, \mathcal{F})$ holds for some $\lambda \in \mathbb{C}$, then we can define a linear operator $R_\lambda : g \mapsto u$ on $\mathcal{F}$ that recovers the unique solution u of (5.1) for a given g. This mapping is, in fact, continuous.

Lemma 5.2. *If condition* $(M_\lambda, \mathcal{F})$ *holds for some* $\lambda \in \mathbb{C}$ *and* $\mathcal{F} \in \{C_0(\Theta; X), C_b(\Theta; X)\}$, *then* R_λ *is bounded on* $\mathcal{F}$.

Proof. Thanks to the Closed Graph Theorem, it suffices to show the closedness of R_λ. Take $g_n, g, u \in \mathcal{F}$ such that $g_n \to g$ and $u_n := R_\lambda g_n \to u$ in $\mathcal{F}$. Since (5.1) holds for u_n and g_n, we have that $u = R_\lambda g$:

$$
\begin{aligned}
u(\theta) = \lim_{n\to\infty} u_n(\theta) &= \lim_{n\to\infty} \left[\Phi_\lambda^t(\varphi^{-t}\theta) u_n(\varphi^{-t}\theta) + \int_0^t \Phi_\lambda^{t-\tau}(\varphi^{\tau-t}\theta) g_n(\varphi^{\tau-t}\theta)\, d\tau \right] \\
&= \Phi_\lambda^t(\varphi^{-t}\theta) u(\varphi^{-t}\theta) + \int_0^t \Phi_\lambda^{t-\tau}(\varphi^{\tau-t}\theta) g(\varphi^{\tau-t}\theta)\, d\tau.
\end{aligned}
$$

□

First, we show by a standard argument that the exponential dichotomy of $\{\Phi^t\}$ implies $(M_\lambda, \mathcal{F})$. Here we do not need to assume that the cocycle is exponentially bounded.

Proposition 5.3. *Let* $\{\Phi^t\}_{t\in\mathbb{R}_+}$ *be a strongly continuous cocycle* $\{\Phi^t\}_{t\in\mathbb{R}_+}$ *over a continuous flow* $\{\varphi^t\}_{t\in\mathbb{R}}$ *on a locally compact metric space* Θ. *Assume that* $\{\Phi^t\}$ *has exponential dichotomy on* Θ. *Then condition* $(M_\lambda, \mathcal{F})$ *holds for* $\mathcal{F} = C_0(\Theta; X)$ *or* $C_b(\Theta; X)$ *and for all* $\lambda = i\xi \in i\mathbb{R}$.

Proof. If $\{\Phi^t\}$ has exponential dichotomy, then $\{\Phi_\lambda^t\}$ with $\lambda = i\xi \in i\mathbb{R}$ also has it, so it is enough to prove the theorem for $\lambda = 0$. Let $P(\cdot)$ be the dichotomy projection and $M, \beta > 0$ be the dichotomy constants from Definition 2.1. Define Green's function $G(\theta, t)$ by

$$
G(\theta, t) = \begin{cases} \Phi_P^t(\theta) = \Phi^t(\theta) P(\theta), & t \geq 0,\ \theta \in \Theta, \\ -\Phi_Q^t(\theta) = -[\Phi_Q^{-t}(\varphi^t\theta) Q(\varphi^t\theta)]^{-1}, & t < 0,\ \theta \in \Theta. \end{cases}
$$

Clearly, $\|G(\theta,t)\| \leq M e^{-\beta|t|}$ and $(\theta, t) \mapsto G(\theta, t)$ is strongly continuous on $\Theta \times (\mathbb{R}\backslash\{0\})$. Moreover, Green's operator $\hat{G}$ defined by

$$
(\hat{G}f)(\theta) = \int_{-\infty}^{\infty} G(\varphi^{-\tau}\theta, \tau) f(\varphi^{-\tau}\theta)\, d\tau \quad \text{for } f \in \mathcal{F} \text{ and } \theta \in \Theta
$$

is a bounded operator on $C_0(\Theta; X)$ and $C_b(\Theta; X)$. To see that $\hat{G}$ indeed maps $C_0(\Theta; X)$ into $C_0(\Theta; X)$, fix $f \in C_0(\Theta; X)$ having support in a compact subset K of Θ, and let $\theta_n \notin K$ tend to ∞. Set

$$
t_n := \sup\{t \geq 0 : \varphi^\tau(\theta_n) \notin K \quad \text{for all} \quad \tau \in [-t, t]\}.
$$

Suppose that $t_n < T < \infty$. Then there exists $\tau_n \in [-T, T]$ such that $\varphi^{\tau_n}\theta_n \in K$ for $n \in \mathbb{N}$. This means that $\theta_n \in \bigcup_{|\tau|\leq T} \varphi^\tau(K)$, a contradiction. As a result, $t_n \to \infty$ and $(\hat{G}f)(\theta_n) \to 0$ due to the estimate $\|G(\theta, \tau)\| \leq M e^{-\beta|\tau|}$.

Let $u = \hat{G}g$ for $g \in \mathcal{F}$. We compute

$$\begin{aligned}
&u(\varphi^t\theta) - \Phi^t(\theta)u(\theta)\\
&= \int_0^\infty \Phi^\tau(\varphi^{t-\tau}\theta)P(\varphi^{t-\tau}\theta)g(\varphi^{t-\tau}\theta)\,d\tau - \Phi^t(\theta)\int_0^\infty \Phi^\tau(\varphi^{-\tau}\theta)P(\varphi^{-\tau}\theta)g(\varphi^{-\tau}\theta)\,d\tau\\
&\quad - \int_{-\infty}^0 \Phi_Q^\tau(\varphi^{t-\tau}\theta)Q(\varphi^{t-\tau}\theta)g(\varphi^{t-\tau}\theta)d\,\tau + \Phi^t(\theta)\int_{-\infty}^0 \Phi_Q^\tau(\varphi^{-\tau}\theta)Q(\varphi^{-\tau}\theta)g(\varphi^{-\tau}\theta)\,d\tau\\
&= \int_{-\infty}^t \Phi^{t-\tau}(\varphi^\tau\theta)P(\varphi^\tau\theta)g(\varphi^\tau\theta)d\tau - \int_{-\infty}^0 \Phi^{t-\tau}(\varphi^\tau\theta)P(\varphi^\tau\theta)g(\varphi^\tau\theta)\,d\tau\\
&\quad - \int_t^\infty \Phi_Q^{t-\tau}(\varphi^\tau\theta)Q(\varphi^\tau\theta)g(\varphi^\tau\theta)\,d\tau + \int_0^\infty \Phi_Q^{t-\tau}(\varphi^\tau\theta)Q(\varphi^\tau\theta)g(\varphi^\tau\theta)\,d\tau\\
&= \int_0^t \Phi^{t-\tau}(\varphi^\tau\theta)P(\varphi^\tau\theta)g(\varphi^\tau\theta)\,d\tau + \int_0^t \Phi^{t-\tau}(\varphi^\tau\theta)Q(\varphi^\tau\theta)g(\varphi^\tau\theta)\,d\tau\\
&= \int_0^t \Phi^{t-\tau}(\varphi^\tau\theta)g(\varphi^\tau\theta)d\tau
\end{aligned}$$

for $t \geq 0$ and $\theta \in \Theta$. This proves that $u = \hat{G}g$ satisfies (5.1) with $\lambda = 0$.

To show uniqueness, take $g = 0$ and let $u \in \mathcal{F}$ satisfy $u(\varphi^t\theta) = \Phi^t(\theta)u(\theta)$ for $\theta \in \Theta$ and $t \geq 0$. Since $\{\Phi^t\}$ has exponential dichotomy, we have

$$P(\theta)u(\theta) = \Phi_P^t(\varphi^{-t}\theta)P(\varphi^{-t}\theta)u(\varphi^{-t}\theta) \quad \text{and} \quad Q(\theta)u(\theta) = [\Phi_Q^t(\theta)]^{-1}Q(\varphi^t\theta)u(\varphi^t\theta)$$

for $t \geq 0$ and $\theta \in \Theta$. Thus, the estimates

$$|P(\theta)u(\theta)| \leq Me^{-\beta t}\|u\|_\infty \quad \text{and} \quad |Q(\theta)u(\theta)| \leq Me^{-\beta t}\|u\|_\infty$$

imply $u = 0$. $\square$

We now address the converse of Proposition 5.3. In the case $\mathcal{F} = C_0(\Theta; X)$, the result is an easy consequence of Corollary 4.11.

Theorem 5.4. *Let $\{\Phi^t\}_{t\in\mathbb{R}_+}$ be an exponentially bounded, strongly continuous cocycle over a continuous flow $\{\varphi^t\}_{t\in\mathbb{R}}$ on a locally compact metric space Θ.*

(a) *Assume that $p(\theta) \geq d_0 > 0$ for all $\theta \in \mathcal{B}(\Theta)$. If condition $(M_\lambda, C_0(\Theta; X))$ holds for all $\lambda = i\xi \in i\mathbb{R}$, then $\{\Phi^t\}$ has exponential dichotomy on Θ.*

(b) *Assume that $\mathcal{B}(\Theta) = \emptyset$. If condition $(M_0, C_0(\Theta; X))$ holds, then $\{\Phi^t\}$ has exponential dichotomy on Θ.*

Proof. Due to Corollary 4.11, we only have to show that $i\mathbb{R} \subset \rho(\Gamma)$ in case (a) and $0 \in \rho(\Gamma)$ in case (b). Fix $g \in C_0(\Theta; X)$. By the assumption, there exists the function $u = R_\lambda g \in C_0(\Theta; X)$. Applying E^h in (5.1), we obtain

$$e^{-\lambda h}(E^h u)(\theta) = \Phi_\lambda^h(\varphi^{-h}\theta)u(\varphi^{-h}\theta) = u(\theta) - \int_0^h \Phi_\lambda^{h-\tau}(\varphi^{\tau-h}\theta)g(\varphi^{\tau-h}\theta)\,d\tau.$$

Therefore,

$$\tfrac{1}{h}[e^{-\lambda h}(E^h u)(\theta) - u(\theta)] = -\frac{1}{h}\int_0^h \Phi_\lambda^\tau(\varphi^{-\tau}\theta)g(\varphi^{-\tau}\theta)\,d\tau,$$

and we conclude that $u \in \mathcal{D}(\Gamma)$ and $(\lambda - \Gamma)u = g$. This means that $\lambda - \Gamma$ is surjective for all $\lambda \in i\mathbb{R}$ in case (a) and for $\lambda = 0$ in case (b).

If $(\Gamma - \lambda I)u = 0$, then $e^{-\lambda t}E^t u = u$ or $u(\varphi^t\theta) = \Phi^t_\lambda(\theta)u(\theta)$ for $\theta \in \Theta$ and $t \geq 0$. So the uniqueness part of the condition $(M_\lambda, C_0(\Theta; X))$ implies the injectivity of $\lambda - \Gamma$ for all $\lambda \in i\mathbb{R}$ in case (a) and for $\lambda = 0$ in case (b). $\square$

The case $\mathcal{F} = C_b(\Theta; X)$ cannot be reduced to the results of the previous sections since the evolution semigroup is not strongly continuous on $C_b(\Theta; X)$. Instead, we use the discrete operators $\pi_\theta(E)$. We start with two preliminary lemmas. The first one can be proved exactly as [20, Lemma 2.4].

Lemma 5.5. *Assume that $\theta \in \Theta$ satisfies $\varphi^1\theta = \theta$. Then, $\pi_\theta(E)$ is hyperbolic on $c_0(\mathbb{Z}; X)$ if and only if $\Phi^1(\theta)$ is hyperbolic on X. Moreover, if $1 \notin \sigma_{ap}(\pi_\theta(E))$, then $\sigma_{ap}(\Phi^1(\theta)) \cap \mathbb{T} = \emptyset$.*

Lemma 5.6. *Assume that condition $(M_\lambda, \mathcal{F})$ holds for $\mathcal{F} \in \{C_0(\Theta; X), C_b(\Theta; X)\}$ and some $\lambda \in \mathbb{C}$. Then, for each periodic point $\theta_0 \in \Theta$ with $p(\theta_0) > 0$, the operator $e^{\lambda p(\theta_0)} - \Phi^{p(\theta_0)}(\theta_0)$ is surjective in X.*

Proof. Fix $y \in X$. Take $\beta \in C([0, p(\theta_0)]; \mathbb{R})$ such that $\beta(0) = 0$, $\beta(p(\theta_0)) = 1/p(\theta_0)$, and $\int_0^{p(\theta_0)} \beta(\tau)d\tau = 1$. On the orbit $\mathcal{O}(\theta_0)$ define

$$g(\varphi^\tau\theta_0) = \beta(\tau)\Phi^\tau_\lambda(\theta_0)y + \left[\tfrac{1}{p(\theta_0)} - \beta(\tau)\right]\Phi^{p(\theta_0)+\tau}_\lambda(\theta_0)y.$$

The function g is continuous on $\mathcal{O}(\theta_0)$. Extend g on Θ continuously to a function $g \in \mathcal{F}$ with a compact support. Let $u = R_\lambda g$. Observe that

$$\Phi^{p(\theta_0)-\tau}_\lambda(\varphi^\tau\theta_0)\Phi^{p(\theta_0)+\tau}_\lambda(\theta_0) = \Phi^{2p(\theta_0)}(\theta_0).$$

Using this fact, (5.1), and $\int_0^{p(\theta_0)}[\frac{1}{p(\theta_0)} - \beta(\tau)]d\tau = 0$, one obtains

$$u(\varphi^{p(\theta_0)}\theta_0) = u(\theta_0) = \Phi^{p(\theta_0)}_\lambda(\theta_0)u(\theta_0) + \Phi^{p(\theta_0)}_\lambda(\theta_0)y.$$

For $x = u(\theta_0)$, we get

$$x - e^{-\lambda p(\theta_0)}\Phi^{p(\theta_0)}(\theta_0)x = e^{-\lambda p(\theta_0)}\Phi^{p(\theta_0)}(\theta_0)y.$$

Therefore, $[e^{\lambda p(\theta_0)} - \Phi^{p(\theta_0)}(\theta_0)](e^{-\lambda p(\theta_0)}(x + y)) = y$. $\square$

Theorem 5.7. *Let $\{\Phi^t\}_{t\in\mathbb{R}_+}$ be an exponentially bounded, strongly continuous cocycle over a continuous flow $\{\varphi^t\}_{t\in\mathbb{R}}$ on a locally compact metric space Θ.*

(a) *Assume that $p(\theta) \geq d_0 > 0$ for all $\theta \in \mathcal{B}(\theta)$. If condition $(M_\lambda, C_b(\Theta, X))$ holds for all $\lambda = i\xi \in i\mathbb{R}$, then $\{\Phi^t\}$ has exponential dichotomy on Θ.*

(b) *Assume that $\mathcal{B}(\Theta) = \emptyset$. If condition $(M_0, C_b(\Theta; X))$ holds, then $\{\Phi^t\}$ has exponential dichotomy on Θ.*

Proof. We will show that $I - \pi_\theta(E)$ is invertible in $\mathcal{L}(c_0(\mathbb{Z}; X))$ and

$$\|(I - \pi_\theta(E))^{-1}\|_{\mathcal{L}(c_0(\mathbb{Z};X))} \leq C \quad \text{for} \quad \theta \in \Theta. \tag{5.2}$$

Then the theorem follows from the Discrete Dichotomy Theorem 3.7 and Proposition 3.6.

Step I. We prove that

$$\inf_{\theta\in\Theta} \|I - \pi_\theta(E)\|_{\bullet, c_0(\mathbb{Z};X)} > 0. \tag{5.3}$$

Suppose (5.3) does not hold; that is, suppose that

$$\inf_{\theta\in\Theta} \|I - \pi_\theta(E)\|_{\bullet,c_0(\mathbb{Z};X)} = 0. \tag{5.4}$$

By Proposition 4.4, there exists a Mañe sequence $\{\theta_m\}$ in Θ. We consider separately the aperiodic and the periodic cases.

(1) Assume that there is a subsequence with $p(\theta_m) \to \infty$ or $\theta_m \notin \mathcal{B}(\Theta)$. An application of Lemma 4.7 gives $f_n \in \mathcal{D}(\Gamma)$ such that $\|f_n\|_\infty = 1$ but $\Gamma f_n \to 0$, where Γ is the generator of the corresponding evolution semigroup on $C_0(\Theta; X)$. Recall that the functions f_n are given explicitly by (4.7) while Γf_n is computed in (4.8). Using (5.1), it is easy to verify that $R_0 \Gamma f_n = -f_n$. This contradicts the continuity of R_0 shown in Lemma 5.2.

(2) It remains to consider the case $\theta_m \in \Omega = \{\theta \in \mathcal{B}C(\Theta) : d_0 \le p(\theta) \le d_1\}$ for some d_1 (use Proposition 4.6). We claim that there is a constant c such that

$$\|zI - \Phi^{p(\theta)}(\theta)\|_{\bullet,X} \ge c > 0 \tag{5.5}$$

for $\theta \in \Omega$ and $z \in \mathbb{T}$. We postpone the proof of this claim. Combining (5.5), the assumptions in part (a), and Lemma 5.6, we can verify the assumptions of Lemma 4.10. But the conclusion of Lemma 4.10 contradicts (5.4) by virtue of the Discrete Dichotomy Theorem.

To prove (5.5), suppose that $\inf\{\|zI - \Phi^{p(\theta)}(\theta)\|_{\bullet,X} : \theta \in \Omega\} = 0$ for some $z \in \mathbb{T}$. Then the conclusion of Lemma 4.8 hold, that is, there is a sequence of functions $f_n \in \mathcal{D}(\Gamma)$ given by (4.10) such that $\|f_n\|_\infty \ge 1/2$ and $\|(\Gamma - i\xi)f_n\|_\infty \to 0$ for some $\xi \in \mathbb{R}$, see (4.11). Set $g_n = \Gamma f_n - i\xi f_n$ and $\lambda = i\xi$. In the next lemma, we show that $f_n = -R_\lambda g_n$. But this is impossible since R_λ is a bounded operator on $C_b(\Theta; X)$ due to condition $(M_\lambda, C_b(\Theta; X))$ and Lemma 5.2.

Lemma 5.8. *The functions f_n defined in* (4.10) *and $g_n = (\Gamma - i\xi)f_n$ defined in* (4.11) *satisfy $f_n = -R_\lambda g_n$; that is, for $t \ge 0$ and $\theta \in \Theta$ one has*

$$f_n(\varphi^t\theta) - \Phi^t_{i\xi}(\theta) f_n(\theta) = -\int_0^t \Phi^{t-\tau}_{i\xi}(\varphi^t\theta) g_n(\varphi^\tau\theta)\, d\tau. \tag{5.6}$$

Proof of Lemma 5.8. Recall that f_n and g_n are supported in $\mathcal{U}_n = \{\varphi^s\sigma : 0 \le s < p(\sigma), \sigma \in \Sigma\}$, see (4.10) and (4.11). Fix $\theta = \varphi^s\sigma \in \mathcal{U}_n$ and $t \ge 0$.

First, assume that $s + t < p(\sigma)$. Then, by (4.10),

$$\begin{aligned} f_n(\varphi^t\theta) - \Phi^t_{i\xi}(\theta) f_n(\theta) &= f_n(\varphi^{t+s}\sigma) - \Phi^t_{i\xi}(\varphi^s\sigma) f_n(\varphi^s\sigma) \\ &= \beta(\sigma)\Phi^{t+s}_{i\xi}(\sigma)\left[\alpha(\tfrac{t+s}{p(\sigma)})\,(x_n - \Phi^{p(\sigma)}_{i\xi} x_n) + \Phi^{p(\sigma)}_{i\xi}(\sigma) x_n\right] \\ &\quad - \beta(\sigma)\Phi^{t+s}_{i\xi}(\sigma)\left[\alpha(\tfrac{s}{p(\sigma)})\,(x_n - \Phi^{p(\sigma)}_{i\xi} x_n) + \Phi^{p(\sigma)}_{i\xi}(\sigma) x_n\right] \\ &= \beta(\sigma)\Phi^{t+s}_{i\xi}(\sigma)\left[\alpha(\tfrac{t+s}{p(\sigma)}) - \alpha(\tfrac{s}{p(\sigma)})\right](x_n - \Phi^{p(\sigma)}_{i\xi} x_n). \end{aligned}$$

On the other hand, (4.11) yields

$$
\begin{aligned}
&- \int_0^t \Phi_{i\xi}^{t-\tau}(\varphi^\tau\theta) g_n(\varphi^\tau\theta)\, d\tau \\
&= \beta(\sigma) \int_0^t \Phi_{i\xi}^{t-\tau}(\varphi^{\tau+s}\sigma)\Phi_{i\xi}^{\tau+s}(\sigma)[x_n - \Phi_{i\xi}^{p(\sigma)}(\sigma)x_n] \frac{1}{p(\sigma)} \alpha'(\tfrac{\tau+s}{p(\sigma)})\, d\tau \\
&= \beta(\sigma)\Phi_{i\xi}^{t+s}(\sigma)(x_n - \Phi_{i\xi}^{p(\sigma)} x_n) \int_0^t \alpha'(\tfrac{\tau+s}{p(\sigma)})\, \frac{d\tau}{p(\sigma)},
\end{aligned}
$$

and (5.6) holds.

Second, assume that $s+t \in [kp(\sigma),(k+1)p(\sigma))$ for $k = 1,2,\ldots$. Now, $\varphi^t\theta = \varphi^{t+s}\sigma = \varphi^{t+s-kp(\sigma)}\sigma$, where $t+s-kp(\sigma) \in [0, p(\sigma))$. The LHS of (5.6) can be written as

$$
\begin{aligned}
&\beta(\sigma)\, \Phi_{i\xi}^{t+s-kp(\sigma)}(\sigma)\, \alpha(\tfrac{t+s-kp(\sigma)}{p(\sigma)})\, [x_n - \Phi_{i\xi}^{p(\sigma)}(\sigma)x_n] + \beta(\sigma)\Phi_{i\xi}^{t+s-(k-1)p(\sigma)}(\sigma)\, x_n \\
&\quad - \beta(\sigma)\, \Phi_{i\xi}^{t+s+p(\sigma)}(\sigma)x_n - \beta(\sigma)\, \Phi_{i\xi}^{t+s}(\sigma)\, \alpha(\tfrac{s}{p(\sigma)})\, [x_n - \Phi_{i\xi}^{p(\sigma)}(\sigma)x_n].
\end{aligned}
$$

We split the integral on the RHS of (5.6) in three integrals and compute

$$
\begin{aligned}
&- \int_0^{p(\sigma)-s} \Phi_{i\xi}^{t-\tau}(\varphi^\tau\theta) g_n(\varphi^\tau\theta)\, d\tau \\
&= \beta(\sigma) \int_0^{p(\sigma)-s} \tfrac{1}{p(\sigma)} \alpha'(\tfrac{\tau+s}{p(\sigma)}) \Phi_{i\xi}^{t-\tau}(\varphi^{s+\tau}\sigma)\Phi_{i\xi}^{\tau+s}(\sigma)[x_n - \Phi_{i\xi}^{p(\sigma)}(\sigma)x_n]\, d\tau \\
&= \beta(\sigma) \Big(1 - \alpha(\tfrac{s}{p(\sigma)})\Big)\, \Phi_{i\xi}^{t+s}(\sigma)[x_n - \Phi_{i\xi}^{p(\sigma)}(\sigma)x_n]
\end{aligned}
$$

since $\alpha(1) = 1$ by the choice of α. Further,

$$
\begin{aligned}
&- \int_{p(\sigma)-s}^{kp(\sigma)-s} \Phi_{i\xi}^{t-\tau}(\varphi^\tau\theta) g_n(\varphi^\tau\theta)\, d\tau \\
&= \beta(\sigma) \sum_{\ell=1}^{k-1} \int_{\ell p(\sigma)-s}^{(\ell+1)p(\sigma)-s} \tfrac{1}{p(\sigma)}\, \alpha'(\tfrac{s+\tau-\ell p(\sigma)}{p(\sigma)})\, \Phi_{i\xi}^{t-\tau}(\varphi^{s+\tau-\ell p(\sigma)}\sigma)\Phi_{i\xi}^{s+\tau-\ell p(\sigma)}(\sigma) \\
&\qquad\qquad \cdot [x_n - \Phi_{i\xi}^{p(\sigma)}(\sigma)x_n]\, d\tau \\
&= \beta(\sigma)[\Phi_{i\xi}^{t+s-(k-1)p(\sigma)}(\sigma)x_n - \Phi_{i\xi}^{t+s}(\sigma)x_n],
\end{aligned}
$$

where we have used $\alpha(0) = 0$, $\alpha(1) = 1$, and the cocycle property. Finally,

$$
\begin{aligned}
&- \int_{kp(\sigma)-s}^{t} \Phi_{i\xi}^{t-\tau}(\varphi^\tau\theta) g_n(\varphi^\tau\theta)\, d\tau \\
&= \beta(\sigma) \int_{kp(\sigma)-s}^{t} \tfrac{1}{p(\sigma)} \alpha'(\tfrac{\tau+s-kp(\sigma)}{p(\sigma)})\, \Phi_{i\xi}^{t-\tau}(\varphi^{\tau+s-kp(\sigma)}\sigma)\Phi_{i\xi}^{\tau+s-kp(\sigma)}(\sigma)[x_n - \Phi_{i\xi}^{p(\sigma)}(\sigma)x_n]\, d\tau \\
&= \beta(\sigma)\, \Phi_{i\xi}^{t+s-kp(\sigma)}(\sigma)\, \alpha(\tfrac{t+s-kp(\sigma)}{p(\sigma)})\, [x_n - \Phi_{i\xi}^{p(\sigma)}(\sigma)x_n].
\end{aligned}
$$

The lemma follows by combining these identities. □

As a result, (5.3) is verified.

Step II. Next, we prove that $I - \pi_{\theta_0}(E)$ is invertible on $c_0(\mathbb{Z}; X)$ for each $\theta_0 \in \Theta$. Together with (5.3), this implies (5.2) and so the theorem is established. We distinguish between three cases.

(1) Assume that $p(\theta_0) = \infty$. For $\gamma \geq 0$, we define the operator

$$D(\gamma) : (x_k) \mapsto (x_k - e^{-\gamma(|k|-|k-1|)}\Phi^1(\varphi^{k-1}\theta_0)x_{k-1})$$

on $c_0(\mathbb{Z}; X)$. Notice that

$$D(\gamma) \;\to\; D(0) = I - \pi_{\theta_0}(E) \quad \text{in} \quad \mathcal{L}(c_0(\mathbb{Z}; X)) \quad \text{as} \quad \gamma \to 0. \tag{5.7}$$

By (5.3), this implies that there exist constants $c, \gamma_0 > 0$ such that

$$\|D(\gamma)\|_{\bullet, c_0(\mathbb{Z};X)} \geq c \quad \text{for all} \;\; 0 \leq \gamma \leq \gamma_0. \tag{5.8}$$

Take $\bar{y} = (y_k) \in c_0(\mathbb{Z}; X)$ with $y_k = 0$ for $|k| \geq n = n(\bar{y})$. Choose a function $\alpha \in C(\mathbb{R})$ with $\operatorname{supp} \alpha \subset (0, 1)$ and $\int_{\mathbb{R}} \alpha(\tau)d\tau = 1$. For $m = n+1, n+2, \cdots$ and $0 \leq \gamma \leq \gamma_0$, define

$$\tilde{g}_m(\varphi^t\theta_0) = \alpha(t - k + 1)\, \Phi^{t-k+1}(\varphi^{k-1}\theta_0)\, e^{\gamma|k-1|}\, y_{k-1}$$

for $t \in [k-1, k)$ and $k = -m+1, \ldots, m$. We can extend $\tilde{g}_m$ to a function $g_m \in C(\Theta; X)$ with compact support and

$$\|g_m\|_\infty \leq \|\alpha\|_\infty \sup_{0 \leq \tau \leq 1,\, \theta \in \Theta} \|\Phi^\tau(\theta)\|\, e^{\gamma_0 n}\, \|\bar{y}\|_\infty\,.$$

Condition $(M_0, C_b(\Theta; X))$ yields the function $u_m = R_0 g_m \in C_b(\Theta; X)$, that is, one has

$$\begin{aligned} u_m(\varphi^k\theta_0) &= \Phi^1(\varphi^{k-1}\theta_0)u_m(\varphi^{k-1}\theta_0) + \int_0^1 \Phi^{1-\tau}(\varphi^{\tau+k-1}\theta_0)g(\varphi^{\tau+k-1}\theta_0)\, d\tau \\ &= \Phi^1(\varphi^{k-1}\theta_0)u_m(\varphi^{k-1}\theta_0) + \Phi^1(\varphi^{k-1}\theta_0)e^{\gamma|k-1|}y_{k-1} \end{aligned}$$

for $k = -m+1, \ldots, m$. Set $v_m = (u_m(\varphi^k\theta_0) + e^{\gamma|k|}y_k)_{k\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z}; X)$ and

$$r_m = (r_{mk})_{k\in\mathbb{Z}} = (I - \pi_{\theta_0}(E))v_m - (e^{\gamma|k|}y_k)_{k\in\mathbb{Z}}$$

for $m > n$. Then, $(r_{mk})_{k\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z}; X)$ and $r_{mk} = 0$ for $|k| \leq m$. Since $y_k = 0$ for $|k| \geq n$, we obtain

$$\|r_m\|_\infty \leq \|I - \pi_{\theta_0}(E)\|_{\mathcal{L}(\ell^\infty(\mathbb{Z};X))}\, \|u_m\|_\infty \leq \|I - \pi_{\theta_0}(E)\|\, \|R_0\|\, \|g_m\|_\infty \leq C,$$

where C depends on $\bar{y}$ but not on m and $\gamma \in [0, \gamma_0]$. Further, $(e^{-\gamma|k|}v_{mk})_{k\in\mathbb{Z}} \in c_0(\mathbb{Z}; X)$ and

$$\begin{aligned} \|D(\gamma)(e^{-\gamma|k|}v_{mk})_{k\in\mathbb{Z}} - \bar{y}\|_\infty &= \|(e^{-\gamma|k|}[(I - \pi_{\theta_0}(E))v_m]_k)_{k\in\mathbb{Z}} - \bar{y}\|_\infty \\ &= \sup_{k\in\mathbb{Z}} |e^{-\gamma|k|}r_{mk}| \leq C\, e^{-\gamma m} \end{aligned}$$

for $m > n$. As a consequence, $D(\gamma)$ has dense range on $c_0(\mathbb{Z}; X)$ and is invertible by (5.8). Now, (5.7) implies the invertibility of $I - \pi_{\theta_0}(E)$ on $c_0(\mathbb{Z}; X)$ in case (1).

(2) Assume that $p(\theta_0) \in (0, \infty)$. We start with a special case.

(2i) Let $p(\theta_0) = 1$. Lemma 5.5 and (5.3) yield $\sigma_{ap}(\Phi^1(\theta_0)) \cap \mathbb{T} = \emptyset$. Further, $e^\lambda - \Phi^1(\theta_0)$ is surjective by Lemma 5.6 for all $\lambda = i\xi \in i\mathbb{R}$ in case (a) and for $\lambda = 0$ in case (b) of the theorem. Therefore, $e^\lambda \in \rho(\Phi^1(\theta_0))$ and the boundary of $\sigma(\Phi^1(\theta_0))$ does not intersect $\mathbb{T}$. As a result, $\Phi^1(\theta_0)$ is hyperbolic and, by Lemma 5.5, $I - \pi_{\theta_0}(E)$ is invertible on $c_0(\mathbb{Z}; X)$.

(2ii) Let $p(\theta_0) = T \in (0, \infty)$. We define the continuous flow $\psi^t(\theta) = \varphi^{tT}(\theta)$ and the exponentially bounded, strongly continuous cocycle $\Psi^t(\theta) = \Phi^{tT}(\theta)$ over $\{\psi^t\}$ on Θ. This cocycle satisfies condition $(M_{\lambda T}, C_b(\Theta; X))$. Also, for the flow $\{\psi^t\}$ one has $p_\psi(\theta_0) = 1$. An application of parts (I) and (II.2.i) to the closed ψ^t-invariant set $\mathcal{O}(\theta_0)$ and the cocycle

Ψ shows that the corresponding operators $I - \pi_\theta(E_\Psi)$ are invertible on $c_0(\mathbb{Z}; X)$ and $\|(I - \pi_\theta(E_\Psi))^{-1}\| \le C$ for $\theta \in \mathcal{O}(\theta_0)$. Therefore Ψ^t has exponential dichotomy on $\mathcal{O}(\theta_0)$ due to the Discrete Dichotomy Theorem 3.7. Hence, Φ^t has exponential dichotomy on $\mathcal{O}(\theta_0)$ which implies the invertibility of $I - \pi_{\theta_0}(E)$.

(3) Assume that $p(\theta_0) = 0$. Then $\Phi^t(\theta_0) = e^{tA}$ is a strongly continuous semigroup on X. For $y \in X$, choose $g \in C_b(\Theta; X)$ with $g(\theta_0) = y$. By condition $(M_\lambda, \mathcal{F})$ (for all $\lambda = i\xi \in i\mathbb{R}$ in case (a) and for $\lambda = 0$ in case (b) of the theorem), there is a function $f \in C_b(\Theta; X)$ such that $f(\theta_0) =: x$ satisfies

$$e^{\lambda t}x - e^{tA}x = \int_0^t e^{(t-\tau)A} e^{\lambda\tau} y \, d\tau.$$

Consequently, $x \in \mathcal{D}(A)$ and $(\lambda - A)x = y$, i.e., $\lambda \notin \sigma_r(A)$. Lemma 5.5 and (5.3) yield $\sigma_{ap}(e^A) \cap \mathbb{T} = \emptyset$. Hence, $\sigma_{ap}(A) \cap i\mathbb{R} = \emptyset$ by the spectral inclusion theorem. In particular, $\lambda \notin \sigma(A)$ and the boundary of $\sigma(A)$ does not intersect $i\mathbb{R}$. So we infer that $\sigma(A) \cap i\mathbb{R} = \emptyset$ and, by the spectral mapping theorem for the residual spectrum, $\sigma_r(e^A) \cap \mathbb{T} = \emptyset$. As a result, $\Phi^1(\theta_0)$ is hyperbolic. So Lemma 5.5 implies $1 \in \rho(\pi_{\theta_0}(E))$. $\square$

We combine the results of this section in the following characterization.

Corollary 5.9. *Let $\{\Phi^t\}_{t\in\mathbb{R}_+}$ be an exponentially bounded, strongly continuous cocycle $\{\Phi^t\}_{t\in\mathbb{R}_+}$ over a continuous flow $\{\varphi^t\}_{t\in\mathbb{R}}$ on a locally compact metric space Θ. Let $\mathcal{F} \in \{C_b(\Theta, X), C_0(\Theta, X)\}$.*

(a) *Assume that $p(\theta) \ge d_0 > 0$ for $\theta \in \mathcal{B}(\theta)$. Then condition $(M_\lambda, \mathcal{F})$ holds for all $\lambda = i\xi \in i\mathbb{R}$ if and only if $\{\Phi^t\}$ has exponential dichotomy on Θ.*

(b) *Assume that $\mathcal{B}(\Theta) = \emptyset$. Then condition $(M_0, \mathcal{F})$ holds if and only if $\{\Phi^t\}$ has exponential dichotomy on Θ.*

Note added in proof.

We remark that F. Räbiger [private communication] recently gave a direct proof of the following fact: If $\mathcal{B}(\Theta) = \emptyset$, then condition $(M_0, C_b(\Theta; X))$ implies condition $(M_0, C_0(\Theta; X))$. In other words, he proved that $g \in C_0(\Theta; X)$ implies $R_0 g \in C_0(\Theta; X)$ assuming condition $(M_0, C_b(\Theta; X))$. The proof does not use operators $\pi_\theta(E)$. It is based on a characterization theorem for evolution semigroups and their generators similar to, e. g., Theorem 3.4 in [38], and on the fact that $\Gamma u = -g$ for $u, g \in C_0(\Theta; X)$ if and only if u and g satisfy the mild integral equation (5.1) with $\lambda = 0$ (compare Lemma 1.1 in [32]). An indirect proof of the fact above is contained in our Theorem 5.7.

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