ALMOST PERIODICITY OF INHOMOGENEOUS PARABOLIC EVOLUTION EQUATIONS

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Abstract. We show the (asymptotic) almost periodicity of the bounded solution to the parabolic evolution equation $u'(t) = A(t)u(t) + f(t)$ on $\mathbb{R}$ (on $\mathbb{R}_+$) assuming that the linear operators $A(t)$ satisfy the 'Acquistapace–Terreni' conditions, that the evolution family generated by $A(\cdot)$ has an exponential dichotomy, and that $R(\omega, A(\cdot))$ and f are (asymptotically) almost periodic.

1. Introduction

In the present work we investigate the almost periodicity of the solutions to the parabolic inhomogeneous evolution equations

$$u'(t) = A(t)u(t) + f(t), \quad t \in \mathbb{R}, \tag{1.1}$$

$$u'(t) = A(t)u(t) + f(t), \quad t > 0, \qquad u(0) = x, \tag{1.2}$$

in a Banach space X. It is assumed that the linear operators $A(t)$ satisfy the 'Acquistapace–Terreni' conditions, that the evolution family U solving the homogeneous problem has an exponential dichotomy, and that the functions $t \mapsto R(\omega, A(t))$, for an $\omega \geq 0$, and f are almost periodic (compare Section 2). In Theorem 4.5 we show that then the unique bounded mild solution $u : \mathbb{R} \to X$ of (1.1) is almost periodic. This fact is a consequence of the almost periodicity of Green's function corresponding to U, established also in Theorem 4.5. In Theorem 5.4 we prove the analogous results for (1.2) on $\mathbb{R}_+$ in the context of asymptotic almost periodicity imposing a necessary compatibility condition on x and f.

The almost periodicity of inhomogeneous problems has been studied by many authors in the autonomous case, where $A(t) = A$, and in the periodic case, where $A(t) = A(t+p)$, see [3], [4], [6], [11], [12], [16], [21], and the references therein. Equations with almost periodic $A(\cdot)$ are treated in, e.g., [8] and [10] for $X = \mathbb{C}^n$ and in [13] for a certain class of parabolic problems, see also [5], [15], [17]. For general evolution families U (but subject to an extra condition not assumed here), it is shown in [14] that U has an exponential dichotomy *with* an almost periodic Green's function if and only if there is a unique almost periodic mild solution u of (1.1) for each almost periodic f, see also [8, Prop.8.3]. Our main theorems extend [8, Prop.8.4], [10, Thm.7.7], and [13, p.240], and complement [14, Thm.5.3] in the case of parabolic evolution equations. The initial value problem (1.2) was not studied in [8], [10], [13], and [14] in the context of asymptotic almost periodicity.

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Our strategy is similar to Henry's approach in [13, §7.6] who derived the almost periodicity of Green's function Γ corresponding to U (and thus of the dichotomy projections $P(\cdot)$) from a formula for $\Gamma(t+\tau, s+\tau) - \Gamma(t,s)$ (compare (4.2)). In the context of [13], this equation allows to verify Bohr's definition of almost periodicity by straightforward estimates in operator norm if τ is a pseudo period of $A(\cdot)$. However, in the present more general situation one obtains such a formula only on a subspace of X so that one cannot proceed in this way. To overcome this difficulty, we employ Yosida approximations. This requires some preparations given in Section 3 and a somewhat delicate limiting process presented in Section 4. We point out that we can *not* estimate the relevant quantities for the approximating problems independently of n, cf. Lemma 4.1. The almost periodicity of the mild solution u of (1.1) then follows from the standard formula (2.11) which expresses u in terms of Γ, see Theorem 4.5. We also give a straightforward application to a second order parabolic equation. The initial value problem (1.2) is treated in Section 5 by essentially the same methods.

In the next section we collect several concepts and preliminary results. We refer to [9] for unexplained notation. By $c = c(\alpha, \beta, \cdots)$ we denote a generic constant only depending on the constants in the hypotheses involved and the quantities $\alpha, \beta, \cdots$.

2. Prerequisites

Let X be a Banach space. A set $U = \{U(t,s) : t \geq s,\ t,s \in \mathbb{R}\}$ of bounded linear operators on X is called an *evolution family* if

(E1) $U(t,s) = U(t,r)U(r,s)$ and $U(s,s) = I$ for $t \geq r \geq s$ and
(E2) $(t,s) \mapsto U(t,s)$ is strongly continuous for $t > s$.

We say that an evolution family U has an *exponential dichotomy* (or is *hyperbolic*) if there are projections $P(t)$, $t \in \mathbb{R}$, being uniformly bounded and strongly continuous in t and constants $\delta > 0$ and $N \geq 1$ such that

(a) $U(t,s)P(s) = P(t)U(t,s)$,
(b) the restriction $U_Q(t,s) : Q(s)X \to Q(t)X$ of $U(t,s)$ is invertible (and we set $U_Q(s,t) := U_Q(t,s)^{-1}$),
(c) $\|U(t,s)P(s)\| \leq Ne^{-\delta(t-s)}$ and $\|U_Q(s,t)Q(t)\| \leq Ne^{-\delta(t-s)}$

for $t \geq s$ and $t,s \in \mathbb{R}$. Here and below we let $Q = I - P$ for a projection P. Exponential dichotomy is a classical concept in the study of the long–term behaviour of evolution equations, see e.g. [7], [8], [9], [13], [15]. If U is hyperbolic, then the operator family

$$\Gamma(t,s) := \begin{cases} U(t,s)P(s), & t \geq s,\ t,s \in \mathbb{R}, \\ -U_Q(t,s)Q(s), & t < s,\ t,s \in \mathbb{R}, \end{cases}$$

is called *Green's function* corresponding to U and $P(\cdot)$. If $P(t) = I$ for $t \in \mathbb{R}$, then U is *exponentially stable*. The evolution family is called *exponentially bounded* if there are constants $M > 0$ and $\gamma \in \mathbb{R}$ such that $\|U(t,s)\| \leq Me^{\gamma(t-s)}$ for $t \geq s$. For computations involving Green's function it is useful to observe that

$$(t,s) \mapsto U_Q(t,s)Q(s) \qquad \text{is strongly continuous for } t,s \in \mathbb{R},\ t \neq s,$$
$$U_Q(t,s)Q(s) = U_Q(t,r)U_Q(r,s)Q(s) \quad \text{for } t,r,s \in \mathbb{R},$$

cf. [9, Lemma VI.9.17].

Let U be an exponentially bounded evolution family on X. It is a well known and important fact that the exponential dichotomy of U persists under small perturbations, see e.g. [7], [8], [9, §VI.9], [13], [20]. Our approach also relies on this property. More precisely, we will use Proposition 2.1 below which is a refinement of [19, Prop.2.3]. This result is based on a characterization of exponential dichotomy by means of the *evolution semigroup*

$$(T_U(t)f)(s) := U(s, s-t)f(s-t), \qquad t \geq 0,\ s \in \mathbb{R},\ f \in C_0(\mathbb{R}, X),$$

on $C_0(\mathbb{R}, X)$ (the space of continuous functions vanishing at infinity). Note that T_U is an exponentially bounded semigroup being strongly continuous on $(0, \infty)$, but not necessarily at $t = 0$. We then have

$$U \text{ has exponential dichotomy} \iff I - T_U(1) \text{ is invertible}, \tag{2.1}$$

see the equivalence (a)⇔(b) in [9, Thm.VI.9.18] or [7, §3.2.3] and the references therein. There it is also shown that then T_U is hyperbolic and that the dichotomy projections of U are given by

$$P(\cdot) = \frac{1}{2\pi i} \int_{|\lambda|=1} R(\lambda, T_U(1))\, d\lambda. \tag{2.2}$$

To be precise, in these works it is assumed that $(t, s) \mapsto U(t, s)$ is strongly continuous for $t \geq s$ (and not only for $t > s$ as in our paper), but for the proof of the above mentioned facts this does not matter, see [20].

Proposition 2.1. *Let U and V be evolution families with $\|U(t,s)\|, \|V(t,s)\| \leq Me^{\gamma(t-s)}$ for $t \geq s$. Assume that U has an exponential dichotomy with projections $P_U(s)$ and constants $N, \delta > 0$ and that*

$$q := \sup_{s\in\mathbb{R}} \|U(s+1, s) - V(s+1, s)\| \leq \frac{(1-e^{-\delta})^2}{8N^2}. \tag{2.3}$$

Then V has an exponential dichotomy with projections $P_V(s)$ and constants $0 < \delta' < -\log(2qN + e^{-\delta})$ and N', where N' only depends on $M, \gamma, N, \delta, \delta'$. Moreover,

$$\|P_U(t) - P_V(t)\| \leq q\, \frac{16N^2}{3(1-e^{-\delta})^2}\,. \tag{2.4}$$

Proof. The result is a consequence of [19, Prop.2.3] and its proof except for the uniformity of N'. Equations (2.6) and (2.7) of [19] combined with (2.3) yield

$$\|R(1, T_V(1))\| \leq \frac{8N}{3(1-e^{-\delta})}\,.$$

The uniformity of N' thus follows from the next lemma (a variant of [18, Lem.4]). □

Lemma 2.2. *Let U be an evolution family on a Banach space X such that $\|U(t,s)\| \leq Me^{\gamma(t-s)}$, $t \geq s$, and $\|R(1, T_U(1))\| \leq C$ for the evolution semigroup $T_U(\cdot)$ on $C_0(\mathbb{R}, X)$. Then U has an exponential dichotomy with constants $N, \delta > 0$ depending only on M, γ, C.*

Proof. We first recall that if $\lambda \in \rho(T_U(1))$, then $\lambda e^{i\xi} \in \rho(T_U(1))$ and $\|R(\lambda e^{i\xi}, T_U(1)\| = \|R(\lambda, T_U(1))\|$, where $\xi \in \mathbb{R}$ and $\lambda \in \mathbb{C}$, see [7, (3.8)] or [9, p.483]. This fact implies

$$\|R(\lambda, T_U(1))\| \leq \tilde{C} := \frac{C}{1-(e^{\delta}-1)C} \tag{2.5}$$

for $|\lambda| = e^{\alpha}$ and $0 \le |\alpha| \le \delta < \log(1+\frac{1}{C})$. By (2.1) and a simple rescaling argument, we obtain the exponential dichotomy of U for every exponent $0 < \delta < \log(1+\frac{1}{C})$. Fix such a δ. If the result were false, then there would exist evolution families U_n on Banach spaces X_n satisfying the asumptions, real numbers t_n and s_n, and elements $x_n \in X_n$ such that $\|x_n\| = 1$ and

$$e^{\delta|t_n-s_n|}\,\|\Gamma_n(t_n,s_n)x_n\| \longrightarrow \infty \qquad \text{as} \;\; n\to\infty, \tag{2.6}$$

where Γ_n is Green's function of U_n. By assumption and (2.2), the projections $P_n(t)$ are uniformly bounded for $t \in \mathbb{R}$ and $n \in \mathbb{N}$. Hence the operators $\Gamma_n(t,s)$, $s \le t \le s+1$, $n \in \mathbb{N}$, are also uniformly bounded. Thus we have either $t_n > s_n + 1$ or $t_n < s_n$ in (2.6).

In the first case, we write $t_n = s_n + k_n + \tau_n$ for $k_n \in \mathbb{N}$ and $\tau_n \in (0,1]$. Otherwise, $t_n = s_n - k_n + \tau_n$ for $k_n \in \mathbb{N}$ and $\tau_n \in (0,1]$. Take continuous functions φ_n with $0 \le \varphi_n \le 1$, $\operatorname{supp}\varphi_n \subset (s_n + \frac{\tau_n}{2}, s_n + \frac{3\tau_n}{2})$, and $\varphi_n(t_n \mp k_n) = 1$ (here $t_n - k_n$ is used in the first case, and $t_n + k_n$ in the second). Set $\lambda = e^{\mp\delta}$ and $f_n(s) = e^{\pm\delta(s-s_n)}\,\varphi_n(s)\,U_n(s,s_n)x_n$. (We let $U(t,s) := 0$ for $t < s$.) Using [9, (VI.9.4)] in the second inequality, we compute

$$\begin{aligned}
[R(\lambda, T_{U_n}(1))f_n](t_n) &= \sum_{k=0}^{\infty} \lambda^{-(k+1)}[T_{U_n}(k)P_n(\cdot)f_n](t_n) - \sum_{k=1}^{\infty} \lambda^{k-1}[T_{U_n,Q}(k)^{-1}Q_n(\cdot)f_n](t_n)\\
&= \sum_{k=0}^{\infty} \lambda^{-(k+1)} U_n(t_n, t_n-k)P_n(t_n-k)f_n(t_n-k)\\
&\quad - \sum_{k=1}^{\infty} \lambda^{k-1} U_{n,Q}(t_n, t_n+k)Q_n(t_n+k)f(t_n+k)\\
&= \sum_{k=0}^{\infty} e^{\pm\delta(k+1)}e^{\pm\delta(t_n-k-s_n)}\varphi_n(t_n-k)U_n(t_n,s_n)P_n(s_n)x_n\\
&\quad - \sum_{k=1}^{\infty} e^{\mp\delta(k-1)}e^{\pm\delta(t_n+k-s_n)}\varphi_n(t_n+k)U_{n,Q}(t_n,s_n)Q_n(s_n)x_n.
\end{aligned}$$

Here exactly one term does not vanish, namely $k = k_n$ in the first sum if $t_n > s_n$ and $k = k_n$ in the second sum if $t_n < s_n$. Therefore

$$R(\lambda, T_{U_n}(1))f_n(t_n) = e^{\pm\delta}e^{\delta|t_n-s_n|}\,\Gamma_n(t_n,s_n)x_n\,,$$

and the assumptions and (2.5) imply

$$\|e^{-\delta}e^{\delta|t_n-s_n|}\Gamma_n(t_n,s_n)x_n\| \le \|R(\lambda, T_{U_n}(1))f_n\|_\infty \le \tilde{C}Me^{2(\gamma+\delta)}.$$

This estimate contradicts (2.6). □

In the present work we study operators $A(t)$ on X subject to the following hypotheses.

(H1) There is an $\omega \ge 0$ such that the operators $A(t)$, $t \in \mathbb{R}$, satisfy $\Sigma_\phi \cup \{0\} \subseteq \rho(A(t)-\omega)$, $\|R(\lambda, A(t)-\omega)\| \le \frac{K}{1+|\lambda|}$, and

$$\|(A(t)-\omega)R(\lambda, A(t)-\omega)\,[R(\omega, A(t)) - R(\omega, A(s))]\| \le L\,|t-s|^{\mu}\,|\lambda|^{-\nu}$$

for $t,s \in \mathbb{R}$, $\lambda \in \Sigma_\phi := \{\lambda \in \mathbb{C}\setminus\{0\} : |\arg\lambda| \le \phi\}$, and constants $\phi \in (\frac{\pi}{2}, \pi)$, $L, K \ge 0$, and $\mu,\nu \in (0,1]$ with $\mu+\nu > 1$.

This assumption was introduced by P. Acquistapace and B. Terreni in [2] (for $\omega = 0$). It implies that there exists a unique evolution family U on X such that $(t,s) \mapsto U(t,s) \in \mathcal{L}(X)$ is continuous for $t > s$, $U(\cdot, s) \in C^1((s,\infty), \mathcal{L}(X))$, $\partial_t U(t,s) = A(t)U(t,s)$, and

$$\|A(t)^k U(t,s)\| \le C\,(t-s)^{-k}, \tag{2.7}$$

$$\|A(t)U(t,s)R(w, A(s))\| \le C, \tag{2.8}$$

$$\|U(t,s)(\omega - A(s))^\alpha x\| \le C\,(\mu-\alpha)^{-1}\,(t-s)^{-\alpha}\,\|x\| \tag{2.9}$$

for $0 < t - s \le 1$, $k = 0, 1$, $0 \le \alpha < \mu$, $x \in D((\omega - A(s))^\alpha)$, and a constant C depending only on the constants in (H1). Moreover, $\partial_s^+ U(t,s)x = -U(t,s)A(s)x$ for $t > s$ and $x \in D(A(s))$ with $A(s)x \in \overline{D(A(s))}$. This follows by an obvious rescaling from [1, Thm.2.3] and [23, Thm.2.1], see also [2], [22]. We say that $A(\cdot)$ *generates* U. Note that U is exponentially bounded by (2.7) with $k = 0$. We further suppose that

(H2) the evolution family U generated by $A(\cdot)$ has an exponential dichotomy with constants $N, \delta > 0$, dichotomy projections $P(t)$, $t \in \mathbb{R}$, and Green's function Γ.

We point out that it is quite difficult to find conditions on $A(\cdot)$ implying (H2). Such results are usually based on (2.1) and variants of it, see [8], [10], [13], [15], [18], [19], [20].

Assume that (H1) and (H2) hold and that $f : \mathbb{R} \to X$ is bounded and continuous. A *classical solution* of (1.1) is a function $u \in C^1(\mathbb{R}, X)$ satisfying $u(t) \in D(A(t))$ for $t \in \mathbb{R}$ and (1.1). It is known that then

$$u(t) = U(t,s)u(s) + \int_s^t U(t,\tau)f(\tau)\,d\tau \qquad \text{for all } \ t \ge s, \tag{2.10}$$

see [1, Prop.3.2, 5.1]. On the other hand, there is a unique bounded continuous function satisfying (2.10), namely

$$u(t) = \int_{\mathbb{R}} \Gamma(t,\tau) f(\tau)\,d\tau, \qquad t \in \mathbb{R}, \tag{2.11}$$

see e.g. (the proof of) [7, Thm.4.28]. We call the function u given by (2.11) the *mild solution* of (1.1). Observe that the mild solution is a classical one if, for instance, f is Hölder continuous due to [2, Thm.6.3] and (2.10).

It remains to introduce the concept of almost periodicity, see e.g. [4], [10], [15], [16]. The next definition due to H.Bohr is the most convenient one for our purposes.

Definition 2.3. *Let Y be a Banach space and $g : \mathbb{R} \to Y$ be continuous. A number $\tau \in \mathbb{R}$ is an ϵ–*almost period* of g if $\|g(t+\tau) - g(t)\| \le \epsilon$ holds for all $t \in \mathbb{R}$ and some $\epsilon > 0$. The function g is called* almost periodic *if for every $\epsilon > 0$ there exist a set $P(\epsilon) \subseteq \mathbb{R}$ of ϵ–pseudo periods of g and a number $\ell(\epsilon) > 0$ such that each interval $(a, a + \ell(\epsilon))$, $a \in \mathbb{R}$, contains an $\tau = \tau_\epsilon \in P(\epsilon)$. The space of almost periodic functions is denoted by $AP(\mathbb{R}, Y)$.*

We recall that $AP(\mathbb{R}, Y)$ is a closed subspace of the space of bounded and uniformly continuous functions, see [15, Chap.1]. Our last assumption reads as follows.

(H3) $R(\omega, A(\cdot)) \in AP(\mathbb{R}, \mathcal{L}(X))$ with pseudo periods $\tau = \tau_\epsilon$ belonging to sets $P(\epsilon, A)$.

It is not difficult to verify that then $R(\omega', A(\cdot)) \in AP(\mathbb{R}, \mathcal{L}(X))$ if $\omega' \ge \omega$.

3. Yosida approximations

Assume that (H1) holds and define the Yosida opproximations $A_n(t) = nA(t)R(n, A(t))$ of $A(t)$ for $n > \omega$ and $t \in \mathbb{R}$. These operators generate an evolution family U_n on X. We want to show that $A_n(\cdot)$ satisfies the same assumptions as $A(\cdot)$. The elementary (but tedious) proof of the next result is omitted, cf. [2, Lem.4.2] or [22, Prop.2.1].

Lemma 3.1. *Assume that (H1) holds. Fix $\omega' > \omega$ and $\phi' \in (\frac{\pi}{2}, \phi)$. Then there are constants $n_0 > \omega$, $L' \geq L$, and $K' \geq K$ (only depending on the constants in (H1), ω', and ϕ') such that the operators $A_n(t)$, $t \in \mathbb{R}$, satisfy (H1) with constants $K', \phi', \omega', L', \mu, \nu$ for all $n \geq n_0$.*

Since (H1), (H2), and (H3) still hold for $A(\cdot)$ with constants K', L', ω', ϕ', we can assume that $A(\cdot)$ and $A_n(\cdot)$ satisfy (H1) with the same constants, denoted by K, L, ω, ϕ.

Lemma 3.2. *If (H1) and (H3) hold, then there is a number $n_1 \geq n_0$ such that $R(\omega, A_n(\cdot)) \in AP(\mathbb{R}, \mathcal{L}(X))$ for $n \geq n_1$, with pseudo periods belonging to $P(\epsilon/\kappa, A)$, where $\kappa := 2 + 4K$.*

Proof. Let $\tau > 0$ and $t \in \mathbb{R}$. We first observe that

$$R(\lambda, A_n(t)) = \tfrac{1}{\lambda+n}(n - A(t))R(\tfrac{\lambda n}{\lambda+n}, A(t)) = \tfrac{n^2}{(\lambda+n)^2}R(\tfrac{\lambda n}{\lambda+n}, A(t)) + \tfrac{1}{\lambda+n} \tag{3.1}$$

if $n \geq n_0$ and $|\arg(\lambda - \omega)| \leq \phi$. Equation (3.1) yields

$$\begin{aligned} R(\omega, A_n(t+\tau)) - R(\omega, A_n(t)) &= \frac{n^2}{(\omega+n)^2}\Big(R\Big(\frac{\omega n}{\omega+n}, A(t+\tau)\Big) - R\Big(\frac{\omega n}{\omega+n}, A(t)\Big)\Big) \\ &= \frac{n^2}{(\omega+n)^2}R(\omega, A(t+\tau))\Big[1 - \frac{\omega^2}{\omega+n}R(\omega, A(t+\tau))\Big]^{-1} \\ &\quad - \frac{n^2}{(\omega+n)^2}R(\omega, A(t))\Big[1 - \frac{\omega^2}{\omega+n}R(\omega, A(t))\Big]^{-1}, \end{aligned} \tag{3.2}$$

where we have used that

$$\Big\|\frac{\omega^2}{\omega+n}R(\omega, A(s))\Big\| \leq \frac{\omega^2}{\omega+n}\frac{K}{1+\omega} \leq \frac{\omega K}{n} \leq \frac{1}{2}$$

for $n \geq n_1 := \max\{n_0, 2\omega K\}$ and $s \in \mathbb{R}$. In particular,

$$\Big\|\Big[1 - \frac{\omega^2}{\omega+n}R(\omega, A(s))\Big]^{-1}\Big\| \leq 2. \tag{3.3}$$

Hence, (3.2) implies

$$\begin{aligned} &\|R(\omega, A_n(t+\tau)) - R(\omega, A_n(t))\| \\ &\qquad \leq 2\,\|R(\omega, A(t+\tau)) - R(\omega, A(t))\| \\ &\qquad\quad + \frac{K}{1+\omega}\,\Big\|\Big[1 - \frac{\omega^2}{\omega+n}R(\omega, A(t+\tau))\Big]^{-1} - \Big[1 - \frac{\omega^2}{(\omega+n)^2}R(\omega, A(t))\Big]^{-1}\Big\|. \end{aligned}$$

Employing (3.3) again, we obtain

$$\begin{aligned}
&\Big\| \Big[1 - \frac{\omega^2}{\omega+n} R(\omega, A(t+\tau))\Big]^{-1} - \Big[1 - \frac{\omega^2}{\omega+n} R(\omega, A(t))\Big]^{-1} \Big\| \\
&\le 4 \Big\| \Big[1 - \frac{\omega^2}{\omega+n} R(\omega, A(t+\tau))\Big] - \Big[1 - \frac{\omega^2}{\omega+n} R(\omega, A(t))\Big] \Big\| \\
&\le 4\omega \, \|R(\omega, A(t+\tau)) - R(\omega, A(t))\|.
\end{aligned}$$

Putting everything together, we arrive at

$$\|R(\omega, A_n(t+\tau)) - R(\omega, A_n(t))\| \le (2+4K)\|R(\omega, A(t+\tau)) - R(\omega, A(t))\| \tag{3.4}$$

for $n \ge n_1$ and $t \in \mathbb{R}$. The assertion thus follows from (H3). □

In order to see that $A_n(\cdot)$ satisfies also (H2), we need the following result which is of interest in itself. For $\omega = 0$ it is shown in [5, Prop.4.4] (note that our result does not simply follow by rescaling). We give here a different, more elementary proof leading to a different rate of convergence.

Proposition 3.3. *Let (H1) hold and fix $0 < t_0 < t_1$. Then $\|U(t,s) - U_n(t,s)\| \le c(t_1, \theta)\, n^{-\theta}$ for $0 < t_0 \le t - s \le t_1$, $n \ge n_2(t_0) := \max\{n_0, t_0^{-2/\mu}\}$, and any $0 < \theta < \min\{\mu/2, 1-\mu/2, \mu(\mu+\nu-1)/2\}$. Moreover, $\|(U(t,s) - U_n(t,s))R(\omega, A(s))\| \le c(t_1, \alpha)\, n^{-\alpha}$ for $0 \le t - s \le t_1$, $n \ge n_0$, and $\alpha \in (0, \mu)$.*

Proof. Let $0 < h < t_0$ and $0 < t_0 \le t - s \le t_1$. Then we have

$$\begin{aligned}
U(t,s) - U_n(t,s) &= (U(t,s+h) - U_n(t,s+h))U(s+h,s) - U_n(t,s+h) \\
&\quad \cdot [U(s+h,s) - e^{hA(s)} + e^{hA(s)} - e^{hA_n(s)} + e^{hA_n(s)} - U_n(s+h,s)] \\
&=: S_1 - S_2.
\end{aligned} \tag{3.5}$$

Due to Lemma 4.3 in [2] and equation (2.6) and Lemma 2.2 in [1], we obtain

$$\|S_2\| \le c(t_1)\,(h^{\mu+\nu-1} + (hn)^{-1}). \tag{3.6}$$

The other term can be transformed into

$$\begin{aligned}
S_1 &= \int_{s+h}^{t} U_n(t,\sigma)(A(\sigma) - A_n(\sigma))U(\sigma,s)\, d\sigma \\
&= \int_{s+h}^{t} U_n(t,\sigma)(\omega - A_n(\sigma))^{\alpha}(\omega - A_n(\sigma))^{1-\alpha}((A(\sigma)-\omega)^{-1} - (A_n(\sigma)-\omega)^{-1}) \\
&\quad \cdot (A(\sigma) - \omega)U(\sigma,s)\, d\sigma.
\end{aligned}$$

where $\alpha \in (0, \mu)$. The estimates (2.7), (2.9), and

$$\begin{aligned}
\|(A_n(\sigma) - \omega)^{-1} - (A(\sigma) - \omega)^{-1}\| = \|\tfrac{1}{\omega+n} A(\sigma) R(\tfrac{\omega n}{\omega+n}, A(\sigma)) A(\sigma) R(\omega, A(\sigma))\| &\le \frac{c}{n}, \\
\|(\omega - A_n(\sigma))^{1-\alpha}\| \le c\, \|\omega - A_n(\sigma)\|^{1-\alpha} &\le c n^{1-\alpha}
\end{aligned} \tag{3.7}$$

(use the moment inequality [9, Thm.II.5.38] in the second line) lead to

$$\|S_1\| \le c(t_1, \alpha)\, h^{-1} n^{-\alpha}. \tag{3.8}$$

Combining (3.5), (3.6), and (3.8), we deduce

$$\|U(t,s) - U_n(t,s)\| \le c(t_1, \alpha)\,((nh)^{-1} + h^{\mu+\nu-1} + n^{-\alpha} h^{-1}).$$

The first assertion now follows if we set $h := n^{-\mu/2}$. The second one can be shown using the formula

$$(U(t,s) - U_n(t,s))R(\omega, A(s)) = \int_s^t U_n(t,\sigma)(\omega - A_n(\sigma))^{\alpha}(\omega - A_n(\sigma))^{1-\alpha}$$
$$\cdot ((A(\sigma) - \omega)^{-1} - (A_n(\sigma) - \omega)^{-1})(A(\sigma) - \omega)U(\sigma, s)R(\omega, A(s))\, d\sigma,$$

and the estimates (2.9), (3.7), and (2.8). ☐

Propositions 2.1 and 3.3 immediately imply that $A_n(\cdot)$ fulfills also (H2).

Corollary 3.4. *Let (H1) and (H2) hold. Then there is a number $n_3 \geq n_2(1)$ such that U_n has an exponential dichotomy for $n \geq n_3$ with constants $\delta' \in (0, \delta)$ and $N' = N'(\delta')$ independent of n. Moreover, the dichotomy projections $P_n(t)$ of U_n satisfy $\|P_n(t) - P(t)\| \leq c(\theta)\, n^{-\theta}$ for $t \in \mathbb{R}$, where $\theta \in (0,1)$ and $n_2(1)$ are given by Proposition 3.3.*

4. Main results for equations on $\mathbb{R}$

We first establish the almost periodicity of Green's function Γ_n for the evolution family U_n generated by the Yosida approximation $A_n(\cdot)$.

Lemma 4.1. *Assume that (H1), (H2), and (H3) hold. Let $n \geq \max\{n_1, n_3\}$, $\eta > 0$, and $\tau \in P(\eta/\kappa, A)$, where n_1, n_3, and κ were given in Section 3. We then have*

$$\|\Gamma_n(t+\tau, s+\tau) - \Gamma_n(t,s)\| \leq c\,\eta\, n^2\, e^{-\frac{\delta}{2}|t-s|}, \quad t,s \in \mathbb{R}.$$

Proof. The operators $\Gamma_n(t,s)$ exist by Corollary 3.4. It is easy to see that

$$\begin{aligned} g_n(\sigma) &:= \frac{d}{d\sigma}\Big(\Gamma_n(t,\sigma)\Gamma_n(\sigma+\tau, s+\tau)\Big) \\ &= \Gamma_n(t,\sigma)(A_n(\sigma+\tau) - A_n(\sigma))\Gamma_n(\sigma+\tau, s+\tau) \\ &= \Gamma_n(t,\sigma)(A_n(\sigma) - \omega)((A_n(\sigma) - \omega)^{-1} - (A_n(\sigma+\tau) - \omega)^{-1}) \\ &\qquad \cdot (A_n(\sigma+\tau) - \omega)\Gamma_n(\sigma+\tau, s+\tau), \end{aligned}$$

for $\sigma \neq t, s$, $\tau \geq 0$, and $n \geq n_3$. Hence,

$$\int_{\mathbb{R}} g_n(\sigma)\, d\sigma = \begin{cases} \int_{-\infty}^s g_n(\sigma)d\sigma + \int_s^t g_n(\sigma)d\sigma + \int_t^\infty g_n(\sigma)d\sigma, & t \geq s, \\ \int_{-\infty}^t g_n(\sigma)d\sigma + \int_t^s g_n(\sigma)d\sigma + \int_s^\infty g_n(\sigma)d\sigma, & t < s, \end{cases}$$

$$= \begin{cases} -\Gamma_n(t,s)Q_n(s+\tau) + P_n(t)\Gamma_n(t+\tau, s+\tau) - \Gamma_n(t,s)P_n(s+\tau) + Q_n(t)\Gamma_n(t+\tau, s+\tau), \\ P_n(t)\Gamma_n(t+\tau, s+\tau) - \Gamma_n(t,s)Q_n(s+\tau) + Q_n(t)\Gamma_n(t+\tau, s+\tau) - \Gamma_n(t,s)P_n(s+\tau), \end{cases}$$

$$= \Gamma_n(t+\tau, s+\tau) - \Gamma_n(t,s).$$

We have shown that

$$\Gamma_n(t+\tau, s+\tau) - \Gamma_n(t,s) = \int_{\mathbb{R}} \Gamma_n(t,\sigma)(A_n(\sigma) - \omega)\,[R(\omega, A_n(\sigma+\tau)) - R(\omega, A_n(\sigma))]$$
$$\cdot (A_n(\sigma+\tau) - \omega)\Gamma_n(\sigma+\tau, s+\tau)\, d\sigma \tag{4.1}$$

for $s, t \in \mathbb{R}$ and $n \geq n_3$. Lemma 3.2 and Corollary 3.4 now yield

$$\|\Gamma_n(t+\tau, s+\tau) - \Gamma_n(t,s)\| \leq c\eta n^2 \int_{\mathbb{R}} e^{-\frac{3\delta}{4}|t-\sigma|}\, e^{-\frac{3\delta}{4}|\sigma-s|}\, d\sigma$$

if also $n \geq n_1$ and $\tau \in P(\eta/\kappa, A)$, which gives the asserted estimate. ☐

Lemma 4.2. *Assume that (H1) and (H2) hold. Fix $0 < t_0 < t_1$ and let $\theta > 0$, $n_2(t_0)$, and n_3 be given by Proposition 3.3 and Corollary 3.4. Then $\|\Gamma(t,s) - \Gamma_n(t,s)\| \le c(t_1,\theta)\, n^{-\theta}$ holds for $t_0 \le |t-s| \le t_1$ and $n \ge \max\{n_3, n_2(t_0)\}$.*

Proof. If $t_0 \le t-s \le t_1$, we write

$$\Gamma_n(t,s) - \Gamma(t,s) = (U_n(t,s) - U(t,s))P_n(s) + U(t,s)(P_n(s) - P(s)).$$

So the assertion is a consequence of Proposition 3.3 and Corollary 3.4. For $-t_1 \le t-s \le -t_0$, we have

$$\begin{aligned}\Gamma(t,s) - \Gamma_n(t,s) &= U_{n,Q}(t,s)Q_n(s) - U_Q(t,s)Q(s)\\ &= U_Q(t,s)Q(s)(Q_n(s) - Q(s)) + (Q_n(t) - Q(t))U_{n,Q}(t,s)Q_n(s)\\ &\quad - U_Q(t,s)Q(s)(U_n(s,t) - U(s,t))U_{n,Q}(t,s)Q_n(s).\end{aligned}$$

Again the asserted estimate follows from Proposition 3.3 and Corollary 3.4. □

Employing (4.1), we extend a formula given on [13, p.240] to the present setting.

Corollary 4.3. *Let (H1) and (H2) hold, $t, s, \tau \in \mathbb{R}$, and $x \in D((\omega - A(s))^\beta)$ for some $\beta > 0$ (or x contained in a suitable interpolation space). We then have*

$$\begin{aligned}\Gamma(t+\tau, s+\tau)x - \Gamma(t,s)x = \int_{\mathbb{R}} &\Gamma(t,\sigma)(A(\sigma) - \omega)\,[R(\omega, A(\sigma+\tau)) - R(\omega, A(\sigma))]\\ &\cdot (A(\sigma+\tau) - \omega)\Gamma(\sigma+\tau, s+\tau)x\, d\sigma. \end{aligned} \tag{4.2}$$

Proof. We want to obtain (4.2) by taking the limit as $n \to \infty$ in (4.1) evaluated at $x \in D((\omega - A(s))^\beta)$. The left hand side converges as required due to Lemma 4.2 and Corollary 3.4. For $r \neq \rho$, $n \ge n_3$, and $\alpha \in (1-\nu, \mu)$, we write

$$\Gamma_n(r,\rho)(\omega - A_n(\rho))^\alpha = \begin{cases} P_n(r)U_n(r,\rho)(\omega - A_n(\rho))^\alpha, & \rho < r \le \rho+1,\\ U_n(r,\rho+1)P_n(\rho+1)U_n(\rho+1,\rho)(\omega - A_n(\rho))^\alpha, & \rho+1 \le r,\\ -U_{n,Q}(r,\rho+1)Q_n(\rho+1)U_n(\rho+1,\rho)(\omega - A_n(\rho))^\alpha, & r < \rho,\end{cases}$$

$$A_n(r)\Gamma_n(r,\rho) = \begin{cases} A_n(r)U_n(r,\rho)P_n(\rho), & \rho < r \le \rho+1,\\ A_n(r)U_n(r,r-1)U_n(r-1,\rho)P_n(\rho), & \rho+1 \le r,\\ -A_n(r)U_n(r,r-1)U_{n,Q}(r-1,\rho)Q_n(\rho)x, & r < \rho.\end{cases}$$

By (the proof of) [22, Prop.3.1] and Lemma 4.2 these terms converge strongly to the analogous terms without n as $n \to \infty$. Using also [22, Prop.2.1], one deduces the pointwise convergence of the integrand in (4.1). Moreover, we observe that

$$\begin{aligned}A_n(r)U_n(r,\rho)P_n(\rho)x = A_n(r)U_n(r,\rho)(\omega - A_n(\rho))^{-\beta}\Big(&(\omega - A_n(\rho))^\beta x\\ &- (\omega - A_n(\rho))^\beta U_n(\rho,\rho-1)U_{n,Q}(\rho-1,\rho)Q_n(\rho)x\Big).\end{aligned}$$

Due to (2.7) and [23, Thm.2.1], the norm of the right hand side is smaller than

$$c\,(r-\rho)^{\beta-1}(\|(\omega - A_n(\rho))^\beta x\| + \|x\|).$$

The moment inequality, see e.g. [9, Thm.II.5.38], further yields

$$\|(\omega - A_n(\rho))^\beta x\| = \|\left[(\omega - A_n(\rho))(\omega - A(\rho))^{-1}\right]^\beta (\omega - A(\rho))^\beta x\| \le c\,\|(\omega - A(\rho))^\beta x\|.$$

Combining these estimates with (2.9), [22, Prop.2.1], and (2.7), we obtain an integrable, n–independent bound of the integrand in (4.1) evaluated at $x \in D((\omega - A(s))^\beta)$. The assertion thus follows from the theorem of dominated convergence. $\square$

Though the above formula is quite interesting, the proofs of our main results only use the two preceding lemmas.

Proposition 4.4. *Assume that (H1), (H2), and (H3) hold. Let $\epsilon > 0$ and $h > 0$. Then*

$$\|\Gamma(t+\tau, s+\tau) - \Gamma(t,s)\| \le \epsilon\, e^{-\frac{\delta}{2}|t-s|}$$

holds for $|t-s| \ge h$ and $\tau \in P(\eta/\kappa, A)$, where $\kappa = 2 + 4K$ and $\eta = \eta(\epsilon, h) \to 0$ as $\epsilon \to 0$ and h is fixed.

Proof. Let $\epsilon > 0$ and $h > 0$ be fixed. Then there is a $t_\epsilon > h$ such that

$$\|\Gamma(t+\tau, s+\tau) - \Gamma(t,s)\| \le \epsilon\, e^{-\frac{\delta}{2}|t-s|}$$

for $|t-s| \ge t_\epsilon$. Let $h \le |t-s| \le t_\epsilon$ and $\tau \in P(\eta/\kappa, A)$. Lemmas 4.1 and 4.2 show that

$$\|\Gamma(t+\tau, s+\tau) - \Gamma(t,s)\| \le (c(t_\epsilon) e^{\frac{\delta}{2} t_\epsilon}\, n^{-\theta} + c\eta n^2)\, e^{-\frac{\delta}{2}|t-s|}$$

for $n \ge \max\{n_1, n_2(h), n_3\}$. We now choose first a large n and then a small $\eta > 0$ (depending on ϵ and h) in order to obtain the assertion. $\square$

Theorem 4.5. *Assume that (H1), (H2), and (H3) hold. Then $r \mapsto \Gamma(t+r, s+r)$ belongs to $AP(\mathbb{R}, \mathcal{L}(X))$ for $t, s \in \mathbb{R}$, where we may take the same pseudo periods for t, s with $|t-s| \ge h > 0$. If $f \in AP(\mathbb{R}, X)$, then the unique bounded mild solution $u = \int_{\mathbb{R}} \Gamma(\cdot, s) f(s)\, ds$ of (1.1) is almost periodic.*

Proof. In Lemma 4.1 we have seen that $P_n(\cdot) \in AP(\mathbb{R}, \mathcal{L}(X))$. Corollary 3.4 then shows that $P(\cdot) \in AP(\mathbb{R}, \mathcal{L}(X))$. Thus the first assertion follows from Proposition 4.4. Further, for $\tau, h > 0$ and $t \in \mathbb{R}$, we write

$$\begin{aligned} u(t+\tau) - u(t) &= \int_{\mathbb{R}} \Gamma(t+\tau, s+\tau) f(s+\tau)\, ds - \int_{\mathbb{R}} \Gamma(t,s) f(s)\, ds \\ &= \int_{\mathbb{R}} \Gamma(t+\tau, s+\tau)(f(s+\tau) - f(s))\, ds + \int_{|t-s|\ge h} (\Gamma(t+\tau, s+\tau) - \Gamma(t,s)) f(s)\, ds \\ &\quad + \int_{|t-s|\le h} (\Gamma(t+\tau, s+\tau) - \Gamma(t,s)) f(s)\, ds. \end{aligned}$$

For $\overline{\epsilon} > 0$ let $\eta = \eta(\overline{\epsilon}, h)$ be given by Proposition 4.4. Let $P(\epsilon, A, f)$ be the set of pseudo periods for the almost periodic function $t \mapsto (f(t), R(\omega, A(t)))$, cf. [15, p.6]. Taking $\tau \in P(\eta/\kappa, A, f)$, we deduce from Proposition 4.4 and (H2) that

$$\|u(t+\tau) - u(t)\| \le \tfrac{2N}{\delta\kappa}\, \eta(\overline{\epsilon}, h) + (\tfrac{4}{\delta}\, \overline{\epsilon} + 4Nh)\|f\|_\infty.$$

for $t \in \mathbb{R}$. Given an $\epsilon > 0$, we can take first a small $h > 0$ and then a small $\overline{\epsilon} > 0$ such that $\|u(t+\tau) - u(t)\| \le \epsilon$ for $t \in \mathbb{R}$ and $\tau \in P(\eta/\kappa, A, f) =: P(\epsilon)$. $\square$

Remark 4.6. For $g \in AP(\mathbb{R}, Y)$ and $\lambda \in \mathbb{R}$ the means

$$\lim_{t\to\infty} \frac{1}{2t} \int_{-t}^{t} e^{-i\lambda s} g(s)\, ds$$

exist. They are different from zero for at most countable many λ, which are called the *frequencies* of g, see e.g. [4, §4.5], [15, §2.3]. The *module* of g is the smallest additive subgroup of $\mathbb{R}$ containing all frequencies of g. By [15, p.44] (see also [10, Thm.4.5]) and the proof of Theorem 4.5 the module of the solution u to (1.1) is contained in the joint module of f and $R(\omega, A(\cdot))$ (which is the smallest additive subgroup of $\mathbb{R}$ containing the frequencies of the function $t \mapsto (f(t), R(\omega, A(t))))$. Similarly, the modules of $\Gamma(t+\cdot, s+\cdot)$, $t \neq s$, are contained in that of $R(\omega, A(\cdot))$.

Example 4.7. Consider the parabolic problem

$$\begin{aligned} D_t\, u(t,x) &= \sum_{k,l=1}^{n} D_k\, a_{kl}(t,x)\, D_l u(t,x) + a_0(t,x)u(t,x) + f(t,x), \quad t \in \mathbb{R},\ x \in \Omega, \\ \sum_{k,l=1}^{n} n_k(x)\, a_{kl}(t,x)\, D_l u(t,x) &= 0, \quad t \in \mathbb{R},\ x \in \partial\Omega, \end{aligned} \tag{4.3}$$

on the time interval $\mathbb{R}$. Here $\Omega \subseteq \mathbb{R}^n$ is a bounded domain with C^2–boundary $\partial\Omega$ being locally on one side of Ω, $D_t = d/dt$, $D_k = d/dx_k$, and $n(x)$ is the outer unit normal vector. We assume that the coefficients satisfy

$$\begin{aligned} a_{kl} &\in C_b^\mu(\mathbb{R}, C(\overline{\Omega})) \cap C_b(\mathbb{R}, C^1(\overline{\Omega})) \cap AP(\mathbb{R}, L^n(\Omega)), \quad k,l = 1, \cdots, n, \\ a_0 &\in C_b^\mu(\mathbb{R}, L^n(\Omega)) \cap C_b(\mathbb{R}, C(\overline{\Omega})) \cap AP(\mathbb{R}, L^{n/2}(\Omega)) \end{aligned}$$

for some $\frac{1}{2} < \mu \leq 1$, where $n/2$ is replaced by 1 if $n = 1$. Moreover, (a_{kl}) is supposed to be symmetric and real and to satisfy $\sum_{k,l} a_{kl}(t,x)\, y_k\, y_l \geq \eta\, |y|^2$ for a constant $\eta > 0$, $x \in \overline{\Omega}$, $t \in \mathbb{R}$, $y \in \mathbb{R}^n$. Let

$$A(t,x,D) = \sum_{k,l=1}^{n} D_k\, a_{kl}(t,x)\, D_l\, + a_0(t,x).$$

We then define on $X = L^p(\Omega)$, $1 < p < \infty$, the operator $A(t)\varphi = A(t,\cdot,D)\varphi$ with domain

$$D(A(t)) = \{\varphi \in W^{2,p}(\Omega) : \sum_{k,l=1}^{n} n_k(\cdot)\, a_{kl}(t,\cdot)\, D_l\varphi = 0 \text{ on } \partial\Omega\},$$

where the boundary condition is understood in the sense of traces if necessary. It is shown in [22, §4] that $A(t)$, $t \in \mathbb{R}$, fulfill (H1) for μ and each $\nu \in (0, \frac{1}{2})$. Thus there exists an evolution family U on X solving the Cauchy problem corresponding to (4.3) for $f = 0$. In the same way, one shows that (H3) holds (with the same pseudo periods as the coefficients). Therefore (4.3) has a unique mild solution $u \in AP(\mathbb{R}, X)$ provided that $f \in AP(\mathbb{R}, X)$ and U has an exponential dichotomy. The solution is classical if, e.g., f is also Hölder continuous in time.

5. Main results for equations on $\mathbb{R}_+$

The arguments and results of the previous sections can be extended to the problem (1.2) on $\mathbb{R}_+$. Here we will concentrate on the necessary modifications for the sake of brevity. We first introduce the space

$$AP(\mathbb{R}_+, Y) := \{g : \mathbb{R}_+ \to Y : \exists\, \tilde{g} \in AP(\mathbb{R}, Y) \text{ s.t. } \tilde{g}|\mathbb{R}_+ = g\}$$

of almost periodic functions on $\mathbb{R}_+$. We remark that the function $\tilde{g}$ in the above definition is uniquely determined, cf. [4, Prop.4.7.1]. The following concept is more important for our investigations.

Definition 5.1. *A continuous function $g : \mathbb{R}_+ \to Y$ is called* asymptotically almost periodic *if for every $\epsilon > 0$ there exists a set $P(\epsilon) \subseteq \mathbb{R}_+$ and numbers $s(\epsilon), \ell(\epsilon) > 0$ such that each interval $(a, a + \ell(\epsilon))$, $a \geq 0$, contains an $\tau = \tau_\epsilon \in P(\epsilon)$ and the estimate $\|g(t+\tau) - g(t)\| \leq \epsilon$ holds for all $t \geq s(\epsilon)$ and $\tau \in P(\epsilon)$. The space of asymptotically almost periodic functions is denoted by $AAP(\mathbb{R}_+, Y)$.*

Due to e.g. [4, Thm.4.7.5], these spaces are related by the equality

$$AAP(\mathbb{R}_+, Y) = AP(\mathbb{R}_+, Y) \oplus C_0(\mathbb{R}_+, Y). \tag{5.1}$$

Evolution families and exponential dichotomy on time intervals $[a, \infty)$ are defined by restricting the definitions on $\mathbb{R}$ to parameters $t, s \geq a$. So we can make the following assumptions.

- (H1') The operators $A(t)$, $t \geq -1$, satisfy (H1) for $t, s \geq -1$.
- (H2') The evolution family U generated by $A(\cdot)$ has an exponential dichotomy on $[-1, \infty)$ with projections $P(t)$, $t \geq -1$, constants $N, \delta > 0$, and Green's function Γ.
- (H3') $R(\omega, A(\cdot)) \in AAP(\mathbb{R}_+, \mathcal{L}(X))$ with constants $s(\epsilon, A)$ and sets $P(\epsilon, A)$.

(We have to involve the interval $[-1, \infty)$ for technical reasons, see (5.6). Each interval $[b, \infty)$ with $b < 0$ would do the job.) Assume that (H1') and (H2') hold and let $f : \mathbb{R}_+ \to X$ be bounded and continuous. Then the *mild solution* of (1.2) is given by

$$u(t) := U(t, 0)x + \int_0^t U(t, s) f(s)\, ds, \qquad t \geq 0.$$

This function is a classical solution (i.e., $u \in C^1((0, \infty), X) \cap C(\mathbb{R}_+, X)$, $u(t) \in D(A(t))$, and (1.2) holds for $t > 0$) if $x \in \overline{D(A(0))}$ and, e.g., f is Hölder continuous, see [2, Thm.6.3]. By [1, Prop.3.2,5.1], each classical solution is a mild one if $x \in \overline{D(A(0))}$. Writing $f(s) = P(s)f(s) + Q(s)f(s)$, one sees that a mild solution u satisfies

$$u(t) = U(t, 0)\Big(x + \int_0^\infty U_Q(0, s) Q(s) f(s)\, ds\Big) + \int_0^\infty \Gamma(t, s) f(s)\, ds, \quad t \geq 0. \tag{5.2}$$

Thus u is bounded if and only if the term in brackets belongs to $P(0)X$ if and only if

$$Q(0)x = -\int_0^\infty U_Q(0, s) Q(s) f(s)\, ds. \tag{5.3}$$

In this case the mild solution of (1.2) is given by

$$u(t) = U(t, 0)P(0)x + \int_0^\infty \Gamma(t, s) f(s)\, ds, \qquad t \geq 0. \tag{5.4}$$

Again we consider the Yosida approximations $A_n(t)$, $t \geq 0$, and the evolution family U_n generated by $A_n(\cdot)$. Lemma 3.1 and Proposition 3.3 clearly hold on $\mathbb{R}_+$ if we replace (H1) by (H1'). Since also the estimate (3.4) remains valid for $t, \tau \geq 0$, Lemma 3.2 is still true if we replace (H1) and (H3) by (H1') and (H3') and $AP(\mathbb{R}, \mathcal{L}(X))$ by $AAP(\mathbb{R}_+, \mathcal{L}(X))$. Moreover, we can take $s(\epsilon, A_n) = s(\epsilon, A)$.

However, it is not immediate that Corollary 3.4 can be extended to the half line case because the proof of the perturbation result Proposition 2.1 only works on $\mathbb{R}$. But one can overcome this obstacle using a suitable extension of U.

Lemma 5.2. *Assume that (H1') and (H2') hold. Then there is a number $n_3' \geq n_0$ such that, for $n \geq n_3'$, the evolution family U_n generated by $A_n(\cdot)$ has an exponential dichotomy on $\mathbb{R}_+$ with dichotomy projections $P_n(t)$ and constants $\delta' \in (0,\delta)$ and $N' = N'(\delta')$ independent of n. Moreover, $\|P_n(t) - P(t)\| \leq c\,n^{-\theta}$ for $t \geq 0$, where $\theta \in (0,1)$ is a fixed number given by Proposition 3.3.*

Proof. Let $d \geq \delta$, $n \geq n_0$, and set $R = \delta Q(0) - dP(0)$. We define

$$U^d(t,s) := \begin{cases} U(t,s), & t \geq s \geq 0, \\ U(t,0)e^{-sR}, & t \geq 0 \geq s, \\ e^{(t-s)R}, & 0 \geq t \geq s, \end{cases} \qquad U_n^d(t,s) := \begin{cases} U_n(t,s), & t \geq s \geq 0, \\ U_n(t,0)e^{-sR}, & t \geq 0 \geq s, \\ e^{(t-s)R}, & 0 \geq t \geq s. \end{cases}$$

Clearly, U^d and U_n^d satisfy (E1). Observing that $e^{rR} = e^{-rd}P(0) + e^{r\delta}Q(0)$, it is easy to see that U^d and U_n^d are exponentially bounded independent of d and n and that U^d has an exponential dichotomy on $\mathbb{R}$ with constants N, δ and projections $P^d(t) = P(t)$ for $t \geq 0$ and $P^d(t) = P(0)$ for $t \leq 0$. Moreover, $(t,s) \mapsto U_n^d(t,s)$ is norm continuous for $t \geq s$, $s \mapsto U^d(s+t,s)$ is norm continuous from the left for each $t > 0$, and $t \mapsto U^d(t,s)$ is norm continuous from the left for $t > s$ ((E2) only holds for U^d if $D(A(0))$ is dense). As shown in [20], Proposition 2.1 remains valid under these conditions. To apply this result, we want to find $n_3' \geq n_0$ and $d_n \geq \delta$ such that

$$\|U^{d_n}(s+1,s) - U_n^{d_n}(s+1,s)\| \leq c\,n^{-\theta} \tag{5.5}$$

for $s \in \mathbb{R}$, $n \geq n_3'$, and some $\theta > 0$. Due to Proposition 3.3, we have

$$\|U^d(s+1,s) - U_n^d(s+1,s)\| \leq \begin{cases} c\,n^{-\theta}, & s \geq -\frac{1}{2}, \\ 0, & -1 \geq s, \end{cases}$$

for a fixed $\theta \in (0,1)$ and $n \geq n_2(1/2)$. If $s \in (-1,-1/2)$, then Proposition 3.3 and (2.7) yield

$$\begin{aligned} &\|U^d(s+1,s) - U_n^d(s+1,s)\| \\ &\quad\leq \|(U(s+1,0) - U_n(s+1,0))(R(\omega,A(0))(\omega - A(0))U(0,-1)U_Q(-1,0)Q(0)e^{-s\delta}\| \\ &\qquad + \|(U(s+1,0) - U_n(s+1,0))P(0)e^{sd}\| \\ &\quad\leq c\,(n^{-\theta} + e^{-d/2}) \leq 2c\,n^{-\theta} \end{aligned} \tag{5.6}$$

if we choose a sufficiently large $d =: d_n$. Thus (5.5) holds for $n \geq n_2(1/2) =: n_3'$ and these d_n. The assertions then follow from Proposition 2.1 by restricting U_n^d to $\mathbb{R}_+$. ☐

We can now proceed almost as in the previous section; we only have to take care of certain additional exponentially decaying terms.

Proposition 5.3. *Assume that (H1'), (H2'), and (H3') hold. Let $\epsilon > 0$, $h > 0$, and $|t - s| \geq h$. Then there are numbers $\eta = \eta(\epsilon,h)$ and $\tilde{s}(\epsilon) \geq s(\eta,A)$ such that*

$$\|\Gamma(t+\tau,s+\tau) - \Gamma(t,s)\| \leq \epsilon\, e^{-\frac{\delta}{2}|t-s|}$$

for $\tau \in P(\eta/\kappa, A)$ and $t, s \geq \tilde{s}(\epsilon)$, where $\kappa = 2 + 4K$ and $\eta = \eta(\epsilon, h) \to 0$ as $\epsilon \to 0$ and h is fixed. Moreover, $P(\cdot) \in AAP(\mathbb{R}_+, \mathcal{L}(X))$.

Proof. Let $\epsilon > 0$ and $h > 0$ be fixed. Let $t, s \geq 0$. Then there is a $t_\epsilon > h$ such that

$$\|\Gamma(t+\tau, s+\tau) - \Gamma(t,s)\| \leq \epsilon\, e^{-\frac{\delta}{2}|t-s|}$$

for $|t-s| \geq t_\epsilon$. For $h \leq |t-s| \leq t_\epsilon$ we deduce as in Lemma 4.2 from Proposition 3.3 and Lemma 5.2 that

$$\|\Gamma(t,s) - \Gamma_n(t,s)\| \leq c(t_\epsilon, \theta)\, n^{-\theta} e^{-\frac{\delta}{2}|t-s|} \tag{5.7}$$

if $n \geq \max\{n_3', n_2(h)\}$. Using the same function g_n, the arguments given in the proof of Lemma 4.1 lead to

$$\Gamma_n(t+\tau, s+\tau) - \Gamma_n(t,s) = \Gamma_n(t,a)\Gamma_n(a+\tau, s+\tau) + \int_a^\infty g_n(\sigma)\, d\sigma$$

for $t, s \geq a \geq 0$ and $\tau \geq 0$. Taking $\eta > 0$, $\tau \in P(\eta/\kappa, A)$, and $t, s \geq b \geq a := s(\eta, A)$, this equality yields as in Lemma 4.1

$$\begin{aligned}\|\Gamma_n(t+\tau, s+\tau) - \Gamma_n(t,s)\| &\leq c\eta n^2 \int_a^\infty e^{-\frac{3\delta}{4}|t-\sigma|}\, e^{-\frac{3\delta}{4}|\sigma-s|}\, d\sigma + ce^{-\frac{3\delta}{4}(t-a)}\, e^{-\frac{3\delta}{4}(s-a)} \\ &\leq ce^{-\frac{\delta}{2}|t-s|}(\eta n^2 + e^{-\frac{3\delta}{2}(b-a)}),\end{aligned} \tag{5.8}$$

where we use Lemma 5.2. We first choose a sufficiently large $n = n(\epsilon, h)$, then a small $\eta = \eta(\epsilon, h)$, and finally a large $b =: \tilde{s}(\epsilon) \geq s(\eta, A)$ in order to obtain the asserted estimate for $h \leq |t-s| \leq t_\epsilon$ from (5.7) and (5.8). The last claim is shown as in Theorem 4.5. □

Theorem 5.4. *Assume that (H1'), (H2'), and (H3') hold and that $x \in X$ and $f \in AAP(\mathbb{R}_+, \mathcal{L}(X))$ satisfy* (5.3). *Then the mild solution u of* (1.2) *is asymptotically almost periodic.*

Proof. Using (5.4), we write

$$\begin{aligned}u(t+\tau) - u(t) = (U(t+\tau,t) - I)U(t,0)P(0)x + U(t+\tau,\tau)P(\tau)\int_0^\tau U(\tau,s)P(s)f(s)\, ds \\ + \int_0^\infty \Gamma(t+\tau, s+\tau)(f(s+\tau) - f(s))\, ds \\ + \int_0^\infty (\Gamma(t+\tau, s+\tau) - \Gamma(t,s))f(s)\, ds =: S_1 + S_2 + S_3 + S_4\end{aligned}$$

for $t, \tau \geq 0$. Clearly, $\|S_1\| + \|S_2\| \leq ce^{-\delta t}$. Let $\epsilon > 0$ be given and set $a := s(\epsilon, f)$. For $t \geq a$ and $\tau \in P(\epsilon, f)$, the asymptotic almost periodicity of f yields

$$\begin{aligned}S_3 &= U(t+\tau, a+\tau)P(a+\tau)\int_0^a U(a+\tau, s)P(s)(f(s+\tau) - f(s))\, ds \\ &\quad + \int_a^\infty \Gamma(t+\tau, s+\tau)(f(s+\tau) - f(s))\, ds, \\ \|S_3\| &\leq \frac{2N^2}{\delta}\, \|f\|_\infty\, e^{-\delta(t-a)} + \frac{2N}{\delta}\, \epsilon.\end{aligned}$$

For $\bar\epsilon > 0$ and $h > 0$, let $\eta = \eta(\bar\epsilon, h)$ and $b := \max\{\tilde s(\bar\epsilon), s(\epsilon, f)\}$ be given by Proposition 5.3. Choosing $t \ge b$ and $\tau \in P(\eta/\kappa, A)$, we deduce from Proposition 5.3 that

$$\begin{aligned} S_4 &= U(t+\tau, b+\tau)P(b+\tau)\int_0^b U(b+\tau, s+\tau)P(s)f(s)\,ds \\ &\quad - U(t,b)P(b)\int_0^b U(b,s)P(s)f(s)\,ds \\ &\quad + \int_b^{t-h} (\Gamma(t+\tau, s+\tau) - \Gamma(t,s))f(s)\,ds + \int_{t-h}^{t+h} \cdots\,ds + \int_{t+h}^{\infty} \cdots\,ds, \\ \|S_4\| &\le \Big(\frac{2N^2}{\delta}\,e^{-\delta(t-b)} + \frac{4\bar\epsilon}{\delta} + 4Nh\Big)\|f\|_\infty\,. \end{aligned}$$

We now take first a small $h > 0$ and $\bar\epsilon > 0$ and then a large $\hat s(\epsilon) \ge b$ such that $\|u(t+\tau) - u(t)\| \le c\epsilon$ for $t \ge \hat s(\epsilon)$ and $\tau \in P(\eta/\kappa, A, f)$, the joint set of pseudo periods of f and $R(\omega, A(\cdot))$. ☐

We conclude with some remarks concerning solutions of (1.2) in $AP(\mathbb{R}_+, X)$. Assume that (H1), (H2), and (H3) hold and $f \in AP(\mathbb{R}_+, X)$. Let $\tilde f \in AP(\mathbb{R}, X)$ be the extension of f. By Theorem 4.5 the function

$$\tilde u(t) = \int_{\mathbb{R}} \Gamma(t,s)\tilde f(s)\,ds, \qquad t \in \mathbb{R}, \tag{5.9}$$

belongs to $AP(\mathbb{R}, X)$. Its restriction $u \in AP(\mathbb{R}_+, X)$ satisfies

$$u(t) = U(t,0)\int_{-\infty}^0 U(0,s)P(s)\tilde f(s)\,ds + \int_0^\infty \Gamma(t,s)f(s)\,ds, \quad t \ge 0.$$

In view of (5.2) this function is a mild solution of (1.2) with the initial value

$$x = \tilde u(0) = \int_{\mathbb{R}} \Gamma(0,s)\tilde f(s)\,ds. \tag{5.10}$$

(Note that (5.10) implies (5.3).)

Conversely, if $u \in AP(\mathbb{R}_+, X)$ is a mild solution of (1.2), then it is given by (5.4). The formula (5.4) gives

$$u(t) = U(t,0)\Big(P(0)x - \int_{-\infty}^0 U(0,s)P(s)\tilde f(s)\,ds\Big) + \int_{\mathbb{R}} \Gamma(t,s)\tilde f(s)\,ds, \quad t \ge 0.$$

Since the second summand is almost periodic, the decomposition (5.1) shows that

$$P(0)x = \int_{-\infty}^0 U(0,s)P(s)\tilde f(s)\,ds.$$

On the other hand, (5.3) must hold so that x has to satisfy (5.10).

Theorem 5.5. *Assume that (H1), (H2), and (H3) hold and that $f \in AP(\mathbb{R}_+, X)$. Then* (1.2) *has a mild solution $u \in AP(\mathbb{R}_+, X)$ if and only if the initial value x is given by* (5.10). *In this case u is the restriction of $\tilde u$ defined in* (5.9).

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