ASYMPTOTICALLY AUTONOMOUS PARABOLIC EVOLUTION EQUATIONS

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Abstract. We study the long–term behaviour of the parabolic evolution equation

$$u'(t) = A(t)u(t) + f(t), \quad t > s, \qquad u(s) = x.$$

If $A(t)$ converges to a sectorial operator A with $\sigma(A) \cap i\mathbb{R} = \emptyset$ as $t \to \infty$, then the evolution family solving the homogeneous problem has exponential dichotomy. If also $f(t) \to f_\infty$, then the solution u converges to the 'stationary solution at infinity', i.e.

$$\lim_{t\to\infty} u(t) = -A^{-1}f_\infty =: u_\infty, \qquad \lim_{t\to\infty} u'(t) = 0, \qquad \lim_{t\to\infty} A(t)u(t) = Au_\infty.$$

1. Introduction

In the present work we investigate the asymptotic behaviour of the solution to the parabolic evolution equation

$$u'(t) = A(t)u(t) + f(t), \quad t > s \geq 0, \qquad u(s) = x, \tag{1.1}$$

on a Banach space X. It is assumed that the operators $A(t)$ satisfy the 'Acquistapace–Terreni' conditions and that $A(t)$ converges (in a suitable sense) to a sectorial operator A and $f(t) \to f_\infty$ as $t \to \infty$, cf. hypothesis (P) in Section 3. In Theorem 4.1 we show that the mild solution u of (1.1) tends to the stationary solution of (1.1) at infinity, i.e.,

$$\lim_{t\to\infty} u(t) = -A^{-1}f_\infty =: u_\infty,$$

if the spectrum of A does not intersect $i\mathbb{R}$. Moreover, if f is e.g. Hölder continuous, then u is a classical solution of (1.1),

$$\lim_{t\to\infty} u'(t) = 0, \quad \text{and} \quad \lim_{t\to\infty} A(t)u(t) = Au_\infty.$$

As an essential step we establish in Theorem 3.3 that the spectral condition $\sigma(A) \cap i\mathbb{R} = \emptyset$ implies that the evolution family $U(t, s)$ solving the homogeneous problem

$$u'(t) = A(t)u(t), \quad t > s, \qquad u(s) = x, \tag{1.2}$$

has exponential dichotomy on an interval $[a, \infty)$ for some $a \geq 0$. This is a quite remarkable result in view of the known fact that the location of the spectra $\sigma(A(t))$ does not influence the asymptotic behaviour of $U(\cdot, \cdot)$, in general, as can be seen by the counterexamples in [17]. In Example 4.2 we study a second order parabolic partial differential equation with Neumann boundary conditions. Observe that in this situation the domains $D(A(t))$ depend on the time parameter t.

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H. Tanabe established the assertions sketched above in 1961 in the special case that the operators $A(t)$ and A have a common domain and are of negative type, see [13, §5.8] or [18, §5.6]. We also note that in [5, Prop.5.3], [9, §7.6], [10], [11, Chap.11] the exponential dichotomy of (1.2) for the operators $A(wt)$ and large $w \geq 0$ was shown for certain classes of $A(t)$ asssuming that $A(t) \to A$ in Césaro sense. The generalization of Tanabe's theorem to the present setting first requires the parabolic regularity theory developed by P. Acquistapace and B. Terreni, [1], [2], H. Amann, [3], or A. Yagi, [19], [20]. Second we need a characterization of exponential dichotomy in terms of the spectrum of the evolution semigroup associated with $U(t,s)$, see e.g. [4], [14], [15]. In the next section we present these prerequisites along with some extensions of known results.

2. Preliminaries

Let X be a Banach space and $J \in \{\mathbb{R}, [a,\infty)\}$. A collection $U(t,s)$, $t \geq s$, $t,s \in J$, of bounded linear operators on X is called an *evolution family* if

(a) $U(t,s) = U(t,r)U(r,s), \quad U(s,s) = I,$ and
(b) $(t,s) \mapsto U(t,s)$ is strongly continuous

for $t \geq r \geq s$ and $t,r,s \in J$. It is not difficult to see that the solution u of a 'well–posed' Cauchy problem (1.2) is given by $u(t) = U(t,s)x$, $t \geq s$, for an evolution family $U(\cdot,\cdot)$, cf. [7, §VI.9.a]. Notice that a C_0–semigroup $S(\cdot)$ yields the evolution family $U(t,s) := S(t-s)$ for $t \geq s$ and $J = \mathbb{R}$.

We say that an evolution family $U(\cdot,\cdot)$ has *exponential dichotomy* (or is *hyperbolic*) if there are projections $P(\cdot) \in C_b(J, \mathcal{L}_s(X))$ and constants $N, \delta > 0$ such that

(a) $U(t,s)P(s) = P(t)U(t,s)$,
(b) the restriction $U_Q(t,s) : Q(s)X \to Q(t)X$ of $U(t,s)$ is invertible (and we set $U_Q(s,t) := U_Q(t,s)^{-1}$),
(c) $\|U(t,s)P(s)\| \leq Ne^{-\delta(t-s)}$ and $\|U_Q(s,t)Q(t)\| \leq Ne^{-\delta(t-s)}$

for $t \geq s$ and $t,s \in J$. Here and below we let $Q = I - P$ for a projection P. If $U(\cdot,\cdot)$ is hyperbolic, then the operator family

$$\Gamma(t,s) := \begin{cases} U(t,s)P(s), & t \geq s,\ t,s \in J, \\ -U_Q(t,s)Q(s), & t < s,\ t,s \in J, \end{cases}$$

is called *Green's function* corresponding to $U(\cdot,\cdot)$ and $P(\cdot)$. If $P(t) = I$ for $t \in J$, then $U(\cdot,\cdot)$ is *exponentially stable*. Exponential dichotomy is a classical concept in the study of the long–term behaviour of evolution equations, see [4], [5], [6], [9], [11], [12], [16], where also applications to non–linear problems are treated.

For computations involving Green's function it is useful to observe that

$$J^2 \ni (t,s) \mapsto U_Q(t,s)Q(s) \qquad \text{is strongly continuous,} \tag{2.1}$$

$$U_Q(t,s)Q(s) = U_Q(t,r)U_Q(r,s)Q(s) \quad \text{for } t,r,s \in J, \tag{2.2}$$

see [7, Lemma VI.9.17]. We remark that the projections $P(t)$ are uniquely determined by (a)–(c) if $J = \mathbb{R}$ by [17, Cor.3.3]. The projections are not unique in the case $J = [a,\infty)$, cf. [5, p.16]. If $J = \mathbb{R}$ and $U(t,s) = S(t-s)$, $t \geq s$, for a C_0–semigroup $S(\cdot)$, the above concept of hyperbolicity coincides with the usual one for semigroups (see e.g. [7, §V.1.c]) due to [17, Cor.3.3]. Recall that a semigroup is hyperbolic if and only if the unit circle belongs to resolvent set of $\rho(S(t_0))$ for some/all $t_0 > 0$, and then the (time–invariant) dichotomy projection is given by

$$P = \frac{1}{2\pi i} \int_{\mathbb{T}} R(\lambda, S(t_0))\, d\lambda,$$

where $\mathbb{T} := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. In this situation, we occasionally use the notation $S_Q(t) = e^{tA_Q}$, $t \in \mathbb{R}$, where A generates $S(\cdot)$.

Assume that an evolution family $U(\cdot,\cdot)$ with $J = \mathbb{R}_+$ has exponential dichotomy on $[a,\infty)$ for some $a \geq 0$. Then the question arises whether $U(\cdot,\cdot)$ has exponential dichotomy on $\mathbb{R}_+$. This is of course true if $U(\cdot,\cdot)$ is exponentially stable. Further, if $U(t,s)$ has a locally uniformly bounded inverse $U(s,t)$ for $t \geq s \geq 0$, then it can easily be shown that $U(\cdot,\cdot)$ is hyperbolic on $\mathbb{R}_+$ with $P(t) := U(t,a)P(a)U(a,t)$ for $t \in [0,a]$. In general such an extension is not possible for non–invertible evolution families as the next simple example shows.

Example 2.1. On $X = L^1[0,\infty)$ we consider the semigroups

$$(T_1(t)f)(\xi) = \begin{cases} 0, & 0 \leq \xi \leq t, \\ f(\xi - t), & \xi > t, \end{cases} \quad \text{and} \quad (T_2(t)f)(\xi) = \begin{cases} e^t f(\xi), & 0 \leq \xi \leq 1, \\ e^{-t} f(\xi), & \xi > 1, \end{cases}$$

and define the evolution family

$$U(t,s) := \begin{cases} T_1(t-s), & s \leq t \leq 1, \\ T_2(t-1)T_1(1-s), & s \leq 1 \leq t, \\ T_2(t-s), & 1 \leq s \leq t. \end{cases}$$

Clearly, $U(\cdot,\cdot)$ has exponential dichotomy on $[1,\infty)$ with $P(t)f = f|_{[1,\infty)}$ for $t \geq 1$. However, $U(1,0)f(\xi) = 0$ for $0 \leq \xi \leq 1$ so that condition (b) of the definition of exponential dichotomy cannot be satisfied on $J = [0,\infty)$.

Our approach is based on the following characterization of exponential dichotomy essentially due to R. Rau, [15, Thm.6], see also [4, Thm.6.41] and [14, Thm.1.5]. Let $J = \mathbb{R}$ and $U(\cdot,\cdot)$ be an *exponentially bounded* evolution family on X, i.e., there are constants $M \geq 1$ and $w \in \mathbb{R}$ such that $\|U(t,s)\| \leq Me^{w(t-s)}$ for $t \geq s$. On $E := C_0(\mathbb{R}, X)$ we then define

$$(T(t)f)(s) := U(s, s-t)f(s-t), \quad s \in \mathbb{R},\ t \geq 0,\ f \in E.$$

It is straightforward to verify that $T(\cdot)$ is a C_0–semigroup on E which we call the *evolution semigroup* associated with $U(\cdot,\cdot)$.

Theorem 2.2. *Let $U(\cdot,\cdot)$ be an exponentially bounded evolution family on X with $J = \mathbb{R}$ and let $T(\cdot)$ be the associated evolution semigroup on $E = C_0(\mathbb{R}, X)$. Then $U(\cdot,\cdot)$ has exponential dichotomy if and only if $I - T(t)$ is invertible for some/all $t > 0$. In this case the unit circle $\mathbb{T}$ belongs to the resolvent set of $T(t)$ and the dichotomy projections of $U(\cdot,\cdot)$ are given by*

$$P(\cdot) = \frac{1}{2\pi i} \int_{\mathbb{T}} R(\lambda, T(t))\, d\lambda. \tag{2.3}$$

We refer to the monograph [4] for a detailed account of the theory of evolution semigroups and the relationship between their spectra and the exponential dichotomy of the underlying evolution family. In this work we only use the following straightforward consequence of Theorem 2.2. The first part of Proposition 2.3 is known, see [4, Thm.5.23] or [9, Thm.7.6.10], but assertion (2.5) seems to be new.

Proposition 2.3. *Let $U(\cdot,\cdot)$ and $V(\cdot,\cdot)$ be exponentially bounded evolution families with $J=\mathbb{R}$. Asume that $U(\cdot,\cdot)$ has exponential dichotomy with projections $P(s)$ and constants $N,\delta>0$ and that*

$$q := \sup_{s\in\mathbb{R}} \|U(s+t,s)-V(s+t,s)\| \le \frac{(1-e^{-\delta t})^2}{8N^2} \tag{2.4}$$

for some $t>0$. Then $V(\cdot,\cdot)$ has exponential dichotomy with exponent $0<\delta_V< -\frac{1}{t}\log(2qN+e^{-\delta t})$ and projections $P_V(s)$ satisfying

$$\dim P_V(s)X = \dim P(s)X \quad \text{and} \quad \dim\ker P_V(s) = \dim\ker P(s), \quad s\in\mathbb{R}. \tag{2.5}$$

Proof. For simplicity we consider $t=1$, the proof for arbitrary $t>0$ is the same. Let $T(\cdot)$ and $S(\cdot)$ be the evolution semigroups on $E=C_0(\mathbb{R},X)$ induced by $U(\cdot,\cdot)$ and $V(\cdot,\cdot)$, respectively. Observe that

$$\begin{aligned} R(\lambda,T(1))f &= \sum_{n=0}^{\infty}\lambda^{-(n+1)}T(n)P(\cdot)f - \sum_{n=1}^{\infty}\lambda^{n-1}(T_Q(n)Q(\cdot))^{-1}f \\ &= \sum_{n=0}^{\infty}\lambda^{-(n+1)}U(\cdot,\cdot-n)f(\cdot-n) - \sum_{n=1}^{\infty}\lambda^{n-1}U_Q(\cdot,\cdot+n)f(\cdot+n) \end{aligned}$$

(cf. [7, (VI.9.4)]) which yields

$$\|R(\lambda,T(1))\| \le \frac{2N}{1-e^{-\delta}} \tag{2.6}$$

for $|\lambda|=1$. Therefore (2.4) implies that $\lambda - S(1)$, $|\lambda|=1$, has the inverse

$$R(\lambda,S(1)) = R(\lambda,T(1))\sum_{n=0}^{\infty}[(S(1)-T(1))R(\lambda,T(1))]^n \tag{2.7}$$

so that $V(\cdot,\cdot)$ is hyperbolic due to Theorem 2.2. Considering the evolution families $e^{\pm\mu(t-s)}U(t,s)$ and $e^{\pm\mu(t-s)}V(t,s)$ for $\mu\in(0,\delta)$, we see that $I-e^{\pm\mu}S(1)$ is invertible if

$$e^{\mu}\frac{2qN}{1-e^{(\mu-\delta)}} < 1.$$

This implies the asserted estimate for the exponent δ_V because of Theorem 2.2. Finally, we deduce from (2.3) that

$$\begin{aligned} P_V(\cdot)-P(\cdot) &= \frac{1}{2\pi i}\int_{\mathbb{T}}[R(\lambda,S(1))-R(\lambda,T(1))]\,d\lambda \\ &= \frac{1}{2\pi i}\int_{\mathbb{T}} R(\lambda,S(1))\,[S(1)-T(1)]\,R(\lambda,T(1))\,d\lambda. \end{aligned}$$

Together with (2.4), (2.6), and (2.7), this identity yields $\|P_V(s)-P(s)\|<1$ for $s\in\mathbb{R}$. Assertion (2.5) now follows from [8, Lemma II.4.3]. □

In the present work we study evolution families which solve the Cauchy problem (1.2) for operators $A(t)$, $t \geq 0$, on X subject to

$$\overline{D(A(t))} = X, \quad \Sigma_\phi \cup \{0\} \subset \rho(A(t)), \quad \|R(\lambda, A(t))\| \leq \frac{K}{1+|\lambda|}, \tag{2.8}$$

$$\|A(t)R(\lambda, A(t))\,(A(t)^{-1} - A(s)^{-1})\| \leq L\,|t-s|^\mu\,|\lambda|^{-\nu} \tag{2.9}$$

for $t, s \geq 0$, $\lambda \in \Sigma = \Sigma_\phi := \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg\lambda| \leq \phi\}$, and constants $\phi \in (\frac{\pi}{2}, \pi)$, $L, K \geq 0$, and $\mu, \nu \in (0,1]$ with $\mu + \nu > 1$. We set $A(t) := A(0)$ for $t \leq 0$ and note that the extended operator family $A(t)$, $t \in \mathbb{R}$, also satisfies (2.8) and (2.9) with the same constants. Recall that $A(t)$ generates an analytic semigroup $(e^{\tau A(t)})_{\tau \geq 0}$ on X due to (2.8).

Condition (2.9) was introduced by P. Acquistapace and B. Terreni in [2]. Assumption (2.8) and (2.9) imply that there exists a unique evolution family $U(\cdot,\cdot)$ on X such that $U(\cdot, s) \in C^1((s,\infty), \mathcal{L}(X))$, $\partial_t U(t,s) = A(t)U(t,s)$,

$$\|A(t)^k U(t,s)\| \leq C_1\,(t-s)^{-k}, \quad 0 < t-s \leq 1, \quad k = 0, 1, \tag{2.10}$$

$$\|A(t)U(t,s)A(s)^{-1}\| \leq C_1, \quad 0 \leq t-s \leq 1, \tag{2.11}$$

for a constant C_1 depending only on the constants in (2.8) and (2.9), see [1, Thm.2.3] and also [2], [3], [19], [20]. Note that $U(\cdot,\cdot)$ is exponentially bounded by (2.10). We say that $A(\cdot)$ *generates* $U(\cdot,\cdot)$. We also need the representation formula

$$U(t,s) = e^{(t-s)A(s)} + \int_s^t Z(r,s)\,dr \tag{2.12}$$

given in [1, (2.6)], where the operators $Z(t,s)$ satisfy

$$\|Z(t,s)\| \leq C_2\,(t-s)^{\mu+\nu-2} \tag{2.13}$$

$$\|Z(t,s) - Z(r,s)\| \leq C_2\,(t-r)^\eta\,(r-s)^{\mu+\nu-2-\eta} \tag{2.14}$$

$$\|Z(t,r) - Z(t,s)\| \leq C_2\,(r-s)^\eta\,(t-r)^{\mu+\nu-2-\eta} \tag{2.15}$$

for $\eta \in (0, \mu+\nu-1)$ and $s < r < t \leq s+1$, see [1, Lemma 2.2]. A careful inspection of the proof given in [1] shows that the constant C_2 depends only on η and the constants in (2.8) and (2.9).

An essential tool in our approach are the real interpolation spaces of exponent ∞: For $\alpha \in (0,1)$ and an operator A satisfying (2.8) with $A(t)$ replaced by A, we define

$$\|x\|_\alpha^A := \sup_{\lambda \in \Sigma} \|\lambda^\alpha AR(\lambda, A)x\| \quad \text{and} \quad X_\alpha^A := \{x \in X : \|x\|_\alpha^A < \infty\}$$

and endow X_α^A with the norm $\|\cdot\|_\alpha^A$. Further, $X_1^A := D(A)$ is equipped with the graph norm of A. Recall that

$$X_1^A \hookrightarrow X_\beta^A \hookrightarrow D((-A)^\alpha) \hookrightarrow X_\alpha^A \hookrightarrow X \tag{2.16}$$

for $0 < \alpha < \beta < 1$, where $D((-A)^\alpha)$ is endowed with the graph norm and the norms of the above embeddings only depend on α, β, and the constants in (2.8). We refer to [3, Chap.V], [7, §II.5], [12, Chap.1, 2] for a thorough treatment of interpolation theory in the context of (analytic) semigroups. We further set $X_\alpha^t := X_\alpha^{A(t)}$ and $\|\cdot\|_\alpha^t := \|\cdot\|_\alpha^{A(t)}$ for operators $A(t)$, $t \geq 0$, satisfying (2.8) and (2.9).

The following refinement of (2.10) is needed for the proof of Theorem 4.1, see also [12, Cor.6.1.8] for the case of constant domains.

Proposition 2.4. *Let* (2.8) *and* (2.9) *hold and let* $0 < \alpha < \mu + \nu - 1$*. Then*

$$\|A(t)U(t,s)x\|_\alpha^t \le C_3\,(t-s)^{-1-\alpha}\,\|x\|$$

for $0 < t-s \le 1$*,* $x \in X$*, and a constant* C_3 *depending only on* α *and the constants in* (2.8) *and* (2.9)*.*

Proof. We proceed similarly as in [19, Prop.3.1]. Let $U_n(t,s)$, $t \ge s \ge 0$, be generated by the Yosida approximations $A_n(t) := nA(t)R(n, A(t))$, $n \in \mathbb{N}$. It is straightforward to verify that the operators $A_n(t)$, $t \ge 0$, satisfy (2.8) and (2.9) with constants only depending on the constants in (2.8) and (2.9) for $A(t)$, see [2, Lemma 4.2] or [19, Prop.2.1]. Note that the fractional powers $(-A_n(t))^{-\beta}$, $\beta > 0$, are uniformly bounded in $n \in \mathbb{N}$ and $t \ge 0$ and converge strongly to $(-A(t))^{-\beta}$ as $n \to \infty$. Fix $\alpha \in (0, \mu+\nu-1)$ and set $\theta := \frac{1}{2}(\alpha+\nu+1-\mu) \in (1-\mu, \nu)$. Then we have

$$U_n(t,s) = e^{(t-s)A_n(t)} + \int_s^t A_n(t)e^{(t-\tau)A_n(t)}\,(A_n(t)^{-1} - A_n(\tau)^{-1})A_n(\tau)U_n(\tau,s)\,d\tau,$$

$$(-A_n(t))^{\alpha+1}U_n(t,s) = (-A_n(t))^{\alpha+1}e^{(t-s)A_n(t)} - \int_s^t (-A_n(t))^{2+\alpha-\theta}e^{(t-\tau)A_n(t)} \cdot (-A_n(t))^{\theta}(A_n(t)^{-1} - A_n(\tau)^{-1})\,A_n(\tau)U_n(\tau,s)\,d\tau$$

for $t \ge s \ge 0$ and $n \in \mathbb{N}$. We further define

$$W_n(t,s) := (-A_n(t))^{\alpha+1}U_n(t,s) - (-A_n(t))^{\alpha+1}e^{(t-s)A_n(t)} \quad \text{and} \tag{2.17}$$

$$R_n(t,s) := \int_s^t (-A_n(t))^{2+\alpha}e^{(t-\tau)A_n(t)}\,(A_n(t)^{-1} - A_n(\tau)^{-1})\,A_n(\tau)e^{(\tau-s)A_n(\tau)}\,d\tau.$$

This yields

$$W_n(t,s) = -R_n(t,s) + \int_s^t \big[(-A_n(t))^{2+\alpha-\theta}e^{(t-\tau)A_n(t)} \cdot (-A_n(t))^{\theta}(A_n(t)^{-1} - A_n(\tau)^{-1})\,(-A_n(\tau))^{-\alpha}\big]\,W_n(\tau,s)\,d\tau \tag{2.18}$$

By [13, Thm.2.6.13], (2.9), and (2.16), the kernel $[\cdots]$ of this integral equation can be estimated by $c_1\,(t-\tau)^{\mu+\theta-\alpha-2}$ for $0 < t-\tau \le 1$. As in [19, p.144], one sees that $\|R_n(t,s)\| \le c_2\,(t-s)^{\mu+\theta-\alpha-2}$ for $0 < t-s \le 1$. Here c_1 and c_2 only depend on α and the constants in (2.8) and (2.9). Note that $\mu+\theta-\alpha-2 > -1$. Thus the solution $W_n(t,s)$ can be estimated in the same way due to [3, Thm.II.3.2.2]. The kernel of (2.18) and the operators $R_n(t,s)$ converge strongly to the same expressions without the index n by [19, Prop.2.1]. Since (2.18) is solved by a Neumann series, see [3, §II.3.2], $W_n(t,s)$ also converges strongly as $n \to \infty$. Hence, $(-A_n(t))^{\alpha+1}U_n(t,s)$ has a strong limit $V(t,s)$ and

$$\|V(t,s)\| \le c_3\,(t-s)^{-\alpha-1} \quad \text{for} \;\; 0 < t-s \le 1 \tag{2.19}$$

by (2.17) and $-\alpha-1 < \mu+\theta-\alpha-2$, where $c_3 = c_3(\alpha,\mu,\nu,\phi,K,L)$. Observe that

$$(-A_n(t))^{\alpha+1}U_n(t,s)x = A(t)(-A_n(t))^{\alpha-1}nR(n,A(t))A_n(t)U_n(t,s)x =: A(t)y_n$$

for $x \in X$, $t > s$, and $n \in \mathbb{N}$, and that y_n tends to $-(-A(t))^{\alpha}U(t,s)x$ as $n \to \infty$ due to [19, p.144]. As a consequence,

$$U(t,s)X \subseteq D((-A(t))^{\alpha+1}) \quad \text{and} \quad V(t,s) = (-A(t))^{\alpha+1}U(t,s)$$

for $t > s$. Therefore (2.19) and (2.16) establish the proposition. □

Remark 2.5. Consider operators $A(t)$ such that $A_w(t) := A(t) - w$ satisfy (2.8) and (2.9) for some $w \geq 0$. Then, due to a simple rescaling argument, the above mentioned results also hold for $A(t)$ (where the constants C_k depend additionally on w). Observe that (2.8) and (2.9) remain valid for $A(t) - w'$ and $w' \geq w$.

We finally note some properties of an hyperbolic evolution family which is generated by operators $A(t)$ satisfying the conditions of the above remark, cf. [9, Lemma 7.6.2], [12, §6.3.2]. Let $t > 0$. Then $Q(t)X \subseteq D(A(t))$ because of $Q(t) = U(t,0)U_Q(0,t)Q(t)$ and (2.10). Similarly one sees that

$$\|Q(t)\|_{\mathcal{L}(X,X_1^t)} \leq C_4 \quad \text{and} \quad \|P(t)\|_{\mathcal{L}(X_1^t)} \leq C_4 \tag{2.20}$$

for $t \geq 1$ and a constant C_4 depending only on w, the constants in (2.8) and (2.9), and the dichotomy constants. For $t, s > 0$ we further obtain

$$\begin{aligned} \tfrac{1}{h}\left(U_Q(t+h,s)Q(s) - U_Q(t,s)Q(s)\right) &= \tfrac{1}{h}\left(U(t+h,t) - I\right)Q(t)U_Q(t,s)Q(s) \\ &\longrightarrow A(t)U_Q(t,s)Q(s) \end{aligned}$$

strongly as $h \searrow 0$. Moreover, $A(t)U_Q(t,s) = A(t)U(t,0)U_Q(0,s)Q(s)$ so that $A(\cdot)U_Q(\cdot,s)Q(s)$ is continuous in $\mathcal{L}(X)$. Thus,

$$U_Q(\cdot,s)Q(s) \in C^1((0,\infty),\mathcal{L}(X)) \quad \text{and} \quad \partial_t U_Q(t,s)Q(s) = A(t)U_Q(t,s)Q(s). \tag{2.21}$$

3. Exponential dichotomy

We now formulate our basic hypothesis:

(P) There are operators A and $A(t)$, $t \geq 0$, and a number $w \geq 0$ such that A_w and $A_w(t)$ satisfy (2.8), $A_w(t)$ satisfy (2.9), and $R(w, A(t)) \to R(w, A)$ in $\mathcal{L}(X, X_\alpha^A)$ for some $0 < \alpha \leq 1$.

In view of Remark 2.5, the operators $A(t)$ generate an evolution family $U(\cdot,\cdot)$. Observe that due to the Trotter–Kato approximation theorem, see e.g. [7, Thm.III.4.8], (P) implies

$$R(\mu, A(t)) \to R(\mu, A) \quad \text{strongly as} \quad t \to \infty \quad \text{for all} \quad \operatorname{Re}\mu \geq w. \tag{3.1}$$

We note the following special case of (P).

Remark 3.1. Assume that the operators $A_w(t)$ satisfy (2.8), that $D(A(t)) = D(A(0)) =: Y$, $t \geq 0$, with uniformly equivalent graph norms, and that $A(\cdot) : \mathbb{R}_+ \to \mathcal{L}(Y, X)$ is globally Hölder continuous of exponent $\mu \in (0,1]$. Let $A(t)$ converge in $\mathcal{L}(Y, X)$ to a closed operator A with domain Y as $t \to \infty$. Then (P) holds with $\alpha = \nu = 1$. Moreover, if $w = 0$ then $s(A) < 0$. We omit the straightforward proof.

The next lemma provides us with the essential step for the proof of our main results.

Lemma 3.2. *Assume that (P) holds. Then $U(s+t, s) \to e^{tA}$ in $\mathcal{L}(X)$ as $s \to \infty$ uniformly for $t \in [t_0, t_1] \subset (0,\infty)$.*

Proof. By rescaling we may assume that $w = 0$ in (P). Fix $t_1 > t_0 > 0$. Set

$$o(s) := \sup_{\tau \geq s} \|A(\tau)^{-1} - A^{-1}\|_{\mathcal{L}(X, X_\alpha^A)}\,.$$

For $0 < h < t_0 \leq t \leq t_1$ and $s \geq 0$, we write

$$U(s+t,s) - e^{tA} = [U(s+t,s+h) - e^{(t-h)A}]\, U(s+h,s) \\ + e^{(t-h)A}\, [U(s+h,s) - e^{hA(s)} + e^{hA(s)} - e^{hA}].$$

The first summand yields

$$\begin{aligned}
&(U(s+t,s+h) - e^{(t-h)A})\, U(s+h,s)x \\
&= \lim_{\delta\to 0} (e^{\delta A} U(s+t-\delta,s+h) - e^{(t-h)A})\, U(s+h,s)x \\
&= \lim_{\delta\to 0} \int_{s+h}^{s+t-\delta} [e^{(s+t-\tau)A} A(\tau) U(\tau,s+h) - A e^{(s+t-\tau)A} U(\tau,s+h)]\, U(s+h,s)x\, ds \\
&= \int_{s+h}^{s+t} A e^{(s+t-\tau)A}\, (A^{-1} - A(\tau)^{-1})\, A(\tau) U(\tau,s+h) A(s+h)^{-1}\, A(s+h) U(s+h,s)x\, ds
\end{aligned}$$

for $x \in X$. From [12, Prop.2.2.2], (P), (2.11), and (2.10) now follows

$$\|(U(s+t,s+h) - e^{(t-h)A})\, U(s+h,s)\| \leq c_1\, o(s)\, h^{-1} \int_h^t (t-\tau)^{\alpha-1}\, d\tau \leq c_2\, o(s)\, h^{-1}$$

for constants c_k. Further,

$$\|U(s+h,s) - e^{hA(s)}\| \leq c_3\, h^{\mu+\nu-1}$$

by (2.12) and (2.13). Finally,

$$e^{hA(s)} - e^{hA} = \frac{1}{2\pi i} \int_\Gamma e^{\lambda h} \lambda^{-\alpha}\, \lambda^\alpha A R(\lambda,A)(A^{-1} - A(s)^{-1})\, A(s) R(\lambda, A(s))\, d\lambda$$

for a suitable path Γ in Σ so that (P) implies

$$\|e^{hA(s)} - e^{hA}\| \leq c_4\, o(s) \int_\Gamma e^{h \operatorname{Re}\lambda}\, |\lambda|^{-\alpha}\, |d\lambda| \leq c_5\, h^{\alpha-1}\, o(s)$$

for constants c_k. Altogether we have shown that

$$\|U(s+t,s) - e^{tA}\| \leq c_6\, o(s)(h^{-1} + h^{\alpha-1}) + c_7\, h^{\mu+\nu-1}$$

for $0 < h \leq t_0 \leq t \leq t_1$, $s \geq 0$, and constants c_k independent of h, s, and $t \in [t_0,t_1]$. Given $\varepsilon > 0$, we first choose a small $h > 0$ and then a large s_ε to obtain $\|U(s+t,s) - e^{tA}\| \leq \varepsilon$ for $s \geq s_\varepsilon$ and $t \in [t_0,t_1]$. □

In view of Remark 3.1 the next result generalizes [13, Thm.5.8.1] where the case of exponential stability and constant domains is treated. Note that if $s(A) < 0$, then $U(\cdot,\cdot)$ is exponentially stable and we can take $a = 0$ in Theorem 3.3.

Theorem 3.3. *Let (P) hold and $\sigma(A) \cap i\mathbb{R} = \emptyset$. Then the evolution family $U(\cdot,\cdot)$ generated by $A(\cdot)$ has exponential dichotomy on an interval $[a,\infty) \subseteq \mathbb{R}_+$ with projections $P(t)$ and exponent δ'. Moreover, $P(t) \to P$ strongly as $t \to \infty$, $\dim P(t)X = \dim PX$, $\dim\ker P(t) = \dim\ker P$ for $t \geq a$, and by choosing a large a we can take δ' arbitrarily close to δ, where P and δ are the dichotomy projection and exponent of e^{tA}.*

Proof. (1) We first recall that $\sigma(A)\cap i\mathbb{R}=\emptyset$ implies the exponential dichotomy of e^{tA}, see [7, IV.3.12, V.1.17] or [12, §2.3]. For a given $a\ge 1$, we set

$$U_a(t,s):=\begin{cases} U(t,s), & t\ge s\ge a,\\ U(t,a)e^{(a-s)A}, & t>a>s,\\ e^{(t-s)A}, & a\ge t\ge s.\end{cases} \tag{3.2}$$

Fix $\varepsilon>0$. For $s\in(a-1,a-\frac{1}{2})$ we obtain as in the proof of Lemma 3.2 the identities

$$\begin{aligned} U_a(s+1,s)-e^{A} &= \left(U(s+1,a)-e^{(s+1-a)A}\right)e^{(a-s)A}\\ &= \left(U(s+1,a)-e^{(s+1-a)A}\right)\left(e^{(a-s)A}-U(a,s)\right)\\ &\quad+\int_a^{s+1} A_w e^{(s+1-\tau)A}\left(A_w^{-1}-A_w(\tau)^{-1}\right)A_w(\tau)U(\tau,s)\,d\tau. \end{aligned}$$

From [12, Prop.2.2.2], (P), and (2.10), now follows that

$$\begin{aligned} \|U_a(s+1,s)-e^{A}\| &\le c_1\,\|e^{(a-s)A}-U(a,s)\|+c_2\int_a^{s+1}(s+1-\tau)^{\alpha-1}o(\tau)(\tau-s)^{-1}\,d\tau\\ &\le c_3\left(\|e^{(a-s)A}-U(a,s)\|+o(a)\right) \end{aligned}$$

for constants c_k independent of a and s with $a-s\in(\frac{1}{2},1)$. Using Lemma 3.2 we find $a_1=a_1(\varepsilon)$ such that

$$\|U_a(s+1,s)-e^{A}\|\le\varepsilon \qquad \text{for } \ a\ge a_1 \text{ and } s\in(a-1,a-\tfrac{1}{2}). \tag{3.3}$$

Lemma 3.2 further yields $a_2=a_2(\varepsilon)$ such that

$$\begin{aligned} \|U_a(s+1,s)-e^{A}\| &= \|\left(U(s+1,a)-e^{(s+1-a)A}\right)e^{(a-s)A}\|\\ &\le c_4\,\|U(s+1,a)-e^{(s+1-a)A}\|\le\varepsilon \end{aligned} \tag{3.4}$$

for $a\ge a_2$ and $s\in[a-1/2,a)$. Finally,

$$U_a(s+1,s)-e^{A}=\begin{cases} U(s+1,s)-e^{A}, & s\ge a,\\ 0, & s\le a-1,\end{cases}$$

so that due to Lemma 3.2 there exists $a_3=a_3(\varepsilon)$ with

$$\|U_a(s+1,s)-e^{A}\|\le\varepsilon \qquad \text{for } \ a\ge a_3 \text{ and } s\notin(a-1,a). \tag{3.5}$$

Combining (3.3), (3.4), (3.5), we see that there is $a_0=a_0(\varepsilon)$ such that

$$\|U_a(s+1,s)-e^{A}\|\le\varepsilon \qquad \text{for } \ a\ge a_0 \text{ and } s\in\mathbb{R}.$$

Thus Proposition 2.3 establishes the theorem except for the convergence of $P(t)$.

(2) Given $t\ge a+1$ and $x\in D(A(t))$ we check the integrability of the function

$$u(\tau)=A_w\Gamma_A(t-\tau)\left(A_w^{-1}-A_w(\tau)^{-1}\right)A_w(\tau)\Gamma(\tau,t)x, \quad \tau\ne t,$$

on $[a,\infty)$, where $u(t):=0$ and Γ_A and Γ are Green's function for e^{tA} and $U(t,s)$, respectively. Let $\delta'':=\delta+\delta'$. For $a\le\tau\le t-1$ we have

$$\begin{aligned} \|u(\tau)\| &= \|A_w e^{A}e^{(t-\tau-1)A}P\left(A_w^{-1}-A_w(\tau)^{-1}\right)A_w(\tau)Q(\tau)U_Q(\tau,t)Q(t)x\|\\ &\le c_5\,o(\tau)\,e^{-\delta''|t-\tau|}\,\|x\|, \end{aligned}$$

where we have used the analyticity of e^{tA}, (P), (2.16), and (2.20). Further,

$$\|u(\tau)\| = \|PA_w e^{(t-\tau)A}\,(A_w^{-1} - A_w(\tau)^{-1})\,A_w(\tau)Q(\tau)U_Q(\tau,t)Q(t)x\|$$
$$\le c_6\,o(\tau)\,(t-\tau)^{\alpha-1}\,\|x\|$$

for $\tau \in (t-1,t)$ due to [12, Prop.2.2.2], (P), and (2.20). Next,

$$\|u(\tau)\| = \|A_w e^A e^{(t-\tau-1)A_Q} Q\,(A_w^{-1} - A_w(\tau)^{-1})\,A_w(\tau)U(\tau,t)P(t)x\|$$
$$\le c_7\,o(\tau)\,\|A_w(t)P(t)x\| \le c_8\,o(\tau)\,\|x\|_1^t$$

for $\tau \in (t,t+1]$ by (P), (2.16), (2.11), and (2.20). Finally, (P), (2.16), and (2.10) yield

$$\|u(\tau)\| = \|A_w e^A e^{(t-\tau-1)A_Q} Q\,(A_w^{-1} - A_w(\tau)^{-1})\,A_w(\tau)U(\tau,\tau-1)U(\tau-1,t)P(t)x\|$$
$$\le c_9\,o(\tau)\,e^{-\delta''|t-\tau|}\,\|x\|$$

for $\tau \ge t+1$. Altogether we have derived the estimate

$$\int_a^\infty \|u(\tau)\|\,d\tau \le \int_a^\infty k(t-\tau)o(\tau)\,d\tau\,\|x\|_1^t \tag{3.6}$$

for the function

$$k(t) := \begin{cases} c\,e^{-\delta''|t|}, & t \ge 1 \text{ and } t < 0, \\ c\,t^{\alpha-1}, & 0 < t < 1, \end{cases}$$

where c is a constant. On the other hand, (2.21) shows that

$$\begin{aligned}\int_a^\infty u(\tau)\,d\tau &= \lim_{\varepsilon\to 0}\int_a^{t-\varepsilon} \big[Ae^{(t-\tau)A}PU_Q(\tau,t)Q(t)x - e^{(t-\tau)A}PA(\tau)U_Q(\tau,t)Q(t)x\big]\,d\tau \\ &\quad + \int_t^\infty \big[Ae^{(t-\tau)A_Q}QU(\tau,t)P(t)x - e^{(t-\tau)A_Q}QA(\tau)U(\tau,t)P(t)x\big]\,d\tau \\ &= \lim_{\varepsilon\to 0} -e^{\varepsilon A}PU_Q(t-\varepsilon,t)Q(t)x + e^{(t-a)A}PU_Q(a,t)Q(t)x + QP(t)x \\ &= -Px + P(t)x + e^{(t-a)A}PU_Q(a,t)Q(t)x.\end{aligned}$$

Combining this formula with (3.6), we arrive at

$$\|P(t)x - Px\| \le \int_a^\infty k(t-\tau)o(\tau)\,d\tau\,\|x\|_1^t + c'\,e^{-\delta''(t-a)}\,\|x\|. \tag{3.7}$$

for $x \in D(A(t))$. Now take $x \in X$ and $\varepsilon > 0$. Fix $\mu > 0$ such that $\|x - \mu R(\mu,A)x\| \le \varepsilon$. Set $f(t) = \mu R(\mu, A(t))x$. Then,

$$\overline{\lim_{t\to\infty}}\,\|P(t)x - Px\| \le \overline{\lim_{t\to\infty}}\,\Big(\|P(t)(x-f(t))\| + \|P(t)f(t) - Pf(t)\| + \|P(f(t)-x)\|\Big)$$
$$\le (\|P\| + \|P(\cdot)\|_\infty)\,\varepsilon$$

by (3.1), (3.7), and $o(\tau) \to 0$ as $\tau \to \infty$. As a result, $P(t) \to P$ strongly as $t \to \infty$. $\square$

The following simple example shows that strong convergence of $A_w(t)^{-1}$ to A_w^{-1} does not suffice in Theorem 3.3.

Example 3.4. Let $X = \ell^2$, $A = -I$, and

$$A(t)(x_k) = -(x_1, \cdots, x_{n-1}, (t-n+1)x_n, 0, \cdots) \quad \text{for} \quad n-1 \le t \le n.$$

Then, $A(t)x \to -x$ for $x = (x_k) \in X$ as $t \to \infty$ and $s(A) = -1$. But $U(t,0)x = x$ if $0 \le t \le n$ and $x_k = 0$ for $k = 1, \cdots, n$, so that $\|U(n,0)\| = 1$.

4. The inhomogeneous problem

We now apply the results of the previous section to study the evolution equation

$$u'(t) = A(t)u(t) + f(t), \quad t > a, \tag{4.1}$$

$$u(a) = x, \tag{4.2}$$

for $f \in C_b([a,\infty), X)$ and $x \in X$ supposing the assumptions of Theorem 3.3. Since (2.8) and (2.9) hold, there exists a unique differentiable solution u of (4.1) and (4.2) provided that f is regular enough, see [2, §6]. We want to establish the convergence of $u(t)$ as $t \to \infty$ assuming that $f(t) \to f_\infty$. It is known that the solution of (4.1) can be represented as

$$\begin{aligned} u(t) &= U(t,a)y + \int_a^\infty \Gamma(t,\tau)f(\tau)\,d\tau \qquad (4.3)\\ &= U(t,a)y + \int_a^t U(t,\tau)P(\tau)f(\tau)\,d\tau - \int_t^\infty U_Q(t,\tau)Q(\tau)f(\tau)\,d\tau, \end{aligned}$$

where $y \in X$ and Γ is Green's function of $U(t,s)$, $t \geq s \geq a$, which exists by Theorem 3.3. Observe that the integral in (4.3) is a bounded function of t so that u is bounded if and only if $y \in P(a)X$. On the other hand, the initial condition (4.2) requires that

$$x = u(a) = y - \int_a^\infty U_Q(a,\tau)Q(\tau)f(\tau)\,d\tau.$$

Therefore a bounded solution u of (4.1) and (4.2) exists only if

$$Q(a)x = -\int_a^\infty U_Q(a,\tau)Q(\tau)f(\tau)\,d\tau. \tag{4.4}$$

(See [12, Prop.6.3.6] for these results in a special case.) We thus assume that (4.4) holds. The function u defined in (4.3) is then given by

$$u(t) = U(t,a)P(a)x + \int_a^\infty \Gamma(t,\tau)f(\tau)\,d\tau =: v_1(t) + v_2(t) \tag{4.5}$$

We call this function u the *mild solution* of (4.1) and (4.2). Our next result extends Theorem 5.6.1 in [18] where assumption (a) below in the situation of Remark 3.1 with $w = 0$ was studied. Recall that we can choose $a = 0$ if $s(A) < 0$.

Theorem 4.1. *Let (P) hold and $\sigma(A) \cap i\mathbb{R} = \emptyset$. Fix $a \geq 0$ as obtained in Theorem 3.3. Assume that $f \in C([a,\infty), X)$ converges to f_∞ in X as $t \to \infty$ and that $x \in X$ satisfies (4.4). Then the mild solution u of (4.1) and (4.2) converges to $u_\infty := -A^{-1}f_\infty$ as $t \to \infty$. If, in addition, either*

(a) $f \in C_b^\beta([a,\infty), X)$ *for some* $\beta \in (0,1)$ *or*
(b) $\sup_{t \geq a} \|f(t)\|_\beta^t < \infty$ *for some* $\beta \in (0,1)$,

then $u \in C^1((a,\infty), X)$, $u(t) \in D(A(t))$ for $t > a$, and (4.1) holds. Moreover, $u'(t) \to 0$ and $A(t)u(t) \to Au_\infty$ as $t \to \infty$.

Proof. (1) We use the functions v_k defined in (4.5). As noticed above,

$$u(a) = P(a)x - \int_a^\infty U_Q(a,\tau)Q(\tau)f(\tau)\,d\tau = x$$

by (4.4). Recall that $v_1 \in C^1((a,\infty),X) \cap C([a,\infty),X)$, $v_1(t) \in D(A(t))$, and $v_1'(t) = A(t)v_1(t)$ for $t > a$. Moreover, $v_1(t) \to 0$ and

$$\|v_1'(t)\| = \|A(t)v_1(t)\| \le \|A(t)U(t,t-1)\| \, \|U(t-1,a)P(a)x\| \ \longrightarrow\ 0$$

as $t \to \infty$ due to (2.10). In Lemma 3.2 and Theorem 3.3 we have seen that $U(s+t,s) \to e^{tA}$ and $P(s) \to P$ strongly as $s \to \infty$, where P is the dichotomy projection of e^{tA}. Extending f by 0 to $\mathbb{R}$, we obtain

$$v_2(t) = \int_0^\infty U(t,t-\tau)P(t-\tau)f(t-\tau)\,d\tau - \int_0^\infty U_Q(t,t+\tau)Q(t+\tau)f(t+\tau)\,d\tau$$

for $t \ge a$. The first integrand converges to $e^{\tau A}Pf_\infty$ as $t \to \infty$. Concerning the second integrand, we observe that

$$\begin{aligned} &U_Q(t,t+\tau)Q(t+\tau)f(t+\tau) - e^{-\tau A_Q}Qf_\infty \\ &\quad = U_Q(t,t+\tau)Q(t+\tau)\left[Q(t+\tau)f(t+\tau) - Qf_\infty\right] + U_Q(t,t+\tau)Q(t+\tau) \\ &\qquad \cdot \left[e^{\tau A} - U(t+\tau,t)\right] e^{-\tau A_Q}Qf_\infty + \left[Q(t+\tau) - Q\right] e^{-\tau A_Q}Qf_\infty \end{aligned}$$

tends to 0 as $t \to \infty$. Hence,

$$\begin{aligned} \lim_{t\to\infty} u(t) = \lim_{t\to\infty} v_2(t) &= \int_0^\infty e^{\tau A_P}Pf_\infty\,d\tau - \int_0^\infty e^{-\tau A_Q}Qf_\infty\,d\tau \\ &= -A_P^{-1}f_\infty + (-A_Q^{-1})f_\infty = -A^{-1}f_\infty \end{aligned} \tag{4.6}$$

by the theorem of dominated convergence and standard semigroup theory.

(2) Assume in addition that (a) holds. For $t \ge r \ge a$ we have

$$\begin{aligned} v_2(t) &= \int_r^t U(t,\tau)P(\tau)f(\tau)\,d\tau + U(t,r)\int_a^r U(r,\tau)P(\tau)f(\tau)\,d\tau \\ &\quad + \int_r^t U(t,\tau)Q(\tau)f(\tau)\,d\tau - U(t,r)\int_r^\infty U_Q(r,\tau)Q(\tau)f(\tau)\,d\tau \\ &= \int_r^t U(t,\tau)f(\tau)\,d\tau + U(t,r)v_2(r). \end{aligned} \tag{4.7}$$

Taking $r = a$ in (4.7), [2, Thm.6.1(i)] implies that $v_2 \in C^1((a,\infty),X)$, $v_2(t) \in D(A(t))$, and $v_2'(t) = A(t)v_2(t) + f(t)$ for $t > a$, so that u solves (4.1) in this case. The other assertions now easily follow from

$$\lim_{t\to\infty} v_2'(t) = 0. \tag{4.8}$$

To establish (4.8), we first want to show that v_2' is uniformly Hölder continuous on $[a+1,\infty)$. In view of (4.7), v_2' is given by

$$v_2'(t) = A(t)U(t,r)v_2(r) + A(t)\int_r^t U(t,\tau)f(\tau)\,d\tau + f(t) \tag{4.9}$$

for $t > r \ge 0$. For $t \ge s > r + \frac{1}{2}$, we infer from $A(t)U(t,s) = \partial_t U(t,s)$ and (2.12) that

$$\begin{aligned} &A(t)U(t,r)v_2(r) - A(s)U(s,r)v_2(r) \\ &\qquad = A(r)(e^{(t-r)A(r)} - e^{(s-r)A(r)})v_2(r) + (Z(t,r) - Z(s,r))v_2(r) \\ &\qquad = (e^{(t-s)A(r)} - I)\,A(r)e^{(s-r)A(r)}v_2(r) + (Z(t,r) - Z(s,r))v_2(r). \end{aligned}$$

So [12, Prop.2.2.4, 2.2.2] and (2.14) yield

$$\begin{aligned}\|A(t)U(t,r)v_2(r) - A(s)U(s,r)v_2(r)\| &\le c_1\,(t-s)^\beta\,((s-r)^{-1-\beta} + (s-r)^{\mu+\nu-2-\beta})\,\|v_2\|_\infty \\ &\le c_2\,(t-s)^\beta\,\|f\|_\infty\end{aligned}$$

for constants c_k independent of t, s, r satisfying $r+1 \ge t \ge s \ge r+\frac{1}{2}$. Further, assumption (a) and [2, Thm.6.1(ii)] show that

$$\left\| A(t)\int_r^t U(t,\tau)f(\tau)d\tau - A(s)\int_r^s U(s,\tau)f(\tau)d\tau \right\| \le c_3\,(t-s)^\beta\,\|f\|_{C^\beta}$$

if $r+1 \ge t \ge s \ge r+\frac{1}{2}$, where c_3 does not depend on t, s, r (since the constant in [2, Thm.6.1(ii)] only depends on β, w, and the constants in (2.8) and (2.9) as can be deduced from the proof given there). As a result,

$$\|v_2'(t) - v_2'(s)\| \le c_4\,(t-s)^\beta\,\|f\|_{C^\beta}$$

where c_4 is independent of $t \ge s$ provided we take $t, s \in [r+\frac{1}{2}, r+1]$. This can be achieved by choosing $r = t-1 \ge a$ and $0 \le t-s \le \frac{1}{2}$. Since $v_2'(t)$ is bounded for $t \ge a+1$ due to (4.9) (for $r = t-1$), (2.10), assumption (a) and [2, Thm.6.1(ii)], we obtain that $v_2' \in C_b^\beta([a+1,\infty), X)$.

Set $\tilde{v}_2(t) := v_2(t) - u_\infty$. The interpolation result [12, Prop.1.2.19] and [12, Prop.0.2.2] then imply

$$\begin{aligned}\sup_{n\le t\le n+1}\|v_2'(t)\| \le \|\tilde{v}_2\|_{C^\gamma([n,n+1],X)} &\le c_5\,\|\tilde{v}_2\|_{C([n,n+1],X)}^{1-\theta}\,\|\tilde{v}_2\|_{C^{1+\beta}([n,n+1],X)}^{\theta} \\ &\le c_6 \sup_{n\le t\le n+1}\|v_2(t) - u_\infty\|^{1-\theta}\end{aligned}$$

for some $\gamma \in (1, 1+\beta)$, $\theta := \frac{\gamma}{1+\beta}$, and a constant independent of n. This verifies (4.8).

(3) Now assume that (b) holds. Using [2, Thm.6.2(i)] instead of [2, Thm.6.1(i)], we obtain again from (4.7) that $v_2 \in C^1((a,\infty), X)$, $v_2(t) \in D(A(t))$, and $v_2'(t) = A(t)v_2(t) + f(t)$ for $t > a$. Thus it remains to show (4.8) also in the present situation. This time we want to employ

$$\sup_{t\ge a+1}\|v_2'(t)\|_\beta^t < \infty. \tag{4.10}$$

This fact is an immediate consequence of (4.9) (for $r = t-1$), Proposition 2.4, assumption (b), and [2, Thm.6.2(iii)] (and its proof). On the other hand, (4.1) implies

$$R(w, A(t))v_2'(t) = -v_2(t) + wR(w, A(t))v_2(t) + R(w, A(t))f(t).$$

The right hand side converges to 0 as $t \to \infty$ because of (4.6) and (P), i.e.,

$$\lim_{t\to\infty} R(w, A(t))v_2'(t) = 0. \tag{4.11}$$

Using the moment inequality (see e.g. [7, Thm.II.5.34]), (2.16), and (4.10), we derive

$$\begin{aligned}\|v_2'(t)\| &\le c_7\,\|(w-A(t))^{-1}v_2'(t)\|^{1-\theta}\,\|(w-A(t))^\gamma v_2'(t)\|^\theta \\ &\le c_8\,\|R(w, A(t))v_2'(t)\|^{1-\theta}\end{aligned}$$

for $\gamma \in (0, \beta)$, $\theta := \frac{1}{1+\gamma}$, and constants c_k independent of t. Hence, (4.8) follows from (4.11). ☐

We conclude this paper with a straightforward application to second order elliptic operators $A(t)$ in divergence form. In view of [1, §6], more general elliptic operators could be treated in a similar way.

Example 4.2. Consider the initial–boundary value problem

$$\partial_t u(t,x) = \sum_{k,l=1}^{n} \partial_k a_{kl}(t,x)\, \partial_l u(t,x) + a_0(t,x)u(t,x) + f(t,x), \quad t > s \geq 0,\ x \in \Omega,$$

$$\sum_{k,l=1}^{n} n_k(x)\, a_{kl}(t,x)\, \partial_l u(t,x) = 0, \quad t > s \geq 0,\ x \in \partial\Omega,$$

$$u(s,\cdot) = \varphi.$$

Here $\Omega \subseteq \mathbb{R}^n$ is a bounded domain with boundary $\partial\Omega$ of class C^2 being locally on one side of Ω and $n(x)$ is the outer unit normal vector. We assume that the coefficients satisfy

$$a_{kl} \in C_b^{\mu}(\mathbb{R}_+, C(\overline{\Omega})) \cap C_b(\mathbb{R}_+, C^1(\overline{\Omega})), \quad k,l = 1, \cdots, n,$$
$$a_0 \in C_b^{\mu}(\mathbb{R}_+, L^n(\Omega)) \cap C_b(\mathbb{R}_+, C(\overline{\Omega}))$$

for some $\frac{1}{2} < \mu \leq 1$ and that (a_{kl}) is symmetric, real, and uniformly elliptic, i.e.,

$$\sum_{k,l=1}^{n} a_{kl}(t,x)\, v_k\, v_l \geq \eta\, |v|^2$$

for a constant $\eta > 0$, $x \in \overline{\Omega}$, $t \geq 0$, $v \in \mathbb{R}^n$.

On $X_p := L^p(\Omega)$, $1 < p < \infty$, and $X_\infty := C(\overline{\Omega})$ we introduce the realizations $A_p(t)$, $1 < p < \infty$, and $A_\infty(t)$ of the differential operator

$$A(t,x,D) := \sum_{k,l=1}^{n} \partial_k\, a_{kl}(t,x)\, \partial_l \; + a_0(t,x)$$

with domains

$$D(A_p(t)) := \{f \in W^{2,p}(\Omega) : \sum_{k,l=1}^{n} n_k(\cdot)\, a_{kl}(t,\cdot)\, \partial_l f = 0 \text{ on } \partial\Omega\},$$

$$D(A_\infty(t)) := \{f \in \bigcap_{p>1} W^{2,p}(\Omega) : A(t,\cdot,D)f \in C(\overline{\Omega}),\ \sum_{k,l=1}^{n} n_k(\cdot)\, a_{kl}(t,\cdot)\, \partial_l f = 0 \text{ on } \partial\Omega\},$$

where the boundary condition is understood in the sense of traces if necessary. It is shown in [19, §4] that $A_p(t)$, $1 < p \leq \infty$, $t \geq 0$, fulfill (2.9) for μ and each $\nu \in (0, \frac{1}{2})$. Thus there exists an evolution family $U(\cdot,\cdot)$ on X_p solving the above partial differential equation for $f = 0$ on X_p.

We further suppose that there are functions $\tilde{a}_{kl} \in C^1(\overline{\Omega})$, $k,l = 1, \cdots, n$, and $\tilde{a}_0 \in C(\overline{\Omega})$ such that $(\tilde{a}_{kl})$ is real, symmetric, and uniformly elliptic, and

$$a_{kl}(t,x) \to \tilde{a}_{kl}(x) \quad \text{and} \quad a_0(t,x) \to \tilde{a}_0(x)$$

uniformly in $x \in \overline{\Omega}$ as $t \to \infty$. For the coefficients $\tilde{a}_{kl}$ and $\tilde{a}_0$ we define A_p in the same way as $A_p(t)$, $1 < p \leq \infty$. One can verify (P) for $\alpha \in (0, \frac{1}{2})$ by means of the arguments from [19, §4]. Thus $U(\cdot,\cdot)$ has an exponential dichotomy on X_p for a time interval $[a,\infty)$ if $\sigma(A_p) \cap i\mathbb{R} = \emptyset$, $1 < p \leq \infty$. (We note that the spectra of A_p do not depend on $p \in (1,\infty]$

due to [7, Prop.IV.2.17], standard elliptic regularity results, and the Sobolev embedding theorem.) If φ and f further satisfy the assumptions of Theorem 4.1, then the solution of (4.1) and (4.2) converges as asserted in this theorem.

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