CENTER MANIFOLDS AND DYNAMICS NEAR EQUILIBRIA OF QUASILINEAR PARABOLIC SYSTEMS WITH FULLY NONLINEAR BOUNDARY CONDITIONS

YURI LATUSHKIN, JAN PRÜSS, AND ROLAND SCHNAUBELT

Abstract. We study quasilinear systems of parabolic partial differential equations with fully nonlinear boundary conditions on bounded or exterior domains. Our main results concern the asymptotic behavior of the solutions in the vicinity of an equilibrium. The local center, center–stable, and center–unstable manifolds are constructed and their dynamical properties are established using nonautonomous cutoff functions. Under natural conditions, we show that each solution starting close to the center manifold converges to a solution on the center manifold.

1. Introduction

The investigation of the long term behavior of solutions starting near an equilibrium is an essential step in the study of the qualitative properties of a nonlinear evolution equation. In many cases, the structure of the flow in a neighborhood of a steady state u_* is largely determined by the spectrum of the linearization at u_*, see e.g. [4], [6], [13], [14], [16], [18], [19], [22]. In the current work we construct local invariant manifolds for a class of nonlinear equations utilizing *nonautonomous* cutoff functions.

In this paper we treat parabolic systems with nonlinear boundary conditions and we construct local invariant C^1–manifolds consisting of solutions to the nonlinear problem. These local center, center–stable, and center–unstable manifolds are tangent at u_* to the corresponding spectral subspaces of the linearization. We also show that, under natural conditions, each solution starting close to the center manifold converges exponentially to a solution living on the center manifold. In this sense, in a vicinity of u_* the dynamics of the system is reduced to the dynamics on the center manifold which is governed by an ordinary differential equation. To be more precise, we consider the equations

$$\begin{aligned} \partial_t u(t) + A(u(t))u(t) &= F(u(t)), \quad \text{on } \Omega,\ \ t > 0,\\ B_j(u(t)) &= 0, \quad \text{on } \partial\Omega,\ \ t \ge 0,\ \ j = 1, \cdots, m,\\ u(0) &= u_0, \quad \text{on } \Omega, \end{aligned} \tag{1.1}$$

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on a (possibly unbounded) domain Ω in $\mathbb{R}^n$ with compact boundary $\partial\Omega$, where the solution $u(t,x)$ takes values in $\mathbb{C}^N$. The main part of the differential equation is given by a linear differential operator $A(u)$ of order $2m$ (with $m \in \mathbb{N}$) whose matrix–valued coefficients depend on the derivatives of u up to order $2m-1$, and F is a general nonlinear reaction term acting on the derivatives of u up to order $2m-1$. Therefore the differential equation is quasilinear. Our analysis focusses on the fully nonlinear boundary conditions

$$[B_j(u)](x) := b(x, u(x), \nabla u(x), \cdots, \nabla^{m_j} u(x)) = 0, \quad x \in \partial\Omega, \;\; j = 1, \cdots, m,$$

for the partial derivatives of u up to order $m_j \leq 2m-1$. We assume mild local regularity of the coefficients and that the linearization at a given steady state u_* is normally elliptic and satisfies the Lopatinskii–Shapiro condition (see Section 2). For illustration, we give a simple example where $N = 1$ and $m = 2$ (see e.g. [3] or [12, §6] for the system case $N > 1$). In the case of the quasilinear heat equation with a nonlinear Dirichlet boundary condition

$$\begin{aligned} \partial_t u(t) - a(u(t))\Delta u(t) &= f(u(t)), \quad \text{on } \Omega, \;\; t > 0, \\ b(u(t)) &= 0, \quad \text{on } \partial\Omega, \;\; t \geq 0, \\ u(0) &= u_0, \quad \text{on } \Omega, \end{aligned}$$

we have to require that $a, f \in C^1(\mathbb{R})$, $b \in C^3(\mathbb{R})$ are real, and that there is a steady state $u_* \in W_p^2(\Omega)$ with $a(u_*) \geq \delta > 0$, $|b'(u_*)| \geq \delta > 0$, and $p > n+2$.

Fully nonlinear boundary conditions appear naturally in the treatment of free boundary problems, see e.g. [5] or [9], and in the study of diffusion through interfaces, see e.g. [11]. The results of the present paper do not directly cover such problems, but we think that our methods can be generalized in order to deal with moving boundaries and transmission problems in future work. We note that the recent work [15] already contains the linear spectral analysis which is necessary for applications of center manifold theory to the Stefan problem with surface tension. **Roland: We believe that our methods can also be generalized to the investigation of the vicinity of a periodic orbit $u_*(t)$ in which case the linearization will become nonautonomous.**

We look for solutions u of (1.1) in the space $\mathbb{E}_1 = L_p([0,T]; W_p^{2m}(\Omega;\mathbb{C}^N)) \cap W_p^1([0,T]; L_p(\Omega;\mathbb{C}^N))$ for a fixed finite exponent $p > n + 2m$. The terms of highest order are thus contained in L_p spaces. The solution space $\mathbb{E}_1$ is continuously embedded into $C([0,T]; X_p)$ for the Slobodetskii space $X_p = W_p^{2m-2m/p}(\Omega;\mathbb{C}^N)$, and X_p is the smallest space with this property. Since also $X_p \hookrightarrow BC^{2m-1}(\overline{\Omega};\mathbb{C}^N)$ by Sobolev's embedding theorem, the nonlinear terms in (1.1) are continuous in (t,x) up to $t = 0$, and thus the initial condition can be understood in classical sense. In particular, the initial value u_0 of (1.1) has to belong to X_p and must fulfill the boundary conditions $B_j(u_0) = 0$ by continuity. Moreover, the solution u is continuous in X_p on $[0,T]$, and the norm of X_p is the natural norm for our work. So our nonlinear phase space is the C^1 manifold in X_p given by

$$\mathcal{M} = \{u_0 \in X_p : B_1(u_0) = 0, \cdots, B_m(u_0) = 0\}.$$

In our previous work [12] we have established the local wellposedness and certain smoothing properties of (1.1), and we have constructed the local stable and unstable manifolds at the steady state u_* assuming that the spectrum of the linearization of (1.1) at u_* does not intersect $i\mathbb{R}$. At first glance, we followed an approach that appears to be quite "standard". One introduces a new function $v(t) = u(t) - u_*$ in order to transform (1.1) into the problem **Roland:**

$$\begin{aligned} \partial_t v(t) + A_* v(t) = G(v(t)) \qquad & \textbf{on } \Omega, \quad \textbf{a.e. } t > 0, \\ B_{j*} v(t) = H_j(v(t)) \qquad & \textbf{on } \partial\Omega, \quad t \geq 0, \quad j \in \{1, \cdots, m\}, \\ v(0) = v_0, \qquad & \textbf{on } \Omega. \end{aligned} \tag{1.2}$$

see (2.21) **below. The problem** (1.2) **involves the linearizations** A_* **and** B_{j*}**, as well as the nonlinearities** G **and** H_j**which have the same order as the linear part but vanish at** $v = 0$ **together with their derivatives.** The stable, resp. unstable, manifold consist of initial values of solutions $u = v - u_*$ to (1.1) which belong to $\mathbb{E}_1$–type spaces of exponentially decaying functions on $\mathbb{R}_+$, resp. $\mathbb{R}_-$, see (2.14). Such functions v are obtained as fixed points of a Lyapunov–Perron map composed of the solution operator of the linearized inhomogeneous initial(final)–boundary value problem on $\mathbb{R}_+$ ($\mathbb{R}_-$) and of the substitution operators given by the nonlinearities in (1.2), cf. (4.5). The relevant definitions and results are briefly repeated in Section 2. Modifying our methods from [12], in Theorem 4.1 we construct the stable and unstable local manifolds $\mathcal{M}_{\mathrm{s}}$ and $\mathcal{M}_{\mathrm{u}}$ of (1.1) under the assumption that the linearization has spectral gaps in the left and the right open half plane, respectively. We point out that such gaps always exist if the underlying spatial domain Ω is bounded.

However, the actual implementation of this "standard" approach faces a fundamental difficulty already for the stable manifold: The nonlinear compatibility condition defining the solution manifold $\mathcal{M}$ obstructs a direct application of the usual methods. It turns out that one has to construct $\mathcal{M}_{\mathrm{s}}$ as a graph of a map defined on the (linearly) stable part of the tangent space X_p^0 of $\mathcal{M}$. This leads to an additional term in the fixed point problem, see the additive term in (4.5). Moreover, since we are not merely dealing with a semilinear problem, we need maximal regularity for the linearized initial–boundary value problem. This regularity property is known for compact time intervals (see [8] and the references therein). Using the spectral decompositions and semigroup theory, we could extend this result to unbounded intervals in [12, §3], cf. Propositions 2.5 and 2.6 below. We point out that in the maximal regularity result the boundary data must be contained in spaces involving fractional space and time regularity, see (2.14).

Yet another principal difficulty occurs when one tries to construct in a similar way the local center–unstable and center–stable manifolds $\mathcal{M}_{\mathrm{cu}}$ and $\mathcal{M}_{\mathrm{cs}}$ of (1.1), which should complement the stable and unstable manifolds $\mathcal{M}_{\mathrm{s}}$ and $\mathcal{M}_{\mathrm{u}}$ under the spectral assumptions of Theorem 4.1. Here, already in the linear case, $\mathcal{M}_{\mathrm{cu}}$ and $\mathcal{M}_{\mathrm{cs}}$ may contain exponentially growing functions. In the corresponding function spaces substitution operators behave badly; in particular, they are locally Lipschitz only under very restrictive conditions. A well known trick to overcome this difficulty is to multiply the nonlinearities in the transformed problem (1.2) by a suitable cutoff function, called $\digamma(t, v)$ below, which is equal to 1 if v is small and equal to 0 if v is large in a suitable norm (see e.g. [4], [6], [13], [14], [16], [18], [19], [21], [22]). But here we run into severe troubles. The space for the boundary data has to involve (fractional) time regularity which we can only control by means of the full $\mathbb{E}_1$–norm of v, say, on small time intervals. As a result, the cutoff must contain *nonlocal* terms of the form $\|v\|_{\mathbb{E}_1([t-a,t+a])}$, see (3.2), and becomes *nonautonomous.* This fact leads to many technical problems, but most importantly, it changes the nature of our evolution equation drastically: It becomes nonlocal and even noncausal after introducing the cutoff, see (3.7). We treat these rather delicate questions in Section 3 in detail. We add that one also needs an additional argument (taken from [21]) in order to upgrade the invariant manifolds from being merely Lipschitz to the class C^1.

Our main results concern local center manifolds, where we use similar methods as for $\mathcal{M}_{\rm cs}$ and $\mathcal{M}_{\rm cu}$ (working on the time interval $\mathbb{R}$ instead of $\mathbb{R}_+$ or $\mathbb{R}_-$). In the center case, we assume that the linearization has spectral gaps in both the left and the right open halfplanes, see (2.34). It is well known that local center manifolds are not uniquely determined, in general. (On a technical level, the nonuniqueness arises from possible modifications of the cutoff.) We show that 'our' center manifold $\mathcal{M}_{\rm c}$ is a C^1 manifold in X_p tangent to the center subspace of the linearized problem at u_* and that it is Lipschitz in the smaller Sobolev space $X_1 = W_p^{2m}(\Omega;\mathbb{C}^N)$. Moreover, $\mathcal{M}_{\rm c} = \mathcal{M}_{\rm cs} \cap \mathcal{M}_{\rm cu}$ and $\mathcal{M}_{\rm c} \cap \mathcal{M}_{\rm s} = \mathcal{M}_{\rm c} \cap \mathcal{M}_{\rm u} = \{u_*\}$. Also, $\mathcal{M}_{\rm c}$ is locally invariant under the flow of (1.1) and it contains all small global solutions of (1.1) on $\mathbb{R}$. These facts are presented in Theorem 4.2 and Corollary 5.3. Analogous results for the center–stable and center–unstable manifolds are proved in Theorem 5.1 and 5.2.

In Section 6 we additonally assume that there is no unstable spectrum and that the center subspace of the linearization is finite dimensional. Moreover, u_* is assumed to be (Lyapunov) stable with respect to the flow on the (finite dimensional) center manifold. Under these assumptions we show that each solution starting sufficiently close to the center manifold converges exponentially to a solution living on the center manifold; the latter solution is given by the ordinary differential equation (4.7). In particular, u_* is stable with respect to the full problem (1.1). Our proof is inspired by the arguments in [13, §9.3]. However, in contrast to [13], we cannot work with the cutoff problem because of its nonlocality. We managed to avoid the use of the cutoff by means of a careful analysis controlling the norms of all relevant functions in the proof. In these calculations we need the fact that the center manifold is Lipschitz in $X_1 = W_p^{2m}(\Omega;\mathbb{C}^N)$ which follows from an additional local regularity property of (1.1) established in the Appendix.

Center manifolds for fully nonlinear parabolic problems with linear boundary conditions were constructed and investigated in [6], [13], and [14]. Quasilinear equations with quasilinear boundary conditions were treated in [16] and [19]. We emphasize that in these works inhomogeneous boundary values do not appear explicitly in the analysis so that the above mentioned difficulties are not present in these papers. We note that in [10] the stability of a simplified Stefan type moving boundary problem was established by means of the results from [19]. We also refer to [12] for further literature concerning (1.1).

Notation. We set $D_k = -i\partial_k = -i\partial/\partial x_k$ and use the multi index notation. The k–tensor of the partial derivatives of order k is denoted by ∇^k, and we let $\underline{\nabla}^k u = (u, \nabla u, \cdots, \nabla^k u)$. For an operator A on a Banach space we write $\mathrm{dom}(A)$, $\ker(A)$, $\mathrm{ran}(A)$, $\sigma(A)$, and $\rho(A)$ for its domain, kernel, range, spectrum, and resolvent set, respectively. $\mathcal{B}(X,Y)$ is the space of bounded linear operators between two Banach spaces X and Y, and $\mathcal{B}(X) := \mathcal{B}(X,X)$. A ball in X with the radius r and center at u will be denoted by $B_X(u,r)$. For an open set $U \subset \mathbb{R}^n$ with (sufficiently regular) boundary ∂U, $C^k(U)$ (resp., $BC^k(U)$, $BUC^k(U)$, $C_0^k(U)$) are the spaces of k–times continuously differentiable functions u on U (such that u and its derivatives up to order k are bounded, bounded and uniformly continuous, vanish at ∂U and at infinity (if U is unbounded), respectively), where $BC^k(U)$ is endowed with its canonical norm. For $C^k(\overline{U})$, $BC^k(\overline{U})$, $BUC^k(\overline{U})$, we require in addition that u and its derivatives up to order k have a continuous extension to ∂U. For unbounded U, we write $C_0^k(\overline{U})$ for the space of $u \in C^k(\overline{U})$ such that u and its derivatives up to order k vanish at infinity. By $W_p^k(U)$ we denote the Sobolev spaces, see e.g. [1, Def.3.1], and by $W_p^s(U)$ the Slobodetskii spaces endowed with the norm

$$|v|^p_{W_p^s(U)} = |v|^p_{L_p(U)} + \sum_{|\alpha|=k} [\partial^\alpha v]^p_{W_p^\sigma(U)}\,, \quad [w]^p_{W_p^\sigma(U)} = \iint_{U^2} \frac{|w(y)-w(x)|^p}{|y-x|^{n+\sigma p}}\,dx\,dy,$$

for $s = k+\sigma$ with $k \in \mathbb{N}_0$ and $\sigma \in (0,1)$, see [1, Thm.7.48] or [20, Rem.4.4.1.2]. Finally, $J \subset \mathbb{R}$ is a closed interval with nonempty interior, c is a generic constant, and $\varepsilon : \mathbb{R}_+ \to \mathbb{R}_+$ is a generic nondecreasing function with $\varepsilon(r) \to 0$ as $r \to 0$.

2. Setting and preliminaries

We introduce the setting of our paper; more details can be found in [12]. Let $\Omega \subset \mathbb{R}^n$ be an open connected set with a compact boundary $\partial\Omega$ of class C^{2m} and outer unit normal $\nu(x)$, where $m \in \mathbb{N}$ is given by (2.5) below. Throughout this paper, we fix a finite exponent p with

$$p > n + 2m. \tag{2.1}$$

Let $E = \mathbb{C}^N$ with $\mathcal{B}(E) = \mathbb{C}^{N\times N}$ for some fixed $N \in \mathbb{N}$. We put

$$X_0 = L_p(\Omega;\mathbb{C}^N), \quad X_1 = W_p^{2m}(\Omega;\mathbb{C}^N), \quad X_p = W_p^{2m(1-1/p)}(\Omega;\mathbb{C}^N),$$

and denote the norms of these spaces by $|\cdot|_0$, $|\cdot|_1$, and $|\cdot|_p$, respectively. Recall that the spatial trace operator γ at $\partial\Omega$ induces continuous maps

$$\gamma : W_p^s(\Omega;\mathbb{C}^N) \to W_p^{s-1/p}(\partial\Omega;\mathbb{C}^N) \tag{2.2}$$

for $1/p < s \le 2m$ if $s - 1/p$ is not an integer. We set

$$\begin{aligned}
&Y_0 = L_p(\partial\Omega;\mathbb{C}^N), \quad Y_{j1} = W_p^{2m\kappa_j}(\partial\Omega;\mathbb{C}^N), \quad Y_{jp} = W_p^{2m\kappa_j - 2m/p}(\partial\Omega;\mathbb{C}^N),\\
&Y_1 = Y_{11} \times \cdots \times Y_{m1}\,, \quad Y_p = Y_{1p} \times \cdots \times Y_{mp}
\end{aligned}$$

for $j \in \{1,\cdots,m\}$, $m_j \in \{0,\cdots,2m-1\}$ given by (2.5), and the numbers

$$\kappa_j = 1 - \frac{m_j}{2m} - \frac{1}{2mp}\,. \tag{2.3}$$

Here the Sobolev–Slobodetskii spaces on $\partial\Omega$ are defined via local charts, see [1, Thm.7.53], [20, Def.3.6.1]. We observe that $X_1 \hookrightarrow X_p \hookrightarrow X_0$, $Y_{j1} \hookrightarrow Y_{jp} \hookrightarrow Y_0$,

$$X_p \hookrightarrow C_0^{2m-1}(\overline{\Omega};\mathbb{C}^N), \quad \text{and} \quad Y_{jp} \hookrightarrow C^{2m-1-m_j}(\partial\Omega;\mathbb{C}^N) \tag{2.4}$$

by (2.1), (2.3), and standard properties of Sobolev spaces, cf. [20, §4.6.1]. Our basic equations (1.1) involve the operators given by

$$\begin{aligned}
[A(u)v](x) &= \sum_{|\alpha|=2m} a_\alpha(x, u(x), \nabla u(x), \cdots, \nabla^{2m-1}u(x))\, D^\alpha v(x), \quad x \in \Omega,\\
[F(u)](x) &= f(x, u(x), \nabla u(x), \cdots, \nabla^{2m-1}u(x)), \quad x \in \Omega, \qquad (2.5)\\
[B_j(u)](x) &= b_j(x, (\gamma u)(x), (\gamma\nabla u)(x), \cdots, (\gamma\nabla^{m_j}u)(x)), \quad x \in \partial\Omega,
\end{aligned}$$

for $j \in \{1,\cdots,m\}$ and functions $u \in X_p$ and $v \in X_1$, where the integers $m \in \mathbb{N}$ and $m_j \in \{0,\cdots,2m-1\}$ fixed. We set $B = (B_1,\cdots,B_m)$. We assume throughout that the coefficients in (2.5) satisfy:

(R) $a_\alpha \in C^1(E\times E^n\times\cdots\times E^{(n^{2m-1})}; BC(\overline{\Omega};\mathcal{B}(E)))$ for $\alpha\in\mathbb{N}_0^n$ with $|\alpha|=2m$,
$a_\alpha(x,0)\longrightarrow a_\alpha(\infty)$ in $\mathcal{B}(E)$ as $x\to\infty$, if Ω is unbounded,
$f\in C^1(E\times E^n\times\cdots\times E^{(n^{2m-1})}; BC(\overline{\Omega};E))$,
$b_j\in C^{2m+1-m_j}(\partial\Omega\times E\times E^n\times\cdots\times E^{(n^{m_j})};E)$ for $j\in\{1,\cdots,m\}$.

Occasionally, we will need one more degree of smoothness of the coefficients as recorded in the following hypothesis:

(RR) $a_\alpha \in C^2(E\times E^n\times\cdots\times E^{(n^{2m-1})}; BC(\overline{\Omega};\mathcal{B}(E)))$ for $\alpha\in\mathbb{N}_0^n$ with $|\alpha|=2m$,
$f\in C^2(E\times E^n\times\cdots\times E^{(n^{2m-1})}; BC(\overline{\Omega};E))$,
$b_j\in C^{2m+2-m_j}(\partial\Omega\times E\times E^n\times\cdots\times E^{(n^{m_j})};E)$ for $j\in\{1,\cdots,m\}$.

In view of (2.4), only continuous functions will be inserted into the nonlinearites. Thus we will omit the trace γ in $B_j(u)$ and in similar expressions. We fix a numbering of the components of ∇^k so that a partial derivative $\partial^\beta u(x)$ of order $|\beta|=k$ is inserted at a fixed position called $l(\beta,k)$ into the functions a_α, f, and b_j. It is not diffcult to see that

$$A\in C^1(X_p;\mathcal{B}(X_1,X_0)) \quad\text{and}\quad F\in C^1(X_p;X_0) \tag{2.6}$$

with the locally bounded derivatives

$$\begin{aligned}
[F'(u)v](x) &= \sum_{k=0}^{2m-1}\sum_{|\beta|=k} i^k\,(\partial_{l(\beta,k)}f)(x,u(x),\nabla u(x),\cdots,\nabla^{2m-1}u(x))\,D^\beta v(x),\\
[A'(u)w]v(x) &= A'(u)[v,w](x) \qquad\qquad (2.7)\\
&= \sum_{|\alpha|=2m}\sum_{k=0}^{2m-1}\sum_{|\beta|=k}(\partial_{l(\beta,k)}a_\alpha)(x,u(x),\cdots,\nabla^{2m-1}u(x))\,[\partial^\beta v(x),D^\alpha w(x)]
\end{aligned}$$

for $x\in\Omega$, $u,v\in X_p$, and $w\in X_1$, see [12, (25)] and the text before it. (Observe that $(\partial_{l(\beta,k)}a_\alpha)(x,z):E^2\to E$ is bilinear.) We further have

$$B_j\in C^1(X_p;Y_{jp})\cap C^1(X_1;Y_{j1}),\qquad j\in\{1,\cdots,m\}, \tag{2.8}$$

with the locally bounded derivatives

$$[B_j'(u)v](x)=\sum_{k=0}^{m_j}\sum_{|\beta|=k} i^k\,(\partial_{l(\beta,k)}b_j)(x,u(x),\nabla u(x),\cdots,\nabla^{m_j}u(x))\,D^\beta v(x),$$

where $x\in\partial\Omega$ and $u,v\in X_p$, resp. $u,v\in X_1$. The continuous differentiability of $B_j:X_p\to Y_{jp}$ was shown in [12, Cor.12], and $B_j\in C^1(X_1;Y_{1p})$ can be proved by the arguments used in step (4) and (5) of the proof of [12, Prop.10], see in particular inequality (69) in [12]. We set $B'(u)=(B_1'(u),\cdots,B_m'(u))$.

The symbols of the principal parts of the linear differential operators are the matrix–valued functions given by

$$\mathcal{A}_\#(x,z,\xi)=\sum_{|\alpha|=2m}a_\alpha(x,z)\,\xi^\alpha,\quad \mathcal{B}_{j\#}(x,z,\xi)=\sum_{|\beta|=m_j}i^{m_j}\,(\partial_{l(\beta,m_j)}b_j)(x,z)\,\xi^\beta$$

for $x\in\overline{\Omega}$, $z\in E\times\cdots\times E^{(n^{2m-1})}$ and $\xi\in\mathbb{R}^n$, resp. $x\in\partial\Omega$, $z\in E\times\cdots\times E^{(n^{m_j})}$ and $\xi\in\mathbb{R}^n$. We further set $\mathcal{A}_\#(\infty,\xi)=\sum_{|\alpha|=2m}a_\alpha(\infty)\,\xi^\alpha$ if Ω is unbounded. We introduce the *normal ellipticity* and the *Lopatinskii–Shapiro condition* for $A(u_0)$ and $B'(u_0)$ at a function $u_0\in X_p$ as follows:

(E) $\sigma(\mathcal{A}_\#(x, \underline{\nabla}^{2m-1}u_0(x), \xi)) \subset \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\} =: \mathbb{C}_+$ and (if Ω is unbounded) $\sigma(\mathcal{A}_\#(\infty, \xi)) \subset \mathbb{C}_+$, for $x \in \overline{\Omega}$ and $\xi \in \mathbb{R}^n$ with $|\xi| = 1$.

(LS) Let $x \in \partial\Omega$, $\xi \in \mathbb{R}^n$, and $\lambda \in \overline{\mathbb{C}_+}$ with $\xi \perp \nu(x)$ and $(\lambda, \xi) \neq (0, 0)$. The function $\varphi = 0$ is the only solution in $C_0(\mathbb{R}_+; \mathbb{C}^N)$ of the ode system

$$\lambda\varphi(y) + \mathcal{A}_\#(x, \underline{\nabla}^{2m-1}u_0(x), \xi + i\nu(x)\partial_y)\varphi(y) = 0, \quad y > 0, \tag{2.9}$$

$$\mathcal{B}_{j\#}(x, \underline{\nabla}^{m_j}u_0(x), \xi + i\nu(x)\partial_y)\varphi(0) = 0, \quad j \in \{1, \cdots, m\}. \tag{2.10}$$

We refer to [3], [7], [8], and the references therein for more information concerning these conditions. We can now state our basic hypothesis.

Hypothesis 2.1. *Condition (R) holds, and (E), (LS) hold at a steady state $u_* \in X_1$ of (1.1), i.e., $A(u_*)u_* = F(u_*)$ on Ω, $B(u_*) = 0$ on $\partial\Omega$.*

For the investigation of (1.1), we need several spaces of functions on $J \times \Omega$ and $J \times \partial\Omega$, where $J \subset \mathbb{R}$ is a closed interval with a nonempty interior. The base space and solution space of (1.1) are

$$\mathbb{E}_0(J) = L_p(J; L_p(\Omega; \mathbb{C}^N)) = L_p(J; X_0),$$

$$\mathbb{E}_1(J) = W_p^1(J; L_p(\Omega; \mathbb{C}^N)) \cap L_p(J; W_p^{2m}(\Omega; \mathbb{C}^N)) = W_p^1(J; X_0) \cap L_p(J; X_1),$$

respectively. We equip $\mathbb{E}_0(J)$ with the usual p–norm and $\mathbb{E}_1(J)$ with the norm

$$\|u\|_{\mathbb{E}_1(J)} = \Big[\, \|u\|^p_{\mathbb{E}_0(J)} + \|\dot{u}\|^p_{\mathbb{E}_0(J)} + \sum_{|\alpha|=2m} \|\partial^\alpha u\|^p_{\mathbb{E}_0(J)} \Big]^{\frac{1}{p}}.$$

Very often we use the crucial embeddings

$$\mathbb{E}_1(J) \hookrightarrow BUC(J; X_p) \hookrightarrow BUC(J; C_0^{2m-1}(\overline{\Omega}; \mathbb{C}^N)), \tag{2.11}$$

see [2, Thm.III.4.10.2] for the first and (2.4) for the second embedding. We denote by $c_0 = c_0(J)$ the norm of the first embedding in (2.11), which is uniform for J of length greater than a fixed $\ell > 0$. Observe that (2.11) implies that the trace operator γ_0 at time $t = 0$ is continuous from $\mathbb{E}_1(J)$ to X_p if $0 \in J$. The boundary data of our linearized equations will be contained in the spaces

$$\begin{aligned} \mathbb{F}_j(J) &= W_p^{\kappa_j}(J; L_p(\partial\Omega; \mathbb{C}^N)) \cap L_p(J; W_p^{2m\kappa_j}(\partial\Omega; \mathbb{C}^N)) \\ &= W_p^{\kappa_j}(J; Y_0) \cap L_p(J; Y_{j1}), \qquad j \in \{1, \cdots, m\}, \end{aligned} \tag{2.12}$$

endowed with their natural norms, where $\mathbb{F}(J) := \mathbb{F}_1(J) \times \cdots \times \mathbb{F}_m(J)$. We have

$$\mathbb{F}_j(J) \hookrightarrow BUC(J; Y_{jp}) \hookrightarrow BUC(J \times \partial\Omega) \quad \text{and} \quad \gamma_0 \in \mathcal{B}(\mathbb{F}_j(J), Y_{jp}) \tag{2.13}$$

if $0 \in J$, see [8, §3] and [12, §2]. For $\alpha, \beta \in \mathbb{R}$, we set $e_\alpha(t) = e^{\alpha t}$ for $t \in \mathbb{R}$ and define the function $e_{\alpha,\beta}$ by setting $e_{\alpha,\beta}(t) = e_\alpha(t)$ for $t \le 0$ and $e_{\alpha,\beta}(t) = e_\beta(t)$ for $t \ge 0$. Then we introduce the weighted spaces

$$\begin{aligned} \mathbb{E}_k(\mathbb{R}_\pm, \alpha) &= \{v : e_\alpha v \in \mathbb{E}_k(\mathbb{R}_\pm)\}, \qquad \mathbb{F}(\mathbb{R}_\pm, \alpha) = \{v : e_\alpha v \in \mathbb{F}(\mathbb{R}_\pm)\}, \\ \mathbb{E}_k(\alpha, \beta) &= \{v : e_{\alpha,\beta} v \in \mathbb{E}_k(\mathbb{R})\}, \qquad \mathbb{F}(\alpha, \beta) = \{v : e_{\alpha,\beta} v \in \mathbb{F}(\mathbb{R})\}, \end{aligned} \tag{2.14}$$

where $k = 0, 1$, endowed with the canonical norms $\|v\|_{\mathbb{E}_0(\mathbb{R}_+,\alpha)} = \|e_\alpha v\|_{\mathbb{E}_0(\mathbb{R}_+)}$ etc. We also use the analogous norms on compact intervals J.

We assume that Hypothesis 2.1 holds. Due to (2.6) and (2.8), we can linearize the problem (1.1) at the steady state $u_* \in X_1$ obtaining the operators defined by

$$\begin{aligned} A_* &= A(u_*) + A'(u_*)u_* - F'(u_*) \in \mathcal{B}(X_1, X_0), \\ B_{j*} &= B_j'(u_*) \in \mathcal{B}(X_p, Y_{jp}) \cap \mathcal{B}(X_1, Y_{j1}). \end{aligned} \tag{2.15}$$

We set $B_* = (B_{1*}, \cdots, B_{m*})$. We further define the nonlinear maps

$$\begin{aligned} &G \in C^1(X_1; X_0) \quad \text{ and } \quad H_j \in C^1(X_p; Y_{jp}) \cap C^1(X_1; Y_{j1}) \\ &\text{with } \; G(0) = H_j(0) = 0 \quad \text{ and } \quad G'(0) = H_j'(0) = 0 \end{aligned} \tag{2.16}$$

for $j \in \{1, \cdots, m\}$ by setting

$$\begin{aligned} G(v) &= \big(A(u_*)v - A(u_* + v)v\big) - \big(A(u_* + v)u_* - A(u_*)u_* - [A'(u_*)u_*]v\big) \\ &\quad + \big(F(u_* + v) - F(u_*) - F'(u_*)v\big), \end{aligned} \tag{2.17}$$

$$H_j(v) = B_j'(u_*)v - B_j(u_* + v), \tag{2.18}$$

for $v \in X_1$, resp. $v \in X_p$. Again, we put $H(v) = (H_1(v), \cdots, H_m(v))$. The corresponding Nemytskii operators are denoted by

$$\mathbb{G}(v)(t) = G(v(t)), \qquad \mathbb{H}_j(v)(t) = H_j(v(t)), \qquad \mathbb{H}(v)(t) = H(v(t)) \tag{2.19}$$

for $v \in \mathbb{E}_1^{\mathrm{loc}}(J)$ (which is the space of $v : J \to X_0$ such that $v \in \mathbb{E}_1([a,b])$ for all intervals $[a,b] \subset J$). We recall a part of Proposition 10 from [12] describing the mapping properties of $\mathbb{G}$ and $\mathbb{H}$.

Proposition 2.2. *Let (R) hold. Define $\mathbb{G}$ and $\mathbb{H}$ by (2.17), (2.18), (2.19) for some $u_* \in X_1$ with $B(u_*) = 0$. Take $\delta \geq 0$. Then we have:*

$$\begin{aligned} \mathbb{G} &\in C^1(\mathbb{E}_1([a,b]); \mathbb{E}_0([a,b])), \quad \mathbb{G} \in C^1(\mathbb{E}_1(\mathbb{R}_\pm, \pm\delta); \mathbb{E}_0(\mathbb{R}_\pm, \pm\delta)), \\ \mathbb{H} &\in C^1(\mathbb{E}_1([a,b]); \mathbb{F}([a,b])), \quad \mathbb{H} \in C^1(\mathbb{E}_1(\mathbb{R}_\pm, \pm\delta); \mathbb{F}(\mathbb{R}_\pm, \pm\delta)). \end{aligned}$$

Moreover, $\mathbb{G}(0) = 0$, $\mathbb{G}'(0) = 0$, $\mathbb{H}(0) = 0$, *and* $\mathbb{H}'(0) = 0$.

Theorem 14 of [12] shows that (1.1) generates a local semiflow on the solution manifold

$$\mathcal{M} = \{u_0 \in X_p : B(u_0) = 0\}. \tag{2.20}$$

In particular, a function u_0 is the initial value of the (unique) solution $u \in \mathbb{E}_1([0,T])$ of (1.1) for some $T > 0$ if and only if $u_0 \in \mathcal{M}$. Setting $v = u - u_*$ and $v_0 = u_0 - u_*$, we further see that $u_0 \in \mathcal{M}$ if and only if $v_0 \in X_p$ and $B_* v_0 = H(v_0)$ and that $u \in \mathbb{E}_1([0,T])$ solves (1.1) if and only if $v \in \mathbb{E}_1([0,T])$ satisfies

$$\begin{aligned} \partial_t v(t) + A_* v(t) &= G(v(t)) \qquad \text{on } \Omega, \quad \text{a.e. } t > 0, \\ B_{j*} v(t) &= H_j(v(t)) \qquad \text{on } \partial\Omega, \quad t \geq 0, \quad j \in \{1, \cdots, m\}, \\ v(0) &= v_0, \qquad \text{on } \Omega. \end{aligned} \tag{2.21}$$

Remark 2.3. Theorem 14(a) of [12] implies the following facts: For each given $T > 0$, there is a radius $\rho = \rho(T) > 0$ such that for every $u_0 = u_* + v_0 \in \mathcal{M}$ with $|v_0|_p \leq \rho$ there exists a unique solution $u = u_* + v$ of (1.1) on $[0,T]$, and $\|v\|_{\mathbb{E}_1([0,T])} \leq c_* \, |v_0|_p$ with a constant $c_* = c_*(T)$ independent of u_0 in this ball. $\Diamond$

We now recast and extend some results from [12] regarding the solvability of the inhomogeneous linear problem

$$\begin{aligned} \partial_t v(t) + A_* v(t) &= g(t) \qquad \text{on } \Omega, \quad \text{a.e. } t \in J, \\ B_* v(t) &= h(t) \qquad \text{on } \partial\Omega, \quad t \in J, \\ v(0) &= v_0, \qquad \text{on } \Omega, \end{aligned} \tag{2.22}$$

in weighted function spaces on the unbounded interval $J \in \{\mathbb{R}_+, \mathbb{R}_-, \mathbb{R}\}$. We assume that Hypothesis 2.1 holds. (Actually, when dealing only with (2.22) we do not have to assume that $u_* \in X_1$ is a steady state of (1.1).) We recall from [8, Thm.2.1] that on a bounded interval $J = [a, b]$ the boundary value problem obtained by combining the first two lines of (2.22) with the initial condition $v(a) = v_0$ has a unique solution $v \in \mathbb{E}_1([a, b])$ if and only if $g \in \mathbb{E}_0([a, b])$, $h \in \mathbb{F}([a, b])$, $v_0 \in X_p$, and $B_* v_0 = h(a)$. A solution $v \in \mathbb{E}_1^{\mathrm{loc}}(J)$ of (2.22) on J will be denoted by $v = S(v_0, g, h)$, where $J \subset \mathbb{R}$ is any closed interval containing 0. We stress that this notation incorporates the compatibility condition $B_* v_0 = h(0)$ because of the second line in (2.22) and (2.11). Moreover, the solution $S(v_0, g, h)$ is unique if $J = \mathbb{R}_+$, but uniqueness may fail on $J = \mathbb{R}_-$.

As in [12, (31)], we define $A_0 = A_*|_{\ker B_*}$ with the domain $\mathrm{dom}(A_0) = \{u \in X_1 : B_{j*} u = 0, j = 1, \dots, m\}$, and denote by $T(\cdot)$ the analytic semigroup on X_0 generated by $-A_0$. We further need the extrapolated semigroup $T_{-1}(\cdot)$ and its generator A_{-1} acting on the extrapolation space X_{-1} of A_0; here, X_{-1} is the completion of X_0 with respect to the norm $|u_0|_{-1} = |(\mu + A_0)^{-1} u_0|_0$ for some fixed $\mu \in \rho(-A_0)$. We further employ the map

$$\Pi = (\mu + A_{-1}) \mathcal{N}_1 \in \mathcal{B}(Y_1, X_1) \tag{2.23}$$

where $\mathcal{N}_1 \in \mathcal{B}(Y_1, X_1)$ is the solution operator, $\mathcal{N}_1 : \varphi \mapsto u$, of the elliptic boundary value problem $(\mu + A_*)u = 0$ on Ω, $B_* u = \varphi$ on $\partial\Omega$, see [12, Prop.5]. Also, by the same Proposition 5 in [12], there exists a right inverse $\mathcal{N}_p \in \mathcal{B}(Y_p, X_p)$ of B_*. Due to [12, Prop.6], the function $v = S(v_0, g, h)$ is a solution of (2.22) if and only if $v \in \mathbb{E}_1^{\mathrm{loc}}(J)$, $v(0) = v_0$, and the variation of constants formula

$$v(t) = T(t - \tau) v(\tau) + \int_\tau^t [T(t - s) g(s) + T_{-1}(t - s) \Pi h(s)] \, ds \tag{2.24}$$

holds for all $t \geq \tau$ in J. If $J = \mathbb{R}_+$ (or $J = [0, T]$), it suffices to take $\tau = 0$ in (2.24), and $v(0) = v_0$ follows from (2.24).

In order to treat solutions of (2.22) on the intervals $J = \mathbb{R}_\pm$, we assume that the (rescaled) semigroup $\left\{e^{\delta t} T(t)\right\}_{t \geq 0}$ is *hyperbolic* for $\delta \in [\delta_1, \delta_2]$ for some segment $[\delta_1, \delta_2] \subset \mathbb{R}$ (i.e., $\sigma(-A_0 + \delta) \cap i\mathbb{R} = \emptyset$). Let P be the (stable) spectral projection for $-A_0 + \delta$ corresponding to the part of $\sigma(-A_0 + \delta)$ in the open left halfplane, and set $Q = I - P$. Then $T(t)$ is invertible on QX_0 with the inverse $T_Q(-t)Q$, and $\|e^{t\delta} T(t) P\|$, $\|e^{-t\delta} T_Q(-t) Q\| \leq c e^{-\epsilon t}$ for $t \geq 0$ and some $\epsilon > 0$.

Remark 2.4. If $e_\delta T(\cdot)$ is hyperbolic on X_0 then $e_\delta T_{-1}(\cdot)$ is hyperbolic on X_{-1} with projections P_{-1} and $Q_{-1} = I - P_{-1}$ being the extensions of P and Q, respectively. Moreover, Q_{-1} maps X_{-1} into $\mathrm{dom}(A_0)$, and P leaves invariant X_p, X_1, and $\mathrm{dom}(A_0)$. (See [12, §2] for these facts.) $\Diamond$

Given $(w_0, g, h) \in X_p \times \mathbb{E}_0(\mathbb{R}_+, \delta) \times \mathbb{F}(\mathbb{R}_+, \delta)$, resp. $(w_0, g, h) \in X_0 \times \mathbb{E}_0(\mathbb{R}_-, \delta) \times \mathbb{F}(\mathbb{R}_-, \delta)$, we can then define

$$\begin{aligned} L^+_{P,A_0}(w_0, g, h)(t) = T(t) w_0 &+ \int_0^t [T(t - s) P g(s) + T_{-1}(t - s) P_{-1} \Pi h(s)] \, ds \\ &- \int_t^\infty T_Q(t - s) Q [g(s) + \Pi h(s)] \, ds, \qquad t \geq 0, \end{aligned} \tag{2.25}$$

$$\phi_0^+ = -\int_0^\infty T_Q(-s) Q [g(s) + \Pi h(s)] \, ds, \qquad \text{resp.,} \tag{2.26}$$

$$L^-_{P,A_0}(w_0, g, h)(t) = T_Q(t) Q w_0 + \int_{-\infty}^t [T(t - s) P g(s) + T_{-1}(t - s) P_{-1} \Pi h(s)] \, ds$$

$$-\int_t^0 T_Q(t-s)Q[g(s)+\Pi h(s)]\,ds, \qquad t\le 0, \tag{2.27}$$

$$\phi_0^- = \int_{-\infty}^0 [T(-s)Pg(s)+T_{-1}(-s)P_{-1}\Pi h(s)]\,ds. \tag{2.28}$$

(We drop the subscript '-1' in the Q–integrals.) As in [12, §3], one verifies that these integrals in fact exist. Clearly, a function $v\in\mathbb{E}_1^{\rm loc}(J)$ solves (2.22) if and only if $\tilde v = e_\delta v\in\mathbb{E}_1^{\rm loc}(J)$ is a solution of the rescaled problem

$$\begin{aligned} \partial_t\tilde v(t)+(A_*-\delta)\tilde v(t) &= e^{\delta t}g(t) \qquad \text{on } \Omega, \quad \text{a.e. } t\in J,\\ B_*\tilde v(t) &= e^{\delta t}h(t) \qquad \text{on } \partial\Omega, \quad t\in J,\\ \tilde v(0) &= v_0, \qquad \text{on } \Omega, \end{aligned}$$

whose solution operator will be denoted by $S_{A_0-\delta}$. We characterize the solvability of (2.22) at first in the case $J=\mathbb{R}_+$. Using (2.24), (2.25) and (2.26), we infer that

$$\begin{aligned} e_\delta S_{A_0}(v_0,g,h) &= S_{A_0-\delta}(v_0,e_\delta g,e_\delta h) &(2.29)\\ &= e_\delta T(\cdot)[Qv_0-\phi_0^+] + L^+_{P,A_0-\delta}(Pv_0,e_\delta g,e_\delta h) &(2.30)\\ &= e_\delta T(\cdot)[Qv_0-\phi_0^+] + e_\delta L^+_{P,A_0}(Pv_0,g,h). &(2.31) \end{aligned}$$

Proposition 2.5. *Assume that Hypothesis 2.1 holds and that for $\delta\in[\delta_1,\delta_2]\subset\mathbb{R}$ the semigroup $e_\delta T(\cdot)$ is hyperbolic with the stable projection P, and let $Q=I-P$. Suppose that $(v_0,g,h)\in X_p\times\mathbb{E}_0(\mathbb{R}_+,\delta)\times\mathbb{F}(\mathbb{R}_+,\delta)$ and $B_*v_0=h(0)$. Using the above notations, the following assertions are equivalent:*

(a) $S_{A_0}(v_0,g,h)\in\mathbb{E}_0(\mathbb{R}_+,\delta)$.
(b) $L^+_{P,A_0}(v_0-\phi_0^+,g,h)\in\mathbb{E}_0(\mathbb{R}_+,\delta)$.
(c) $\phi_0^+=Qv_0$.

If these assertions hold, then $S_{A_0}(v_0,g,h)=L^+_{P,A_0}(Pv_0,g,h)\in\mathbb{E}_1(\mathbb{R}_+,\delta)$, and we have the maximal regularity estimate

$$\|S_{A_0}(v_0,g,h)\|_{\mathbb{E}_1(\mathbb{R}_+,\delta)} \le c\,(|v_0|_p+\|g\|_{\mathbb{E}_0(\mathbb{R}_+,\delta)}+\|h\|_{\mathbb{F}(\mathbb{R}_+,\delta)}), \tag{2.32}$$

where c does not depend on v_0, g, h, or δ.

Proof. Using rescaling as in (2.29) and (2.31), it suffices to prove the proposition for $\delta=0$. For this case, assertions (b)$\Leftrightarrow$(c)$\Rightarrow$(a) and the last statement have been proved in [12, Prop.8], and (a)$\Rightarrow$(c) follows from (2.30). □

The corresponding result for $J=\mathbb{R}_-$ looks a bit different since in (2.27) we have to write $T(t)Qw_0$ rather than $T(t)w_0$ for negative t. Moreover, Proposition 2.6 does not require a compatibility condition since it deals with a final value problem on $J=\mathbb{R}_-$. The next result follows by rescaling from Proposition 9 of [12].

Proposition 2.6. *Assume that Hypothesis 2.1 holds and that for $\delta\in[\delta_1,\delta_2]\subset\mathbb{R}$ the semigroup $e_\delta T(\cdot)$ is hyperbolic with the stable projection P, and let $Q=I-P$. Suppose that $(v_0,g,h)\in X_0\times\mathbb{E}_0(\mathbb{R}_-,\delta)\times\mathbb{F}(\mathbb{R}_-,\delta)$. Using the above notations, there is a solution $v=S_{A_0}(v_0,g,h)$ of (2.22) in $\mathbb{E}_0(\mathbb{R}_-,\delta)$ if and only if $Pv_0=\phi_0^-$. In this case, this solution is unique, $v=L^-_{P,A_0}(v_0,g,h)\in\mathbb{E}_1(\mathbb{R}_-,\delta)$, and*

$$\|S_{A_0}(v_0,g,h)\|_{\mathbb{E}_1(\mathbb{R}_-,\delta)} \le c\,(|Qv_0|_0+\|g\|_{\mathbb{E}_0(\mathbb{R}_-,\delta)}+\|h\|_{\mathbb{F}(\mathbb{R}_-,\delta)}), \tag{2.33}$$

where c does not depend on v_0, g, h, or δ.

In order to treat the interval $J = \mathbb{R}$, we assume that $T(\cdot)$ has an *exponential trichotomy*, i.e., there is a splitting

$$\begin{aligned}
&\sigma(-A_0) = \sigma_s \cup \sigma_c \cup \sigma_u \qquad \text{with} \qquad\qquad (2.34)\\
&\max \operatorname{Re} \sigma_s < -\omega_s < -\underline{\omega}_c < \min \operatorname{Re} \sigma_c \le 0 \le \max \operatorname{Re} \sigma_c < \overline{\omega}_c < \omega_u < \min \operatorname{Re} \sigma_u\,.
\end{aligned}$$

(If Ω is bounded, $\sigma(-A_0)$ is discrete and thus (2.34) automatically holds with $\sigma_u \subset i\mathbb{R}$ and arbitrarily small $\underline{\omega}_c = \overline{\omega}_c$.) We take numbers $\alpha \in [\underline{\omega}_c, \omega_s]$ and $\beta \in [\overline{\omega}_c, \omega_u]$ and denote by P_k the spectral projections for $-A_0$ corresponding to σ_k, $k = s, c, u$. We set $P_{\mathrm{cs}} = P_{\mathrm{s}} + P_{\mathrm{c}}$, $P_{\mathrm{cu}} = P_{\mathrm{c}} + P_{\mathrm{u}}$, and $P_{\mathrm{su}} = P_{\mathrm{s}} + P_{\mathrm{u}}$. Then the rescaled semigroups $e_\alpha T(\cdot)$ and $e_{-\beta} T(\cdot)$ are hyperbolic on X_0 with stable projections P_{s} and P_{cs}, respectively. The restriction of $T(t)$ to $P_k X_0$ yields a group denoted by $T_k(t)$, $t \in \mathbb{R}$, where $k = c, u, cu$. For $g \in \mathbb{E}_0(\alpha, -\beta)$, $h \in \mathbb{F}(\alpha, -\beta)$ and $w_0 \in X_0$, we can then define

$$\begin{aligned}
L_{A_0}(w_0, g, h)(t) = T_c(t) P_{\mathrm{c}} w_0 &+ \int_0^t T_c(t-s) P_{\mathrm{c}} [g(s) + \Pi h(s)]\, ds \\
&+ \int_{-\infty}^t [T(t-s) P_{\mathrm{s}} g(s) + T_{-1}(t-s) P_{\mathrm{s},-1} + \Pi h(s)]\, ds \\
&- \int_t^\infty T_u(t-s) P_{\mathrm{u}} [g(s) + \Pi h(s)]\, ds, \qquad t \in \mathbb{R}, \qquad (2.35)\\
\phi_0 = \int_{-\infty}^0 &[T(-s) P_{\mathrm{s}} g(s) + T_{-1}(-s) P_{\mathrm{s},-1} \Pi h(s)]\, ds \\
&- \int_0^\infty T_u(-s) P_{\mathrm{u}} [g(s) + \Pi h(s)]\, ds. \qquad (2.36)
\end{aligned}$$

The trichotomy and the assumptions on the data imply that the integrals are well–defined. The next result then easily follows from Propositions 2.5 and 2.6.

Proposition 2.7. *Assume that Hypothesis 2.1 holds and that $T(\cdot)$ has a trichotomy as in* (2.34)*. Take $\alpha \in [\underline{\omega}_c, \omega_s]$ and $\beta \in [\overline{\omega}_c, \omega_u]$ and denote by P_k the spectral projections corresponding to σ_k, $k = s, c, u$. Suppose that $(v_0, g, h) \in X_0 \times \mathbb{E}_0(\alpha, -\beta) \times \mathbb{F}(\alpha, -\beta)$. Using the above notations, there is a solution $v = S_{A_0}(v_0, g, h)$ of* (2.22) *in $\mathbb{E}_0(\alpha, -\beta)$ if and only if $P_{\mathrm{su}} v_0 = \phi_0$. In this case, this solution is unique, and we have $v = L_{A_0}(v_0, g, h) \in \mathbb{E}_1(\alpha, -\beta)$ and*

$$\|S_{A_0}(v_0, g, h)\|_{\mathbb{E}_1(\alpha,-\beta)} \le c\,(|P_{\mathrm{c}} v_0|_0 + \|g\|_{\mathbb{E}_0(\alpha,-\beta)} + \|h\|_{\mathbb{F}(\alpha,-\beta)}), \qquad (2.37)$$

where c does not depend on v_0, g, h, α, or β.

3. The cutoff problem and the corresponding Nemytskii operators

In this section we introduce a nonlocal and (if $J = \mathbb{R}$) time–invariant cutoff for (2.21) and discuss the mapping properties of the corresponding Nemytskii operators. The cutoff depends on a parameter $\eta > 0$ to be fixed in the following sections. For $t \in \mathbb{R}$ and $n \in \mathbb{Z}$, we set

$$\begin{aligned}
J(t) = [t - \tfrac{3}{2}, t + \tfrac{3}{2}], \quad J_n = [n, n+1], \quad J_n^* = [n - \tfrac{1}{8}, n + \tfrac{9}{8}],\\
J_n' = [n - \tfrac{1}{4}, n + \tfrac{5}{4}], \quad \text{and} \quad J_n'' = [n-2, n+3].
\end{aligned}$$

We further introduce

$$N(t, v) = \|v\|_{\mathbb{E}_1(J(t))} \quad \text{for} \quad v \in \mathbb{E}_1^{\mathrm{loc}}(\mathbb{R}). \qquad (3.1)$$

Given an $\eta > 0$, we take even functions $\chi, \gamma \in C^\infty(\mathbb{R})$ such that $0 \le \chi \le 1$, $\chi(t) = 1$ for $t \in [-\eta, \eta]$, $\operatorname{supp} \chi \subset (-2\eta, 2\eta)$, $\|\chi'\|_\infty \le 2/\eta$, and such that $\gamma \ge 0$, $\int_{\mathbb{R}} \gamma(t)\, dt = 1$, $\operatorname{supp} \gamma \subseteq (-1/4, 1/4)$. We now define the cutoff

$$\digamma_{\mathbb{R}}(t,v) = \digamma(t,v) := (\gamma * \chi(N(\cdot\,, v)))(t) = \int_{\mathbb{R}} \gamma(t-s)\chi(\|v\|_{\mathbb{E}_1([s-3/2,s+3/2])})\, ds \tag{3.2}$$

for $t \in \mathbb{R}$ and $v \in \mathbb{E}_1^{\mathrm{loc}}(\mathbb{R})$. Observe that the integrand is continuous in s and that $\digamma(t,v)$ depends on the restriction of v to $(t - 7/4, t + 7/4)$. For functions $v \in \mathbb{E}_1^{\mathrm{loc}}(J)$, we define $\digamma(t,v)$ as in (3.2) for $t \in [\frac{7}{4} + \inf J, -\frac{7}{4} + \sup J]$, where J is a closed interval of length greater than $7/2$.

In order to treat $v \in \mathbb{E}_1^{\mathrm{loc}}(\mathbb{R}_+)$ or $w \in \mathbb{E}_1^{\mathrm{loc}}(\mathbb{R}_-)$, we further fix the extension operators $R_\pm : \mathbb{E}_1^{\mathrm{loc}}(\mathbb{R}_\pm) \to \mathbb{E}_1^{\mathrm{loc}}(\mathbb{R})$ given by

$$(R_+v)(t) = \begin{cases} v(t), & t \ge 0, \\ (1+t)v(-t), & t \in [-1,0], \\ 0, & t \le -1, \end{cases} \qquad (R_-w)(t) = \begin{cases} w(t), & t \le 0, \\ (1-t)w(-t), & t \in [0,1], \\ 0, & t \ge 1. \end{cases}$$

Occasionally, we use the notation $v_{\mathbb{R}} := R_\pm v$ in both cases. We need the elementary estimates

$$\|R_+v\|_{\mathbb{E}_1([-1,1])} \le c_R \|v\|_{\mathbb{E}_1([0,1])}\,, \qquad \|R_-v\|_{\mathbb{E}_1([-1,1])} \le c_R \|v\|_{\mathbb{E}_1([-1,0])}\,, \tag{3.3}$$

$$\|v\|_{\mathbb{E}_1([0,T])} \le c_E \|v\|_{\mathbb{E}_1(\mathbb{R}_+,-\alpha)}\,, \qquad \|v\|_{\mathbb{E}_1([-T,0])} \le c_E \|v\|_{\mathbb{E}_1(\mathbb{R}_-,\alpha)} \tag{3.4}$$

for constants c_R and c_E and for all $T > 0$ and $\alpha \ge 0$, where c_E depends on T and is uniform for α in compact intervals. We then define the cutoffs

$$\digamma_{\mathbb{R}_\pm}(t,v) = \digamma_\pm(t,v) := \digamma(t, R_\pm v) = (\gamma * \chi(N(\cdot\,, v_{\mathbb{R}})))(t) \tag{3.5}$$

for $t \in \mathbb{R}$ and $v \in \mathbb{E}_1^{\mathrm{loc}}(\mathbb{R}_\pm)$.

Finally, for $v \in \mathbb{E}_1^{\mathrm{loc}}(J)$ and $J \in \{\mathbb{R}, \mathbb{R}_+, \mathbb{R}_-\}$, we define the Nemytskii operators

$$\mathbb{G}_{\digamma J}(v)(t) = \digamma_J(t,v) G(v(t)) \quad \text{and} \quad \mathbb{H}_{\digamma J}(v)(t) = \digamma_J(t,v) H(v(t)), \quad t \in J, \tag{3.6}$$

for the cutoffs of the nonlinear maps G and H defined in (2.17) and (2.18), where we assume that (R) holds and that $u_* \in X_1$ satisfies $B(u_*) = 0$. We also abbreviate $\mathbb{G}_\digamma = \mathbb{G}_{\digamma\mathbb{R}}$, $\mathbb{G}_{\digamma\pm} = \mathbb{G}_{\digamma\mathbb{R}_\pm}$, $\mathbb{H}_\digamma = \mathbb{H}_{\digamma\mathbb{R}}$, and $\mathbb{H}_{\digamma\pm} = \mathbb{H}_{\digamma\mathbb{R}_\pm}$. If Hypothesis 2.1 holds, we consider the cutoff version of the initial-boundary value problem (2.21) given by

$$\begin{aligned} \partial_t v(t) + A_* v(t) &= \mathbb{G}_{\digamma J}(v)(t) \qquad \text{on } \Omega, \quad \text{a.e. } t \in J, \\ B_* v(t) &= \mathbb{H}_{\digamma J}(v)(t) \qquad \text{on } \partial\Omega, \quad t \in J, \\ v(0) &= v_0, \qquad \text{on } \Omega, \end{aligned} \tag{3.7}$$

where $J \in \{\mathbb{R}, \mathbb{R}_+, \mathbb{R}_-\}$. We stress that the cutoff problem (3.7) is not local in time. In particular, even for $J = \mathbb{R}_+$ it is not a well–posed Cauchy problem. In fact, we will only solve (3.7) globally in function spaces on J. By definition, a function $v \in \mathbb{E}_1^{\mathrm{loc}}(J)$ solves (3.7) if and only if $v = S(v_0, g, h)$ is a fixed point of the solution operator $S = S_{A_0}$ of the linear problem (2.22) with $g = \mathbb{G}_{\digamma J}(v)$ and $h = \mathbb{H}_{\digamma J}(v)$. Hence, the compatibility condition $B_* v_0 = h(0) = \mathbb{H}_{\digamma J}(v)(0)$ must hold.

We now collect several properties of cutoffs (3.2) and (3.5) for $J \in \{\mathbb{R}, \mathbb{R}_+, \mathbb{R}_-\}$. The first remark implies that a solution $v \in \mathbb{E}_1^{\mathrm{loc}}(J)$ of (3.7) in fact satisfies (2.21) on $[a,b] \subset J$ if $\|v\|_{\mathbb{E}_1([t-2,t+2]\cap J)}$ is sufficiently small for each $t \in [a,b]$.

Remark 3.1. If $v \in \mathbb{E}_1^{\mathrm{loc}}(J)$ satisfies $\|v\|_{\mathbb{E}_1([t-2,t+2])} \le \eta$ for some $t \in J$ (where $|t| \ge 2$ if $J = \mathbb{R}_\pm$), then $\digamma_J(t,v) = 1$. This fact follows from the properties of γ and χ in (3.2). If $J = \mathbb{R}_\pm$ and $t \in J \cap [-2,2]$, then $\|v\|_{\mathbb{E}_1([t-2,t+2]\cap J)} \le (1+c_R)^{-1}\eta$ implies $\digamma_\pm(t,v) = 1$. Indeed, for $J = \mathbb{R}_+$, $t \in [0,2]$, and $s \in [t-\frac{1}{4}, t+\frac{1}{4}]$ we have

$$\begin{aligned}\|R_+v\|_{\mathbb{E}_1(J(s))} &\le \|R_+v\|_{\mathbb{E}_1(J(s)\cap\mathbb{R}_+)} + \|R_+v\|_{\mathbb{E}_1(J(s)\cap\mathbb{R}_-)} \\ &\le \|v\|_{\mathbb{E}_1(J(s)\cap\mathbb{R}_+)} + c_R\,\|v\|_{\mathbb{E}_1(J(s)\cap[0,1])} \\ &\le (1+c_R)\,\|v\|_{\mathbb{E}_1(J(s)\cap\mathbb{R}_+)} \le \eta\end{aligned}$$

due to (3.3) and the fact that $[0,1] \subset J(s)$ if $J(s)\cap\mathbb{R}_- \ne \emptyset$. The case $J = \mathbb{R}_-$ can be treated in the same way. $\Diamond$

Remark 3.2. For $v \in \mathbb{E}_1^{\mathrm{loc}}(\mathbb{R})$ and $J = \mathbb{R}$ the cutoff is time invariant. Indeed,

$$\digamma(t+t_0,v) = \int_{\mathbb{R}} \gamma(t-s)\chi(\|v\|_{\mathbb{E}_1([s+t_0-3/2,s+t_0+3/2])})\,ds = \digamma(t, v(\cdot+t_0)) \tag{3.8}$$

for $t,t_0 \in \mathbb{R}$. As a result, if v solves the cutoff problem (3.7) on $J = \mathbb{R}$ with $v(0) = v_0$, then $w = v(\cdot+t_0)$ solves the cutoff problem on $\mathbb{R}$ with $w(0) = v(t_0)$. In contrast to the case $J = \mathbb{R}$, for $J = \mathbb{R}_\pm$ the problem (3.7) is not translation invariant. $\Diamond$

Remark 3.3. Let us suppose that $\digamma(t_0,v) \ne 0$ for some $v \in \mathbb{E}_1^{\mathrm{loc}}(\mathbb{R})$, $t_0 \in J_n$, and $n \in \mathbb{Z}$. Then there exists a $t \in J_n'$ such that $\chi(N(t,v)) \ne 0$, and hence $\|v\|_{\mathbb{E}_1(J(t))} \le 2\eta$. As a result, $\|v\|_{\mathbb{E}_1(J_n')} \le 2\eta$ since $J_n' \subset J(t)$ for each $t \in J_n'$. Similarly, if $\digamma(t_0,v) \ne 0$ for some $t_0 \in J_n^*$, then $\|v\|_{\mathbb{E}_1(J_n^*)} \le 2\eta$. $\Diamond$

Remark 3.4. Assume that $v,u \in \mathbb{E}_1^{\mathrm{loc}}(J)$ and $t,s \in J$. Temporarily, we set $v = v_{\mathbb{R}}$ if $J = \mathbb{R}$. Then (3.2) and (3.3) imply the Lipschitz estimates

$$\begin{aligned}|\digamma_J(t,v) - \digamma_J(t,u)| &\le \sup_{|t-s|\le 1/4} |\chi(\|v_{\mathbb{R}}\|_{\mathbb{E}_1(J(s))}) - \chi(\|u_{\mathbb{R}}\|_{\mathbb{E}_1(J(s))})| \\ &\le 2\eta^{-1}\,\|v_{\mathbb{R}} - u_{\mathbb{R}}\|_{\mathbb{E}_1([t-7/4,t+7/4])} \le c\eta^{-1}\,\|v-u\|_{\mathbb{E}_1(J\cap[t-7/4,t+7/4])},\end{aligned} \tag{3.9}$$

$$|\digamma_J(t,v) - \digamma_J(s,v)| = \Big|\int_{\mathbb{R}} (\gamma(t-\tau) - \gamma(s-\tau))\chi(N(\tau,v_{\mathbb{R}}))\,d\tau\Big| \le c\,|t-s|, \tag{3.10}$$

where c does not depend on t,s,u,v or η. $\Diamond$

Remark 3.5. Let $v \in \mathbb{E}_1^{\mathrm{loc}}(\mathbb{R}_+)$, resp. $v \in \mathbb{E}_1^{\mathrm{loc}}(\mathbb{R}_-)$. Then $\digamma_+(t,v) = \digamma(t,v)$ for $t \ge 7/4$ and $\digamma_-(t,v) = \digamma(t,v)$ for $t \le -7/4$, respectively. Moreover, (3.8) holds for $t+t_0 \ge 7/4$, resp. $t+t_0 \le -7/4$, and $t,t_0 \in \mathbb{R}$. (Here $v(\cdot+t_0)$ is defined on $[-t_0,\infty)$, resp. on $(-\infty,-t_0]$.) $\Diamond$

We now consider the maps $\mathbb{G}_{\digamma J}$ and $\mathbb{H}_{\digamma J}$, see (3.6), on the spaces $\mathbb{E}_1(\mathbb{R}_\pm, \mp\alpha)$ and $\mathbb{E}_1(\alpha,-\beta)$, where $\alpha,\beta \ge 0$ (these values of α,β were not treated in Proposition 2.2). We start with a preliminary result concerning the Lipschitz properties.

Proposition 3.6. *Assume that (R) holds and $u_* \in X_1$ satisfies $B(u_*) = 0$. Take $\eta \in (0,d]$ and $\alpha,\beta \in [0,d]$ for some $d > 0$. Then the maps $\mathbb{G}_{\digamma_\pm} : \mathbb{E}_1(\mathbb{R}_\pm,\mp\alpha) \to \mathbb{E}_0(\mathbb{R}_\pm,\mp\alpha)$, $\mathbb{G}_{\digamma} : \mathbb{E}_1(\alpha,-\beta) \to \mathbb{E}_0(\alpha,-\beta)$, $\mathbb{H}_{\digamma_\pm} : \mathbb{E}_1(\mathbb{R}_\pm,\mp\alpha) \to \mathbb{F}(\mathbb{R}_\pm,\mp\alpha)$, and $\mathbb{H}_{\digamma} : \mathbb{E}_1(\alpha,-\beta) \to \mathbb{F}(\alpha,-\beta)$ are (globally) Lipschitz with the Lipschitz constant $\varepsilon(\eta)$ for a nondecreasing function ε converging to 0 as $\eta \to 0$ which does not depend on α or β. Moreover, $\mathbb{G}_{\digamma J}(0) = 0$ and $\mathbb{H}_{\digamma J}(0) = 0$ for $J \in \{\mathbb{R}_+,\mathbb{R}_-,\mathbb{R}\}$.*

Proof. We show the result only for the interval $J = \mathbb{R}_+$. The case $J = \mathbb{R}_-$ then follows by reflection, whereas the case $J = \mathbb{R}$ can be treated in a similar way as $J = \mathbb{R}_+$. In this proof we write $\digamma$ instead of $\digamma_+$ and v instead of R_+v etc. We take $\alpha, \eta \in (0, d]$ and $u, v \in \mathbb{E}_1(\mathbb{R}_+, -\alpha)$. In this proof ε and c do not depend on α, u, v, and c does not depend on η.

(a) We first address the Lipschitz property of $\mathbb{G}_\digamma$. We consider an interval J_n, $n \in \mathbb{N}_0$, and estimate $\mathbb{G}_\digamma(v) - \mathbb{G}_\digamma(u)$ on this interval. We may assume that $\digamma(t_0, v) \neq 0$ for some $t_0 \in J_n$, thus

$$\|v\|_{\mathbb{E}_1(J_n)} \leq \|v\|_{\mathbb{E}_1(J_n')} \leq 2\eta \tag{3.11}$$

by Remark 3.3. For $t \in J_n$, one obtains

$$\begin{aligned}|\mathbb{G}_\digamma(v)(t) - \mathbb{G}_\digamma(u)(t)|_0 \leq{}& |\digamma(t, v) - \digamma(t, u)|\ |G(v(t))|_0 \\ &+ |\digamma(t, u)|\ |G(v(t)) - G(u(t))|_0.\end{aligned}$$

In the second term in the right-hand side of the last inequality we may assume that $\digamma(t_0, u) \neq 0$ for some $t_0 \in J_n$ since otherwise this term is equal to zero on J_n. Remark 3.3 then shows that $\|u\|_{\mathbb{E}_1(J_n)} \leq \|u\|_{\mathbb{E}_1(J_n')} \leq 2\eta$. Estimate (3.9) in Remark 3.4 and Proposition 2.2 thus imply

$$\begin{aligned}&\int_{J_n} |\mathbb{G}_\digamma(v)(t) - \mathbb{G}_\digamma(u)(t)|_0^p e^{-\alpha t p}\, dt \\ &\leq c\eta^{-p} e^{-\alpha n p}\, \|u - v\|^p_{\mathbb{E}_1(J_n'')} \int_n^{n+1} |G(v(t))|_0^p\, dt + e^{-\alpha n p} \int_n^{n+1} |G(v(t)) - G(u(t))|_0^p\, dt \\ &\leq c\eta^{-p}\, \|e_{-\alpha}(v - u)\|^p_{\mathbb{E}_1(J_n'')}\, \varepsilon(\eta)^p \eta^p \\ &\qquad + e^{-\alpha n p} \sup_{\|w\|_{\mathbb{E}_1(J_n)} \leq 2\eta} \|\mathbb{G}'(w)\|^p_{\mathcal{B}(\mathbb{E}_1(J_n), \mathbb{E}_0(J_n))} \|v - u\|^p_{\mathbb{E}_1(J_n)} \\ &\leq c\varepsilon(\eta)^p\, \|e_{-\alpha}(v - u)\|^p_{\mathbb{E}_1(J_n'')},\end{aligned}$$

where c and ε do not depend on n. Now the Lipschitz estimate for $\mathbb{G}_\digamma$ easily follows, using also (3.3) and (3.4).

(b) We establish the Lipschitz property of $\mathbb{H}_\digamma$. We deduce the inequality

$$\|e_{-\alpha}(\mathbb{H}_\digamma(v) - \mathbb{H}_\digamma(u))\|_{L_p(\mathbb{R}_+; Y_1)} \leq c\varepsilon(\eta) \|v - u\|_{\mathbb{E}_1(\mathbb{R}_+, -\alpha)} \tag{3.12}$$

similarly to the proof given in part (a). In order to estimate $e_{-\alpha}(\mathbb{H}_\digamma(v) - \mathbb{H}_\digamma(u))$ in $W^{\kappa_j}(\mathbb{R}_+; Y_0)$, we fix a number $j \in \{1, \cdots, m\}$, and write H, $\mathbb{F}$ and κ instead of H_j, $\mathbb{F}_j$ and κ_j. Let $t \in J_n$, $n \in \mathbb{N}_0$, and $|t - s| \leq 1/8$. Again, we may assume that $\digamma(t_0, v) \neq 0$ for some $t_0 \in J_n$, so that (3.11) holds by Remark 3.3. Note that $s \in J_n^* \subset J_n'$. We further split:

$$\begin{aligned}\Delta(t, s) :={}& \mathbb{H}_\digamma(v)(t) - \mathbb{H}_\digamma(v)(s) - (\mathbb{H}_\digamma(u)(t) - \mathbb{H}_\digamma(u)(s)) \\ ={}& \big[\digamma(t, v) - \digamma(s, v) - (\digamma(t, u) - \digamma(s, u))\big]\, H(v(t)) \\ &+ \digamma(t, u)\, \big[H(v(t)) - H(v(s)) - (H(u(t)) - H(u(s)))\big] \\ &+ \big(\digamma(s, v) - \digamma(s, u)\big)\, \big(H(v(t)) - H(v(s))\big) \\ &+ \big(\digamma(t, u) - \digamma(s, u)\big)\, \big(H(v(s)) - H(u(s))\big) =: S_1 + S_2 + S_3 + S_4.\end{aligned}$$

In the expression S_1 the term in square brackets satisfies the estimate

$$\Big| \int_{\mathbb{R}} (\gamma(t - \tau) - \gamma(s - \tau))(\chi(N(\tau, v)) - \chi(N(\tau, u)))\, d\tau \Big|$$

$$
\begin{aligned}
&\le c\eta^{-1}|t-s| \sup_{\substack{|\tau-t|\le 1/4\\ |\tau-s|\le 1/4}} |N(\tau,v)-N(\tau,u)| \\
&\le c\eta^{-1}|t-s| \sup_{\tau\in[n-1/2,n+3/2]} \|v-u\|_{\mathbb{E}_1(J(\tau))} \le c\eta^{-1}|t-s|\,\|v-u\|_{\mathbb{E}_1(J_n'')}.
\end{aligned}
$$

By means of (2.13), (3.11), and Proposition 2.2, we estimate:

$$
\sup_{t\in J_n} |H(v(t))|_{Y_0} \le c\,\|\mathbb{H}(v)\|_{\mathbb{F}(J_n)} \le c\varepsilon(\eta)\,\|v\|_{\mathbb{E}_1(J_n)} \le c\varepsilon(\eta)\eta.
$$

As a result,

$$
|S_1|_{Y_0} \le c\varepsilon(\eta)|t-s|\,\|v-u\|_{\mathbb{E}_1(J_n'')},
$$

and thus

$$
\begin{aligned}
\Big(\iint\limits_{\substack{|t-s|\le 1/8\\ t\in J_n}} e^{-\alpha tp} \frac{|S_1|_{Y_0}^p}{|t-s|^{1+\kappa p}}\,dt\,ds\Big)^{\frac{1}{p}} &\le c\varepsilon(\eta)e^{-\alpha n}\,\|v-u\|_{\mathbb{E}_1(J_n'')} \\
&\le c\varepsilon(\eta)\,\|e_{-\alpha}(v-u)\|_{\mathbb{E}_1(J_n'')}.
\end{aligned}
$$

Next, we treat S_2. We may assume that $\digamma(t_0,u)\neq 0$ for some $t_0\in J_n$ (otherwise $S_2=0$). Hence, $\|u\|_{\mathbb{E}_1(J_n')}\le 2\eta$ by Remark 3.3. Using also (3.11) and Proposition 2.2, we derive

$$
\begin{aligned}
\Big(\iint\limits_{\substack{|t-s|\le 1/8\\ t\in J_n}} e^{-\alpha tp} \frac{|S_2|_{Y_0}^p}{|t-s|^{1+\kappa p}}\,dt\,ds\Big)^{\frac{1}{p}} &\le ce^{-\alpha n}\Big[\mathbb{H}(v)-\mathbb{H}(u)\Big]_{W_p^\kappa(J_n';Y_0)} \\
&\le ce^{-\alpha n} \sup_{\|w\|_{\mathbb{E}_1(J_n')}\le 2\eta} \|\mathbb{H}'(w)\|_{\mathcal{B}(\mathbb{E}_1(J_n'),\mathbb{F}(J_n'))}\,\|v-u\|_{\mathbb{E}_1(J_n')} \\
&\le c\varepsilon(\eta)\,\|e_{-\alpha}(v-u)\|_{\mathbb{E}_1(J_n')}.
\end{aligned}
$$

Dealing with S_3, we note that Remark 3.4 further yields:

$$
|S_3|_{Y_0} \le c\eta^{-1}\,\|v-u\|_{\mathbb{E}_1(J_n'')}\,|H(v(t))-H(v(s))|_0\,.
$$

Therefore, we obtain

$$
\begin{aligned}
\Big(\iint\limits_{\substack{|t-s|\le 1/8\\ t\in J_n}} e^{-\alpha tp} \frac{|S_3|_{Y_0}^p}{|t-s|^{1+\kappa p}}\,dt\,ds\Big)^{\frac{1}{p}} &\le ce^{-\alpha n}\eta^{-1}\,\|v-u\|_{\mathbb{E}_1(J_n'')}\big[\mathbb{H}(v)\big]_{W_p^\kappa(J_n';Y_0)} \\
&\le c\eta^{-1}\,\|e_{-\alpha}(v-u)\|_{\mathbb{E}_1(J_n'')}\ \varepsilon(\eta)\,\|v\|_{\mathbb{E}_1(J_n')} \le c\varepsilon(\eta)\,\|e_{-\alpha}(v-u)\|_{\mathbb{E}_1(J_n'')}
\end{aligned}
$$

due to (3.11) and Proposition 2.2. Finally, we estimate the expression S_4. We may assume that $\digamma(t_0,u)\neq 0$ for some $t_0\in J_n^*$ (otherwise $S_4=0$). Then $\|u\|_{\mathbb{E}_1(J_n^*)}\le 2\eta$ due to Remark 3.3. So (3.10), (2.13), (3.11), and Proposition 2.2 lead to

$$
\begin{aligned}
\Big(\iint\limits_{\substack{|t-s|\le 1/8\\ t\in J_n}} e^{-\alpha tp} \frac{|S_4|_{Y_0}^p}{|t-s|^{1+\kappa p}}\,dt\,ds\Big)^{\frac{1}{p}} &\le ce^{-\alpha n} \sup_{s\in J_n^*} |H(v(s))-H(u(s))|_{Y_0} \\
&\le ce^{-\alpha n}\,\|\mathbb{H}(v)-\mathbb{H}(u)\|_{\mathbb{F}(J_n^*)} \\
&\le ce^{-\alpha n} \sup_{\|w\|_{\mathbb{E}_1(J_n^*)}\le 2\eta} \|\mathbb{H}'(w)\|_{\mathcal{B}(\mathbb{E}_1(J_n^*),\mathbb{F}(J_n^*))}\,\|v-u\|_{\mathbb{E}_1(J_n^*)} \\
&\le c\varepsilon(\eta)\,\|e_{-\alpha}(v-u)\|_{\mathbb{E}_1(J_n^*)}.
\end{aligned}
$$

Summing up, we arrive at the inequality

$$\Big(\iint\limits_{\substack{|t-s|\le 1/8\\ t\in J_n}} e^{-\alpha tp} \frac{|\Delta(t,s)|^p_{Y_0}}{|t-s|^{1+\kappa p}}\, dt\, ds\Big)^{\frac1p} \le c\varepsilon(\eta)\, \|e_{-\alpha}(v-u)\|_{\mathbb{E}_1(J_n'')}.$$

Therefore, a variant of Lemma 11 of [12] and estimates (3.12), (3.3) and (3.4) imply:

$$\begin{aligned}
&\Big[e_{-\alpha}(\mathbb{H}_F(v)-\mathbb{H}_F(u))\Big]_{W^\kappa_p(\mathbb{R}_+;Y_0)}\\
&\quad\le c\,\|e_{-\alpha}(\mathbb{H}_F(v)-\mathbb{H}_F(u))\|_{L_p(\mathbb{R}_+;Y_0)} + c\Big(\sum_{n=0}^{\infty} \iint\limits_{\substack{|t-s|\le 1/8\\ t\in J_n}} e^{-\alpha tp} \frac{|\Delta(t,s)|^p_{Y_0}}{|t-s|^{1+\kappa p}}\, dt\, ds\Big)^{\frac1p}\\
&\quad\le c\varepsilon(\eta)\, \|v-u\|_{\mathbb{E}_1(\mathbb{R}_+,-\alpha)} + \Big(\sum_{n=0}^{\infty} c\varepsilon(\eta)^p \|e_{-\alpha}(v-u)\|^p_{\mathbb{E}_1(J_n'')}\Big)^{\frac1p}\\
&\quad\le c\varepsilon(\eta)\, \|v-u\|_{\mathbb{E}_1(\mathbb{R}_+,-\alpha)}. \qquad (3.13)
\end{aligned}$$

The Lipschitz property of $\mathbb{H}_F$ is a direct consequence of (3.12) and (3.13). □

Remark 3.7. Let $J\subset\mathbb{R}_+$ be a closed interval of length larger than 2 and $\delta\in[a,b]$ for some $b>a\ge 0$. Given $r>0$, we consider functions $v\in\mathbb{E}_1(J,\delta)$ such that $\|v\|_{\mathbb{E}_1([s,s+2])}\le r$ for all intervals $[s,s+2]\subset J$. For such v and w, we have

$$\|\mathbb{G}(w)-\mathbb{G}(v)\|_{\mathbb{E}_0(J,\delta)} \le \varepsilon(r)\|w-v\|_{\mathbb{E}_1(J,\delta)}\,,\ \ \|\mathbb{H}(w)-\mathbb{H}(v)\|_{\mathbb{F}(J,\delta)} \le \varepsilon(r)\|w-v\|_{\mathbb{E}_1(J,\delta)}\,,$$

where ε is a nondecreasing function with $\varepsilon(r)\to 0$ as $r\to 0$ and ε does not depend on v, w, J, or δ. Indeed, to show this fact, one extends v to $\tilde v\in\mathbb{E}_1(\mathbb{R}_+,\delta)$ such that $\|\tilde v\|_{\mathbb{E}_1(\mathbb{R}_+,\delta)}\le c\,\|v\|_{\mathbb{E}_1(J,\delta)}$ and $\|\tilde v\|_{\mathbb{E}_1([s,s+2])}\le cr$ for all $[s,s+2]\subset J$, where the constant c does not depend on J, δ, v, or s. As in the proof of Proposition 3.6 one now treats the intervals J_n separately using Proposition 2.2.

Next, we want to establish the continuous differentiability of $\mathbb{G}_F$ and $\mathbb{H}_F$ in certain spaces. We start with the differentiability of F. We first observe that, for a measure space (M,μ), the map $q(u)=\int_M |u|^p d\mu$ belongs to $C^1(L_p(M,d\mu))$, and that its derivative at $u\in L_p(M,d\mu)$ is given by $\langle v, q'(u)\rangle = \int_M p\,\mathrm{Re}(u\overline{v})|u|^{p-2}\,d\mu$. This fact implies that for $t\in\mathbb{R}$ the map $u\longmapsto N(t,u)=\|u\|_{\mathbb{E}_1(J(t))}$, see (3.1), is continuously differentiable on $\mathbb{E}_1(J(t))\setminus\{0\}$, and its derivative $N'(t,u)\in\mathbb{E}_1(J(t))^*$ is given by

$$\begin{aligned}
\langle v, N'(t,u)\rangle = \|u\|^{1-p}_{\mathbb{E}_1(J(t))}\Big[&\int_{J(t)}\int_\Omega \mathrm{Re}(u\overline{v})\,|u|^{p-2}dx\,ds + \int_{J(t)}\int_\Omega \mathrm{Re}(\dot u\dot{\overline{v}})\,|\dot u|^{p-2}dx\,ds\\
&+\sum_{|\alpha|=2m}\int_{J(t)}\int_\Omega \mathrm{Re}(\partial^\alpha u\,\partial^\alpha\overline{v})\,|\partial^\alpha u|^{p-2}\,dx\,ds\Big],
\end{aligned}$$

where $v\in\mathbb{E}_1(J(t))$. Observe that $\|N'(t,u)\|_{\mathbb{E}_1(J(t))^*}\le c$ for a constant depending only on m. Take $u,v\in\mathbb{E}_1([t-2,t+2])$ and $|t-s|\le 1/4$, where $u\ne 0$ and $\|v\|_{\mathbb{E}_1}<\|u\|_{\mathbb{E}_1}$. Denoting the restrictions of u and v to $J(s)\subset[t-2,t+2]$ by the same symbols, we further deduce

$$\begin{aligned}
|N(s,u+v)-N(s,u)-\langle N'(s,u),v\rangle| &= \Big|\int_0^1 \langle N'(s,u+\theta v)-N'(s,u),v\rangle\, d\theta\Big|\\
&\le c\varepsilon(\|v\|_{\mathbb{E}_1(J(s))})\|v\|_{\mathbb{E}_1(J(s))} \le c\varepsilon(\|v\|_{\mathbb{E}_1([t-2,t+2])})\,\|v\|_{\mathbb{E}_1([t-2,t+2])}.
\end{aligned}$$

Here c and ε do not depend on t and s since $N(\tau,u) = N(0,u(\cdot+\tau))$. As a result, the map $\mathbb{E}_1([t-2,t+2]) \ni u \mapsto \digamma(t,u)$ is C^1 with the derivative $\digamma'(t,u) = [\gamma * \chi'(N(\cdot,u))N'(\cdot,u)](t)$, and the maps

$$\Gamma_0 : u \mapsto \chi(N(\cdot,u)) \quad \text{and} \quad \Gamma_1 : u \mapsto \digamma(\cdot,u) \quad \text{belong to} \quad C^1(\mathbb{E}_1(J''); C(J)) \tag{3.14}$$

where $J = [a,b]$ and $J'' = [a-2, b+2]$. (Here we set $N'(t,0) = 0$ and note that $\digamma(t,u) = 1$ and thus $\digamma'(t,u) = 0$ provided $\|u\|_{\mathbb{E}_1([t-2,t+2])} < \eta$.) We further have

$$|\langle \digamma'(t,u), v\rangle| \le c\eta^{-1} \|v\|_{\mathbb{E}_1([t-2,t+2])}, \tag{3.15}$$

$$|\langle \digamma'(t,u) - \digamma'(s,u), v\rangle| \le c\eta^{-1}|t-s| \, \|v\|_{\mathbb{E}_1([t-2,t+2])} \tag{3.16}$$

for $u,v \in \mathbb{E}_1([t-2,t+2])$, $t \in \mathbb{R}$, $|t-s| \le 1/4$, and constants c independent of t,s,u,v,η. Observe that the cutoffs $\digamma_\pm(t,v) = \digamma(t, R_\pm v)$ on $\mathbb{R}_\pm$ have the analogous differentiability properties.

Given $\alpha, \beta \ge 0$ and $u \in \mathbb{E}_1(\alpha, -\beta)$, we introduce the linear operators $\mathbb{G}'_\digamma$ and $\mathbb{H}'_\digamma$ acting on $v \in \mathbb{E}_1(\alpha,-\beta)$ by the formulas

$$[\mathbb{G}'_\digamma(u)v](t) = \langle v, \digamma'(t,u)\rangle \, G(u(t)) + \digamma(t,u)G'(u(t))v(t), \tag{3.17}$$

$$[\mathbb{H}'_\digamma(u)v](t) = \langle v, \digamma'(t,u)\rangle \, H(u(t)) + \digamma(t,u)H'(u(t))v(t). \tag{3.18}$$

Here G and H were defined in (2.17) and(2.18), and the brackets denote the scalar product in $\mathbb{E}_1(J(t))$ applied to the restriction of v to the interval $J(t)$. We also set

$$[\mathbb{G}'_{\digamma\pm}(u)v](t) = [\mathbb{G}'_\digamma(R_\pm u)R_\pm v](t), \qquad [\mathbb{H}'_{\digamma\pm}(u)v](t) = [\mathbb{H}'_\digamma(R_\pm u)R_\pm v](t) \tag{3.19}$$

for $t \in \mathbb{R}_\pm$ and $u,v \in \mathbb{E}_1(\mathbb{R}_+, -\alpha)$ in the case $J = \mathbb{R}_+$, respectively, $u,v \in \mathbb{E}_1(\mathbb{R}_-, \alpha)$ in the case $J = \mathbb{R}_-$.

The maps $\mathbb{G}_\digamma$ and $\mathbb{H}_\digamma$ are not differentiable if the range space has the *same* weight function. But, as we will see in the next proposition, they become C^1 maps with the derivatives $\mathbb{G}'_\digamma$ and $\mathbb{G}'_\digamma$ given in (3.17), (3.18) if we take a *smaller* weight function in the range space, cf. [21].

Proposition 3.8. *Assume that (R) holds and that $u_* \in X_1$ satisfies $B(u_*) = 0$. Let $\eta \in (0,d]$, $0 \le \alpha \le \beta \le d$, and $0 \le \alpha' \le \beta' \le d$ for some $d > 0$. Define the operators $\mathbb{G}'_\digamma$, $\mathbb{G}'_{\digamma\pm}$ $\mathbb{H}'_\digamma$, and $\mathbb{H}'_{\digamma\pm}$ by (3.17), (3.18) and (3.19), respectively, where η is the paramater for the cutoff $\digamma$. Then the following assertions hold.*

(a) The operators $\mathbb{G}'_\digamma(u) : \mathbb{E}_1(\alpha,-\alpha') \to \mathbb{E}_0(\beta,-\beta')$, $\mathbb{H}'_\digamma(u) : \mathbb{E}_1(\alpha,-\alpha') \to \mathbb{F}(\beta,-\beta')$, $\mathbb{G}'_{\digamma\pm}(u) : \mathbb{E}_1(\mathbb{R}_\pm, \mp\alpha) \to \mathbb{E}_0(\mathbb{R}_\pm, \mp\beta)$, and $\mathbb{H}'_{\digamma\pm}(u) : \mathbb{E}_1(\mathbb{R}_\pm, \mp\alpha) \to \mathbb{F}(\mathbb{R}_\pm, \mp\beta)$ are all bounded with the norms $\varepsilon(\eta)$, where ε is a nondecreasing function converging to 0 as $\eta \to 0$ which does not depend on $u, \alpha, \alpha', \beta, \beta'$.

(b) If $\beta > \alpha$ and $\beta' > \alpha'$, then the maps $\mathbb{G}_\digamma : \mathbb{E}_1(\alpha,-\alpha') \to \mathbb{E}_0(\beta,-\beta')$, $\mathbb{H}_\digamma : \mathbb{E}_1(\alpha,-\alpha') \to \mathbb{F}(\beta,-\beta')$, $\mathbb{G}_{\digamma\pm} : \mathbb{E}_1(\mathbb{R}_\pm,\mp\alpha) \to \mathbb{E}_0(\mathbb{R}_\pm,\mp\beta)$, and $\mathbb{H}_{\digamma\pm} : \mathbb{E}_1(\mathbb{R}_\pm,\mp\alpha) \to \mathbb{F}(\mathbb{R}_\pm,\mp\beta)$ are continuously differentiable and the operators $\mathbb{G}'_\digamma$ $\mathbb{H}'_\digamma$, $\mathbb{G}'_{\digamma\pm}$, and $\mathbb{H}'_{\digamma\pm}$, respectively, are their derivatives. Moreover, $\mathbb{G}'_{\digamma J}(0) = 0$ and $\mathbb{H}'_{\digamma J}(0) = 0$ for $J \in \{\mathbb{R}_+, \mathbb{R}_-, \mathbb{R}\}$.

Proof. As in the proof of Proposition 3.6 we restrict ourselves to the case $J = \mathbb{R}_+$, and we write $\digamma$ instead of $\digamma_+$ and v instead of $R_+ v$.

(a) Norm estimates for $\mathbb{G}'_\digamma$ and $\mathbb{H}'_\digamma$. Since the spaces with the exponential weights form a scale, it is enough to give a proof for $a := \beta = \alpha \in (0,d]$. Let $t \in J_n$ for some $n \in \mathbb{N}_0$ and $\eta \in (0,d]$. If $\digamma(t_0,u) = 0$ and $\digamma'(t_0,u) = 0$ for all $t_0 \in J_n$,

then $\mathbb{G}_F'(u)v = 0$ on J_n. Otherwise, we have $\|u\|_{\mathbb{E}_1(J_n')} \le 2\eta$, cf. Remark 3.3. So Proposition 2.2, (3.15) and (3.3) yield

$$\|\mathbb{G}_F'(u)v\|_{\mathbb{E}_0(J_n)} \le c\eta^{-1}\|v\|_{\mathbb{E}_1(J_n'')}\,\varepsilon(\eta)\eta + \varepsilon(\eta)\,\|v\|_{\mathbb{E}_1(J_n)} \le c\varepsilon(\eta)\,\|v\|_{\mathbb{E}_1(J_n'')},$$

$$\begin{aligned}\|\mathbb{G}_F'(u)v\|^p_{\mathbb{E}_0(\mathbb{R}_+,-a)} &\le \sum_{n=0}^{\infty} e^{-anp}\,\|\mathbb{G}_F'(u)v\|^p_{\mathbb{E}_0(J_n)} \le c\varepsilon(\eta)^p \sum_{n=0}^{\infty} e^{-anp}\,\|v\|^p_{\mathbb{E}_1(J_n'')} \\ &\le c\varepsilon(\eta)^p\,\|v\|^p_{\mathbb{E}_1(\mathbb{R}_+,-a)}, \end{aligned} \tag{3.20}$$

proving assertion (a) for G_F'. Here and below in the proof of assertion (a) all constants are uniform for u, v, a, and η, but may depend on d. Starting the proof for $\mathbb{H}_F'$, as in (3.20), one obtains that

$$\|e_{-a}\mathbb{H}_F'(u)v\|^p_{L_p(\mathbb{R}_+;Y_1)} \le c\varepsilon(\eta)\,\|v\|_{\mathbb{E}_1(\mathbb{R}_+,-a)}\,.$$

Further, let $|t-s| \le 1/8$ and $t \in J_n$. We fix $j \in \{1,\cdots,m\}$ and write H, $\mathbb{F}$ and κ instead of H_j, $\mathbb{F}_j$ and κ_j. It holds

$$\begin{aligned}\Delta(t,s) &:= \mathbb{H}_F'(u)v(t) - \mathbb{H}_F'(u)v(s) \\ &= \langle F'(t,u) - F'(s,u), v\rangle H(u(t)) + \langle F'(s,u), v\rangle[H(u(t)) - H(u(s))] \\ &\quad + [F(t,u) - F(s,u)]H'(u(t))v(t) + F(s,u)[H'(u(t))v(t) - H'(u(s))v(s)] \\ &=: S_1 + S_2 + S_3 + S_4. \end{aligned}$$

As before we can assume that $\|u\|_{\mathbb{E}_1(J_n^*)} \le 2\eta$, cf. Remark 3.3. By means of (3.16), (3.15), (3.10), (2.13) and Proposition 2.2, we estimate

$$\begin{aligned}&|S_1|_{Y_0} \le c\eta^{-1}|t-s|\,\|v\|_{\mathbb{E}_1(J_n'')}\|\mathbb{H}(u)\|_{C(J_n;Y_0)} \le c\varepsilon(\eta)|t-s|\,\|v\|_{\mathbb{E}_1(J_n'')}\,, \\ &|S_2|_{Y_0} \le c\eta^{-1}\,\|v\|_{\mathbb{E}_1(J_n'')}\,|H(u(t)) - H(u(s))|_{Y_0}\,, \\ &|S_3|_{Y_0} \le c\,|t-s|\,\|\mathbb{H}'(u)v\|_{C(J_n;Y_0)} \le c\,|t-s|\,\|\mathbb{H}'(u)v\|_{\mathbb{F}(J_n)} \le c\varepsilon(\eta)|t-s|\,\|v\|_{\mathbb{E}_1(J_n)}, \\ &|S_4|_{Y_0} \le c\,|H'(u(t))v(t) - H'(u(s))v(s)|_{Y_0}. \end{aligned}$$

Using Proposition 2.2 once more, these inequalities lead to

$$\begin{aligned}&\Big(\int_{J_n}\int_{|t-s|\le 1/8} e^{-atp}\,\frac{|\Delta(t,s)|^p_{Y_0}}{|t-s|^{1+\kappa p}}\,ds\,dt\Big)^{1/p} \\ &\quad \le e^{-an}\big[c\varepsilon(\eta)\,\|v\|_{\mathbb{E}_1(J_n'')} + c\eta^{-1}\,\|v\|_{\mathbb{E}_1(J_n'')}[\mathbb{H}(u)]_{W_p^\kappa(J_n^*;Y_0)} + c\,\|\mathbb{H}'(u)v\|_{W_p^\kappa(J_n^*;Y_0)}\big] \\ &\quad \le c\varepsilon(\eta)\|e_{-a}v\|_{\mathbb{E}_1(J_n'')}. \end{aligned}$$

A slight variation of Lemma 11 from [12] and (3.3) now imply that

$$[e_{-a}\mathbb{H}_F'(u)v]_{W_p^\kappa(\mathbb{R}_+;Y_0)} \le c\varepsilon(\eta)\,\|v\|_{\mathbb{E}_1(\mathbb{R}_+,-a)},$$

concluding the proof of assertion (a) in Proposition 3.8.

(b) We now assume that $\beta > \alpha$. We fix $u \in \mathbb{E}_1(\mathbb{R}_+,-\alpha)$ and $\eta > 0$. The constants below do not depend on $v \in \mathbb{E}_1(\mathbb{R}_+,-\alpha)$, but possibly on u, η, or $\beta - \alpha$.

(1) Differentiability of $\mathbb{G}_F : \mathbb{E}_1(\mathbb{R}_+,-\alpha) \to \mathbb{E}_0(\mathbb{R}_+,-\beta)$. We have to estimate

$$\begin{aligned}\Delta_G(t) &= F(t,u+v)G(u(t)+v(t)) - F(t,u)G(u(t)) \\ &\quad - \langle F'(t,u), v\rangle G(u(t)) - F(t,u)G'(u(t))v(t) \end{aligned}$$

for $t \ge 0$ and $v \in \mathbb{E}_1(\mathbb{R}_+,-\alpha)$. We first consider $t \ge n_0 \ge 2$ for some $n_0 \in \mathbb{N}$ to be fixed below. Proposition 3.6 then yields

$$\|e_{-\beta}\Delta_G\|_{\mathbb{E}_0([n_0,\infty))} \le e^{(\alpha-\beta)n_0}\big[\|e_{-\alpha}(F(\cdot,u+v)\mathbb{G}(u+v) - F(\cdot,u)\mathbb{G}(u))\|_{\mathbb{E}_0(\mathbb{R}_+)}$$

$$+ \|e_{-\alpha}\langle F'(\cdot,u),v\rangle\mathbb{G}(u) + e_{-\alpha}F(\cdot,u)\mathbb{G}'(u)v\|_{\mathbb{E}_0([n_0,\infty))}\big]$$
$$\le e^{(\alpha-\beta)n_0}\big[\varepsilon(\eta)\,\|v\|_{\mathbb{E}_0(\mathbb{R}_+,-\alpha)} + \|S_1+S_2\|_{\mathbb{E}_0([n_0,\infty))}\big],$$

where we set $S_1 = e_{-\alpha}\langle F'(\cdot,u),v\rangle\mathbb{G}(u)$ and $S_2 = e_{-\alpha}F(\cdot,u)\mathbb{G}'(u)v$. Let $t\in J_n$ for some $n\ge n_0$. If $F(t_0,u)=0$ and $F'(t_0,u)=0$ for all $t_0\in J_n$, then $S_1=S_2=0$ on J_n. Otherwise, we have $\|u\|_{\mathbb{E}_1(J_n')}\le 2\eta$, cf. Remark 3.3. So we deduce from (3.15) and Proposition 2.2 that

$$\|S_1+S_2\|_{\mathbb{E}_0(J_n)} \le ce^{-\alpha n}\varepsilon(\eta)\,\|v\|_{\mathbb{E}_1(J_n'')} \le c\,\|e_{-\alpha}v\|_{\mathbb{E}_0(J_n'')}.$$

As a result,

$$\|e_{-\beta}\Delta_G\|_{\mathbb{E}_0([n_0,\infty))} \le ce^{(\alpha-\beta)n_0}\|v\|_{\mathbb{E}_1(\mathbb{R}_+,-\alpha)}, \tag{3.21}$$

where c does not depend on n_0. Let $\epsilon>0$ be given. Recalling that $\beta>\alpha$, we fix $n_0=n_0(\epsilon)\ge 2$ such that the right hand side of (3.21) is less than $\epsilon\|v\|_{\mathbb{E}_1(\mathbb{R}_+,-\alpha)}$.

Second, we treat the interval $[0,n_0]$ for the number n_0 just fixed. Using Proposition 2.2, (3.14),(3.9) and (3.3), we infer that

$$\begin{aligned}
\|e_{-\beta}\Delta_G\|_{\mathbb{E}_0([0,n_0])} &\le \|\mathbb{G}(u+v)-\mathbb{G}(u)-\mathbb{G}'(u)v\|_{\mathbb{E}_0([0,n_0])}\\
&\quad + \|(F(\cdot,u+v)-F(\cdot,u)-\langle F'(\cdot,u),v\rangle)\,\mathbb{G}(u)\|_{\mathbb{E}_0([0,n_0])}\\
&\quad + \|(F(\cdot,u+v)-F(\cdot,u))\,\mathbb{G}'(u)v\|_{\mathbb{E}_0([0,n_0])}\\
&\le c\varepsilon(\|v\|_{\mathbb{E}_1([0,n_0+2])})\,\|v\|_{\mathbb{E}_1([0,n_0+2])}\\
&\le c\varepsilon(\|v\|_{\mathbb{E}_1(\mathbb{R}_+,-\alpha)})\,\|v\|_{\mathbb{E}_1(\mathbb{R}_+,-\alpha)}\,,
\end{aligned} \tag{3.22}$$

where c and ε may depend on n_0. Here and below we often use the boundedness of the restriction operator from $\mathbb{E}_1(\mathbb{R}_+,-\alpha)$ to $\mathbb{E}_1([0,b])$, see (3.4). The asserted differentiability of $\mathbb{G}_F$ now follows from (3.21) and (3.22).

(2) Differentiability of $\mathbb{H}_F:\mathbb{E}_1(\mathbb{R}_+,-\alpha)\to\mathbb{F}(\mathbb{R}_+,-\beta)$. This time we set

$$\begin{aligned}
\Delta_H(t) &= F(t,u+v)H(u(t)+v(t)) - F(t,u)H(u(t))\\
&\quad - \langle F'(t,u),v\rangle H(u(t)) - F(t,u)H'(u(t))v(t)
\end{aligned}$$

for $t\ge 0$ and $v\in\mathbb{E}_1(\mathbb{R}_+,-\alpha)$. As above in part (1), we obtain

$$\|e_{-\beta}\Delta_H\|_{L_p(\mathbb{R}_+;Y_1)} \le c\epsilon\,\|v\|_{\mathbb{E}_1(\mathbb{R}_+,-\alpha)} \tag{3.23}$$

for each given $\epsilon>0$ and all v with $\|v\|_{\mathbb{E}_1(\mathbb{R}_+,-\alpha)}\le r_\epsilon$. In the estimates for $[\,\cdot\,]_{W_p^{\kappa_j}}$, cf. (2.12), we fix $j\in\{1,\cdots,m\}$ and write H, $\mathbb{F}$ and κ instead of H_j, $\mathbb{F}_j$ and κ_j.

(i) We first consider $t\ge n_0$ and $|t-s|\le 1/8$ for some $n_0\in\mathbb{N}$ with $n_0\ge 2$ to be fixed below, and split:

$$\begin{aligned}
\Delta_H(t)-\Delta_H(s) &= [F(t,u+v)H(u(t)+v(t)) - F(s,u+v)H(u(s)+v(s))\\
&\qquad - (F(t,u)H(u(t)) - F(s,u)H(u(s)))]\\
&\quad - [F(t,u)H'(u(t))v(t) - F(s,u)H'(u(s))v(s)]\\
&\quad - [\langle F'(t,u),v\rangle H(u(t)) - \langle F'(s,u),v\rangle H(u(s))] =: S_1+S_2+S_3.
\end{aligned}$$

The Lipschitz estimate in Proposition 3.6 shows that

$$\begin{aligned}
&\Big(\int_{n_0}^\infty\int_{|t-s|\le 1/8} e^{-\beta tp}\frac{|S_1|_{Y_0}^p}{|t-s|^{1+\kappa p}}\,ds\,dt\Big)^{\frac1p}\\
&\qquad\le e^{(\alpha-\beta)n_0}[e_{-\alpha}(\mathbb{H}_F(u+v)-\mathbb{H}_F(u))]_{W_p^\kappa(\mathbb{R}_+;Y_0)} \le ce^{(\alpha-\beta)n_0}\,\|v\|_{\mathbb{E}_1(\mathbb{R}_+,-\alpha)}.
\end{aligned}$$

Take $t \in J_n$ for some $n \geq n_0$. If $F(t_0, u) = 0$ (resp. $F'(t_0, u) = 0$) for all $t_0 \in J_n^*$, then $S_2 = 0$ (resp. $S_3 = 0$) on J_n^*. Otherwise, $\|u\|_{\mathbb{E}_1(J_n*)} \leq 2\eta$ for J_n^*, cf. Remark 3.3. We then deduce from Proposition 2.2, (2.13), (3.10), (3.16) and (3.15) that

$$
\begin{aligned}
|e^{-\beta t} S_2|_{Y_0} &\leq e^{(\alpha-\beta)n_0}\big\{e^{-\alpha n}\,|F(t,u) - F(s,u)|\;\|\mathbb{H}'(u)v\|_{C(J_n;Y_0)} \\
&\qquad + e^{-\alpha n}\,[H'(u(t))v(t) - H'(u(s))v(s)]\big\} \\
&\leq e^{(\alpha-\beta)n_0}\big\{c|t-s|\,\|e_{-\alpha}v\|_{\mathbb{E}_1(J_n)} + e^{-\alpha n}\,[H'(u(t))v(t) - H'(u(s))v(s)]\big\},
\end{aligned}
$$

$$
\begin{aligned}
\Big(\int_{J_n}\int_{|t-s|\leq 1/8} & e^{-\beta t}\frac{|S_2|_{Y_0}^p}{|t-s|^{1+\kappa p}}\,ds\,dt\Big)^{\frac{1}{p}} \\
&\leq ce^{(\alpha-\beta)n_0}\big(\|e_{-\alpha}v\|_{\mathbb{E}_1(J_n)} + e^{-\alpha n}\,[\mathbb{H}'(u)v]_{W_p^\kappa(J_n^*;Y_0)}\big) \\
&\leq ce^{(\alpha-\beta)n_0}\,\|e_{-\alpha}v\|_{\mathbb{E}_1(J_n^*)}\,,
\end{aligned}
$$

$$
\begin{aligned}
|e^{-\beta t} S_3|_{Y_0} &\leq e^{(\alpha-\beta)n_0}e^{-\alpha n}\big(|\langle F'(t,u) - F'(s,u), v\rangle|\;\|\mathbb{H}(u)\|_{C(J_n;Y_0)} \\
&\qquad + |\langle F'(s,u), v\rangle|\;|H(u(t)) - H(u(s))|_{Y_0}\big) \\
&\leq ce^{(\alpha-\beta)n_0}\,\|e_{-\alpha}v\|_{\mathbb{E}_1(J_n'')}\big(|t-s| + |H(u(t)) - H(u(s))|_{Y_0}\big),
\end{aligned}
$$

$$
\begin{aligned}
\Big(\int_{J_n}\int_{|t-s|\leq 1/8} & e^{-\beta tp}\frac{|S_3|_{Y_0}^p}{|t-s|^{1+\kappa p}}\,ds\,dt\Big)^{\frac{1}{p}} \\
&\leq ce^{(\alpha-\beta)n_0}\|e_{-\alpha}v\|_{\mathbb{E}_1(J_n'')}\,(1 + [\mathbb{H}(u)]_{W_p^\kappa(J_n^*;Y_0)}) \\
&\leq ce^{(\alpha-\beta)n_0}\,\|e_{-\alpha}v\|_{\mathbb{E}_1(J_n'')}.
\end{aligned}
$$

These inequalities imply the estimate

$$
\Big(\int_{n_0}^\infty e^{-\beta tp}\int_{|t-s|\leq 1/8}\frac{|\Delta_H(t) - \Delta_H(s)|_{Y_0}^p}{|t-s|^{1+\kappa p}}\,ds\,dt\Big)^{\frac{1}{p}} \leq ce^{(\alpha-\beta)n_0}\|v\|_{\mathbb{E}_1(\mathbb{R}_+,-\alpha)}\,, \tag{3.24}
$$

where c does not depend on n_0 or v. Since $\beta > \alpha$, for a given $\epsilon > 0$ we can fix $n_0 = n_0(\epsilon) \geq 2$ such that $ce^{(\alpha-\beta)n_0} \leq \epsilon$ in (3.24).

(ii) Second, we take $t, s \in [0, n_0]$ and $|t-s| \leq 1/4$ for this n_0, and infer:

$$
\begin{aligned}
|\Delta_H(t) - \Delta_H(s)|_{Y_0} &\leq |H(u(t)+v(t)) - H(u(t)) - H'(u(t))v(t) \\
&\qquad\qquad - (H(u(s)+v(s)) - H(u(s)) - H'(u(s))v(s))|_{Y_0} \\
&\quad + |F(t,u+v) - F(s,u+v)|\,|H(u(s)+v(s)) - H(u(s)) - H'(u(s))v(s)|_{Y_0} \\
&\quad + |(F(t,u+v) - F(t,u) - \langle F'(t,u), v\rangle) \\
&\qquad\qquad - (F(s,u+v) - F(s,u) - \langle F'(s,u), v\rangle)|\,|H(u(t))|_{Y_0} \\
&\quad + |F(s,u+v) - F(s,u) - \langle F'(s,u), v\rangle|\,|H(u(t)) - H(u(s))|_{Y_0} \\
&\quad + |F(t,u+v) - F(t,u) - (F(s,u+v) - F(s,u))|\,|H'(u(t))v(t)|_{Y_0} \\
&\quad + |F(s,u+v) - F(s,u)|\,|H'(u(t))v(t) - H'(u(s))v(s)|_{Y_0} =: S_1 + \cdots + S_6.
\end{aligned}
$$

We set $\Delta_H' := \mathbb{H}(u+v) - \mathbb{H}(u) - \mathbb{H}'(u)v$. In the remainder of this proof, we use Proposition 2.2, (2.13) and (3.3) without further notice, and c and ε may depend on n_0. In the following integrals it is always understood that $s \geq 0$. We first obtain:

$$
\begin{aligned}
\Big(\int_0^{n_0}\int_{|t-s|\leq 1/4} & e^{-\beta tp}\frac{|S_1|_{Y_0}^p}{|t-s|^{1+\kappa p}}\,ds\,dt\Big)^{\frac{1}{p}} \leq [\Delta_H']_{W_p^\kappa([0,n_0+1/4];Y_0)} \\
&\leq \varepsilon(\|v\|_{\mathbb{E}_1([0,n_0+1/4])})\,\|v\|_{\mathbb{E}_1([0,n_0+1/4])} \leq c\varepsilon(\|v\|_{\mathbb{E}_1(\mathbb{R}_+,-\alpha)})\,\|v\|_{\mathbb{E}_1(\mathbb{R}_+,-\alpha)}.
\end{aligned}
$$

Similarly, (3.10) yields

$$\Big(\int_0^{n_0}\int_{|t-s|\le 1/4} e^{-\beta tp}\frac{|S_2|_{Y_0}^p}{|t-s|^{1+\kappa p}}\,ds\,dt\Big)^{\frac{1}{p}}$$
$$\le c\,\|\Delta_H'\|_{C([0,n_0+1/4];Y_0)} \le c\varepsilon(\|v\|_{\mathbb{E}_1(\mathbb{R}_+,-\alpha)})\,\|v\|_{\mathbb{E}_1(\mathbb{R}_+,-\alpha)}.$$

Next, from (3.14) we deduce

$$\begin{aligned}|S_3| \le c\,\|\mathbb{H}(u)\|_{C([0,n_0];Y_0)}\,|t-s| \sup_{-1\le\tau\le n_0+1} &|\chi(N(\tau,u+v)) - \chi(N(\tau,u))\\ &- \chi'(N(\tau,u))\langle N'(\tau,u),v\rangle|\\ \le c|t-s|\,\varepsilon(\|v\|_{\mathbb{E}_1([0,n_0+3])})\,&\|v\|_{\mathbb{E}_1([0,n_0+3])},\end{aligned}$$

$$\Big(\int_0^{n_0}\int_{|t-s|\le 1/4} e^{-\beta tp}\frac{|S_3|_{Y_0}^p}{|t-s|^{1+\kappa p}}\,ds\,dt\Big)^{\frac{1}{p}}$$
$$\le c\varepsilon(\|v\|_{\mathbb{E}_1([0,n_0+3])})\,\|v\|_{\mathbb{E}_1([0,n_0+3])} \le c\varepsilon(\|v\|_{\mathbb{E}_1(\mathbb{R}_+,-\alpha)})\,\|v\|_{\mathbb{E}_1(\mathbb{R}_+,-\alpha)}.$$

In the same way the inequality

$$|S_4|_{Y_0} \le \varepsilon(\|v\|_{\mathbb{E}_1([0,n_0+3])})\,\|v\|_{\mathbb{E}_1([0,n_0+3])}\,|H(u(t)) - H(u(s))|_{Y_0}$$

implies that

$$\Big(\int_0^{n_0}\int_{|t-s|\le 1/4} e^{-\beta tp}\frac{|S_4|_{Y_0}^p}{|t-s|^{1+\kappa p}}\,ds\,dt\Big)^{\frac{1}{p}} \le c\varepsilon(\|v\|_{\mathbb{E}_1(\mathbb{R}_+,-\alpha)})\,\|v\|_{\mathbb{E}_1(\mathbb{R}_+,-\alpha)}.$$

Definition (3.2) of the cutoff leads to the estimate

$$\begin{aligned}|S_5|_{Y_0} &\le c\,|t-s| \sup_{-1\le\tau\le n_0+1} |N(\tau,u+v) - N(\tau,u)|\,\|\mathbb{H}'(u)v\|_{C([0,n_0];Y_0)}\\ &\le c\,|t-s|\,\|v\|^2_{\mathbb{E}_1([0,n_0+3])} \le c|t-s|\,\|v\|^2_{\mathbb{E}_1(\mathbb{R}_+,-\alpha)}\end{aligned}$$

so that

$$\Big(\int_0^{n_0}\int_{|t-s|\le 1/4} e^{-\beta tp}\frac{|S_5|_{Y_0}^p}{|t-s|^{1+\kappa p}}\,ds\,dt\Big)^{\frac{1}{p}} \le c\,\|v\|^2_{\mathbb{E}_1(\mathbb{R}_+,-\alpha)}.$$

The term S_6 can be treated similarly. Therefore we have shown that

$$\Big(\int_0^{n_0}\int_{|t-s|\le 1/4} e^{-\beta tp}\frac{|\Delta_H(t)-\Delta_H(s)|_{Y_0}^p}{|t-s|^{1+\kappa p}}\,ds\,dt\Big)^{\frac{1}{p}} \le c\varepsilon(\|v\|_{\mathbb{E}_1(\mathbb{R}_+,-\alpha)})\,\|v\|_{\mathbb{E}_1(\mathbb{R}_+,-\alpha)}.$$

Putting together the estimates obtained in (i) and (ii), we conclude that for each $\epsilon > 0$ there exists a $r'_\epsilon \le r_\epsilon$ such that if $\|v\|_{\mathbb{E}_1(\mathbb{R}_+,-\alpha)} \le r'_\epsilon$, then

$$\Big(\int_0^{\infty}\int_{|t-s|\le 1/4} e^{-\beta tp}\frac{|\Delta_H(t)-\Delta_H(s)|_{Y_0}^p}{|t-s|^{1+\kappa p}}\,ds\,dt\Big)^{\frac{1}{p}} \le \epsilon\,\|v\|_{\mathbb{E}_1(\mathbb{R}_+,-\alpha)}.$$

Using [12, Lem.11] and (3.23), we obtain $[e_{-\beta}\Delta_H]_{W_p^\kappa(\mathbb{R}_+;Y_0)} \le c\epsilon\|v\|_{\mathbb{E}_1(\mathbb{R}_+,-\alpha)}$ for $\|v\|_{\mathbb{E}_1(\mathbb{R}_+,-\alpha)} \le r'_\epsilon$, finishing the proof of the differentiability of $\mathbb{H}_{\mathcal{F}}$.

It remains to establish the continuity of the maps $u \mapsto \mathbb{G}'_{\mathcal{F}}(u)$ and $u \mapsto \mathbb{H}'_{\mathcal{F}}(u)$ in $\mathcal{B}(\mathbb{E}_1(\mathbb{R}_+ - \alpha), \mathbb{E}_0(\mathbb{R}_+ - \beta))$ and $\mathcal{B}(\mathbb{E}_1(\mathbb{R}_+,-\alpha), \mathbb{F}(\mathbb{R}_+,-\beta))$, respectively. This can be done by similar arguments as above. $\square$

4. Stable, unstable and center manifolds

We first construct and study the *local stable manifold* $\mathcal{M}_{\rm s}$, resp. the *local unstable manifold* $\mathcal{M}_{\rm u}$, assuming that $\sigma(-A_0)$ has a spectral gap in the open left, resp. right, halfplane. These manifolds are of class C^1 in X_p, and are tangent at u_* to $P_{\rm s}X_p^0$ and $P_{\rm u}X_0$, respectively. These results are established in Theorem 4.1 which is actually a somewhat simpler variant of Theorem 17 in [12] where the hyperbolic case $i\mathbb{R} \subset \rho(-A_0)$ has been addressed. Next, in our main Theorem 4.2, we consider the case of trichotomy, assuming that $\sigma(-A_0)$ has spectral gaps in both the left and the right open halfplanes, cf. (2.34).

We choose the formulation of the spectral conditions for Theorem 4.1 in view of the situation in Theorem 4.2. We assume the existence of numbers $\omega_s, \omega_u, \omega_{cu}, \omega_{cs} > 0$ such that at least one of the following assertions holds:

$$\sigma(-A_0) = \sigma_s \cup \sigma_{cu} \quad \text{with} \quad \max \operatorname{Re} \sigma_s < -\omega_s < -\omega_{cu} < \min \operatorname{Re} \sigma_{cu}\,, \tag{4.1}$$

$$\sigma(-A_0) = \sigma_{cs} \cup \sigma_u \quad \text{with} \quad \max \operatorname{Re} \sigma_{cs} < \omega_{cs} < \omega_u < \min \operatorname{Re} \sigma_u\,. \tag{4.2}$$

We denote by P_k the spectral projections for $-A_0$ corresponding to σ_k, $k \in \{s, cs, cu, u\}$. As noted in Remark 2.4, we have $P_{\rm u}X_0 \subset P_{\rm cu}X_0 \subset \operatorname{dom}(A_0)$, and thus on $P_{\rm cu}X_0$ the norms in X_0, X_p and X_1 are equivalent. Finally, we recall the notation $X_p^0 = \{z_0 \in X_p : B_* z_0 = 0\}$ for the tangent space at u_* to the nonlinear phase space $\mathcal{M} = \{u_0 \in X_p : B(u_0) = 0\}$ for (1.1), and that $\mathcal{P} = I - \mathcal{N}_p B_*$ projects X_p onto X_p^0, see the remarks before Theorem 14 in [12].

Theorem 4.1. *Assume Hypothesis 2.1. Then there are numbers $r \geq \rho > 0$ and $\rho_0 > 0$ such that the following assertions hold.*

(a) Let (4.1) hold and take any $\alpha \in (\omega_{cu}, \omega_s)$. Then there are BC^1-maps

$$\phi_s : P_{\rm s}X_p^0 \cap B_{X_p}(0, \rho_0) \to P_{\rm cu}X_0 \quad \text{and} \quad \vartheta_s : P_{\rm s}X_p^0 \cap B_{X_p}(0, \rho_0) \to P_{\rm s}X_p\,,$$

such that $\phi_s(0) = \vartheta_s(0) = 0$, $\phi_s'(0) = \vartheta_s'(0) = 0$, and

$$\begin{aligned}\mathcal{M}_{\rm s} &:= \Big\{u_0 = u_* + z_0 + \vartheta_s(z_0) + \phi_s(z_0) \in B_{X_p}(u_*, \rho) : z_0 \in P_{\rm s}X_p^0 \cap B_{X_p}(0, \rho_0)\Big\} \\ &= \Big\{u_0 \in \mathcal{M} \cap B_{X_p}(u_*, \rho) : \exists \text{ solution } u \text{ of } (1.1) \text{ on } \mathbb{R}_+ \text{ with} \\ &\qquad |u(t) - u_*|_p < r \;\; \forall\, t \geq 0 \;\; \text{and} \;\; |u(t) - u_*|_1 \leq c e^{-\alpha t} \;\; \forall\, t \geq 1\Big\}. \end{aligned} \tag{4.3}$$

In (4.3) we can take $c = \overline{c}\,|u(0) - u_*|_p$ for a constant $\overline{c}$ independent of u_0, t, α, and we have $u = u_* + \Phi_s(P_{\rm s}\mathcal{P}(u_0 - u_*))$ for a map $\Phi_s \in BC^1(P_{\rm s}X_p^0 \cap B_{X_p}(0, \rho_0); \mathbb{E}_1(\mathbb{R}_+, \alpha))$. If $u_0 \in \mathcal{M}_{\rm s}$ and the forward solution u of (1.1) stays in $B(u_*, \rho)$ on $[0, t]$ for some $t > 0$, then $u(t) \in \mathcal{M}_{\rm s}$. If $u_0 \in \mathcal{M}_{\rm s}$ and there is a backward solution u of (1.1) staying in $B(u_*, \rho)$ on $[t, 0]$ for some $t < 0$, then $u(t) \in \mathcal{M}_{\rm s}$.

(b) Let (4.2) hold and take any $\beta \in (\omega_{cs}, \omega_u)$. Then there is a BC^1-map

$$\phi_u : P_{\rm u}X_0 \cap B_{X_p}(0, \rho_0) \to P_{\rm cs}X_p$$

such that $\phi_u(0) = 0$, $\phi_u'(0) = 0$, and

$$\begin{aligned}\mathcal{M}_{\rm u} &:= \Big\{u_0 = u_* + z_0 + \phi_u(z_0) \in B_{X_p}(u_*, \rho) : z_0 \in P_{\rm u}X_0 \cap B_{X_p}(0, \rho_0)\Big\} \\ &= \Big\{u_0 \in \mathcal{M} \cap B_{X_p}(u_*, \rho) : \exists \text{ solution } u \text{ of } (1.1) \text{ on } \mathbb{R}_- \text{ with } |u(t) - u_*|_p < r \\ &\qquad \text{and} \;\; |u(t) - u_*|_1 \leq c e^{\beta t} \;\; \forall\, t \leq 0\Big\}. \end{aligned} \tag{4.4}$$

In (4.4) we can take $c = \overline{c}\,|u(0) - u_*|_0$ for a constant $\overline{c}$ independent of u_0, t, β, and we have $u = u_* + \Phi_u(P_{\rm u}(u_0 - u_*))$ for a map $\Phi_u \in BC^1(P_{\rm u}X_0 \cap B_{X_p}(0,\rho_0); \mathbb{E}_1(\mathbb{R}_-, -\beta))$. The dimension of $\mathcal{M}_{\rm u}$ is equal to the dimension of $P_{\rm u}X_0$. If $u_0 \in \mathcal{M}_{\rm u}$ and the forward solution u of (1.1) stays in $B(u_*,\rho)$ on $[0,t]$ for some $t > 0$, then $u(t) \in \mathcal{M}_{\rm u}$. If $u_0 \in \mathcal{M}_{\rm u}$ and the solution u from (4.4) stays in $B(u_*,\rho)$ on $[t,0]$ for some $t < 0$, then $u(t) \in \mathcal{M}_{\rm u}$. Moreover, if $\sigma_u \neq \emptyset$, then u_* is (Lyapunov) unstable in X_p for (1.1). In addition, if (RR) holds, then the map $\phi_u : P_{\rm u}X_0 \cap B_{X_p}(0,\rho_0) \to P_{\rm cs}X_1$ is Lipschitz.

Proof. We provide only a sketch of the proof referring to [12, Thm.17] for missing details. The basic idea is to look for solutions v of (2.21) on $\mathbb{R}_+$ and $\mathbb{R}_-$ which satisfy the decay properties of (4.3) and (4.4), respectively. The maps ϕ_s, ϑ_s and ϕ_u will then be defined in terms of the initial, respectively, final value $v(0)$ of (2.21).

(a) We define the Lyapunov-Perron map $\mathcal{L}_s : P_{\rm s}X_p^0 \times \mathbb{E}_1(\mathbb{R}_+,\alpha) \to \mathbb{E}_1(\mathbb{R}_+,\alpha)$ by

$$\mathcal{L}_s(z_0, v) = L^+_{P_s,A_0}(z_0 + P_s\mathcal{N}_p\gamma_0\mathbb{H}(v), \mathbb{G}(v), \mathbb{H}(v)), \tag{4.5}$$

cf. (2.25), the discussion after (2.23), and also [12, (82)]. By means of Propositions 2.2 and 2.5, (2.13) and the implicit function theorem, we find numbers $r_0, \rho_0 > 0$ and a BC^1–map $\Phi_s : P_{\rm s}X_p^0 \cap B_{X_p}(0,\rho_0) \to \mathbb{E}_1(\mathbb{R}_+,\alpha)$ such that $\Phi_s(0) = 0$ and $\Phi_s(z_0)$ is the unique solution of $v = \mathcal{L}_s(z_0, v)$ for $\|v\|_{\mathbb{E}_1(\mathbb{R}_+,\alpha)} < r_0$ and $|z_0|_p < \rho_0$. Set $v = \Phi_s(z_0)$, $u = v + u_*$, and $u_0 = v(0) + u_*$. Then u solves (1.1) on $\mathbb{R}_+$ with $u(0) = u_0 = u_* + \Phi_s(z_0)(0)$. Using also [12, Prop.15] and (2.11), we see that u satisfies the properties listed in (4.3) with $c = \overline{c}\,|u(0) - u_*|_p$ and $r = c'\,|z_0|_p$ for some constants $\overline{c}, c' > 0$. We further define the maps

$$\vartheta_s(z_0) = P_s\mathcal{N}_p\gamma_0\mathbb{H}(\Phi_s(z_0)) \qquad \text{and}$$

$$\phi_s(z_0) = P_{\rm cu}\gamma_0\Phi_s(z_0) = -\int_0^\infty T_{cu}(-\tau)P_{\rm cu}[G(\Phi_s(z_0))(\tau) + \Pi H(\Phi_s(z_0))(\tau)]\,d\tau,$$

which are of class BC^1 from $P_{\rm s}X_p^0 \cap B_{X_p}(0,\rho_0)$ to $P_{\rm s}X_p$ and $P_{\rm cu}X_0$, respectively, and fulfill $\phi_s(0) = \vartheta_s(0) = 0$ and $\phi_s'(0) = \vartheta_s'(0) = 0$. Observe that $z_0 = P_{\rm s}(u_0 - u_* - \mathcal{N}_pH(u_0 - u_*)) = P_{\rm s}\mathcal{P}(u_0 - u_*)$ since $H(u_0 - u_*) = B_*(u_0 - u_*)$.

Let u be a solution of (1.1) with $|u_0 - u_*|_p < \rho$, $|u(t) - u_*|_p < r$, and $|u(t) - u_*|_p \le ce^{-\bar{\alpha}t}$ for all $t \ge 0$ and some $c > 0$ and $\bar{\alpha} \in (\omega_{cu}, \omega_s)$. Put $v = u - u_*$. Take $\sigma > 0$ with $\bar{\alpha} - \sigma \in (\omega_{cu}, \omega_s)$. Hence, $d = \sigma/(2\bar{\alpha}) \in (0,1)$. For $N \in \mathbb{N}$ and sufficiently small $r > 0$, Remark 2.3 yields

$$\begin{aligned}\|e_{\bar{\alpha}-\sigma}v\|^p_{\mathbb{E}_1([0,N])} &\le \sum_{n=0}^{N-1} ce^{(\bar{\alpha}-\sigma)np}\,\|v\|^p_{\mathbb{E}_1(J_n)} \\ &\le \sum_{n=0}^{N-1} ce^{(\bar{\alpha}-\sigma)np}\,|v(n)|_p^{dp}\,|v(n)|_p^{(1-d)p} \le cr^{dp},\end{aligned}$$

where the constants do not depend on N. Therefore $v \in \mathbb{E}_1(\mathbb{R}_+, \bar{\alpha} - \sigma)$ with norm less than cr^d, and so $v = L^+_{P_s,A_0}(P_{\rm s}v(0), \mathbb{G}(v), \mathbb{H}(v))$ due to Proposition 2.5. Set $z_0 = P_s(v(0) - \mathcal{N}_pH(v(0)))$. Then $|z_0|_p \le c\,|v(0)|_p \le c\rho < \rho_0$ for sufficiently small ρ. We thus have the solution $w = \Phi_s(z_0) \in \mathbb{E}_1(\mathbb{R}_+,\alpha)$ of the equation $w = \mathcal{L}_s(z_0, w)$. As in the proof of assertion (ii) of [12, Thm.17] we infer that $v = w$ for sufficiently small r. As a result, (4.3) holds. The invariance properties of $\mathcal{M}_{\rm s}$ follow from (4.3).

(b) All but two last assertions for $\mathcal{M}_{\rm u}$ can be shown in a similar way using (2.27) and Proposition 2.6, cf. [12, Thm.17]. The remaining two are proved as follows.

If $\sigma_u \neq \emptyset$, then there exists a function $u_0 \in \mathcal{M}_{\mathrm{u}} \setminus \{u_*\}$ with a corresponding solution u of (1.1) on $\mathbb{R}_-$ from (4.4). Hence, for each $\delta > 0$ there is a $t = t(\delta) > 0$ such that $|u(-t) - u_*|_p \le ce^{-\beta t} < \delta$. Let $\epsilon = |u_0 - u_*|_p/2 > 0$ and set $w = u(\cdot - t)$. Then w solves (1.1) on $[0,t]$, and we have $|w(0) - u_*|_p = |u(-t) - u_*|_p < \delta$ but $|w(t) - u_*|_p = |u_0 - u_*|_p \ge \epsilon$. As a result, u_* is (Lyapunov) unstable in X_p.

Let (RR) hold and take $z_0, \overline{z}_0 \in P_{\mathrm{u}}X_0 \cap B_{X_p}(0,\rho_0)$. Then we have solutions $u = v + u_*$ and $\overline{u} = \overline{v} + u_*$ of (1.1) on $\mathbb{R}_-$ given by $v = \Phi_u(z_0)$ and $\overline{v} = \Phi_u(\overline{z}_0)$ for a BC^1–map Φ_u from $P_{\mathrm{u}}X_0 \cap B_{X_p}(0,\rho_0)$ to $\mathbb{E}_1(\mathbb{R}_-,-\beta)$. Employing Theorem A.1 and (2.11), we then obtain

$$\begin{aligned}|\phi_u(z_0) - \phi_u(\overline{z}_0)|_1 &= |P_{\mathrm{cs}}(v(0) - \overline{v}(0))|_1 \le c\,|v(-1) - \overline{v}(-1)|_p \\ &\le c\,\|\Phi_u(z_0) - \Phi_u(\overline{z}_0)\|_{\mathbb{E}_1(\mathbb{R}_-,-\beta)} \le c\,|z_0 - \overline{z}_0|_p\end{aligned}$$

for constants independent of z_0 and $\overline{z}_0$, possibly after decreasing $\rho_0 > 0$. □

We now establish the main result of this paper where we construct a *local center manifold* $\mathcal{M}_{\mathrm{c}}$ and show some of its basic properties. In particular, $\mathcal{M}_{\mathrm{c}}$ is a C^1–manifold in X_p being tangent to $P_{\mathrm{c}}X_0$ at u_*. Further properties of $\mathcal{M}_{\mathrm{c}}$ are described in Corollary 5.3 and Theorem 6.1. We assume that the spectrum of $-A_0$ has the decomposition described in (2.34), and recall that this assumption automatically holds if the spatial domain Ω is bounded.

Theorem 4.2. *Assume that Hypothesis 2.1 and* (2.34) *hold. Let the projections P_k and the numbers ω_k be given by* (2.34)*. Take any $\alpha \in (\underline{\omega}_c, \omega_s)$ and $\beta \in (\overline{\omega}_c, \omega_u)$. Then there is a number $\eta_c > 0$ such that for each $\eta \in (0,\eta_c]$ there exists a radius $\rho = \rho(\eta) > 0$ such that the following assertions hold, where the cutoff $\digamma$ is defined in* (3.2) *for the chosen $\eta \in (0,\eta_c]$.*

(a) There exists a map $\phi_c \in C^1(P_{\mathrm{c}}X_0; P_{\mathrm{su}}X_p)$ with a bounded derivative such that $\phi_c(0) = 0$, $\phi_c'(0) = 0$, and

$$\begin{aligned}\widetilde{\mathcal{M}}_c &:= \big\{u_0 = u_* + z_0 + \phi_c(z_0) : z_0 \in P_{\mathrm{c}}X_0\big\} \\ &= \big\{u_0 = u_* + v(0) : \exists \text{ solution } v \in \mathbb{E}_1(\alpha,-\beta) \text{ of } (3.7) \text{ on } J = \mathbb{R}\big\}. \qquad (4.6)\end{aligned}$$

If $u_0 \in \widetilde{\mathcal{M}}_c$, then $u_* + v(t) \in \widetilde{\mathcal{M}}_c$ for each $t \in \mathbb{R}$ and $v = \Phi_c(P_{\mathrm{c}}(u_0 - u_*)) = P_{\mathrm{c}}v + \phi_c(P_{\mathrm{c}}v)$ for a map $\Phi_c \in C^1(P_{\mathrm{c}}X_0; \mathbb{E}_1(\alpha,-\beta))$ having a bounded derivative, where v is the solution of the cutoff problem (3.7) *given by* (4.6)*.*

(b) We define $\mathcal{M}_{\mathrm{c}} = \widetilde{\mathcal{M}}_c \cap B_{X_p}(u_*,\rho)$. Let $u_0 \in \mathcal{M}_{\mathrm{c}}$ and v be given by (4.6) *with $u_0 = v(0) + u_*$. Then $\digamma(t,v) = 1$ and v solves the original equation* (2.21) *(at least) for $t \in [-3,3]$, so that $\mathcal{M}_{\mathrm{c}} \subset \mathcal{M}$. The dimension of $\mathcal{M}_{\mathrm{c}}$ is equal to* $\dim P_{\mathrm{c}}X_0$.

(c) Let $u_0 \in \mathcal{M}_{\mathrm{c}}$ and v be given by (4.6)*. If the forward solution u of* (1.1) *exists and stays in $B_{X_p}(u_*,\rho)$ on $[0,t_0]$ for some $t_0 > 0$, then $u(t) = v(t) + u_* \in \mathcal{M}_{\mathrm{c}}$ for $0 \le t \le t_0$. If the function $\hat{u} = v + u_*$ stays in $B_{X_p}(u_*,\rho)$ on $[t_0,0]$ for some $t_0 < 0$, then $\hat{u}(t) \in \mathcal{M}_{\mathrm{c}}$ and $\hat{u}$ solves* (1.1) *for $t_0 \le t \le 0$.*

(d) Let $u_0 = u_* + v_0 \in \mathcal{M}_{\mathrm{c}}$ and let v be given by (4.6)*. Assume that $v(t) + u_* \in \mathcal{M}_{\mathrm{c}}$ for all $t \in (a,b)$ and some $a < 0 < b$. Then $y = P_{\mathrm{c}}v$ satisfies the equations*

$$\begin{aligned}&\dot{y}(t) = -A_0P_{\mathrm{c}}y(t) + P_{\mathrm{c}}\Pi H(y(t) + \phi_c(y(t))) + P_{\mathrm{c}}G(y(t) + \phi_c(y(t))), \\ &y(0) = P_{\mathrm{c}}(u_0 - u_*),\end{aligned} \qquad (4.7)$$

on $P_{\mathrm{c}}X_0$ for $t \in (a,b)$. Moreover, $v \in C((a,b);X_1)$ and

$$B_*\phi_c(P_{\mathrm{c}}v_0) = B_*v_0 = H(v_0), \qquad (4.8)$$

$$P_{\rm su}(A_*v_0 - G(v_0)) = \phi_c'(P_{\rm c}v_0)P_{\rm c}(A_*v_0 - G(v_0)). \tag{4.9}$$

(e) If u solves (1.1) on $\mathbb{R}$ with $|u(t)-u_*|_p < \rho$ for all $t\in\mathbb{R}$, then $u(t)\in\mathcal{M}_{\rm c}$ for all $t\in\mathbb{R}$.

(f) In addition, assume that (RR) holds. Then there is a $\rho_0>0$ such that the map $\phi_c : P_{\rm c}X_0 \cap B_{X_p}(0,\rho_0) \to P_{\rm su}X_1$ is Lipschitz.

Proof. We first construct a manifold $\widetilde{\mathcal{M}}_c$ consisting of solutions on $\mathbb{R}$ in a weighted $\mathbb{E}_1$–space, similarly to Theorem 4.1. However, since $T_c(\cdot)$, in general, is an *unbounded* group, we must work in spaces containing exponentially growing functions. Therefore we have to treat the modified problem (3.7) with the cutoff $\digamma$. The desired center manifold $\mathcal{M}_{\rm c}$ is then obtained by restriction to small balls.

(a) We define the Lyapunov-Perron map $\mathcal{L}_c : P_{\rm c}X_0 \times \mathbb{E}_1(\alpha,-\beta) \to \mathbb{E}_1(\alpha,-\beta)$ by

$$\mathcal{L}_c(z_0,v) = L_{A_0}(z_0, \mathbb{G}_\digamma(v), \mathbb{H}_\digamma(v)),$$

where the operators L_{A_0}, $\mathbb{G}_\digamma$ and $\mathbb{H}_\digamma$ are given by (2.35) and (3.6). Due to Propositions 2.7 and 3.8, the map $\mathcal{L}^0 : v\mapsto \mathcal{L}_c(z_0,v)$ is C^1 from $\mathbb{E}_1(\alpha',-\beta')$ to $\mathbb{E}_1(\alpha,-\beta)$ for $\alpha'\in(\underline{\omega}_c,\alpha)$ and $\beta'\in(\overline{\omega}_c,\beta)$ and the derivative of $\mathcal{L}^0$ is bounded by $c_1\varepsilon(\eta)$ in the norm of both $\mathcal{B}(\mathbb{E}_1(\alpha',-\beta'))$ and $\mathcal{B}(\mathbb{E}_1(\alpha,-\beta))$, independent of $z_0\in P_cX_0$. Moreover, $\mathcal{L}^0$ is Lipschitz in $\mathbb{E}_1(\alpha',-\beta')$ with Lipschitz constant $c_1\varepsilon(\eta)$ independent of $z_0\in P_{\rm c}X_0$ by Proposition 3.6. Finally, the map $z_0\mapsto \mathcal{L}_c(z_0,v)$ is affine from $P_{\rm c}X_0$ to $\mathbb{E}_1(\alpha',-\beta')$ with the derivative $T(\cdot)P_{\rm c}$.

We now fix $\eta=\eta_c>0$ such that $c_1\varepsilon(\eta)\le 1/2$. (Note that this estimate holds for every $\eta'\in(0,\eta)$.) Then Theorem 3 of [21] (with $Y_0=Y=\mathbb{E}_1(\alpha',-\beta')$ and $Y_1=\mathbb{E}_1(\alpha,-\beta)$) shows that for each $z_0\in P_{\rm c}X_0$ there exists a unique solution $v=\Phi_c(z_0)\in\mathbb{E}_1(\alpha',-\beta')$ of the equation $v=\mathcal{L}_c(z_0,v)$, where $\Phi_c\in C^1(P_{\rm c}X_0;\mathbb{E}_1(\alpha,-\beta))$ and $\Phi_c(0)=0$. Employing [21, (4.4)], it is easy to check that $\Phi_c'(z_0)\in\mathcal{B}(P_{\rm c}X_0,\mathbb{E}_1(\alpha,-\beta))$ is bounded uniformly in z_0. We further define

$$\begin{aligned}\phi_c(z_0) = \gamma_0 P_{\rm su}\Phi_c(z_0) = &\int_{-\infty}^0 T_{-1}(-\tau)P_{\rm s}[G_\digamma(\Phi_c(z_0))(\tau) + \Pi H_\digamma(\Phi_c(z_0))(\tau)]\,d\tau \\ &- \int_0^\infty T_c(-\tau)P_{\rm u}[G_\digamma(\Phi_c(z_0))(\tau) + \Pi H_\digamma(\Phi_c(z_0))(\tau)]\,d\tau\end{aligned}$$

for $z_0\in P_{\rm c}X_0$. Taking also into account (2.13), we see that $\phi_c\in C^1(P_{\rm c}X_0;P_{\rm su}X_p)$, that ϕ_c' is bounded, and that $\phi_c(0)=0$ and $\phi_c'(0)=0$. Equality (4.6) follows from Proposition 2.7. If $u_0\in\widetilde{\mathcal{M}}_c$ with the corresponding solution v of (3.7) and $t\in\mathbb{R}$, then $w=v(\cdot+t)$ solves (3.7) with the initial condition $w(0)=v(t)$ thanks to Remark 3.2. This means that $u_*+v(t)\in\widetilde{\mathcal{M}}_c$, and thus $v(t)=P_{\rm c}v(t)+\phi_c(P_{\rm c}v(t))$.

(b) Let $u_0\in\widetilde{\mathcal{M}}_c\cap B_{X_p}(u_*,\rho)$. Set $v_0=u_0-u_*$, $z_0=P_{\rm c}v_0$, and $v=\Phi_c(z_0)$. From (3.4) and part (a) we infer that

$$\begin{aligned}\|v\|_{\mathbb{E}_1([-5,5])} &\le c\,\|v\|_{\mathbb{E}_1(\alpha,-\beta)} = c\,\|\Phi_c(z_0)-\Phi_c(0)\|_{\mathbb{E}_1(\alpha,-\beta)} \\ &\le c\,|z_0|_0 \le c\,|v_0|_p \le c'\rho.\end{aligned} \tag{4.10}$$

If we take $\rho\le\eta/c'$, Remark 3.1 implies that $\digamma(t,v)=1$ for $t\in[-3,3]$, so that v solves (2.21) on $[-3,3]$ in this case.

(c) Take $u_0\in\mathcal{M}_{\rm c}$ and let u be the forward solution of (1.1). Part (b) and the uniqueness of (1.1) yield that $u=v+u_*$ on $[0,2]$, where v is given by (4.6). Thus $u(t)=v(t)+u_*\in\widetilde{\mathcal{M}}_c$ by part (a) for $t\in[0,2]$. If $u(t)\in B_{X_p}(u_*,\rho)$ for $t\in[0,t_0]$ and $t_0\le 2$, we thus obtain $u(t)\in\mathcal{M}_{\rm c}$ for $t\in[0,t_0]$. If $t_0>2$, this argument can be iterated as long as u stays in $B_{X_p}(u_*,\rho)$. The assertions concerning the backward invariance of $\mathcal{M}_{\rm c}$ are direct consequences of parts (a) and (b).

(d) Let $u_0 = u_* + v_0 \in \mathcal{M}_{\rm c}$ and let v be given by (4.6). By parts (a)–(c), the function $v = y + \phi_c(y)$ solves (2.21) on (a,b). Theorem 14 of [12] thus shows that v is continuous in X_1. Moreover,

$$\begin{aligned}
\dot y(t) &= P_{\rm c}\dot v(t) = P_{\rm c}[-A_*v(t) + G(v(t))]\\
&= -P_{\rm c}(\mu + A_*)(v(t) - \mathcal{N}_1 H(v(t))) + \mu P_{\rm c} v(t) + P_{\rm c} G(v(t))\\
&= -A_0 P_{\rm c} y(t) + P_{\rm c}\Pi H(y(t) + \phi_c(y(t))) + P_{\rm c} G(y(t) + \phi_c(y(t))).
\end{aligned}$$

Equality (4.8) is clear since $B_*P_{\rm c} = 0$ and $v_0 \in \mathcal{M}$. We further have

$$\begin{aligned}
\dot v(t) &= -A_*v(t) + G(v(t)),\\
\dot v(t) &= P_{\rm c}(-A_*v(t) + G(v(t))) + \phi_c'(P_{\rm c}v(t))P_{\rm c}(-A_*v(t) + G(v(t))),
\end{aligned}$$

so that (4.9) follows by taking $t = 0$.

(e) For a global solution u of (1.1) staying in $B_{X_p}(u_*,\rho)$, Remark 2.3 implies that $\|u\|_{\mathbb{E}_1([t-2,t+2])} \le c_*\rho$ for each $t \in \mathbb{R}$ (possibly after decreasing $\rho > 0$). In particular, $u \in \mathbb{E}_1(\alpha,-\beta)$. Taking $\rho \le \eta/c_*$, we further deduce that $v = u - u_*$ solves (3.7) on $J = \mathbb{R}$ using Remark 3.1. So (e) follows from the definition of $\mathcal{M}_{\rm c}$.

(f) We first note that (4.10) and (2.11) imply $|v(0)|_p \le c\,|z_0|_0$ for $u_0 = v(0)+u_* \in \widetilde{\mathcal{M}}_c$ with $v = \Phi_c(z_0)$. Hence, there is a number ρ_0 such that $u_0 = u_* + z_0 + \phi_c(z_0) \in \mathcal{M}_{\rm c}$ if $z_0 \in P_{\rm c}X_0 \cap B_{X_p}(u_*,\rho_0)$. Then v solves (2.21) on $[-1,0]$. So we can show assertion (f) as the final assertion in Theorem 4.1(b). □

Remark 4.3. Given $r \ge \rho > 0$, the manifolds $\mathcal{M}_{\rm s}$ and $\mathcal{M}_{\rm u}$ from Theorem 4.1 are uniquely determined by (4.3) and (4.4) as sets of initial values of exponentially decaying solutions of (1.1). There is no such description for $\mathcal{M}_{\rm c}$ from Theorem 4.2. In fact, there are simple ODEs in dimension two admitting infinitely many locally invariant manifolds which are tangent to $P_{\rm c}X_0$ at u_* and satisfy $\mathcal{M}_{\rm c} \cap \mathcal{M}_{\rm s} = \mathcal{M}_{\rm c} \cap \mathcal{M}_{\rm u} = \{u_*\}$ (cf. Corollary 5.3). However, if u_* is stable in forward and backward time, then Theorem 4.2(e) implies that our $\mathcal{M}_{\rm c}$ is the only manifold in $B_{X_p}(u_*,\rho)$ with these properties. ♢

5. Center stable and center unstable manifolds

In this section we go back to the situation of Theorem 4.1. In Theorem 5.1 we construct a *local center–stable manifold* $\mathcal{M}_{\rm cs}$ assuming (4.2), and in Theorem 5.2 we construct a *local center–unstable manifold* $\mathcal{M}_{\rm cu}$ assuming (4.1). These manifolds are of class C^1 in X_p, and are tangent to $P_{\rm cs}X_p^0$, resp. to $P_{\rm cu}X_0$, at u_*. They will be used to prove further properties of the center manifold in Corollary 5.3. Recall that $\mathcal{P} = I - \mathcal{N}_p B_*$.

Theorem 5.1. *Assume Hypothesis 2.1 and* (4.2)*. Take any $\beta \in (\omega_{cs},\omega_u)$. Then there is a number $\eta_{cs} > 0$ such that for each $\eta \in (0,\eta_{cs}]$ there exists a radius $\rho = \rho(\eta) > 0$ such that the following assertions hold, where the cutoff F_+ is defined in* (3.5) *for the chosen $\eta \in (0,\eta_{cs}]$.*

(a) There exist maps $\phi_{cs} \in C^1(P_{\rm cs}X_p^0; P_{\rm u}X_0)$ and $\vartheta_{cs} \in C^1(P_{\rm cs}X_p^0; P_{\rm cs}X_p)$ with bounded derivatives such that $\phi_{cs}(0) = \vartheta_{cs}(0) = 0$, $\phi_{cs}'(0) = \vartheta_{cs}'(0) = 0$, and

$$\begin{aligned}
\widetilde{\mathcal{M}}_{cs} &:= \{u_0 = u_* + z_0 + \vartheta_{cs}(z_0) + \phi_{cs}(z_0) : z_0 \in P_{\rm cs}X_p^0\}\\
&= \{u_0 = u_* + v(0) : \exists \textit{ solution } v \in \mathbb{E}_1(\mathbb{R}_+,-\beta) \textit{ of } (3.7) \textit{ on } J = \mathbb{R}_+\}. \quad (5.1)
\end{aligned}$$

Moreover, the function v in (5.1) *is given by $v = \Phi_{cs}(P_{\rm cs}\mathcal{P}(u_0 - u_*))$ for a map $\Phi_{cs} \in C^1(P_{\rm cs}X_p^0; \mathbb{E}_1(\mathbb{R}_+, -\beta))$ having a bounded derivative.*

(b) We define $\mathcal{M}_{\rm cs} = \widetilde{\mathcal{M}}_{cs} \cap B_{X_p}(u_*, \rho)$. Let $u_0 \in \mathcal{M}_{\rm cs}$ and v be the function from (5.1) *with $u_0 = v(0) + u_*$. Then $\digamma_+(t, v) = 1$ and v solves the original equation* (2.21) *(at least) for $t \in [0, 4]$.*

(c) Let $u_0 \in \mathcal{M}_{cs}$ and v be given by (5.1). *Assume that a forward or a backward solution u of* (1.1) *exists and stays in $B_{X_p}(u_*, \rho)$ on $[0, t_0]$ or on $[-t_0, 0]$ for some $t_0 > 0$. Set $v(t) = u(t) - u_*$ for $-t_0 \le t \le 0$ in the second case. Then $u(t) = u_* + v(t) = u_* + P_{\rm cs}v(t) + \phi_{cs}(P_{\rm cs}\mathcal{P}v(t)) + \vartheta_{cs}(P_{\rm cs}\mathcal{P}v(t)) \in \mathcal{M}_{\rm cs}$ for $0 \le t \le t_0$ or $-t_0 \le t \le 0$, respectively.*

(d) We have $\mathcal{M}_{\rm cs} \cap \mathcal{M}_{\rm u} = \{u_*\}$.

Proof. We follow the strategy of the construction of the stable manifold in Theorem 4.1, but now we must work in the space $\mathbb{E}_1(\mathbb{R}_+, -\beta)$ containing exponentially growing functions. Thus, as in Theorem 4.2, we have to involve the cutoff $\digamma_+$ which leads to various technical difficulties.

(a) We define the map $\mathcal{L}_{cs} : P_{\rm cs}X_p^0 \times \mathbb{E}_1(\mathbb{R}_+, -\beta) \to \mathbb{E}_1(\mathbb{R}_+, -\beta)$ by

$$\mathcal{L}_{cs}(z_0, v) = L^+_{P_{\rm cs}, A_0}(z_0 + P_{\rm cs}\mathcal{N}_p\gamma_0\mathbb{H}_{\digamma_+}(v), \mathbb{G}_{\digamma_+}(v), \mathbb{H}_{\digamma_+}(v)), \tag{5.2}$$

where the operators $L^+_{P_{\rm cs}, A_0}$, $\mathbb{G}_{\digamma_+}$ and $\mathbb{H}_{\digamma_+}$ are given by (2.25) and (3.6). Observe that the semigroup $e_{-\beta}T(\cdot)$ is hyperbolic with the stable projection $P_{\rm cs}$. Due to Propositions 2.5 and 3.8 and the embedding (2.13), the map $\mathcal{L}^0 : v \mapsto \mathcal{L}_{cs}(z_0, v)$ is C^1 from $\mathbb{E}_1(\mathbb{R}_+, -\beta')$ to $\mathbb{E}_1(\mathbb{R}_+, -\beta)$ for $\beta' \in (\omega_{cs}, \beta)$ and the derivative of $\mathcal{L}^0$ is bounded by $c_1\varepsilon(\eta)$ in the norm of both $\mathcal{B}(\mathbb{E}_1(\mathbb{R}_+, -\beta'))$ and $\mathcal{B}(\mathbb{E}_1(\mathbb{R}_+, -\beta))$, independent of $z_0 \in P_{\rm cs}X_p^0$. Moreover, $\mathcal{L}^0$ is Lipschitz in $E_1(\mathbb{R}_+, -\beta')$ with the Lipschitz constant $c_1\varepsilon(\eta)$ independent of $z_0 \in P_{\rm cs}X_p^0$ by Proposition 3.6. Finally, the map $z_0 \mapsto \mathcal{L}_{cs}(z_0, v)$ is affine from $P_{\rm cs}X_p^0$ to $\mathbb{E}_1(\mathbb{R}_+, -\beta')$ with the derivative $T(\cdot)P_{\rm cs}$.

We now fix $\eta = \eta_{cs} > 0$ such that $c_1\varepsilon(\eta) \le 1/2$. (Note that this inequality also holds for each $\eta' \in (0, \eta)$.) Then Theorem 3 of [21] (with $Y_0 = Y = \mathbb{E}_1(\mathbb{R}_+, -\beta')$ and $Y_1 = \mathbb{E}_1(\mathbb{R}_+, -\beta)$) shows that for each $z_0 \in P_{\rm cs}X_p^0$ there exists a unique solution $v = \Phi_{cs}(z_0) \in \mathbb{E}_1(\mathbb{R}_+, -\beta')$ of the equation $v = \mathcal{L}_{cs}(z_0, v)$, where $\Phi_{cs} \in C^1(P_{\rm cs}X_p^0; \mathbb{E}_1(\mathbb{R}_+, -\beta))$ and $\Phi_{cs}(0) = 0$. Due to [21, (4.4)], the derivatives $\Phi'_{cs}(z_0) \in \mathcal{B}(P_{\rm cs}X_p^0, \mathbb{E}_1(\mathbb{R}_+, -\beta))$ are bounded uniformly in z_0. We then introduce

$$\vartheta_{cs}(z_0) = P_{\rm cs}\mathcal{N}_p\gamma_0\mathbb{H}_{\digamma_+}(\Phi_{cs}(z_0)) \qquad \text{and}$$

$$\phi_{cs}(z_0) = \gamma_0 P_{\rm u}\Phi_{cs}(z_0) = -\int_0^\infty T_u(-\tau)P_{\rm u}[G_{\digamma_+}(\Phi_{cs}(z_0))(\tau) + \Pi H_{\digamma_+}(\Phi_{cs}(z_0))(\tau)]d\tau,$$

for $z_0 \in P_{\rm cs}X_p^0$. Taking also into account (2.13), we see that $\phi_{cs} \in C^1(P_{\rm cs}X_p^0; P_{\rm u}X_p)$ and $\vartheta_{sc} \in C^1(P_{\rm cs}X_p^0; P_{\rm cs}X_p)$ with bounded derivatives and that $\phi_{cs}(0) = \vartheta_{cs}(0) = 0$ and $\phi'_{cs}(0) = \vartheta'_{cs}(0) = 0$. The inclusion '$\subset$' in (5.1) is clear by the above definitions, with $v = \Phi_{cs}(z_0)$. Moreover, $z_0 = P_{\rm cs}\mathcal{P}v(0) = P_{\rm cs}\mathcal{P}(u_0 - u_*)$. Conversely, let $v \in \mathbb{E}_1(\mathbb{R}_+, -\beta)$ solve (3.7) on $J = \mathbb{R}_+$. Proposition 2.5 then implies $v = L^+_{P_{\rm cs}, A_0}(P_{\rm cs}v(0), \mathbb{G}_{\digamma_+}(v), \mathbb{H}_{\digamma_+}(v))$. Setting $z_0 = P_{\rm cs}\mathcal{P}v(0)$ and using $B_*v(0) = \mathbb{H}_{\digamma_+}(v)(0)$, we obtain $P_{\rm cs}v(0) = z_0 + P_{\rm cs}\mathcal{N}_p\gamma_0\mathbb{H}_{\digamma_+}(v)$. Therefore $v = \mathcal{L}_{cs}(z_0, v)$ which entails $v = \Phi_{cs}(z_0)$. This fact leads to $v(0) + u_* \in \widetilde{\mathcal{M}}_{cs}$.

(b) Take $u_0 \in \widetilde{\mathcal{M}}_{cs} \cap B_{X_p}(u_*, \rho)$ for some $\rho > 0$ and the corresponding solution v of (3.7) on $J = \mathbb{R}_+$ given by (5.1). From part (a) we deduce that

$$\|v\|_{\mathbb{E}_1(\mathbb{R}_+,-\beta)} = \|\Phi_{cs}(z_0) - \Phi_{cs}(0)\|_{\mathbb{E}_1(\mathbb{R}_+,-\beta)} \le c\,|z_0|_p \le c\,|v(0)|_p \tag{5.3}$$

for constants independent of v. So (3.4) yields

$$\|v\|_{\mathbb{E}_1([0,6])} \le c'\rho \tag{5.4}$$

for a constant c' that does not depend on v and ρ. We take

$$\rho \le \rho_1 := \frac{\eta}{c'(1+c_R)}, \tag{5.5}$$

cf. (3.3). Then $\digamma_+(t,v) = 1$ for $0 \le t \le 4$ by Remark 3.1. As a result, v solves the original problem (2.21) on $[0,4]$.

(c.i) Let $u_0 \in \mathcal{M}_{\mathrm{cs}}$ and denote by u the solution of (1.1) on $[0,t_0]$ with $u(0) = u_0$, for some $t_0 > 0$. We set $w = u - u_*$. Let $v \in \mathbb{E}_1(\mathbb{R}_+,-\beta)$ be the solution of (3.7) with $v(0) = u_0 - u_*$ given by (5.1). We assume that $|w(t)|_p < \rho$ for $0 \le t \le t_0$. We want to show that $w(t) = v(t)$ and $u(t) \in \mathcal{M}_{\mathrm{cs}}$ for $0 \le t \le t_0$. First we consider the case when $t_0 \le 2$. Part (b) shows that $\digamma_+(t,v) = 1$ and that v solves (2.21) for $0 \le t \le t_0$. Then $w(t) = v(t)$ for $0 \le t \le t_0$ by the uniqueness of (1.1). We further set $\tilde{v}(t) = v(t+t_0)$ for $t \ge 0$. Remark 3.5 yields that $\digamma_+(t,\tilde{v}) = \digamma_+(t+t_0,v)$ for $t \ge 2$. Further, we have $\|\tilde{v}\|_{\mathbb{E}_1([t-2,t+2]\cap\mathbb{R}_+)} \le \|v\|_{\mathbb{E}_1([0,6])} \le \eta/(1+c_R)$ for $0 \le t \le 2$ due to (5.4) and (5.5). Remark 3.1 thus implies that $\digamma_+(t,\tilde{v}) = 1$ for $0 \le t \le 2$. Finally, $\digamma_+(t+t_0,v) = 1$ for $0 \le t \le 2$ by part (b). Therefore $\digamma_+(t,\tilde{v}) = \digamma_+(t+t_0,v)$ for all $t \ge 0$, and so $\tilde{v} \in \mathbb{E}_1(\mathbb{R}_+,-\beta)$ solves (3.7) on $J = \mathbb{R}_+$ with $\tilde{v}(0) = v(t_0)$. This means that $v(t_0) + u_* \in \widetilde{\mathcal{M}}_{cs} \cap B_{X_p}(u_*,\rho) = \mathcal{M}_{\mathrm{cs}}$. Since we can replace here t_0 by $t \in [0,t_0]$, (the proof of) part (a) yields $u(t) = u_* + v(t) = u_* + P_{\mathrm{cs}}v(t) + \phi_{cs}(P_{\mathrm{cs}}\mathcal{P}v(t)) + \vartheta_{cs}(P_{\mathrm{cs}}\mathcal{P}v(t)) \in \mathcal{M}_{\mathrm{cs}}$ for $0 \le t \le t_0$. If $t_0 > 2$, we obtain the assertion by a finite iteration of this argument.

(c.ii) Let $u_0 \in \mathcal{M}_{\mathrm{cs}}$ and assume that there is a solution u of (1.1) on $[-t_0,0]$ with $u(0) = u_0$, for some $t_0 > 0$. We set $w(t) = u(t) - u_*$ and assume that $|w(t)|_p < \rho$ for $-t_0 \le t \le 0$. Let $v \in \mathbb{E}_1(\mathbb{R}_+,-\beta)$ be the solution of (3.7) with $v(0) = u_0 - u_*$ given by (5.1). We want to show that $u(t) \in \mathcal{M}_{\mathrm{cs}}$ for $-t_0 \le t \le 0$. To this aim, we set $w(t) = v(t)$ and $z(t) = w(t-t_0)$ for $t \ge 0$. Clearly, $z \in \mathbb{E}_1(\mathbb{R}_+,-\beta)$, $z(0) = w(-t_0)$, and z satisfies the first two equations in (2.21) on $[0,t_0+2]$ since w and v solve (2.21) on $[-t_0,0]$ and $[0,2]$, respectively. Take $t \in [0,t_0+2]$ and s with $|t-s| \le 1/4$. Note that $[0,1] \subset J(s)$ if $J(s) \cap \mathbb{R}_- \ne \emptyset$. We thus deduce from (3.3) that

$$\begin{aligned}
\|R_+z\|_{\mathbb{E}_1(J(s))} &\le \|z\|_{\mathbb{E}_1(J(s)\cap[0,t_0])} + \|z\|_{\mathbb{E}_1(J(s)\cap[t_0,\infty))} + \|R_+z\|_{\mathbb{E}_1(J(s)\cap[-1,0])}\\
&\le (1+c_R)\big(\|w(\cdot-t_0)\|_{\mathbb{E}_1(J(s)\cap[0,t_0])} + \|v(\cdot-t_0)\|_{\mathbb{E}_1(J(s)\cap[t_0,\infty))}\big)\\
&= (1+c_R)\big(\|w\|_{\mathbb{E}_1(J(s-t_0)\cap[-t_0,0])} + \|v\|_{\mathbb{E}_1(J(s-t_0)\cap\mathbb{R}_+)}\big).
\end{aligned}$$

Since w solves (2.21) on $J(s-t_0) \cap [-t_0,0] =: [a,b]$ and $|w(a)|_p < \rho$, Remark 2.3 with $T = 3$ yields $\|w\|_{\mathbb{E}_1([a,b])} \le c_*\rho$ as soon as $\rho > 0$ is sufficiently small. Moreover, $\|v\|_{\mathbb{E}_1(J(s-t_0)\cap\mathbb{R}_+)} \le c'\rho$ because of (5.4) and $J(s-t_0) \cap \mathbb{R}_+ \subset [0,4]$. As a result,

$$\|R_+z\|_{\mathbb{E}_1(J(s))} \le (1+c_R)(c_*+c')\rho \le \eta \quad \text{for} \quad \rho \le \rho_2 := \frac{\eta}{(1+c_R)(c_*+c')}, \tag{5.6}$$

and so $\digamma_+(t,z) = 1$ for $0 \le t \le t_0+2$. (Observe that ρ_2 is less than the number ρ_1 given by (5.5).) The function z thus satisfies (3.7) for $0 \le t \le t_0+2$. For $t \ge t_0+2$, we have $\digamma_+(t,z) = \digamma_+(t-t_0,v)$ by Remark 3.5. In particular, z

fulfills the equations (3.7) also for $t \geq t_0 + 2$. Summing up, z solves (3.7) on $\mathbb{R}_+$ and so $u_* + z(0) = u(-t_0) \in \widetilde{\mathcal{M}}_{cs} \cap B_{X_p}(u_*, \rho) = \mathcal{M}_{\rm cs}$. Replacing here $-t_0$ by $t \in [-t_0, 0]$ and writing $v(t) = w(t)$ for $-t_0 \leq t \leq 0$, we arrive at $u(t) = u_* + v(t) = u_* + P_{\rm cs}v(t) + \phi_{cs}(P_{\rm cs}\mathcal{P}v(t)) + \vartheta_{cs}(P_{\rm cs}\mathcal{P}v(t)) \in \mathcal{M}_{\rm cs}$ for $-t_0 \leq t \leq 0$.

(d) Assume that $u_0 = u_* + v_0 \in \mathcal{M}_{\rm cs} \cap \mathcal{M}_{\rm u}$. We take $\beta + \epsilon \in (\beta, \omega_u)$. Let $v \in \mathbb{E}_1(\mathbb{R}_+, -\beta)$ be the solution of (3.7) with $v(0) = v_0$ given by (5.1). Due to Theorem 4.1(b), there is a solution w of (2.21) on $\mathbb{R}_-$ with $w(0) = v_0$ satisfying

$$|w(t)|_p \leq \overline{c}e^{(\beta+\epsilon)t}|v_0|_p \leq \overline{c}\rho \tag{5.7}$$

for all $t \leq 0$ if $\rho > 0$ is sufficiently small. We choose $\rho \leq \rho_3 := \rho_2/\overline{c}$ (see (5.6)) and take $t \leq 0$. Then part (c.ii) of the proof implies that $u_* + w(t) \in \mathcal{M}_{\rm cs}$ and that the function $z_t \in \mathbb{E}_1(\mathbb{R}_+, -\beta)$ given by $z_t(\tau) = w(t + \tau)$ for $\tau \in [0, -t]$ and $z_t(\tau) = v(t + \tau)$ for $\tau \geq -t$ solves (3.7) on $J = \mathbb{R}_+$. From (5.3) we deduce that

$$\|z_t\|_{\mathbb{E}_1(\mathbb{R}_+,-\beta)} \leq c\,|z_t(0)|_p = c\,|w(t)|_p\,, \tag{5.8}$$

where the constants do not depend on $t \leq 0$. Using (2.11), (5.8) and (5.7), we have

$$|v_0|_p = e^{-\beta t}|e^{-\beta(-t)}z_t(-t)|_p \leq ce^{-\beta t}\,\|z_t\|_{\mathbb{E}_1(\mathbb{R}_+,-\beta)} \leq ce^{-\beta t}|w(t)|_p \leq ce^{\epsilon t}|v(0)|_p$$

with the constants independent of t. Letting $t \to -\infty$, we have $u_0 - u_* = v_0 = 0$. $\square$

Theorem 5.2. *Assume Hypothesis 2.1 and* (4.1)*. Take any $\alpha \in (\omega_{cu}, \omega_s)$. Then there is a number $\eta_{cu} > 0$ such that for each $\eta \in (0, \eta_{cu}]$ there exists a radius $\rho = \rho(\eta) > 0$ such that the following assertions hold for the cutoff $\digamma_-$ defined for the chosen $\eta \in (0, \eta_{cu}]$.*

(a) There exists a map $\phi_{cu} \in C^1(P_{\rm cu}X_0; P_{\rm s}X_p)$ with a bounded derivative such that $\phi_{cu}(0) = 0$, $\phi'_{cu}(0) = 0$, and

$$\begin{aligned}\widetilde{\mathcal{M}}_{cu} &:= \big\{u_0 = u_* + z_0 + \phi_{cu}(z_0) : z_0 \in P_{\rm cu}X_0\big\} \\ &= \{u_0 = u_* + v(0) : \exists \text{ solution } v \in \mathbb{E}_1(\mathbb{R}_-, \alpha) \text{ of } (3.7) \text{ on } J = \mathbb{R}_-\}.\end{aligned} \tag{5.9}$$

Moreover, the function v in (5.9) *is given by $v = \Phi_{cu}(P_{\rm cu}(u_0 - u_*))$ for a map $\Phi_{cu} \in C^1(P_{\rm cu}X_0; \mathbb{E}_1(\mathbb{R}_-, \alpha))$ having a bounded derivative.*

(b) Define $\mathcal{M}_{\rm cu} = \widetilde{\mathcal{M}}_{cu} \cap B_{X_p}(u_*, \rho)$. Let $u_0 \in \mathcal{M}_{\rm cu}$ and v be the function from (5.9) *with $u_0 = v(0) + u_*$. Then $\digamma_-(t, v) = 1$ and v solves the original equation* (2.21) *(at least) for $t \in [-4, 0]$. The dimension of $\mathcal{M}_{\rm cu}$ is equal to* $\dim P_{\rm cu}X_0$.

(c) Let $u_0 \in \mathcal{M}_{\rm cu}$ and v be given by (5.9)*. If the forward solution u of* (1.1) *exists and stays in $B_{X_p}(u_*, \rho)$ on $[0, t_0]$ for some $t_0 > 0$, then $u(t) = u_* + v(t) \in \mathcal{M}_{\rm cu}$ for $0 \leq t \leq t_0$. If the function $\hat{u} = u_* + v$ stays in $B_{X_p}(u_*, \rho)$ on $[t_0, 0]$ for some $t_0 < 0$, then $\hat{u}(t) = u_* + v(t) \in \mathcal{M}_{\rm cu}$ and $\hat{u}$ solves* (1.1) *for $t_0 \leq t \leq 0$. In particular, $v(t) = P_{\rm cu}v(t) + \phi_{cu}(P_{\rm cu}v(t))$ for $t \in [0, t_0]$, resp. $t \in [t_0, 0]$.*

(d) We have $\mathcal{M}_{\rm cu} \cap \mathcal{M}_{\rm s} = \{u_*\}$.

(e) Assume, in addition, that (RR) holds. Then there is a $\rho_0 > 0$ such that the map $\phi_{cu} : P_{\rm cu}X_0 \cap B_{X_p}(0, \rho_0) \to P_{\rm s}X_1$ is Lipschitz.

Proof. Parts (a)–(d) of the following proof are similar to the proof of the previous theorem so we can omit some details and focus on the differences.

(a) We define the Lyapunov–Perron map $\mathcal{L}_{cu} : P_{\rm cu}X_0 \times \mathbb{E}_1(\mathbb{R}_-, \alpha) \to \mathbb{E}_1(\mathbb{R}_-, \alpha)$ by setting $\mathcal{L}_{cu}(z_0, v) = L^-_{P_{\rm s},A_0}(z_0, \mathbb{G}_{\digamma_-}(v), \mathbb{H}_{\digamma_-}(v))$, where the operators $L^-_{P_{\rm cu},A_0}$, $\mathbb{G}_{\digamma_-}$ and $\mathbb{H}_{\digamma_-}$ are given by (2.27) and (3.6). Using Propositions 2.6, 3.6 and 3.8, we find $\eta_{cu} > 0$ such that the assumptions of Theorem 3 of [21] hold for the cutoff $\digamma_-$ with the parameter $\eta \in (0, \eta_{cu}]$. As a result, for each $z_0 \in P_{\rm cu}X_0$ there exists a unique solution $v = \Phi_{cu}(z_0) \in \mathbb{E}_1(\mathbb{R}_-, \alpha')$ of the equation $v = \mathcal{L}_{cu}(z_0, v)$, where $\Phi_{cu} \in C^1(P_{\rm cu}X_0; \mathbb{E}_1(\mathbb{R}_-, \alpha))$, $\Phi_{cu}(0) = 0$, and the derivatives $\Phi'_{cu}(z_0) \in \mathcal{B}(P_{\rm cu}X_0, \mathbb{E}_1(\mathbb{R}_-, \alpha))$ are bounded uniformly in z_0. We then introduce the map

$$\phi_{cu}(z_0) = \gamma_0 P_{\rm s}\Phi_{cu}(z_0) = \int_{-\infty}^{0} T(-\tau)P_{\rm s}[G_{\digamma_-}(\Phi_{cu}(z_0))(\tau) + \Pi H_{\digamma_-}(\Phi_{cu}(z_0))(\tau)]\, d\tau,$$

for $z_0 \in P_{\rm cu}X_0$. Due to (2.13), we obtain that $\phi_{cu} \in C^1(P_{\rm cu}X_0; P_{\rm s}X_p)$ with a bounded derivative and that $\phi_{cu}(0) = 0$ and $\phi'_{cu}(0) = 0$. Equality (5.9) follows from Proposition 2.6, where $v = \Phi_{cu}(z_0)$ and $z_0 = P_{\rm cu}(u_0 - u_*)$.

(b) Take $u_0 \in \widetilde{\mathcal{M}}_{cu} \cap B_{X_p}(u_*, \rho)$ for some $\rho > 0$ and the corresponding solution v of (3.7) given by (5.9). From (3.4) and part (a) we deduce

$$\|v\|_{\mathbb{E}_1([-6,0])} \le c_E\, \|v\|_{\mathbb{E}_1(\mathbb{R}_-,\alpha)} \le cc_E\, |z_0|_p \le c'\, |v(0)|_p \le c'\rho \tag{5.10}$$

with the constants independent of v and α. We take

$$\rho \le \rho_1 := \frac{\eta}{c'(1 + c_R)}, \tag{5.11}$$

cf. (3.3). Then $\digamma_-(t, v) = 1$ for $-4 \le t \le 0$ by Remark 3.1. As a result, v solves the original problem (2.21) on $[-4, 0]$.

(c.i) Take $u_0 \in \mathcal{M}_{\rm cu}$ such that the solution u of (1.1) on $[0, t_0]$ with $u(0) = u_0$ stays in $B_{X_p}(u_*, \rho)$ for some $\rho, t_0 > 0$. We set $w = u - u_*$. Let $v \in \mathbb{E}_1(\mathbb{R}_-, \alpha)$ be the solution of (3.7) on $J = \mathbb{R}_-$ with $v(0) = u_0 - u_*$ given by (5.9). We further define $w(t) = v(t)$ and $z(t) = w(t + t_0)$ for $t \le 0$. Clearly, $z \in \mathbb{E}_1(\mathbb{R}_-, \alpha)$, $z(0) = w(t_0)$, and z satisfies the first two equations in (1.1) on $[-t_0 - 2, 0]$ since w and v solve (1.1) on $[0, t_0]$ and $[-2, 0]$, respectively. Take $t \in [-t_0 - 2, 0]$ and s with $|t - s| \le 1/4$. As in part (c.ii) if the proof of Theorem 5.2, we deduce from (3.3) that

$$\begin{aligned}\|R_- z\|_{\mathbb{E}_1(J(s))} &\le (1 + c_R)\big(\|w(\cdot + t_0)\|_{\mathbb{E}_1(J(s)\cap[-t_0,0])} + \|v(\cdot + t_0)\|_{\mathbb{E}_1(J(s)\cap(-\infty,-t_0])}\big)\\ &= (1 + c_R)\big(\|w\|_{\mathbb{E}_1(J(s+t_0)\cap[0,t_0])} + \|v\|_{\mathbb{E}_1(J(s+t_0)\cap\mathbb{R}_-)}\big).\end{aligned}$$

Remark 2.3 shows that $\|w\|_{\mathbb{E}_1([a,b])} \le c_*\rho$ for sufficiently small $\rho > 0$ since w solves (1.1) on $J(s + t_0) \cap [0, t_0] =: [a, b]$. Using $(J(s + t_0) \cap \mathbb{R}_-) \subset [-4, 0]$ and (5.10), we estimate $\|v\|_{\mathbb{E}_1(J(s+t_0)\cap\mathbb{R}_-)} \le c'\rho$. Consequently,

$$\|R_- z\|_{\mathbb{E}_1(J(s))} \le (1 + c_R)(c_* + c')\rho \le \eta \quad \text{for} \quad \rho \le \rho_2 := \frac{\eta}{(1 + c_R)(c_* + c')}\,, \tag{5.12}$$

and hence $\digamma_-(t, z) = 1$ for $-t_0 - 2 \le t \le 0$. The function z thus satisfies (3.7) for $-t_0 - 2 \le t \le 0$. Moreover, Remark 3.5 yields that $\digamma_-(t, z) = \digamma_-(t + t_0, v)$ for $t \le -t_0 - 2$; and so z fulfills the equations (3.7) for $t \le -t_0 - 2$. Summing up, we have shown that z solves (3.7) on $\mathbb{R}_-$, and so $u_* + z(0) = u(t_0) \in \mathcal{M}_{\rm cu}$.

(c.ii) Let $u_0 \in \mathcal{M}_{\rm cu}$ and v be given by (5.9). Assume that $\hat{u} = u_* + v$ stays in $B_{X_p}(u_*, \rho)$ on $[t_0, 0]$ for some $t_0 < 0$. We first consider the case when $t_0 \in [-2, 0)$. Part (b) shows that $\digamma_-(t, v) = 1$ and v solves (2.21) on $[t_0, 0]$. We further set $\tilde{v}(t) = v(t + t_0)$ for $t \le 0$. From Remark 3.5 it follows that $\digamma_-(t, \tilde{v}) = \digamma_-(t + t_0, v)$ for $t \le -2$.. Since $\|\tilde{v}\|_{\mathbb{E}_1([t-2,t+2]\cap\mathbb{R}_-)} \le \|v\|_{\mathbb{E}_1([-6,0])} \le \eta/(1 + c_R)$ for $-2 \le t \le 0$ by (5.3), Remark 3.1 yields $\digamma_-(t, \tilde{v}) = 1$ for $-2 \le t \le 0$. Finally, $\digamma_-(t + t_0, v) = 1$ for $-2 \le t \le 0$ due to part (b); so that $\digamma_-(t, \tilde{v}) = \digamma_-(t + t_0, v)$ for all $t \le 0$. As a result, $\tilde{v} \in \mathbb{E}_1(\mathbb{R}_-, \alpha)$ solves (3.7) on $J = \mathbb{R}_-$ with $\tilde{v}(0) = v(t_0)$. This means that $v(t) + u_* \in \mathcal{M}_{\rm cu}$ for each $t \in [t_0, 0]$, as asserted. The general case $t_0 < -2$ is then established by repeating the arguments for the first case finitely many times.

(d) Assume that $u_0 = u_* + v_0 \in \mathcal{M}_{\rm cu} \cap \mathcal{M}_{\rm s}$. Let $v \in \mathbb{E}_1(\mathbb{R}_-, \alpha)$ be the solution of (3.7) with $v(0) = v_0$ given by (5.9). For $\alpha + \epsilon \in (\alpha, \omega_s)$, there is a solution w of (2.21) on $\mathbb{R}_+$ with $w(0) = v_0$ satisfying $|w(t)|_p \le \bar{c}e^{-(\alpha+\epsilon)t}|v_0|_p \le \bar{c}\rho$ for all $t \ge 0$ if $\rho > 0$ sufficiently small, due to Theorem 4.1(a). Set $w(t) = v(t)$ for $t \le 0$. If we choose $\rho \le \rho_3 := \rho_2/\bar{c}$ (see (5.12)), then part (c.i) of the proof shows that $u_* + w(t) \in \mathcal{M}_{\rm cu}$ for $t \ge 0$ and that the function $z_t = w(\cdot + t) \in \mathbb{E}_1(\mathbb{R}_-, \alpha)$ solves (3.7) on $J = \mathbb{R}_-$. So estimate (5.10) yields $\|z_t\|_{\mathbb{E}_1(\mathbb{R}_-,\alpha)} \le c\,|w(t)|_p$, where the constant does not depend on $t \ge 0$. Using also (2.11), we arrive at

$$|v_0|_p = e^{\alpha t}|e^{\alpha(-t)}z_t(-t)|_p \le ce^{\alpha t}\,\|z_t\|_{\mathbb{E}_1(\mathbb{R}_-,\alpha)} \le ce^{\alpha t}|w(t)|_p \le ce^{-\epsilon t}|v(0)|_p$$

for constants independent of $t \ge 0$. Letting $t \to \infty$, we deduce $u_0 - u_* = v_0 = 0$.

(e) Assertion (e) can be shown as the last assertion in Theorem 4.1(b). □

Corollary 5.3. *Assume that Hypothesis 2.1 and* (2.34) *hold. Then there is a number* $\overline{\rho} > 0$ *such that* $\mathcal{M}_{\rm c} \cap B_{\overline{\rho}} = \mathcal{M}_{\rm cs} \cap \mathcal{M}_{\rm cu} \cap B_{\overline{\rho}}$, $\mathcal{M}_{\rm c} \cap \mathcal{M}_{\rm s} \cap B_{\overline{\rho}} = \{u_*\}$, *and* $\mathcal{M}_{\rm c} \cap \mathcal{M}_{\rm u} \cap B_{\overline{\rho}} = \{u_*\}$. *Here,* $B_{\overline{\rho}} = B_{X_p}(u_*, \overline{\rho})$ *and* $\mathcal{M}_k$, $k \in \{\rm s, c, cs, cu, u\}$, *are the manifolds obtained in Theorems 4.1, 4.2, 5.1, and 5.2.*

Proof. We set $\eta = \min\{\eta_c, \eta_{cs}, \eta_{cu}\} > 0$ and let ρ' be less than or equal to the minimum of the numbers $\rho(\eta)$ obtained in Theorems 4.2, 5.1, and 5.2. For $u_0 \in \mathcal{M}_{\rm c} \cap B_{X_p}(u_*, \rho')$, there exists the function v from (4.6) with $v(0) = u_0 - u_*$, where $\digamma(t, v) = 1$ for $|t| \le 2$. For $s \in [0, 9/4]$ and $\sigma \in [-9/4, 0]$, we have $\|R_+v\|_{\mathbb{E}_1(J(s))} \le c'_R\,\|v\|_{\mathbb{E}_1([0,4])}$ and $\|R_-v\|_{\mathbb{E}_1(J(\sigma))} \le c'_R\,\|v\|_{\mathbb{E}_1([-4,0])}$ for some constant c'_R. In view of (4.10), we can decrease $\rho' > 0$ in order to obtain $\digamma_+(t, v) = 1$ for $t \in [0, 2]$ and $\digamma_-(t, v) = 1$ for $t \in [-2, 0]$. Thus $\digamma(t, v) = \digamma_\pm(t, v)$ for $t \in \mathbb{R}_\pm$ by Remark 3.5, and so the restrictions of v to $\mathbb{R}_+$ and $\mathbb{R}_-$ belong to $\widetilde{\mathcal{M}}_{cs}$ and $\widetilde{\mathcal{M}}_{cu}$ by (5.1) and (5.9), respectively. As a result, $u_0 \in \mathcal{M}_{\rm cs} \cap \mathcal{M}_{\rm cu}$. The converse inclusion can be shown similarly, thereby fixing a possibly smaller $\rho' =: \overline{\rho}$. The last two equalities then follow from Theorems 5.1 and 5.2. □

Remark 5.4. We now sketch an alternative construction of a local center manifold $\widehat{\mathcal{M}}_c$ as the intersection of $\mathcal{M}_{\rm cs}$ and $\mathcal{M}_{\rm cu}$, cf. [4]. Let the assumptions of Theorem 4.2 hold. Then Theorems 5.1 and 5.2 can be proved as above so that we have local center–stable and center–unstable manifolds $\mathcal{M}_{\rm cs}$ and $\mathcal{M}_{\rm cu}$ with corresponding maps ϕ_{cs}, ϑ_{cs} and ϕ_{cu}. For technical reasons, we need another description of $\mathcal{M}_{\rm cs}$. To this aim, we solve the fixed point problem

$$v = L^+_{P_{\rm cs},A_0}(z_0 + P_{\rm s}\mathcal{N}_p\gamma_0\mathbb{H}_{\digamma_+}(v), \mathbb{G}_{\digamma_+}(v), \mathbb{H}_{\digamma_+}(v)) \tag{5.13}$$

for $z_0 \in P_{\rm cs}X^0_p$ and $v \in \mathbb{E}_1(\mathbb{R}_+, -\beta)$. As in Theorem 5.1, for sufficiently small $\eta \le \eta_{cs}$ we obtain a solution map $\Phi_{c|s} : z_0 \mapsto v$ for (5.13), and so we can define $\vartheta_{c|s}(z_0) = P_{\rm s}\mathcal{N}_p\gamma_0\mathbb{H}_{\digamma_+}(\Phi_{c|s}(z_0)) \in P_{\rm s}X_p$ and $\phi_{c|s}(z_0) = \gamma_0\Phi_{c|s}(z_0) - z_0 - \vartheta_{c|s}(z_0) \in P_{\rm u}X_0$ for $z_0 \in P_{\rm cs}X^0_p$. We now fix the same η in the construction of $\mathcal{M}_{\rm cs}$. It is possible to show that $\widetilde{\mathcal{M}}_{cs} = \{u_0 = u_* + z_0 + \vartheta_{c|s}(z_0) + \phi_{c|s}(z_0) : z_0 \in P_{\rm cs}X^0_p\}$. For $x_0 \in P_{\rm s}X^0_p$, $y \in P_{\rm c}X_0$ and $z \in P_{\rm u}X_0$ with norms less than $\rho_0 > 0$, we further set

$$\Psi(y, (x_0, z)) = \Big(x_0 - \phi_{cu}(y + z) + P_{\rm s}\mathcal{N}_pH(y + z + \phi_{cu}(y + z)), z - \phi_{c|s}(x_0 + y)\Big).$$

Observe that $B_*\phi_{cu}(y + z) = B_*(y + z + \phi_{cu}(y + z)) = H(y + z + \phi_{cu}(y + z)) = B_*P_{\rm s}\mathcal{N}_pH(y + z + \phi_{cu}(y + z))$ since $v_0 = y + z + \phi_{cu}(y + z))$ is the final value of a solution v to the cutoff problem satisfying $\digamma_-(0, v) = 1$ if $\rho_0 > 0$ is small enough. Hence, Ψ maps into the set $V := P_{\mathrm{s}}X_p^0 \times P_{\mathrm{u}}X_0$. Because of $\Psi(0,0) = 0$ and $D_2\Psi(0,0) = I_V$, there exist $r, \rho > 0$ and $\psi = (\psi_s, \psi_u) \in C^1(B_{X_p}(0,\rho) \cap P_{\mathrm{c}}X_0, B_{X_p}(0,r) \cap V)$ such that $(x_0, z) = \psi(y)$ is the unique solution of the equation $\Psi(y,(x_0,z)) = 0$ in these balls. We now introduce

$$\widehat{\phi}_c(y) = \psi_s(y) + \vartheta_{c|s}(y + \psi_s(y)) + \psi_u(y),$$
$$\widehat{\mathcal{M}}_c = \{u_* + y + \widehat{\phi}_c(y) : y \in B_{X_p}(0,\rho) \cap P_{\mathrm{c}}X_p\}.$$

Then $\widehat{\phi}_c \in C^1(B_{X_p}(0,\rho) \cap P_{\mathrm{c}}X_0; P_{\mathrm{su}}X_p)$, $\widehat{\phi}_c(0) = 0$, $\widehat{\phi}_c'(0) = 0$, and one can check that $\widehat{\mathcal{M}}_c = \mathcal{M}_{\mathrm{cs}} \cap \mathcal{M}_{\mathrm{cu}} \cap B_{X_p}(u_*, \rho)$ for a sufficiently small $\rho > 0$. (Here it is useful to work with the new description of $\mathcal{M}_{\mathrm{cs}}$.) Finally it can be seen that $\widehat{\mathcal{M}}_c$ has analogous properties as those stated for $\mathcal{M}_{\mathrm{c}}$ in Theorem 4.2(b)–(f) and Corollary 5.3. We point out that in this approach $\widehat{\mathcal{M}}_c$ is not constructed as the restriction of a global object such as $\widetilde{\mathcal{M}}_c$ in Theorem 4.2(a). In particular, for $\widehat{\mathcal{M}}_c$ there is no counterpart for the description given by (4.6) and the invariance property of $\widetilde{\mathcal{M}}_c$ stated in Theorem 4.2(a). $\Diamond$

6. Stability and attractivity of the center manifold

We now investigate the stability of the steady state u_* of (1.1) and the attractivity of $\mathcal{M}_{\mathrm{c}}$. As in Theorem 4.2, we assume that Hypothesis 2.1 and (2.34) hold. In parabolic problems, the center–unstable manifold is finite dimensional in many cases; e.g., if the spatial domain Ω is bounded. Moreover, there are important applications where $\mathcal{M}_{\mathrm{cu}}$ consists of equilibria only, see e.g. [10, Prop.6.4], [15]. Thus it is quite possible that one can check the stability of u_* with respect to the semiflow on $\mathcal{M}_{\mathrm{cu}}$ generated by (1.1) without knowing a priori that u_* is stable with respect to the full semiflow of (1.1) on $\mathcal{M}$. In Theorem 6.1 below we show that u_* is stable on $\mathcal{M}$ under the following conditions: $s(-A_0) \le 0$, u_* is stable on $\mathcal{M}_{\mathrm{cu}} = \mathcal{M}_{\mathrm{c}}$, $P_{\mathrm{cu}} = P_{\mathrm{c}}$ has finite rank, and the additional regularity assumption (RR) holds. In fact, we establish a stronger result saying that each solution starting sufficiently close to u_* converges exponentially to a solution on $\mathcal{M}_{\mathrm{c}}$. Here we can assume that $s(-A_0) \le 0$ without loss of generality since by Theorem 4.1 $-A_0$ has no spectrum in the open right halfplane if u_* is stable and P_{cu} has finite rank.

Theorem 6.1. *Let Hypothesis 2.1 and (RR) hold. Assume that the spectrum of $-A_0$ admits a splitting $\sigma(-A_0) = \sigma_s \cup \sigma_c$ corresponding to the spectral projections P_{s} and P_{c} such that P_{c} has finite rank, $\sigma_c \subset i\mathbb{R}$, and there is a number α with $\max \operatorname{Re} \sigma_s < -\alpha < 0$. Suppose that for each $r > 0$ there is a $\rho > 0$ such that for $u_0 \in \mathcal{M}_{\mathrm{c}}$ with $|P_{\mathrm{c}}(u_0 - u_*)|_0 < \rho$ the solution u of (1.1) exists and $u(t) \in \mathcal{M}_{\mathrm{c}} \cap B_{X_p}(u_*, r)$ for all $t \ge 0$. Then there is a $\overline{\rho} > 0$ such that for every $u_0 = u_* + v_0 \in \mathcal{M}$ with $|v_0|_p \le \overline{\rho}$ the solution $u = u_* + v$ of (1.1) exists on $\mathbb{R}_+$ and there is a solution $\overline{u}$ of (1.1) on $\mathbb{R}_+$ such that $\overline{u}(t) \in \mathcal{M}_{\mathrm{c}}$ for all $t \ge 0$ and*

$$|u(t) - \overline{u}(t)|_1 \le ce^{-\alpha t}\, |P_{\mathrm{s}}v_0 - \phi_c(P_{\mathrm{c}}v_0)|_p \tag{6.1}$$

for $t \ge 1$ and a constant c independent of u_0. As a result, u_* is stable for (1.1), i.e.: For each $r > 0$ there exists a $\rho' > 0$ such that for every $u_0 \in \mathcal{M} \cap B_{X_p}(u_*, \rho')$ the solution u of (1.1) exists on $\mathbb{R}_+$ and $u(t) \in B_{X_p}(u_*, r)$ for all $t \ge 0$.

Proof. Let $u = u_* + v$ solve (1.1) with the initial value $u_0 = u_* + v_0 \in \mathcal{M}$. We proceed in three steps: First, we derive a forward evolution equation in $P_{\mathrm{s}}X_p$ for the function $w = P_{\rm s}v - \phi_c(P_{\rm c}v)$ on a certain interval $[0,T]$ and estimate w employing this equation. Second, we take the solution z on $\mathcal{M}_{\rm c}$ with $P_{\rm c}z(T) = P_{\rm c}v(T)$ and estimate the function $y = P_{\rm c}(v - z)$ on an interval $[t_0, T]$ by means of a backward evolution equation in $P_{\rm c}X_0$ for y. Third, using the stability of $\mathcal{M}_{\rm c}$ we show that these estimates hold for all $T \ge t_0 \ge 1$ and construct the desired solution $\overline{u} = u_* + \overline{z}$ on $\mathcal{M}_{\rm c}$ by letting $T \to \infty$.

Step 1. Set $-\omega_s = \max \mathrm{Re}\, \sigma_s < 0$ and take constants $N \ge 1$ and $\delta \in (0, \omega_s)$ such that $\|e^{-tA_0P_{\rm c}}P_{\rm c}\|_{\mathcal{B}(X_0)} \le Ne^{-\delta t}$ for all $t \le 0$. Using Theorem 4.2, we fix a radius $\rho_c > 0$ such that ϕ_c is globally Lipschitz with the Lipschitz constant ℓ as a map from $P_cX_0 \cap \overline{B}_{X_0}(0, \rho_c)$ to X_1 and X_p, and such that $\|\phi_c'(\xi)\|_{\mathcal{B}(X_0)} \le \ell$ for $\xi \in P_cX_0 \cap \overline{B}_{X_0}(0, \rho_c)$. We set

$$\varepsilon_1(R) = \max_{x \in X_1, |x|_1 \le R} \{\|G'(x)\|_{\mathcal{B}(X_1,X_0)}\,, \|H'(x)\|_{\mathcal{B}(X_1,Y_1)}\}. \tag{6.2}$$

Due to (2.16), we can fix a (small) number $R > 0$ such that

$$d := N\varepsilon_1(R)(1 + \|P_{\rm c}\Pi\|_{\mathcal{B}(Y_1,X_0)})(1 + \ell\, \|P_{\rm c}\|_{\mathcal{B}(X_0,X_1)}) < \omega_s - \delta, \tag{6.3}$$

$$R\, \|P_{\rm c}\|_{\mathcal{B}(X_0,X_1)} \le \rho_c\,. \tag{6.4}$$

Then there exists a number $r > 0$ having the following properties:

(a) $r(1 + \ell)\|P_{\rm c}\|_{\mathcal{B}(X_p,X_1)} \le R/2$ and $r\, \|P_{\rm c}\|_{\mathcal{B}(X_p,X_0)} \le \rho_c$.

(b) If $|x_0|_p \le r$ or $|P_{\rm c}x_0|_0 \le r$, then the solution z on $\mathcal{M}_{\rm c}$ with $P_{\rm c}x_0 = P_{\rm c}z(0)$ exists on $[-2, \infty)$ and $|z(t)|_1 \le R$ for all $t \ge -2$.

(c) If $w_0 + u_* \in \mathcal{M}$ and $|w_0|_p \le r$, then the solution w of (2.21) with $w(0) = w_0$ exists for $t \in [0, 2]$, $\|w\|_{\mathbb{E}_1([0,2])} \le c_*r$ and $|w(1)|_1 \le R$ (where c_* is the constant given by Remark 2.3 with $T = 2$).

(d) $\hat{c}\varepsilon_2(c_*c_Pr) \le 1/2$, where $c_P = (1 + \ell)(\|P_{\rm c}\|_{\mathcal{B}(X_1)} + \|P_{\rm c}\|_{\mathcal{B}(X_0)})$, $\hat{c}$ is defined below in (6.8) and $\varepsilon_2(\cdot)$ is the Lipschitz constant from Remark 3.7.

(To obtain (b) and (c), we use the stability of u_* in $\mathcal{M}_{\rm c}$, Theorem 4.2, and [12, Prop.15].) Take $u_0 = v_0 + u_* \in \mathcal{M}$ with $|v_0|_p \le \rho \le \rho_1 \le r$, where $\rho_1 > 0$ is chosen such that the solution v of (2.21) exists on $[0, 4]$ and $|v(t)|_p \le r$ for $0 \le t \le 4$. (Use Remark 2.3 and (2.11).) Hence, $|v(t)|_1 \le R$ for $1 \le t \le 4$ and $|P_{\rm c}v(t)|_0 \le \rho_c$ for $0 \le t \le 4$ by Properties (c) and (a). Let $T \ge 4$ be the supremum of all t' such that the solution v exists on $[0, t']$ and $|v(t)|_p \le r$ for all $t \in [0, t']$. Seeking a contradiction, we suppose that $T < \infty$. Then T is in fact the maximum of all t' as above, and $|v(t)|_1 \le R$ for $1 \le t \le T$, due to Property (c). Define

$$w = P_{\rm s}v - \phi_c(P_{\rm c}v), \quad w_0 = w(0), \qquad x = v - w = P_{\rm c}v + \phi_c(P_{\rm c}v) \tag{6.5}$$

on $[0, T]$. Observe that $u_* + x(t) \in \mathcal{M}_{\rm c}$ and $P_{\rm c}x(t) = P_{\rm c}v(t)$ for $t \in [0, T]$ and that, in general, x is not a solution of (2.21). Recall the definition of A_0, $\mathcal{N}_1$ and Π, cf. (2.23). Using (2.21), (4.8), (4.9), (6.5), we deduce that

$$B_*w(t) = H(v(t)) - B_*\phi_c(P_{\rm c}x(t)) = H(v(t)) - H(x(t)) =: h(t), \tag{6.6}$$

$$\begin{aligned}
\dot{w}(t) &= P_{\rm s}(-A_*v(t) + G(v(t))) - \phi_c'(P_{\rm c}v(t))P_{\rm c}[G(v(t)) - A_*v(t)] \\
&\quad - \phi_c'(P_{\rm c}x(t))P_{\rm c}[A_*x(t) - G(x(t))] + P_{\rm s}(A_*x(t) - G(x(t))) \\
&= -P_{\rm s}(A_0 + \mu)(w(t) - \mathcal{N}_1h(t)) + \mu P_{\rm s}w(t) + P_{\rm s}(G(v(t)) - G(x(t))) \\
&\quad + \phi_c'(P_{\rm c}v(t))P_{\rm c}[(A_0 + \mu)(w(t) - \mathcal{N}_1h(t)) - \mu w(t) + G(x(t)) - G(v(t))] \\
&= -A_0P_{\rm s}w(t) + P_{\rm s}\Pi h(t) + P_{\rm s}(G(v(t)) - G(x(t))) \\
&\quad - \phi_c'(P_{\rm c}v(t))P_{\rm c}[\Pi h(t) + G(v(t)) - G(x(t))]
\end{aligned}$$

for $t \in (0,T]$, where we also employed (6.6) in the second part and $\dot{w}(t)$ exists in X_0. Setting $g = G(v) - G(x) - \phi_c'(P_c v)P_c[\Pi h + G(v) - G(x)]$, we obtain

$$w(t) = T(t-\tau)P_s w(\tau) + \int_\tau^t T_{-1}(t-\sigma)P_s[g(\sigma) + \Pi h(\sigma)]\, d\sigma$$

for $0 \le \tau \le t \le T$. We take $\tau \in [0, T-2]$ and $\alpha \in (0, \omega_s)$. In view of (6.6) and the exponential stability of $e_\alpha T(\cdot)P_s$, we can argue as in the proof of Proposition 8 in [12] (see inequality (43)) and estimate:

$$e^{-\alpha\tau}\|w\|_{\mathbb{E}_1([\tau,T],\alpha)} \le \hat{c}_0 \left[|w(\tau)|_p + e^{-\alpha\tau}\|g\|_{\mathbb{E}_0([\tau,T],\alpha)} + e^{-\alpha\tau}\|h\|_{\mathbb{F}([\tau,T],\alpha)}\right], \quad (6.7)$$

with a constant $\hat{c}_0$ independent of τ, T, r, ρ, and chosen uniformly for α contained in compact intervals in $(0,\omega_s)$. Since $|P_c v(t)|_0 \le \rho_c$ for $t \in [0,T]$ by (a), formula (6.5) yields $\|x\|_{\mathbb{E}_1(J)} \le c_P \|v\|_{\mathbb{E}_1(J)}$ for intervals $J \subset [0,T]$, where $c_P = (1+\ell)(\|P_c\|_{\mathcal{B}(X_1)} + \|P_c\|_{\mathcal{B}(X_0)})$. So we conclude from (6.7) and Remarks 2.3 and 3.7 that

$$e^{-\alpha\tau}\|w\|_{\mathbb{E}_1([\tau,T],\alpha)} \le \hat{c}_0|w(\tau)|_p + \hat{c}\varepsilon_2(c_* c_P r)e^{-\alpha\tau}\|w\|_{\mathbb{E}_1([\tau,T],\alpha)}\,,$$

where c_* is the constant given by Remark 2.3 and

$$\hat{c} := \hat{c}_0\,[2 + \ell(\|P_c\|_{\mathcal{B}(X_0)} + \|P_c\Pi\|_{\mathcal{B}(Y_1,X_0)})]. \quad (6.8)$$

Hence, for $0 \le \tau \le t \le T$ with $T - \tau \ge 2$, Property (d) above and (2.11) imply that

$$\begin{aligned} e^{-\alpha\tau}\|w\|_{\mathbb{E}_1([\tau,T],\alpha)} &\le \hat{c}_0|w(\tau)|_p + \tfrac{1}{2}\, e^{-\alpha\tau}\|w\|_{\mathbb{E}_1([\tau,T],\alpha)}\,, \\ e^{\alpha(t-\tau)}|w(t)|_p &\le c_0 e^{-\alpha\tau}\|w\|_{\mathbb{E}_1([\tau,T],\alpha)} \le 2c_0\hat{c}_0\,|w(\tau)|_p\,. \end{aligned} \quad (6.9)$$

Step 2. By Property (b), there exists a number $a \le T-2$ and a solution $z(\cdot; T, P_c v(T)) = z = P_c z + \phi_c(P_c z)$ on $\mathcal{M}_c$ of (2.21) on $[a,T]$ with $P_c z(T) = P_c v(T)$. Also, there is a minimal number $t_0 \in [1, T-2]$ such that $z(t) \in \mathcal{M}_c$ exists and $|z(t)|_1 \le R$ for $t_0 \le t \le T$. We set $y = P_c(v - z)$ and note that

$$v - z = y + w + \phi_c(P_c v) - \phi_c(P_c z). \quad (6.10)$$

Since v and z solve (2.21), we obtain

$$\begin{aligned} B_*(v(t) - z(t)) &= H(v(t)) - H(z(t)) =: h_1(t), \\ \dot{y}(t) &= P_c(-A_*(v(t) - z(t))) + P_c(G(v(t)) - G(z(t))) \\ &= -P_c[(A_0 + \mu)(v(t) - z(t) - \mathcal{N}_1 h_1(t)) - \mu(v(t) - z(t))] + P_c g_1(t) \\ &= -A_0 P_c y(t) + P_c\Pi h_1(t) + P_c g_1(t) \end{aligned} \quad (6.11)$$

for $t \in [t_0, T]$, where $g_1(t) := G(v(t)) - G(z(t))$. Since $y(T) = 0$ and $|v(t)|_1, |z(t)|_1 \le R$, equation (6.11) implies (cf. (6.2)) that

$$\begin{aligned} y(t) &= -\int_t^T e^{-(t-\tau)A_0 P_c} P_c(g_1(\tau) + \Pi h_1(\tau))\, d\tau, \\ |y(t)|_0 &\le \int_t^T N e^{-\delta(t-\tau)}(1 + \|P_c\Pi\|_{\mathcal{B}(Y_1,X_0)})\varepsilon_1(R)\,|v(\tau) - z(\tau)|_1\, d\tau\,. \end{aligned}$$

Recalling the definition of d in (6.3) and setting $d_0 = d\,(1 + \ell\,\|P_c\|_{\mathcal{B}(X_0,X_1)})^{-1}$, we then deduce from (6.10) and (6.4) that

$$e^{\delta t}|y(t)|_0 \le d\int_t^T e^{\delta\tau}|y(\tau)|_0\, d\tau + d_0\int_t^T e^{\delta\tau}\,|w(\tau)|_1\, d\tau.$$

Gronwall's inequality and Fubini's theorem thus yield

$$\begin{aligned}
e^{\delta t}|y(t)|_0 &\le d_0 \int_t^T e^{\delta\tau}\,|w(\tau)|_1\,d\tau + dd_0 \int_t^T e^{d(\tau-t)} \int_\tau^T e^{\delta\sigma}|w(\sigma)|_1\,d\sigma\,d\tau \\
&= d_0 \int_t^T e^{\delta\tau}\,|w(\tau)|_1\,d\tau + dd_0 \int_t^T e^{\delta\sigma}|w(\sigma)|_1 \int_t^\sigma e^{d(\tau-t)}\,d\tau\,d\sigma \\
&= d_0 \int_t^T e^{d(\sigma-t)} e^{\delta\sigma}|w(\sigma)|_1\,d\sigma.
\end{aligned}$$

There is an $\alpha \in (d+\delta, \omega_s)$ due to (6.3). Hölder's inequality and (6.9) thus lead to

$$|y(t)|_0 \le d_0 \int_t^T e^{(d+\delta)(\sigma-t)}|w(\sigma)|_1\,d\sigma \le ce^{-\alpha t}\,\|w\|_{\mathbb{E}_1([t,T],\alpha)} \le c\,|w(t)|_p \tag{6.12}$$

for $t \in [t_0, T]$ with $T - t \ge 2$. Here and below the constants c do not depend on t, t_0, T, v, ρ. Observe that $z = P_{\rm c}(v-y) + \phi_c(P_{\rm c}(v-y))$. Employing (6.4), $|v(t_0)|_p \le r$, Property (a), (6.12) (6.9) and (6.5), we then estimate:

$$\begin{aligned}
|z(t_0)|_1 &\le (1+\ell)\,(\|P_{\rm c}\|_{\mathcal{B}(X_0,X_1)}\,|y(t_0)|_0 + \|P_{\rm c}\|_{\mathcal{B}(X_p,X_1)}\,|v(t_0)|_p) \\
&\le c\,|w(t_0)|_p + R/2 \le c\,|w(0)|_p + R/2 \le c\,|v_0|_p + R/2.
\end{aligned}$$

So we can find $\rho_2 \in (0, \rho_1]$ such that $|z(t_0)|_1 \le 3R/4$ if $|v_0|_p \le \rho \le \rho_2$. As a result, $t_0 = 1$ and

$$|P_{\rm c}z(1)|_0 \le |y(1)|_0 + |P_{\rm c}v(1)|_0 \le c\,(|w(1)|_p + |v(1)|_p) \le c\,|v(1)|_p \le \overline{c}\,|v_0|_p\,, \tag{6.13}$$

where we used (6.12), (6.5), Remark 2.3, and (2.11). In view of (6.13) and the assumed stability of $\mathcal{M}_{\rm c}$, there exists a $\rho_3 \in (0, \rho_2]$ such that $|z(T)|_p \le r/2$ if $|v_0|_p \le \rho \le \rho_3$. From (6.10), (6.12) and (6.9), we then deduce

$$|v(T)|_p \le |z(T)|_p + |y(T)|_p + \ell\,|y(T)|_0 + |w(T)|_p \le \tfrac{r}{2} + c\,|w(0)|_p \le \tfrac{r}{2} + c\,|v_0|_p < r,$$

if we take $|v_0|_p \le \rho \le \rho_4$ for a sufficently small $\rho_4 \in (0, \rho_3]$. This fact contradicts the choice of r so that $T = \infty$; i.e., v solves (2.21) on $\mathbb{R}_+$ and $|v(t)|_p \le r$ for all $t \ge 0$. Therefore (6.9) and (6.12) hold for all $T \ge 4$ with uniform constants.

Step 3. In (6.13) we have seen that $P_{\rm c}z(1) = P_{\rm c}z(1; T, P_{\rm c}v(T))$ is bounded by $\overline{c}\,|v_0|_p$ for all $T \ge 4$. We fix $\overline{\rho} \in (0, \rho_4]$ with $\overline{c}\,\overline{\rho} \le r$ and take v_0 with $|v_0|_p \le \rho \le \overline{\rho}$. Since $P_{\rm c}$ has finite rank, there are $T_n \to \infty$ such that $P_{\rm c}z(1; T_n, P_{\rm c}v(T_n))$ converges to some $\zeta \in P_{\rm c}X_0$ with $|\zeta|_0 \le \overline{c}\rho \le r$. Let $\overline{z}$ be the solution on $\mathcal{M}_{\rm c}$ with $P_{\rm c}\overline{z}(1) = \zeta$. By Property (b) and (6.4), $\overline{z}(t) \in \mathcal{M}_{\rm c}$ exists for all $t \ge 0$ and $|P_c\overline{z}(t)|_0 \le \rho_c$. The functions $P_{\rm c}\overline{z}$ and $P_{\rm c}z(\cdot; T_n, P_{\rm c}v(T_n))$ satisfy the ode (4.7) so that

$$P_{\rm c}\overline{z}(t) = \lim_{n\to\infty} P_{\rm c}z(t; 1, P_{\rm c}z(1; T_n, P_{\rm c}v(T_n))) = \lim_{n\to\infty} P_{\rm c}z(t; T_n, P_{\rm c}v(T_n)).$$

Estimates (6.12) and (6.9) thus yield

$$|P_{\rm c}(v(t) - \overline{z}(t))|_0 = \lim_{n\to\infty} |P_{\rm c}(v(t) - z(t; T_n, P_{\rm c}v(T_n)))|_0 \le c\,|w(t)|_p \le ce^{-\alpha t}|w_0|_p$$

for $t \ge 1$. Combining this inequality with (6.5) and (6.9), we also obtain

$$|P_{\rm s}(v(t) - \overline{z}(t))|_p \le |w(t)|_p + |\phi_c(P_{\rm c}v(t)) - \phi_c(P_{\rm c}\overline{z}(t))|_p \le ce^{-\alpha t}|w_0|_p.$$

Inequality (6.1) now follows from the two preceeding estimates and Theorem A.1. Moreover, $|v(t)|_p \le |v(t) - \overline{z}(t)|_p + |\overline{z}(t)|_p \le c\rho + |\overline{z}(t)|_p$ for $t \ge 0$. Since $|\zeta|_0 \le \overline{c}\rho$, the stability of u_* is a consequence of the stability of $\mathcal{M}_{\rm c}$. $\square$

Appendix A. An additional regularity result

We now establish an improved version of Proposition 15 of [12] needed to show that the center, center-unstable and unstable manifolds are Lipschitz in X_1.

Theorem A.1. *Assume hypothesis (RR), and that (E) and (LS) hold at a function $u_0 \in X_p$ with $B(u_0) = 0$. Fix a number $T > 0$ which is strictly smaller than the maximal existence time $t^+(u_0)$ of the solution u of (1.1) such that conditions (E) and (LS) hold at the function $u(t)$ for each $t \in [0, T]$. Then there exists a $\rho > 0$ such that for each initial value $v_0 \in \mathcal{M}$ with $|v_0 - u_0|_p \le \rho$ the solution v of (1.1) with $v(0) = v_0$ satisfies*

$$\|t(\dot{v} - \dot{u})\|_{\mathbb{E}_1([0,T])} \le c\,|v_0 - u_0|_p, \tag{A.1}$$

where the constant c is independent of v_0 but may depend on u, T, ρ. In particular, for each $\tau \in (0, T)$ we have

$$\|v - u\|_{C^{1-1/p}([\tau,T];X_1)} \le c(\tau)\,|v_0 - u_0|_p. \tag{A.2}$$

Proof. The existence of a solution v with the initial value $v_0 \in B_{X_p}(u_0, \rho)$ was shown in [12, Thm.14] for sufficiently small $\rho > 0$, whereas the number $T > 0$ exists due to Remark 1 of [12]. Similarly, there is an $\epsilon \in (0, 1/2)$ such that $T' = (1+\epsilon)T < t^+(u_0)$ and (E), (LS) hold at all functions $u(t)$ for $t \in J' = [0, T']$. We set $z(t) = v(t)-u(t)$, $z_0 = v_0 - u_0$, and $w_\lambda(t) = v(\lambda t) - u(\lambda t)$ for $t \in J = [0, T]$ and $\lambda \in (1 - \epsilon, 1 + \epsilon)$. As in Section 2 of [12], we define $A_*(t)$, $B_*(t)$, $G(t, \cdot)$, and $H(t, \cdot)$ as in (2.15), (2.17), and (2.18) replacing u_* by $u(t)$. Note that z solves the resulting version of equation (2.21) with A_* replaced by $A_*(t)$ and B_* replaced by $B_*(t)$. Moreover, we denote by S the solution operator of the corresponding version of equation (2.22), see [12, Thm.2]. Then w_λ satisfies:

$$\begin{aligned}
\partial_t w_\lambda(t) &= \lambda(\dot{v}(\lambda t) - \dot{u}(\lambda t)) \\
&= \lambda(-A_*(\lambda t)(v(\lambda t) - u(\lambda t)) + G(\lambda t, v(\lambda t) - u(\lambda t))), \quad \text{on } \Omega, t > 0, \\
B_*(\lambda t) w_\lambda(t) &= H(\lambda t, v(\lambda t) - u(\lambda t)), \quad \text{on } \partial\Omega, t > 0, \\
w_\lambda(0) &= z_0, \quad \text{on } \Omega.
\end{aligned}$$

So w_λ solves the initial-boundary value problem

$$\begin{aligned}
\partial_t w(t) + A_*(t) w(t) &= \mathbb{G}(\lambda, w)(t), \quad \text{on } \Omega, t > 0, \\
B_*(t) w(t) &= \mathbb{H}(\lambda, w)(t), \quad \text{on } \partial\Omega, t > 0, \\
w_0 &= z_0, \quad \text{on } \Omega,
\end{aligned} \tag{A.3}$$

where we introduced the maps

$$\begin{aligned}
\mathbb{G}(\lambda, w)(t) &= (A_*(t) - \lambda A_*(\lambda t)) w(t) + \lambda G(\lambda t, w(t)), \\
\mathbb{H}(\lambda, w)(t) &= (B_*(t) - B_*(\lambda t)) w(t) + H(\lambda t, w(t)),
\end{aligned}$$

for $w \in \mathbb{E}_1(J)$, $t \in J$, and $\lambda \in (1 - \epsilon, 1 + \epsilon)$. We observe that

$$\begin{aligned}
\mathbb{G}(\lambda, w)(t) = A(u(t))w(t) &+ A'(u(t))[w(t), u(t)] - F'(u(t))w(t) \\
&- \lambda A(u(\lambda t) + w(t))(u(\lambda t) + w(t)) + \lambda A(u(\lambda t))u(\lambda t) \\
&+ \lambda F(u(\lambda t) + w(t)) - \lambda F(u(\lambda t)),
\end{aligned} \tag{A.4}$$

$$\mathbb{H}(\lambda, w)(t) = B'(u(t))w(t) - B(u(\lambda t) + w(t)). \tag{A.5}$$

We claim that the map $\lambda \mapsto u(\lambda\cdot)$ belongs to $C^1((1-\epsilon,1+\epsilon),\mathbb{E}_1(J))$. Indeed, for $\mu,\lambda \in (1-\epsilon,1+\epsilon) \subseteq (1/2,3/2)$ we have:

$$\begin{aligned}
u(\mu t) - u(\lambda t) - (\mu-\lambda)tu'(\lambda t) &= \frac{\mu-\lambda}{\lambda}\int_0^1 \lambda t(u'(\lambda t + \theta(\mu-\lambda)t) - u'(\lambda t))\, d\theta \\
&= \frac{\mu-\lambda}{\lambda}\int_0^1 \Big(((\lambda+\theta(\mu-\lambda))t)u'((\lambda+\theta(\mu-\lambda))t) - \lambda t u'(\lambda t)\Big)\, d\theta \qquad \text{(A.6)} \\
&\quad - \frac{(\mu-\lambda)^2}{\lambda}\int_0^1 \Big[\frac{\theta t}{(\lambda+\theta(\mu-\lambda))t}\Big](\lambda t + \theta(\mu-\lambda)t)u'(\lambda t+\theta(\mu-\lambda)t)\, d\theta.
\end{aligned}$$

We note that the expression in the square brackets in the last interval is contained in $[0,2]$, and recall that $tu' \in \mathbb{E}_1(J')$ due to [12, Thm.14]. Moreover, the dilation operators T_a given by $T_a f(t) = f(at)$ on $\mathbb{E}_1(\mathbb{R}_+)$ are strongly continuous in $a>0$. (Below, we extend u from $\mathbb{E}_1(J')$ to $\mathbb{E}_1(\mathbb{R}_+)$ to use the strong continuity.) Thus (A.6) yields

$$\begin{aligned}
&\|u(\mu\cdot) - u(\lambda\cdot) - (\mu-\lambda)tu'(\lambda\cdot)\|_{\mathbb{E}_1(J)} \\
&\qquad \le 2\,|\mu-\lambda|\,\varepsilon(\theta|\mu-\lambda|) + c\,|\mu-\lambda|^2 \le |\mu-\lambda|\,\varepsilon(|\mu-\lambda|),
\end{aligned}$$

showing that $\partial_\lambda u(\lambda\cdot) = tu'(\lambda\cdot)$ in $\mathbb{E}_1(J)$. Since also

$$tu'(\lambda t) - tu'(\mu t) = \lambda^{-1}(\lambda t u'(\lambda t) - \mu t u'(\mu t)) + (\lambda^{-1}-\mu^{-1})\mu t u'(\mu t),$$

the map $\lambda \mapsto u(\lambda\cdot) \in \mathbb{E}_1(J)$ is continuously differentiable. Combining this fact with the observations in [12, §2], we see that the map $(\lambda,w) \mapsto \mathbb{G}(\lambda,w) \in \mathbb{E}_0(J)$ is continuously differentiable with $\mathbb{G}(1,w) = \mathbb{G}(w)$, $\partial_2\mathbb{G}(1,w) = \mathbb{G}'(w)$, and

$$\begin{aligned}
\partial_1\mathbb{G}(1,w) = (A(u) - A(u+w))u - A(u+w)w + F(u+w) - F(u) \\
+ (A'(u) - A'(u+w))[tu',u] - A'(u+w)[tu',w] \qquad \text{(A.7)} \\
+ (A(u) - A(u+w))tu' + (F'(u+w) - F'(u))tu'.
\end{aligned}$$

We claim that $B'(u) \in \mathcal{B}(\mathbb{E}_1(J),\mathbb{F}(J))$. Indeed, due to Proposition 10(Ib) of [12] with $u_* = 0$ we only need to check that $B'(0) \in \mathcal{B}(\mathbb{E}_1(J),\mathbb{F}(J))$ which follows from (16) and (17) in [12] and [17, Thm.4.6.4.1]. Proposition 10 of [12] then implies that the map $v \mapsto B(v)$ belongs to $C^1(\mathbb{E}_1(J),\mathbb{F}(J))$. Therefore the map $(\lambda,w) \mapsto \mathbb{H}(\lambda,w)$ is contained in $C^1((1-\epsilon,1+\epsilon)\times\mathbb{E}_1(J),\mathbb{F}(J))$ with

$$\begin{aligned}
\mathbb{H}(1,w) = \mathbb{H}(w), \quad \partial_2\mathbb{H}(1,w) = \mathbb{H}'(w) \\
\partial_1\mathbb{H}(1,w) = -B'(u+w)tu' = (B'(u) - B'(u+w))tu',
\end{aligned} \qquad \text{(A.8)}$$

using that $B(u(\lambda t)) = 0$, and hence $0 = \frac{d}{d\lambda}B(u(\lambda\cdot)) = B'(u(\lambda\cdot))tu'$. In order to solve (A.3), we set

$$\mathcal{L}(\lambda,w) = w - S(z_0 - \mathcal{N}_p H(0,z_0) + \mathcal{N}_p\gamma_0\mathbb{H}(\lambda,w), \mathbb{G}(\lambda,w), \mathbb{H}(\lambda,w)), \qquad \text{(A.9)}$$

where $\mathcal{N}_p \in \mathcal{B}(Y_p,X_p)$ is a right inverse of $B'(u_0) = B_*(0)$ (see [12, Prop.5]). Because of $B(u_0+z_0) = 0$, we infer:

$$B_*(0)[z_0 - \mathcal{N}_p H(0,z_0) + \mathcal{N}_p\gamma_0\mathbb{H}(\lambda,w))] = \mathbb{H}(\lambda,w)(0).$$

Therefore Theorem 2 of [12], (2.13), and the properties of $\mathbb{G}$ and $\mathbb{H}$, established above, show that $\mathcal{L} \in C^1((1-\epsilon,1+\epsilon)\times\mathbb{E}_1(J),\mathbb{E}_1(J))$ and that

$$\begin{aligned}
\mathcal{L}(1,z) &= z - S(z_0,\mathbb{G}(z),\mathbb{H}(z)) = 0, \\
\partial_2\mathcal{L}(1,z) &= I - S(\mathcal{N}_p\gamma_0\mathbb{H}'(z),\mathbb{G}'(z),\mathbb{H}'(z)).
\end{aligned}$$

Since $\|z\|_{\mathbb{E}_1(J)} \le c|z_0|_p \le c\rho$ by [12, Thm.14], Theorem 2 and Proposition 10 of [12] and (2.13) imply that $\partial_2\mathcal{L}(1,z)$ is invertible if ρ is sufficiently small. So we obtain a function $\Psi \in C^1((1-\hat\epsilon, 1+\hat\epsilon), \mathbb{E}_1(J))$ for some $0<\hat\epsilon<\epsilon$ satisfying $\Psi(1)=z$ and $\mathcal{L}(\lambda,\Psi(\lambda))=0$. Set $w_0(\lambda)=\Psi(\lambda)(0)$. Using [12, Cor.12] in the estimate, we derive

$$\begin{aligned}
w_0(\lambda)-z_0 &= \mathcal{N}_p(\mathbb{H}(\lambda,\Psi(\lambda))(0)-H(0,z_0)) \qquad\qquad \text{(A.10)}\\
&= \mathcal{N}_p(B'(u_0)w_0(\lambda)-B(u_0+w_0(\lambda))-B'(u_0)z_0+B(u_0+z_0))\\
&= -\mathcal{N}_p(B(u_0+w_0(\lambda))-B(u_0+z_0)-B'(u_0+z_0)(w_0(\lambda)-z_0))\\
&\qquad +\mathcal{N}_p(B'(u_0)-B'(u_0+z_0))(w_0(\lambda)-z_0),\\
|w_0(\lambda)-z_0|_p &\le c\varepsilon(|z_0-w_0(\lambda)|_p)\,|w_0(\lambda)-z_0|_p + c\varepsilon(|z_0|_p)\,|w_0(\lambda)-z_0|_p.
\end{aligned}$$

Observe that $|z_0-w_0(\lambda)|_p \le c\,\|z-\Psi(\lambda)\|_{\mathbb{E}_1(J)}$ by (2.11). Decreasing $\hat\epsilon>0$ and $\rho>0$ if necessary, we thus conclude that $w_0(\lambda)=z_0$. Hence, $\Psi(\lambda)$ solves (A.3) due to (A.10) and (A.9). Possibly after decreasing $\hat\epsilon>0$ once more, we deduce that $\Psi(\lambda)=w_\lambda$ from (A.3) and Theorem 2 and Proposition 10 of [12]. As a result,

$$\begin{aligned}
t(\dot v-\dot u) &= \Psi'(1) = -\partial_2\mathcal{L}(1,z)^{-1}\partial_1\mathcal{L}(1,z)\\
&= \partial_2\mathcal{L}(1,z)^{-1}S(\mathcal{N}_p\gamma_0\partial_1\mathbb{H}(1,z),\partial_1\mathbb{G}(1,z),\partial_1\mathbb{H}(1,z)).
\end{aligned}$$

Theorem 2 of [12], (A.7), (A.8), (2.7), (RR), and Lemma A.2 below now yield

$$\begin{aligned}
\|t(\dot v-\dot u)\|_{\mathbb{E}_1(J)} &\le c\,(\|\partial_1\mathbb{G}(1,z)\|_{\mathbb{E}_0(J)}+\|\partial_1\mathbb{H}(1,z)\|_{\mathbb{F}(J)})\\
&\le c\,\|z\|_{\mathbb{E}_1} \le c\,|v_0-u_0|_p\,,
\end{aligned}$$

which is (A.1). Finally, for $\tau\in(0,T)$ Sobolev's embedding theorem implies that

$$\begin{aligned}
\|v-u\|_{C^{1-1/p}([\tau,T];X_1)} &\le c\|v-u\|_{W^1_p([\tau,T];X_1)} \le c(\tau)\|t\dot v-t\dot u\|_{\mathbb{E}_1(J)}+c\|v-u\|_{\mathbb{E}_1(J)}\\
&\le c(\tau)\,|v_0-u_0|_p\,. \qquad\square
\end{aligned}$$

The proof of the following lemma is omitted. It uses arguments from the proof of Proposition 10 in [12].

Lemma A.2. *Assume that (RR) holds and $J=[0,T]$. Then the map $v\mapsto B'(v)\in\mathcal{B}(\mathbb{E}_1(J),\mathbb{F}(J))$ is locally Lipschitz on $\mathbb{E}_1(J)$.*

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Y. Latushkin, Department of Mathematics, University of Missouri, Columbia, MO 65211, USA.
Email address: `yuri@math.missouri.edu`

J. Prüss, FB Mathematik und Informatik, Martin–Luther–Universität, 06099 Halle, Germany.
Email address: `pruess@mathematik.uni-halle.de`

R. Schnaubelt, Department of Mathematics, University of Karlsruhe, 76128 Karlsruhe, Germany.
Email address: `schnaubelt@math.uni-karlsruhe.de`