A STRUCTURALLY DAMPED PLATE EQUATION WITH DIRICHLET-NEUMANN BOUNDARY CONDITIONS

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Abstract. We investigate sectoriality and maximal regularity in L^p-L^q-Sobolev spaces for the structurally damped plate equation with Dirichlet-Neumann (clamped) boundary conditions. We obtain unique solutions with optimal regularity for the inhomogeneous problem in the whole space, in the half-space, and in bounded domains of class C^4. It turns out that the first-order system related to the scalar equation on $\mathbb{R}^n$ is sectorial only after a shift in the operator. On the half-space one has to include zero boundary conditions in the underlying function space in order to obtain sectoriality of the shifted operator and maximal regularity for the case of homogeneous boundary conditions. We further show that the semigroup solving the problem on bounded domains is exponentially stable.

1. Introduction and preliminaries

In this paper, we study the linear structurally damped plate equation with inhomogeneous Dirichlet-Neumann (clamped) boundary conditions given by

$$\begin{aligned} \partial_t^2 u + \Delta^2 u - \rho\Delta\partial_t u &= f, && (t,x) \in (0,\infty)\times G,\\ u &= g_0, && (t,x) \in (0,\infty)\times \partial G,\\ \partial_\nu u &= g_1, && (t,x) \in (0,\infty)\times \partial G,\\ u|_{t=0} &= \varphi_0, && x\in G,\\ \partial_t u|_{t=0} &= \varphi_1, && x\in G. \end{aligned} \tag{1.1}$$

Here, $\rho > 0$ is a fixed parameter and ∂_ν stands for the normal derivative with respect to the outer unit normal. We treat the full space $G = \mathbb{R}^n$ (where we drop the boundary conditions), the half-space $G = \mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x_n > 0\}$, and bounded domains $G \subset \mathbb{R}$ with a boundary of class C^4. We establish maximal regularity of type L^p for the inhomogeneous problem (1.1) and discuss sectoriality of the operator matrix governing the associated first order system. The generated semigroup is exponentially stable for bounded G.

The undamped plate equation with $\rho = 0$ occurs as a linear model for vibrating stiff objects where the potential energy involves curvature-like terms which lead to the Bi-Laplacian $(-\Delta)^2$ as the main 'elastic' operator B, see

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e.g. Chapter 12 of [25] or [27]. (In the one-dimensional case one obtains the Euler-Bernoulli beam equation.) In this model, energy dissipation is neglected and the equation has no smoothing effect as the governing semigroup is unitary on the canonical L^2–based phase space. One adds damping terms to incorporate the loss of energy. Structural damping describes a situation where higher frequencies are more strongly damped than low frequencies. Here the damping term has 'half of the order' of the leading elastic term, as proposed in Russell's seminal paper [27]. Such systems have been studied in detail also from the viewpoint of dynamical systems and control theory, see e.g. [5], [20], [23], [29] and the references therein. In the L^2 case, the basic generation results were already obtained in [6]. It turned out that the underlying semigroup is analytic, which is false if the damping operator is a fractional power of the elastic operator with exponent strictly less than 1/2. In this sense, structural damping is a borderline case. The case of strong damping (where the elastic operator is bounded by the damping operator) is easier as it can be handled by perturbation arguments, see e.g. Section VI.3.a of [14].

Structurally damped plate and wave equations can also be considered in L^p-based spaces for $p \neq 2$ (in contrast to the weaker damping given by $-\rho\partial_t u$), which is very convenient for the treatment of nonlinear terms in the framework of parabolic evolution equations, see e.g. [4], [7] and [28]. However, in this context the available existence results are restricted to the very special case that the damping operator is a multiple of the square root $B^{1/2}$ of the elastic operator B (which we call the *square root case*). On the other hand, in L^2 one can treat much more general problems, [6]; but these results use the numerical range in an essential way and seem to be restricted to the L^2 case. In our problem (1.1), the damping operators is a multiple of $B^{1/2}$ only if $G = \mathbb{R}^n$. For other domains the square root case corresponds to the boundary conditions $u = \Delta u = 0$ on ∂G. In the square root case one can easily compute the resolvent of the associated generator in terms of the given operators and show its sectoriality, see [16] and the references therein, as well as [4], [7], [8], [15], [28] for more recent contributions. Moreover, Theorem 4.1 of [7] shows maximal regularity in the square root case if the elastic operator B has an '$\mathcal{R}$-bounded H^∞-calculus' (which can be applied to our case if $G = \mathbb{R}^n$). In these papers, inhomogeneous boundary data have not been considered.

In our work we establish a fairly complete well-posedness and regularity theory for (1.1) with inhomogeneous boundary conditions in an L^p context, where $p \in (1, \infty)$. We have chosen the (arguably most basic) situation of a clamped plate (i.e., having Dirichlet and Neumann boundary conditions) governed by the Bi-Laplacian and the Laplacian. We believe that our methods also apply to analogous general systems with coefficients and other boundary conditions, provided that appropriate ellipticity and Lopatinski-Shapiro conditions hold, cf. e.g. [10]. For conciseness we do not investigate such generalizations here.

The problem (1.1) on a bounded domain is reduced to corresponding equations on the full and half-space by localization, transformation and perturbation, see Section 5. In our approach we use ideas from [10] and [11] where different, more standard parabolic systems have been treated. We rewrite (1.1) as a system of first order in time with the new state $v = (u, \partial_t u)^\top$, which is governed by an operator matrix $A(D)$ on $\mathbb{R}^n$ or $A_{p,0}$ on $\mathbb{R}^n_+$, see (2.2) and (4.1), respectively. This has the advantage that one works in the framework of well developed theories for operator semigroups, dynamical systems (cf. [5]) and control problems (cf. [23]). We further see that our problem leads to a mixed-order boundary value problem in the sense of Douglis-Nirenberg, see e.g. Proposition 3.4 and [9]. The full and half-space problems are then solved via Laplace transform in time and Fourier transform in space. To invert these transforms, we mainly use Michlin's theorem and employ its operator-valued version due to Weis, [31], for the inversion of the Laplace transform. This step requires recently developed methods from operator-valued harmonic analysis briefly indicated at the end of this section.

The full space problem is solved in Theorem 2.5. In Section 2 we however focus on a detailed study of regularity properties of the resolvent of $A(D)$ needed later on, see Theorem 2.3. These results are based on an analysis of the symbols associated with (1.1) which play an essential role in our approach. We thus present detailed proofs although some of the results could also be deduced from e.g. [7] and [16]. In Section 3 we derive the crucial solution formula for the parameter-dependent elliptic boundary value problem (3.1) corresponding to (1.1) on $\mathbb{R}^n_+$ and establish the core estimates on the operators appearing there, see Theorem 3.5 and Corollary 3.6. These facts rely on a thorough investigation of the relevant symbols in Lemma 3.2. We further show in Proposition 3.4 that the operator matrix $A(D)$ with Dirichlet–Neumann boundary conditions is not sectorial in $H^2_p(\mathbb{R}^n_+) \times L^p(\mathbb{R}^n_+)$ even if we allow shifts. The resolvent still exists but it does not satisfy the sectoriality estimates. This is actually a general phenemenon of such elliptic systems if the state space allows traces relevant to the boundary conditions, see [9].

Theorem 4.4 then shows that the restriction $A_{p,0}$ of $A(D)$ to $H^2_{p,0}(\mathbb{R}^n_+) \times L^p(\mathbb{R}^n_+)$ is sectorial after applying a shift. To derive the resolvent estimate, one has to exploit the additional zero boundary conditions of the right-hand side, which is done using the Hardy-type Lemma 4.1. Such techniques may also be applied to other Douglis–Nirenberg systems on state spaces involving regularity in future work. In Theorem 4.5 and 4.6 we then deduce well-posedness and maximal regularity of (1.1) on $\mathbb{R}^n_+$ from the previous results combined with semigroup theory and operator-valued harmonic analysis. In the last section, we finally treat the case of bounded domains. Here we can omit many details which are similar to, e.g., [10] and [11]. We further use standard spectral theory of analytic semigroups to show that the semigroup solving (1.1) on a bounded domain is exponentially stable. (This fact was recently shown in the square root case, [15].) We thus obtain maximal regularity on $(0,\infty)$ and not just on bounded time intervals as for the full and half-space.

We will investigate maximal regularity in the sense of well-posedness in L^p-L^q-Sobolev spaces for equation (1.1). For this, we will make use of the concept of $\mathcal{R}$-boundedness and vector-valued Fourier multiplier theorems which has become kind of standard for L^p-theory of boundary value problems. We give a short summary of these tools, for a more detailed exposition we refer to [10] and [22].

Let X and Y be Banach spaces, and let $L(X,Y)$ be the space of all bounded linear operators from X to Y. For an interval $J=(0,T)$ with $T\in(0,\infty]$, we denote by $L^q(J;X)$ the X-valued L^q-space, by $H_q^k(J;X)$, $k\in\mathbb{N}_0$, the X-valued Sobolev space, and by $W_q^s(J;X) := B_{qq}^s(J;X)$, $s\in(0,\infty)\setminus\mathbb{N}$, the X-valued Sobolev-Slobodeckii space (which coincides with the Besov space). Moreover, $(\cdot,\cdot)_{\theta,q}$ stands for the real interpolation functor. Throughout, we let $p\in(1,\infty)$.

A family $\mathcal{T}\subset L(X,Y)$ of operators is $\mathcal{R}$-*bounded* if there exists a constant $C>0$ such that for all $m\in\mathbb{N}$, $(T_k)_{k=1,\dots,m}\subset\mathcal{T}$, and $(x_k)_{k=1,\dots,m}\subset X$ we have

$$\Big\|\sum_{k=1}^m r_kT_kx_k\Big\|_{L^p([0,1];Y)} \le C\,\Big\|\sum_{k=1}^m r_kx_k\Big\|_{L^p([0,1];X)}.$$

Here the Rademacher functions r_k, $k\in\mathbb{N}$, are given by $r_k\colon[0,1]\to\{-1,1\}, t\mapsto \operatorname{sign}(\sin(2^k\pi t))$. If two families $\mathcal{T}_j\subset L(X_j,Y_j)$, $j\in\{1,2\}$, are $\mathcal{R}$–bounded, then also $\mathcal{T}_1+\mathcal{T}_2$ (if $X_1=X_2$ and $Y_1=Y_2$) and $\mathcal{T}_2\mathcal{T}_1$ (if $Y_1=X_2$) are $\mathcal{R}$–bounded.

Domains of closed operators are endowed with the graph norm. A densely defined, closed operator $A\colon D(A)\subset X\to X$ is said to have *maximal* L^q-*regularity*, $1<q<\infty$, in the interval $J=(0,T)$ if the Cauchy problem

$$\begin{aligned}\partial_t u(t)+Au(t)&=f(t), \qquad t\in J,\\ u|_{t=0}&=u_0,\end{aligned}$$

has, for every $f\in L^q(J;X)$ and $u_0\in(X,D(A))_{1-1/q,q}$, a unique locally integrable solution $u:J\to D(A)$ such that $\partial_t u, Au\in L^q(J;X)$ and

$$\|\partial_t u\|_{L^q(J;X)}+\|Au\|_{L^q(J;X)}\le C\big(\|f\|_{L^q(J;X)}+\|u_0\|_{(X,D(A))_{1-1/q,q}}\big)$$

with a constant C independent of f and u_0. If J is bounded or A is invertible, this property is equivalent to the isomorphy

$$\Big(\partial_t+A,\gamma_{0,t}\Big)\colon H_q^1(J;X)\cap L^q(J;D(A))\to L^q(J;X)\times(X,D(A))_{1-1/q,q},$$

where $\gamma_{0,t}\colon u\mapsto u|_{t=0}$ denotes the time trace. It is known that $-A$ generates an analytic C_0–semigroup if it has maximal L^q-regularity. If this semigroup is exponentially stable, then one even obtains maximal L^q-regularity on $(0,\infty)$.

In the following, we use the notation $\Sigma_\vartheta := \{z\in\mathbb{C}\setminus\{0\} : |\arg z|<\vartheta\}$ for $\vartheta\in(0,\pi]$. Recall that a closed operator $A\colon D(A)\subset X\to X$ is called $(\mathcal{R})$-*sectorial* if A has dense domain and dense range, and if there exists an angle

$\vartheta \in (0,\pi)$ such that $\rho(-A) \supset \Sigma_{\pi-\vartheta}$ and the set $\{\lambda(\lambda+A)^{-1} : \lambda \in \Sigma_{\pi-\vartheta}\}$ is $(\mathcal{R})$-bounded. In this case, the angle of $(\mathcal{R})$-boundedness is defined as the infimum of all ϑ for which this holds.

A Banach space X is called of class HT if the vector-valued Hilbert transform is continuous in $L^q((0,\infty);X)$ for some (and then any) $q \in (1,\infty)$. Sobolev–Slobodeckii spaces with $p \in (1,\infty)$ are of class HT, as well as their X–valued analogues if X is of class HT. It was shown by Weis in [31] that a sectorial operator in a Banach space of class HT has maximal L^q-regularity for all $q \in (1,\infty)$ if and only if the set $\{\lambda(\lambda+A)^{-1} : \operatorname{Re}\lambda \geq 0,\, \lambda \neq 0\}$ is $\mathcal{R}$-bounded.

2. The full space case

In this section we solve (1.1) in the whole space $G = \mathbb{R}^n$ (omitting the boundary conditions). Let us remark that in this case (1.1) can be treated by an operator-theoretic approach as it can be written in the form

$$\begin{aligned} \partial_t^2 u + \rho B^{1/2}\partial_t u + Bu &= f, \qquad t \in (0,\infty),\\ u|_{t=0} &= \varphi_0,\\ \partial_t u|_{t=0} &= \varphi_1 \end{aligned} \tag{2.1}$$

with the operator $B\colon D(B) \subset L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ being defined by $D(B) := H_p^4(\mathbb{R}^n)$ and $Bu := (-\Delta)^2 u$. Therefore, (2.1) is related to the quadratic operator pencil $V\colon H_p^4(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$,

$$V(\lambda) := \lambda^2 + \lambda\rho B^{1/2} + B = (\alpha_+\lambda + B^{1/2})(\alpha_-\lambda + B^{1/2}),$$

where

$$\alpha_\pm = \begin{cases} \frac{\rho}{2} \pm \sqrt{\frac{\rho^2}{4} - 1}, & \rho \geq 2,\\ \frac{\rho}{2} \pm i\sqrt{1 - \frac{\rho^2}{4}}, & 0 < \rho < 2. \end{cases}$$

Defining the angle $\vartheta = \vartheta(\rho)$ by

$$\vartheta(\rho) := \begin{cases} \arctan \frac{2}{\rho}\sqrt{1 - \frac{\rho^2}{4}}, & 0 < \rho < 2,\\ 0, & 2 \leq \rho < \infty, \end{cases}$$

we can write $\alpha_\pm = e^{\pm i\vartheta}$ for $\rho \leq 2$ and $\alpha_\pm > 0$ as $\rho \geq 2$. Note that $\arg\alpha_\pm = \pm\vartheta(\rho)$ and $\vartheta(\rho) \nearrow \frac{\pi}{2}$ for $\rho \searrow 0$.

By the theory of quadratic operator pencils and second-order Cauchy problems, we can invert the operator $V(\lambda)$ and show maximal L^p-regularity, see Theorem 3.4 of [16] and and Theorem 4.1 of [7], as well as [4] and [28]. However, a more detailed investigation of the related first-order system will be useful for the analysis of the half-space. To this aim, we set $v = (u, \partial_t u)^\top$ and re-write (1.1) with $G = \mathbb{R}^n$ as

$$\partial_t v + A(D)v = \begin{pmatrix} 0 \\ f \end{pmatrix}, \qquad (t,x) \in (0,\infty) \times \mathbb{R}^n,$$

$$v|_{t=0} = \begin{pmatrix} \varphi_0 \\ \varphi_1 \end{pmatrix}, \qquad x \in \mathbb{R}^n,$$

with $A(D) := \mathscr{F}^{-1} A(\xi) \mathscr{F}$, where $\mathscr{F}$ denotes the Fourier transform in $\mathbb{R}^n$ and the matrix-valued symbol $A(\xi)$ is given by

$$A(\xi) := \begin{pmatrix} 0 & -1 \\ |\xi|^4 & \rho|\xi|^2 \end{pmatrix}.$$

Note that the Fourier transform is defined by

$$(\mathscr{F}\phi)(\xi) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\xi} \phi(x) dx, \qquad \xi \in \mathbb{R}^n,$$

for Schwartz functions $\phi \in \mathscr{S}(\mathbb{R}^n)$ and extended by duality to tempered distributions. Here and in the following, we use the standard multi-index notation and put $D = -i\nabla = -i(\partial_1, \dots, \partial_n)^\top$. We also set

$$A(\xi, \lambda) := \lambda + A(\xi) = \begin{pmatrix} \lambda & -1 \\ |\xi|^4 & \lambda + \rho|\xi|^2 \end{pmatrix}.$$

We thus have

$$A(D) = \begin{pmatrix} 0 & -I \\ (-\Delta)^2 & -\rho\Delta \end{pmatrix} \qquad \text{and} \qquad A(D, \lambda) = \begin{pmatrix} \lambda & -I \\ (-\Delta)^2 & \lambda - \rho\Delta \end{pmatrix}. \tag{2.2}$$

Employing the spaces

$$\mathbb{E} := H_p^2(\mathbb{R}^n) \times L^p(\mathbb{R}^n),$$
$$\mathbb{F} := H_p^4(\mathbb{R}^n) \times H_p^2(\mathbb{R}^n),$$

we introduce the unbounded operator $A_p \colon D(A_p) \subset \mathbb{E} \to \mathbb{E}$ by $D(A_p) := \mathbb{F}$ and $A_p u := A(D)u$. Note that for the weight matrix

$$S_1(\xi) := \begin{pmatrix} 1 + |\xi|^2 & 0 \\ 0 & 1 \end{pmatrix}$$

the operator $S_1(D) := \mathscr{F}^{-1} S_1(\xi) \mathscr{F}$ defines an isomorphism of $\mathbb{E}$ onto $L^p(\mathbb{R}^n; \mathbb{C}^2)$, and we thus have the equivalence of norms $\|f\|_{\mathbb{E}} \cong \|S_1(D)f\|_{L^p}$. Setting $S_2(\xi) := (1 + |\xi|^2) S_1(\xi)$, one obtains $S_2(D) \in L_{\mathrm{Isom}}(\mathbb{F}, L^p(\mathbb{R}^n; \mathbb{C}^2))$ and $\|u\|_{\mathbb{F}} \cong \|S_2(D)u\|_{L^p}$.

Remark 2.1. *Below we will use Michlin's theorem in the following variant: Let* $b \colon (\mathbb{R}^n \times \overline{\Sigma}_{\pi-\vartheta-\varepsilon}) \setminus \{0\} \to \mathbb{C}$, $(\xi, \lambda) \mapsto b(\xi, \lambda)$, *be infinitely smooth and homogeneous in* $(\xi, \lambda^{1/2})$ *of degree* 0*. Then* $\xi^\beta \partial_\xi^\beta \lambda^\gamma \partial_\lambda^\gamma b$ *is uniformly bounded for* $(\xi, \lambda) \in (\mathbb{R}^n \times \overline{\Sigma}_{\pi-\vartheta-\varepsilon}) \setminus \{0\}$, *for each* $\beta \in \mathbb{N}_0^n$ *and* $\gamma \in \mathbb{N}_0^2$ *(where we identify* $\mathbb{C}$ *with* $\mathbb{R}^2$*). Michlin's theorem then implies that* $\|\lambda^\gamma \partial_\lambda^\gamma \mathscr{F}^{-1} b(\cdot, \lambda) \mathscr{F}\|_{L(L^p(\mathbb{R}^n))} \le C$ *with a constant* C *not depending on* λ *(see e.g. Theorem* 5.2.7 *of* [18] *and the remarks preceding it). In fact, in this situation the family of operators*

$$\left\{ \lambda^\gamma \partial_\lambda^\gamma \mathscr{F}^{-1} b(\cdot, \lambda) \mathscr{F} : \lambda \in \overline{\Sigma}_{\pi-\vartheta-\varepsilon} \right\} \subset L(L^p(\mathbb{R}^n))$$

is even $\mathcal{R}$-bounded by Corollary 3.3 *in* [17]*. This applies to symbols of the form*

$$\frac{\lambda^{(s-|\alpha|)/2}\xi^\alpha}{(\lambda+|\xi|^2)^{s/2}}$$

with $s\in\mathbb{N}$ and $|\alpha|\in\{0,\dots,s\}$. We will tacitly make use of these facts in the estimates below.

We first show that $A_p+\lambda$ is invertible for all λ in the above setting, but that A_p fails to be sectorial. Later we will see that $A_p+\lambda_0$ is $\mathcal{R}$-sectorial for every positive shift λ_0.

Proposition 2.2. *a) For $\vartheta=\vartheta(\rho)$ and all $\lambda\in\Sigma_{\pi-\vartheta}$, the operator $A_p+\lambda:\mathbb{F}\to\mathbb{E}$ is invertible.*

b) The operator A_p is not sectorial in $\mathbb{E}$ for any angle and, consequently, $-A_p$ does not generate a bounded C_0-semigroup on $\mathbb{E}$.

Proof. a) Due to the definition of the spaces, the operator $A_p+\lambda$ belongs to $L(\mathbb{F},\mathbb{E})$ for every $\lambda\in\mathbb{C}$. Let $\lambda\in\Sigma_{\pi-\vartheta}$. From the identity

$$\det A(\xi,\lambda)=\lambda^2+\lambda\rho|\xi|^2+|\xi|^4=(\alpha_+\lambda+|\xi|^2)(\alpha_-\lambda+|\xi|^2)$$

and $\alpha_\pm\lambda\in\Sigma_\pi$, we deduce that $A(\xi,\lambda)$ is invertible with inverse

$$A(\xi,\lambda)^{-1}=\frac{1}{(\alpha_+\lambda+|\xi|^2)(\alpha_-\lambda+|\xi|^2)}\begin{pmatrix}\lambda+\rho|\xi|^2 & 1\\ -|\xi|^4 & \lambda\end{pmatrix}. \tag{2.3}$$

To show that $(A_p+\lambda)^{-1}$ exists in $L(\mathbb{E},\mathbb{F})$, we have to establish $M(D,\lambda)\in L(L^p(\mathbb{R}^n;\mathbb{C}^2))$ for the matrix-valued multiplier symbol

$$M(\xi,\lambda):=S_2(\xi)A(\xi,\lambda)^{-1}S_1(\xi)^{-1}.$$

Direct calculations lead to

$$\begin{aligned}M(\xi,\lambda)&=\frac{1}{\det A(\xi,\lambda)}S_2(\xi)\begin{pmatrix}\lambda+\rho|\xi|^2 & 1\\ -|\xi|^4 & \lambda\end{pmatrix}S_1(\xi)^{-1}\\ &=\frac{1}{(\alpha_+\lambda+|\xi|^2)(\alpha_-\lambda+|\xi|^2)}\begin{pmatrix}(1+|\xi|^2)(\lambda+\rho|\xi|^2) & (1+|\xi|^2)^2\\ -|\xi|^4 & \lambda(1+|\xi|^2)\end{pmatrix}.\end{aligned}$$

For every fixed $\lambda\in\Sigma_{\pi-\vartheta}$, each of the terms

$$\frac{1+|\xi|^2}{\alpha_\pm\lambda+|\xi|^2},\quad\frac{\lambda}{\alpha_\pm\lambda+|\xi|^2},\quad\text{and}\quad\frac{|\xi|^2}{\alpha_\pm\lambda+|\xi|^2}$$

can be estimated by a constant depending only on λ and ρ. Similarly, the k-th derivatives in ξ of each term are bounded by a constant times $|\xi|^{-k}$, where the constants depend on λ, ρ and k. Michlin's theorem then implies $M(D,\lambda)\in L(L^p(\mathbb{R}^n;\mathbb{C}^2))$. Clearly, $M(D,\lambda)$ is the inverse of $S_1(D)A(D,\lambda)S_2(D)^{-1}$ in $L(L^p(\mathbb{R}^n;\mathbb{C}^2))$, and thus assertion a) holds.

b) Assume that A_p is sectorial in $\mathbb{E}$ of some angle, i.e., $\|\lambda(A_p+\lambda)^{-1}\|_{L(\mathbb{E})}\le C$ for all $\lambda\in(0,\infty)$ with some constant C independent of λ. Similarly to a), this property is equivalent to the uniform boundedness of the operator $M_0(D,\lambda) \in L(L^p(\mathbb{R}^n;\mathbb{C}^2))$ with the symbol

$$
\begin{aligned}
M_0(\xi,\lambda) &:= \lambda S_1(\xi) A(\xi,\lambda)^{-1} S_1(\xi)^{-1} \\
&= \frac{1}{(\alpha_+\lambda + |\xi|^2)(\alpha_-\lambda + |\xi|^2)} \begin{pmatrix} \lambda(\lambda+\rho|\xi|^2) & \lambda(1+|\xi|^2) \\ -\frac{\lambda|\xi|^4}{1+|\xi|^2} & \lambda^2 \end{pmatrix}.
\end{aligned} \tag{2.4}
$$

Since every L^p-Fourier multiplier is an L^∞-function (see e.g. Proposition 3.17 in [10]), we derive

$$
\Big| \frac{\lambda(1+|\xi|^2)}{(\alpha_+\lambda + |\xi|^2)(\alpha_-\lambda + |\xi|^2)} \Big| \le C
$$

for all $\lambda > 0$ and $\xi \in \mathbb{R}^n$, where the constant C does not depend on λ or ξ. However, setting $\lambda = k^{-2}$ and $|\xi| = k^{-1}$ with $k \in \mathbb{N}$, the expression on the left-hand side equals $\frac{k^2+1}{(\alpha_++1)(\alpha_-+1)}$ which tends to ∞ as $k \to \infty$. □

Although A_p is not sectorial, certain λ-dependent estimates for the inverse operator are valid in each sector $\Sigma_{\pi-\vartheta-\varepsilon}$ with $\varepsilon > 0$. One could formulate the next result more concisely within homogeneous Sobolev spaces, but for simplicity we avoid this setting. We often denote the vector-valued space $L^p(\mathbb{R}^n;\mathbb{C}^m)$ also by $L^p(\mathbb{R}^n)$, for any $m \in \mathbb{N}$.

Theorem 2.3. *Let $\varepsilon \in (0, \pi-\vartheta)$, $\lambda \in \Sigma_{\pi-\vartheta-\varepsilon}$, and $h = (h_1,h_2)^\top \in \mathbb{E}$. Set $v := (v_1,v_2)^\top := (A_p+\lambda)^{-1}h$. Let $k \in \{0,1,2\}$, $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = k$, $\gamma \in \mathbb{N}_0^2$, and $\delta \in \mathbb{N}_0^n$ with $|\delta| = 2$. Then there is a constant $C_\varepsilon > 0$ such that*

$$
\Big\| \lambda^{1-\frac{k}{2}} \begin{pmatrix} D^\alpha D^\delta v_1 \\ D^\alpha v_2 \end{pmatrix} \Big\|_{L^p(\mathbb{R}^n)} \le C_\varepsilon \big(\|\Delta h_1\|_{L^p(\mathbb{R}^n)} + \|h_2\|_{L^p(\mathbb{R}^n)} \big), \tag{2.5}
$$

$$
\|\lambda^{2-\frac{k}{2}} D^\alpha v_1\|_{L^p(\mathbb{R}^n)} \le C_\varepsilon \big(\|\lambda h_1\|_{L^p(\mathbb{R}^n)} + \|h_2\|_{L^p(\mathbb{R}^n)} \big). \tag{2.6}
$$

Moreover, the families of operators

$$
\Big\{ \lambda^\gamma \partial_\lambda^\gamma \Big[\lambda^{1-\frac{k}{2}} \begin{pmatrix} D^\alpha D^\delta & 0 \\ 0 & D^\alpha \end{pmatrix} A(D,\lambda)^{-1} \Big] : \lambda \in \Sigma_{\pi-\vartheta-\varepsilon} \Big\} \tag{2.7}
$$

in $L(\mathbb{E}, L^p(\mathbb{R}^n))$ and

$$
\Big\{ \lambda^\gamma \partial_\lambda^\gamma \Big[\lambda^{1-\frac{k}{2}} \begin{pmatrix} D^\alpha D^\delta & 0 \\ 0 & D^\alpha \end{pmatrix} A(D,\lambda)^{-1} \begin{pmatrix} (\lambda-\Delta)^{-1} & 0 \\ 0 & 1 \end{pmatrix} \Big] : \lambda \in \Sigma_{\pi-\vartheta-\varepsilon} \Big\}, \tag{2.8}
$$

$$
\Big\{ \lambda^\gamma \partial_\lambda^\gamma \Big[(\lambda-\Delta)^{2-\frac{k}{2}} \begin{pmatrix} D^\alpha & 0 \end{pmatrix} A(D,\lambda)^{-1} \begin{pmatrix} (\lambda-\Delta)^{-1} & 0 \\ 0 & 1 \end{pmatrix} \Big] : \lambda \in \Sigma_{\pi-\vartheta-\varepsilon} \Big\} \tag{2.9}
$$

in $L(L^p(\mathbb{R}^n))$ are $\mathcal{R}$–bounded.

Proof. We proceed as in the proof of Proposition 2.2, where we replace the matrices $S_i(\xi)$ by

$$
\dot{S}_1(\xi) := \begin{pmatrix} |\xi|^2 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \dot{S}_{(2+k)/2}(\xi) := |\xi|^k \dot{S}_1(\xi)
$$

and use the symbols

$$\begin{aligned}\dot M_k(\xi,\lambda) &:= \lambda^{1-\frac{k}{2}}\dot S_{(2+k)/2}(\xi)A(\xi,\lambda)^{-1}\dot S_1(\xi)^{-1}\\ &= \frac{\lambda^{1-\frac{k}{2}}\,|\xi|^k}{(\alpha_+\lambda+|\xi|^2)(\alpha_-\lambda+|\xi|^2)}\begin{pmatrix}\lambda+\rho\,|\xi|^2 & |\xi|^2\\ -|\xi|^2 & \lambda\end{pmatrix}\end{aligned}$$

for $k\in\{0,1,2\}$, cf. (2.3). We fix $\varepsilon\in(0,\pi-\vartheta)$ and take $\lambda\in\Sigma_{\pi-\vartheta-\varepsilon}$ and $\xi\in\mathbb{R}^n$. Observe that then the expressions

$$\frac{\lambda}{\alpha_\pm\lambda+|\xi|^2}\quad\text{and}\quad\frac{|\xi|^2}{\alpha_\pm\lambda+|\xi|^2}$$

are uniformly bounded. Moreover, $2\,|\lambda|^{\frac12}\,|\xi|\le|\lambda|+|\xi|^2$ and $\nabla|\xi|=\xi\,|\xi|^{-1}$. Therefore the terms $\xi^\beta\partial_\xi^\beta\,\lambda^\gamma\partial_\lambda^\gamma\,\dot M_k(\xi,\lambda)$ are bounded by a constant depending on $|\alpha|$, $|\gamma|$ and ε, but not on $\lambda\in\Sigma_{\pi-\vartheta-\varepsilon}$ and $\xi\in\mathbb{R}^n$. A result by Girardi and Weis (Corollary 3.3 in [17]) now says that the family of operators

$$\{\lambda^\gamma\partial_\lambda^\gamma\dot M_k(D,\lambda):\lambda\in\Sigma_{\pi-\vartheta-\varepsilon}\}\subset L(L^p(\mathbb{R}^n))$$

is $\mathcal{R}$-bounded for each $\varepsilon>0$. Since the symbols $\xi^\alpha\,|\xi|^{-|\alpha|}$ and $|\xi|^2(1+|\xi|^2)^{-1}$ also satisfy the assumptions of Michlin's theorem, the estimate (2.5) and the assertion about (2.7) follow.

In the definition of $\dot M_k$ one can replace $\dot S_1(\xi)^{-1}$ by the symbol

$$\begin{pmatrix}(\lambda+|\xi|^2)^{-1} & 0\\ 0 & 1\end{pmatrix}$$

and then establish the $\mathcal{R}$–boundedness of the operator family (2.8) as above. By means of the symbols

$$\left(\frac{\lambda^{1-k/2}\xi^\alpha(\lambda+\rho|\xi|^2)}{(\alpha_+\lambda+|\xi|^2)(\alpha_-\lambda+|\xi|^2)},\ \frac{\lambda^{2-k/2}\xi^\alpha}{(\alpha_+\lambda+|\xi|^2)(\alpha_-\lambda+|\xi|^2)}\right),$$
$$\frac{(\lambda+|\xi|^2)^{2-k/2}\xi^\alpha}{(\alpha_+\lambda+|\xi|^2)(\alpha_-\lambda+|\xi|^2)}\ \left(\frac{\lambda+\rho|\xi|^2}{\lambda+|\xi|^2},\ 1\right),$$

we finally derive (2.6) and the $\mathcal{R}$–boundedness of (2.9) from (2.3) and Michlin's theorem as before. □

Although the operator A_p is not sectorial, the above theorem contains precise resolvent estimates. By the next result, the singularity for $\lambda\to0$ disappears if we consider the shifted operator $A_p+\lambda_0$ with $\lambda_0>0$.

Proposition 2.4. *For every $\lambda_0>0$, the operator $A_p+\lambda_0$ is $\mathcal{R}$-sectorial with $\mathcal{R}$-angle $\vartheta(\rho)$.*

Proof. As in the proof of Proposition 2.2 b), we have to consider $M_0(\xi,\lambda)$ from (2.4) with $\xi\in\mathbb{R}^n$ and $\lambda\in\lambda_0+\Sigma_{\pi-\vartheta-\varepsilon}$ for fixed $\lambda_0>0$ and $\varepsilon\in(0,\pi-\vartheta)$.

However, as $\alpha_\pm\lambda$ cannot approach zero, now the term

$$\frac{\lambda(1+|\xi|^2)}{(\alpha_+\lambda+|\xi|^2)(\alpha_-\lambda+|\xi|^2)}$$

is uniformly bounded for $\lambda \in \lambda_0 + \Sigma_{\pi-\vartheta-\varepsilon}$. The same holds for all other terms of $M_0(\xi,\lambda)$ and for $\xi^\beta \partial_\xi^\beta M_0(\xi,\lambda)$ with $\beta \in \mathbb{N}_0^n$. Using Corollary 3.3 in [17], we deduce that $A_p + \lambda_0$ is $\mathcal{R}$-sectorial in $\mathbb{E}$. □

Proposition 2.4 allows us to solve (1.1) in optimal regularity. Part b) of the next result would also follow from Theorems 2.1 and 4.1 of [7].

Theorem 2.5. *a) The operator $-A_p$ generates an analytic C_0-semigroup on $\mathbb{E}$ and has maximal L^q-regularity on bounded time intervals for every $q \in (1,\infty)$.*

b) Let $f \in L^p((0,T);L^p(\mathbb{R}^n)) =: \mathcal{E}$ for some $T>0$, $\varphi_0 \in W_p^{4-2/p}(\mathbb{R}^n)$ and $\varphi_1 \in W_p^{2-2/p}(\mathbb{R}^n)$. Then there is a unique solution

$$u \in H_p^2((0,T);L^p(\mathbb{R}^n)) \cap L^p((0,T);H_p^4(\mathbb{R}^n)) =: \mathcal{F}$$

of (1.1) on $G=\mathbb{R}^n$, and there is a constant $C_p(T)>0$ such that

$$\|u\|_{\mathcal{F}} \le C_p(T)\big(\|f\|_{\mathcal{E}} + \|\varphi_0\|_{W_p^{4-2/p}(\mathbb{R}^n)} + \|\varphi_1\|_{W_p^{2-2/p}(\mathbb{R}^n)}\big).$$

c) Let $f=0$, $\varphi_0 \in H_p^2(\mathbb{R}^n)$ and $\varphi_1 \in L^p(\mathbb{R}^n)$. Then there exists a unique solution u of (1.1) on $G=\mathbb{R}^n$ with

$$\partial_t^2 u,\ \partial_t\nabla^2 u,\ \nabla^4 u \in C([\varepsilon,\infty),L^p(\mathbb{R}^n))$$

for each $\varepsilon>0$ and

$$\partial_t u,\ \nabla^2 u \in C([0,\infty),L^p(\mathbb{R}^n)).$$

If $\varphi_0 \in H_p^4(\mathbb{R}^n)$ and $\varphi_1 \in H_p^2(\mathbb{R}^n)$, we can take $\varepsilon=0$.

Proof. Assertion a) follows from Proposition 2.4, Theorem 4.2 in [31] and rescaling, since we have $\vartheta(\rho) < \frac{\pi}{2}$. In the context of part b) we thus obtain a unique solution $v=(v_1,v_2)^\top \in H_p^1((0,T);\mathbb{E}) \cap L^p((0,T);\mathbb{F}) =: \mathcal{X}$ of the first-order problem

$$\begin{aligned} \partial_t v + A(D)v &= (0,f)^\top, \qquad t>0,\\ v(0) &= (\varphi_0,\varphi_1)^\top. \end{aligned} \tag{2.10}$$

Moreover, $\|v\|_{\mathcal{X}} \le C_p(T)\,(\|f\|_{\mathcal{E}(T)} + \|(\varphi_0,\varphi_1)\|_{W_p^{4-2/p}(\mathbb{R}^n)\times W_p^{2-2/p}(\mathbb{R}^n)})$ for some constant $C_p(T)>0$. (See e.g. Theorems 1.14.5 and 2.4.2/2 in [30] for the relevant properties of real interpolation spaces.) We set $u := v_1$. The first component of (2.10) then yields $\partial_t u = v_2$ which easily implies that u belongs to $\mathcal{F}$, solves (1.1) and satisfies the estimate in b). Conversely, if $u\in\mathcal{F}$ solves (1.1), then $v := (u,\partial_t u)^\top$ belongs to $H_p^1((0,T);\mathbb{E}) \cap L^p((0,T);\mathbb{F})$ and fulfills (2.10). We recall that $\mathcal{F} \hookrightarrow H_p^1(J;H_p^2(\mathbb{R}^n))$. (This fact can be found, e.g., in Lemma 4.3 of [12].) Hence, assertion b) holds. Part c) can similarly be shown using that $-A_p$ generates an analytic C_0-semigroup on $\mathbb{E}$. □

3. The stationary problem in the half-space case

In this section we treat the model problem in the half-space $\mathbb{R}^n_+$. We start with a homogeneous right-hand side and inhomogeneous boundary conditions. We thus study the parameter-dependent boundary value problem

$$\begin{aligned} A(D,\lambda)v &= 0 && \text{in } \mathbb{R}^n_+, \\ v_1 &= g_0 && \text{on } \mathbb{R}^{n-1}, \\ -\partial_n v_1 &= g_1 && \text{on } \mathbb{R}^{n-1}, \end{aligned} \tag{3.1}$$

for $\lambda \in \Sigma_{\pi-\vartheta}$ and given functions g_0 and g_1 on $\mathbb{R}^{n-1}$, say in the Schwartz class.

Following a standard approach in parameter-elliptic theory, we apply the partial Fourier transform $\mathscr{F}'$ in the tangential variables $x' := (x_1, \dots, x_{n-1})^\top$. We set $w(x_n) := w(\xi', x_n, \lambda) := (\mathscr{F}'v)(\xi', x_n, \lambda)$ and

$$A(\xi', D_n, \lambda) = \begin{pmatrix} \lambda & -1 \\ (|\xi'|^2 - \partial_n^2)^2 & \lambda + \rho(|\xi'|^2 - \partial_n^2) \end{pmatrix}.$$

Problem (3.1) then leads to the family of ordinary differential equations

$$A(\xi', D_n, \lambda)w(x_n) = 0, \qquad x_n > 0, \tag{3.2}$$

$$w_1(0) = (\mathscr{F}'g_0)(\xi'), \tag{3.3}$$

$$-\partial_n w_1(0) = (\mathscr{F}'g_1)(\xi'), \tag{3.4}$$

on the half-line $\mathbb{R}_+$, where $\xi' \in \mathbb{R}^{n-1}$. Equation (3.2) gives $w_2 = \lambda w_1$ for the solution w_1 of

$$\lambda^2 w_1(x_n) + \lambda\rho(|\xi'|^2 - \partial_n^2)w_1(x_n) + (|\xi'|^2 - \partial_n^2)^2 w_1(x_n) = 0, \qquad x_n > 0. \tag{3.5}$$

To solve this equation, we consider its characteristic polynomial

$$P(\tau) := \lambda^2 + \lambda\rho(|\xi'|^2 - \tau^2) + (|\xi'|^2 - \tau^2)^2.$$

Straightforward calculations show that the roots of this polynomial are given by $\tau = \pm\sqrt{|\xi'|^2 + \alpha_\pm \lambda}$. We know from the beginning of Section 2 that $\arg \alpha_\pm = \pm\vartheta$, and hence $|\xi'|^2 + \alpha_\pm\lambda \notin (-\infty, 0)$ for $\lambda \in \Sigma_{\pi-\vartheta}$. The above square root is thus well-defined. The roots with positive real part are given by

$$\tau_1 = \tau_1(\xi', \lambda) := \sqrt{|\xi'|^2 + \alpha_+\lambda} \quad \text{and} \quad \tau_2 = \tau_2(\xi', \lambda) := \sqrt{|\xi'|^2 + \alpha_-\lambda}.$$

We have $\tau_1 \neq \tau_2$ for $\rho \neq 2$, while in the case $\rho = 2$ the root $\tau_1 = \tau_2$ has multiplicity 2. For fixed $\varepsilon > 0$, we obtain $\operatorname{Re}\tau_j \geq C|\tau_j|$ and

$$C(|\xi'|^2 + |\lambda|)^{1/2} \leq |\tau_j(\xi', \lambda)| \leq C'(|\xi'|^2 + |\lambda|)^{1/2} \tag{3.6}$$

for all $\xi' \in \mathbb{R}^{n-1}$ and $\lambda \in \Sigma_{\pi-\vartheta-\varepsilon}$. Our arguments below also involve the points $\tau(r, \xi', \lambda) = \tau(r) := \tau_1 + r(\tau_2 - \tau_1) \in \Sigma_{(\pi-\varepsilon)/2}$, $r \in [0,1]$, on the straight line between τ_1 and τ_2, which also satisfy

$$C(|\xi'|^2 + |\lambda|)^{1/2} \leq |\tau(r, \xi', \lambda)| \leq C'(|\xi'|^2 + |\lambda|)^{1/2} \tag{3.7}$$

for all $r \in [0,1]$, $\xi' \in \mathbb{R}^{n-1}$, and $\lambda \in \Sigma_{\pi-\vartheta-\varepsilon}$. Here, the upper inequality directly follows from (3.6). For the lower one, the above estimates yield

$$\begin{aligned} |\tau(r)| &\geq \operatorname{Re}\tau(r) = (1-r)\operatorname{Re}\tau_1 + r\operatorname{Re}\tau_2 \geq C\left((1-r)\,|\tau_1| + r\,|\tau_2|\right) \\ &\geq C(|\xi'|^2 + |\lambda|)^{1/2}. \end{aligned}$$

Here and below, $C, C', \dots$ stand for generic constants which may be different in each appearance and which are independent of ξ', λ, and y_n (but which may depend on ε and ρ).

Lemma 3.1. *Let $\xi' \in \mathbb{R}^{n-1}$ and $\lambda \in \Sigma_{\pi-\vartheta}$. We define the fundamental solutions $\omega^{(i)} = (\omega_j^{(i)}(\xi',\cdot,\lambda))_{j=1,2} \colon (0,\infty) \to \mathbb{C}^2$ for $i \in \{0,1\}$ by*

$$\begin{aligned} \omega_1^{(0)}(\xi', x_n, \lambda) &= \tfrac{1}{\tau_1-\tau_2}(-\tau_2 e^{-\tau_1 x_n} + \tau_1 e^{-\tau_2 x_n}), \\ \omega_1^{(1)}(\xi', x_n, \lambda) &= \tfrac{1}{\tau_1-\tau_2}(-e^{-\tau_1 x_n} + e^{-\tau_2 x_n}), \\ \omega_2^{(i)} &= \lambda\omega_1^{(i)} \end{aligned}$$

for $\rho \neq 2$. For $\rho = 2$ we set

$$\begin{aligned} \omega_1^{(0)}(\xi', x_n, \lambda) &= (1+\tau x_n)e^{-\tau x_n}, \\ \omega_1^{(1)}(\xi', x_n, \lambda) &= x_n e^{-\tau x_n}, \\ \omega_2^{(i)} &= \lambda\omega_1^{(i)}, \end{aligned}$$

where $\tau := \tau_1 = \tau_2$. Then $\omega^{(i)}$ is a solution of (3.2) *with the initial values*

$$\omega_1^{(0)}(0) = 1, \quad \partial_n\omega_1^{(0)}(0) = 0$$

and

$$\omega_1^{(1)}(0) = 0, \quad \partial_n\omega_1^{(1)}(0) = 1,$$

respectively. In particular, $\{\omega^{(0)}, \omega^{(1)}\}$ is a basis of the space of all stable solutions of (3.2)*.*

Proof. We first consider the case $\rho \neq 2$. Then every stable solution of (3.2) has the form $\omega(x_n) = (\omega_1(x_n), \omega_2(x_n))^\top$ with $\omega_2(x_n) = \lambda\omega_1(x_n)$ and $\omega_1(x_n) = c_1 e^{-\tau_1 x_n} + c_2 e^{-\tau_2 x_n}$. The initial values are given by

$$\omega(0) = c_1 + c_2 \quad \text{and} \quad (\partial_n\omega)(0) = -\tau_1 c_1 - \tau_2 c_2.$$

The formulas for the fundamental solutions now follow directly from the initial conditions.

Similarly, in the case $\rho = 2$, we have a double root $\tau = \tau_1 = \tau_2 = \sqrt{|\xi'|^2 + \lambda}$, and every stable solution is of the form $\omega_1(x_n) = (c_1 + x_n c_2)e^{-\tau x_n}$. The initial conditions $\omega_1(0) = c_1$ and $(\partial_n\omega_1)(0) = -\tau c_1 + c_2$ then yield the asserted expression for the fundamental solutions. □

The following technical result will be the basis for the a priori estimate of the solutions of the half-space problems.

Lemma 3.2. *a) For fixed $\varepsilon > 0$, $k \in \mathbb{N}$ and $\ell \in \mathbb{Z}$, we define the function $f_{k,\ell}\colon \mathbb{R}^{n-1} \times (0,\infty) \times \Sigma_{\pi-\vartheta-\varepsilon} \to \mathbb{C}$ by*

$$f_{k,\ell}(\xi', x_n, \lambda) := \begin{cases} \frac{x_n^k}{\tau_1-\tau_2}\,(\tau_1^\ell e^{-\tau_1 x_n} - \tau_2^\ell e^{-\tau_2 x_n}), & \rho \neq 2, \\ x_n^{k+1}\tau^\ell e^{-\tau x_n}, & \rho = 2 \quad (\text{with } \tau = \tau_1 = \tau_2). \end{cases}$$

Then for all $\gamma \in \mathbb{N}_0^2$ and $\beta' \in \mathbb{N}_0^{n-1}$ we obtain

$$\Big|\lambda^\gamma \partial_\lambda^\gamma (\xi')^{\beta'} \partial_{\xi'}^{\beta'} f_{k,\ell}(\xi', x_n, \lambda)\Big| \leq C\big(|\xi'|^2 + |\lambda|\big)^{(\ell-k-1)/2}.$$

b) Let $\omega^{(i)}$, $i \in \{0,1\}$, be the fundamental solutions from Lemma 3.1*. Further, let $\varepsilon > 0$, $k \in \{0,1,2,3,4\}$ and $\alpha = (\alpha', \alpha_n) \in \mathbb{N}_0^n$ with $|\alpha| = k$. Then for all $\gamma \in \mathbb{N}_0^2$, $\beta' \in \mathbb{N}_0^{n-1}$, $x_n > 0$, $\lambda \in \Sigma_{\pi-\vartheta-\varepsilon}$, $m \in \mathbb{N}_0$, and $\xi' \in \mathbb{R}^{n-1}$ the inequality*

$$\Big|\lambda^\gamma \partial_\lambda^\gamma (\xi')^{\beta'} \partial_{\xi'}^{\beta'} \Big[\lambda^{2-\frac{k}{2}} (\xi')^{\alpha'} x_n^{m+1} \partial_n^{\alpha_n+j} \omega_1^{(i)}(\xi', x_n, \lambda)(\lambda + |\xi'|^2)^{(i-j+m-3)/2}\Big]\Big| \leq C$$

holds for $j \in \{0,1\}$.

Proof. a) We only consider $\rho \neq 2$, the case $\rho = 2$ is treated in the same way (it is actually a bit simpler). We define

$$\varphi\colon \Sigma_{(\pi-\varepsilon)/2} \to \mathbb{C}; \quad \tau \mapsto x_n^k \tau^\ell e^{-\tau x_n}.$$

Recall that $\tau(r) = \tau_1 + r(\tau_2 - \tau_1) \in \Sigma_{(\pi-\varepsilon)/2}$ for $r \in [0,1]$. We start with the case $|\gamma| = |\beta'| = 0$. Using the elementary estimate $|(\tau x_n)^m e^{-\tau x_n}| \leq C$ for $\tau \in \Sigma_{(\pi-\varepsilon)/2}$ and $x_n > 0$, we obtain

$$\begin{aligned} |f_{k,\ell}(\xi', x_n, \lambda)| &= \Big|\frac{\varphi(\tau_1) - \varphi(\tau_2)}{\tau_1 - \tau_2}\Big| = \Big|\int_0^1 \varphi'(\tau_1 + r(\tau_2 - \tau_1))\,dr\Big| \\ &\leq C \sup_{r\in[0,1]} \Big[|(x_n\tau(r))^k e^{-\tau(r)x_n}| + |(x_n\tau(r))^{k+1} e^{-\tau(r)x_n}|\Big] |\tau(r)|^{\ell-k-1} \\ &\leq C \sup_{r\in[0,1]} |\tau(r)|^{\ell-k-1} \leq C(|\xi'|^2 + |\lambda|)^{(\ell-k-1)/2}. \end{aligned}$$

In the last step we employed inequality (3.7). The statement in the case $\beta' \neq 0$ and $\gamma = 0$ follows iteratively from the recursion formula

$$\partial_{\xi_j} f_{k,\ell} = \xi_j\Big(\frac{f_{k,\ell}}{\tau_1\tau_2} + \ell f_{k,\ell-2} - f_{k+1,\ell-1}\Big).$$

This formula can directly be checked observing that $\partial_{\xi_j}\tau = \frac{\xi_j}{\tau}$ for $\tau = \tau_1, \tau_2$.

For the λ–derivatives we note that $\partial_{\lambda_1}\tau = \frac{\alpha_\pm}{2\tau}$ and $\partial_{\lambda_2}\tau = \frac{i\alpha_\pm}{2\tau}$. We compute

$$\begin{aligned} \partial_{\lambda_1} f_{k,\ell} &= \partial_{\lambda_1} \int_0^1 \varphi'(\tau_1 + r(\tau_2 - \tau_1))\,dr \\ &= \int_0^1 \varphi''(\tau_1 + r(\tau_2 - \tau_1))\Big(\frac{\alpha_+}{2\tau_1} + \frac{r\alpha_-}{2\tau_2} - \frac{r\alpha_+}{2\tau_1}\Big)dr. \end{aligned}$$

We set $\sigma = (|\xi'|^2 + |\lambda|)^{1/2}$. Estimate (3.7) yields

$$|\lambda_1 \partial_{\lambda_1} f_{k,\ell}| \le C \frac{|\lambda_1|}{\sigma} \sup_{0\le r\le 1} \sum_{j=0}^{2} |y_n^{k+j} \tau(r)^{\ell+j-2} e^{-\tau(r) y_n}| \le C\sigma \sup_{0\le r\le 1} |\tau(r)|^{\ell-k-2}$$
$$\le C(|\xi'|^2 + |\lambda|)^{(\ell-k-1)/2}.$$

The λ_2–derivative is treated in the same way so that we have shown a) for $|\gamma| = 1$ and $\beta' = 0$. The remaining cases can now be established by recursion.

b) For $\rho \ne 2$ and $i = 0$, we write

$$\begin{aligned}\omega_1^{(0)}(\xi', x_n, \lambda) &= -\frac{\tau_2}{\tau_1 - \tau_2} e^{-\tau_1 x_n} + \frac{\tau_1}{\tau_1 - \tau_2} e^{-\tau_2 x_n} \\ &= (1 - \tfrac{\tau_1}{\tau_1 - \tau_2}) e^{-\tau_1 x_n} + (1 + \tfrac{\tau_2}{\tau_1 - \tau_2}) e^{-\tau_2 x_n} \\ &= (e^{-\tau_1 x_n} + e^{-\tau_2 x_n}) - f_{0,1}(\xi', x_n, \lambda).\end{aligned}$$

It follows

$$\begin{aligned} x_n^{m+1} \partial_n^{\alpha_n + j} \omega_1^{(0)}(\xi', x_n, \lambda) = (-1)^{\alpha_n + j} \Big(& x_n^{m+1} \tau_1^{\alpha_n + j} e^{-\tau_1 x_n} \\ & + x_n^{m+1} \tau_2^{\alpha_n + j} e^{-\tau_2 x_n} - f_{m+1, \alpha_n + j + 1}(\xi', x_n, \lambda) \Big). \end{aligned} \tag{3.8}$$

The first term on the right hand side can be estimated by

$$\begin{aligned} \big| x_n^{m+1} \tau_1^{\alpha_n + j} e^{-\tau_1 x_n} \big| &= |\tau_1|^{\alpha_n + j - m - 1} \big| (\tau_1 x_n)^{m+1} e^{-\tau_1 x_n} \big| \\ &\le C(|\xi'|^2 + |\lambda|)^{(\alpha_n + j - m - 1)/2}. \end{aligned}$$

Derivatives with respect to ξ' and λ can be handled as in a), and we infer

$$\Big| \lambda^\gamma \partial_\lambda^\gamma (\xi')^{\beta'} \partial_{\xi'}^{\beta'} \big[x_n^{m+1} \tau_1^{\alpha_n + j} e^{-\tau_1 x_n} \big] \Big| \le C(|\xi'|^2 + |\lambda|)^{(\alpha_n + j - m - 1)/2}. \tag{3.9}$$

The same inequality holds for the second term in (3.8), and due to part a) also for the third one.

For $\rho \ne 2$ and $i = 1$, we have $\omega_1^{(1)}(\xi', x_n, \lambda) = f_{0,0}(\xi', x_n, \lambda)$ and hence

$$x_n^{m+1} \partial_n^{\alpha_n + j} \omega_1^{(1)}(\xi', x_n, \lambda) = (-1)^{\alpha_n + j} f_{m+1, \alpha_n + j}(\xi', x_n, \lambda).$$

Assertion a) then implies

$$\Big| \lambda^\gamma \partial_\lambda^\gamma (\xi')^{\beta'} \partial_{\xi'}^{\beta'} \big[x_n^{m+1} \partial_n^{\alpha_n + j} \omega_1^{(1)}(\xi', x_n, \lambda) \big] \Big| \le C(|\xi'|^2 + |\lambda|)^{(\alpha_n + j - m - 2)/2}.$$

In the case $\rho = 2$ (where $\tau_1 = \tau_2 = \tau$) the situation is similar. For $\omega_1^{(0)}(\xi', x_n, \lambda) = (1 + \tau x_n) e^{-\tau x_n}$, Leibniz' formula yields

$$\begin{aligned} \big| x_n^{m+1} \partial_n^{\alpha_n + j} \omega_1^{(0)}(\xi', x_n, \lambda) \big| &= \big| x_n^{m+1} \tau^{\alpha_n + j} (1 - \alpha_n - j + \tau x_n) e^{-\tau x_n} \big| \\ &\le C(|\xi'|^2 + |\lambda|)^{(\alpha_n + j - m - 1)/2}. \end{aligned}$$

The derivatives with respect to ξ' and λ can then be controlled as in (3.9). In the same way, we estimate $\omega_1^{(1)}(\xi', x_n, \lambda) = x_n e^{-\tau x_n}$. In all cases, we have established

$$\Big| \lambda^\gamma \partial_\lambda^\gamma (\xi')^{\beta'} \partial_{\xi'}^{\beta'} \big[x_n^{m+1} \partial_n^{\alpha_n + j} \omega_1^{(i)}(\xi', x_n, \lambda) \big] \Big| \le C(|\xi'|^2 + |\lambda|)^{(\alpha_n + j - i - m - 1)/2}.$$

The statement in b) now follows from Leibniz' rule and the observation

$$\big|\lambda^\gamma \partial_\lambda^\gamma (\xi')^{\beta'} \partial_{\xi'}^{\beta'} [(\xi')^{\alpha'} \lambda^{2-k/2}]\big| \le C(|\xi'|^2 + |\lambda|)^{(|\alpha'|+4-k)/2}. \qquad \square$$

In the next result, we introduce the solution operators $L_j^{(i)}(\lambda)$ for the parameter–dependent boundary value problem (3.1) and establish the crucial a priori bounds for these operators. For $s \ge 0$ and $\lambda \in \mathbb{C}$ we will use the parameter-dependent shift operators $(\lambda - \Delta')^s = (\mathscr{F}')^{-1}(\lambda + |\xi'|^2)^s \mathscr{F}'$ on $\mathbb{R}^{n-1}$ and $(\lambda - \Delta)^s = (\mathscr{F})^{-1}(\lambda + |\xi|^2)^s \mathscr{F}$ on $\mathbb{R}^n$.

Proposition 3.3. *For $i, j \in \{0, 1\}$ and $\lambda \in \Sigma_{\pi-\vartheta}$, we define the operator $L_j^{(i)}(\lambda)$ by*

$$(L_j^{(i)}(\lambda)\phi)(\cdot, x_n) := -\int_0^\infty (\mathscr{F}')^{-1} \partial_n^j \omega_1^{(i)}(\cdot, x_n + y_n, \lambda)(\mathscr{F}'\phi)(\cdot, y_n) dy_n, \quad x_n > 0,$$

for all functions $\phi : \mathbb{R}^n_+ \to \mathbb{C}$ which are restrictions of Schwartz functions on $\mathbb{R}^n$. Here the 'dot' refers to x' or ξ' in $\mathbb{R}^{n-1}$. Then the following assertions hold.

a) Set $v_1^{(i)} = L_0^{(i)}(\lambda)\partial_n \phi + L_1^{(i)}(\lambda)\phi$ and $v^{(i)} = (v_1^{(i)}, \lambda v_1^{(i)})^\top$ for $i \in \{0, 1\}$. Then $v_1^{(i)}(\cdot, x_n) = (\mathscr{F}')^{-1}\omega_1^{(i)}(\cdot, x_n, \lambda)(\mathscr{F}'\phi)(\cdot, 0)$ for $x_n > 0$ and

$$\begin{aligned} A(D, \lambda)v^{(i)} &= 0 \qquad \text{in } \mathbb{R}^n_+, \quad i = 0, 1, \\ v_1^{(0)}(\cdot, 0) &= \phi(\cdot, 0), \quad \partial_n v_1^{(0)}(\cdot, 0) = 0 \qquad \text{on } \mathbb{R}^{n-1}, \\ v_1^{(1)}(\cdot, 0) &= 0, \quad \partial_n v_1^{(1)}(\cdot, 0) = \phi(\cdot, 0) \qquad \text{on } \mathbb{R}^{n-1}. \end{aligned}$$

b) Let $\varepsilon \in (0, \pi - \vartheta)$, $\gamma \in \mathbb{N}_0^2$, $k \in \{0, 1, 2, 3, 4\}$ and $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = k$. Then the set of operators in $L(L^p(\mathbb{R}^n_+))$

$$\big\{\lambda^\gamma \partial_\lambda^\gamma \big[\lambda^{2-k/2} D^\alpha L_j^{(i)}(\lambda)(\lambda - \Delta')^{(i-j-3)/2}\big] : \lambda \in \Sigma_{\pi-\vartheta-\varepsilon}, \big\}$$

is (well-defined and) $\mathcal{R}$-bounded.

Proof. a) Integrating by parts in the integral defining $L_j^{(i)}(\lambda)$, we obtain the first assertion. The properties of $\omega_1^{(i)}$ shown in Lemma 3.1 then yield the second part of assertion a).

b) Let $x_n, y_n > 0$, $\lambda \in \Sigma_{\pi-\vartheta-\varepsilon}$, $\xi' \in \mathbb{R}^{n-1}$, $\gamma \in \mathbb{N}_0^2$, $k \in \mathbb{N}_0$, $\alpha \in \mathbb{N}_0^n$ and $\beta' \in \mathbb{N}_0^{n-1}$. Lemma 3.2 b) yields with $m = 0$

$$\big|\lambda^\gamma \partial_\lambda^\gamma (\xi')^{\beta'} \partial_{\xi'}^{\beta'} \big[\lambda^{2-\frac{k}{2}} (\xi')^{\alpha'} \partial_n^{\alpha_n + j} \omega_1^{(i)}(\xi', x_n + y_n, \lambda)(\lambda + |\xi'|^2)^{(i-j-3)/2}\big]\big| \le \frac{C}{x_n + y_n},$$

where C does not depend on x_n, y_n, λ or ξ'. The Michlin-type Corollary 3.2 in [17] thus shows that the family of operators

$$\big\{(\mathscr{F}')^{-1} \lambda^\gamma \partial_\lambda^\gamma \big[\lambda^{2-\frac{k}{2}} (\xi')^{\alpha'} \partial_n^{\alpha_n + j} \omega_1^{(i)}(\xi', x_n + y_n, \lambda)(\lambda + |\xi'|^2)^{(i-j-3)/2}\big] \mathscr{F}' : \\ \lambda \in \Sigma_{\pi-\vartheta-\varepsilon}\big\} \subset L(L^p(\mathbb{R}^{n-1}))$$

is $\mathcal{R}$-bounded with $\mathcal{R}$-bound not greater than $\frac{C}{x_n + y_n}$, for all $x_n, y_n > 0$. As the scalar integral operator in $L^p(\mathbb{R}_+)$ with kernel $\frac{1}{x_n + y_n}$ is bounded, we can apply Proposition 4.12 in [10] to derive the statement. $\square$

Based on the above result, we now investigate the inhomogeneous parameter-dependent boundary value problem

$$
\begin{aligned}
A(D,\lambda)v &= h && \text{in } \mathbb{R}^n_+,\\
v_1 &= g_0 && \text{on } \mathbb{R}^{n-1},\\
-\partial_n v_1 &= g_1 && \text{on } \mathbb{R}^{n-1},
\end{aligned}
\tag{3.10}
$$

for $\lambda \in \Sigma_{\pi-\vartheta}$ and given functions $h = (h_1, h_2)^\top$ in $\mathbb{R}^n_+$ and g_0, g_1 on $\mathbb{R}^{n-1}$. Due to the structure of the matrix $A(D)$, the natural choice of spaces is

$$
\begin{aligned}
\mathbb{E}_+ &:= H^2_p(\mathbb{R}^n_+) \times L^p(\mathbb{R}^n_+),\\
\mathbb{F}_+ &:= H^4_p(\mathbb{R}^n_+) \times H^2_p(\mathbb{R}^n_+),\\
\mathbb{G} &:= W^{4-1/p}_p(\mathbb{R}^{n-1}) \times W^{3-1/p}_p(\mathbb{R}^{n-1}).
\end{aligned}
$$

We remark that (3.10) is a mixed-order boundary value problem in the sense of Douglis-Nirenberg, see e.g. [1]. The boundary conditions can be written in matrix form as $B(D)v = g$ where

$$
B(D) := \gamma_0 \begin{pmatrix} 1 & 0 \\ -\partial_n & 0 \end{pmatrix}.
$$

Here $\gamma_0 \colon v \mapsto v|_{\mathbb{R}^{n-1}}$ denotes the trace onto the boundary $\mathbb{R}^{n-1}$ of $\mathbb{R}^n_+$. By standard trace results (see e.g. Theorem 2.9.1 in [30]), the operator $(A(D,\lambda), B(D)) \colon \mathbb{F}_+ \to \mathbb{E}_+ \times \mathbb{G}$ is continuous. As usual, the L^p-realization $A_{p,+} \colon D(A_{p,+}) \subset \mathbb{E}_+ \to \mathbb{E}_+$ of the boundary value problem $(A(D), B(D))$ is defined by

$$
D(A_{p,+}) := \{v \in \mathbb{F}_+ : B(D)v = 0\} \quad \text{and} \quad A_{p,+}v := A(D)v.
$$

Note that we can write the domain of this operator in the form $D(A_{p,+}) = (H^4_p(\mathbb{R}^n_+) \cap H^2_{p,0}(\mathbb{R}^n_+)) \times H^2_p(\mathbb{R}^n_+)$, where for $k \in \mathbb{N}$ we define

$$
H^k_{p,0}(\mathbb{R}^n_+) := \{u \in H^k_p(\mathbb{R}^n_+) : \gamma_0 u = \gamma_0 \partial_n u = \cdots = \gamma_0 \partial_n^{k-1} u = 0\}.
$$

Before stating precise a priori estimates for the solution, we note that $\lambda_0 + A_{p,+}$ is not sectorial on $\mathbb{E}_+$ for any shift $\lambda_0 \geq 0$.

Proposition 3.4. *For each $\lambda_0 \geq 0$, the operator $A_{p,+} + \lambda_0$ is not sectorial in $\mathbb{E}_+$ and, consequently, does not generate a C_0-semigroup.*

Proof. The mixed-order system $(A(D) + \lambda_0, B(D))$ fits into the framework of Section 3.2 of [9] with the Douglis-Nirenberg structure $(s_1, s_2) = (0, 2)$, $(m_1, m_2) = (2, 0)$, and $(r_1, r_2) = (-2, -1)$. By Theorem 3.8 in [9], for every $h \in \mathbb{E}_+$ and $v_\lambda \in D(A_{p,+})$ with $A(D,\lambda)v_\lambda = h$ for $\lambda \in (0, \infty)$, the estimate $\sup_{\lambda \in (0,\infty)} \|\lambda v_\lambda\|_{\mathbb{E}_+} < \infty$ implies $\gamma_0 h_1 = \gamma_0 \partial_n h_1 = 0$. Therefore, the desired resolvent estimate does not hold for $h \in \mathbb{E}_+$ with $B(D)h \neq 0$. □

The proof of the last result indicates that zero boundary conditions have to be included in the basic space $\mathbb{E}_+$. In Section 4 we will indeed obtain a sectorial operator in this way.

To solve the inhomogeneous boundary value problem (3.10), we make use of restriction and extension operators. Let $e_0\colon L^p(\mathbb{R}^n_+) \to L^p(\mathbb{R}^n)$ denote the trivial extension by zero and $r_+\colon L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n_+)$ the restriction onto $\mathbb{R}^n_+$. Instead of the trivial extension e_0, we will also consider a global coretraction e_+ of r_+ which satisfies $e_+ \in L(H^s_p(\mathbb{R}^n_+), H^s_p(\mathbb{R}^n))$ and $r_+e_+ = \mathrm{id}_{H^s_p(\mathbb{R}^n_+)}$ for all $s \in \mathbb{N}_0$ (see e.g. Section 4.4 of [3]). A parameter-dependent extension operator from $\mathbb{R}^{n-1}$ to $\mathbb{R}^n_+$ is defined by

$$(E_\lambda \phi)(\cdot, x_n) := (\mathscr{F}')^{-1} \exp\big(-(\lambda + |\xi'|^2)^{1/2} x_n \big) \mathscr{F}'\phi \quad (x_n > 0).$$

This extension was studied in [2] and [19], for instance. In particular, Proposition 2.3 of [2] yields (after a minor modification) that E_λ belongs to $L(W_p^{k-1/p}(\mathbb{R}^{n-1}), H^k_p(\mathbb{R}^n_+))$ and that $\gamma_0 E_\lambda = \mathrm{id}_{W_p^{k-1/p}(\mathbb{R}^{n-1})}$ for all $k \in \mathbb{N}$ and $\lambda \in \Sigma_{\pi-\vartheta}$. We further deduce that

$$\partial_n E_\lambda \phi = -(\lambda - \Delta')^{1/2} E_\lambda \phi, \qquad \phi \in W_p^{1-1/p}(\mathbb{R}^{n-1}). \tag{3.11}$$

Theorem 3.5. *For all $\lambda \in \Sigma_{\pi-\vartheta}$, $h \in \mathbb{E}_+$ and $g \in \mathbb{G}$, there exists a unique solution $v \in \mathbb{F}_+$ of* (3.10)*. Moreover, this solution can be written in the form*

$$v = R(\lambda) e_+ h + T(\lambda) E_\lambda g, \quad T(\lambda) = T^{(0)}(\lambda)\partial_n + T^{(1)}(\lambda)$$

with operators $R(\lambda)$ and $T^{(j)}(\lambda)$, $j = 0, 1$, which have the following $\mathcal{R}$-boundedness property: Let $\varepsilon > 0$. Then for all $k \in \{0, 1, 2\}$, $|\alpha| = k$, $|\delta| = 2$, and $\gamma \in \mathbb{N}_0^2$ the families of operators

$$\left\{ \lambda^\gamma \partial_\lambda^\gamma \left[\lambda^{1-\frac{k}{2}} D^\alpha \begin{pmatrix} D^\delta & 0 \\ 0 & 1 \end{pmatrix} R(\lambda) \begin{pmatrix} (\lambda - \Delta)^{-1} & 0 \\ 0 & 1 \end{pmatrix} \right] : \lambda \in \Sigma_{\pi-\vartheta-\varepsilon} \right\}$$

in $L(L^p(\mathbb{R}^n), L^p(\mathbb{R}^n_+))$ and

$$\left\{ \lambda^\gamma \partial_\lambda^\gamma \left[\lambda^{1-\frac{k}{2}} D^\alpha \begin{pmatrix} D^\delta & 0 \\ 0 & 1 \end{pmatrix} T^{(j)}(\lambda) \begin{pmatrix} (\lambda - \Delta')^{(-j-3)/2} & 0 \\ 0 & (\lambda - \Delta')^{(-j-2)/2} \end{pmatrix} \right] : \lambda \in \Sigma_{\pi-\vartheta-\varepsilon} \right\}$$

in $L(L^p(\mathbb{R}^n_+))$ are $\mathcal{R}$-bounded.

Proof. **(i)** Let $\lambda \in \Sigma_{\pi-\vartheta}$, $h \in \mathbb{E}_+$ and $g \in \mathbb{G}$. We set $v' := r_+(A_p+\lambda)^{-1} e_+ h \in \mathbb{F}_+$ (see Proposition 2.2 a)) and write $v = v' + v''$. Then v'' has to solve the boundary value problem

$$\begin{aligned} A(D, \lambda) v'' &= 0 \qquad \text{in } \mathbb{R}^n_+, \\ B(D) v'' &= g - B(D) v' \qquad \text{on } \mathbb{R}^{n-1}. \end{aligned} \tag{3.12}$$

The function $\widetilde{g} := (\widetilde{g}_0, \widetilde{g}_1)^\top := E_\lambda g - \left(\begin{smallmatrix} v_1' \\ -\partial_n v_1' \end{smallmatrix}\right)$ is an extension of $g - B(D)v'$ to $\mathbb{R}^n_+$. By Proposition 3.3, a solution of (3.12) is given by

(3.13)
$$v'' = T(\lambda)\widetilde{g} := \sum_{j=0}^{1} T^{(j)}(\lambda)\partial_n^{1-j}\widetilde{g} \quad \text{with } T^{(j)}(\lambda) := \begin{pmatrix} L_j^{(0)}(\lambda) & -L_j^{(1)}(\lambda) \\ \lambda L_j^{(0)}(\lambda) & -\lambda L_j^{(1)}(\lambda) \end{pmatrix}$$

We remark that the operators $L_j^{(i)}(\lambda)$ were defined in Proposition 3.3 for restrictions of Schwartz functions to $\mathbb{R}^n_+$, but Proposition 3.3 b) shows that $L_j^{(i)}(\lambda)\partial_n^{1-j}$ can continuously be extended to an operator in $L(H_p^{4-i}(\mathbb{R}^n_+), H_p^4(\mathbb{R}^n_+))$ for $i, j \in \{0, 1\}$. In the same proposition, the equalities $A(D,\lambda)T(\lambda)\widetilde{g} = 0$ and $B(D)T(\lambda)\widetilde{g} = \gamma_0\widetilde{g}$ were shown for restrictions of Schwartz functions, and by continuity this identities also hold for the extended operators. As a result, the function $v := v' + v'' \in \mathbb{F}_+$ solves (3.10).

If $z \in \mathbb{F}_+$ is another solution of (3.10), then $\varphi := v - z \in \mathbb{F}_+$ solves this problem with $h = 0$ and $g = 0$. In particular, φ_1 belongs to $H^2_{p,0}(\mathbb{R}^n_+)$ so that $\varphi_2 = \lambda\varphi_1 \in H^2_{p,0}(\mathbb{R}^n_+)$. Therefore, $e_0\varphi$ is contained in $H^2_p(\mathbb{R}^n;\mathbb{C}^2) \subset \mathbb{E}$ and satisfies $A(D,\lambda)e_0\varphi = 0$. This means that $\Delta^2 e_0\varphi_1 = (\rho\Delta - \lambda)e_0\varphi_2$ in $\mathbb{R}^n$ which yields $e_0\varphi_1 \in H^4_p(\mathbb{R}^n)$ and hence $e_0\varphi \in \mathbb{F}$. Proposition 2.2 a) now implies $e_0\varphi = 0$ and thus the uniqueness of the solution of (3.10).

(ii) In this part we fix $\varepsilon > 0$ and consider $\lambda \in \Sigma_{\pi-\vartheta-\varepsilon}$. We have seen in part (i) of the proof that the unique solution v of (3.10) is given by $v = R(\lambda)e_+h + T(\lambda)E_\lambda g$ where $T(\lambda)$ is defined in (3.13) and

$$R(\lambda) := r_+(A_p + \lambda)^{-1} - T(\lambda)\begin{pmatrix} 1 & 0 \\ -\partial_n & 0 \end{pmatrix} r_+(A_p+\lambda)^{-1}. \tag{3.14}$$

Let $|\alpha| = k \in \{0, 1, 2\}$, $|\delta| = 2$ and $\gamma \in \mathbb{N}_0^2$. By Theorem 2.3, the family

$$\left\{\lambda^\gamma \partial_\lambda^\gamma \Big[\lambda^{1-k/2} D^\alpha \begin{pmatrix} D^\delta & 0 \\ 0 & 1 \end{pmatrix} (A_p+\lambda)^{-1} \begin{pmatrix} (\lambda-\Delta)^{-1} & 0 \\ 0 & 1 \end{pmatrix}\Big] : \lambda \in \Sigma_{\pi-\vartheta-\varepsilon}\right\}$$

in $L(L^p(\mathbb{R}^n))$ is $\mathcal{R}$-bounded, i.e., the first term in (3.14) is $\mathcal{R}$-bounded as asserted in the theorem. For the second term, we use (3.13) and write

$$\begin{aligned} T(\lambda)&\begin{pmatrix} 1 & 0 \\ -\partial_n & 0 \end{pmatrix} r_+(A_p+\lambda)^{-1} \\ &= \sum_{j=0}^{1} \begin{pmatrix} L_j^{(0)}(\lambda) & L_j^{(1)}(\lambda) \\ \lambda L_j^{(0)}(\lambda) & \lambda L_j^{(1)}(\lambda) \end{pmatrix} \begin{pmatrix} (\lambda-\Delta')^{(-j-3)/2} & 0 \\ 0 & (\lambda-\Delta')^{(-j-2)/2} \end{pmatrix} \\ &\quad\times \begin{pmatrix} \partial_n^{1-j}(\lambda-\Delta')^{(j+3)/2} & 0 \\ \partial_n^{2-j}(\lambda-\Delta')^{(j+2)/2} & 0 \end{pmatrix} r_+(A_p+\lambda)^{-1}. \end{aligned}$$

Let $i, j \in \{0,1\}$, $|\alpha| = k \in \{0, \ldots, 4\}$ and $\gamma \in \mathbb{N}_0^2$. The desired statement about the $\mathcal{R}$-boundedness for the second term in (3.14) now follows from Leibniz' rule, from the $\mathcal{R}$-boundedness of the family

$$\Big\{\lambda^\gamma \partial_\lambda^\gamma\big[\lambda^{2-k/2} D^\alpha L_j^{(i)}(\lambda)(\lambda-\Delta')^{(i-j-3)/2}\big] : \lambda \in \Sigma_{\pi-\vartheta-\varepsilon}\Big\} \tag{3.15}$$

in $L(L^p(\mathbb{R}^n_+))$, see Proposition 3.3 b), and from the $\mathcal{R}$-boundedness of the family

$$\left\{\lambda^\gamma \partial_\lambda^\gamma \Big[\big((\lambda-\Delta)^{2-k/2} D^\alpha \quad 0\big)\,(A_p+\lambda)^{-1} \begin{pmatrix} (\lambda-\Delta)^{-1} & 0 \\ 0 & 1 \end{pmatrix} \Big] : \lambda \in \Sigma_{\pi-\vartheta-\varepsilon}\right\}$$

in $L(L^p(\mathbb{R}^n))$, see Theorem 2.3.

The $\mathcal{R}$-boundedness of the second operator family in the theorem is deduced from Proposition 3.3 b) and (3.11) in the same way. □

Corollary 3.6. *For each $\varepsilon > 0$ and $\lambda_0 > 0$ there exists a constant $C = C(\varepsilon, \lambda_0)$ sucht that for all $|\alpha| = k \in \{0,1,2\}$, $|\delta| = 2$ and all $\lambda \in \lambda_0 + \Sigma_{\pi-\vartheta-\varepsilon}$ the estimate*

$$\begin{aligned}\|\lambda^{1-k/2} D^\alpha v\|_{\mathbb{E}_+} \le C\Big(&\|h\|_{\mathbb{E}_+} + \|g\|_{\mathbb{G}} + |\lambda|\,\|h_1\|_{L^p(\mathbb{R}^n_+)} \\ &+ |\lambda|^{2-\frac{1}{p}}\|g_0\|_{L^p(\mathbb{R}^{n-1})} + |\lambda|^{\frac{3}{2}-\frac{1}{p}}\|g_1\|_{L^p(\mathbb{R}^{n-1})}\Big)\end{aligned}$$

holds for all $h = (h_1, h_2)^\top \in \mathbb{E}_+$ and $g = (g_0, g_1)^\top \in \mathbb{G}$, where v is the unique solution of (3.10)*.*

Proof. We use the parameter-dependent norms $|\!|\!|\phi|\!|\!|_{s,p,\mathbb{R}^n_+} := \|\phi\|_{H^s_p(\mathbb{R}^n_+)} + |\lambda|^{s/2}\|\phi\|_{L^p(\mathbb{R}^n_+)}$, $\phi \in H^s_p(\mathbb{R}^n_+)$, for $s \in [0,\infty)$ and its analogues in $\mathbb{R}^n$ and $\mathbb{R}^{n-1}$. By Michlin's theorem the norm $|\!|\!|\phi|\!|\!|_{s,p,\mathbb{R}^n}$ is equivalent to $\|(\lambda-\Delta)^{s/2}\phi\|_{L^p(\mathbb{R}^n)}$ where the constants of the equivalence may be chosen independent of $\lambda \in \lambda_0 + \Sigma_{\pi-\vartheta-\varepsilon}$.

Due to Theorem 3.5, the problem (3.10) has a solution v satisfying

$$\begin{aligned}\|\lambda^{1-k/2} D^\alpha v\|_{\mathbb{E}_+} &\le C\Big(\|(\lambda-\Delta)e_+h_1\|_{L^p(\mathbb{R}^n)} + \|e_+h_2\|_{L^p(\mathbb{R}^n)} + \|h_1\|_{H^2_p(\mathbb{R}^n_+)} \\ &\qquad + \|(\lambda-\Delta')^2 E_\lambda g_0\|_{L^p(\mathbb{R}^n_+)} + \|(\lambda-\Delta')^{3/2} E_\lambda g_1\|_{L^p(\mathbb{R}^n_+)}\Big) \\ &\le C\Big(|\!|\!|h_1|\!|\!|_{2,p,\mathbb{R}^n_+} + \|h\|_{\mathbb{E}_+)} + |\!|\!|E_\lambda g_0|\!|\!|_{4,p,\mathbb{R}^n_+} + |\!|\!|E_\lambda g_1|\!|\!|_{3,p,\mathbb{R}^n_+}\Big).\end{aligned}$$

(We also use the equation $\lambda v_1 = v_2 + h_1$ and the lower bound for $|\lambda|$ in the shifted sector to deal with zero order part of the norm in $\mathbb{E}_+$.) Now the statement follows from the fact that E_λ is continuous with respect to the parameter-dependent norms in the sense that $|\!|\!|E_\lambda \phi|\!|\!|_{s,p,\mathbb{R}^n_+} \le C_s |\!|\!|\phi|\!|\!|_{s-1/p,p,\mathbb{R}^{n-1}}$ for all $\lambda \in \Sigma_{\pi-\vartheta-\varepsilon} + \lambda_0$, $s \in \mathbb{N}$ and $\phi \in W_p^{s-1/p}(\mathbb{R}^{n-1})$, cf. Proposition 2.3 of [2]. □

4. Sectoriality and maximal regularity of the evolution equation on the half-space

In this section we solve the inhomogeneous problem (1.1) on $\mathbb{R}^n_+$ in optimal regularity. As a first step we discuss the sectoriality of the operator matrix A_p governing the associated first order system.

We have seen in the previous section that the operator $A_{p,+}$ is not sectorial in the basic space $\mathbb{E}_+$. As indicated in Theorem 3.8 of [9], see the proof of Proposition 3.4, one has to include zero boundary conditions already in the basic spaces. We thus use the spaces

$$\mathbb{E}_0 := H^2_{p,0}(\mathbb{R}^n_+) \times L^p(\mathbb{R}^n_+),$$
$$\mathbb{F}_0 := \big(H^4_p(\mathbb{R}^n_+) \cap H^2_{p,0}(\mathbb{R}^n_+)\big) \times H^2_{p,0}(\mathbb{R}^n_+).$$

We will see below that it is advantageous to replace the 0–extension operator e_0 from $\mathbb{E}_0$ to $\mathbb{E}$ by the odd extension $e_s \in L(\mathbb{E}_0, \mathbb{E})$ which is defined by

$$(e_s f)(x) := \begin{cases} f(x), & \text{if } x_n \geq 0, \\ -f(x', -x_n), & \text{if } x_n < 0. \end{cases}$$

The L^p-realization $A_{p,0}\colon D(A_{p,0}) \subset \mathbb{E}_0 \to \mathbb{E}_0$ of the boundary value problem $(A(D), B(D))$ in the space $\mathbb{E}_0$ is defined by

$$D(A_{p,0}) := \mathbb{F}_0 \quad \text{and} \quad A_{p,0}v := A(D)v. \tag{4.1}$$

For the analysis of this operator, we start with a Hardy-type result.

Lemma 4.1. *Let X be a Banach space and let M be the operator of multiplication with t, i.e., $(Mf)(t) := tf(t)$ for functions $f\colon (0,\infty) \to X$. For all $f \in H^2_{p,0}((0,\infty);X)$ we then obtain $M^{-2}f \in L^p((0,\infty);X)$ and*

$$\|M^{-2}f\|_{L^p((0,\infty);X)} \leq C\|f''\|_{L^p((0,\infty);X)}.$$

In particular, $M^{-2} \in L(H^2_{p,0}((0,\infty);X), L^p((0,\infty);X))$.

Proof. As $f(0) = f'(0) = 0$, we can write $f(t) = \int_0^t \int_0^s f''(r)dr\,ds$ and compute

$$\begin{aligned}
\|M^{-2}f\|_{L^p((0,\infty);X)} &= \Big(\int_0^\infty t^{-2p}\|f(t)\|_X^p dt\Big)^{1/p} \\
&\leq \Big(\int_0^\infty \Big(t^{-2p}\Big(\int_0^t \int_0^s \|f''(r)\|_X\,dr\,ds\Big)^p dt\Big)^{1/p} \\
&\leq \Big(\int_0^\infty \Big(\int_0^t \int_0^s \|f''(r)\|_X \,\frac{dr}{s}\,\frac{ds}{t}\Big)^p dt\Big)^{1/p} \\
&= \Big(\int_0^\infty \Big(\int_0^t \int_0^1 \|f''(\rho s)\|_X \,d\rho\,\frac{ds}{t}\Big)^p dt\Big)^{1/p} \\
&= \Big(\int_0^\infty \Big(\int_0^1 \int_0^1 \|f''(\rho\sigma t)\|_X \,d\rho\,d\sigma\Big)^p dt\Big)^{1/p},
\end{aligned}$$

where we substituted $\rho = r/s$ and $\sigma = s/t$. With Minkowski's inequality, we conclude

$$\begin{aligned}
\|M^{-2}f\|_{L^p((0,\infty);X)} &\leq \int_0^1 \int_0^1 \Big(\int_0^\infty \|f''(\rho\sigma t)\|_X^p\,dt\Big)^{1/p} d\rho\,d\sigma \\
&= \int_0^1 \int_0^1 \Big(\int_0^\infty \|f''(\tau)\|_X^p\,d\tau\Big)^{1/p} \frac{d\rho}{\rho^{1/p}}\,\frac{d\sigma}{\sigma^{1/p}}
\end{aligned}$$

$$= \Big(\frac{p}{p-1}\Big)^2 \|f''\|_{L^p((0,\infty);X)}. \qquad \square$$

Remark 4.2. *Let M_n denote the operator of multiplication with x_n. Then for every $f \in H^2_{p,0}(\mathbb{R}^n_+)$ we have $M_n^{-2} f \in L^p(\mathbb{R}^n_+)$ by Lemma 4.1. This gives additional information on the Fourier transform of $e_s f$ because of $\partial_n^2 \mathscr{F} e_s M_n^{-2} f = -\mathscr{F} e_s f$. To see this equality, we may assume that $f \in \mathscr{D}(\mathbb{R}^n_+)$ by density, and write*

$$\partial^2_{\xi_n} \int_{\mathbb{R}} e^{-i\xi_n x_n} \frac{1}{x_n^2} (\mathscr{F}' e_s f)(\xi', x_n) dx_n = - \int_{\mathbb{R}} e^{-i\xi_n x_n} (\mathscr{F}' e_s f)(\xi', x_n) dx_n.$$

We exploit the above observation in the next lemma which will provide the main step of the proof of the following sectoriality result.

Lemma 4.3. *Let $\varepsilon \in (0, \pi - \vartheta)$ and $b\colon (\mathbb{R}^n \times \overline{\Sigma}_{\pi-\vartheta-\varepsilon}) \setminus \{0\} \to \mathbb{C}$ be infinitely smooth and homogeneous of degree 0 in $(\xi, \lambda^{1/2})$. We set*

$$\begin{aligned} b_0(\xi,\lambda) &:= -(\lambda + |\xi'|^2)\partial_n^2 b(\xi,\lambda), \\ b_1(\xi,\lambda) &:= -2i(\lambda + |\xi'|^2)^{1/2}\partial_n b(\xi,\lambda), \\ b_2(\xi,\lambda) &:= b(\xi,\lambda) \end{aligned}$$

for $(\xi,\lambda) \in (\mathbb{R}^n \times \overline{\Sigma}_{\pi-\vartheta-\varepsilon}) \setminus \{0\}$. We then obtain

$$r_+ \mathscr{F}^{-1} b(\cdot,\lambda) \mathscr{F} e_s f = \sum_{\ell=0}^{2} M_n^\ell (\lambda - \Delta')^{-1+\ell/2} r_+ \mathscr{F}^{-1} b_\ell(\cdot,\lambda) \mathscr{F} e_s M_n^{-2} f$$

for all $f \in H^2_{p,0}(\mathbb{R}^n_+)$ and

$$\|r_+ \mathscr{F}^{-1} b_\ell(\cdot,\lambda)\mathscr{F} e_s\|_{L(L^p(\mathbb{R}^n_+))} \le C \quad (\ell = 0,1,2).$$

Moreover, the operator families

$$\big\{\lambda^\gamma \partial_\lambda^\gamma r_+ \mathscr{F}^{-1} b_\ell(\cdot,\lambda)\mathscr{F} e_s : \lambda \in \overline{\Sigma}_{\pi-\vartheta-\varepsilon}\big\} \subset L(L^p(\mathbb{R}^n_+))$$

are $\mathcal{R}$-bounded for every $\gamma \in \mathbb{N}_0^2$ and $\ell = 0,1,2$.

Proof. Set $f^{[2]} := M_n^{-2} f \in L^p(\mathbb{R}^n_+)$ for $f \in \mathscr{D}(\mathbb{R}^n_+)$. Let $x_n > 0$. Using Remark 4.2 and integrating by parts, we deduce

$$\begin{aligned} &\mathscr{F}'\big[r_+ \mathscr{F}^{-1} b(\cdot,\lambda)\mathscr{F} e_s f\big](\cdot, x_n) \\ &\quad = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i x_n \xi_n} b(\cdot,\xi_n,\lambda)(\mathscr{F} e_s f)(\cdot,\xi_n)\, d\xi_n \\ &\quad = -\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i x_n \xi_n} b(\cdot,\xi_n,\lambda)\partial_n^2(\mathscr{F} e_s f^{[2]})(\cdot,\xi_n)\, d\xi_n \\ &\quad = -\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \partial_n^2\big[e^{i x_n \xi_n} b(\cdot,\xi_n,\lambda)\big](\mathscr{F} e_s f^{[2]})(\cdot,\xi_n)\, d\xi_n \\ &\quad = -\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i x_n \xi_n} \Big[\partial_n^2 b(\cdot,\xi_n,\lambda) + 2i x_n \partial_n b(\cdot,\xi_n,\lambda) \\ &\qquad\qquad - x_n^2 b(\cdot,\xi_n,\lambda)\Big](\mathscr{F} e_s f^{[2]})(\cdot,\xi_n) d\xi_n \end{aligned}$$

$$= \sum_{\ell=0}^{2} x_n^\ell \mathscr{F}'(\lambda-\Delta')^{-1+\ell/2}\big[r_+\mathscr{F}^{-1}b_\ell(\cdot,\lambda)\mathscr{F}e_s f^{[2]}\big](\cdot,x_n).$$

By density the first assertion follows. As b is homogeneous of degree 0, the same holds for b_ℓ with $\ell = 0,1,2$. Remark 2.1 thus yields the remaining assertions. □

We now establish the sectoriality of the shifted operator matrix on $\mathbb{E}_0$ which governs the associated first order system.

Theorem 4.4. *For every $\lambda_0 > 0$, the operator $A_{p,0}+\lambda_0$ is $\mathcal{R}$-sectorial in $\mathbb{E}_0$ with $\mathcal{R}$-angle $\vartheta(\rho)$.*

Proof. Let $h \in \mathbb{E}_0$, $\varepsilon \in (0,\pi-\vartheta)$ and $\lambda \in \lambda_0 + \Sigma_{\pi-\vartheta-\varepsilon}$. As in part (i) of the proof of Theorem 3.5, one sees that the equation $A(D,\lambda)v = h$ with boundary condition $B(D)v = 0$ has the unique solution v given by

$$v = R(\lambda)e_s h = r_+(A_p+\lambda)^{-1}e_s h - T(\lambda)\begin{pmatrix}1 & 0\\ -\partial_n & 0\end{pmatrix} r_+(A_p+\lambda)^{-1}e_s h. \tag{4.2}$$

To check the asserted $\mathcal{R}$-bound, we can restrict ourselves to h belonging to the dense subset $\mathscr{D}(\mathbb{R}^n_+)$ of $\mathbb{E}_0$. As $e_s h \in \mathbb{E}$, the function $\widetilde{v} := (A_p+\lambda)^{-1}e_s h$ belongs to $\mathbb{F}$ and solves the equation $A(D,\lambda)\widetilde{v} = e_s h$ in $\mathbb{R}^n$. Since $e_s h$ is odd, also the map $x \mapsto -\widetilde{v}(x',-x_n)$ satisfies this equation. Because of uniqueness, the function $\widetilde{v}$ is odd, and we obtain $\gamma_0 v' = 0$ for $v' := r_+\widetilde{v}$. Therefore, we may assume that $\widetilde{g}_0 = 0$ in (3.13) and replace the second term in (4.2) by

(4.3)

$$\sum_{j=0}^{1}\begin{pmatrix}0 & -L_j^{(1)}(\lambda)\\ 0 & -\lambda L_j^{(1)}(\lambda)\end{pmatrix}\partial_n^{1-j}\begin{pmatrix}1 & 0\\ -\partial_n & 0\end{pmatrix} r_+(A_p+\lambda)^{-1}e_s h$$
$$= \begin{pmatrix}1\\ \lambda\end{pmatrix}\sum_{j=0}^{1}\sum_{k=1}^{2} L_j^{(1)}(\lambda)\partial_n^{2-j} r_+\mathscr{F}^{-1}\widetilde{a}_{1k}(\cdot,\lambda)\mathscr{F}e_s h_k =: S_1(\lambda)h_1 + S_2(\lambda)h_2,$$

where we denote the first line of $A(\xi,\lambda)^{-1}$ by $(\widetilde{a}_{11}(\xi,\lambda),\widetilde{a}_{12}(\xi,\lambda))$, i.e.,

$$\big(\widetilde{a}_{11}(\xi,\lambda),\widetilde{a}_{12}(\xi,\lambda)\big) := \Big(\tfrac{\lambda+\rho|\xi|^2}{(\alpha_+\lambda+|\xi|^2)(\alpha_-\lambda+|\xi|^2)},\ \tfrac{1}{(\alpha_+\lambda+|\xi|^2)(\alpha_-\lambda+|\xi|^2)}\Big),$$

see (2.3). Since $\mathscr{F}^{-1}\widetilde{a}_{1k}(\cdot,\lambda)\mathscr{F}e_s h_k$ is a Schwartz function, we can write

$$L_j^{(1)}(\lambda)\partial_n^{2-j} r_+\mathscr{F}^{-1}\widetilde{a}_{1k}(\xi,\lambda)\mathscr{F}e_s h_k \tag{4.4}$$
$$= L_j^{(1)}(\lambda)(\lambda-\Delta')^{1-k-j/2} r_+\mathscr{F}^{-1}(i\xi_n)^{2-j}(\lambda+|\xi'|^2)^{k-1+j/2}\widetilde{a}_{1k}(\xi,\lambda)\mathscr{F}e_s h_k$$

for $j \in \{0,1\}$ and $k \in \{1,2\}$. The functions

$$g_{kj}(\xi,\lambda) := (i\xi_n)^{2-j}(\lambda+|\xi'|^2)^{k-1+j/2}\widetilde{a}_{1k}(\xi,\lambda)$$
$$= \begin{cases} \dfrac{(\lambda+\rho|\xi|^2)(i\xi_n)^{2-j}(\lambda+|\xi'|^2)^{j/2}}{(\alpha_+\lambda+|\xi|^2)(\alpha_-\lambda+|\xi|^2)} & \text{if } k=1,\\[2ex] \dfrac{(i\xi_n)^{2-j}(\lambda+|\xi'|^2)^{1+j/2}}{(\alpha_+\lambda+|\xi|^2)(\alpha_-\lambda+|\xi|^2)} & \text{if } k=2,\end{cases}$$

are smooth and homogeneous of degree 0 in $(\xi, \lambda^{1/2})$ and therefore satisfy Michlin's condition. As a result, for $k = 2$ the set

$$\big\{\lambda^\gamma \partial_\lambda^\gamma r_+ \mathscr{F}^{-1} g_{2j}(\cdot, \lambda) \mathscr{F} e_s : \lambda \in \Sigma_{\pi-\vartheta-\varepsilon}\big\} \subset L(L^p(\mathbb{R}^n_+)) \tag{4.5}$$

is $\mathcal{R}$-bounded for $j \in \{0, 1\}$ and $\gamma \in \mathbb{N}_0^2$.

As we will see below, in the case $k = 1$ we need a more refined representation formula which exploits that $h_1 \in H^2_{p,0}(\mathbb{R}_+)$ and not only that $h_1 \in H^2_p(\mathbb{R}_+)$. To this aim, we apply Lemma 4.3 and obtain as above that

$$\begin{aligned} L_j^{(1)}(\lambda) &\partial_n^{2-j} r_+ \mathscr{F}^{-1} \widetilde{a}_{11}(\cdot, \lambda) \mathscr{F} e_s h_1 \\ &= L_j^{(1)}(\lambda)(\lambda - \Delta')^{-j/2} r_+ \mathscr{F}^{-1} g_{1j}(\cdot, \lambda) \mathscr{F} e_s h_1 \\ &= \sum_{\ell=0}^{2} L_j^{(1)}(\lambda)(\lambda - \Delta')^{-1+\ell/2-j/2} M_n^\ell r_+ \mathscr{F}^{-1} g_{1j\ell}(\cdot, \lambda) \mathscr{F} e_s h_1^{[2]} \end{aligned} \tag{4.6}$$

where $h_1^{[2]} := M_n^{-2} h_1$, the functions $g_{1j\ell}$ are given by

$$g_{1j\ell}(\xi, \lambda) := (i\xi_n)^{2-j} (\lambda + |\xi'|^2)^{j/2} \widetilde{a}_{11\ell}(\xi, \lambda),$$

and $\widetilde{a}_{11\ell}$ are defined as b_ℓ in Lemma 4.3 with b replaced by $\widetilde{a}_{11}$. By homogeneity, for the corresponding Fourier multipliers the set of operators

$$\big\{\lambda^\gamma \partial_\lambda^\gamma r_+ \mathscr{F}^{-1} g_{1j\ell}(\cdot, \lambda) \mathscr{F} e_s : \lambda \in \Sigma_{\pi-\vartheta-\varepsilon}\big\} \subset L(L^p(\mathbb{R}^n_+)) \tag{4.7}$$

is $\mathcal{R}$-bounded for $\ell \in \{0, 1, 2\}$, $j \in \{0, 1\}$ and $\gamma \in \mathbb{N}_0^2$.

To prove the theorem, we have to estimate $\lambda v = \lambda R(\lambda) e_s h$ in the space $\mathbb{E}_0$. For the first term in (4.2), the $\mathcal{R}$-boundedness of $\{r_+ \lambda (A_p + \lambda)^{-1} e_s : \lambda \in \lambda_0 + \Sigma_{\pi-\vartheta-\varepsilon}\}$ in $L(\mathbb{E}_0)$ follows directly from Proposition 2.4. To treat the second term in (4.2), we first use (4.3) and (4.4). For the summands with $k = 2$, Proposition 3.3 and (4.5) imply that $\{S_2(\lambda) : \lambda \in \lambda_0 + \Sigma_{\pi-\vartheta-\varepsilon}\}$ is $\mathcal{R}$-bounded in $L(L^p(\mathbb{R}^n_+), \mathbb{E}_0)$.

It remains to consider the summands with $k = 1$ in (4.3). In view of the definition of the space $\mathbb{E}_0$, the representation (4.6) and the $\mathcal{R}$-bound (4.7), we have to show that

$$\big\{\lambda^{2-|\alpha|/2} D^\alpha L_j^{(1)}(\lambda)(\lambda - \Delta')^{-1+\ell/2-j/2} M_n^\ell : \lambda \in \lambda_0 + \Sigma_{\pi-\vartheta-\varepsilon}\big\} \tag{4.8}$$

in $L(L^p(\mathbb{R}^n_+))$ is $\mathcal{R}$-bounded for $|\alpha| \le 2$, $\ell \in \{0, 1, 2\}$ and $j \in \{0, 1\}$.

For $\ell = 0$, this fact is stated in Proposition 3.3 b). For $\ell > 0$, we follow the lines of the proof of Proposition 3.3 and write

$$\begin{aligned} \lambda^{2-|\alpha|/2} D^\alpha L_j^{(1)}(\lambda)&(\lambda - \Delta')^{-1+\ell/2-j/2} M_n^\ell \phi(\cdot, x_n) \\ &= -\int_0^\infty (\mathscr{F}')^{-1} m(\cdot, x_n + y_n, \lambda)(\mathscr{F}'\phi)(\cdot, y_n) dy_n \end{aligned}$$

with

$$m(\xi', x_n + y_n, \lambda) := \lambda^{2-|\alpha|/2} (\xi')^{\alpha'} (\lambda + |\xi'|^2)^{-1+\ell/2-j/2} y_n^\ell \partial_n^{\alpha_n + j} w_1^{(1)}(\xi', x_n + y_n, \lambda)$$

for $x_n, y_n > 0$, $\alpha = (\alpha', \alpha_n)$, and $\xi' \in \mathbb{R}^{n-1}$. Since $y_n^\ell < (x_n + y_n)^\ell$ for $x_n > 0$, Lemma 3.2 b) shows that $(x_n + y_n)m(\cdot, x_n + y_n, \lambda)$ satisfies Michlin's condition. The $\mathcal{R}$-boundedness of (4.8) can thus be established as in the proof of Proposition 3.3. □

The $\mathcal{R}$-boundedness results above enable us to solve the instationary problem (1.1) on $\mathbb{R}^n_+$ with inhomogeneous right-hand sides, i.e.,

$$
\begin{aligned}
\partial_t^2 u + \Delta^2 u - \rho \partial_t \Delta u &= f \quad \text{in } J \times \mathbb{R}^n_+, \\
\gamma_0 u &= g_0 \quad \text{on } J \times \mathbb{R}^{n-1}, \\
\gamma_0 \partial_\nu u &= g_1 \quad \text{on } J \times \mathbb{R}^{n-1}, \\
u|_{t=0} &= \varphi_0 \quad \text{in } \mathbb{R}^n_+, \\
\partial_t u|_{t=0} &= \varphi_1 \quad \text{in } \mathbb{R}^n_+.
\end{aligned} \tag{4.9}
$$

Here $J = (0, T)$, $T \in (0, \infty)$, is a finite time interval, and we recall that $\rho > 0$ is fixed. The natural spaces for the right-hand sides are given by

$$
\begin{aligned}
f \in \mathcal{E}_+ &:= L^p(J; L^p(\mathbb{R}^n_+)), \\
g_0 \in \mathcal{G}_0 &:= W_p^{2-1/(2p)}(J; L^p(\mathbb{R}^{n-1})) \cap L^p(J; W_p^{4-1/p}(\mathbb{R}^{n-1})), \\
g_1 \in \mathcal{G}_1 &:= W_p^{3/2-1/(2p)}(J; L^p(\mathbb{R}^{n-1})) \cap L^p(J; W_p^{3-1/p}(\mathbb{R}^{n-1})), \\
\varphi_0 \in Y_0 &:= W_p^{4-2/p}(\mathbb{R}^n_+), \\
\varphi_1 \in Y_1 &:= W_p^{2-2/p}(\mathbb{R}^n_+).
\end{aligned}
$$

The analogues of these spaces for the time interval $\mathbb{R}$ are denoted by $\mathcal{E}_+(\mathbb{R})$ etc. The data have to satisfy the compatibility conditions

$$
\begin{aligned}
g_0|_{t=0} &= \gamma_0 \varphi_0, \\
g_1|_{t=0} &= \gamma_0 \partial_\nu \varphi_0, \\
\partial_t g_0|_{t=0} &= \gamma_0 \varphi_1 \qquad \text{if } p > \tfrac{3}{2}, \\
\partial_t g_1|_{t=0} &= \gamma_0 \partial_\nu \varphi_1 \qquad \text{if } p > 3.
\end{aligned} \tag{4.10}
$$

The solution will belong to the space

$$
u \in \mathcal{F}_+ := H_p^2(J; L^p(\mathbb{R}^n_+)) \cap L^p(J; H_p^4(\mathbb{R}^n_+)).
$$

We recall that $\mathcal{F}_+ \hookrightarrow H_p^1(J; H_p^2(\mathbb{R}^n_+))$. This is stated, e.g., in Lemma 4.3 of [12] for $\mathbb{R}^n$ instead of $\mathbb{R}^n_+$, and follows for $\mathbb{R}^n_+$ by the existence of a universal extension operator (see Lemma 2.9.1/1 in [30]). For $i \in \{0, 1\}$, we will write ${}_0\mathcal{G}_i$ for the subspace of all $g_i \in \mathcal{G}_i$ which satisfy (4.10) with $\varphi_0 = \varphi_1 = 0$.

We first state the result for homogeneous boundary conditions which follows from Theorem 4.4 as in the proof of Theorem 2.5. (For the initial values in part a) one now needs an interpolation result essentially due to Grisvard, see e.g. Theorem 4.9.1 and Example 4.9.3 in [3].)

Theorem 4.5. *a) The operator $-B_{p,0}$ generates an analytic C_0-semigroup on $\mathbb{E}_0$ with maximal L^q-regularity on bounded time intervals for every $q \in (1, \infty)$.*

b) Let $f \in \mathcal{E}_+$, $g = 0$, and let $\varphi_0 \in Y_0$, $\varphi_1 \in Y_1$ satisfy (4.10) with $g = 0$. Then there is a unique solution $u \in \mathcal{F}_+$ of (4.9), and there is a constant $C_p(T) > 0$ such that

$$\|u\|_{\mathcal{F}_+} \leq C_p(T)\Big(\|f\|_{\mathcal{E}_+} + \|\varphi_0\|_{Y_0} + \|\varphi_1\|_{Y_1}\Big).$$

c) Let $f = 0$, $g = 0$, $\varphi_0 \in H^2_{p,0}(\mathbb{R}^n_+)$ and $\varphi_1 \in L^p(\mathbb{R}^n_+)$. Then there exists a unique solution u of (4.9) with

$$\partial_t^2 u,\ \partial_t \nabla^2 u,\ \nabla^4 u \in C([\varepsilon, \infty), L^p(\mathbb{R}^n_+))$$

for each $\varepsilon > 0$ and

$$\partial_t u,\ \nabla^2 u \in C([0, \infty), L^p(\mathbb{R}^n_+)).$$

If $\varphi_0 \in H^4_p(\mathbb{R}^n_+) \cap H^2_{p,0}(\mathbb{R}^n_+)$ and $\varphi_1 \in H^2_{p,0}(\mathbb{R}^n_+)$, we can take $\varepsilon = 0$.

Based on Theorems 4.5 and 3.5, we can now solve (4.9) by inverting the Banach space valued Fourier transform in time, where we proceed as in [11], for instance.

Theorem 4.6. *Let $T \in (0, \infty)$ and $p \in (1, \infty)$ with $p \notin \{3/2, 3\}$. Then for every $(f, g_0, g_1, \varphi_0, \varphi_1) \in \mathcal{E}_+ \times \mathcal{G}_0 \times \mathcal{G}_1 \times Y_0 \times Y_1$ satisfying the compatibility conditions (4.10), there exists a unique solution $u \in \mathcal{F}_+$ of (4.9). Conversely, if $u \in \mathcal{F}_+$ is a solution of (4.9), then the right-hand sides of (4.9) belong to the spaces indicated above and satisfy the compatibility conditions (4.10). Finally, there is a constant $C_p(T) > 0$ such that*

$$\|u\|_{\mathcal{F}_+} \leq C_p(T)\big(\|f\|_{\mathcal{E}_+} + \|\varphi_0\|_{Y_0} + \|\varphi_1\|_{Y_1} + \|g_0\|_{\mathcal{G}_0} + \|g_1\|_{\mathcal{G}_1}\big).$$

Proof. The necessity of the regularity and compatibility conditions (4.10) follows from standard spatial and temporal trace theorems, see e.g. Corollary 2.8 in [21] in a more general setting. The uniqueness is a consequence of Theorem 4.5.

To show existence, let data $(f, g_0, g_1, \varphi_0, \varphi_1) \in \mathcal{E}_+ \times \mathcal{G}_0 \times \mathcal{G}_1 \times Y_0 \times Y_1$ be given which satisfy (4.10). Extending f, φ_0 and φ_1 to $\mathbb{R}^n$ and applying Theorem 2.5, we obtain a solution $u' \in \mathcal{F}_+$ of

$$\begin{aligned} \partial_t^2 u' + \Delta^2 u' - \rho \partial_t \Delta u' &= f \quad \text{in } J \times \mathbb{R}^n_+, \\ u'|_{t=0} &= \varphi_0 \quad \text{in } \mathbb{R}^n_+, \\ \partial_t u'|_{t=0} &= \varphi_1 \quad \text{in } \mathbb{R}^n_+ \end{aligned}$$

which satisfies the asserted estimate with $g_0 = g_1 = 0$. We set $\widetilde{g}_0 = g_0 - \gamma_0 u'$ and $\widetilde{g}_1 = g_1 - \gamma_0 \partial_\nu u'$. Again standard trace theory and (4.10) yield that $\widetilde{g}_k \in {}_0\mathcal{G}_k$ for $k \in \{0, 1\}$. Moreover, $\|\widetilde{g}_k\|_{\mathcal{G}_k} \leq C_p(T)\,(\|g_k\|_{\mathcal{G}_k} + \|u'\|_{\mathcal{F}_+})$.

Considering $u - u'$, we may therefore assume in the following that the data in (4.9) satisfy $f = 0$, $\varphi_0 = \varphi_1 = 0$ and $g_k \in {}_0\mathcal{G}_k$ for $k \in \{0, 1\}$. Note that test functions on $(0, \infty) \times \mathbb{R}^{n-1}$ are dense in ${}_0\mathcal{G}_k$, see Theorem 4.7.1 in [3]. Since we will show that the solution operator $g = (g_0, g_1)^\top \mapsto u$ is continuous from

${}_0\mathcal{G}_0 \times {}_0\mathcal{G}_1 \to \mathcal{F}_+$, we may restrict ourselves to test functions g_0 and g_1. We extend them by 0 to functions on $\mathbb{R}$, using the same symbol. We now employ similar arguments as in Proposition 4.5 of [11] (see also the proof of Lemma 3.4 of [26] for a more detailed exposition in a somewhat different situation).

Let $\mathscr{F}_t$ be the temporal Fourier transform and put $\hat{g} := \mathscr{F}_t g$. In view of Theorem 3.5, setting $\lambda = i\tau$ with $\tau \in \mathbb{R}$, we define $\hat{v}(i\tau) := T(i\tau)E_{i\tau}\hat{g}$ and recall that $\hat{v}_2(i\tau) = i\tau\hat{v}_1(i\tau)$ for $\tau \in \mathbb{R}$. We write $v := \mathscr{F}_t^{-1}\hat{v}$ and $u := v_1$, observing also that $\partial_t u = \mathscr{F}_t^{-1}(i\tau\hat{v}_1(i\tau)) = v_2$. Taking into acount (3.13), (3.11) and that $E_{i\cdot}\hat{g}$ is rapidly decaying, we can compute

$$\begin{aligned}
\hat{v}(i\tau) &= \sum_{j=0}^{1} T^{(j)}(\lambda) \begin{pmatrix} (i\tau - \Delta')^{(-j-3)/2} & 0 \\ 0 & (i\tau - \Delta')^{(-j-2)/2} \end{pmatrix} \mathscr{F}_t \\
&\qquad \cdot \mathscr{F}_t^{-1} \left[\begin{pmatrix} (i\cdot - \Delta')^{(j+3)/2} & 0 \\ 0 & (i\cdot - \Delta')^{(j+2)/2} \end{pmatrix} \partial_n^{1-j} E_{i\cdot}\hat{g} \right](\tau) \\
&= \sum_{j=0}^{1} T^{(j)}(\lambda) \begin{pmatrix} (i\tau - \Delta')^{(-j-3)/2} & 0 \\ 0 & (i\tau - \Delta')^{(-j-2)/2} \end{pmatrix} \mathscr{F}_t \\
&\qquad \cdot (-1)^{j+1} \mathscr{F}_t^{-1} \left[\begin{pmatrix} (i\cdot - \Delta')^{2} & 0 \\ 0 & (i\cdot - \Delta')^{3/2} \end{pmatrix} E_{i\cdot}\hat{g} \right](\tau).
\end{aligned} \tag{4.11}$$

We further note that $E_{i\tau}\hat{g}(\cdot, x_n) = e^{-x_n(i\tau-\Delta')^{1/2}}\hat{g}(i\tau,\cdot)$ for $x_n > 0$ and $\tau \in \mathbb{R}$ since the Dunford calculus for sectorial operators and Fourier multipliers coincide here. The operator $L = \partial_t - \Delta'$ with domain $H^1_p(\mathbb{R}, L^p(\mathbb{R}^{n-1})) \cap L^p(\mathbb{R}, H^2_p(\mathbb{R}^{n-1}))$ is sectorial of angle $\pi/2$ in $L^p(\mathbb{R}, L^p(\mathbb{R}^{n-1}))$, hence $-L^{1/2}$ generates an analytic semigroup. Because of $\mathscr{F}_t^{-1}(\lambda + i\cdot - \Delta')^{-1}\mathscr{F}_t = (\lambda + L)^{-1}$ for $\operatorname{Re}\lambda < 0$, we can use the Dunford calculus to deduce

$$\mathscr{F}_t^{-1} \left[\begin{pmatrix} (i\cdot - \Delta')^{2} & 0 \\ 0 & (i\cdot - \Delta')^{3/2} \end{pmatrix} E_{i\cdot}\hat{g} \right](x_n) = \begin{pmatrix} Le^{-x_n L^{1/2}} L g_0 \\ L^{1/2} e^{-x_n L^{1/2}} L g_1 \end{pmatrix}.$$

The norm in $\mathcal{E}_+$ of these functions is bounded by $C\,(\|g_0\|_{\mathcal{G}_0} + \|g_1\|_{\mathcal{G}_1})$. Here, for the first component we use Lemma 3.5 of [11] and for the second that $Lg_1 \in (L^p(\mathbb{R}_+, L^p(\mathbb{R}^{n-1})), D(L))_{\frac{1}{2}-\frac{1}{2p},p} = (L^p(\mathbb{R}_+, L^p(\mathbb{R}^{n-1})), D(L^{1/2}))_{1-\frac{1}{p},p}$ by Lemma 3.1 of [11] and the reiteration theorem, see e.g. Theorems 1.10.2 and 1.15.2 in [30]. In the first part of (4.11) we employ our Proposition 3.3 and the operator-valued Fourier multiplier theorem (Theorem 3.4 of [31]) and conclude

$$\|\partial_t^2 u\|_{\mathcal{E}_+(\mathbb{R})} + \|\nabla^4 u\|_{\mathcal{E}_+(\mathbb{R})} \le c\,(\|g_0\|_{\mathcal{G}_0} + \|g_1\|_{\mathcal{G}_1}). \tag{4.12}$$

Since g_k have support in $(0,\infty)$ and since the symbols involved have a holomorphic extension to the half-plane $\{\tau \in \mathbb{C} : \operatorname{Im}\tau < 0\}$, all Fourier multipliers (with respect to t) have the Volterra property in the sense of Section 2 in [13]. Hence, the function u vanishes on $(-\infty, 0)$, so that u and $\partial_t u$ have trace 0 at $t = 0$. In particular, (4.12) implies that $\|u\|_{\mathcal{E}_+(J)} \le c(T)\,(\|g_0\|_{\mathcal{G}_0} + \|g_1\|_{\mathcal{G}_1})$ which yields the asserted estimate $\|u\|_{\mathcal{F}_+(J)} \le c(T)\,(\|g_0\|_{\mathcal{G}_0} + \|g_1\|_{\mathcal{G}_1})$. Finally, $\hat{v}(i\tau)$ solves

(3.10) with $\lambda = i\tau$ and boundary data $\hat{g}(i\tau)$. So the first component $u = v_1$ of the inverse Fourier transform in time of $\hat{v}$ is the desired solution of (4.9). $\square$

5. The evolution equation in a bounded domain

In this section we consider a bounded domain $G \subset \mathbb{R}^n$ with boundary of class C^4. We use the analogous spaces as in the previous section, replacing $\mathbb{R}^n_+$ by G, which we denote by $\mathcal{E}(G)$ etc. Moreover, we allow for $T = \infty$ in the time intervals. As before, we define $D(A_{p,0}) = \mathbb{F}_0(G)$ and $A_{p,0}v = A(D)v$.

Theorem 5.1. *Let $G \subset \mathbb{R}^n$ be a bounded domain with boundary of class C^4 and $\rho > 0$. The operator $A_{p,0}$ is $\mathcal{R}$-sectorial of angle $\vartheta(\rho)$ in $\mathbb{E}_0(G)$. Moreover, $-A_{p,0}$ generates an exponentially stable, analytic C_0-semigroup on $\mathbb{E}_0(G)$ with maximal L^q-regularity on $(0, \infty)$ for every $q \in (1, \infty)$.*

Proof. The $\mathcal{R}$-sectoriality of $\lambda_1 + A_{p,0}$ for sufficiently large $\lambda_1 \geq 0$ is shown by a standard localization argument based on the $\mathcal{R}$–bounds shown in Theorems 2.3 and 3.5. For details we refer to Section 8 of [10]. Via localization, transformation to the half-space and perturbation, one can reduce the problem to equations on $\mathbb{R}^n$ and $\mathbb{R}^n_+$ having constant coefficients and no lower order terms. Choosing appropriate transformations, these model problems turn out to be those studied in Theorems 2.3 and 3.5, cf. p. 102 of [10]. In this argument plenty of lower order terms appear which can be absorbed adding a large $\lambda_1 \geq 0$. There are also top-order perturbations both in G and in the boundary conditions which are treated by means of the continuity of the coefficients of the transformed operators and by choosing sufficiently small neighborhoods in the localization. Here one has to exploit the full power of the regularity results in Theorems 2.3 and 3.5.

As in the proof of Theorem 2.5, it now follows that $-A_{p,0}$ generates an analytic semigroup on $\mathbb{E}_0(G)$ with maximal L^q-regularity on bounded time intervals. Because of standard theory of analytic semigroups, it thus remains to show that the spectrum of $-A_{p,0}$ is contained in the open right half-plane. Since $\mathbb{F}_0(G)$ is compactly embedded in $\mathbb{E}_0(G)$, the spectrum is a discrete set of eigenvalues contained in the complement of $\lambda_1 + \Sigma_{\pi-\vartheta}$. If v is an eigenfunction for $A_{q,0}$ and some $q \in (1, \infty)$, then it is also an eigenvalue for $A_{p,0}$ for all $p \in (1, q)$ and the same eigenvalue. The case of $p > q$ is treated by a standard bootstrap argument using the invertibility of $\mu + A_{r,0}$ for large $\mu > 0$ and $r > q$. We can thus restrict ourselves to $p = 2$. We then define the scalar product in $\mathbb{E}_0(G)$ by

$$\langle v, w\rangle_{\mathbb{E}_0(G)} := \langle \Delta v_1, \Delta w_1\rangle_{L^2(G)} + \langle v_1, w_1\rangle_{L^2(G)} + \langle v_2, w_2\rangle_{L^2(G)}, \quad v, w \in \mathbb{E}_0(G).$$

Let $\lambda v + A_{2,0}v = 0$ for some $\lambda \in \mathbb{C}$ and $0 \neq v = (v_1, v_2)^\top \in D(A_{2,0})$. Taking the scalar product with v in $\mathbb{E}_0(G)$, integrating by parts and taking the real part, we deduce

$$0 = \operatorname{Re}\langle \lambda v + A_{2,0}v, v\rangle_{\mathbb{E}_0(G)} = (\operatorname{Re}\lambda) \int_G (|\Delta v_1|^2 + |v_2|^2)\, dx + \rho \int_G |\nabla v_2|^2\, dx$$

thanks to the boundary conditions. Hence, $\operatorname{Re}\lambda$ is non-positive. If $\operatorname{Re}\lambda = 0$, then $v_2 \in H^2_{2,0}(G)$ has to vanish, so that $(-\Delta)^2 v_1 = 0$ because of $\lambda v + A_{2,0} v = 0$. Since $v_1 \in H^4_2(G) \cap H^2_{2,0}(G)$, we obtain $v_1 = 0$ and the contradiction $v = 0$. As a result, $\operatorname{Re}\lambda < 0$. □

We can now state our final result on the solvability and regularity of the inhomogeneous damped plate equation (1.1).

Theorem 5.2. *Let $G \subset \mathbb{R}^n$ be a bounded domain with boundary of class C^4 and $\rho > 0$. Then the following assertions hold.*

a) Let $f = 0$, $g_0 = g_1 = 0$, $\varphi_0 \in H^2_{p,0}(G)$ and $\varphi_1 \in L^p(G)$. Then there exists a unique solution u of (1.1) *with*

$$\partial_t^2 u,\ \partial_t \nabla^2 u,\ \nabla^4 u \in C_0([\varepsilon, \infty), L^p(G))$$

for each $\varepsilon > 0$ and

$$\partial_t u,\ \nabla^2 u \in C_0([0, \infty), L^p(G)).$$

If $\varphi_0 \in H^4_p(G) \cap H^2_{p,0}(G)$ and $\varphi_1 \in H^2_{p,0}(G)$, we can take $\varepsilon = 0$.

b) Let $T \in (0, \infty]$ and $p \in (1, \infty)$ with $p \notin \{3/2, 3\}$. Then for every $(f, g_0, g_1, \varphi_0, \varphi_1) \in \mathcal{E}(G) \times \mathcal{G}_0(G) \times \mathcal{G}_1(G) \times Y_0(G) \times Y_1(G)$ satisfying the compatibility conditions (4.10) *on G, there exists a unique solution $u \in \mathcal{F}(G)$ of* (1.1)*. Conversely, if $u \in \mathcal{F}(G)$ is a solution of* (1.1)*, then the right-hand sides of* (1.1) *belong to the spaces indicated above and satisfy the compatibility conditions* (4.10)*. Finally, there is a constant $C_p > 0$ such that*

$$\|u\|_{\mathcal{F}(G)} \le C_p \big(\|f\|_{\mathcal{E}(G)} + \|\varphi_0\|_{Y_0(G)} + \|\varphi_1\|_{Y_1(G)} + \|g_0\|_{\mathcal{G}_0(G)} + \|g_1\|_{\mathcal{G}_1(G)} \big).$$

Proof. We omit the details the proof which follows a fairly standard pattern, based on our results above. Assertion a), the uniqueness in b) and the case $g_0 = g_1 = 0$ in b) follow from Theorem 5.1 and standard semigroup theory. The necessity in b) is a consequence of trace theorems again. The main step of the proof is the existence part of b) for $f = 0$ and $\varphi_0 = \varphi_1 = 0$ on finite time intervals. This can be done by localization, transformation to the half-space and perturbation as in Section 5 of [11], using the $\mathcal{R}$–bounds of Theorems 2.3 and 3.5. Since $-A_{p,0}$ generates an exponentially stable analytic semigroup by Theorem 5.1, one can extend the existence statement and the maximal regularity estimate to the time interval $(0, \infty)$ as in Proposition 8 of [24]. □

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