# ASYMPTOTIC BEHAVIOUR OF PARABOLIC PROBLEMS WITH DELAYS IN THE HIGHEST ORDER DERIVATIVES

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Abstract. We use semigroup methods to investigate the partial functional differential equation $u'(t) = Au(t) + \int_{-r}^{0} dB(\theta)u(t+\theta)$ for a sectorial operator $A$ on a Banach space $X$ and a function $B : [-r, 0] \to \mathcal{L}(D(A), X)$ of bounded variation having no mass at 0. Using a perturbation theorem due to Weiss and Staffans, we construct the solution semigroup on a product space in order to solve the delay equation in a classical sense. Employing the spectrum of the semigroup and its generator, we then study exponential dichotomy and stability of solutions. If $X$ is a Hilbert space, these properties can be characterized by estimates on $(\lambda - A - \widehat{dB}(\lambda))^{-1} \in \mathcal{L}(X, D(A))$. Related results on stability also hold for general Banach spaces. The case $B = \eta A$ with scalar valued $\eta$ is treated in some detail.

## 1. Introduction

Perturbations of parabolic evolution equations may lead to a loss of regularity if the perturbing term is of the same order as the undelayed part and contains retardations. This already happens in the case of the simple delay equation

$$u'(t) = Au(t) + \beta Au(t-r), \quad t \geq 0, \qquad u(t) = \varphi(t), \quad t \in [-r, 0], \tag{1.1}$$

where $A$ generates an analytic $C_0$–semigroup on a Banach space $X$, $\beta \in \mathbb{R}\setminus\{0\}$, $r > 0$, and $\varphi : [-r, 0] \to D(A)$. One can write this problem as a Volterra equation by transforming the initial history $\varphi$ into an inhomogeneity. In this framework, Prüss' monograph [21] provides a detailed study of the regularity of solutions of more general perturbation problems with operator valued kernels, see Sections 7.4–7.6 of [21]. In particular, the solution operator of the equation can be discontinuous at points $t > 0$ which are determined by the perturbation, [21, Thm.7.6]. In (1.1) this happens at $t = kr$, $k \in \mathbb{N}$, see [21, Ex.1.1].

Related investigations in a semigroup context were carried out for the problem

$$\begin{aligned} u'(t) &= Au(t) + A_1 u(t-r) + \int_{-r}^{0} a(s) A_2 u(t+s)\, ds, \quad t \geq 0, \\ u(t) &= \varphi(t), \quad t \in [-r, 0], \end{aligned} \tag{1.2}$$

for instance in [7], [8], [9], [16], [17], [18], [19], [26, Ch.8]. (Here the operators $A_1$ and $A_2$ are $A$-bounded and $a \in L^2[-r, 0]$.) The very recent work [6] allows kernels $K \in L^1([-r, 0], \mathcal{L}(D(A), X))$ instead of $a(\cdot)A_2$, but assumes further mapping properties of $K$

1991 *Mathematics Subject Classification.* 34K20, 34K30, 47D06.

*Key words and phrases.* Partial functional differential equation, history function space, maximal regularity, feedback, exponential dichotomy and stability, Gearhart's spectral mapping theorem, Weis–Wrobel theorem, Fourier type.

We like to thank Jan Prüss (Halle) for stimulating discussions. A. B. was supported by the OTKA grant Nr. F034840 and by the FKFP grant Nr. 0049/2001.

and $A_1$ with respect to extrapolated norms. All these authors write (1.2) as a problem on a suitable product space $\mathcal{X}$, as discussed in Chapter 3, which allows to incorporate the initial history function $\varphi$ into the new state space. On this space one constructs a semigroup $\mathcal{T}_L$ which gives the solution of (1.2). The semigroup corresponding to (1.1) turns out to be discontinuous in operator norm at any $t \geq 0$, cf. [8, Thm.3.1]. A different semigroup approach for problems arising in viscoelasticity was developped in [24].

The lack of regularity of (1.1) leads to severe problems in the study of the long term behaviour of solutions. In the semigroup framework, it is not clear whether $\mathcal{T}_L$ satisfies the spectral mapping theorem

$$\sigma(\mathcal{T}_L(t)) \setminus \{0\} = \exp(t\sigma(\mathcal{A}_L)), \tag{1.3}$$

($\mathcal{A}_L$ is the generator of $\mathcal{T}_L$) since $\mathcal{T}_L$ is not eventually norm continuous, cf. [11, §IV.3]. We will indeed show in Example 5.8 that (1.3) is violated in the case of (1.1) for certain $A = A^* < 0$ on a Hilbert space $X$ provided that $|\beta| > 1$.

In this paper we study the more general problem

$$u'(t) = Au(t) + \int_{-r}^{0} dB(\theta)u(t+\theta), \quad t \geq 0, \qquad u(t) = \varphi(t), \quad t \in [-r, 0], \tag{1.4}$$

for a map $B : [-r, 0] \to \mathcal{L}(D(A), X)$ of bounded variation $b$ with $db([-t, 0]) \to 0$ as $t \searrow 0$. Imposing additional regularity assumptions on $\theta \mapsto B(\theta)$, the solution semigroup $\mathcal{T}_L$ for (1.4) becomes eventually norm continuous. Then (1.3) holds so that the spectrum of the generator $\mathcal{A}_L$ determines to a large extend the asymptotic behaviour of $\mathcal{T}_L$. This fact is exploited in [9, §5] or [17] for (1.2) with $A_1 = 0$, in [2], [3] for $B$ defined on $D((w - A)^\alpha)$ with $0 \leq \alpha < 1$, and in [11, §VI.6], [32] for $B$ defined on $X$. On the other hand, the monograph [21] comprehensively treats the long term behaviour of a large class of problems under various regularity assumptions on $\theta \mapsto B(\theta) \in \mathcal{L}(D(A), X)$. Thus we focus on functions $B$ just being of bounded variation, where most of the results on asymptotics of [21] do not apply.

In order to obtain positive results for such irregular problems, one may replace pure spectral criteria by estimates on the resolvent of the generator $\mathcal{A}_L$: On Hilbert spaces Gearhart's spectral mapping theorem characterizes in this way the exponential stability and dichotomy of a $C_0$–semigroup. This fact was already used in [9] and [18] to deduce exponential stability for the equation (1.2). On general Banach spaces theorems due to Weis, Wrobel and van Neerven give sufficient conditions for the exponential stability of classical solutions. See [20, §4.2] and the next section. On the other hand, Prüss has established a version of Gearhart's theorem for Volterra equations on Hilbert spaces in [21, §11.2, 12.3] imposing norm estimates that extend Gearhart's conditions on the resolvent.

In the present paper we treat (1.4) by means of a semigroup approach. We proceed in this way mainly for two reasons: First, we can apply semigroup results as the Weis–Wrobel theorem not available for Volterra equations. Second, an exponential dichotomy on $X$ cannot obtained for the Volterra equation since the initial data of (1.4) are $X$–valued functions and not elements of $X$. However, the history function space $\mathcal{X}$ allows to work with exponential dichotomies. Another motivation for the semigroup approach can be seen in [24], [26, Ch.8]: It makes it possible to transform (with some effort) control problems for delay equations into the standard setting of system theory. Having made the decision to work with semigroups, we strive for arguments staying entirely inside this setting. Thus our paper gives an alternative approach to some of the existence results in Chapter 7 of [21] which were obtained by more direct perturbation arguments. But certain drawbacks of the semigroup technique should be noted: On unbounded history intervals the semigroup approach does not work well when investigating asymptotic properties. It is further restricted to perturbation problems (1.4) for a generator $A$ and, as we see below, we will need an extra condition on $A$ if $X$ is not a Hilbert space.

In Section 3, we construct the solution semigroup directly by means of a perturbation theorem for semigroups, which was established by Weiss, [31], and Staffans, [25], for the study of feedback problems in control theory. The resulting semigroup then yields the classical solutions of (1.4). To our knowledge, so far purely semigroup theoretic approaches to (1.4) have been restricted to cases where $B$ is more regular, see e.g. [2], [11], and the references therein. In [24] a somewhat different situation with scalar valued kernels defined on $D(A)$ was studied. In contrast to our method, in [7], [16], [26], [32] the problem (1.2) is first solved by fixed point arguments and then the solutions are used to construct the semigroup.

As pointed out in [21, §13.6], in the present setting one can define the solution semigroup for (1.4) only on history function spaces which enjoy the property of 'maximal regularity' with respect to $A$. This works for instance if $A$ generates an analytic semigroup and if one takes Hölder or Besov spaces for the history functions. In this paper we use the history function space $L^p([-r,0],X_1)$, $1<p<\infty$, and are thus forced to suppose that $A$ has 'maximal regularity of type $L^p$', as introduced below. This property implies that $A$ generates an analytic semigroup. If $X$ is a Hilbert space, then each generator of an analytic semigroup has maximal regularity of type $L^p$. (This is the framework of [7], [16], [26].) For $X=L^q(\Omega)$, $q\in(1,\infty)$, maximal $L^p$ regularity has been established for large classes of operators arising from pdes. These facts and other prerequisites are presented in the next section.

Our main results are contained in Section 4, where we employ the semigroup $\mathcal{T}_L$ to study exponential stability and dichotomy of the solutions to (1.4). The perturbation approach of Section 3 immediately yields a first robustness result for the exponential dichotomy of (1.4) valid in every Banach space $X$, see Proposition 4.2. We further use the spectrum of the generator $\mathcal{A}_L$, which we determine in Proposition 4.3. Our formula for $\sigma(\mathcal{A}_L)$ seems to be a new result; even for (1.1) only certain inclusions were proven in the literature, see e.g. [9, Prop.4.2], [19, Thm.9]. In particular, the spectrum of $\mathcal{A}_L$ coincides with the spectrum of (1.4) as defined in [21, (11.7)], i.e., $\rho(\mathcal{A}_L)=\{\lambda\in\mathbb{C}:\exists\, H(\lambda):=(\lambda-A-L_\lambda)^{-1}\in\mathcal{L}(X,D(A))\}$, where $L_\lambda x=\int dB(\theta)\,e^{\lambda\theta}x$. In [21] the norm $H(\lambda)$ in $\mathcal{L}(X,D(A))$ is employed to determine asymptotic properties of solutions. We show in Proposition 4.3 that the norm of $R(\lambda,\mathcal{A}_L)$ is essentially equivalent to the norm of $H(\lambda)$ in $\mathcal{L}(X,D(A))$. Combined with the results on the asymptotic behaviour of semigroups mentioned above, Proposition 4.3 thus yields conditions for exponential stability and dichotomy of (1.4) in terms of the growth of $H(\lambda)$. Our criteria are necessary and sufficient in the Hilbert space case, see Theorems 4.4 and 4.5. These theorems lead to refined robustness results in Corollary 4.6. So far we have dealt with the properties of the solution $u(t)$ and its history $u(t+\cdot)$ at the same time. If one only considers $u(t)$, one can weaken the assumptions on $H(\lambda)$, see Theorem 4.7. Our results for Banach spaces $X$ have no counterpart in [21]. After Theorem 4.4 we discuss the relationship between our theorems and those in [21] for the Hilbert space case.

In the final section, we then treat the special case $dB(\theta) = A\,d\eta(\theta)$, where $\eta$ is scalar valued and $X$ is supposed to be a Hilbert space. The robustness conditions of Section 4 now translate into various sector conditions for the range of the Laplace transform of $d\eta$ which hence imply exponential stability or dichotomy of solutions. In some cases these conditions turn out to be sharp: In Proposition 5.5 we show that if $-d\eta$ is positive and $A$ is sectorial with angle $\pi$, then growth and spectral bound of $\mathcal{A}_L$ coincide and can be computed quite explicitly. Nevertheless, the spectral mapping theorem may be violated, as shown in Example 5.8. These discussions extend and improve considerably the corresponding results from [9] and [18] for the exponential stability of problem (1.2).

## 2. Assumptions and background

We first specify our assumptions for the problem (1.4). Let $X$ be a Banach space with norm $\|x\|$, $X_1$ be the domain $D(A)$ of the operator $A$ on $X$ with non–empty resolvent set $\rho(A)$, $r > 0$, $1 < p < \infty$, and $\varphi \in C([-r,0], X_1)$. Throughout we fix a number $w$ in $\rho(A)$ of $A$ and endow $X_1$ with the norm $\|x\|_1 = \|(w-A)x\|$. We further set $R(\lambda, A) = (\lambda - A)^{-1}$ for $\lambda \in \rho(A)$ and denote by $\sigma(A) = \mathbb{C} \setminus \rho(A)$ the spectrum of $A$. Let $\mathcal{L}(Y,Z)$ be the space of bounded linear operators between two Banach spaces $Y$ and $Z$, where $\mathcal{L}(Z) = \mathcal{L}(Z,Z)$. The symbol $\hookrightarrow$ means a continuous embedding. We designate by $a^+$, $a^-$, $a \wedge b$, $a \vee b$ the positive and negative part and the minimum and maximum of real numbers, respectively. Our main hypothesis reads as follows.

(H) $A$ is a sectorial operator in $X$ of type $(\phi, K, d)$, where $\phi \in (\pi/2, \pi]$, $d \in \mathbb{R}$, $K > 0$, with dense domain $X_1 = D(A)$, $A$ has maximal regularity of type $L^q$, and the operator valued function $B : [-r, 0] \to \mathcal{L}(X_1, X)$ has bounded variation $b$ with $db([-t,0]) \to 0$ as $t \searrow 0$.

*Sectoriality of type* $(\phi, K, d)$ means that the sector $\Sigma_{\phi,d} = \{\lambda \in \mathbb{C}\setminus\{d\} : |\arg(\lambda - d)| < \phi\}$ belongs to $\rho(A)$ and $\|R(\lambda, A)\| \le K\,|\lambda - d|^{-1}$ for $\lambda \in \Sigma_{\phi,d}$. It is well known that a sectorial, densely defined operator $A$ with $\phi > \pi/2$ generates an analytic $C_0$–semigroup $T = (T(t))_{t\ge 0}$ on $X$. The operator $A$ has *maximal regularity of type* $L^q$ if, for some $q \in (1,\infty)$ and $a > 0$ and every $f \in L^q([0,a], X)$, the mild solution $u(t) = \int_0^t T(t-s)f(s)\,ds$ of the inhomogeneous problem

$$u'(t) = Au(t) + f(t), \quad t \ge 0, \qquad u(0) = 0, \tag{2.1}$$

actually belongs to $L^q([0,a], X_1) \cap W^{1,q}([0,a], X)$ and solves (2.1) for a.e. $t \in [0,a]$. If $A$ has this property, then it holds in fact for all $q \in (1,\infty)$ and $a > 0$. It can be seen that sectoriality of angle $\phi > \pi/2$ is a necessary condition for maximal $L^q$-regularity and that it is also sufficient if $X$ is a Hilbert space. Analytic semigroups on $X = L^s(\Omega)$, $1 < s < \infty$, arising from pdes typically possess maximal $L^q$–regularity, see e.g. [5]. Recently, Weis obtained a breakthrough in this subject, [27]. He characterized those operators having maximal regularity of type $L^q$ by a 'randomized' version of the notion of sectoriality if $X$ is a UMD-space (e.g., $X = L^s(\Omega)$, $1 < s < \infty$). We refer the reader to [1], [5], [10], [21], [27] for these and related facts.

We further need the real interpolation space $Y = (X, X_1)_{1-1/p,p}$ for some fixed $p \in (1, \infty)$. If $X$ is a Hilbert space and $p = 2$, then $Y$ is again (isomorphic to) a Hilbert space (see (2.2)). It is well known that $X_1 \hookrightarrow Y \hookrightarrow X$ and that

$$x \in Y \iff T(\cdot)x \in L^p_{loc}(\mathbb{R}_+, X_1), \qquad \|x\|_Y \le c\,\|T(\cdot)x\|_{L^p([0,a],X_1)} \le c'\,\|x\|_Y\,, \tag{2.2}$$

where the constants may depend on $a$ and $p$, cf. [1, §I.2]. Moreover, the parts of $A$ in $X_1$ and $Y$ (also denoted by $A$) are again sectorial with the same $d$ and $\phi$. The analytic $C_0$–semigroups generated by the parts of $A$ are just the restrictions of $T(t)$ to $Y$ and $X_1$ (thus also denoted by $T(t)$). See [1, Thm.V.2.1.3] for these properties. Moreover, the parts of $A$ have the same spectrum as $A$ by [11, Prop.IV.2.17], and hence the spectra of $T(t)$ and its restrictions also coincide due to the spectral mapping theorem, see below in (2.7). We also need the continuous embedding

$$W^{1,p}([0,a], X) \cap L^p([0,a], X_1) \hookrightarrow C([0,a], Y), \qquad a > 0, \tag{2.3}$$

proved in [1, Thm.III.4.10.2].

Our main interest is directed to the asymptotic behaviour of solutions, as described by the following concept (see e.g. Sections IV.2, IV.3, V.1 in [11]). Let $C$ be the generator of a $C_0$–semigroup $U$ on a Banach space $Z$. We say that $U$ has an *exponential dichotomy* if there is a bounded linear projection $P$ on $Z$ commuting with $U(t)$, $t \ge 0$, such that $U(t) : QZ \to QZ$ has the inverse $U_Q(-t)$ and $\|U(t)P\| \le Ne^{-\delta t}$, $\|U_Q(-t)Q\| \le Ne^{-\delta t}$ for $t \ge 0$ and some constants $N, \delta > 0$, where $Q = I - P$. This property is equivalent to the fact that the unit circle $\mathbb{T}$ belongs to the resolvent set of $U(t)$ for some/all $t > 0$. Moreover, the dichotomy projection is given by

$$P = \frac{1}{2\pi i}\int_{\mathbb{T}} R(\lambda, U(t))\,d\lambda, \qquad t > 0. \tag{2.4}$$

We say that $PZ$ and $QZ$ are the *stable* and *unstable subspaces*, respectively. If $U$ has an exponential dichotomy, then it is easily verified that the operator

$$R(is)x = \int_0^\infty e^{-ist}T(t)P\,dt - \int_0^\infty e^{ist}T_Q(-t)Q\,dt \tag{2.5}$$

is the inverse of $is - A$ for $s \in \mathbb{R}$. This shows that

$$U \text{ has an exponential dichotomy} \quad \Longrightarrow \quad is \in \rho(C),\ \sup_{s\in\mathbb{R}} \|R(is, C)\| < \infty. \tag{2.6}$$

The converse implication holds if $X$ is a Hilbert space due to Gearhart's theorem, [20, Thm.2.2.4], but not in general, see e.g. [11, V.1.12]. If the spectral mapping theorem

$$\sigma(U(t)) \setminus \{0\} = \exp(t\sigma(C)), \tag{2.7}$$

is valid, then already $i\mathbb{R} \subset \rho(C)$ implies the exponential dichotomy of $U$. If $t \mapsto U(t) \in \mathcal{L}(Z)$ is continuous at some $t > 0$, then (2.7) is true. In particular, the analytic semigroup $T$ generated by $A$ satisfies the spectral mapping theorem. Thus, if $\sigma(A) \cap i\mathbb{R} = \emptyset$, then we have exponential dichotomies for $T$ on $X$, $Y$, and $X_1$, and the corresponding projections coincide on $X_1$ due to (2.4).

*Exponential stability* is just the special case of exponential dichotomy where $P = I$ or the spectral radius of $U(t)$, $t > 0$, is strictly less than 1. We further set

$$\omega_\alpha(C) = \inf\{a \in \mathbb{R} : \|U(t)(\gamma - C)^{-\alpha}\| \le M_a e^{at}, \quad t \ge 0\}$$

for $\alpha \in [0,1]$ and some fixed $\gamma > \omega_0(C)$. One has

$$s(C) := \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(C)\} \le \omega_1(C) \le \omega_\alpha(C) \le \omega_\beta(C) \le \omega_0(C)$$

for $1 \ge \alpha \ge \beta \ge 0$, and strict inequalities may occur, see e.g. Examples 1.2.4 and 4.2.9 in [20]. Motivated by Gearhart's theorem, one also introduces

$$s_0(C) = \inf\{a \in \mathbb{R} : R(\lambda, C) \text{ exists and is uniformly bounded for } \operatorname{Re}\lambda > a\}.$$

Similar as in (2.5) one sees that $s_0(C) \le \omega_0(C)$, but again there are generators satisfying $s_0(C) < \omega_0(C)$, [11, V.1.12]. In Hilbert spaces, however, Gearhart's theorem implies that $\omega_0(C) = s_0(C)$. This can be extended as follows: A Banach space $Z$ has *Fourier type* $q \in [1,2]$ if the Fourier transform maps $L^q(\mathbb{R}; Z)$ continuously into $L^{q'}(\mathbb{R}, Z)$, where $1/q + 1/q' = 1$. Clearly, any Banach space has at least Fourier type 1 and Hilbert spaces have Fourier type 2. If $Z$ has Fourier type $q$, then $L^s(\Omega, Z)$ also has Fourier type $q$ if $q \le s \le q'$ by [12, Prop.2.3] ($\Omega$ is a $\sigma$–finite measure space). The Weis–Wrobel theorem now says that

$$\omega_{\frac{1}{q}-\frac{1}{q'}}(C) \le s_0(C) \tag{2.8}$$

if $Z$ has Fourier type $q$. This covers both the stability case in Gearhart's theorem and the inequality $\omega_1(C) \le s_0(C)$ valid for every Banach space. The number $1/q - 1/q'$ in (2.8) cannot be improved in general, see [20, Ex.4.2.9]. However, if (2.7) holds, then all quantities $\omega_\alpha(A)$, $\alpha \ge 0$, $s(A)$, $s_0(A)$ coincide.

Our existence Theorem 3.6 is based on a perturbation result due to Weiss, [31], in Hilbert spaces and due to Staffans, [25], in Banach spaces; see also [23] for corresponding work on non–autonomous problems. These authors work in the framework of so–called regular systems in control theory. For the reader's convenience, we collect here the concepts and facts needed below. (In fact, [25] and [31] contain much more general results.)

Let $Z$ and $U$ be Banach spaces, $1 < q < \infty$, and $S$ be a $C_0$–semigroup on $Z$ generated by $G$. An *output system* for $S$ is a family of bounded linear operators $\Psi_t : Z \to L^q([0,t], U)$, $t \ge 0$, satisfying

$$[\Psi_{t+s} z]\,(\tau) = [\Psi_t S(s) z]\,(\tau - s) \tag{2.9}$$

for $\tau \in [s, s+t]$, $z \in Z$, and $t, s \ge 0$. Given $\Psi_t$, there is a bounded *output operator* $C : D(G) \to U$ such that $(\Psi_t z)(\tau) = CS(\tau)z$ for $z \in D(G)$ and $0 \le \tau \le t$. Moreover, there exists the *Yosida extension* $\tilde{C}$ of $C$ defined by

$$\tilde{C}z = \lim_{s\to\infty} sCR(s,G)z \qquad \text{for } \ z \in D(\tilde{C}) = \{z \in Z : \text{this limit exists in } U\}.$$

It can be shown that $T(\tau)z \in D(\tilde{C})$ and

$$(\Psi_t z)(\tau) = \tilde{C}S(\tau)z, \qquad z \in Z, \quad \text{a.e. } \tau \in [0,t]. \tag{2.10}$$

These results are contained in [28], see in particular Theorem 4.5 and Proposition 4.7.

An *input system* for $S$ is a family of bounded linear operators $\Phi_t : L^q([0,t], U) \to Z$, $t \ge 0$, such that

$$\Phi_{t+s} u = \Phi_t(u(\cdot + s)|[0,t]) + S(t)\Phi_s(u|[0,s]) \tag{2.11}$$

for $u \in L^q([0, s+t], U)$ and $t, s \ge 0$. To represent $\Phi_t$, we need the *extrapolation space* $Z_{-1}$ of $Z$ for $G$ which is the completion of $Z$ with respect to the norm $\|R(w,G)z\|$ for some fixed $w \in \rho(G)$, see [11, §II.5] for the definitions and properties of these spaces. One can uniquely extend $G$ to a bounded operator $G_{-1} : Z \to Z_{-1}$ which generates in $Z_{-1}$ a $C_0$–semigroup $S_{-1}$ extending $S$. Every input system can be written as

$$\Phi_t u = \int_0^t S_{-1}(t-s)Bu(s)\,ds \tag{2.12}$$

where the integral exists in $Z_{-1}$ and the *input operator* $B \in \mathcal{L}(U, Z_{-1})$ is given by

$$Bu_0 = \lim_{t\to 0} \tfrac{1}{t}\,\Phi_t u_0 \qquad (\text{in } Z_{-1}) \tag{2.13}$$

for $u_0 \in U$ (also denoting the corresponding constant function). These facts can be found in [29], in particular in Theorem 3.9.

Given $S(t)$, $\Psi_t$ and $\Phi_t$, we call bounded operators $\mathbb{F}_t$ on $L^q([0,t],U)$, $t \geq 0$, *input–output operators* if

$$[\mathbb{F}_{t+s}u]\,(\tau) = [\mathbb{F}_t(u(\cdot+s)|[0,t])]\,(\tau-s) + [\Psi_t\Phi_s(u|[0,s])]\,(\tau-s) \tag{2.14}$$

for $\tau \in [s, s+t]$, $t, s \geq 0$, and $u \in L^q([0,s+t],U)$. The system $\Sigma = (S(t), \Phi_t, \Psi_t, \mathbb{F}_t; t \geq 0)$ is called *regular* if

$$\lim_{t\to 0} \frac{1}{t}\int_0^t [\mathbb{F}_t u_0](\tau)\,d\tau = 0, \tag{2.15}$$

see [30]. It can be then shown that $\mathbb{F}_t u(\tau) = \tilde{C}\int_0^\tau T_{-1}(\tau-\sigma)Bu(\sigma)\,d\sigma$ for $u \in L^q([0,t],U)$ and a.e. $\tau \in [0,t]$, but we will not use this fact.

A bounded operator $K$ on $U$ is called an *admissible feedback* for a regular system $\Sigma$ if $I - \mathbb{F}_t K$ is invertible on $L^q([0,t],U)$ on some (and hence all) $t > 0$. Then

$$G_K = G_{-1} + BK\tilde{C} \qquad \text{with } \; D(G_K) = \{z \in D(\tilde{C}) : G_K z \in Z\}$$

(the sum is defined in $Z_{-1}$) generates the unique $C_0$–semigroup $S_K$ on $Z$ such that

$$S_K(t)z = S(t)z + \int_0^t S_{-1}(t-s)BK\tilde{C}S_K(s)z\,ds$$

for $z \in Z$ and $t \geq 0$. Here one has $\tilde{C}S_K(\cdot)z = (I - \mathbb{F}_t K)^{-1}\Psi_t z$ for $z \in Z$. Observe that the perturbation $BK\tilde{C}$ acts from an intermediate space between $D(G)$ and $Z$ into a space larger than $Z$. This perturbation theorem was proved by Weiss, [31, §6+7], in the Hilbert space setting; the above version is due to Staffans, [25, Chap.7].

## 3. Existence of solutions

We want to recast (1.4) as a perturbation problem for a semigroup on the product space

$$\mathcal{X} = (X, X_1)_{1-\frac{1}{p},p} \times L^p([-r,0],X_1) = Y \times L^p([-r,0],X_1)$$

endowed with the norm $\|\binom{x}{\varphi}\|_{\mathcal{X}} = \|x\|_Y + \|\varphi\|_{L^p([-r,0],X_1)}$. Throughout we suppose that (H) holds and fix some $p \in (1,\infty)$. We first define

$$(T_t x)\,(\theta) = \begin{cases} T(t+\theta)x, & \theta \in [-r,0],\ t+\theta \geq 0, \\ 0, & \theta \in [-r,0],\ t+\theta < 0, \end{cases}$$

$$(S(t)\varphi)\,(\theta) = \begin{cases} \varphi(t+\theta), & \theta \in [-r,0],\ t+\theta < 0, \\ 0, & \theta \in [-r,0],\ t+\theta \geq 0, \end{cases}$$

for $x \in X$, $\varphi \in L^1([-r,0],X)$, and $t \ge 0$. Observe that $S(t)$ yields the left translation semigroup $S$ on $L^p([-r,0],X_1)$ generated by $D_0\varphi = \frac{d}{d\theta}\varphi$ with domain $D(D_0) = \{\varphi \in W^{1,p}([-r,0],X_1) : \varphi(0) = 0\}$. Clearly, $\rho(D_0) = \mathbb{C}$ and

$$R(\lambda, D_0)\varphi\,(\theta) = \int_\theta^0 e^{\lambda(\theta-s)}\varphi(s)\,ds = \int_\theta^0 e^{\lambda s}\varphi(\theta - s)\,ds, \qquad \theta \in [-r,0]. \tag{3.1}$$

We further set $D\varphi = \frac{d}{d\theta}\varphi$ for $\varphi \in W^{1,1}([-r,0],X)$. Employing these mappings, we introduce the matrix operators

$$\mathcal{T}(t) = \begin{pmatrix} T(t) & 0 \\ T_t & S(t) \end{pmatrix} \qquad \text{and} \qquad \mathcal{A} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$$

on $\mathcal{X}$ with $D(\mathcal{A}) = \{\binom{x}{\varphi} \in X_1 \times W^{1,p}([-r,0],X_1) : \varphi(0) = x\}$. Finally, we set $(e_\lambda x)\,(\theta) = e^{\lambda\theta}x$ for $x \in X$, $\lambda \in \mathbb{C}$, and $\theta \in [-r,0]$.

**Lemma 3.1.** *Assume that $A$ generates an analytic $C_0$–semigroup $T$ on $X$. Then $\mathcal{A}$ generates the $C_0$–semigroup $\mathcal{T} = (\mathcal{T}(t))_{t\ge 0}$ on $\mathcal{X}$. Moreover, $\omega_0(\mathcal{A}) = \omega_0(A)$, $\sigma(\mathcal{A}) = \sigma(A)$, and*

$$R(\lambda,\mathcal{A}) = \begin{pmatrix} R(\lambda,A) & 0 \\ e_\lambda R(\lambda,A) & R(\lambda,D_0) \end{pmatrix} =: \mathcal{R}_\lambda, \qquad \lambda \in \rho(A).$$

*Proof.* Due to (2.2), $\mathcal{T}(t)$ is a bounded operator on $\mathcal{X}$ and $\mathcal{T}(t)\binom{x}{\varphi} \to \binom{x}{\varphi}$ in $\mathcal{X}$ as $t \to 0$. It is easy to verify that $\mathcal{T}(\cdot)$ is a semigroup with growth bound $\omega_0(A)$, cf. [2]. Thus $\mathcal{T}$ is a $C_0$–semigroup whose generator is denoted by $\tilde{\mathcal{A}}$. Taking the Laplace transform of $\mathcal{T}$, we see that $R(\lambda,\tilde{\mathcal{A}}) = \mathcal{R}_\lambda$ for $\operatorname{Re}\lambda > \omega_0(A)$.

Let $\lambda \in \rho(A)$. One can check in a straightforward way that $\mathcal{R}_\lambda\mathcal{X} \subseteq D(\mathcal{A})$, $(\lambda-\mathcal{A})\mathcal{R}_\lambda = I$, and $\mathcal{R}_\lambda(\lambda-\mathcal{A})\binom{x}{\varphi} = \binom{x}{\varphi}$ for $\binom{x}{\varphi} \in D(\mathcal{A})$, cf. [2]. Hence, $\lambda \in \rho(\mathcal{A})$ and $R(\lambda,\mathcal{A}) = \mathcal{R}_\lambda$ for $\lambda \in \rho(A)$. This fact also shows that $\mathcal{A} = \tilde{\mathcal{A}}$.

Assume that $\lambda \in \rho(\mathcal{A})$. If $(\lambda - A)x = 0$ for some $x \in X_1$, then $\binom{x}{e_\lambda x} \in D(\mathcal{A})$ and $(\lambda-\mathcal{A})\binom{x}{e_\lambda x} = 0$; so that $x = 0$. If $y \in Y$ is given, then there exists $\binom{x}{\varphi} \in D(\mathcal{A})$ such that $(\lambda-\mathcal{A})\binom{x}{\varphi} = \binom{y}{0}$. In particular, $x \in X_1$ and $(\lambda - A)x = y$, and hence $Ax \in Y$. Therefore $\lambda$ belongs to the resolvent set of the part of $A$ in $Y$, so that $\lambda \in \rho(A)$. ☐

Using the matrix $\mathcal{A}$, one can rewrite (1.4) as the evolution equation

$$\binom{u'(t)}{v'(t)} = \mathcal{A}\binom{u(t)}{v(t)} + \mathcal{L}\binom{u(t)}{v(t)}, \quad t \ge 0, \qquad \binom{u(0)}{v(0)} = \binom{\varphi(0)}{\varphi}, \tag{3.2}$$

on $\mathcal{X}$, where $\varphi \in W^{1,p}([-r,0],X_1) \hookrightarrow C([-r,0],X_1)$,

$$L\varphi = \int_{-r}^0 dB(\theta)\varphi(\theta), \quad \text{and} \quad \mathcal{L} = \begin{pmatrix} 0 & L \\ 0 & 0 \end{pmatrix},$$

compare the proof of Theorem 3.6. Observe that the perturbation $\mathcal{L}$ maps $Y \times C([-r,0],X_1)$ into $X \times L^p([-r,0],X_1)$; that is, it acts from an intermediate space between $D(\mathcal{A})$ and $\mathcal{X}$ into a space larger than $\mathcal{X}$. In order to deal with this difficulty, we want to use the Weiss–Staffans theory introduced in the previous section. The basic idea is to factorize $\mathcal{L} = \mathcal{BC}$ via the space $\mathcal{U} = X_1 \times L^p([-r,0],X_1)$ and the continuous operators

$$\mathcal{B} = \begin{pmatrix} w-A & 0 \\ 0 & 0 \end{pmatrix} : \mathcal{U} \to X \times L^p([-r,0],X_1),$$

$$\mathcal{C} = \begin{pmatrix} 0 & R(w,A)L \\ 0 & 0 \end{pmatrix} : X \times C([-r,0],X_1) \to \mathcal{U}$$

(recall that $w \in \rho(A)$ was fixed). Observe that $\mathcal{C} \in \mathcal{L}(D(\mathcal{A}),\mathcal{U})$ if we endow $D(\mathcal{A})$ with the graph norm. For technical reasons, we work primarily with the corresponding input and output maps defined by

$$\Phi_t \begin{pmatrix} u \\ f \end{pmatrix} := \begin{pmatrix} \int_0^t T(t-s)(w-A)u(s)\,ds \\ \int_0^t T_{t-s}(w-A)u(s)\,ds \end{pmatrix}, \qquad \begin{pmatrix} u \\ f \end{pmatrix} \in L^p([0,t],\mathcal{U}),\ t \geq 0, \tag{3.3}$$

$$\left[\Psi_t \begin{pmatrix} x \\ \varphi \end{pmatrix}\right](s) := \mathcal{C}\mathcal{T}(s)\begin{pmatrix} x \\ \varphi \end{pmatrix} = \begin{pmatrix} R(w,A)Lg_{\varphi,x}(s) \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} x \\ \varphi \end{pmatrix} \in D(\mathcal{A}),\ 0 \leq s \leq t, \tag{3.4}$$

where $[g_{\varphi,x}(s)](\theta) = T(s+\theta)x$ if $s+\theta \geq 0$ and $[g_{\varphi,x}(s)](\theta) = \varphi(s+\theta)$ if $s+\theta < 0$, for $\theta \in [-r,0]$ and $s \in [0,t]$. Let $t,s \geq 0$ and $\theta \in [-r,0]$. Observe that $[T_{t-s}(w-A)u(s)](\theta) = 0$ if $t-s+\theta \leq 0$, and

$$[T_{t-s}(w-A)u(s)](\theta) = T(t+\theta-s)(w-A)u(s)$$

if $t-s+\theta \geq 0$. Hence, the second component of $\Phi_t\binom{u}{f}$ is equal to

$$\int_0^{(t+\theta)^+} T((t+\theta)^+ - s)(w-A)u(s)\,ds, \qquad \theta \in [-r,0].$$

In what follows, we do not distinguish in notation between a function defined on an interval and its restrictions to subintervals.

**Lemma 3.2.** *Assume that (H) holds. Then $\Psi_t$ can be extended to a bounded operator from $\mathcal{X}$ into $L^p([0,t],\mathcal{U})$ such that $\|\Psi_t\binom{x}{\varphi}\|_{L^p([0,t],\mathcal{U})} \leq C_1\,\|B\|_{BV}\,\|\binom{x}{\varphi}\|_{\mathcal{X}}$ for $\binom{x}{\varphi} \in \mathcal{X}$, $t \in [0,r]$, and a constant $C_1$ only depending on $p$, $w$, $r$, and the type of $A$. Moreover $\Phi_t$ and $\Psi_t$, $t \geq 0$, are input and output systems for $\mathcal{T}$.*

*Proof.* 1) By maximal regularity and the embedding (2.3), $\Phi_t$ maps contiuously into $\mathcal{X}$. Let $\binom{u}{f} \in L^p_{loc}(\mathbb{R}_+,\mathcal{U})$ and $t,s \geq 0$. Then

$$\mathcal{T}(t)\Phi_s\begin{pmatrix} u \\ f \end{pmatrix} = \begin{pmatrix} \int_0^s T(t+s-\tau)(w-A)u(\tau)\,d\tau \\ I(t,s) \end{pmatrix}, \qquad \text{where}$$

$$I(t,s)\,(\theta) = \begin{cases} \int_0^s T(t+\theta+s-\tau)(w-A)u(\tau)\,d\tau, & t+\theta \geq 0,\ \theta \in [-r,0], \\ \int_0^{(t+\theta+s)^+} T((t+\theta+s)^+ - \tau)(w-A)u(\tau)\,d\tau, & t+\theta < 0,\ \theta \in [-r,0]. \end{cases}$$

On the other hand,

$$\Phi_t\begin{pmatrix} u(\cdot+s) \\ f(\cdot+s) \end{pmatrix} = \begin{pmatrix} \int_s^{s+t} T(t+s-\tau)(w-A)u(\tau)\,d\tau \\ \int_s^{s+(t+\theta)^+} T((t+\theta)^+ + s - \tau)(w-A)u(\tau)\,d\tau \end{pmatrix}$$

so that (2.11) is verified.

2) Identity (2.9) is clear on $D(\mathcal{A})$. Let $\binom{x}{\varphi} \in D(\mathcal{A})$, $0 \le t \le r$, and $1/p + 1/p' = 1$. Using Hölder's inequality, Fubini's theorem, and (2.2), we estimate

$$
\begin{aligned}
\|\Psi_t \tbinom{x}{\varphi}\|_{L^p[0,t],\mathcal{U})} &= \Big[\int_0^t \|R(w,A)L(T_s x + S(s)\varphi)\|_{X_1}^p \, ds\Big]^{\frac{1}{p}} \\
&\le \Big[\int_0^t \Big[\int_{-s}^0 \|T(s+\theta)x\|_{X_1}\, db(\theta)\Big]^p ds\Big]^{\frac{1}{p}} + \Big[\int_0^t \Big[\int_{-r}^{-s} \|\varphi(s+\theta)\|_{X_1}\, db(\theta)\Big]^p ds\Big]^{\frac{1}{p}} \\
&\le \|B\|_{BV}^{\frac{1}{p'}} \Big(\Big[\int_0^t \int_{-s}^0 \|T(s+\theta)x\|_{X_1}^p \, db(\theta)\, ds\Big]^{\frac{1}{p}} + \Big[\int_0^t \int_{-r}^{-s} \|\varphi(s+\theta)\|_{X_1}^p \, db(\theta)\, ds\Big]^{\frac{1}{p}}\Big) \\
&\le \|B\|_{BV}^{\frac{1}{p'}} \Big(\Big[\int_{-t}^0 \int_{-\theta}^t \|T(s+\theta)x\|_{X_1}^p \, ds\, db(\theta)\Big]^{\frac{1}{p}} + \Big[\int_{-r}^0 \int_0^{-\theta\wedge t} \|\varphi(s+\theta)\|_{X_1}^p \, ds\, db(\theta)\Big]^{\frac{1}{p}}\Big) \\
&\le \|B\|_{BV}^{\frac{1}{p'}} \Big(\Big[\int_{-t}^0 c\,\|x\|_Y^p \, db(\theta)\Big]^{\frac{1}{p}} + \Big[\int_{-r}^0 \|\varphi\|_{L^p([-r,0],X_1)}^p \, db(\theta)\Big]^{\frac{1}{p}}\Big) \\
&\le c\,\|B\|_{BV}\, \|\tbinom{x}{\varphi}\|_{\mathcal{X}} .
\end{aligned}
$$

The assertions for $\Psi_t$ can now easily be deduced, cf. [23, Lem. 2.3]. □

In the next lemma we compute the Yosida extension $\tilde{\mathcal{C}}$ of $\mathcal{C}$ for a class of vectors which is large enough for our purposes, namely

$$\mathcal{D}_C = \{\tbinom{x}{\varphi} \in X_1 \times C([-r,0],X_1) : \varphi(0) = x\}.$$

**Lemma 3.3.** *Assume that (H) holds. If* $\binom{x}{\varphi} \in \mathcal{D}_C$*, then* $\binom{x}{\varphi} \in D(\tilde{\mathcal{C}})$ *and* $\tilde{\mathcal{C}}\binom{x}{\varphi} = \mathcal{C}\binom{x}{\varphi} = (R(w,A)L\varphi, 0)^T$*. Moreover,* $\mathcal{T}(t)\mathcal{D}_C \subseteq \mathcal{D}_C$ *for* $t \ge 0$*.*

*Proof.* By approximation, we see that $\Psi_t\binom{x}{\varphi} = (R(w,A)Lg_{x,\varphi}, 0)^T$ for $\binom{x}{\varphi} \in \mathcal{D}_C$, cf. (3.4). Moreover, the function $s \mapsto g_{x,\varphi}(s) \in C([-r,0],X_1)$ is continuous so that $s \mapsto \Psi_t\binom{x}{\varphi}(s) \in \mathcal{U}$ is a continuous for $s \ge 0$. This implies the first assertion thanks to [28, Prop.4.7]. The second one is clear. □

In order to define the corresponding input–output operator, we use the space $X_2 = D(A^2)$ and introduce

$$[\mathbb{F}_t\tbinom{u}{f}]\,(s) = \mathcal{C}\Phi_s\tbinom{u}{f} \quad \text{for} \quad \tbinom{u}{f} \in L^p_{loc}(\mathbb{R}_+, X_2 \times L^p([-r,0],X_1)) \;\text{ and }\; 0 \le s \le t.$$

Observe that for these inputs we have $\Phi_s\binom{u}{f} \in \mathcal{D}_C$.

**Lemma 3.4.** *Assume that (H) holds. Then* $\mathbb{F}_t$*,* $t \ge 0$*, can be extended to bounded operators* $\mathbb{F}_t : L^p([0,t],\mathcal{U}) \to L^p([0,t],\mathcal{U})$*, where* $\|\mathbb{F}_t\| \le C_2\, db([-t,0])$ *for* $t \le r$ *and a constant* $C_2$ *only depending on* $A$*,* $p$*,* $w$*,* $r$*. These extensions are regular input–output operators for* $S(t)$*,* $\Phi_t$ *and* $\Psi_t$*.*

*Proof.* Identity (2.14) is an easy consequence of (2.10) and Lemmas 3.2 and 3.3 if $\binom{u}{f} \in L^p_{loc}(\mathbb{R}_+, X_2 \times L^p([-r,0],X_1))$. For $0 \le t \le r$ and using Hölder's inequality, Fubini's theorem, and the maximal regularity of $A$, we further estimate

$$
\begin{aligned}
\|\mathbb{F}_t\tbinom{u}{f}\|_{L^p([0,t],\mathcal{U})}^p &= \int_0^t \Big\|L\int_0^s T_{s-\tau}(w-A)u(\tau)\,d\tau\Big\|_X^p ds \\
&\le \int_0^t \Big[\int_{-s}^0 \Big\|\int_0^{s+\theta} T(s+\theta-\tau)(w-A)u(\tau)\,d\tau\Big\|_{X_1} db(\theta)\Big]^p ds
\end{aligned}
$$

$$\leq \int_0^t b(-s)^{p-1} \int_{-s}^0 \Big\| \int_0^{s+\theta} T(s+\theta-\tau)(w-A)u(\tau)\,d\tau \Big\|_{X_1}^p db(\theta)\,ds$$
$$\leq b(-t)^{p-1} \int_{-t}^0 \int_0^{t+\theta} \Big\| \int_0^{\sigma} T(\sigma-\tau)(w-A)u(\tau)\,d\tau \Big\|_{X_1}^p d\sigma\,db(\theta)$$
$$\leq c\,b(-t)^{p-1} \int_{-t}^0 \int_0^{t+\theta} \|u(\tau)\|_{X_1}^p\,d\tau\,db(\theta)$$
$$\leq c\,b(-t)^p\,\|u\|_{L^p([0,t],X_1)}^p\,.$$

Thus $\mathbb{F}_t$ can continuously be extended to a bounded operator in $L^p([0,t],\mathcal{U})$ if $t \leq r$. Due to (2.14) and Lemma 3.2, this extension can be carried out iteratively for all $t \geq 0$ and (2.14) remains valid. To check the regularity property (2.15), we first compute

$$\Phi_s \binom{x}{\varphi} = \begin{pmatrix} (w-A)\int_0^s T(\tau)x\,d\tau \\ (w-A)\int_0^{(s+\cdot)^+} T(\tau)x\,d\tau \end{pmatrix} = w\Phi_s \begin{pmatrix} R(w,A)x \\ \varphi \end{pmatrix} + \begin{pmatrix} x - T(s)x \\ x - T((s+\cdot)^+)x \end{pmatrix}$$

for $\binom{x}{\varphi} \in \mathcal{U}$ (also denoting the corresponding constant function). Approximating $x$ in $X_1$ by $x_n \in X_2$, we see that

$$\mathbb{F}_t \binom{x}{\varphi}(s) = \begin{pmatrix} wR(w,A)L\int_0^{(s+\cdot)^+} T(\tau)x\,d\tau \\ 0 \end{pmatrix} + \begin{pmatrix} R(w,A)L(x - T((s+\cdot)^+)x) \\ 0 \end{pmatrix}$$

for $0 \leq s \leq t$. Therefore $\mathbb{F}_t\binom{x}{\varphi}(s) \to 0$ in $\mathcal{U}$ as $s \to 0$, and $\mathbb{F}_t$ is regular. □

So we have established that $(\mathcal{T}(t), \Phi_t, \Psi_t, \mathbb{F}_t)$ is a regular system. But we still have to identify the operator defined in (2.13), which we temporarily denote by $\tilde{\mathcal{B}} : \mathcal{U} \to \mathcal{X}_{-1}$. Besides the extrapolation space $\mathcal{X}_{-1}$ for $\mathcal{A}$, we need the space $\mathcal{Z} = X \times L^1([-r,0],X_1)$ endowed with its natural norm. Note that $\mathcal{B}$ maps $\mathcal{U}$ into $\mathcal{Z}$.

**Lemma 3.5.** *Assume that (H) holds. Then $\mathcal{X} \hookrightarrow \mathcal{Z} \hookrightarrow \mathcal{X}_{-1}$ and $\mathcal{B} = \tilde{\mathcal{B}}$.*

*Proof.* 1) One can extend $R(w,\mathcal{A})$ to a continuous and injective operator $\mathcal{R} : \mathcal{Z} \to \mathcal{X}$ (having the same representation) due to Lemma 3.1. Let $v \in \mathcal{Z}$. Then there are $v_n \in \mathcal{X}$ converging to $v$ in $\mathcal{Z}$. Since

$$\|v_n - v_m\|_{\mathcal{X}_{-1}} = \|\mathcal{R}(v_n - v_m)\|_{\mathcal{X}} \leq c\,\|v_n - v_m\|_{\mathcal{Z}}\,,$$

$(v_n)$ is a Cauchy sequence in $\mathcal{X}_{-1}$. We set $Jv = (v_n) + \mathcal{N}$, where $\mathcal{N}$ is the space of null sequences in $(\mathcal{X}, \|\cdot\|_{\mathcal{X}_{-1}})$. Observe that $J : \mathcal{Z} \to \mathcal{X}_{-1}$ is well defined, linear, and bounded. If $Jv = 0$ for some $v \in \mathcal{Z}$, then $v_n \to 0$ in $\mathcal{X}_{-1}$, i.e., $\mathcal{R}v_n \to 0$ in $\mathcal{X}$. Hence, $\mathcal{R}v = 0$ which shows that $v = 0$. So we have embedded $\mathcal{Z}$ into $\mathcal{X}_{-1}$.

2) It remains to compute the limit defining $\tilde{\mathcal{B}}$. For $t \in (0,r]$ and $\binom{x}{\varphi} \in \mathcal{U}$, one has

$$R(w,\mathcal{A})\Phi_t \binom{x}{\varphi} = \begin{pmatrix} R(w,A) & 0 \\ e_w R(w,A) & R(w,D_0) \end{pmatrix} \begin{pmatrix} \int_0^t T(t-s)(w-A)x\,ds \\ \int_0^{(t+\cdot)^+} T((t+\cdot)^+ - s)(w-A)x\,ds \end{pmatrix}$$
$$= \begin{pmatrix} \int_0^t T(s)x\,ds \\ e_w \int_0^t T(s)x\,ds + I_t \end{pmatrix}, \qquad \text{where}$$

$$
\begin{aligned}
I_t(\theta) &= \int_{-t\vee\theta}^{0} e^{w(\theta-\tau)}(w-A)\int_0^{t+\tau} T(s)x\,ds\,d\tau \\
&= \int_{-t\vee\theta}^{0} e^{w(\theta-\tau)}\Big[w\int_0^{t+\tau} T(s)x\,ds + x - T(t+\tau)x\Big]\,d\tau, \\
\|I_t(\theta)\|_{X_1} &\le ct\,(t\|x\|_{X_1} + \sup_{0\le\sigma\le t}\|T(\sigma)x - x\|_{X_1}).
\end{aligned}
$$

As a result,

$$
R(w,\mathcal{A})\frac{1}{t}\Phi_t\begin{pmatrix}x\\ \varphi\end{pmatrix} \to \begin{pmatrix}x\\ e_w x\end{pmatrix} = R(w,\mathcal{A})\begin{pmatrix}(w-A)x\\ 0\end{pmatrix} = R(w,\mathcal{A})\mathcal{B}\begin{pmatrix}x\\ \varphi\end{pmatrix}
$$

in $\mathcal{X}$ as $t\to 0$, which means that $\frac{1}{t}\Phi_t\binom{x}{\varphi} \to \mathcal{B}\binom{x}{\varphi}$ in $\mathcal{X}_{-1}$. □

We now come to the main result in this section, where we construct the solution semigroup for (1.4) using the above set-up.

**Theorem 3.6.** *Assume that (H) holds. Then the operator*

$$
\begin{aligned}
\mathcal{A}_L &= \begin{pmatrix}A & L\\ 0 & D\end{pmatrix} \qquad \textit{with} \\
D(\mathcal{A}_L) &= \{\tbinom{x}{\varphi} \in X_1\times W^{1,p}([-r,0],X_1) : \varphi(0)=x,\ L\varphi + Ax \in Y\}
\end{aligned}
\tag{3.5}
$$

*generates the $C_0$–semigroup $\mathcal{T}_L$ on $\mathcal{X}$ satisfying*

$$
\mathcal{T}_L(t)\begin{pmatrix}x\\ \varphi\end{pmatrix} = \mathcal{T}(t)\begin{pmatrix}x\\ \varphi\end{pmatrix} + \int_0^t \mathcal{T}_{-1}(t-s)\mathcal{B}\mathcal{C}\mathcal{T}_L(s)\begin{pmatrix}x\\ \varphi\end{pmatrix}ds \tag{3.6}
$$

$$
= \begin{pmatrix}T(t)x\\ T_t x + S(t)\varphi\end{pmatrix} + \begin{pmatrix}\int_0^t T(t-s)Lv(s)ds\\ \int_0^t T_{t-s}Lv(s)\,ds\end{pmatrix} \tag{3.7}
$$

*for $v(t) = [\mathcal{T}_L(t)\binom{x}{\varphi}]_2$, $\binom{x}{\varphi}\in D(\mathcal{A}_L)$, and $t\ge 0$. Moreover,*

$$
\|\tilde{\mathcal{C}}\mathcal{T}_L(\cdot)\tbinom{x}{\varphi}\|_{L^p([0,t],\mathcal{U})} \le 2C_1\,\|B\|_{BV}\,\|\tbinom{x}{\varphi}\|_{\mathcal{X}} \tag{3.8}
$$

*for $t\in[0,t_0]$, $\binom{x}{\varphi}\in\mathcal{X}$, the constant $C_1$ of Lemma 3.2, and a number $t_0\in(0,r]$ such that $db([-t_0,0]) \le (2C_2)^{-1}$, where $C_2$ is given by Lemma 3.4. If $\varphi\in W^{1,p}([-r,0],X_1)$, then $u = [\mathcal{T}_L(\cdot)\binom{\varphi(0)}{\varphi}]_1$ belongs to $C^1(\mathbb{R}_+,Y)\cap C(\mathbb{R}_+,X_1)\cap W^{1,p}_{loc}(\mathbb{R}_+,X_1)$ and solves (1.4). Moreover, $[v(t)](\theta) = u(t+\theta)$ where we set $u(t)=\varphi(t)$ for $-r\le t\le 0$. The solution of (1.4) is unique in the class $L^p_{loc}(\mathbb{R}_+,X_1)\cap W^{1,p}_{loc}(\mathbb{R}_+,X)$.*

*Proof.* Lemmas 3.1, 3.2, 3.4, and 3.5 show that $\Sigma = (\mathcal{T}(t),\Phi_t,\Psi_t,\mathbb{F}_t)$ is a regular well–posed system with input and output operators $\mathcal{B}$ and $\mathcal{C}$. Lemma 3.4 also yields that $I-\mathbb{F}_t$ is invertible on $L^p([0,t],\mathcal{U})$ and $\|(I-\mathbb{F}_t)^{-1}\| \le 2$ if $0\le t\le t_0$. Therefore we can apply Theorems 7.1.2, 7.1.8, 7.5.3(iii), and Remark 7.1.3 of [25]* to deduce that the operator

$$
\tilde{\mathcal{A}}_L := \mathcal{A}_{-1} + \mathcal{B}\tilde{\mathcal{C}} \ \text{ with } \ D(\tilde{\mathcal{A}}_L) := \{\tbinom{x}{\varphi}\in D(\tilde{\mathcal{C}}) : \tilde{\mathcal{A}}_L\tbinom{x}{\varphi}\in\mathcal{X}\} \tag{3.9}
$$

(the sum is defined in $\mathcal{X}_{-1}$) generates a $C_0$–semigroup $\mathcal{T}_L$ on $\mathcal{X}$ satisfying (3.6) with $\mathcal{C}$ replaced by $\tilde{\mathcal{C}}$ and that

$$
\tilde{\mathcal{C}}\mathcal{T}_L(\cdot)\tbinom{x}{\varphi} = (I-\mathbb{F}_t)^{-1}\Psi_t\tbinom{x}{\varphi} \tag{3.10}
$$

*The feedthrough operator $D$ in [25] equals 0 in our setting.

for $\binom{x}{\varphi} \in \mathcal{X}$. Estimate (3.8) thus follows from Lemma 3.2. Using the representation of $D(\tilde{\mathcal{A}}_L)$ established below and Lemma 3.3, we also see that we can replace $\tilde{\mathcal{C}}$ by $\mathcal{C}$ in (3.6). This equation further yields (3.7) because of (2.12) and Lemma 3.5.

We have to verify that $\mathcal{A}_L = \tilde{\mathcal{A}}_L$. Take $\binom{x}{\varphi} \in D(\mathcal{A}_L)$. Due to Lemma 3.1 and (3.1), we obtain

$$\begin{aligned}\mathcal{A}\lambda R(\lambda, \mathcal{A})\begin{pmatrix} x \\ \varphi \end{pmatrix} &= \begin{pmatrix} \lambda AR(\lambda, A)x \\ De_\lambda \lambda R(\lambda, A)x + \lambda DR(\lambda, D_0)\varphi \end{pmatrix} \\ &= \begin{pmatrix} \lambda AR(\lambda, A)x \\ \lambda e_\lambda(\lambda R(\lambda, A)x - x) + \lambda R(\lambda, D_0)D\varphi \end{pmatrix} \longrightarrow \begin{pmatrix} Ax \\ D\varphi \end{pmatrix}\end{aligned}$$

in $\mathcal{Z}$ as $\lambda \to \infty$. This limit also exists in $\mathcal{X}_{-1}$ by Lemma 3.5 so that $\mathcal{A}_{-1}\binom{x}{\varphi} = \binom{Ax}{D\varphi} \in \mathcal{Z}$ on $D(\mathcal{A}_L)$. Moreover, $D(\mathcal{A}_L) \subseteq D(\tilde{\mathcal{C}})$ and $\tilde{\mathcal{C}} = \mathcal{C}$ on $D(\mathcal{A}_L)$ by Lemma 3.3. Thus the operator $\tilde{\mathcal{A}}_L$ defined by (3.9) extends $\mathcal{A}_L$ given in (3.5). The two operators coincide if $\mu - \mathcal{A}_L$ is surjective for some $\mu > w_0(\tilde{\mathcal{A}}_L)$. So let $\binom{y}{\psi} \in \mathcal{X}$ and set $\varphi = e_\mu x + R(\mu, D_0)\psi$ for $x \in X_1$ and $\mu > w_0(\tilde{\mathcal{A}}_L) \vee d \vee 0$ to be determined. Using $(\mu - D)\varphi = \psi$, we see that

$$\binom{x}{\varphi} \in D(\mathcal{A}_L),\ (\mu - \mathcal{A}_L)\binom{x}{\varphi} = \binom{y}{\psi} \iff x - R(\mu, A)Le_\mu x = R(\mu, A)(LR(\mu, D_0)\psi + y). \tag{3.11}$$

We have to find $\mu$ and $x$ satisfying the right hand side of this equivalence. Observe that $\|R(\mu, A)\|_{\mathcal{L}(X,X_1)} \le 1 + 2K$ for $\mu \ge \mu_0 \ge 0$ and a sufficiently large $\mu_0$. We further estimate

$$\begin{aligned}\|Le_\mu x\|_X &\le \int_{-\varepsilon}^0 e^{\mu\theta} db(\theta)\, \|x\|_{X_1} + \int_{-r}^{-\varepsilon} e^{\mu\theta} db(\theta)\, \|x\|_{X_1} \\ &\le (db([-\varepsilon, 0]) + \|B\|_{BV}\, e^{-\mu\varepsilon})\, \|x\|_{X_1} \\ &\le \frac{1}{2(1 + 2K)}\, \|x\|_{X_1} \end{aligned} \tag{3.12}$$

for sufficiently small $\varepsilon \in (0, r)$ and large $\mu \ge \mu_0$ (here we use (H)). We can thus invert $I - R(\mu, A)Le_\mu$ in $\mathcal{L}(X_1)$ for a fixed $\mu$ to obtain $x \in X_1$ satisfying (3.11). So we arrive at the asserted representation for $\tilde{\mathcal{A}}_L$.

Finally, take $\binom{x}{\varphi} \in D(\mathcal{A}_L)$. Then the function $\binom{u}{v} = \mathcal{T}_L(\cdot)\binom{x}{\varphi}$ belongs to $C^1(\mathbb{R}_+, \mathcal{X})$, $\binom{u(t)}{v(t)} \in D(\mathcal{A}_L)$, and we have $\binom{u'(t)}{v'(t)} = \mathcal{A}_L\binom{u(t)}{v(t)}$. In particular, $u \in C^1(\mathbb{R}_+, Y)$ and $v \in C(\mathbb{R}_+, W^{1,p}([-r, 0], X_1))$ fulfill the equations $u'(t) = Au(t) + Lv(t)$ (the sum is taken in $X$) and $v'(t) = Dv(t)$. Thus $u \in C(\mathbb{R}_+, X_1)$ and the equation for $v$ yields

$$[v(t)]\,(\theta) = \begin{cases} u(t + \theta), & t + \theta \ge 0,\ \theta \in [-r, 0], \\ \varphi(t + \theta), & t + \theta < 0,\ \theta \in [-r, 0], \end{cases} \tag{3.13}$$

because $[v(t)](0) = u(t)$ and $v(0) = \varphi$. As a result, we have solved (1.4) in the required regularity class for the initial history $\varphi$. To prove uniqueness, assume that $u \in L^p_{loc}(\mathbb{R}_+, X_1) \cap W^{1,p}_{loc}(\mathbb{R}_+, X)$ solves (1.4) with $\varphi = 0$ and extend $u$ to $[-r, 0]$ by 0. Let $\tau \in (0, r]$. Then we estimate

$$\int_0^\tau \|Lu_s\|_X^p ds \le \int_0^\tau \Big[\int_{-s}^0 db(\theta)\, \|u(s + \theta)\|_{X_1}\Big]^p ds \le (db[-\tau, 0])^p \int_0^\tau \|u(s)\|_{X_1}^p\, ds$$

using Young's inequality for measures. The maximal regularity of $A$ and (1.4) thus imply

$$\int_0^\tau \|u(s)\|_{X_1}^p \, ds \le c\,(db[-\tau,0])^p \int_0^\tau \|u(s)\|_{X_1}^p \, ds$$

for a constant $c > 0$ only depending on $A$ and $r$. As a result, $u = 0$ on $[0,\tau]$ for some small $\tau > 0$. Iterating this argument, we obtain $u = 0$ on $\mathbb{R}_+$. □

Theorems 7.4-7.6 in [21] imply similar results on a.e. solutions of (1.4) assuming less regularity for the initial data. In fact, the problem is studied in [21] in full detail. The main difference is that Prüss achieves (with much effort) to work on $X$ itself, whereas we only come arbitrarily close to $X$ letting $p \to 1$. For later use we further establish a result on the inhomogeneous problem with more general initial data using the above setting.

**Proposition 3.7.** *Assume that (H) holds. Let $\binom{x}{\varphi} \in \mathcal{X}$ and $f \in L^p(J,Y)$ for $J = [0,T]$ and some $T > 0$. Then there is a unique $u \in L^p(J,X_1) \cap W^{1,p}(J,X_1) \cap C(J,Y)$ solving*

$$u'(t) = Au(t) + f(t) + \int_{-r}^0 dB(\theta)u(t+\theta), \text{ a.e. } t \ge 0, \quad u(t) = \varphi(t), \text{ a.e. } t \in [-r,0], \quad (3.14)$$

*which is given by*

$$\begin{pmatrix} u(t) \\ u_t \end{pmatrix} = \mathcal{T}_L(t)\begin{pmatrix} x \\ \varphi \end{pmatrix} + \int_0^t \mathcal{T}_L(t-s)\begin{pmatrix} f(s) \\ 0 \end{pmatrix} ds, \quad t \ge 0. \quad (3.15)$$

*Proof.* The uniqueness of solutions can be proved as in Theorem 3.6. Denote by $\binom{u(t)}{v(t)}$ the right hand side of (3.15) for given $\binom{x}{\varphi} \in \mathcal{X}$ and $f \in L^p(J,Y)$, and extend $u$ by setting $u(t) = \varphi(t)$ for $t \in [-r,0]$.

First take regular data $\binom{x}{\varphi} \in D(\mathcal{A}_L)$ and $f \in C^1(J,Y)$. In this case $\binom{u}{v} \in C(\mathbb{R}_+, D(\mathcal{A}_L)) \cap C^1(\mathbb{R}_+,\mathcal{X})$ solves

$$\binom{u'(t)}{v'(t)} = \mathcal{A}_L\binom{u(t)}{v(t)} + \binom{f(t)}{0}, \quad t \ge 0, \qquad \binom{u(0)}{v(0)} = \binom{x}{\varphi},$$

due to standard semigroup theory. Therefore $v(t) = u_t$ and $u \in C^1(\mathbb{R}_+,Y) \cap C(\mathbb{R}_+,X_1)$ satisfies (3.14). This shows that

$$u(t) = T(t)x + \int_0^t T(t-s)(f(s) + Lu_s)\, ds, \qquad t \ge 0. \quad (3.16)$$

Second, we approximate given $\binom{x}{\varphi} \in \mathcal{X}$ and $f \in L^p(J,Y)$ in these spaces by $\binom{x_n}{\varphi_n} \in D(\mathcal{A}_L)$ and $f \in C^1(J,Y)$. Denote by $\binom{u}{u}$ and $\binom{u_n}{v_n}$ the right hand side of (3.15) for these given and approximating data, respectively. Then (3.15) implies that $u_n$ converges to $u$ in $L^p([-r,T],X_1) \cap C([-r,T],Y)$ and $v(t) = u_t$. Observe that $s \mapsto Lu_s$ belongs to $L^p(J,X)$. Thus we can pass to the limit also in (3.16) so that (3.16) holds for general data, too. The assertion then follows from the maximal regularity of $A$. □

## 4. Asymptotic behaviour of solutions

We start with a robustness result for exponential dichotomy based on (3.6) and the following lemma.

**Lemma 4.1.** *Assume that (H) holds and that $\sigma(A) \cap i\mathbb{R} = \emptyset$. Then $\mathcal{T}$ has an exponential dichotomy on $\mathcal{X}$ with the same exponent, and the dimension of its unstable subspace coincides with that of $T$.*

*Proof.* By assumption, $T$ has an exponential dichotomy on $X$ and $Y$ with projections $P$ and $Q = I - P$ and constants $N, \delta > 0$. Moreover, $Q = T(1)T_Q(-1)Q$ so that $Q \in \mathcal{L}(X, X_1)$ and $QX = QY$. We define

$$\mathcal{Q} = \begin{pmatrix} Q & 0 \\ T_Q(\cdot)Q & 0 \end{pmatrix}.$$

Clearly, $\mathcal{Q}$ is a bounded projection on $\mathcal{X}$, $\dim \mathcal{Q}\mathcal{X} = \dim QY = \dim QX$, and $\mathcal{T}(t)\mathcal{Q} = \mathcal{Q}\mathcal{T}(t)$. The inverse of $\mathcal{T}(t)$, $t \geq 0$, on $\mathcal{Q}\mathcal{X}$ is given by

$$\mathcal{T}_Q(-t) = \begin{pmatrix} T_Q(-t) & 0 \\ T_Q(-t+\cdot)Q & 0 \end{pmatrix}.$$

This formula implies that $\|\mathcal{T}_Q(-t)\mathcal{Q}\| \leq N_1 \, e^{-\delta t}$ for $t \geq 0$. For $t > r$ and $\mathcal{P} = I - \mathcal{Q}$, it holds

$$\mathcal{T}(t)\mathcal{P} = \begin{pmatrix} T(t) & 0 \\ T_t & 0 \end{pmatrix} \begin{pmatrix} P & 0 \\ -T_Q(\cdot)Q & I \end{pmatrix} = \begin{pmatrix} T(t)P & 0 \\ T_tP & 0 \end{pmatrix}$$

so that $\|\mathcal{T}(t)\mathcal{P}\| \leq N_2 \, e^{-\delta t}$ for $t \geq 0$ and a constant $N_2$. □

We remark that one can prove that $\mathcal{T}$ is eventually norm continuous using (2.2) so that $\mathcal{T}$ satisfies the spectral mapping theorem (2.7). Combined with Lemma 3.1, this fact also shows that $\mathcal{T}$ inherits the exponential dichotomy of $T$; but this argument does not yield the preservation of the dimension of the unstable subspace.

The following results describe the asymptotic behaviour of $\mathcal{T}_L$. Note that the exponential stability of $\mathcal{T}_L$ implies that

$$\|u(t)\|_Y + \|u_t\|_{L^p([-r,0],X_1)} \leq Ne^{-\delta t} \left(\|\varphi(0)\|_Y + \|\varphi\|_{L^p([-r,0],X_1)}\right), \qquad t \geq 0,$$

for the solutions of (1.4). Further, the exponential dichotomy of $\mathcal{T}_L$ leads to an dichotomy on the level of the history functions $u_t \in L^p([-r,0], X_1)$.

**Proposition 4.2.** *Assume that (H) holds and that $\sigma(A) \cap i\mathbb{R} = \emptyset$. If $\|B\|_{BV}$ is sufficiently small, then $\mathcal{T}_L$ has an exponential dichotomy on $\mathcal{X}$ and the dimension of its unstable subspace coincides with that of $T$. If, in addition, $T$ is exponentially stable, then also $\mathcal{T}_L$ is exponentially stable.*

*Proof.* Due to (3.6), (3.8), (2.12), and Lemma 3.2 we have $\|\mathcal{T}_L(t_0) - \mathcal{T}(t_0)\| \leq c\|B\|_{BV}$ where $c$ does not depend on $B$ and $t_0 > 0$ is given by Theorem 3.6. Thus Lemma 4.1 and [22, Prop.2.3] show the assertion provided that $\|B\|_{BV}$ is sufficiently small. □

In the above proof we have used Proposition 2.3 of [22] which deals with non–autonomous problems. In the present situation one can in fact replace this result by simpler arguments based on standard spectral theory, cf. Corollary 4.6. We have not specified the smallness condition for $\|B\|_{BV}$ in Proposition 4.2. Inspecting the proofs, one sees that it involves $r$, $p$, $w$, the dichotomy constants $N, \delta$ of $T$, the type of $A$, a constant related with the maximal regularity of $A$, and the decay of $db([-t, 0])$ as $t \searrow 0$. But the approach of Proposition 4.2 does not give very sharp conditions and is of course restricted to operators $A$ having an exponential dichotomy. To obtain further results, we now employ the spectrum of $\mathcal{A}_L$. We first compute $\sigma(\mathcal{A}_L)$ thereby improving and extending various facts established in [9]. See also [19] for an investigation of the (generalized) eigenfunction spaces of $\mathcal{A}_L$ for (1.2), and [2], [11], [32] for the case of more regular $B$. We set $L_\lambda = Le_\lambda : X_1 \to X$ and $\mathbb{C}_a = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq a\}$.

**Proposition 4.3.** *Assume that (H) holds. Then*

$$\rho(\mathcal{A}_L) = \{\lambda \in \mathbb{C} : \exists\, H(\lambda) := (\lambda - A - L_\lambda)^{-1} \in \mathcal{L}(X, X_1)\},$$

$$R(\lambda, \mathcal{A}_L) = \begin{pmatrix} H(\lambda) & H(\lambda)LR(\lambda, D_0) \\ e_\lambda H(\lambda) & (e_\lambda H(\lambda)L + I)R(\lambda, D_0) \end{pmatrix} =: \mathcal{R}_\lambda,$$

$$\|R(\lambda, \mathcal{A}_L)\|_{\mathcal{L}(\mathcal{X})} \leq c_a \, \|H(\lambda)\|_{\mathcal{L}(X,X_1)},$$

*for $\lambda \in \rho(\mathcal{A}_L)$ with* $\operatorname{Re}\lambda \geq a$ *for some $a \in \mathbb{R}$. Moreover, if $\|H(\lambda)\|_{\mathcal{L}(X,X_1)}$ is unbounded on a closed subset of $\mathbb{C}_a \cap \rho(\mathcal{A}_L)$ for some $a \in \mathbb{R}$, then $\|R(\lambda, \mathcal{A}_L)\|_{\mathcal{L}(\mathcal{X})}$ is unbounded on the same set.*

*Proof.* 1) Assume that $H(\lambda) = (\lambda - A - L_\lambda)^{-1} \in \mathcal{L}(X, X_1)$ exists for some $\lambda \in \mathbb{C}$. Then it is clear that $\mathcal{R}_\lambda$ is a bounded operator on $\mathcal{X}$. More precisely, since $\|e_\lambda\|_p$ and the norm of $R(\lambda, D_0) : L^p([-r, 0], X_1) \to C([-r, 0], X_1)$ are uniformly bounded for $\operatorname{Re}\lambda \geq a$, one has the estimate $\|\mathcal{R}_\lambda\|_{\mathcal{L}(\mathcal{X})} \leq c_a \, \|H(\lambda)\|_{\mathcal{L}(X,X_1)}$ if $\operatorname{Re}\lambda \geq a$. Take $\binom{x}{\varphi} \in \mathcal{X}$. Then $\binom{y}{\psi} = \mathcal{R}_\lambda \binom{x}{\varphi}$ belongs to $X_1 \times W^{1,p}([-r, 0], X_1)$, $\psi(0) = y$, and

$$\begin{aligned} Ay + L\psi &= (A + L_\lambda)H(\lambda)x + (A + L_\lambda)H(\lambda)LR(\lambda, D_0)\varphi + LR(\lambda, D_0)\varphi \\ &= \lambda H(\lambda)x - x + \lambda H(\lambda)LR(\lambda, D_0)\varphi \\ &= \lambda y - x \;\in\; Y. \end{aligned}$$

This shows that $\mathcal{R}_\lambda \binom{x}{\varphi} \in D(\mathcal{A}_L)$ and $(\lambda - \mathcal{A}_L)\mathcal{R}_\lambda = I$ (notice that $(\lambda - D)e_\lambda = 0$). Thus $\lambda - \mathcal{A}_L$ is surjective. Let $(\lambda - \mathcal{A}_L)\binom{x}{\varphi} = 0$ for some $\binom{x}{\varphi} \in D(\mathcal{A}_L)$. Then $\varphi = e_\lambda x$ and hence $(\lambda - A - L_\lambda)x = 0$, so that $x = 0$ and $\varphi = 0$. As a result, $\lambda \in \rho(\mathcal{A}_L)$ and $\mathcal{R}_\lambda = R(\lambda, \mathcal{A}_L)$.

2) Conversely, let $\lambda \in \rho(\mathcal{A}_L)$. We extend $\mathcal{A}_L$ to an operator on $\mathcal{W} = X \times L^p([-r, 0], X_1)$ by setting

$$\tilde{\mathcal{A}}_L = \begin{pmatrix} A & L \\ 0 & D \end{pmatrix}, \quad D(\tilde{\mathcal{A}}_L) = \{\tbinom{x}{\varphi} \in X_1 \times W^{1,p}([-r, 0], X_1) : \varphi(0) = x\}.$$

Observe that $\mathcal{A}_L$ is the part of $\tilde{\mathcal{A}}_L$ in $\mathcal{X}$. As in (3.12) we can choose a sufficiently large $\mu > \omega_0(\mathcal{A}_L) \vee d$ such that $\|L_\mu R(\mu, A)\|_{\mathcal{L}(X)} \leq 1/2$. Hence $\mu - A - L_\mu$ has the inverse

$$H(\mu) = R(\mu, A)(I - L_\mu R(\mu, A))^{-1} \in \mathcal{L}(X, X_1). \tag{4.1}$$

As in the first step, one sees that $\mu \in \rho(\tilde{\mathcal{A}}_L)$ and $R(\mu, \tilde{\mathcal{A}}_L) = \mathcal{R}_\mu$. Since $\mathcal{R}_\mu : \mathcal{W} \to \mathcal{X}$ is bounded, we also obtain that the graph norm of $\tilde{\mathcal{A}}_L$ in $\mathcal{W}$ dominates the norm of $\mathcal{X}$. Consequently, $\rho(\tilde{\mathcal{A}}_L) = \rho(\mathcal{A}_L) \ni \lambda$ by [11, Prop.IV.2.17].

If $(\lambda - A - L_\lambda)x = 0$ for some $x \in X_1$, then $\binom{x}{e_\lambda x}$ belongs to the kernel of $\lambda - \tilde{\mathcal{A}}_L$, so that $x = 0$. Given $y \in X$, there exists $\binom{x}{\varphi} \in D(\tilde{\mathcal{A}}_L)$ with $(\lambda - \tilde{\mathcal{A}}_L)\binom{x}{\varphi} = \binom{y}{0}$. This implies that $\varphi = e_\lambda x$ and $(\lambda - A - L_\lambda)x = y$. Summing up, $\lambda - A - L_\lambda : X_1 \to X$ is bijective and continuous, and thus invertible.

3) Assume that $a_n := \|H(\lambda_n)\|_{\mathcal{L}(X,X_1)} \to \infty$ as $n \to \infty$ for some $\lambda_n$ in a closed subset of $\mathbb{C}_a \cap \rho(\mathcal{A}_L)$. Then $|\lambda_n| \to \infty$ and thus $\lambda_n \in \Sigma_{\phi,d}$ for $n \geq n_0$ and some $n_0 \geq 1$. Moreover, $H(\lambda) \in \mathcal{L}(X, X_1)$ is uniformly bounded on a half plane $\mathbb{C}_\gamma$ for a sufficiently large $\gamma \geq a$ due to (4.1). Hence, $\operatorname{Re}\lambda_n$ is uniformly bounded. Take unit vectors $x_n \in X$ such that

$\|H(\lambda_n)x_n\|_{X_1} \geq a_n - 1/n$. Define $y_n = R(\lambda_n, A)x_n$ and $b_n = a_n^{p'/2}$ for $1/p + 1/p' = 1$. Observe that the vectors $y_n$ are uniformly bounded in $X_1$ for $n \geq n_0$. We further take $n_1 \geq n_0$ such that $b_n \geq 1/r$ for $n \geq n_1$ and set

$$\varphi_n(\theta) = b_n \, e^{\lambda_n \theta} \, \mathbb{1}_{[-b_n^{-1},0]}(\theta), \qquad \theta \in [-r, 0],$$

for $n \geq n_1 \geq n_0$. We then obtain $\|\varphi_n\|_p \leq c \, b_n^{\frac{1}{p'}}$ and, using (3.1),

$$R(\lambda_n, D_0)\varphi_n y_n = e_{\lambda_n}\psi_n y_n, \qquad \text{where} \quad \psi_n(\theta) := \begin{cases} -b_n\theta, & 0 \geq \theta \geq -1/b_n, \\ 1, & -b_n^{-1} \geq \theta \geq -r. \end{cases}$$

This equality yields

$$\begin{aligned} [R(\lambda_n, \mathcal{A}_L)\tbinom{0}{\varphi_n y_n}]_2 &= e_{\lambda_n}(H(\lambda_n)L_{\lambda_n} y_n + y_n) + e_{\lambda_n}H(\lambda_n)L e_{\lambda_n}(\psi_n - 1)y_n + e_{\lambda_n}(\psi_n - 1)y_n \\ &=: S_1 + S_2 + S_3\,. \end{aligned}$$

First, note that $S_1 = e_{\lambda_n}H(\lambda_n)x_n$. Using $|\psi_n - 1| \leq \mathbb{1}_{[-b_n^{-1},0]}$, we estimate

$$\|S_2\|_{L^p([-r,0],X_1)} \leq c_1 \, db([-b_n^{-1}, 0]) \, a_n \quad \text{and} \quad \|S_3\|_{L^p([-r,0],X_1)} \leq c_2 \, b_n^{-\frac{1}{p}}$$

for some constants $c_k$ independent of $n$. Combining these facts, we arrive at

$$\begin{aligned} c_3 \, b_n^{\frac{1}{p'}} \, \|R(\lambda_n, \mathcal{A}_L)\|_{\mathcal{L}(\mathcal{X})} &\geq \|R(\lambda_n, \mathcal{A}_L)\tbinom{0}{\varphi_n y_n}\|_{\mathcal{X}} \\ &\geq c_4(a_n - 1/n) - c_1 \, db([-b_n^{-1}, 0]) \, a_n - c_2 \, b_n^{-\frac{1}{p}} \geq c_5 \, a_n \end{aligned}$$

for sufficiently large $n$ and constants $c_k > 0$. As a result, $\|R(\lambda_n, \mathcal{A}_L)\| \geq c\sqrt{a_n}$. □

Our following main theorems are immediate consequences of the above proposition and the results from spectral theory of semigroups recalled in Section 2.

**Theorem 4.4.** *Assume that (H) holds and that $X$ is Hilbert space. Take $p = 2$.*

(a) *$\mathcal{T}_L$ has an exponential dichotomy on $\mathcal{X}$ if and only if $H(is)$ exists and is uniformly bounded in $\mathcal{L}(X, X_1)$ for $s \in \mathbb{R}$.*
(b) *$\mathcal{T}_L$ is exponentially stable on $\mathcal{X}$ if and only if $H(\lambda)$ exists and is uniformly bounded in $\mathcal{L}(X, X_1)$ for $\operatorname{Re}\lambda \geq 0$.*

*Proof.* Observe that $\mathcal{X}$ is a Hilbert space if $X$ is a Hilbert space and $p = 2$. Thus Gearhart's theorem, [20, Thm.2.2.4], characterizes exponential dichotomy and stability of $\mathcal{T}_L$ by the uniform boundedness of $R(\lambda, \mathcal{A}_L)$ for $\lambda \in i\mathbb{R}$ and $\operatorname{Re}\lambda \geq 0$, respectively. The assertions hence follow from Proposition 4.3. □

We compare the above result with Prüss' monograph [21]. Assume that $\mathcal{A}_L$ has an exponential dichotomy. Then for every $f \in L^2(\mathbb{R}, Y)$ there is a unique $\binom{u}{v} \in L^2(\mathbb{R}, \mathcal{X}) \cap C(\mathbb{R}, \mathcal{X})$ such that

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \mathcal{T}_L(t-s)\begin{pmatrix} u(s) \\ v(s) \end{pmatrix} + \int_s^t \mathcal{T}_L(t-\tau)\begin{pmatrix} f(\tau) \\ 0 \end{pmatrix} d\tau, \quad t \geq s,$$

see e.g. [4, Thm.4.33]. Proposition 3.7 further shows that $u \in L^2_{loc}(\mathbb{R}, X_1) \cap W^{1,2}_{loc}(\mathbb{R}, X) \cap L^2(\mathbb{R}, Y)$ and

$$u'(t) = Au(t) + Lu_t + f(t)$$

for a.e. $t \in \mathbb{R}$. This property is called 'admissibility of $L^2(\mathbb{R}, Y)$'. Admissibility of $L^2(\mathbb{R}, X)$ was obtained in [21, Prop.12.1] (with slightly less local regularity) also assuming that $H(is) : X \to X_1$ is bounded for $s \in \mathbb{R}$ and that $X$ is a Hilbert space. Conversely, admissibility of $BUC(\mathbb{R}, X)$ implies boundedness of $H(is) : X \to X_1$ for $s \in \mathbb{R}$ by [21, Prop.11.5]. Prüss' results are in fact more general and flexible than ours. He can allow for spectrum on $i\mathbb{R}$ and unboundedness of $\|H(is)\|_{\mathcal{L}(X,X_1)}$ imposing regularity and spectral conditions on the class of inhomogeneities $f$. On the other hand, it is not clear whether admissibility of $L^2(\mathbb{R}, X)$ or $L^2(\mathbb{R}, Y)$ in turn implies the exponential dichotomy of $\mathcal{A}_L$ on $\mathcal{X}$. (Roughly speaking, admissibility and exponential dichotomy are equivalent for non retarded problems, [4], and for retarded equations with $B \in BV([-r, 0], \mathcal{L}(X))$, [14].)

In the following two theorems we extend part (b) of Theorem 4.4 to general Banach spaces. These results have no counterpart in [21].

**Theorem 4.5.** *Assume that (H) holds and that $X$ has Fourier type $q \in [1, 2]$. Take $p \in [q, q'] \cap (1, \infty)$. If $H(\lambda)$ exists and is uniformly bounded in $\mathcal{L}(X, X_1)$ for $\operatorname{Re}\lambda \geq 0$, then there are constants $N, \delta > 0$ such that $\|\mathcal{T}_L(t)\binom{x}{\varphi}\|_{\mathcal{X}} \leq Ne^{-\delta t}\,\|(\gamma - \mathcal{A}_L)^{1/q-1/q'}\binom{x}{\varphi}\|$ for $1/q + 1/q' = 1$ and $\binom{x}{\varphi} \in D((\gamma - \mathcal{A}_L)^{1/q-1/q'})$ (where $\gamma > \omega_0(\mathcal{A}_L)$ is fixed).*

*Proof.* It is clear that $X_1$ has Fourier type $q$. Let us check that also $Y$ has Fourier type $q$. We may assume that $q > 1$ since every Banach space has Fourier type 1. For $1 < q \leq p \leq q'$, the estimate in (2.2) and the integral version of Jessen's inequality (cf. [15, §202, 203]) imply

$$\begin{aligned}
\|\mathcal{F}f\|_{L^{q'}(\mathbb{R},Y)} &\leq c\Big[\int_{\mathbb{R}}\Big[\int_0^1 \|T(\tau)(\mathcal{F}f)(t)\|_{X_1}^p\,d\tau\Big]^{\frac{q'}{p}}dt\Big]^{\frac{1}{q'}}\\
&\leq c\Big[\int_0^1\Big[\int_{\mathbb{R}} \|T(\tau)(\mathcal{F}f)(t)\|_{X_1}^{q'}\,dt\Big]^{\frac{p}{q'}}d\tau\Big]^{\frac{1}{p}}\\
&= c\Big[\int_0^1\Big[\int_{\mathbb{R}} \|[\mathcal{F}(T(\tau)f)](t)\|_{X_1}^{q'}\,dt\Big]^{\frac{p}{q'}}d\tau\Big]^{\frac{1}{p}}\\
&\leq c'\Big[\int_0^1\Big[\int_{\mathbb{R}} \|T(\tau)f(s)\|_{X_1}^{q}\,ds\Big]^{\frac{p}{q}}d\tau\Big]^{\frac{1}{p}}\\
&\leq c'\Big[\int_{\mathbb{R}}\Big[\int_0^1 \|T(\tau)f(s)\|_{X_1}^{p}\,d\tau\Big]^{\frac{q}{p}}ds\Big]^{\frac{1}{q}} \leq c''\,\|f\|_{L^q(\mathbb{R},Y)}\,,
\end{aligned}$$

where the $\mathcal{F}$ denotes the Fourier transform and $f$ is an $X_1$–valued Schwartz function, say. Moreover, $L^p([-r, 0], X_1)$ and thus $W^{1,p}([-r, 0], X_1)$ have Fourier type $q$ by [12, Prop.2.3]. Therefore $\mathcal{X}$ has Fourier type $q$. On the other hand, $s_0(\mathcal{A}_L) < 0$ by the assumption and Proposition 4.3. The theorem is now a consequence of the Weis–Wrobel theorem (2.8). □

We can now partly improve the robustness result Proposition 4.2.

**Corollary 4.6.** *Assume that (H) holds, that $A$ is invertible, and that $X$ has Fourier type $q \in [1, 2]$. Take $p \in [q, q'] \cap (1, \infty)$ and $w = 0$, i.e., $\|x\|_{X_1} = \|Ax\|_X$.*

(a) *If $s(A) < 0$ and $\|B\|_{BV} < \frac{1}{1+K}$, then the assertion of Theorem 4.5 holds. In particular, $\mathcal{T}_L$ is exponentially stable if $X$ is a Hilbert space.*

(b) *If $X$ is a Hilbert space (i.e., $q = p = 2$), $i\mathbb{R} \subseteq \rho(A)$, and $\|B\|_{BV} < \inf_{s\in\mathbb{R}}(1 + \|sR(is,A)\|_{\mathcal{L}(X)})^{-1}$, then $\mathcal{T}_L$ has an exponential dichotomy and the unstable subspaces of $\mathcal{T}_L$ and $T$ have the same dimension.*

*Proof.* (a) For $s(A) < 0$ and $\operatorname{Re}\lambda \geq 0$, we have

$$\|L_\lambda R(\lambda, A)\|_{\mathcal{L}(X)} \leq \|B\|_{BV} \|R(\lambda, A)\|_{\mathcal{L}(X,X_1)} \leq \|B\|_{BV}(1 + K) =: c < 1$$

by the assumption. Thus there exists

$$H(\lambda) = R(\lambda, A)\,[I - L_\lambda R(\lambda, A)]^{-1} \in \mathcal{L}(X, X_1)\,.$$

and $\|H(\lambda)\| \leq (1 + K)/(1 - c)$. Therefore assertion (a) follows from Theorem 4.5.

(b) Let $\sigma(A) \cap i\mathbb{R} = \emptyset$. As above one verifies that, if the estimate for $B$ holds,

$$\|L_{is}R(is, A)\|_{\mathcal{L}(X)} \leq c' < 1 \quad \text{and} \quad \|H(is)\|_{\mathcal{L}(X,X_1)} \leq c''.$$

for $s \in \mathbb{R}$. Thus $\mathcal{A}_L$ has an exponential dichotomy by Theorem 4.4. In order to check the assertion concerning the dimensions, we introduce the perturbations $\alpha B$ for $\alpha \in [0, 1]$. Since the hypotheses hold for $\alpha B$ with the same constants, we obtain the corresponding exponentially dichotomic semigroups $\mathcal{T}_{\alpha L} =: \mathcal{T}_\alpha$, where $\mathcal{T}_0 = \mathcal{T}$ and $\mathcal{T}_1 = \mathcal{T}_L$. In view of [13, Lem.II.4.3] and Lemma 4.1, we have to show that the projections $\mathcal{Q}_\alpha$ for $\mathcal{T}_\alpha$ depend continuously on $\alpha$ in operator norm. This is the case if $\alpha \mapsto \mathcal{T}_\alpha(t_0) \in \mathcal{L}(\mathcal{X})$ is continuous for some $t_0 > 0$, because of formula (2.4). We now employ more results from feedback theory in order to establish the continuity of this map. We fix the input operator $\mathcal{B}$ and the output operator $\mathcal{C}$. Then we obtain the semigroup $\mathcal{T}_\alpha$ from $\mathcal{T}_0$ by the admissible feedback $\alpha I : \mathcal{U} \to \mathcal{U}$, cf. Theorem 3.6. In fact, there exists another regular system $\Sigma^\alpha = (\mathcal{T}_\alpha(t), \Phi_t^\alpha, \Psi_t^\alpha, \mathbb{F}_t^\alpha)$ with $\Psi_t^\alpha = (I - \alpha\mathbb{F}_t)^{-1}\Psi_t$, see [25, Thm.7.1.2]. Moreover, $(\beta - \alpha)I : \mathcal{U} \to \mathcal{U}$, $\beta \in [0, 1]$, is an admissible feedback for $\Sigma^\alpha$ producing $\Sigma^\beta$ due to [25, Lem.7.1.7]. In particular, $\mathcal{T}_\beta(t_0) - \mathcal{T}_\alpha(t_0) = (\beta - \alpha)\Phi_{t_0}^\alpha \Psi_{t_0}^\beta$ for the number $t_0 > 0$ given in Theorem 3.6. This expression tends to 0 in norm as $\beta \to \alpha$ by Lemma 3.4. □

If we restrict ourselves to decay of $u(t)$ in $Y$, then it suffices that $H$ is bounded in a weaker norm. As a sample of possible consequences of Sections 4.3–4.5 in [20], we present a result on exponential stability not involving geometric properties of $X$. We denote by $\Pi_1 : \mathcal{X} \to Y$ the projection onto the first component of $\mathcal{X}$.

**Theorem 4.7.** *Assume that (H) holds and that $H(\lambda) : X \to Y$ is bounded for $\operatorname{Re}\lambda > -\varepsilon$ and some $\varepsilon > 0$. Then*

$$\|\Pi_1 \mathcal{T}_L(t)\tbinom{x}{\varphi}\|_Y \leq Ne^{-\delta t}\,\|(w - \mathcal{A}_L)\tbinom{x}{\varphi}\|_{\mathcal{X}}$$

*for $\binom{x}{\varphi} \in D(\mathcal{A}_L)$, $t \geq 0$, and some constants $N, \delta > 0$.*

*Proof.* Due to Proposition 4.3 the operator $\Pi_1 R(\lambda, \mathcal{A}_L + \varepsilon) : \mathcal{X} \to Y$ has a bounded analytic extension to the halfplane $\operatorname{Re}\lambda > 0$. Thus Theorem 4.5.2 of [20] shows that $\|\Pi_1 e^{\varepsilon t}\mathcal{T}_L(t)\| \leq M(1 + t)$, which yields the assertion. □

The reader will have noticed that our results on exponential dichotomy in the non Hilbert space case are restricted to Proposition 4.2. This is due to the fact that, so far, we have no theorem of Weis–Wrobel type for exponential dichotomy.

## 5. Scalar–valued kernels

In this section we treat the case that $B = \eta A$ for a scalar valued function $\eta$ in the space $BV_0([-r,0])$ of functions with total variation $b$ satisfying $db([-t,0]) \to 0$ as $t \searrow 0$. We normalize $\eta$ by $\eta(0) = 0$. Throughout we assume that $X$ is Hilbert space and that $A$ is sectorial with $\phi > \pi/2$ and dense domain $X_1 \neq X$. Thus (H) holds. We set $\mathbb{C}_a = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq a\}$, $\Sigma_\phi = \Sigma_{\phi,0}$ , and

$$\widehat{d\eta}(\lambda) = \int_{-r}^{0} e^{\lambda\theta}\, d\eta(\theta), \qquad \lambda \in \mathbb{C}.$$

Observe that $\lambda - A - L_\lambda = \lambda - (1 + \widehat{d\eta}(\lambda))A$. Hence, $\lambda \in \rho(\mathcal{A}_L)$ if and only if

$$1 + \widehat{d\eta}(\lambda) \neq 0 \quad \text{and} \quad \mu(\lambda) := \lambda(1 + \widehat{d\eta}(\lambda))^{-1} \in \rho(A),$$

and then $H(\lambda) = (1+\widehat{d\eta}(\lambda))^{-1}R(\mu(\lambda), A)$. We start with an easy case where the spectrum of $A$ and the range of $\widehat{d\eta}$ do not interfere.

**Proposition 5.1.** *Let $X$ be a Hilbert space. Let $A$ be sectorial with $d = 0$ and $\phi > \pi/2$, and let $\eta \in BV_0([-r,0])$. If there are $\phi' \in (0, \phi - \pi/2)$ and $a > 0$ such that*

$$\overline{1 + \widehat{d\eta}(\mathbb{C}_a)} \subseteq \Sigma_{\phi'}\,, \tag{5.1}$$

*then $\omega(\mathcal{A}_L) < a$. If $d < 0$ and (5.1) is valid for $a = 0$, then $\mathcal{T}_L$ is exponentially stable. Condition (5.1) holds in particular, if $\phi = \pi$ and*

$$\sup_{\operatorname{Re}\lambda \geq a} |\widehat{d\eta}(\lambda)| < 1. \tag{5.2}$$

*Proof.* Let $\operatorname{Re}\lambda \geq a > 0$. Then (5.1) implies $1+\widehat{d\eta}(\lambda) \neq 0$ and $|1+\widehat{d\eta}(\lambda)|^{-1} \leq c$ for some constant $c$. Observe that $\mu(\lambda) \in \Sigma_\phi$ by (5.1). Hence $H(\lambda)$ exists and

$$\|H(\lambda)\|_{\mathcal{L}(X,X_1)} = |1 + \widehat{d\eta}(\lambda)|^{-1}\|(w - A)R(\mu(\lambda), A)\|_{\mathcal{L}(X)} \leq wK|\lambda|^{-1} + c(K + 1) \leq c'.$$

Due to Proposition 4.3, the resolvent of $\mathcal{A}_L$ is bounded on $\mathbb{C}_a$ and thus it is bounded on $\mathbb{C}_{a'}$ for some $a' < a$. Therefore Gearhart's theorem [20, Thm.2.2.4] shows the first assertion. If $d < 0$, this argument also works for $a = 0$ so that second assertion is also proved. The last assertion is clearly true. □

Condition (5.2) was already used in [9, Thm.4.5] and [18, Thm.2] for special classes of $\eta$ and $A$. The estimates for $\omega_0(\mathcal{A}_L)$ become more complicated if we have $d > 0$ or $a < 0$.

**Proposition 5.2.** *Let $X$ be a Hilbert space, $A$ be sectorial with $\phi > \pi/2$ and $d \geq 0$, and let $\eta \in BV_0([-r,0])$. Assume that there are $\phi' \in (0, \phi - \pi/2)$ and $a \geq 0$ such that (5.1) holds. Hence $\operatorname{Re}(1 + \widehat{d\eta}(\lambda))^{-1} \geq r$ for $\operatorname{Re}\lambda \geq a$ and some $r > 0$. If $ar > d$, then $\omega_0(\mathcal{A}_L) < a$.*

*Proof.* Let $\lambda = \rho + i\tau$ for $\rho \geq a > 0$ and $\tau \in \mathbb{R}$, and let $(1 + \widehat{d\eta}(\lambda))^{-1} = x + iy$ for $x \geq r$, $y \in \mathbb{R}$. Observe that $|(1+\widehat{d\eta}(\lambda))^{-1}| \leq c$ and $|y| \leq x \tan\phi'$ by (5.1). For $y, \tau \geq 0$, we have

$$\begin{aligned} -\operatorname{Re}(\mu(\lambda) - ar) &= ar - \rho x + \tau y \leq a(r - x) + \tau y \leq \tau y \\ &\leq (\tau x + \rho y)\tan\phi' = \tan(\phi')\operatorname{Im}(\mu(\lambda) - ar), \end{aligned}$$

i.e., $\mu(\lambda) - ar \in \overline{\Sigma}_{\phi'+\pi/2}$. This fact can be proved in a similar way if $y$ and $\tau$ have other signs. Using $ar > d$, the assertion can now be shown as in the previous proposition. $\square$

**Proposition 5.3.** *Let $X$ be a Hilbert space, $A$ be sectorial with $\phi = \pi$ and $d \leq 0$, and let $\eta \in BV_0([-r, 0])$. Assume that there are $\phi' \in (0, \pi/2)$ and $a \leq 0$ such that (5.1) holds and that*

$$s := \sup\{x + y^2/x : x + iy \in (1 + \widehat{d\eta}(\mathbb{C}_a))^{-1},\ x, y \in \mathbb{R}\} < \infty.$$

*If $as > d$, then $\omega_0(\mathcal{A}_L) < a$.*

*Proof.* We use the notation of the previous proof. In view of Proposition 5.1 there is nothing to prove if $\operatorname{Re}\lambda \geq 0$. So let $0 > \rho \geq a$. Since $\mu(\lambda) = \rho x - \tau y + i(\tau x + \rho y)$, it is easy to see that $\mu(\lambda) \in \Sigma_\phi$ for some $\phi < \pi$ and large $|\tau|$. Moreover, if $\operatorname{Im}\mu(\lambda) = 0$, then $\tau = -\rho y/x$ and thus $\mu(\lambda) = \rho(x + y^2/x) \geq as$. Combining these facts, we see that $\mu(\lambda) \in \overline{\Sigma}_{\psi,as}$ for some $\psi < \pi$. The assertion now follows as above. $\square$

**Corollary 5.4.** *Let $X$ be a Hilbert space. Let $A$ be sectorial with $\phi = \pi$ and let $\eta \in BV_0([-r, 0])$. Assume that there is $a \in \mathbb{R}$ such that*

$$\sup_{\operatorname{Re}\lambda \geq a} |\widehat{d\eta}(\lambda)| \leq q < 1.$$

*(a) If $a, d \geq 0$ and $a/(1 + q) > s(A)$, then $\omega_0(\mathcal{A}_L) < a$.*
*(b) If $d \leq 0$ and $a/(1 - q) > s(A)$, then $\omega_0(\mathcal{A}_L) < a$.*

*Proof.* Observe that $(1 + \widehat{d\eta}(\mathbb{C}_a))^{-1}$ is contained in the ball in $\mathbb{C}$ which is symmetric with respect to $\mathbb{R}$ and whose boundary intersects $\mathbb{R}$ at the points $1/(1 + q)$ and $1/(1 - q)$. In Propositions 5.2 and 5.3 we thus have $r = 1/(1 + q)$ and $s = 1/(1 - q)$, which yields both assertions. $\square$

The next propsition shows that the seemingly rough smallness condition (5.2) gives a precise equality for $\omega_0(\mathcal{A}_L)$ if $\eta$ is decreasing, i.e., the measure $-d\eta$ is positive. (See also [18, Thm.2] for (1.2) with $A = A_1 = A_2$.) In fact, we even obtain $\omega_0(\mathcal{A}_L) = s(\mathcal{A}_L)$. We point out that standard results on positive semigroups, see e.g. [20], are not applicable here since we require no order properties for $X$ and $T(t)$; and even if we did so: $Y$ and $X_1$ would not be Banach lattices in most applications. Using $\eta \in BV_0([-r, 0])$, one checks that the function $\mathbb{R} \to \mathbb{R}_+$, $a \mapsto -\widehat{d\eta}(a)$, is strictly decreasing and bijective if $-d\eta \geq 0$ and $d\eta \neq 0$.

**Proposition 5.5.** *Let $X$ be a Hilbert space, $\eta \in BV_0([-r, 0])$, and $A$ be sectorial with $d \leq 0$, $\phi = \pi$, and $X_1 \neq X$. Assume that $-d\eta$ is a positive measure and not equal to 0. Then exists a unique $\omega \in \mathbb{R}$ such that $\widehat{d\eta}(\omega) = -1$. Denote by $a_0$ the infimum of the numbers $a > \omega$ such that $\alpha(1 + \widehat{d\eta}(\alpha))^{-1} > s(A)$ for all $\alpha \geq a$. Then $s(\mathcal{A}_L) = \omega_0(\mathcal{A}_L) = \max\{\omega, a_0\}$.*

*Proof.* Let $\operatorname{Re}\lambda \geq a > \max\{\omega, a_0\}$. Then

$$|\widehat{d\eta}(\lambda)| \leq -\int_{-r}^{0} e^{a\theta}\, d\eta(\theta) =: q < -\widehat{d\eta}(\omega) = 1.$$

Thus condition (5.2) holds and

$$\frac{a}{1-q} = \frac{a}{1+\widehat{d\eta}(a)} > s(A).$$

Corollary 5.4(b) now yields $\omega_0(\mathcal{A}_L) \le \max\{\omega, a_0\}$. On the other hand, $\omega - A - L_\omega = \omega$ since $\widehat{d\eta}(\omega) = -1$. Observe that $a_0(1+\widehat{d\eta}(a_0))^{-1} = s(A)$ if $a_0 > \omega$, so that in this case

$$a_0 - A - L_{a_0} = (1+\widehat{d\eta}(a_0))(s(A) - A).$$

This shows that $H(\omega)$ and $H(a_0)$ either do not exist or do not belong to $\mathcal{L}(X, X_1)$, so that $\max\{\omega, a_0\} \le s(\mathcal{A}_L) \le \omega_0(\mathcal{A}_L)$ by Proposition 4.3. □

**Example 5.6.** Consider equation (1.1) on a Hilbert space $X$ with $\beta \in \mathbb{R} \setminus \{0\}$, i.e., $\eta = -\beta \mathbb{1}_{\{-r\}}$, $d\eta = \beta\,\delta_{-r}$, and $\widehat{d\eta}(\lambda) = \beta e^{-\lambda r}$. Assume that $A$ is sectorial with $\phi = \pi$ and $X \ne X_1$. We denote by $a_0$ (resp., $a_1$) the infimum of the numbers $a > \frac{1}{r}\log|\beta|$ such that $\alpha(1+\beta e^{-\alpha r})^{-1} > s(A)$ (resp., $\alpha(1+|\beta|e^{-\alpha r})^{-1} > s(A)$) for $\alpha \ge a$.

(a) Let $d \le 0$ and $|\beta| \ge 1$. Then $s(\mathcal{A}_L) = \omega_0(\mathcal{A}_L) = \frac{1}{r}\log|\beta|$. In fact, Proposition 5.5 shows these equalities if $\beta \le -1$. If $\beta \ge 1$, then $|\widehat{d\eta}(\lambda)| \le c_a < 1$ for $\operatorname{Re}\lambda \ge a > \frac{1}{r}\log\beta$. Thus, $\omega_0(\mathcal{A}_L) \le \frac{1}{r}\log\beta$ by Proposition 5.1. On the other hand, $\widehat{d\eta}(\frac{1}{r}(i\pi + \log\beta)) = -1$ so that $s(\mathcal{A}_L) \ge \frac{1}{r}\log\beta$ by Proposition 4.3.

(b) Let $d \le 0$ and $-1 < \beta < 0$. Then Proposition 5.5 directly implies that $s(\mathcal{A}_L) = \omega_0(\mathcal{A}_L) = \max\{\frac{1}{r}\log|\beta|, a_0\}$.

(c) If $d \ge 0$, then $\omega_0(\mathcal{A}_L) \le \max\{a_1, \frac{1}{r}\log|\beta|\}$ by Corollary 5.4. Further, if $\beta > 0$ we have $s(\mathcal{A}_L) = \omega_0(\mathcal{A}_L) = \max\{a_1, \frac{1}{r}\log\beta\}$ since $\omega - A - L_\omega = \omega$ for $\omega := \frac{1}{r}\log\beta$ and

$$a_1 - A - L_{a_1} = (1+\beta e^{-a_1 r})(s(A) - A) \qquad \text{if } \; a_1 > \omega.$$

We now turn our attention to exponential dichotomy. For simplicity we concentrate on the case $\phi = \pi$.

**Proposition 5.7.** *Let $X$ be a Hilbert space. Let $A$ be invertible and sectorial with angle $\phi = \pi$, and let $\eta \in BV_0([-r,0])$. If*

$$\overline{1+\widehat{d\eta}(i\mathbb{R})} \subseteq \Delta := \{\lambda \in \mathbb{C}\setminus\{0\} : |\arg\lambda| < \psi \text{ or } |\arg(-\lambda)| < \psi\} \tag{5.3}$$

*for some $\psi < \pi/2$, then $\mathcal{T}_L$ has an exponential dichotomy. Condition* (5.3) *holds in particular, if* $\sup_{s\in\mathbb{R}} |\widehat{d\eta}(is)| < 1$.

*Proof.* From (5.3) we deduce that $|1+\widehat{d\eta}(is)|^{-1} \le c$ and that $\mu(is)$ belongs to the rotated double sector $i\Delta$, for $s \in \mathbb{R}$. This easily implies that

$$(w - A)H(is) = (1+\widehat{d\eta}(is))^{-1}(w-A)R(\mu(is), A)$$

is uniformly bounded for $s \in \mathbb{R}$. The assertion is then a consequence of Theorem 4.4. □

**Example 5.8.** Let $d\eta = \beta\delta_{-r}$ for some $\beta \in \mathbb{R}$.

(a) Let $A$ be sectorial with $\phi > \pi/2$. Then $\mathcal{A}_L$ does not have an exponential dichotomy if $|\beta| = 1$ or if $|\beta| > 1$ and $\sigma(A) \cap \mathbb{R}_-$ is unbounded. (This fact indeed holds for every Banach space $X$.)

(b) Let $|\beta| > 1$. Then there exists a self adjoint and negative definite operator $A$ on $X = \ell^2$ such that $i\mathbb{R}$ belongs to $\rho(\mathcal{A}_L)$.

In view of (a), the semigroup $\mathcal{T}_L$ obtained in (b) violates the spectral mapping theorem (2.7). Nevertheless, it satisfies $\omega_0(\mathcal{A}_L) = s(\mathcal{A}_L)$ by Example 5.6. We prove (a) and (b).

(a) If $|\beta| = 1$, then there is $\tau \in \mathbb{R}$ such that $\beta e^{-i\tau r} = -1$. Hence, $i\tau - A - L_{i\tau} = i\tau$ and $i\tau \in \sigma(\mathcal{A}_L)$ by Proposition 4.3. So (2.6) shows that $\mathcal{T}_L$ does not have an exponential dichotomy. Let $|\beta| > 1$. Then $1 + \beta e^{-i\tau r} \neq 0$ and

$$\frac{i\tau}{1+\beta e^{-i\tau r}} = \frac{-\tau\beta\sin(\tau r) + i\tau(1+\beta\cos(\tau r))}{|1+\beta e^{-i\tau r}|^2}$$

for all $\tau \in \mathbb{R}$. Since $|\beta| > 1$, there are $\tau_n = \tau_0 + 2\pi n/r$ such that $n \in \mathbb{N}_0$, $\tau_0 \in (0, 2\pi/r)$, $\cos\tau_n r = -1/\beta$, and $c_1 = \beta\sin\tau_0 r > 0$. Setting $c_2 = |1+\beta e^{-i\tau_0 r}|^{-1}$, we obtain

$$\mu_n := \frac{i\tau_n}{1+\beta e^{-i\tau_n r}} = -c_1\, c_2^2\,(\tau_0 + 2n\pi). \tag{5.4}$$

If $\mu_n \in \sigma(A)$ for some $n \in \mathbb{N}$, then $i\tau_n \in \sigma(\mathcal{A}_L)$ due to Proposition 4.3, and again $\mathcal{T}_L$ does not have an exponential dichotomy by (2.6). Otherwise, there exists the operator

$$H(i\tau_n) = (1+\beta e^{-i\tau_0 r})^{-1} R(\mu_n, A).$$

For sufficiently large $n$ we obtain

$$\begin{aligned}\|H(i\tau_n)\|_{\mathcal{L}(X,X_1)} &= c_2\,\|(w-A)R(\mu_n,A)\|_{\mathcal{L}(X)} \geq c_2\,(\|(w-\mu_n)R(\mu_n,A)\| - 1)\\ &\geq c_2\Big(\frac{|\mu_n - w|}{d(\mu_n,\sigma(A))} - 1\Big).\end{aligned}$$

Now take a sequence $\lambda_k \in \sigma(A)\cap\mathbb{R}$ tending to $-\infty$. We always find an index $n_k$ such that $|\lambda_k - \mu_{n_k}| \leq 2c_1c_2^2\pi$ (at least if $k$ is sufficiently large). This shows that $\|H(i\tau_n)\|_{\mathcal{L}(X,X_1)} \geq c\,n_k \to \infty$ as $k \to \infty$. Consequently, (a) holds due to Proposition 4.3 and (2.6).

(b) Observe that the numbers 0 and $\mu_n$ in (5.4) with $n \in \mathbb{Z}$ are the only values of $\mu(i\mathbb{R})$ on the real axis. Hence we only have to choose negative numbers $a_j \to -\infty$ different from $\mu_n$ and to take the diagonal operator $A = (a_j)$.

## References


[1] H. Amann, *Linear and Quasilinear Parabolic Problems. Volume 1: Abstract Linear Theory,* Birkhäuser, 1995.

[2] A. Bátkai, S. Piazzera, *Semigroups for Delay Equations in $L^p$-Phase Spaces,* book manuscript, submitted.

[3] A. Bátkai, S. Piazzera, *A semigroup method for delay equations with relatively bounded operators in the delay term*, Semigroup Forum **64** (2002), 71–89.

[4] C. Chicone, Y. Latushkin, *Evolution Semigroups in Dynamical Systems and Differential Equations,* Amer. Math. Soc., 1999.

[5] R. Denk, M. Hieber, J. Prüss, *R-Boundedness, Fourier Multipliers and Problems of Elliptic and Parabolic Type,* Memoirs Amer. Math. Soc. **166**, nr. 788 (2003).

[6] G. Di Blasio, *Delay differential equations with unbounded operators acting on delay terms,* Nonlinear Anal. **52** (2003), 1–18.

[7] G. Di Blasio, K. Kunisch, E. Sinestrari, *$L^2$–regularity for parabolic partial integro–differential equations with delay in the highest-order derivatives,* J. Math. Anal. Appl. **102** (1984), 38–57.

[8] G. Di Blasio, K. Kunisch, E. Sinestrari, *Retarded abstract equations in Hilbert spaces,* in: "Infinite-dimensional systems" (Proceedings Retzhof 1983), Lecture Notes in Math. **1076**, Springer, 1984, pp. 71–77.

[9] G. Di Blasio, K. Kunisch, E. Sinestrari, *Stability for abstract linear functional differential equations,* Israel J. Math. **50** (1985), 231–263.

[10] G. Dore, *$L^p$ regularity for abstract differential equations,* in: H. Komatsu (Ed.), "Functional Analysis and Related Topics" (Proceedings Kyoto 1991), Lecture Notes in Math. **1540**, Springer–Verlag, 1993, pp. 25–38.

[11] K.J. Engel, R. Nagel, *One–Parameter Semigroups for Linear Evolution Equations,* Springer–Verlag, 2000.

[12] M. Girardi, L. Weis, *Operator–valued Fourier multiplier theorems on Besov spaces,* Math. Nachr. **251** (2003), 34–51.

[13] I. Gohberg, S. Goldberg, M.A. Kaashoek, *Classes of Linear Operators Vol. I,* Birkhäuser, 1990.

[14] G. Gühring, F. Räbiger, R. Schnaubelt, *A characteristic equation for non–autonomous partial functional differential equations,* J. Differential Equations **181** (2002), 439–462.

[15] G.H. Hardy, J.E. Littlewood, G. Pòlya, *Inequalities,* Cambridge University Press, Reprint of 2nd Edition, 1999.

[16] J.-M. Jeong, S. Nakagiri, H. Tanabe, *Sturctural operators and semigroups associated with functional differential equations in Hilbert spaces,* Osaka J. Math. **30** (1993), 365–395.

[17] M. Mastinšek, *Norm continuity and stability for a functional differential equation in a Hilbert space,* J. Math. Anal. Appl. **269** (2002), 770–783.

[18] M. Mastinšek, *Stability conditions for abstract functional differential equations in a Hilbert space,* Semigroup Forum **66** (2003), 140–150.

[19] S. Nakagiri, H. Tanabe, *Structural operators and eigenmanifold decompositions for functional differential equations in Hilbert spaces,* J. Math. Anal. Appl. **204** (1996), 554–581.

[20] J.M.A.M. v. Neerven, *The Asymptotic Behaviour of Semigroups of Linear Operators,* Oper. Theory Adv. Appl. **88**, Birkhäuser, 1996.

[21] J. Prüss, *Evolutionary Integral Equations and Applications,* Birkhäuser, 1993.

[22] R. Schnaubelt, *Asymptotically autonomous parabolic evolution equations,* J. Evol. Equ. **1** (2001), 19-37.

[23] R. Schnaubelt, *Feedbacks for non–autonomous regular linear systems,* SIAM J. Control Optim. **41** (2002), 1141–1165.

[24] O.J. Staffans, *Well–posedness and stabilizability of a viscoelastic equation in energy space.* Trans. Amer. Math. Soc. **345** (1994), 527–575.

[25] O.J. Staffans, *Well–Posed Linear Systems Part I:General Theory,* book manuscript dated August 1, 2003; available at: http://www.abo.fi/˜staffans.

[26] H. Tanabe, *Functional Analytic Methods for Partial Differential Equations,* Marcel Dekker, 1997.

[27] L. Weis, *Operator–valued Fourier multiplier theorems and maximal $L^p$-regularity,* Math. Ann. **319** (2001), 735–758.

[28] G. Weiss, *Admissible observation operators for linear semigroups,* Israel J. Math. **65** (1989), 17–43.

[29] G. Weiss, *Admissibility of unbounded control operators,* SIAM J. Control Optim. **27** (1989), 527–545.

[30] G. Weiss, *The representation of regular linear systems on Hilbert spaces,* in: F. Kappel, K. Kunisch, W. Schappacher (Eds.), "Control and Estimation of Distributed Parameter Systems" (Proceedings Vorau 1988), Birkhäuser, 1989, pp. 401–416.

[31] G. Weiss, *Regular linear systems with feedback,* Math. Control Signals Systems **7** (1994), 23–57.

[32] J. Wu, *Theory and Applications of Partial Functional Differential Equations,* Springer–Verlag, 1996.

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