TIME–DEPENDENT PARABOLIC PROBLEMS ON NON–CYLINDRICAL DOMAINS WITH INHOMOGENEOUS BOUNDARY CONDITIONS

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To the memory of Ralph Phillips, inspiring mathematician and human being, collaborator and friend

Abstract. We study the relationship between the Dirichlet problem and the Cauchy problem with inhomogeneous boundary conditions for local operators. Our results are applied to non–autonomous parabolic problems on non–cylindrical domains.

1. Introduction

The results in this paper on the parabolic problems mentioned in the title are obtained by interconnecting semigroup theory (such as the Lumer–Phillips theorem and the mean ergodic theorem), local operator theory, and methods from potential theory (such as parabolic maximum principles and barriers). The results of Section 2 on the relationship between the Dirichlet and the Cauchy problem for a given operator A, which we apply in Section 3, permit now to complete extensive earlier research in [17] and [22] where the solvability of Cauchy problems with homogeneous boundary conditions was derived from a barrier condition. Combining our present work with the previous one (and classical PDE results such as in [15], [16]), we reach a quite general theory covering a wide class of parabolic problems, which we give here in full extent (see Theorems 3.3, 3.4 and Propositions 3.5, 3.6). Since what is needed from earlier work is just recalled, stated with references when and where needed, we keep the present paper rather short and essentially self–contained. We treat several applications in classical context, cf. (3.2) and (3.4), extending in particular a recent result on autonomous problems by W. Arendt and P. Bénilan from [4], see Theorem 3.9 (also in this connection Example 3.7 and Remark 3.8), and obtaining the regularity result Theorem 3.10.

It may be useful to say that the generality of our approach allows to deal with parabolic partial differential equations in non–divergence form on non–cylindrical domains with merely continuous coefficients, partial differential operators which may be singular or degenerate at the boundary, and little regularity (in the PDE sense) for boundaries (compare e.g. with [6], [9], [10], [15], and the more recent references [1], [5], [14], [16]). Also in that context the (potential theoretic) regularity of the boundary (thus the solvability) is not determined only by the geometry of the domain but also by the partial differential operator involved. All this is covered by our barrier condition. (We discuss some of these points in more detail, and give some relevant examples, in Section 4.)

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To be more specific, let us recall two results for elliptic partial differential operators A on a bounded open subset V of $\mathbb{R}^N$. First, Arendt and Bénilan showed that if A is in divergence form and uniformly elliptic, then the Dirichlet problem

$$Au = 0 \quad \text{on} \quad V, \qquad u = \varphi \quad \text{on} \quad \partial V, \tag{1.1}$$

has a unique solution $u \in D(A) \cap C(\overline{V})$ for each $\varphi \in C(\partial V)$ if and only if the part A_0 of A in $C_0(V)$ generates a semigroup (i.e., the Cauchy problem for A on V with zero boundary conditions is well–posed), see [4, Thm.4.1] combined with [4, Thm.4.10]. Second, there are degenerate elliptic operators A such that A_0 is a generator but (1.1) cannot be solved for all $\varphi \in C(\partial V)$, see Example 2.8. For such (regular or degenerate) and other (elliptic or parabolic) situations, we are thus looking for conditions on A ensuring that the generator property of A_0 is equivalent to the solvability of (1.1).

This is achieved in Section 2 in the following quite general setting. Let V be a bounded open subset of a locally compact metric space Ω and A be a closed, dissipative, linear operator on $C(V)$. Following [18], we define on $C(\overline{V})$ the operator A_1 with Ventcel boundary conditions, cf. (2.1). Then the Dirichlet problem (1.1) has a unique solution $u \in D(A) \cap C(\overline{V})$ for each $\varphi \in C(\partial V)$ provided that A_0 is invertible on $C_0(V)$ and A_1 is densely defined on $C(\overline{V})$, see Corollary 2.7. In fact, these properties characterize the solvability of (1.1) (under a mild extra assumption) by Theorem 2.13. These results are formulated and proved in the framework of semigroup theory. Different semigroup approaches to autonomous inhomogeneous initial boundary value problem are developed in e.g. [3], [4], [12].

In Proposition 3.6 we give a simple sufficient condition for the density of $D(A_1)$ in terms of a regularity property of A. This yields a new proof of the above mentioned characterization from [4] for $A = \Delta$ (in fact, for uniformly elliptic operators in non–divergence form), see Example 3.7. Another application to a problem of singular multiplicative perturbation is exposed in Proposition 3.1.

In a next step, these results on the abstract Dirichlet problem (1.1) are applied in Theorem 3.3 to the non–autonomous parabolic initial boundary value problem[1]

$$Lu = F \quad \text{on} \quad \underline{V}_0, \qquad u = f \quad \text{on} \quad \Gamma \cup V(0), \tag{1.2}$$

on a non–cylindrical domain $\underline{V}_0 \subseteq (0,T] \times \Omega$ with lateral boundary Γ and bottom $V(0)$. So we can solve (1.2) for continuous F and f if L_0 is invertible (or, equivalently, a generator) on $C_0(\underline{V}_0)$ and L_1 is densely defined on $C(\overline{\underline{V}_0})$ (where L_k are defined as A_k above). In Section 3 we describe the class of operators L considered in (1.2), but here we note that it contains parabolic partial differential operators L under the conditions mentioned in the second paragraph above, where Ω can be an open subset of $\mathbb{R}^N$ or a network, see [22, §6]. Using Proposition 3.6, we finally show in the non–degenerate case that (1.2) is solvable if and only if L_0 is invertible, see Theorem 3.9 which extends [4].

In order to apply these theorems to concrete partial differential equations, one needs of course a criterion for the invertibility of L_0. Here we can rely on the theory developed in [17], [19], and [22]. We remark that L_0 is a generator if and only if the homogeneous problem (1.2) with $F = 0$ and $f = 0$ on Γ is 'well–posed', see [22, Thm.4.19] for a precise

[1] One can consider (1.2) as a parabolic Dirichlet problem. Correspondingly our Cauchy barriers (CB) may be regarded as parabolic barriers as opposed to the classcial barriers for elliptic Dirichlet problems.

statement. But more importantly, we can characterize the generator property of L_0 by the barrier condition (CB) for L and $\underline{V}_0$ under some mild additional assumptions which are easy to check in the applications, see Theorem 3.4. In Example 4.2 we construct Cauchy barriers for several non–cylindrical domains.

2. Dirichlet and Cauchy problems

Let Ω be a locally compact Hausdorff space having a countable base, Ω^* be its one–point compactification, and $V \neq \emptyset$ be a fixed open subset of Ω. We denote the closure and boundary of V in Ω^* by $\overline{V^*}$ and ∂V^*, respectively. Let $C(V)$ be endowed with the topology of uniform convergence on compact subsets of V. We also use the Banach space $C(\overline{V^*})$ equipped with the sup–norm $\|\cdot\|$ and its closed subspace $C_0(V)$ of functions vanishing on ∂V^*. We identify $C(\overline{V^*})$ with a subspace of $C(V)$ by restriction. Let $A : D(A) \subseteq C(V) \to C(V)$ be a linear operator. Our basic assumption reads as follows.

(H1) A is closed in $C(V)$. If $x \in V$ and $f \in D(A)$ with $|f(x)| = \|f\|$, then $\mathrm{Re}[(Af)(x)\,\overline{f(x)}] \leq 0$.

This hypothesis will lead to closedness and dissipativity of the operators

$$\begin{aligned} D(A_0) &= \{f \in D(A) \cap C_0(V) : Af \in C_0(V)\}, \quad A_0 f = Af, \\ D(A_1) &= \{f \in D(A) \cap C(\overline{V^*}) : Af \in C_0(V)\}, \quad A_1 f = Af. \end{aligned} \tag{2.1}$$

Throughout A_0 and A_1 are considered as operators on $C_0(V)$ and $C(\overline{V^*})$, respectively.

Lemma 2.1. *Let (H1) hold. Then A_0 and A_1 are closed and dissipative in $C_0(V)$ and $C(\overline{V^*})$, respectively.*

Proof. We only consider A_1, the operator A_0 can be treated in the same way. Let $D(A_1) \ni f_n \to f$ and $A_1 f_n \to g$ in $C(\overline{V^*})$. Since A is closed in $C(V)$, f belongs to $D(A)$ and $Af = g \in C_0(V)$ so that A_1 is closed in $C(\overline{V^*})$. Given $f \in D(A_1)$, there is $x \in \overline{V^*}$ such that $\|f\| = |f(x)|$. Set $\phi = \overline{f(x)}\,\delta_x \in C(\overline{V^*})'$. Then $\|\phi\|^2 = \|f\|^2 = \phi(f)$ and the quantity

$$\mathrm{Re}\langle A_1 f, \phi\rangle = \begin{cases} 0 & \text{if } x \in \partial V^*, \\ \mathrm{Re}[Af(x)\,\overline{f(x)}] & \text{if } x \in V, \end{cases}$$

is negative due to (H1). Thus A_1 is dissipative. □

The resolvent set of a linear operator B is denoted by $\rho(B)$ and its resolvent by $R(\lambda, B)$.

Lemma 2.2. *Let (H1) hold and $0 < \lambda \in \rho(A_1)$. Then $\lambda \in \rho(A_0)$ and $R(\lambda, A_1)f = R(\lambda, A_0)f$ for $f \in C_0(V)$.*

Proof. Given $g \in C_0(V)$, set $f = R(\lambda, A_1)g \in D(A_1)$. Then $\lambda f = A_1 f + g \in C_0(V)$ so that $f \in D(A_0)$ and $(\lambda - A_0)f = g$. Lemma 2.1 now yields the assertion. □

The above results allow to relate the generation properties of A_0 and A_1. Example 2.8 below shows that the density of $D(A_1)$ cannot be omitted in assertions (a) and (c) of the next theorem, in general.

Theorem 2.3. *If (H1) holds, the following assertions are equivalent.*

(a) *$(\lambda - A_0)D(A_0)$ is dense in $C_0(V)$ for some $\lambda > 0$ and $D(A_1)$ is dense in $C(\overline{V^*})$.*

(b) A_1 *generates a* C_0*–semigroup* $T_1(\cdot)$ *on* $C(\overline{V^*})$.

(c) A_0 *generates a* C_0*–semigroup* $T_0(\cdot)$ *on* $C_0(V)$ *and* $D(A_1)$ *is dense in* $C(\overline{V^*})$.

If this is the case, $T_0(t)$ *and* $T_1(t)$ *are contractions,* $T_1(t)f = T_0(t)f$ *for* $f \in C_0(V)$ *and* $(T_1(t)f)(x) = f(x)$ *for* $x \in \partial V^*$, $f \in C(\overline{V^*})$, *and* $t \geq 0$.

Proof. (a)⇒(b): If A_1 were not a generator, then there would exist a finite Borel measure $\mu \neq 0$ on $\overline{V^*}$ with

$$\int_{\overline{V^*}} (\lambda f - A_1 f)\, d\mu = 0 \qquad \text{for all } \ f \in D(A_1)$$

because of Lemma 2.1 and the Lumer–Phillips theorem. Using (a), we see that μ must vanish on V. Thus

$$0 = \int_{\partial V^*} (\lambda f - A_1 f)\, d\mu = \int_{\partial V^*} \lambda f\, d\mu$$

for all $f \in D(A_1)$. The second condition in (a) now implies $\mu = 0$ so that (b) is verified.

(b)⇒(c): In view of Lemma 2.1 and 2.2 and the Lumer–Phillips theorem, one only has to show that $D(A_0)$ is dense in $C_0(V)$. This fact follows from

$$\lambda R(\lambda, A_0) f = \lambda R(\lambda, A_1) f \to f$$

as $\lambda \to \infty$ for $f \in C_0(V)$.

The implication '(c)⇒(a)' is clear. By Lemma 2.1, $T_0(t)$ and $T_1(t)$ are contractions, and by Lemma 2.2 both coincide on $C_0(V)$. The identity

$$T_1(t)f - f = A_1 \int_0^t T_1(s) f\, ds \ \in \ C_0(V), \qquad f \in C(\overline{V^*}),\ t \geq 0,$$

establishes the last assertion. □

As an immediate consequence we can solve the Cauchy problem for A with inhomogeneous boundary conditions. In [3, Thm.6.5] the case of uniformly elliptic partial differential operators A in divergence form was treated allowing for time depending boundary values. Non–autonomous extensions of the next result are given in Theorem 3.3 and 3.9.

Corollary 2.4. *Let (H1) hold and* $f \in D(A_1)$*. If (a)–(c) of Theorem 2.3 hold, then there is a unique* $u \in C^1(\mathbb{R}_+, C(\overline{V^*}))$ *such that* $u(t) \in D(A_1)$ *for* $t \geq 0$ *and*

$$\begin{aligned} \tfrac{d}{dt} u(t,\cdot) &= Au(t,\cdot), && t \geq 0, \quad \text{on } V, \\ u(t,x) &= f(x), && t \geq 0,\ x \in \partial V^*, \\ u(0,x) &= f(x), && x \in V. \end{aligned} \tag{2.2}$$

Proof. The function $u(t,x) = (T_1(t)f)(x)$ on $\mathbb{R}_+ \times \overline{V^*}$ solves (2.2) due to Theorem 2.3. The solution is unique since A_0 is a generator. □

We strengthen (H1) in order to obtain positive solutions:

(H2) A is closed in $C(V)$ and real (i.e., if $f \in D(A)$, then $\bar{f} \in D(A)$ and $A\bar{f} = \overline{Af}$). If $x \in V$ and $f \in D(A)$ is real–valued with $0 < f(x) = \sup_V f$, then $(Af)(x) \leq 0$.

As in [22, Cor.2.10] one proves that (H2) implies (H1). The converse can be shown if A is given by a real local operator that satisfies (H1) on all open subsets of V, see [22, Cor.2.10] and the definitions given in the next section. The following result can easily be deduced from R.S. Phillips' characterization of generators of positive contraction semigroups [25, Thm.2.1] (compare the proof of Lemma 2.1).

Proposition 2.5. *If (H2) holds and A_1 is a generator, then $T_1(\cdot)$ is positive. Thus the solution of* (2.2) *is positive if $f \geq 0$.*

The results established so far are essentially stated in [18] without proof. In this work as well as in [3] further results were derived assuming that the Dirichlet problem (1.1) is solvable. Here we are looking for conditions on A_0 and A_1 characterizing the solvability of (1.1). Our main tool is the mean ergodic theorem which we apply to $T_1(\cdot)$.

Theorem 2.6. *Assume that (H1) holds, A_0 is invertible on $C_0(V)$, and A_1 is densely defined on $C(\overline{V^*})$. Then $T_1(\cdot)$ is mean ergodic with projection P onto $\ker A_1 = \operatorname{fix} T_1(\cdot)$. Moreover, $\|P\| \leq 1$ and $Pf(x) = f(x)$ for $x \in \partial V^*$ and $f \in C(\overline{V^*})$. If in addition (H2) holds, then P is positive and $\|T_1(t) - P\| \leq Me^{-\varepsilon t}$ for $t \geq 0$ and constants $M, \varepsilon > 0$.*

Proof. By Theorem 2.3 the operators A_k generate the contraction semigroups $T_k(\cdot)$, $k = 0, 1$. For $\lambda > 0$ and $f \in D(A_1)$, we obtain

$$\lambda R(\lambda, A_1)f = R(\lambda, A_1)A_1 f + f = R(\lambda, A_0)A_1 f + f \\ \to -A_0^{-1}A_1 f + f \tag{2.3}$$

as $\lambda \to 0$ due to Lemma 2.2. The mean ergodicity of $T_1(\cdot)$ follows from (2.3) and [7, Thm.5.1]. The other assertions except for the convergence of $T_1(\cdot)$ are straightforward consequences of Theorem 2.3 and Proposition 2.5. Let (H2) hold. Note that $T_0(\cdot)$ is positive and thus uniformly exponentially stable by [23, Thm.B-IV.1.4]. We now deduce

$$\|T_1(t)f - Pf\| = \|T_0(t)(f - Pf)\| \leq 2Ne^{-\varepsilon t}\,\|f\|$$

from $T_1(t)Pf = Pf$, for $t \geq 0$, $f \in C(\overline{V^*})$, and some constants $N, \varepsilon > 0$. ☐

Observe that $Pf = f - A_0^{-1}A_1 f$ for $f \in D(A_1)$ due to (2.3).

Corollary 2.7. *Assume that (H1) holds, A_0 is invertible, and A_1 is densely defined. Then for each $\varphi \in C(\partial V^*)$ there exists a unique solution $u \in C(\overline{V^*}) \cap D(A)$ of*

$$Au = 0 \quad \text{on } V, \qquad u|\partial V^* = \varphi. \tag{2.4}$$

If (H2) holds and $\varphi \geq 0$, then $u \geq 0$.

Proof. The function $u = Pf$ solves (2.4), where P is given by Theorem 2.6 and f is any extension of φ to $C(\overline{V^*})$. If u satisfies (2.4) for $\varphi = 0$, then $u \in \ker A_0$ so that $u = 0$. ☐

We refer to [11] for a detailed account on the Dirichlet problem for elliptic operators. The following two examples show that Theorem 2.6 and Corollary 2.7 are not valid if one only assumes that A_0 is an invertible generator or that A_1 is a generator and A_0 is injective.

Example 2.8. For $V = (0,1)^2$ and $\Omega = \mathbb{R}^2$, we define

$$Af(x,y) = \partial_{xx} f(x,y) + y^2\,\partial_{yy} f(x,y) - f(x,y)$$

with $D(A) = \{f \in C(V) \cap W^2_{p,loc}(V) : Af \in C(V)\}$ for one (and hence all) $p > 2$. It can be checked that (H2) holds, cf. [24]. Clearly, A_0 is densely defined. Further, the function

$$h : V \to \mathbb{R}_+; \quad h(x, y) = x(1 - x)(1 - y)\sqrt{y}$$

is a Cauchy barrier in the sense of (CB) below. Therefore A_0 generates a contraction semigroup on $C_0(V)$ due to [24, Thm.1] or Theorem 3.4. The operator A_0 is invertible since $A_0 + 1$ is a dissipative generator by the same reasons. However, following the arguments of [11, §6.6, p.116], one shows that the Dirichlet problem corresponding to A cannot be solved for $\varphi \in C(\partial V)$ vanishing for $x = 0, 1$ but not for $y = 0$. Therefore the operator A_1 is not densely defined on $C(\overline{V})$ due to the above corollary.

Example 2.9. Let $\Omega = (0, \infty)$, $V = (1, \infty)$, and $Af = f''$ with $D(A) = C^2(V)$. Then (H2) holds and A_0 is an injective generator on $C_0(V)$. Further, $D(A_1)$ is dense in $C([1, \infty])$. Hence, A_1 is a generator by Theorem 2.3. However, (2.4) has no solution for φ with $\varphi(0) \neq \varphi(\infty)$, and so $T_1(\cdot)$ is not mean ergodic in view of the proof of Corollary 2.7.

Concluding this section, we want to characterize the Dirichlet regularity of V in terms of A_0 and A_1. Here we use the following notion, cf. [17].

Definition 2.10. *The set V is called* weakly (A–)Dirichlet regular *if* (2.4) *has a solution for each $\varphi \in C(\partial V^*)$ and* (A–)Dirichlet regular *if, in addition, the solution is unique.*

Lemma 2.11. *If V is weakly Dirichlet regular and $D(A_0)$ is dense in $C_0(V)$, then $D(A_1)$ is dense in $C(\overline{V^*})$.*

Proof. For $f \in C(\overline{V^*})$, there is $v \in C(\overline{V^*}) \cap D(A)$ such that $Av = 0$ on V and $v = f$ on ∂V^*. By assumption, there exist $u_n \in D(A_0)$ converging to $f - v$ in $C_0(V)$. Hence, $u_n + v \in D(A_1)$ tends to f in $C(\overline{V^*})$. □

Lemma 2.12. *If V is weakly Dirichlet regular and $C_0(V) \subseteq A[D(A) \cap C(\overline{V^*})]$, then A_0 is surjective.*

Proof. For $g \in C_0(V)$, there is $u \in D(A) \cap C(\overline{V^*})$ with $Au = g$. By assumption, there exists $v \in D(A) \cap C(\overline{V^*})$ such that $Av = 0$ and $u = v$ on ∂V^*. Then $f = u - v$ belongs to $D(A_0)$ and $Af = Au = g$. □

The range condition used above holds if A is a local operator on Ω such that A is invertible on $C_0(\Omega)$, cf. [22, Prop.3.4] and the next section.

Theorem 2.13. *Assume that (H1) holds, $D(A_0)$ is dense in $C_0(V)$, and $C_0(V) \subseteq A[D(A) \cap C(\overline{V^*})]$. Then the following assertions are equivalent.*

(a) *V is A–Dirichlet regular.*
(b) *A_0 is invertible on $C_0(V)$ and $D(A_1)$ is dense in $C(\overline{V^*})$.*
(c) *A_1 generates a mean ergodic semigroup on $C(\overline{V^*})$.*

Proof. Assertion (b) follows from (a) in view of Lemmas 2.1, 2.11, 2.12, and the uniqueness of (2.4). The implication '(b)⇒(c)' was shown in Theorem 2.6. If (c) holds, then V is weakly Dirichlet regular due to the proof of Corollary 2.7. The operator A_0 is thus surjective by Lemma 2.12. This implies the injectivity of A_0 since $T_0(\cdot)$ is also mean ergodic, cf. [7, Thm.5.1]. Therefore solutions of (2.4) are unique. □

3. Parabolic problems on non–cylindrical domains

In this section we make use of the theory of local operators, see [17], [22], and the references therein. A *local operator* A is a collection of linear operators $A^V : D(A, V) \subseteq C(V) \to C(V)$ such that, for $f \in D(A, V)$ and $W \subseteq V$, one has $f|W \in D(A, W)$ and $(A^V f)|W = A^W(f|W)$, where W and V belong to the set $\mathcal{O}(\Omega)$ of non–empty open subsets of Ω. Usually we omit the superscript V. We introduce further notions which are explained below in the context of partial differential operators.

(a) A is *locally closed u.c.* if, for $f, g \in C(V)$, the existence of $\mathcal{O}(\Omega) \ni V_n \uparrow V$ and $f_n \in D(A, V_n)$ such that $f_n \to f$ and $Af_n \to g$ uniformly on compact subsets of V implies that $f \in D(A, V)$ and $A^V f = g$.

(b) A is *locally dissipative* if, for all compact $K \subset V$ and $f \in D(A, V)$ with $\sup_{V\setminus K} |f| < \sup_V |f|$, there is $x_0 \in V$ such that $|f(x_0)| = \sup_V |f|$ and $\mathrm{Re}[(Af)(x_0)\,\overline{f(x_0)}] \le 0$.

(c) A is *real* if $f \in D(A, V)$ implies that $\overline{f} \in D(A, V)$ and $A\overline{f} = \overline{Af}$ on V.

Here $V \in \mathcal{O}(\Omega)$. If we assume that (a)–(c) hold and that

(S) for V contained in a base of $\mathcal{O}(\Omega)$ the part A_V of A^V in $C_0(V)$ is densely defined,[2]

then Lemma 2.9 and 2.10 of [22] show that $A = A^V$ satisfies (H2) (and (H1) if A is not supposed to be real). In view of Section 2 we define the operator

$$\tilde{A}_V f = \begin{cases} Af & \text{on } V, \\ 0 & \text{on } \partial V^*, \end{cases} \qquad D(\tilde{A}_V) = \{f \in C(\overline{V^*}) \cap D(A, V) : Af \in C_0(V_0)\},$$

on $C(\overline{V^*})$. We write $A_0 = A_V$ and $A_1 = \tilde{A}_V$ if there is no danger of confusion.

The results in the previous section allow to treat a problem of degenerate multiplicative perturbation. For a local operator A and $p \in C(\Omega)$ we define the local operator $(pA)f = pAf$ with domain $D(pA, V) = D(A, V)$, cf. [17, §6].

Proposition 3.1. *Assume that A is a locally dissipative, locally closed u.c., local operator satisfying (S) on $\mathcal{O}(\Omega)$. Let $p \in C(\Omega)$ be bounded and non–negative and $V \in \mathcal{O}(\Omega)$ such that the set $W = V \setminus \{x \in V : p(x) = 0\}$ is dense in V. Further suppose that $\tilde{A}_V$ is densely defined on $C(\overline{V^*})$. Then $D(\widetilde{(pA)}_W)$ is dense in $C(\overline{W^*})$. Thus there exists a unique solution of* (2.2) *for pA on W if $(pA)_W$ is a generator on $C_0(W)$.*

Proof. Note that $(pA)^W$ satisfies (H1) on W. For $f \in C(\overline{W^*}) = C(\overline{V^*})$ take $f_n \in D(\tilde{A}_V)$ converging to f in sup–norm. Since $pAf_n \in C_0(W)$, the functions $f_n|W$ belong to $D(\widetilde{(pA)}_W)$. Corollary 2.4 now implies the second assertion. □

The above result can be applied to the following situation. (One can deal with more general examples in view of [17, Thm.6.4], see also [21].)

Example 3.2. Let $V = B_1$ be the open unit ball in $\mathbb{R}^N$, $W = B_1 \setminus \{0\}$, and $p(x) = |x|^2$. Let $A = \Delta$ be the local operator with domains $D(A, U) = \{f \in C(U) \cap W^2_{p,loc}(U) : Af = \Delta f \in C(U)\}$ for $U \in \mathcal{O}(\mathbb{R}^N)$ and one (and hence all) $p > N$. Then A satisfies the assumptions of Proposition 3.1, see [24], and $(pA)_W$ is a generator on $C_0(W)$ by e.g. the remarks after [17, Thm.6.10]. $D(\tilde{A}_V)$ is dense in $C(\overline{B_1})$ because of standard existence

[2] In [22] we used a somewhat weaker condition (S) instead of the present one.

results for the Dirichlet problem for the Laplacian and Lemma 2.11. Thus we can solve (2.2) for $r^2\Delta$ on $B_1 \setminus \{0\}$.

Turning to the parabolic case, we let $\underline{\Omega} = [S,T] \times \Omega$ and, for $\underline{V} \in \mathcal{O}(\underline{\Omega})$ and $t \in [S,T]$,

$$V(t) = \{x \in \Omega : (t,x) \in \underline{V}\}, \;\; \underline{V}(t) = \{t\} \times V(t), \;\; \underline{V}_t = \{(s,x) \in \underline{V} : s > t\},$$
$$\underline{V}_0 = \underline{V}_S, \;\; \underline{V}'_t = \{(s,x) \in \underline{V} : s \geq t\}, \;\; I_{\underline{V}} = \{t \in [S,T] : V(t) \neq \emptyset\}.$$

Further, $\overline{\underline{V}^*}$ and $\partial \underline{V}^*$ designate the closure and boundary of $\underline{V}$ in $\underline{\Omega}^* = [S,T] \times \Omega^*$, respectively. Observe that $\partial \underline{V}^*_0 = \partial \underline{V}^* \cup \underline{V}(S)$, [22, Prop.4.1]. We write $\underline{x} = (t,x)$ for a generic point of $\underline{\Omega}$ or $\underline{\Omega}^*$ and consider $C(\overline{\underline{V}^*})$ and $C(\underline{V}_0)$ as subspaces of $C(\underline{V})$.

A local operator L defined on $\mathcal{O}(\underline{\Omega})$ is called *parabolic* if, for $F \in D(L,\underline{V})$, $\underline{V} \in \mathcal{O}(\underline{\Omega})$, and $\varphi \in C^1(I_{\underline{V}})$, we have $\varphi F \in D(L,\underline{V})$ and $L(\varphi F) = \varphi\, LF - \varphi' F$.

Let $\underline{V} \in \mathcal{O}(\underline{\Omega})$ satisfy $V(t) \neq \emptyset$ for $t \in [S,T]$ (to avoid trivial situations). Take $F \in C(\overline{\underline{V}^*})$, $f \in C(\overline{V(S)^*})$, and $g \in C(\partial \underline{V}^*)$ with $g(S,x) = f(x)$ for $x \in \partial V(S)^*$. We are looking for solutions $u \in C(\overline{\underline{V}^*}) \cap D(L,\underline{V}_0)$ of the problem

$$\begin{aligned} Lu &= F \qquad \text{on } \underline{V}_0, \\ u &= g \qquad \text{on } \partial \underline{V}^*, \\ u(S) &= f. \end{aligned} \tag{3.1}$$

We first give our basic example for the above abstract setting: the parabolic differential operator

$$L(\underline{x}, D) = \sum_{k,l=1}^{N} a_{kl}(t,x)\partial_{kl} + \sum_{k=1}^{N} b_k(t,x)\partial_k + c(t,x) - \partial_t \tag{3.2}$$

on an open subset Ω of $\mathbb{R}^N$, where $\partial_t = \frac{\partial}{\partial t}$ and $\partial_k = \frac{\partial}{\partial x_k}$. The coefficients a_{kl},b_k, and $c \leq 0$ are supposed to be merely continuous on $\underline{\Omega}$, real–valued, and to satisfy

$$\sum_{k,l=1}^{N} a_{kl}(t,x)\, y_k\, y_l > 0 \tag{3.3}$$

for $y \in \mathbb{R}^N$ and $(t,x) \in \underline{\Omega}$. The operator may be singular or degenerate at the boundary. Set $\underline{\Omega}_{00} = (S,T) \times \Omega$ and $\underline{W}_{00} = \underline{W} \cap \underline{\Omega}_{00}$ for $\underline{W} \in \mathcal{O}(\underline{\Omega})$. As in [22, §6.1], we define

$$\begin{aligned} &D(L,\underline{W}_0) = \{F \in C(\underline{W}_0) \cap W^{1,2}_{p,loc}(\underline{W}_0) : L(\underline{x},D)F = G \text{ a.e. on } \underline{W}_{00} \text{ for } G \in C(\underline{W}_0)\}, \\ &\qquad LF = G \qquad \text{on } \underline{W}_0. \end{aligned} \tag{3.4}$$

Here, $\underline{W} \in \mathcal{O}(\underline{\Omega})$, $p > N+2$ is fixed, and $W^{1,2}_{p,loc}(\underline{W}_0)$ ($W^{1,2}_{p,loc}(\underline{W})$) denotes the space of functions F such that $F|\underline{U}_{00}$ belongs to the Sobolev space

$$W^{1,2}_p(\underline{U}_{00}) = \{F \in L^p(\underline{U}_{00}) : \partial_k F,\; \partial_{kl} F,\; \partial_t F \in L^p(\underline{U}_{00}),\; k,l = 1,\dots,N\}$$

for all relatively compact open subsets $\underline{U}$ of $\underline{W}_0$ (of $\underline{W}$).

In [22, §6.1] it is shown that $D(L,\underline{W}_0)$ does not depend on the choice of $p > N + 2$. It is rather obvious that L is a real, parabolic, local operator satisfying (S). The local dissipativity corresponds to the parabolic maximum principle on $W^{1,2}_p$. The local closedness u.c. is a straightforward consequence of the standard interior a priori estimates proved in [15, §IV.10]. This property essentially says that we have chosen the 'right' domains $D(L,\underline{W}_0)$ (otherwise one would have to replace L by its 'local closure' $\overline{L}$, see

[22, Thm.2.12]). We point out that in this case (3.1) is just a parabolic partial differential equation whose solution $u \in D(L, \underline{V}_0)$ possesses in the interior the regularity one can expect in view of the classical theory, cf. [15, §IV.9] or [16, Chap.7]. See [22, §6] for all this and for analogous problems on networks.

We come back to the general situation and state a straightforward consequence of the theory developed in the previous section.

Theorem 3.3. *Let L be a locally dissipative, locally closed u.c., parabolic, local operator defined on $\mathcal{O}(\underline{\Omega}_0)$ and satisfying (S), let $\underline{V} \in \mathcal{O}(\underline{\Omega})$ satisfy $V(t) \neq \emptyset$ for $t \in [S,T]$, and let $F \in C(\overline{\underline{V}^*})$, $f \in C(\overline{V(S)^*})$, and $g \in C(\partial \underline{V}^*)$ with $g(S,x) = f(x)$ for $x \in \partial V(S)^*$. Assume that $L_{\underline{V}_0}$ is a generator on $C_0(\underline{V}_0)$ and that $D(\tilde{L}_{\underline{V}_0})$ is dense in $C(\overline{\underline{V}^*})$. Then there is a unique solution $u \in C(\overline{\underline{V}^*}) \cap D(L, \underline{V}_0)$ of (3.1). If L is real and $-F, g, f \geq 0$, then $u \geq 0$.*

Proof. Uniqueness and positivity of solutions follow from the parabolic maximum principle [22, Thm.2.29, 2.30]. By Theorem 2.3, L_1 generates a contraction semigroup on $C(\overline{\underline{V}^*})$. Since $e^{tL_0} = 0$ for $t \geq T - S$ by [22, Thm.4.9], Theorem 2.6 provides us with a projection $P = P_{\underline{V}}$ mapping $C(\overline{\underline{V}^*})$ onto $\ker L_1$ such that $Pw(\underline{x}) = w(\underline{x})$ for $\underline{x} \in \partial \underline{V}_0^*$ and $w \in C(\overline{\underline{V}^*})$. Set $\hat{F}(t,x) = e^{-t} F(t,x)$, $\hat{v} = (L_1 - 1)^{-1} \hat{F}$, and $v(t,x) = e^t \hat{v}(t,x)$ on $\overline{\underline{V}^*}$. Then $v \in D(L, \underline{V}_0) \cap C(\overline{\underline{V}^*})$ and

$$Lv = -e^t \hat{v} + e^t L\hat{v} = -e^t \hat{v} + e^t (\hat{v} + \hat{F}) = F$$

on $\underline{V}_0$ since L is parabolic. Take $G \in C(\overline{\underline{V}^*})$ with $G(S) = f - v(S)$ on $\underline{V}(S)$ and $G = g - v$ on $\partial \underline{V}^*$. The function $u = v + PG$ clearly solves (3.1). □

As observed in the above proof, L_0 is invertible if it is generator, and the converse holds by virtue of the Lumer–Phillips theorem (under the assumptions of Theorem 3.3).

We can now involve our previous work in [17] and [22] on homogeneous initial boundary value problems for real, locally dissipative, locally closed u.c., local operators A defined on $\mathcal{O}(\Omega)$ and satisfying (S). (Here parabolicity is not needed so that we write A instead of L and $V \subseteq \Omega$ instead of $\underline{V} \subseteq \underline{\Omega}$.) We say that $V \in \mathcal{O}(\Omega)$ possesses a *Cauchy barrier* with respect to A if

(CB) there exists a compact subset K of V and a function $h \in D(A, V \setminus K)$ such that $h > 0$ and $(A - \lambda)h \leq 0$ on $V \setminus K$ for some $\lambda \geq 0$, and for all $\varepsilon > 0$ there is a compact set K_ε with $K \subseteq K_\varepsilon \subseteq V$ and $0 \leq h \leq \varepsilon$ on $V \setminus K_\varepsilon$.

It is not difficult to see that V admits a Cauchy barrier (with $K = \emptyset$) if A_V is a generator on $C_0(V)$, see [22, Prop.3.2] or [17, Thm.5.4]. Conversely, the existence of a Cauchy barrier implies that A_V is a generator if A satisfies additionally one of the following hypotheses.

(I) There are relatively compact $V_n \in \mathcal{O}(\Omega)$, $n \in \mathbb{N}$, such that $V_n \subseteq V_{n+1}$, $\bigcup_n V_n = V$, and A_{V_n} is a generator on $C_0(V_n)$. $D(A_V)$ is dense in $C_0(V)$.

(O) A_Ω is a generator on $C_0(\Omega)$. Let $W \in \mathcal{O}(\Omega)$ and $u \in D(A, W)$. If u has compact support, then $u^\# \in D(A, \Omega)$ and $A^\Omega u^\# = (A^W u)^\#$. If $x \in W$ and $u \geq 0$, then there is $0 \leq u_1 \in D(A, W)$ such that u_1 has compact support and $u = u_1$ on a neighbourhood of x in W.

(Here $u^\# \in C_0(\Omega)$ denotes the extension by 0 of $u \in C_0(W)$.) The first condition is used in [17, Thm.5.4] or [22, Thm.3.27] to approximate V from inside by the regular domains V_n. The second one gives us the semigroup e^{tA_Ω} on the larger domain Ω and allows us to localize it to the domain V. This was done in [22] and required considerable efforts involving the theory of Feller semigroups, so–called space–time semigroups, and localization methods from potential theory. In our application (3.2) and (3.4), assumption (O) can easily be checked in the non–degenerate case (as formulated in Theorem 3.9 below) employing [15, Thm.IV.9.1], see the discussion before [22, Thm.6.1]. Assumption (I) is useful for degenerate problems, see [22, Prop.6.5–6.7]. For later reference here and elsewhere, we state Theorems 3.25 and 3.27 of [22].

Theorem 3.4. *Let A be a real, locally dissipative, locally closed u.c., local operator defined on $\mathcal{O}(\Omega)$ and satisfying (S), and let $V \in \mathcal{O}(\Omega)$. Assume that either (I) or (O) holds. Then A_V is a generator on $C_0(V)$ if and only if V admits a Cauchy barrier w.r.t. A.*

In Section 4 we further discuss Cauchy barriers and give examples. At this point we recall [22, Cor.3.26] which says that under condition (O) the generator property of A_V persists under finite intersections of domains. In [19, Thm.3.3] one finds an analogous result in the context of hypothesis (I).

Proposition 3.5. *Let A be a real, locally dissipative, locally closed u.c., local operator satisfying (S) and (O). Let $V, V_k \in \mathcal{O}(\Omega)$, $k = 1, \cdots, m$. If each V_k admits a Cauchy barrier w.r.t. A and $V = \bigcap V_k$, then A_V is a generator on $C_0(V)$.*

Coming back to the main line of argument, we now present a sufficient condition for the density of $D(A_1)$.

Proposition 3.6. *Let $\mu \geq 0$ be a finite regular Borel measure on Ω and $V \in \mathcal{O}(\Omega)$ be relatively compact with $\mu(\partial V) = 0$. Let A be a local operator on Ω such that A_Ω is densely defined and invertible on $C_0(\Omega)$ and $\|A_\Omega^{-1} f\|_{C_0(\Omega)} \leq c\|f\|_{L^q(\Omega,\mu)}$ for $f \in C_0(\Omega)$ and some $q \in [1, \infty)$. Then $\tilde{A}_V$ is densely defined on $C(\overline{V})$.*

Proof. Note that there are relatively compact sets $W_n \in \mathcal{O}(\Omega)$ such that $\partial V \subset W_{n+1} \subset \overline{W_{n+1}} \subset W_n$ and $\mu(W_n) \leq 1/n$ for $n \in \mathbb{N}$. We extend a given $f \in C(\overline{V})$ to a function $\tilde{f} \in C_0(\Omega)$. For $\varepsilon > 0$ there is $\tilde{u} \in D(A_\Omega)$ such that $\|\tilde{f} - \tilde{u}\|_\infty \leq \varepsilon$. Let $\tilde{g} = A\tilde{u}$ and take continuous functions $\alpha_n : \Omega \to [0, 1]$ vanishing on W_{n+1} and being equal to 1 off W_n. Set $\tilde{g}_n = \alpha_n \tilde{g}$ and $v_n = (A_\Omega^{-1} \tilde{g}_n)|\overline{V}$. Then $v_n \in D(\tilde{A}_V)$, $\tilde{g}_n \to \tilde{g}$ in q–norm, and

$$\|f - v_n\|_{C(\overline{V})} \leq \|\tilde{f} - \tilde{u}\|_{C_0(\Omega)} + \|A_\Omega^{-1}(\tilde{g} - \tilde{g}_n)\|_{C_0(\Omega)} \leq \varepsilon + c\,\|\tilde{g} - \tilde{g}_n\|_{L^q(\Omega,\mu)} \leq 2\varepsilon$$

for large n. □

This simple criterion allows to reprove part of the characterizations given in [4, Thm.2.4], where we restrict ourselves to the Laplacian for simplicity (the same arguments apply to the autonomous analogue of the setting of Theorem 3.9, cf. [24]).

Example 3.7. Let V be a bounded open subset of $\mathbb{R}^N$ having a boundary with Lebesgue measure 0 and A be the local operator induced by the Laplacian Δ on $\mathbb{R}^N$, see Example 3.2. Taking a large open ball Ω containing $\overline{V}$, we can check the conditions of Proposition 3.6 and Theorem 2.13. Thus, V is Δ–Dirichlet regular if and only if A_0 is invertible on $C_0(V)$.

Remark 3.8. The equivalence stated in the previous example can also be verified using classical potential theory and Cauchy barriers. Indeed, in view of Lemma 2.12, one only has to show the sufficiency part of this characterization. If A_0 is a generator on $C_0(V)$, then A has a Cauchy barrier by [17, Thm.5.4] or [22, Prop.3.2]. But this already implies the Dirichlet regularity of V by the theorem of Bouligand, see [13, Thm.8.18, 8.22].

We extend the above results to non–degenerate parabolic problems in the classical context (3.2) and (3.4).

Theorem 3.9. *Let L be given by* (3.2) *and* (3.4)*, where Ω is a bounded open subset of $\mathbb{R}^N$ with boundary of class C^2 and the coefficients belong to $C(\overline{\underline{\Omega}}, \mathbb{R})$, $c \le 0$, and* (3.3) *holds on $\overline{\underline{\Omega}} = [-1, T] \times \overline{\Omega}$. Take a relatively compact subset $\underline{V} \Subset \mathcal{O}([0,T] \times \Omega)$ such that $V(t) \neq \emptyset$ for $0 \le t \le T$ and $\partial \underline{V}$ has Lebesgue measure 0 in $\underline{\Omega}$. Then the following assertions are equivalent.*

(a) $\tilde{L}_{\underline{V}_0}$ *is a generator in $C(\overline{\underline{V}})$.*
(b) $L_{\underline{V}_0}$ *is a generator in $C_0(\underline{V}_0)$.*
(c) $\underline{V}_0$ *has a Cauchy barrier for L.*

In this case the conclusions of Theorem 3.3 are true.

Proof. Using [15, Thm.IV.9.1] we see that $L_{\underline{\Omega}_0}$ is invertible (and densely defined) on $C_0(\underline{\Omega}_0)$ and that its inverse maps $(C_0(\underline{\Omega}_0), \|\cdot\|_q)$ continuously into $W^{1,2}_{q,0}(\underline{\Omega}_0) \hookrightarrow C_0(\underline{\Omega}_0)$ if $q > N/2 + 1$. So the theorem follows from Proposition 3.6, Theorem 2.3, Theorem 3.4 (see the remarks before [22, Thm.6.1]), and Theorem 3.3. □

We briefly indicate (expliciting only the case $A = \Delta$) another method to verify the density of $D(L_1)$ using L–harmonic functions. Consider $A = \Delta$, $L = \Delta - \frac{d}{dt}$, and $\underline{V} = [0,T] \times (0,1)^2$. Assume that L_1 is not densely defined in $C(\overline{\underline{V}})$. Then there is a regular Borel measure $\mu \neq 0$ such that

$$\int_{\overline{\underline{V}}} u \, d\mu = 0 \qquad \text{for all} \ \ u \in D(L_1).$$

Since L_0 is densely defined on $C_0(\underline{V}_0)$, the support of μ must be contained in $\partial \underline{V}_0$. The function $u(t,x,y) = (T_0(t)f)(x,y)$ belongs to $C_0(\underline{V})$ and to the kernel of L, where $f \in D(A_0)$ and $T_0(\cdot)$ is the semigroup generated by the Dirichlet Laplacian A_0 on $C_0(V)$. This implies that $\mathrm{supp}\mu \subset [0,T] \times \partial V$. Further let $\alpha \in C^1_c(0,1)$, $k \in \mathbb{N}_0$, and $g(t,x,y)$ be equal to $e^{-kt}\alpha(x)$ for $y = 1$ and $(t,x) \in [0,T] \times [0,1]$ and be equal to 0 on the other faces of the lateral boundary. Extend α to a periodic odd function on $\mathbb{R}$ with Fourier coefficients a_n. Define

$$u_m(t,x,y) = e^{-kt} \sum_{n=1}^{m} \frac{a_n}{\sin \sqrt{k - 4\pi^2 n^2}} \sin(2\pi n x) \sin(y\sqrt{k - 4\pi^2 n^2})$$

for $t \in [0,T]$, $x, y \in \mathbb{R}$, and $m \in \mathbb{N}$. Then $Lu_m = 0$, $u_m = 0$ on the faces $x = 0, 1$ and $y = 0$, and $u_m(t,x,1) \to e^{-kt}\beta(x)$ uniformly for t and x. Thus $\int g \, d\mu = 0$. By an approximation argument we now deduce that μ vanishes on $[0,T] \times (0,1) \times \{1\}$. The other faces can be treated in the same way so that μ is supported in the lateral edges. In a last step one sees in a similar manner that this is impossible. Therefore $\mu = 0$ and $D(L_1)$ is dense.

We finally want to study the regularity of the solutions to the homogeneous problem

$$\begin{aligned} Lu &= 0 \qquad \text{on } \underline{V}_s, \\ u &= 0 \qquad \text{on } \partial\underline{V}_s^* \setminus \underline{V}(s), \\ u(s) &= f. \end{aligned} \tag{3.5}$$

for $s \in (S,T)$, $f \in C_0(V(s))$, and the operator L given by (3.2) and (3.4) supposing that $L_0 = L_{\underline{V}_0}$ is a generator on $C_0(\underline{V}_0)$. In [22, Thm.4.13, 6.1] we proved that the solution $u \in C_0(\underline{V}'_s) \cap \bigcap_{p>1} W^{1,2}_{p,loc}(\underline{V}_s)$ of (3.5) is given by $u(t,\cdot) = U(t,s)f$, $t \in [s,T]$, for a variable space propagator $U(t,s) : C_0(V(s)) \to C_0(V(t))$, $S < s \le t \le T$. This means that

$$U(s,s) = I_{C_0(V(s))}, \qquad U(t,r)U(r,s) = U(t,s), \quad \text{and} \tag{3.6}$$

$$(t,s) \mapsto (U(t,s)F(s))^{\#} \quad \text{is continuous in } C_0(\Omega) \tag{3.7}$$

for $S < s \le r \le t \le T$ and $F \in C_0(\underline{V}_0)$, cf. [22, Def.4.8]. Moreover, $U(t,s)$ is a positive contraction.

In the following we consider initial functions f belonging locally to Slobodeckij spaces $W^\alpha_p(U)$, where U is a bounded open subset of $\mathbb{R}^N$ with a smooth boundary, $\alpha \in [0,2]$, and $1 < p < \infty$. These spaces can be defined, e.g., by Hölder estimates of L^p–type, see [26, §4.4]. Real interpolation of $W^\alpha_p(U)$ yields

$$(W^\alpha_p(U), W^\beta_p(U))_{\theta,p} = W^\gamma_p(U) \tag{3.8}$$

if $\gamma = \alpha(1-\theta) + \beta\theta \notin \mathbb{N}$ and $\theta \in (0,1)$ due to Theorem 4.3.1.1 and formula (2.4.2.16) of [26]. Taking $\alpha = 0$ and $\beta = 2$ in (3.8), [2, Thm.III.4.10.2] leads to

$$W^{1,2}_p([a,b] \times U) = L^p([a,b], W^2_p(U)) \cap W^1_p([a,b], L^p(U)) \hookrightarrow C([a,b], W_p^{2-\frac{2}{p}}(U)) \tag{3.9}$$

if $p \neq 2$. Finally, from [26, Thm.4.6.1, 4.6.2] we deduce

$$W_q^{2-\frac{2}{q}}(U) \hookrightarrow W_p^{2-\frac{2}{r}}(U) \hookrightarrow W_p^{2-\frac{2}{p}}(U) \tag{3.10}$$

if $1 < p < r < q < \infty$. The following result yields in the interior of $\underline{V}'_s$ the regularity at $t = s$ which we can expect in view of [15, §IV.9].

Theorem 3.10. *Let L be given by (3.2) and (3.4), where Ω is an open subset of $\mathbb{R}^N$ and the coefficients belong to $C(\Omega, \mathbb{R})$, $c \le 0$, and (3.3) holds. Moreover, let $\underline{V} \in \mathcal{O}(\underline{\Omega})$ satisfy $V(t) \neq \emptyset$ for $S \le t \le T$. Assume that $L_{\underline{V}_0}$ is a generator on $C_0(\underline{V}_0)$. Given $s \in (S,T)$ and $f \in C_0(V(s)) \cap \bigcap_{p>1} W_{p,loc}^{2-\frac{2}{p}}(V(s))$, we set $u(t,x) = (U(t,s)f)(x)$ for $(t,x) \in \underline{V}'_s$. Then $u \in C_0(\underline{V}'_s) \cap \bigcap_{p>1} W^{1,2}_{p,loc}(\underline{V}'_s)$.*

Proof. Fix $2 < p < r < q < \infty$, $\delta > 0$, and open, relatively compact subsets U, W, W' of $V(s)$ with smooth boundaries such that $\overline{U} \subseteq W \subseteq \overline{W} \subseteq W'$ and $[s-\delta, s+\delta] \times \overline{W'} \subseteq \underline{V}_0$. Choose $F \in C_0(\underline{V}_0)$ with $\|F\|_\infty \le \|f\|_\infty$, $F(s) = f$, $F(t,x) = f(x)$ for $x \in \overline{W'}$ and $|t-s| \le \delta$. We proceed in two steps.

(1) Take $\tau \in (s-\delta, s)$ and set $\underline{W} = [\tau, \tau+\delta] \times W$ and $\underline{W'} = [\tau, \tau+\delta] \times W'$. Using (3.8), (3.10), (3.9), and the interior apriori estimate [15, (IV.10.12)], we calculate

$$\begin{aligned}
&\|U(s,\tau)F(\tau) - f\|_{W_p^{2-\frac{2}{p}}(W)} \\
&\qquad \le c\,\|U(s,\tau)F(\tau) - f\|_{L^p(W)}^{1-\theta}\,\|U(s,\tau)F(\tau) - f\|_{W_p^{2-\frac{2}{r}}(W)}^{\theta} \\
&\qquad \le c\,\|(U(s,\tau)F(\tau))^\# - F(s)^\#\|_{C_0(\Omega)}^{1-\theta}\,\big(\|f\|_{W_q^{2-\frac{2}{q}}(W)} + \|U(s,\tau)F(\tau)\|_{W_q^{2-\frac{2}{q}}(W)}\big)^{\theta} \\
&\qquad \le c\,\|(U(s,\tau)F(\tau))^\# - F(s)^\#\|_{\infty}^{1-\theta}\,\big(\|f\|_{W_q^{2-\frac{2}{q}}(W)} + \|U(\cdot,\tau)F(\tau)\|_{W_q^{1,2}(\underline{W})}\big)^{\theta} \\
&\qquad \le c\,\|(U(s,\tau)F(\tau))^\# - F(s)^\#\|_{\infty}^{1-\theta}\,\big(\|f\|_{W_q^{2-\frac{2}{q}}(W')} + \|U(\cdot,\tau)F(\tau)\|_{C(\overline{\underline{W'}})}\big)^{\theta} \\
&\qquad \le c\,\|(U(s,\tau)F(\tau))^\# - F(s)^\#\|_{\infty}^{1-\theta} \qquad (3.11)
\end{aligned}$$

where $\theta = (1-\frac{1}{p})(1-\frac{1}{r})^{-1}$ and the constants c do not depend on τ,

(2) Now let $\tau_n = s - \frac{1}{n}$ for large n, $u_n = U(\cdot,\tau_n)F(\tau_n)$ on $\underline{V'_s}$, $\underline{U} = [s, s+\delta] \times U$, $\underline{W} = [s, s+\delta] \times W$, and $\underline{W'} = [s, s+\delta] \times W'$. By [22, Thm.4.13, 6.1] we have $u_n \in \bigcap_{p>1} W_{p,loc}^{1,2}(\underline{V'_s})$ since u_n solves (3.5) with initial value $F(\tau_n)$ at time τ_n. Further, $u_n \to u$ in $C_0(\underline{V'_s})$ due to (3.7). An application of (3.6), [15, (IV.10.12)], and (3.11) yields

$$\begin{aligned}
&\|u_n - u_m\|_{W_p^{1,2}(\underline{U})} = \|U(\cdot,s)\,[U(s,\tau_n)F(\tau_n) - U(s,\tau_m)F(\tau_m)]\|_{W_p^{1,2}(\underline{U})} \\
&\le c\,\big(\|u_n - u_m\|_{C(\overline{\underline{W}})} + \|U(s,\tau_n)F(\tau_n) - U(s,\tau_m)F(\tau_m)\|_{W_p^{2-\frac{2}{p}}(W)}\big) \\
&\le c\,\big(\|u_n - u_m\|_\infty + \|(U(s,\tau_n)F(\tau_n))^\# - F(s)^\#\|_\infty^{1-\theta} + \|(U(s,\tau_m)F(\tau_m))^\# - F(s)^\#\|_\infty^{1-\theta}\big).
\end{aligned}$$

Thus (u_n) is a Cauchy sequence in $W_p^{1,2}(\underline{U})$ by the strong continuity of $U(t,s)$. As a result, $u_n \to u$ in $W_p^{1,2}(\underline{U})$, and the assertion follows. □

4. Notes, examples, and comments

In the first part of this section we give an example, 4.1, and recall facts from the literature, showing that indeed in our context the solvability of the parabolic problem does not in general depend solely on the geometry of the domain but also on the operator involved, as covered by our barrier condition. Next, besides the applications given in the main text, we treat in Example 4.2 several non–cylindrical parabolic problems which are reasonably 'nasty' in the sense that they present most of the irregular behaviour mentioned in the second paragraph of our introduction.[3] In Example 4.2 the construction of the parabolic barriers is explicit and easy, illustrating the solution of a non–cylindrical problem with little regularity in regard to coefficients, boundary and degeneracy, just using Theorem 3.4, Proposition 3.5, Theorem 3.9, and the construction of a parabolic barrier 'by hand' without any other classical investment (besides the elliptic result [11, p.26]). In this example we have $L = a(t,\underline{x})\Delta - \partial_t$. This type of a multiplicatively perturbed operator naturally arises from problems with a probabilistic time change which can be quite irregular.

[3]We choose to treat in Example 4.2 the problems we do, among many other often more complicated ones that can be handled by the same or similar ideas, for clarity and concision.

We point out that going to merely continuous coefficients for operators in non–divergence form changes the properties of the problem significantly. This happens already for autonomous problems on cylindrical domains $[S,T]\times V$. In this situation V is Δ–Dirichlet regular if and only if it is A–Dirichlet regular provided that A is in divergence form or has Dini–continuous coefficients. But this equivalence fails if A is in non–divergence form with merely continuous coefficients. See the discussion and the references given in the notes of [11, Chap.6]. These examples can be transferred into the context of (autonomous) parabolic problems using that A has a Cauchy barrier on V if and only if $L = A - \partial_t$ has a Cauchy barrier on $[S,T]\times V$, cf. Example 4.1.

For parabolic operators on non–cylindrical domains this equivalence even breaks down for $L = \Delta - \partial_t$ and $L = \alpha\Delta - \partial_t$, $\alpha > 1$, due to [8, Thm.8.1]. Therefore, in our setting, the Dirichlet regularity of a domain will depend on the operator involved, in general. This fact is also reflected by Wiener type characterizations of boundary regularity which involve the capacity of level sets of the heat kernel corresponding to L, see [9] for $L = \Delta - \partial_t$ and [5] for operators in divergence form. Also most relevant to our context, the following example shows the dependance of the (potential theoretic) boundary regularity on the operator in the degenerate case.

Example 4.1. We consider the situation of Example 3.2 and the local operator $A = \Delta$ introduced there. Define the local operators $L^{(1)} = |x|^2\Delta - \partial_t$ and $L^{(2)} = \Delta - \partial_t$ on $\underline{\Omega} = [0,T]\times W$ as in (3.2) and (3.4). As noted in Example 3.2, there is a Cauchy barrier h for $|x|^2\Delta$ on W. Clearly, $H(t,x) = th(x)$ is then a Cauchy barrier for $L^{(1)}$. On the other hand, if $L^{(2)}$ has a Cauchy barrier, then there would exist a propagator $U(t,s) : C_0(W) \to C_0(W)$ solving (3.5) for $L^{(2)}$ by [22, Thm.4.15]. It is easy to see that $U(t,s) = e^{(t-s)B}$ for a C_0–semigroup on $C_0(W)$. Thus the semigroup $T(\cdot)$ on $C_0(\underline{\Omega}_0)$ generated by $L_0^{(2)}$ is given by $(T(t)u)(\tau,\cdot) = e^{tB}u(t-\tau,\cdot)$ if $t-\tau \in (0,T]$ and $(T(t)u)(\tau,\cdot) = 0$ otherwise, see [22, Thm.4.9]. This representation implies that the function $u(t,x) = \alpha(t)f(x)$ for $f \in D(B)$ and $\alpha \in C_0^1((0,T])$ belongs to $D(L_0^{(2)})$ and $L^{(2)}u = \alpha Bf - \alpha' f$. Consequently, $B \subseteq A_0$ so that A_0 is a generator. But this contradicts Corollaire 6.7 of [17].

Example 4.2. Let $\underline{\Omega} = [0,T]\times\mathbb{R}^2$, $a \in C(\underline{\Omega})$ with $a(\underline{x}) \geq a_0 > 0$ on $\underline{\Omega}$, and define $L = a\Delta - \partial_t$ as in (3.2) and (3.4). We set $\phi(t,x) = \exp(-\frac{\alpha(t)}{x})$ for $x > 0$ and $0 \leq t \leq T$, $\phi(t,x) = 0$ for $x \leq 0$ and $0 \leq t \leq T$, where $\alpha \in C^1([0,T])$, $\alpha > 0$, and $\alpha(0) = 1$.

First, take $\underline{V}_1 = \{(t,x,y) \in \underline{\Omega} : x^2 + y^2 < r(t)\}$ for some $r \in C^1([0,T])$ with $r(0) = 1/2$ and $r' \geq 0$. Then $H(t,x,y) = t(r(t) - x^2 - y^2)$ is a Cauchy barrier for L and $(\underline{V}_1)_0$.

Second, let $\underline{V}_2 = \{(t,x,y) \in \underline{\Omega} : x < 1,\ |y| < \phi(t,x)\}$. Clearly, $\underline{V}_2$ is the intersection of three bounded domains with smooth boundaries such that no tangent plane to the boundary is of the form $t = const$. These regular domains have a Cauchy barrier for L by e.g. [22, Prop.6.4]. Thus there exists a Cauchy barrier on $(\underline{V}_2)_0$ for L by Proposition 3.5.

Third, we define $\underline{V}_3 = [0,T]\times W$ for $W = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 < 1,\ |y| > \phi(0,x)$ if $x \geq 0\}$. Then W is Δ–Dirichlet regular by [11, p.26] or [13] and thus possesses a Cauchy barrier h (with $K = \emptyset$) for $A = \Delta$ by Example 3.7 and Theorem 3.4. Clearly, $H(t,x,y) = th(x,y)$ is a Cauchy barrier on $(\underline{V}_3)_0$ for L.

Hence Proposition 3.5 and Theorem 3.9 imply the solvability of (3.1) for L on all three domains and their intersections. For instance, $\underline{V}_1 \cap \underline{V}_3$ is a 'funnel' in space–time from which one has removed a sharp inward pointing wedge. With respect to the space variables, this lateral boundary is as bad as one can expect in view of classical potential theory.

If a vanishes on the boundary of one of these domains, then the above Cauchy barriers still work on $\underline{V}_1$ and $\underline{V}_3$ (see also [17, §6], [21, §7], [22, §6] for degeneracies in the interior). One can check condition (I) in this case by means of the above arguments. Thus Theorem 3.4 shows that L_0 is invertible on $C_0(\underline{W}_0)$ so that the function $u = L_0^{-1}F$ solves (3.1) for $f = g = 0$ and $F \in C_0(\underline{W}_0)$, where $\underline{W} = \underline{V}_1, \underline{V}_3$ (or $\underline{W} = \underline{V}_1 \cap \underline{V}_3$ in view of [19, Thm.3.3]). Using an approximation argument, it is also possible to treat non–zero initial values $f \in C_0(V(0))$, see [22, Thm.4.14]. Further, consider two of above domains touching each other at a part of the boundary such that a equals 0 only at the boundary of one subdomain. We can then solve (3.1) with non–zero boundary conditions on the subdomain where a does not degenerate.

The general theory presented in this paper opens promising perspectives of new applications and new research, such as developing further results for easier and more flexible construction of barriers (via infima of generalized parabolic subharmonic functions, local barriers), approximation of very irregular domains by more regular ones having barriers (possibly using recent or future improved knowledge of parabolic Harnack inequalities, cf. [14], [16]), improved handling of degenerate problems, and extensions to semilinear problems beyond earlier work in [22] (, relevant for instance to population models in time–dependent habitats).

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