

STABLE AND UNSTABLE MANIFOLDS FOR QUASILINEAR PARABOLIC PROBLEMS WITH FULLY NONLINEAR DYNAMICAL BOUNDARY CONDITIONS

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ABSTRACT. We develop a wellposedness and regularity theory for a large class of quasilinear parabolic problems with fully nonlinear dynamical boundary conditions. Moreover, we construct and investigate stable and unstable local invariant manifolds near a given equilibrium. In a companion paper we treat center, center–stable and center–unstable manifolds for such problems and investigate their stability properties. This theory applies e.g. to reaction–diffusion systems with dynamical boundary conditions and to the two–phase Stefan problem with surface tension.

1. INTRODUCTION

In this paper we develop a wellposedness and regularity theory for a large class of quasilinear parabolic problems with fully nonlinear dynamical boundary conditions. In this framework we construct and investigate stable and unstable local invariant manifolds near a given equilibrium. In the companion paper [37] we treat center, center–stable and center–unstable manifolds for such systems and investigate their stability properties. This theory applies e.g. to reaction–diffusion problems with dynamical boundary conditions and to the two–phase Stefan problem with surface tension.

Quasilinear parabolic problems have been studied successfully from various perspectives. An important, widely used theory was created by Amann in e.g. [1] and [2]. This approach applies in particular to quasilinear problems with conormal boundary conditions, which are understood in a weak sense on the state space of the resulting flow (typically W_p^1). In this framework a dynamical theory was developed which covers center manifolds or bifurcation, for instance, see e.g. [13], [38] or [39].

A somewhat different approach to such systems is based on maximal regularity of type L_p of linearized equations, see [30] for a detailed exposition and also [33], [35]. In our previous papers [17], [19], [20], and [25] we have investigated quasilinear parabolic problems with fully nonlinear *static* boundary conditions

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using maximal L_p regularity of the linearizations, see also [33]. Here the equations at the boundary are understood classically and the evolution equation in the spatial domain holds in L_p sense. We have established a theory of local well-posedness, smoothing properties, invariant manifolds near equilibria and asymptotic stability of certain invariant manifolds and of periodic orbits.

For fully nonlinear problems, it seems that one has to work in a framework of higher regularity as it is presented e.g. in the monograph [22], where also invariant manifolds and bifurcation are discussed, see also [4] or [29]. In this setting one obtains classical solutions, but this enforces additional compatibility conditions. Moreover, such problems do not exhibit the usual parabolic smoothing, in general.

The development of the theory has been much influenced by the study of free boundary problems. Some of them can be treated in the framework of Amann's theory, for instance the Mullins–Sekerka system which is a quasistationary Stefan problem, see e.g. [13]. Others require the theory of fully nonlinear problems, see e.g. [4]. But a large class of systems comprising in particular the Stefan problem with surface tension fits very well to a different approach introduced in [12] which is based on maximal L_p regularity of the linearizations. A detailed analysis of the Stefan problem with surface tension was then carried out in particular in [31] and [34]. Analytic solutions of the classical Stefan problem have been constructed in [32] within this L_p approach. We refer to [15], [21], [26] and the references in [12] and [34] for other approaches.

The methods of [12] had inspired the paper [8] which established a theory of maximal L_p regularity for a class of inhomogeneous linear systems with dynamical boundary conditions. In this paper we investigate a corresponding class of nonlinear problems whose linearizations fit to [8], thereby covering the Stefan problem with surface tension (see Example 2.2) and also a wide range of quasilinear parabolic problems with nonlinear dynamical boundary conditions (see Example 2.1). We refer to [5], [11], [14], [16] [23], [36], [41], [42] and the references therein for various approaches to specific reaction diffusion systems or phase field models with dynamical boundary conditions. We treat the system

$$\begin{aligned} \partial_t u(t) + \mathcal{A}(u(t), \rho(t))u(t) &= \mathcal{R}(u(t), \rho(t), \dot{\rho}(t)), & \text{on } \Omega, \quad t > 0, \\ \partial_t \rho(t) + \mathcal{D}_0(u(t), \rho(t)) &= 0, & \text{on } \Omega, \quad t > 0, \\ \mathcal{D}_j(u(t), \rho(t)) &= 0, & \text{on } \Sigma, \quad t \geq 0, \quad j = 1, \dots, m, \\ u(0) &= u_0, & \text{on } \Omega, \\ \rho(0) &= \rho_0, & \text{on } \Sigma, \end{aligned} \tag{1.1}$$

on a spatial domain Ω which either has the smooth boundary Σ (one phase setting) or is the disjoint union of two domains whose boundary consists of the common part Σ and possibly of further disjoint ‘outer parts’ (two phase setting). On these outer parts we impose linear boundary conditions not shown in (1.1). The solutions u and ρ take values in finite dimensional vector spaces.

In Ω act the main quasilinear diffusion type operator \mathcal{A} of (differential) order $2m$ and the lower order perturbation \mathcal{R} . On the boundary we have a dynamical boundary condition governed by the nonlinear term \mathcal{D}_0 and static boundary condition governed by $\mathcal{D}_1, \dots, \mathcal{D}_m$. One can also consider this system as an

evolution equation for the function $w = (u, \rho)$, where u and ρ are directly coupled via the nonlinearities and also via the static boundary condition. In the operators \mathcal{D}_j the orders with respect to u are strictly less than $2m$. However, the orders in ρ are not bounded a priori. The solution space for ρ has to be adapted to the degree of unboundedness of these operators. We will assume that the nonlinearities are C^1 on the solution spaces of the linear theory and that the resulting linearized boundary value problems are normally elliptic and satisfy Lopatinsky–Shapiro conditions. Our setting is described Section 2, where we also recall the necessary theory from [8].

Another main difficulty in (1.1) is the occurrence of a time derivative of the second component ρ in the evolution equation for u . Such terms arise if one transforms a problem with moving boundaries to a fixed domain, cf. Example 2.2. In (1.1) this term can be treated as a perturbation, which requires extra time regularity of $\partial_t \rho$ provided by the solution space of ρ . However, it is not so clear how to incorporate such terms into the spectral theory of the linearization which is crucial for our results on the longterm behavior. To deal with this difficulty, we insert the second line into the first thereby eliminating the extra time derivative. The resulting perturbation problem is solved in Corollary 2.6.

In Section 3 we establish the local wellposedness of (1.1) in a strong sense in which the equations at the boundary are understood classically and the evolution equation in Ω holds in L_p , see Theorem 3.3. Propositions 3.4 and 3.5 further show a smoothing effect of the solution with corresponding estimates which give extra regularity of some of the invariant manifolds, see e.g. Theorem 5.1(e). This property is crucial for the convergence analysis in [37]. Another important issue is the description of the nonlinear solution manifold \mathcal{M} given in Lemma 3.2 based on its linear counterpart Corollary 2.7. This manifold incorporates first the static boundary conditions (which give a constraint by a nonlinear equation) and second a ‘dynamical’ regularity constraint coming from the dynamical boundary condition. The latter arises because $\partial_t \rho(t)$ possesses extra space regularity which must also be fulfilled by $\mathcal{D}_0(u(t), \rho(t))$.

In Section 4 we develop the linear analysis needed for the asymptotic theory and also for [37]. The main difficulty is that the maximal regularity theory from [8] fits very well to nonlinear theory of (1.1), in the sense that it provides the needed extra time regularity for $\partial_t \rho$. However, the corresponding semigroup lives on a smaller state space. See Theorem 2.5 and Corollary 2.6. Moreover, the extra time regularity induces the additional ‘dynamical’ compatibility condition which does not really fit into the semigroup framework. The latter point can be seen e.g. when dealing with the spectral decomposition which is crucial for our asymptotic theory, see the proof of Proposition 4.5. We can deal with these problems using the parametrizations of \mathcal{M} mentioned above and also extrapolation theory for semigroups. For purely static conditions we have developed the relevant techniques in [19]. However, it required much work and several new arguments to extend these methods to dynamical boundary conditions.

In our second main result, Theorem 5.1, we construct local stable and unstable manifolds \mathcal{M}_s and \mathcal{M}_u near an equilibrium w_* such that the linearization Λ_0

at this equilibrium has no spectrum on $i\mathbb{R}$. We further characterize these manifolds as spaces of initial values near w_* such that the corresponding solutions stay close to w_* for all $t \geq 0$, resp. $t \leq 0$. Actually, the solutions then converge to w_* exponentially. In addition, these manifolds are locally invariant, have trivial intersection and are tangential to the corresponding spectral subspaces of Λ_0 . We use the implicit function theorem, the differentiability of the nonlinear maps, the regularity properties of the linearized inhomogeneous problem, see (2.25), and modifications of known techniques in dynamical systems.

Notation. We set $D_k = -i\partial_k = -i\partial/\partial x_k$ and use multi index notation. The time derivative is denoted by $\partial_t u = \dot{u}$. For a linear operator A on a Banach space we write $D(A)$, $\sigma(A)$ and $\rho(A)$ for its domain, spectrum and resolvent set, respectively. For Banach spaces X , Y and Z , $\mathcal{L}(X, Y)$ is the space of bounded linear operators, where $\mathcal{L}(X) := \mathcal{L}(X, X)$, and $\mathcal{L}_2(X \times Y, Z)$ is the space of bounded bilinear operators. A ball in X with the radius r and center at u will be designated by $B_X(u, r)$. For an open set $U \subset \mathbb{R}^n$ with (sufficiently regular) boundary ∂U , $C^k(U)$ (resp., $C_b^k(U)$, $C_{ub}^k(U)$, $C_0^k(U)$) are the spaces of k -times continuously differentiable functions u on U (such that u and its derivatives up to order k are bounded, bounded and uniformly continuous, vanish at ∂U and at infinity (if U is unbounded), respectively), where $C_b^k(U)$ is endowed with its canonical norm. For $C^k(\overline{U})$, $C_b^k(\overline{U})$, $C_{ub}^k(\overline{U})$, we require in addition that u and its derivatives up to order k have a continuous extension to ∂U . For unbounded U , we write $C_0^k(\overline{U})$ for the space of $u \in C^k(\overline{U})$ such that u and its derivatives up to order k vanish at infinity. By $W_p^k(U)$ we denote the standard Sobolev spaces and by $W_p^s(U)$ the Slobodetskii spaces endowed with the norm

$$|v|_{W_p^s(U)}^p = |v|_{L_p(U)}^p + \sum_{|\alpha|=k} [\partial^\alpha v]_{W_p^\sigma(U)}^p, \quad [w]_{W_p^\sigma(U)}^p = \iint_{U^2} \frac{|w(y) - w(x)|^p}{|y - x|^{n+\sigma p}} dx dy,$$

for $s = k + \sigma$ with $k \in \mathbb{N}_0$ and $\sigma \in (0, 1)$, see Remark 4.4.1.2 in [40]. The Sobolev-Slobodetskii spaces on ∂U are defined via local charts, see Definition 3.6.1 in [40]. In some exceptional cases we also use the Besov spaces $B_{pp}^k(\partial U)$ for $k \in \mathbb{N}$, see Definition 3.6.1 in [40], where $B_{pp}^s(\partial U) = W_p^s(\partial U)$ for non-integer $s > 0$. We write $\mathbb{C}_+ = \{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda > 0\}$ and J for a real interval with nonempty interior. Finally, c is a generic constant and $\varepsilon : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a generic nondecreasing function with $\varepsilon(r) \rightarrow 0$ as $r \rightarrow 0$.

2. SETTING AND PRELIMINARIES

We fix numbers $m \in \mathbb{N}$, $m_j \in \{0, 1, 2m - 1\}$, and $k_j \in \mathbb{N}_0 \cup \{-\infty\}$ for $j \in \{0, 1, \dots, m\}$, describing the order of the differentiable operators appearing in (1.1), where $k_j = -\infty$ if \mathcal{D}_j does not depend on ρ , see (R) and (2.21) below. We consider two different types of domains.

In the *one phase setting*, let $\Omega \subset \mathbb{R}^n$ be an open connected set with a compact boundary $\partial\Omega$ of class $C^{2m+\ell-m_0}$ and outer unit normal $\nu(x)$, where $\ell \in \{m_0, m_0 + 1, \dots\}$ is given by (2.8) below. We set $\Sigma := \partial\Omega$ and $\Gamma_1 = \Gamma_2 := \emptyset$.

In the *two phase setting*, let $\Omega = \Omega_1 \dot{\cup} \Omega_2$ for two open subsets $\Omega_j \subseteq \mathbb{R}^n$ having compact boundaries of class $C^{2m+\ell-m_0}$, where $\partial\Omega_j = \Sigma \dot{\cup} \Gamma_j$ for $j = 1, 2$,

$\partial\Omega_1 \cap \partial\Omega_2 = \Sigma$, and Γ_j may be empty. In this case, $\nu(x)$ is the outer normal of the interface Σ with respect to Ω_1 .

Since we will impose fixed linear homogeneous boundary conditions on Γ_j , in both settings Σ is the important part of the boundary. Throughout, we fix

$$p \in (n + 2m, \infty) \quad (2.1)$$

Let V_u and V_ρ be finite dimensional Banach spaces with norms $|\cdot|$, being the range spaces of the solutions to (1.1). As function spaces on Ω we use

$$\begin{aligned} X &= L_p(\Omega; V_u) \quad \text{in the one and in the two phase case;} \\ X_1 &= W_p^{2m}(\Omega; V_u), \quad X_\gamma = W_p^{2m(1-1/p)}(\Omega; V_u), \quad \text{in the one phase case;} \\ X_1 &= \{v \in W_p^{2m}(\Omega; V_u) \mid B^0 v = 0\}, \quad X_\gamma = (X, X_1)_{1-\frac{1}{p}, p} \quad \text{in the two phase case,} \end{aligned}$$

where B^0 is an m -tuple of fixed linear boundary operators on $\Gamma_1 \cup \Gamma_2$ which are given by (2.22) and satisfy (LS) below. Moreover, by interpolation we have $X_\gamma \subseteq \{v \in W_p^{2m(1-1/p)}(\Omega; V_u) \mid B^0 v = 0\}$ in the two phase case,* using also that X_1 is dense in X_γ and that $B^0 \in \mathcal{B}(W_p^{2m(1-1/p)}(\Omega; V_u), X)$. We note that in the two phase case X , X_γ and X_1 can be identified with the product of the corresponding spaces on Ω_1 and Ω_2 .

Recall that the spatial trace operator γ_Θ induces continuous maps

$$\gamma_\Theta : W_p^s(\Theta; V_u) \rightarrow W_p^{s-1/p}(\partial\Theta; V_u) \quad (2.2)$$

for $1/p < s \leq k$ if $s - 1/p$ is not an integer and Θ has a compact boundary of class C^k . At the boundary we employ the spaces

$$\begin{aligned} Y_u &= L_p(\Sigma; V_u), \quad Y_{j\gamma} = W_p^{2m\kappa_j - 2m/p}(\Sigma; V_u), \quad Y_{j1} = W_p^{2m\kappa_j}(\Sigma; V_u), \\ Y_\rho &= L_p(\Sigma; V_\rho), \quad Y_{0\gamma} = W_p^{2m\kappa_0 - 2m/p}(\Sigma; V_\rho), \quad Y_{01} = W_p^{2m\kappa_0}(\Sigma; V_\rho), \\ Y_k &= Y_{0k} \times \cdots \times Y_{mk}, \quad \widehat{Y}_k = Y_{1k} \times \cdots \times Y_{mk}, \end{aligned}$$

for $j \in \{1, \dots, m\}$, $k \in \{1, \gamma\}$, and the numbers

$$\kappa_j := 1 - \frac{m_j}{2m} - \frac{1}{2mp}, \quad j = 0, 1, \dots, m. \quad (2.3)$$

These spaces are thus determined by the orders m_j of the differentiable (trace type) operators in (1.1) mapping the solution u from Ω to the boundary. We observe that $X_1 \hookrightarrow X_\gamma \hookrightarrow X$, $Y_{j1} \hookrightarrow Y_{j\gamma} \hookrightarrow Y_u$, $Y_{01} \hookrightarrow Y_{0\gamma} \hookrightarrow Y_\rho$,

$$\begin{aligned} X_\gamma &\hookrightarrow C_0^{2m-1}(\overline{\Omega}; V_u) \quad \text{in the one phase case,} \\ X_\gamma &\hookrightarrow C_0^{2m-1}(\overline{\Omega_1}; V_u) \times C_0^{2m-1}(\overline{\Omega_2}; V_u) \quad \text{in the two phase case;} \\ Y_{j\gamma} &\hookrightarrow C^{2m-1-m_j}(\Sigma; V_u), \quad \text{and} \quad Y_{0\gamma} \hookrightarrow C^{2m-1-m_0}(\Sigma; V_\rho) \end{aligned} \quad (2.4)$$

for $j = 1, \dots, m$ due to (2.1), (2.3), and standard embeddings, cf. §4.6.1 in [40].

*Here it should hold equality due to our assumption (LS). Unfortunately, it is not so easy to find this result in the literature in the full generality. We do not need this equality.

For the investigation of (1.1), we need several spaces of functions on $J \times \Omega$ and $J \times \Sigma$, where $J \subset \mathbb{R}$ is an interval with a nonempty interior. The base space and solution space for u in (1.1) are

$$\mathbb{E}(J) = L_p(J; L_p(\Omega; V_u)) = L_p(J; X) \quad \text{and}$$

$$\mathbb{E}_u(J) = W_p^1(J; X) \cap L_p(J; X_1) \subseteq W_p^1(J; L_p(\Omega; V_u)) \cap L_p(J; W_p^{2m}(\Omega; V_u)),$$

respectively, where the last inclusion is an equality in the one phase case. These Banach spaces are endowed with their natural norms. We may omit J in the notation if the interval is irrelevant or clear from the context. If J not compact, we write $\mathbb{E}_{\text{loc}}(J)$ for the space of functions whose their restrictions to each interval $[a, b] \subseteq J$ belong to $\mathbb{E}([a, b])$. Analogous notations are used for \mathbb{E}_u and the other function spaces introduced below.

We denote by $\gamma_t : u \mapsto u(t)$ the trace operator at time $t \in \bar{J}$ (if it is defined). Very often we use the crucial facts

$$\mathbb{E}_u(J) \hookrightarrow C_{ub}(J; X_\gamma) \hookrightarrow C_{ub}(J; C_0^{2m-1}(\bar{\Omega}; V_u)) \quad (\text{one phase}), \quad (2.5)$$

$$\mathbb{E}_u(J) \hookrightarrow C_{ub}(J; X_\gamma) \hookrightarrow C_{ub}(J; C_0^{2m-1}(\bar{\Omega}_1; V_u) \times C_0^{2m-1}(\bar{\Omega}_2; V_u)) \quad (\text{two phase});$$

$\gamma_t : \mathbb{E}_u(J) \rightarrow X_\gamma$ is continuous and has a bounded right inverse

for all $t \in \bar{J}$, cf. Theorem III.4.10.2 in [3] and (2.4). The norms of the first embeddings in (2.5) are uniform for J of length greater than a fixed $d_0 > 0$. For functions vanishing at $t = \inf J$, this constant can be chosen independent of J (see e.g. Theorem 4.2 of [27]).

In the one phase setting, the spatial trace and derivatives extend to continuous operators

$$\gamma_\Omega \partial^\beta : \mathbb{E}_u(J) \rightarrow W_p^{1-\frac{k}{2m}-\frac{1}{2mp}}(J; Y_u) \cap L_p(J; W_p^{2m-k-\frac{1}{p}}(\partial\Omega; V_u)) \quad (2.6)$$

for $0 \leq |\beta| \leq k < 2m$, where the trace has a bounded right inverse. See §3 of [7] and also Lemma 3.4 and Theorem 4.5 of [27]. The natural trace spaces of the solution space \mathbb{E}_u are thus given by

$$\mathbb{F}_j(J) = W_p^{\kappa_j}(J; L_p(\Sigma; V_u)) \cap L_p(J; W_p^{2m\kappa_j}(\Sigma; V_u)) = W_p^{\kappa_j}(J; Y_u) \cap L_p(J; Y_{j1}),$$

$$\mathbb{F}_0(J) = W_p^{\kappa_0}(J; L_p(\Sigma; V_\rho)) \cap L_p(J; W_p^{2m\kappa_0}(\Sigma; V_\rho)) = W_p^{\kappa_0}(J; Y_\rho) \cap L_p(J; Y_{01})$$

for $j \in \{1, \dots, m\}$ endowed with their canonical norms, where we put

$$\mathbb{F}(J) = \mathbb{F}_0(J) \times \dots \times \mathbb{F}_m(J) \quad \text{and} \quad \widehat{\mathbb{F}}(J) = \mathbb{F}_1(J) \times \dots \times \mathbb{F}_m(J).$$

If we replace here V_u or V_ρ by another space W , we write $\mathbb{F}_j(J; W)$ etc. We further have

$$\mathbb{F}_j(J) \hookrightarrow C_{ub}(J; Y_{j\gamma}) \hookrightarrow C_{ub}(J \times \Sigma; V) \quad (2.7)$$

$\gamma_t : \mathbb{F}_j(J) \rightarrow Y_{j\gamma}$ is continuous and has a bounded right inverse

for $t \in \bar{J}$, $j = 0, 1, \dots, m$ and $V = V_u$ if $j \geq 1$ and $V = V_\rho$ if $j = 0$. (See §3 of [7] and also Theorem 4.2 of [27].) The same remarks as after (2.5) apply.

The solution space \mathbb{E}_ρ for ρ in (1.1) is rather sophisticated. It is chosen such that the operators \mathcal{D}_j in (1.1) map \mathbb{E}_ρ into the trace spaces \mathbb{F}_j of the solutions u . It thus strongly depends on the orders k_j of the differential operators acting

on ρ in (1.1), which are not restricted a priori. We follow the presentation in [8] and put $\tilde{\mathcal{J}} = \{j \in \{0, 1, \dots, m\} \mid k_j \neq -\infty\}$ as well as

$$\ell_j = k_j - m_j + m_0, \quad \ell = \max_{j=0,1,\dots,m} \ell_j \geq m_0. \quad (2.8)$$

It is important to note that

$$2m\kappa_j + k_j = 2m\kappa_0 + \ell_j, \quad j = 0, 1, \dots, m. \quad (2.9)$$

We then define

$$\begin{aligned} \mathbb{E}_\rho(J) = & W_p^{1+\kappa_0}(J; L_p(\Sigma; V_\rho)) \cap L_p(J; W_p^{\ell+2m\kappa_0}(\Sigma; V_\rho)) \\ & \cap W_p^1(J; W_p^{2m\kappa_0}(\Sigma; V_\rho)) \cap \bigcap_{j \in \tilde{\mathcal{J}}} W_p^{\kappa_j}(J; W_p^{k_j}(\Sigma; V_\rho)). \end{aligned} \quad (2.10)$$

Observe that ρ has extra space and time regularity compared to u . This is needed in important applications and for the underlying linear theory, see Example 2.2 and Theorem 2.5. One can visualize \mathbb{E}_ρ by the points $(0, 1 + \kappa_0)$, $(\ell + 2m\kappa_0, 0)$, $(2m\kappa_0, 1)$ and (k_j, κ_j) for $j \in \tilde{\mathcal{J}}$, corresponding to the space-time differentiability of the spaces F_i on the right-hand side of (2.10). The *Newton polygon* \mathcal{NP} for \mathbb{E}_ρ is then defined as the convex hull of these points together with $(0, 0)$. The *leading part* $\mathcal{LN}\mathcal{P}$ of \mathcal{NP} is the part of its boundary connecting $(0, 1 + \kappa_0)$ to $(\ell + 2m\kappa_0, 0)$ counterclockwise. We set

$$\mathcal{J} = \{j \in \tilde{\mathcal{J}} \mid \ell_j = \ell \text{ or } (k_j, \kappa_j) \in \mathcal{LN}\mathcal{P}\}.$$

Let F_i and F_j be two different spaces on the right-hand side of (2.10). It is known that $F_i \cap F_j$ embeds into all spaces whose space-time regularity corresponds to the line segment connecting the two points that represent F_i and F_j in \mathcal{NP} , see §2 of [8] and also Proposition 3.2 of [27]. Consequently, the definition of \mathbb{E}_ρ given in (2.10) may contain redundant spaces. Below we discuss nonredundant descriptions of \mathbb{E}_ρ taken from §2 of [8], see also §2 in [28]. From there we will also recall the representations of the temporal trace space Z_γ of \mathbb{E}_ρ and of the temporal trace space Z_γ^1 of the time derivative $\dot{\rho}$ of $\rho \in \mathbb{E}_\rho$. In particular, it holds

$$\begin{aligned} \partial^\beta \in \mathcal{B}(\mathbb{E}_\rho(J), \mathbb{F}_j(J)), \quad \mathbb{E}_\rho(J) \hookrightarrow C_{ub}(J; Z_\gamma), \quad \partial_t \in \mathcal{L}(\mathbb{E}_\rho(J), C_{ub}(J; Z_\gamma^1)) \\ (\gamma_t, \gamma_t \partial_t) \in \mathcal{L}(\mathbb{E}_\rho(J), Z_\gamma \times Z_\gamma^1) \quad \text{has a bounded right inverse,} \\ \gamma_t \in \mathcal{L}(\mathbb{E}_\rho(J), Z_\gamma) \quad \text{has a bounded right inverse,} \end{aligned} \quad (2.11)$$

if $|\beta| \leq k_j$ and $t \in \bar{J}$. The assertion in the second line of (2.11) is shown in §4.1 of [8], and it implies the last one. The same remarks as after (2.5) apply. The trace spaces Z_γ and Z_γ^1 are given by $W_p^s(\Sigma; V_\rho)$ for the numbers $s > 0$ such that $(s, k + 1/p)$ belongs to leading part of \mathcal{NP} for $k = 0$ and $k = 1$, respectively, see §2 of [8] and also Theorem 4.2 of [27].

To state the descriptions of \mathbb{E}_ρ , Z_γ and Z_γ^1 , one has to distinguish between three cases, where we write L_p instead of $L_p(\Sigma; V_\rho)$ etc. and use that $\kappa_j > 1/p$ holds for all j due to (2.1).

Case 1: $\ell = 2m$. One has

$$\mathbb{E}_\rho(J) = W_p^{1+\kappa_0}(J; L_p) \cap L_p(J; W_p^{2m(1+\kappa_0)})$$

since all other spaces in (2.10) correspond to points on or below the straight line $s = 1 + \kappa_0 - r/2m$ from $(0, 1 + \kappa_0)$ to $(2m + 2m\kappa_0, 0)$, due to (2.9). It further holds

$$Z_\gamma = W_p^{2m(\kappa_0+1-1/p)}(\Sigma; V_\rho), \quad Z_\gamma^1 = W_p^{2m(\kappa_0-1/p)}(\Sigma; V_\rho) = Y_{0\gamma}.$$

Case 2: $\ell < 2m$. One has

$$\mathbb{E}_\rho(J) = W_p^{1+\kappa_0}(J; L_p) \cap L_p(J; W_p^{\ell+2m\kappa_0}) \cap W_p^1(J; W_p^{2m\kappa_0})$$

since $(1, 2m\kappa_0)$ lies above the line segment $s = 1 + \kappa_0 - r(1 + \kappa_0)/(\ell + 2m\kappa_0)$ from $(0, 1 + \kappa_0)$ to $(\ell + 2m\kappa_0, 0)$ and all points (κ_j, k_j) are below the line $s = 1 + (2m\kappa_0 - r)/\ell$ connecting $(1, 2m\kappa_0)$ and $(0, \ell + 2m\kappa_0)$. One also obtains the trace spaces

$$Z_\gamma = W_p^{2m\kappa_0+\ell(1-1/p)}(\Sigma; V_\rho), \quad Z_\gamma^1 = W_p^{2m(\kappa_0-1/p)}(\Sigma; V_\rho) = Y_{0\gamma}.$$

Case 3: $\ell > 2m$. Now $(1, 2m\kappa_0)$ belongs to the interior of \mathcal{NP} . The leading part is given by the vertices

$$P_0 = (0, 1 + \kappa_0), \quad P_1 = (k_{j_1}, \kappa_{j_1}), \quad \dots, \quad P_q = (k_{j_q}, \kappa_{j_q}), \quad P_{q+1} = (\ell + 2m\kappa_0, 0)$$

for some $q \in \mathbb{N}$, where these pairs are ordered with increasing k_j and decreasing κ_j . It can be seen that $\ell_{j_i} \in (2m, \ell)$ for $i = 1, \dots, q-1$ and $\ell_q = \ell$. It holds

$$\mathbb{E}_\rho(J) = W_p^{1+\kappa_0}(J; L_p) \cap L_p(J; W_p^{\ell+2m\kappa_0}) \cap \bigcap_{j \in \mathcal{J}} W_p^{\kappa_j}(J; W_p^{k_j}).$$

For later use, see (LS_∞^\pm) below, we define $k_{-1} := 0$, $\kappa_{-1} := 1 + \kappa_0$,

$$\mathcal{J}_{2i} := \{j \in \mathcal{J} \cup \{-1\} \mid (k_j, \kappa_j) = P_i\}, \quad i = 0, \dots, q,$$

$$\mathcal{J}_{2i+1} := \{j \in \mathcal{J} \cup \{-1\} \mid (k_j, \kappa_j) \in \mathcal{NP}_i\}, \quad i = 0, \dots, q,$$

where \mathcal{NP}_i is the edge connecting P_i and P_{i+1} . In case 3 it finally holds

$$Z_\gamma = W_p^{\ell+2m(\kappa_0-1/p)}(\Sigma; V_\rho), \quad Z_\gamma^1 = B_{pp}^{k_{j_1}(\kappa_0-1/p)/(1+\kappa_0-\kappa_{j_1})}(\Sigma; V_\rho).$$

(Recall that $B_{pp}^s = W_p^s$ if $s > 0$ is not an integer.)

We now come back to the general situation. To state our main assumptions, we introduce the spaces

$$Z = W_p^{2m\kappa_0}(\Sigma; V_\rho) = Y_{01}, \quad Z_1 = W_p^{\ell+2m\kappa_0}(\Sigma; V_\rho),$$

$$E = X \times Z, \quad E_1 = X_1 \times Z_1, \quad E_\gamma = X_\gamma \times Z_\gamma, \quad \mathbb{E}_1(J) = \mathbb{E}_u(J) \times \mathbb{E}_\rho(J),$$

for the solutions $(u(t), \rho(t))$. Clearly, $Z_1 \hookrightarrow Z_\gamma \hookrightarrow Z$.

Throughout, W_γ denotes a nonempty convex open subset of E_γ . We define

$$\begin{aligned} W_1 &= \{w_0 \in E_1 \mid w_0 \in W_\gamma\}, \\ \mathbb{W}_1(J) &= \{w \in \mathbb{E}_1(J) \mid w(t) \in W_\gamma \ (\forall t \in J)\} \end{aligned} \tag{2.12}$$

Then, W_1 is convex and open in E_1 , and $\mathbb{W}_1(J)$ is convex and (if J is compact) open in $\mathbb{E}_1(J)$, due to (2.5) and (2.11). The nonlinear maps in (1.1) shall satisfy

(R) $\mathcal{A} \in C^1(W_\gamma; \mathcal{L}(X_1, X))$, $\mathcal{R} \in C^1(W_\gamma \times Y_{0\gamma}; X)$, and $\mathcal{D} = (\mathcal{D}_0, \dots, \mathcal{D}_m) \in C^1(W_1; Y_1)$ induces a map $\mathcal{D} \in C^1(\mathbb{W}_1(J); \mathbb{F}(J))$ for any compact J . The first derivatives of these maps are bounded and uniformly continuous on all closed balls.

We consider $\mathcal{A}'(w)$ as bilinear map from $E_\gamma \times X_1$ to X and $\mathcal{A}'(w)v$ as a bounded linear map from E_γ to X , where $w \in W_\gamma$ and $v \in X_1$. The embeddings (2.5), (2.7), (2.11) and (2.19) then imply that these operators also induce maps

$$\begin{aligned}\mathcal{A} &\in C^1(\mathbb{W}_1(J); C_b(J; \mathcal{L}(X_1, X))) \cap C^1(\mathbb{W}_1(J) \times L_p(J; X_1); \mathbb{E}(J)), \\ \mathcal{R} &\in C^1(\mathbb{W}_1(J); C_b(J; X)), \quad \mathcal{D} \in C^1(W_\gamma; Y_\gamma),\end{aligned}$$

respectively. We set $\widehat{\mathcal{D}} = (\mathcal{D}_1, \dots, \mathcal{D}_m)$. Occasionally, we will need one more degree of smoothness of the operators as recorded in the following hypothesis.

(RR) Condition (R) holds and the maps $\mathcal{A}' : W_\gamma \rightarrow \mathcal{L}_2(E_\gamma \times X_1, X)$, $\mathcal{R}' : W_\gamma \times Y_{0\gamma} \rightarrow \mathcal{L}(E_\gamma \times Y_{0\gamma}, X)$, $\mathcal{D}' : \mathbb{W}_1(J) \rightarrow \mathcal{L}(\mathbb{E}_1(J), \mathbb{F}(J))$ are Lipschitz on closed balls.

We introduce two basic types of examples for such maps covering the three cases $\ell = 2m$, $\ell < 2m$ and $\ell > 2m$. In the first example we use the one phase setting, whereas the second one involves two phases.

Example 2.1. Let $\Omega \subseteq \mathbb{R}^n$ be a domain with a compact boundary of class $C^{2+\ell-m_0}$. Reaction–diffusion equations or phase field models on Ω with dynamical boundary conditions on $\partial\Omega$ lead to the following operators, where $\rho = \gamma_\Omega u$, $V := V_u = V_\rho = \mathbb{C}^N$ and we write the diffusion part in non-divergence form:

$$\begin{aligned}[\mathcal{A}(u)v](x) &= \sum_{|\alpha|=2} a_\alpha(x, u(x), \nabla u(x)) D^\alpha v(x), \quad x \in \Omega, \\ [\mathcal{R}(u)](x) &= f(x, u(x), \nabla u(x)), \quad x \in \Omega, \\ [\mathcal{D}_0(u, \rho)](z) &= b(z, \gamma_\Omega u(z), \gamma_\Omega \nabla^{m_0} u(z)) + c^g(y, \rho(g(y)), \dots, \nabla^{k_0} \rho(g(y))),\end{aligned}\tag{2.13}$$

$z = g(y) \in \Sigma$, for $m_0 \in \{0, 1\}$, $k_0 \in \mathbb{N}_0$, local coordinates g at $x \in \partial\Omega$, functions $u \in X_\gamma$, $v \in X_1$, and $\rho \in Z_1$, where the term $\gamma_\Omega \nabla^{m_0} u$ disappears if $m_0 = 0$ and the terms c^g shall induce a map which does not depend on the coordinates. Finally, we set $\mathcal{D}_1(u, \rho) = \gamma_\Omega u - \rho$, i.e., $k_1 = m_1 = 0$. We assume that

- (a) $a_\alpha \in C^1(V \times V^n; C_b(\overline{\Omega}; \mathcal{L}(V)))$ for $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = 2$, and $a_\alpha(x, 0) \rightarrow a_\alpha(\infty)$ in $\mathcal{L}(V)$ as $x \rightarrow \infty$ if Ω is unbounded;
- (b) $f \in C^1(V \times V^n; C_b(\overline{\Omega}; V))$, and $f(\cdot, 0, 0) \in L_p(\Omega)$ if Ω is unbounded;
- (c) $b \in C^2(\partial\Omega \times V \times V^n; V)$ if $m_0 = 1$ and $b \in C^3(\partial\Omega \times V; V)$ if $m_0 = 0$,
- (d) $c^g \in C^{3-m_0}(\partial\Omega \times V \times \dots \times V^{(n^{k_0})}; V)$ for all local coordinates g .

In view of (2.5) and (2.11), only continuous functions will be inserted into the nonlinearities. One can check (R) as in Proposition 10 of [19], where the case $c = 0$ was treated. Here we have $\ell = \max\{k_0, m_0\}$. For $k_0 = 2$ (surface diffusion) we are in Case 1. If $k_0 = 1$ (surface convection), we are in Case 2. Similarly one can treat higher order problems, cf. [19]. \diamond

Example 2.2. The Stefan problem with Gibbs–Thompson law is a prototypical example for our setting in Case 3, see the introduction for references.

Two phases of a substance occupy at time $t \geq 0$ open subsets $D_i(t)$ of a fixed bounded domain $D \subseteq \mathbb{R}^n$ with $\partial D \in C^2$ and outer unit normal ν_D , where the liquid phase is contained in $D_1(t)$ and the solid one in $D_2(t)$, say. The domains have the compact interface $\Gamma(t) \subseteq D$ so that $D_1(t) \dot{\cup} \Gamma(t) \dot{\cup} D_2(t) = D$. We assume that $\Gamma(t) \cap \partial D = \emptyset$ for all $t \geq 0$. The phases have the temperatures

$u_i(t)$, which are subject to the standard heat equations in $D_i(t)$ with a Neumann boundary condition on ∂D . On the interface the temperature $u_1(t) = u_2(t)$ is proportional to the mean curvature $H(\Gamma(t))$ of $\Gamma(t)$, where the mean curvature is chosen to be negative at $x \in \Gamma(t)$ if $D_1(t)$ is convex near x . (In the classical Stefan problem one assumes that $u_1(t) = u_2(t) = 0$ on the interface.) Moreover, the interface is driven by the Stefan condition saying that its normal velocity $V(t)$ is proportional to the jump in the heat fluxes, where the normal ν of $\Gamma(t)$ is defined with respect to $D_1(t)$. Here the interface and the temperatures are unknown. We thus obtain the system

$$\begin{aligned} \partial_t u_i - d_i \Delta u_i &= 0, & t > 0, x \in D_i(t), \\ \partial_{\nu_D} u_2 &= 0, & t \geq 0, x \in \partial D, \\ u_i &= \sigma H(\Gamma(t)), & t \geq 0, x \in \Gamma(t), \\ d_2 \partial_{\nu} u_2 - d_1 \partial_{\nu} u_1 &= lV, & t \geq 0, x \in \Gamma(t), \\ u_i(0) &= u_0^i, & x \in D_0^i, \quad \Gamma(0) = \Gamma_0, \end{aligned} \quad (2.14)$$

for constants $d_1, d_2, \sigma, l > 0$, initial domains $D_0^i \subseteq D$ and a closed compact C^2 hypersurface $\Gamma_0 \subseteq D$ with $\Gamma_0 = \partial D_0^1$ and $D_0^1 \dot{\cup} \Gamma_0 \dot{\cup} D_0^2 = D$, and initial temperature distributions u_0^i on D_0^i . Actually, this is a simplified model and we refer to [34] for a thermodynamically consistent version allowing for different heat capacities in the phases, kinetic undercooling and coefficients depending on the temperature. This problem could also be treated by the methods in the present paper, but for simplicity we restrict ourselves to the system (2.14).

The approach of [12], [31], and [34] relies on the so called Hanzawa transformation to a fixed domain. One chooses a domain $\Omega_1 \subset \overline{\Omega_1} \subset D$ with smooth boundary $\Sigma = \partial \Omega_1$ being close to Γ_0 in Hausdorff distance. Let $\Omega_2 = D \setminus \overline{\Omega_1}$ and $\Omega = \Omega_1 \dot{\cup} \Omega_2$. Then Ω_2 is connected, but Ω_1 may have finitely many components. One can parametrize a tubular neighborhood around Σ by base points y on Σ and the signed normal distance r to Σ . If the distances are less than a number $2a > 0$ (determined by Σ), this parametrization gives rise to a diffeomorphism. Given a height function $\rho(t) : \Sigma \rightarrow (a, a)$, we can now describe an interface $\Gamma(t)$ at time $t \geq 0$ by the map $\Sigma \mapsto D$, $y \mapsto y + \rho(t, y)\nu_\Sigma(y)$. This map can be extended to a diffeomorphism $\Theta_\rho : \mathbb{R}^n \rightarrow \mathbb{R}^n$ mapping Σ onto $\Gamma(t)$ and Ω_i onto $D_i(t)$, which is constant outside a neighborhood of Σ .

We can now make the transformation $v = u \circ \Theta_\rho =: \Phi_\rho^* u$ with inverse $u = v \circ \Theta_\rho^{-1} =: \Phi_\rho^* v$. Let $\Gamma(\rho) = \Theta_\rho(\Sigma)$ and $d = d_1 \mathbb{1}_{D_1} + d_2 \mathbb{1}_{D_2}$. We then define the transformed operators by $\mathcal{A}^0(\rho) = -\Phi_\rho^*(d\Delta)\Phi_\rho^*$, $H(\rho) = \Phi_\rho^* H(\Gamma(\rho))$, $\mathcal{R}^0(\rho, \partial_t \rho) = \partial_t - \Phi_\rho^* \partial_t \Phi_\rho^*$, and put $C(\rho) = -\beta(\rho)^{-1} \Phi_\rho^*(d_2 \partial_{\nu} - d_1 \partial_{\nu}) \Phi_\rho^*$ with a certain function $\beta(\rho) > 0$. We thus obtain the system

$$\begin{aligned} \partial_t v + \mathcal{A}^0(\rho)v &= \mathcal{R}^0(\rho, \partial_t \rho)v, & t > 0, x \in \Omega, \\ \partial_{\nu_D} v &= 0, & t \geq 0, x \in \partial D, \\ v - \sigma H(\rho) &= 0, & t \geq 0, x \in \Sigma, \\ \partial_t \rho + C(\rho)v &= 0, & t \geq 0, x \in \Sigma, \\ v(0) &= v_0, & x \in \Omega, \quad \rho(0) = \rho_0, & x \in \Sigma, \end{aligned}$$

which is of the form (1.1) with $m = 1$, $m_0 = 1$, $k_0 = -\infty$ (cf. Example 2.4), $k_1 = 2$, $m_1 = 0$, $\ell = 3 > 2m = 2$. We observe that here

$$\begin{aligned}\mathbb{E}_u(J) &= L_p(J, W_p^2(\Omega) \cap N(\partial_{\nu_D})) \cap W_p^1(J, L_p(\Omega)), \\ E_\gamma &= (W_p^{2-\frac{2}{p}}(\Omega) \cap N(\partial_{\nu_D})) \times W_p^{4-\frac{3}{p}}(\Sigma), \\ \mathbb{E}_\rho(J) &= L^p(J, W_p^{4-\frac{1}{p}}(\Sigma)) \cap W_p^{1-\frac{1}{2p}}(J, W_p^2(\Sigma)) \cap W_p^{\frac{3}{2}-\frac{1}{2p}}(J, L^p(\Sigma)), \\ &\hookrightarrow C(J; C^3(\Sigma)) \cap C^1(J; C(\Sigma)),\end{aligned}$$

and the restrictions $v|_{\overline{\Omega_i}}$ belong to $C(J; C^1(\overline{\Omega_i}))$. We set $W_\gamma = \{(v, \rho) \in E_\gamma \mid |\rho|_{C^1} < b\}$ for a sufficiently small $b \in (0, a]$. From the explicit formulas for these maps given e.g. in Section 2 of [34] one can deduce that (RR) (and also (S) below) holds, cf. Lemma 7.4 in [12] or part (iv) in the proof of Theorem 3.5 of [34]. \diamond

We further impose ellipticity conditions on the linearizations of our nonlinear maps \mathcal{A} , \mathcal{R} , \mathcal{D}_j . For functions $w_* = (u_*, \rho_*) \in \mathbb{W}_1(J)$, $t \in J$ and $j \in \{0, 1, \dots, m\}$, we define

$$\begin{aligned}B_j(t) &= \partial_1 \mathcal{D}_j(u_*(t), \rho_*(t)) \in \mathcal{L}(X_1, Y_{j1}) \cap \mathcal{L}(X_\gamma, Y_{j\gamma}), \\ C_j(t) &= \partial_2 \mathcal{D}_j(u_*(t), \rho_*(t)) \in \mathcal{L}(Z_1, Y_{j1}) \cap \mathcal{L}(Z_\gamma, Y_{j\gamma}), \\ A(t) &= \mathcal{A}(w_*(t)) \in \mathcal{L}(X_1, X), \\ A_{*u}(t) &= \mathcal{A}(w_*(t)) + \partial_1 \mathcal{A}(u_*(t), \rho_*(t)) u_*(t) - \partial_1 \mathcal{R}(u_*(t), \rho_*(t), \dot{\rho}_*(t)) \in \mathcal{L}(X_1, X), \\ A_{*\rho}(t) &= \partial_2 \mathcal{A}(u_*(t), \rho_*(t)) u_*(t) - \partial_2 \mathcal{R}(u_*(t), \rho_*(t), \dot{\rho}_*(t)) \in \mathcal{L}(Z_\gamma, X), \\ A_{*\dot{\rho}}(t) &= -\partial_3 \mathcal{R}(u_*(t), \rho_*(t), \dot{\rho}_*(t)) \in \mathcal{L}(Y_{0\gamma}, X), \\ A_*(t) &= (A_{*u}(t), A_{*\rho}(t), A_{*\dot{\rho}}(t)) \in \mathcal{L}(X_1 \times Z_\gamma \times Y_{0\gamma}, X).\end{aligned}\tag{2.15}$$

For a time independent $w_0 = (u_0, \rho_0) \in W_\gamma$, we take some $(u_*, \rho_*) \in \mathbb{W}_1([0, 1])$ with $u_*(0) = u_0$ and $\rho_*(0) = \rho_0$ (e.g. with $\dot{\rho}_*(0) = 0$) and write $A = A(0)$, $B_j = B_j(0)$ and $C_j = C_j(0)$, cf. (2.5), (2.7), (2.11). For $(w_0, y_0) \in W_1 \times Y_{0\gamma}$ we define A_* by inserting (w_0, y_0) instead of $(w_*(t), \dot{\rho}_*(t))$. For an equilibrium w_0 we will always take $y_0 = 0$. For simplicity, we set

$$B = (B_0, \dots, B_m), \quad \widehat{B} = (B_1, \dots, B_m), \quad C = (C_0, \dots, C_m), \quad \widehat{C} = (C_1, \dots, C_m).$$

We also make use of the operator matrices

$$\Lambda = \begin{pmatrix} A & 0 \\ B_0 & C_0 \end{pmatrix}, \quad \Lambda_* = \begin{pmatrix} A_{*u} - A_{*\dot{\rho}} B_0 & A_{*\rho} - A_{*\dot{\rho}} C_0 \\ B_0 & C_0 \end{pmatrix}\tag{2.16}$$

acting from E_1 to E , see (2.15), with given $w_0 \in W_\gamma$ for Λ and $(w_0, y_0) \in W_1 \times Y_{0\gamma}$ for Λ_* . (If we deal with equilibria we put $y_0 = 0$.) We see below that these matrices generate analytic semigroups after restricting them to suitable domains. The semigroup generated by $-\Lambda_*$ will play an important role in the study of asymptotic properties of (1.1) later on. In the third case ($\ell > 2m$) these semigroups do not act on E but on the smaller space E_0 defined by

$$\begin{aligned}Z_0 &= B_{pp}^\varsigma(\Sigma; V_\rho), \quad \text{where } \varsigma = 2m\kappa_0 \text{ if } \ell \leq 2m, \quad \varsigma = \frac{k_{j_1}\kappa_0}{1 + \kappa_0 - \kappa_{j_1}} \text{ if } \ell > 2m, \\ E_0 &= X \times Z_0.\end{aligned}\tag{2.17}$$

The space Z_0 occurs naturally in view of the embedding

$$\mathbb{E}_\rho \hookrightarrow W_p^1(J; Z_0), \quad (2.18)$$

see p.3157 in [8] or Proposition 3.2 of [27]. The trace spaces are ordered as

$$Z_0 \hookrightarrow Z, \quad Z_1 \hookrightarrow Z_\gamma \hookrightarrow Z_0 \hookrightarrow Z_\gamma^1 \hookrightarrow Y_{0\gamma}. \quad (2.19)$$

(For the first and last embedding use (2.9) and $\ell_{j_1} > 2m$ in Case 3.) The domain of the generators will contain compatibility conditions expressed by the spaces

$$\begin{aligned} \tilde{E}_\gamma &= \{(v, \sigma) \in E_\gamma \mid B_0 v + C_0 \sigma \in Z_\gamma^1\}, \\ E_\gamma^0 &= \{(v, \sigma) \in \tilde{E}_\gamma \mid \hat{B}v + \hat{C}\sigma = 0\}, \\ E_1^0 &= \{(v, \sigma) \in E_1 \mid B_0 v + C_0 \sigma \in Z_0, \hat{B}v + \hat{C}\sigma = 0\}, \end{aligned} \quad (2.20)$$

which are Banach spaces endowed with the canonical norms $|(v, \sigma)|_{E_\gamma} + |B_0 v + C_0 \sigma|_{Z_\gamma^1}$ and $|(v, \sigma)|_{E_1} + |B_0 v + C_0 \sigma|_{Z_0}$, respectively, due to (2.15) and (2.19). We equip Λ with the domain $D(\Lambda) = E_1^0$ and denote by Λ_0 the restriction of Λ_* to $D(\Lambda_0) = E_1^0$.

For a given $(u_*, \rho_*) \in \mathbb{W}_1([0, T])$ and any $T > 0$, we will assume that the operators $A(t)$, $B(t)$ and $C(t)$ with $t \in [0, T]$ satisfy the assumptions of [8]; i.e., they are differentiable operators

$$\begin{aligned} A(t)v(x) &= \sum_{|\alpha| \leq 2m} a_\alpha(t, x) D^\alpha v(x), & B_j(t)v(y) &= \sum_{|\beta| \leq m_j} b_{j\beta}(t, y) \gamma_\Omega D^\beta v(y) \\ C_j(t)\sigma \circ g(z) &= \sum_{|\gamma| \leq k_j} c_{j\gamma}^g(t, z) D_{n-1}^\gamma(\sigma \circ g)(z) \end{aligned} \quad (2.21)$$

for $(v, \sigma) \in E_\gamma$, $j \in \{0, 1, \dots, m\}$, $x \in \Omega$, $y \in \Sigma$, $t \in [0, T]$, local coordinates g for Σ and z belonging to the domain of g in \mathbb{R}^{n-1} . Usually we omit the trace operator here. Observe that the coefficients of $C_j(t)$ depend on the local coordinates g , but the operator itself is independent of the choice of coordinates. In the two phase case the term $b_{j\beta}(t, y) \gamma_\Omega D^\beta v(y)$ is understood as $b_{j\beta}^1(t, y) \gamma_\Omega^1 D^\beta v(y) - b_{j\beta}^2(t, y) \gamma_\Omega^2 D^\beta v(y)$ where γ_Ω^i gives the trace of functions on Ω_i to the interface Σ . Still in the two phase case, on the (possibly empty) outside boundaries Γ_1 and Γ_2 we consider boundary operators

$$B_j^0 v(y) = \sum_{|\beta| \leq m_j^0} b_{j\beta}^0(y) \gamma_\Omega D^\beta v(y) \quad (2.22)$$

of order $m_j^0 \in \{0, \dots, 2m-1\}$ for $y \in \Gamma_1 \dot{\cup} \Gamma_2$ and $j = \{1, \dots, m\}$. We set $B^0 = (B_1^0, \dots, B_m^0)$. We start with the regularity assumptions for the coefficients.

(S) The operators $A(t)$, $B(t)$ and $C(t)$ are given by (2.21). If $|\alpha| = 2m$, then $a_\alpha \in C([0, T] \times \bar{\Omega}; \mathcal{L}(V_u))$ and, if Ω is unbounded, $a(t, x)$ converges as $|x| \rightarrow \infty$ to some $a(t, \infty)$ uniformly in $t \in [0, T]$. If $|\alpha| < 2m$, then $a_\alpha \in (L_p + L_\infty)([0, T] \times \Omega; \mathcal{L}(V_u))$. For $\beta \leq m_j$, $|\gamma| \leq k_j$, $j = 0, \dots, m$ and all coordinates g , it holds $b_{j\beta}, c_{j\gamma}^g \in \mathbb{F}_j([0, T]; V)$ in the one phase case as well as $b_{j\beta}^1, b_{j\beta}^2, c_{j\gamma}^g \in \mathbb{F}_j([0, T]; V)$ and $b_{j\beta}^0 \in C^{2m-m_j^0}(\Gamma_1 \dot{\cup} \Gamma_2; V_u)$ (for $j \geq 1$) in the two phase case.

Here we have $V = \mathcal{L}(V_u, V_\rho)$, resp. $V = \mathcal{L}(V_u)$, for $b_{j\beta}$ and $j = 0$, resp. $j \geq 1$, and $V = \mathcal{L}(V_\rho)$, resp. $V = \mathcal{L}(V_\rho, V_u)$, for $c_{j\beta}^g$ and $j = 0$, resp. $j \geq 1$. The function spaces \mathbb{F}_j arise naturally in view of (2.6) and (2.11).

The symbols of the principal parts of the linear differential operators are denoted by $\mathcal{A}_\#$, $\mathcal{B}_{j\#}$ and $\mathcal{C}_{j\#}$, cf. e.g. [6], where we put $C_{j\#}(t) = 0$ if $j \notin \mathcal{J}$. We first assume that the operators $A(t)$ are *normally elliptic*; i.e.,

(E) $\sigma(\mathcal{A}_\#(t, x, \xi)) \subset \mathbb{C}_+$ and (if Ω is unbounded) $\sigma(\mathcal{A}_\#(t, \infty, \xi)) \subset \mathbb{C}_+$, for $x \in \overline{\Omega}$, $t \in [0, T]$ and $\xi \in \mathbb{R}^n$ with $|\xi| = 1$.

To formulate the *Lopatinskii-Shapiro conditions*, at a given point $x \in \Sigma$ we rotate the coordinate system such that $\nu(x) = (0, \dots, 0, -1) \in \mathbb{R}^n$, without changing the notation. Moreover, in the two phase case we reflect the coefficients and functions in Ω_2 in normal direction to Ω_1 . On Ω_1 we thus obtain a system of two components: the given one on Ω_1 and the reflected one from Ω_2 . The latter is still normally elliptic. For $x \in \Sigma$, the conditions below shall refer to this modified system in the two phase setting, where we set $V = V_u$ in the one phase case and $V = V_u^2$ in the two phase case.

(LS) For each $x \in \Sigma$, $t \in [0, T]$, $\lambda \in \overline{\mathbb{C}_+}$ and $\xi' \in \mathbb{R}^{n-1}$ with $|\lambda| + |\xi'| \neq 0$, the ordinary initial value problem

$$\begin{aligned} & (\lambda + \mathcal{A}_\#(t, x, \xi', D_y))v(y) = 0, \quad y > 0, \\ & \mathcal{B}_{0\#}(t, x, \xi', D_y)v(0) + (\lambda + \mathcal{C}_{0\#}^g(t, x, \xi'))\sigma = 0, \\ & \mathcal{B}_{j\#}^\nu(t, x, \xi', D_y)v(0) + \mathcal{C}_{j\#}^g(t, x, \xi')\sigma = 0, \quad j = 1, \dots, m, \end{aligned}$$

has only the trivial solution $(v, \sigma) = 0$ in $C_0([0, \infty); V) \times V_\rho$. Moreover, in the two phase case on the outside boundaries $\Gamma_1 \cup \Gamma_2$ the analogous condition shall hold for A and B_j^0 .

In Cases 2 and 3, the following additional ‘asymptotic’ conditions are required, respectively.

(LS_∞⁻) Let $\ell < 2m$. For each $x \in \Sigma$, $t \in [0, T]$, $\lambda \in \overline{\mathbb{C}_+}$ and $\xi' \in \mathbb{R}^{n-1}$ with $|\lambda| + |\xi'| \neq 0$, the ordinary initial value problem

$$\begin{aligned} & (\lambda + \mathcal{A}_\#(t, x, \xi', D_y))v(y) = 0, \quad y > 0, \\ & \mathcal{B}_{j\#}^\nu(t, x, \xi', D_y)v(0) = 0, \quad j = 1, \dots, m, \end{aligned}$$

and for all $\lambda \in \overline{\mathbb{C}_+}$ and $|\xi'| = 1$ the problem

$$\begin{aligned} & \mathcal{A}_\#(t, x, \xi', D_y)v(y) = 0, \quad y > 0, \\ & \mathcal{B}_{0\#}(t, x, \xi', D_y)v(0) + (\lambda + \mathcal{C}_{0\#}^g(t, x, \xi'))\sigma = 0, \\ & \mathcal{B}_{j\#}(t, x, \xi', D_y)v(0) + \mathcal{C}_{j\#}^g(t, x, \xi')\sigma = 0, \quad j = 1, \dots, m, \end{aligned}$$

only have the trivial solution $(v, \sigma) = 0$ in $C_0([0, \infty); V) \times V_\rho$.

(LS_∞⁺) Let $\ell > 2m$. For each $x \in \Sigma$, $t \in [0, T]$, $\lambda \in \overline{\mathbb{C}_+}$ and $\xi' \in \mathbb{R}^{n-1} \setminus \{0\}$, the ordinary initial value problem

$$\begin{aligned} & (\lambda + \mathcal{A}_\#(t, x, \xi', D_y))v(y) = 0, \quad y > 0, \\ & \mathcal{B}_{j\#}(t, x, \xi', D_y)v(0) + \delta_{j, \mathcal{J}_{2q+1}} \mathcal{C}_{j\#}^g(t, x, \xi')\sigma = 0, \quad j = 0, \dots, m, \end{aligned}$$

and for all $\lambda \in \overline{\mathbb{C}_+} \setminus \{0\}$, $|\xi'| = 1$ and $i = 1, \dots, 2q$, the problem

$$\begin{aligned} & (\lambda + \mathcal{A}_\sharp(t, x, 0, D_y))v(y) = 0, \quad y > 0, \\ & \mathcal{B}_{0\sharp}(t, x, 0, D_y)v(0) + \delta_{-1, \mathcal{J}_i}\lambda\sigma + \delta_{0, \mathcal{J}_i}\mathcal{C}_{0\sharp}^g(t, x, \xi')\sigma = 0, \\ & \mathcal{B}_{j\sharp}(t, x, 0, D_y)v(0) + \delta_{j, \mathcal{J}_i}\mathcal{C}_{j\sharp}^g(t, x, \xi')\sigma = 0, \quad j = 1, \dots, m, \end{aligned}$$

have only the trivial solution $(v, \sigma) = 0$ in $C_0([0, \infty); V) \times V_\rho$. Here, $\delta_{j, \mathcal{J}_i} = 1$ if $j \in \mathcal{J}_i$ and $\delta_{j, \mathcal{J}_i} = 0$ otherwise.

We first give typical examples for these cases.

Example 2.3. We continue to discuss Example 2.1. One can show (S) as in in Proposition 10 of [19], where also the derivatives were computed for $c = 0$. To illustrate the ellipticity conditions, we recall two typical linearizations from Examples 3.6 and 3.2 of [8], respectively. Here $m = 1$ and $\sigma = \gamma_\Sigma v$ is the static boundary condition so that $m_1 = k_1 = 0$. The linear model problems for dynamic boundary conditions with surface convection looks like

$$\begin{aligned} \partial_t v - \Delta v &= g, \quad t > 0, x \in \Omega, \\ \partial_t \sigma + \partial_\nu v + a \cdot \nabla_\Sigma \sigma &= h, \quad t > 0, x \in \Sigma, \end{aligned}$$

for the surface gradient ∇_Σ and a tangential vector field $a \in C^1(\Sigma, \mathbb{R}^{n-1})$. The ellipticity conditions in Case 2 hold with $k_0 = m_0 = 1$. Dynamic boundary conditions with surface diffusion are described by the system

$$\begin{aligned} \partial_t v - \Delta v &= g, \quad t > 0, x \in \Omega, \\ \partial_t \sigma + \partial_\nu v + \Delta_\Sigma \sigma &= h, \quad t > 0, x \in \Sigma, \end{aligned}$$

for the Laplace Beltrami operator Δ_Σ . It satisfies the ellipticity conditions in Case 1 with $m_0 = 1$ and $k_0 = 2$. Similarly one treats the Cahn–Hilliard phase field model, where $m = 2$, see Example 3.3 in [8]. \diamond

Example 2.4. In the framework of Example 2.2 we focus on the most important case for later results, namely the linearization at an equilibrium which is given by a sphere Σ of radius $R > 0$ and by constant temperature. (General initial configurations are similarly treated in e.g. [34].) Here we obtain

$$\begin{aligned} \partial_t v - d\Delta v &= f, \quad t > 0, x \in \Omega, \\ l \partial_t \rho - (d_2 \partial_\nu v_2 - d_1 \partial_\nu v_1) &= g, \quad x \in \Sigma, \\ \partial_\nu v &= 0, \quad x \in \partial D, \\ v + \sigma \left(\frac{1}{R^2} + \frac{1}{n-1} \Delta_\Sigma \right) \rho &= h, \quad t \geq 0, x \in \Sigma, \end{aligned}$$

see equations (1.8) in [31] and (4.3) in [34]. One can check that the ellipticity assumptions hold as in Example 3.4 of [8] for the one phase case. Here, $m_0 = 1$, $m_1 = 0$, $2m = k_1 = 2$, $k_0 = -\infty$ and $\ell = \ell_1 = 3 > 2m$. \diamond

Condition (LS) is analogous to the usual Lopatinski Shapiro conditions, see e.g. [6], [7] and the references therein. The other two conditions have been introduced and discussed in [8]. There it is also shown that (E), (LS) and, if if $\ell \neq 2m$, (LS_∞^\pm) are necessary for the following crucial regularity result. It is taken from Theorems 2.1 and 2.2 of [8], where part (c) can be shown as

Corollary 2.3 of [8]. However, the results in [8] were only proved in the one phase setting. To extend them to the two phase case, we proceed as in Theorem 3.3 of [34]. One first reduces the assertion via localization and perturbation to model problems with constant coefficients on the full space for the interior points x , on the halfspace for x in the outer boundaries $\Gamma_1 \cup \Gamma_2$ and on $\mathbb{R}^{n-1} \times (\mathbb{R} \setminus \{0\})$ for x on the interface Σ (where Ω_2 corresponds to the lower halfplane). The first two problems have been solved in [6] and [7]. To solve the third one, we reflect the functions the lower halfplane to one on the upper halfplane, obtaining a second component. Employing the Lopatinski Shapiro conditions stated above, one can then proceed as in [8] in the one phase case.

Theorem 2.5. *Assume that the operators $A(t)$, $B(t)$, $C(t)$ and B^0 , $t \in J = [0, T]$, are defined for some $w_* \in \mathbb{E}_1([0, T])$ and satisfy (S) , (E) , (LS) , and (LS_∞^-) if $\ell < 2m$ or (LS_∞^+) if $\ell > 2m$. Then the following assertions are true.*

(a) *There is a unique solution $(u, \rho) \in \mathbb{E}_1(J)$ of the problem*

$$\begin{aligned} \partial_t u(t) + A(t)u(t) &= f(t), & \text{on } \Omega, t \in (0, T], \\ \partial_t \rho(t) + B_0(t)u(t) + C_0(t)\rho(t) &= g_0(t), & \text{on } \Sigma, t \in [0, T], \\ B_j(t)u(t) + C_j(t)\rho(t) &= g_j(t), & \text{on } \Sigma, t \in [0, T], j = 1, \dots, m, \\ B^0 u(t) &= 0, & \text{on } \Gamma_1 \cup \Gamma_2, t \in [0, T], \\ (u(0), \rho(0)) &= (u_0, \rho_0), & \text{on } \Omega \times \Sigma, \end{aligned} \quad (2.23)$$

(where we drop the equation $B^0 u(t) = 0$ in the one phase setting) if and only if f , g , u_0 and ρ_0 belong to the data space

$$\begin{aligned} \mathbb{D}(J) := \{ (u_0, \rho_0, f, g) \in X_\gamma \times Z_\gamma \times \mathbb{E}(J) \times \mathbb{F}(J) \mid & B_j(0)u_0 + C_j(0)\rho_0 = g_j(0) \\ \text{for } j = 1, \dots, m; & g_0(0) - B_0(0)u_0 - C_0(0)\rho_0 \in Z_\gamma^1 \}. \end{aligned}$$

The corresponding solution operator $\mathcal{S} : \mathbb{D}(J) \rightarrow \mathbb{E}_1(J)$ is continuous. The norm of \mathcal{S} is bounded uniformly in $T' \in (0, T]$ if we restrict it to the subspace $\mathbb{D}_0([0, T'])$ of $\mathbb{D}([0, T'])$ of data with $g(0) = 0$.

(b) *If the coefficients do not depend on time, the operator $-\Lambda$ with $D(\Lambda) = E_1^0$ generates an analytic C_0 -semigroup in E_0 .*

(c) *There is a $\mu_0 \geq 0$ larger than the growth bound of $-\Lambda$ such that the results on (2.23) hold on $J = \mathbb{R}_+$ if we replace ∂_t by $\partial_t + \mu$ (2.23) for any $\mu \geq \mu_0$, and*

$$\| (u, \rho) \|_{L_p(\mathbb{R}_+; E))} \leq \frac{c}{\mu} \| (u_0, \rho_0, f, g) \|_{\mathbb{D}(\mathbb{R}_+)} \quad (2.24)$$

Here $\mathbb{D}(J)$ is a Banach space endowed with the norm

$$\| f \|_{\mathbb{E}(J)} + \| g \|_{\mathbb{F}(J)} + |(u_0, \rho_0)|_{E_\gamma} + |g_0(0) - B_0(0)u_0 - C_0(0)\rho_0|_{Z_\gamma^1}$$

and \mathbb{D}_0 is closed in it, due to (2.7), (2.15) and (2.19). The assertion concerning \mathbb{D}_0 can be checked as in e.g. Theorem 2.2 of [28]. We note that the *compatibility conditions* expressed by \mathbb{D} are preserved by the solutions of (2.23). Later we mostly need the following variant of the above theorem which involves lower order terms, cf. (2.15) and (2.16).

Corollary 2.6. *Assume that (R) and the assumptions of Theorem 2.5 are valid. Then the assertions (a) and (b) of this theorem still hold for the problem*

$$\begin{aligned} \partial_t u(t) + A_{*u}(t)u(t) + A_{*\rho}(t)\rho(t) + A_{*\dot{\rho}}(t)\dot{\rho}(t) &= f(t), & \text{on } \Omega, t \in (0, T], \\ \partial_t \rho(t) + B_0(t)u(t) + C_0(t)\rho(t) &= g_0(t), & \text{on } \Sigma, t \in [0, T], \\ \widehat{B}(t)u(t) + \widehat{C}(t)\rho(t) &= \widehat{g}(t), & \text{on } \Sigma, t \in [0, T], \\ B^0 u(t) &= 0, & \text{on } \Gamma_1 \cup \Gamma_2, t \in [0, T], \\ (u(0), \rho(0)) &= (u_0, \rho_0), & \text{on } \Omega \times \Sigma, \end{aligned} \quad (2.25)$$

(where we drop the equation $B^0 u(t) = 0$ in the one phase setting) instead of (2.23) and the operator $\Lambda_0 := \Lambda_*|E_1^0$ instead of Λ . (Λ_* is defined for $(w_*, y_*) \in W_1 \times Y_{0\gamma}$, see (2.16) and (2.20).) After increasing μ_0 if needed, for every $(u_0, \rho_0, f, g) \in \mathbb{D}(\mathbb{R}_+)$ there is a unique solution $(u, \rho) \in \mathbb{E}_1(\mathbb{R}_+)$ of the system

$$\begin{aligned} \partial_t u(t) + (A_{*u}(t) + \mu - A_{*\dot{\rho}}(t)B_0(t))u(t) \\ + (A_{*\rho}(t) - A_{*\dot{\rho}}(t)C_0(t))\rho(t) &= f(t), & \text{on } \Omega, t \in (0, T], \\ \partial_t \rho(t) + B_0(t)u(t) + (C_0(t) + \mu)\rho(t) &= g_0(t), & \text{on } \Sigma, t \in [0, T], \\ \widehat{B}(t)u(t) + \widehat{C}(t)\rho(t) &= \widehat{g}(t), & \text{on } \Sigma, t \in [0, T], \\ B^0 u(t) &= 0, & \text{on } \Gamma_1 \cup \Gamma_2, t \in [0, T], \\ (u(0), \rho(0)) &= (u_0, \rho_0), & \text{on } \Omega \times \Sigma, \end{aligned} \quad (2.26)$$

(where we drop the equation $B^0 u(t) = 0$ in the one phase setting) for each $\mu \geq \mu_0$, and it holds $\|(u, \rho)\|_{\mathbb{E}_1(\mathbb{R}_+)} \leq c \|(u_0, \rho_0, f, g)\|_{\mathbb{D}(\mathbb{R}_+)}$.

Proof. (1) To solve the problem (2.25) on $J = [0, T]$, we fix $(u_0, \rho_0, f, g) \in \mathbb{D}(J)$. Let $(v, \sigma), (\bar{v}, \bar{\sigma}) \in \mathbb{E}_1(J)$. We set

$$\tilde{f}(v, \sigma) = \tilde{f} := f + (A - A_{*u})v - A_{*\rho}\sigma - A_{*\dot{\rho}}\dot{\sigma} \in \mathbb{E}(J).$$

Let $\Phi(v, \sigma) \in \mathbb{E}_1(J)$ be the solution (2.23) for the data $(u_0, \rho_0, \tilde{f}, g) \in \mathbb{D}(J)$. The function $\Phi(v, \sigma) - \Phi(\bar{v}, \bar{\sigma})$ then solves (2.23) with $u_0 = 0$, $\rho_0 = 0$ and $g = 0$. Theorem 2.5, (2.15), and the embeddings (2.5), (2.11), (2.19) thus yield

$$\begin{aligned} \|\Phi(v, \sigma) - \Phi(\bar{v}, \bar{\sigma})\|_{\mathbb{E}_1(J)} &\leq c \|\tilde{f}(v, \sigma) - \tilde{f}(\bar{v}, \bar{\sigma})\|_{\mathbb{E}(J)} \\ &\leq cT^{1/p} (\|v - \bar{v}\|_{C(J; X_\gamma)} + \|\sigma - \bar{\sigma}\|_{C(J; Z_\gamma)} + \|\partial_t(\sigma - \bar{\sigma})\|_{C(J; Z_\gamma^1)}) \\ &\leq \tilde{c}T^{1/p} \|(v, \sigma) - (\bar{v}, \bar{\sigma})\|_{\mathbb{E}_1(J)}. \end{aligned}$$

Here we can take the same constants if we replace T by $T' \in (0, T]$ since the involved differences vanish at $t = 0$. For $T' = \min\{T, (2\tilde{c})^{-p}\}$ we obtain a fix point $(u, \rho) = \Phi(u, \rho) \in \mathbb{E}_1([0, T'])$ which solves (2.25). By a finite iteration, the first assertion follows.

(2) Note that $(A_{*\dot{\rho}}(t)B_0(t), A_{*\dot{\rho}}(t)C_0(t)) \in \mathcal{L}(E_\gamma, X)$, and hence $\Lambda - \Lambda_0$ is a lower order perturbation of $(\Lambda, D(\Lambda))$, due to (2.15) and (2.16). So the assertion about Λ_0 holds. If the data belong to $\mathbb{D}(\mathbb{R}_+)$ and $\mu \geq \mu_0$, we further obtain a solution $(u, \rho) \in \mathbb{E}_1^{\text{loc}}$ of (2.26) as in Step (1). Theorem 2.5 then implies that

$$\begin{aligned} \|(u, \rho)\|_{\mathbb{E}_1([0, T])} &\leq c \|(u_0, \rho_0, \bar{f}, g)\|_{\mathbb{D}([0, T])} \\ &\leq \tilde{c} (\|(u_0, \rho_0, f, g)\|_{\mathbb{D}([0, T])} + \|(u, \rho)\|_{L_p([0, T]; E_\gamma)}), \end{aligned}$$

where $\bar{f} = f + A_{*\rho} B_0 u + A_{*\rho} C_0 \rho$. Here and below, the constants do not depend on $T \geq 1$ and $\mu \geq \mu_0$. On the right hand side we can now interpolate X_γ between X and X_1 , as well as Z_γ between Z and Z_1 . By means of Young's inequality and (2.24), we deduce

$$\begin{aligned} \|(u, \rho)\|_{L_p([0, T]; E_\gamma)} &\leq (4\hat{c})^{-1} \|(u, \rho)\|_{L_p([0, T]; E_1)} + c \|(u, \rho)\|_{L_p([0, T]; E)} \\ &\leq (4\hat{c})^{-1} \|(u, \rho)\|_{L_p([0, T]; E_1)} + c/\mu (\|(u_0, \rho_0, f, g)\|_{\mathbb{D}([0, T])} + \|(u, \rho)\|_{L_p([0, T]; E_\gamma)}). \end{aligned}$$

Fixing a sufficiently large $\mu_0 \geq 0$, we conclude that $\|(u, \rho)\|_{\mathbb{E}_1([0, T])} \leq c \|(u_0, \rho_0, f, g)\|_{\mathbb{D}([0, T])}$ for all $T \geq 1$; thus arriving at the asserted estimate. \square

In order to treat the nonlinear compatibility conditions related to (1.1), we need an 'almost right inverse' of the map (B, C) constructed in the next lemma, cf. Proposition 5 in [19] for the simpler case of static boundary conditions.

Corollary 2.7. *Assume that (R) holds. Given $(u_0, \rho_0) \in W_\gamma$, take some $(u_*, \rho_*) \in \mathbb{W}_1([0, T])$ and $T > 0$ with $u_*(0) = u_0$ and $\rho_*(0) = \rho_0$. Assume that the corresponding operators $A(t)$, $B(t)$, $C(t)$ and B^0 , $t \in [0, T]$, satisfy (S), (E), (LS), and (LS_∞^-) if $\ell < 2m$ or (LS_∞^+) if $\ell > 2m$. Put $A = A(0)$, $B = B(0)$ and $C = C(0)$. Then there is a map $\mathcal{N}_\gamma \in \mathcal{L}(Y_\gamma, E_\gamma)$ such that $(\hat{B}, \hat{C})\mathcal{N}_\gamma = I_1$, $(B_0, C_0)\mathcal{N}_\gamma - I_0 \in \mathcal{L}(Y_\gamma, Z_\gamma^1)$, where $I_0(\psi_0, \dots, \psi_m) = \psi_0$ and $I_1(\psi_0, \dots, \psi_m) = (\psi_1, \dots, \psi_m) =: \hat{\psi}$.*

Proof. We first note that $T > 0$ and $(u_*, \rho_*) \in \mathbb{W}_1([0, T])$ with $u_*(0) = u_0$ and $\rho_*(0) = \rho_0$ exist due to (2.5) and (2.11). Using (2.7), for given $\psi \in Y_\gamma$ we find a function $g \in \mathbb{F}([T/2, T])$ with $g(T/2) = \psi$ and $\|g\|_{\mathbb{F}} \leq c|\psi|_{Y_\gamma}$. Set $g(t) = 2tT^{-1}g(T-t)$ for $t \in [0, T/2]$. Note that $\|g\|_{\mathbb{F}([0, T])} \leq c|\psi|_{Y_\gamma}$. Then $(0, 0, 0, g) \in \mathbb{D}([0, T])$ and Theorem 2.5 gives a solution $(v, \sigma) \in \mathbb{E}_1([0, T])$ of (2.23) for this data. Defining $\mathcal{N}_\gamma \psi := (v(T/2), \sigma(T/2))$, we see that $(\hat{B}, \hat{C})\mathcal{N}_\gamma \psi = \hat{\psi}$ and $(B_0, C_0)\mathcal{N}_\gamma \psi - \psi_0 = \dot{\sigma}(T/2) \in Z_\gamma^1$. The asserted continuity follows from

$$|(v(T/2), \sigma(T/2))|_{E_\gamma} + |\dot{\sigma}(T/2)|_{Z_\gamma^1} \leq c\|(v, \sigma)\|_{\mathbb{E}_1} \leq c\|g\|_{\mathbb{F}} \leq c|\psi|_{Y_\gamma},$$

because of Theorem 2.5 and the embeddings (2.5), (2.11). \square

We conclude this section with a simple lemma concerning Slobodeckii spaces.

Lemma 2.8. *Let $a < b < d$, $q \in (1, \infty)$, $\kappa > 1/q$, and V be a Banach space. If $u \in W_q^\kappa((a, b); V)$ and $v \in W_q^\kappa((b, d); V)$ satisfy $u(b) = v(b)$ (where the trace exists by Sobolev's embedding), then the function w given by $w = u$ on $(a, b]$ and $w = v$ on $[b, d)$ belongs to $W_q^\kappa((a, d); V)$ with $\|w\|_{W_q^\kappa} \leq c_w (\|u\|_{W_q^\kappa} + \|v\|_{W_q^\kappa})$.*

Proof. For simplicity, we assume that a and d are finite. Define $u^0 = u - u(b)$ on (a, b) and $v^0 = v - v(b)$ on (b, d) and extend these functions by 0 to (a, d) keeping the notation. It holds that

$$\|u^0\|_{W_q^\kappa((a, d); V)} \leq c\|u^0\|_{W_q^\kappa((a, b); V)} \leq c\|u\|_{W_q^\kappa((a, b); V)},$$

and similarly for v . (The continuity of the 0-extension is shown by interpolation between L^q and W_q^1 with 0 boundary conditions.) The assertion now follows from $w = u^0 + v^0 + 2u(b)$. \square

3. WELLPOSEDNESS AND REGULARITY

In our main results we use the following linearization setup. Assume that (R) holds and define the operators from (2.15) for any given $w_* = (u_*, \rho_*) \in \mathbb{W}_1(J) = \mathbb{W}_1$. We put $\mathbb{W}_1^* = \mathbb{W}_1 - w_*$ and define the nonlinear maps

$$\begin{aligned} F \in C^1(\mathbb{W}_1^*; \mathbb{E}) \quad \text{and} \quad G \in C^1(\mathbb{W}_1^*; \mathbb{F}) \quad \text{with} \\ \text{loc. bdd. derivative, } F(0) = 0, G(0) = 0 \quad \text{and} \quad F'(0) = 0, G'(0) = 0, \end{aligned} \quad (3.1)$$

by setting

$$\begin{aligned} F(v, \sigma) &= (\mathcal{A}(w_*)v - \mathcal{A}(w_* + (v, \sigma))v) \\ &\quad - (\mathcal{A}(w_* + (v, \sigma))u_* - \mathcal{A}(w_*)u_* - [\mathcal{A}'(w_*)u_*](v, \sigma)) \\ &\quad + (\mathcal{R}(w_* + (v, \sigma), \dot{\rho}_* + \dot{\sigma}) - \mathcal{R}(w_*, \dot{\rho}_*) - \mathcal{R}'(w_*, \dot{\rho}_*)(v, \sigma, \dot{\sigma})), \\ G(v, \sigma) &= \mathcal{D}'(w_*)(v, \sigma) + \mathcal{D}(w_*) - \mathcal{D}(w_* + (v, \sigma)), \end{aligned} \quad (3.2)$$

for $(v, \sigma) \in \mathbb{W}_1^*$. We put $\widehat{G} = (G_1, \dots, G_m)$. It holds

$$\begin{aligned} F'(\varphi)(u, \rho) &= [\mathcal{A}(w_*) - \mathcal{A}(w_* + \varphi)]u + [\mathcal{A}'(w_*)u_* - \mathcal{A}'(w_* + \varphi)(u_* + v)](u, \rho) \\ &\quad + [\mathcal{R}'(u_* + v, \rho_* + \sigma, \dot{\rho}_* + \dot{\sigma}) - \mathcal{R}'(u_*, \rho_*, \dot{\rho}_*)](u, \rho, \dot{\sigma}), \\ G'(\varphi)(u, \rho) &= [\mathcal{D}'(w_*) - \mathcal{D}'(w_* + \varphi)](u, \rho) \end{aligned} \quad (3.3)$$

for $\varphi = (v, \sigma) \in \mathbb{W}_1^*$ and $(u, \rho) \in \mathbb{E}_1$. The asserted mapping properties easily follow from (R) and the embeddings (2.5), (2.11), (2.19). Observe that $\widehat{\mathcal{D}}(w_*) = 0$ and $\mathcal{D}_0(w_*)(t) = -\dot{\rho}_*(t) \in Z_\gamma^1$ if w_* solves (1.1) and that also $\mathcal{D}_0(w_*) = 0$ if w_* is an equilibrium of (1.1). Replacing here $\dot{\sigma}$ by $y \in Y_{0\gamma}$ and fixing $t \in J$, we obtain maps

$$\begin{aligned} F(t) &\in C^1((W_1 - w_*(t)) \times Y_{0\gamma}; X), \\ G(t) &\in C^1(W_1 - w_*(t); Y_1) \cap C^1(W_\gamma - w_*(t); Y_\gamma) \end{aligned} \quad (3.4)$$

with the analogous properties as in (3.1).

Let $w_* = (u_*, \rho_*) \in \mathbb{W}_1(J)$ be a solution of (1.1) for some J with $\min J = 0$ and initial values (u_{0*}, ρ_{0*}) . In view of the embeddings (2.5), (2.11), (2.19) and the mapping properties in (R), the initial values and the solutions at time t must belong to the solution manifold

$$\mathcal{M} = \{w_0 = (u_0, \rho_0) \in W_\gamma \mid \widehat{\mathcal{D}}(w_0) = 0, \mathcal{D}_0(w_0) \in Z_\gamma^1\}. \quad (3.5)$$

For $(u_0, \rho_0) \in W_\gamma$ and $w = (u, \rho) \in \mathbb{E}_1(J)$, we put $(v_0, \sigma_0) = (u_0 - u_{0*}, \rho_0 - \rho_{0*})$ and $(v, \sigma) = (u - u_*, \rho - \rho_*)$. Using the linearization described above and (2.15), we see that $(u_0, \rho_0) \in \mathcal{M}$ if and only if (v_0, σ_0) belongs to

$$\begin{aligned} \mathcal{M}^* = \mathcal{M} - (u_{0*}, \rho_{0*}) &= \{(v_0, \sigma_0) \in W_\gamma - (u_{0*}, \rho_{0*}) \mid (\widehat{B}, \widehat{C})(v_0, \sigma_0) = \widehat{G}(v_0, \sigma_0), \\ &\quad B_0 v_0 + C_0 \sigma_0 - G_0(v_0, \sigma_0) \in Z_\gamma^1\}. \end{aligned} \quad (3.6)$$

Moreover, $(u, \rho) \in \mathbb{W}_1$ solves (1.1) if and only if $(v, \sigma) \in \mathbb{W}_1^*$ solves

$$\begin{aligned} \partial_t v(t) + A_*(t)(v(t), \sigma(t), \dot{\sigma}(t)) &= F(v, \sigma)(t), & \text{on } \Omega, t \in (0, T], \\ \partial_t \sigma(t) + B_0(t)v(t) + C_0(t)\sigma(t) &= G_0(v, \sigma)(t), & \text{on } \Sigma, t \in [0, T], \\ \widehat{B}(t)v(t) + \widehat{C}(t)\sigma(t) &= \widehat{G}(v, \sigma)(t), & \text{on } \Sigma, t \in [0, T] \end{aligned}$$

$$\begin{aligned} B^0 v(t) &= 0, & \text{on } \Gamma_1 \cup \Gamma_2, \quad t \in [0, T], \\ (v(0), \sigma(0)) &= (v_0, \sigma_0), & \text{on } \Omega \times \Sigma. \end{aligned} \quad (3.7)$$

Here we drop the equation $B^0 u(t) = 0$ in the one phase setting. This equation is mostly omitted in the following since it is already contained in the domain of $A_*(t)$ and in the solution space. We start with the basic existence and uniqueness result for (1.1).

Proposition 3.1. *Let (R) and (S) be true. Assume that (E) and (LS), as well as (LS_∞[±]) if $\ell \geq 2m$, hold for all functions $(u_{0*}, \rho_{0*}) \in W_\gamma$. Let $w_0 = (u_0, \rho_0) \in \mathcal{M}$. Then there is a number $T = T(w_0) > 0$ such that the problem (1.1) has a unique solution $w = (u, \rho) \in \mathbb{W}_1([0, T]) \hookrightarrow C([0, T]; W_\gamma)$.*

Proof. By (2.5) and (2.11) there exists a function $w_* = (u_*, \rho_*) \in \mathbb{E}_1(J) \hookrightarrow C(J, E_\gamma)$ with $w_*(0) = w_0$ and $\partial_t \rho_*(0) = -\mathcal{D}_0(w_0)$. Since $w_0 \in W_\gamma$ and w_* is continuous in E_γ there is a $T_0 > 0$ with $w_* \in \mathbb{W}_1([0, T_0])$, cf. (2.12). For this w_* , we define $A(t)$, $B(t)$ and $C(t)$ as in (2.15). Consider the problem

$$\begin{aligned} \partial_t \tilde{u}(t) + A(t) \tilde{u}(t) &= \mathcal{R}(w_*(t), \dot{\rho}_*(t)) =: f^0(t), \\ \partial_t \tilde{\rho}(t) + B_0(t) \tilde{u}(t) + C_0(t) \tilde{\rho}(t) &= \mathcal{D}'_0(w_*(t)) w_*(t) - \mathcal{D}_0(w_*(t)) =: g^0(t), \\ \widehat{B}(t) \tilde{u}(t) + \widehat{C}(t) \tilde{\rho}(t) &= \widehat{\mathcal{D}}'(w_*(t)) w_*(t) - \widehat{\mathcal{D}}(w_*(t)) =: \widehat{g}^0(t), \\ (\tilde{u}(0), \tilde{\rho}(0)) &= (u_0, \rho_0), \end{aligned} \quad (3.8)$$

for $t \in [0, T_0]$. Observe that $f^0 \in \mathbb{E}([0, T_0])$, $g^0 \in \mathbb{F}([0, T_0])$, $\widehat{g}^0(0) = \widehat{\mathcal{D}}'(w_0) w_0 - \widehat{\mathcal{D}}(w_0) = \widehat{B}(0) u_0 + \widehat{C}(0) \rho_0$ and $g^0(0) - B_0(0) u_0 - C_0(0) \rho_0 = -\mathcal{D}_0(w_0) \in Z_\gamma^1$. Theorem 2.5 thus yields a solution $\tilde{w} = (\tilde{u}, \tilde{\rho}) \in \mathbb{E}_1([0, T_0])$ of (3.8). As above, we find a $T_1 \in (0, T_0]$ such that $\tilde{w} \in \mathbb{W}_1([0, T_1])$. It further holds $\partial_t \tilde{\rho}(0) = -\mathcal{D}_0(w_0) = \partial_t \rho_*(0)$.

There is an $r_1 > 0$ such that the closed ball in $\mathbb{E}_1([0, T_1])$ with center \tilde{w} and radius r_1 belongs to $\mathbb{W}_1([0, T_1])$. We now define the space

$$\Sigma(T, r) = \{w \in \mathbb{E}_1([0, T]) \mid w(0) = w_0, \|w - \tilde{w}\|_{\mathbb{E}_1([0, T])} \leq r\} \subseteq \mathbb{W}_1([0, T_1])$$

for any $r \in (0, r_1]$ and $T \in (0, T_1]$. The set $\Sigma(T, r)$ is closed in $\mathbb{E}_1([0, T])$. For a given $w = (u, \rho) \in \Sigma(T, r)$, we look at the linear problem

$$\begin{aligned} \partial_t v(t) + A(t) v(t) &= \mathcal{R}(w(t), \dot{\rho}(t)) + A(t) u(t) - \mathcal{A}(w(t)) u(t), \\ \partial_t \sigma(t) + B_0(t) v(t) + C_0(t) \sigma(t) &= \mathcal{D}'_0(w_*(t)) w(t) - \mathcal{D}_0(w(t)), \\ \widehat{B}(t) v(t) + \widehat{C}(t) \sigma(t) &= \widehat{\mathcal{D}}'(w_*(t)) w(t) - \widehat{\mathcal{D}}(w(t)), \\ (v(0), \sigma(0)) &= (u_0, \rho_0), \end{aligned} \quad (3.9)$$

for $t \in [0, T]$. As above, Theorem 2.5 yields a solution $\varphi = (v, \sigma) =: \Phi(w) \in \mathbb{E}_1([0, T])$ of (3.9). Notice that $w = (u, \rho) \in \Sigma(T, r)$ solves (1.1) if and only if $w = \Phi(w)$. To obtain such a fixed point, we show that Φ is a strict contraction on $\Sigma(T, r)$ provided that $T, r > 0$ are small enough.

Let $(z, \tau) = \Phi(w) - \tilde{w} \in \mathbb{E}_1([0, T])$. We set $g = \mathcal{D}'(w_*)(w - w_*) - \mathcal{D}(w) + \mathcal{D}(w_*)$. We observe that problems (3.8) and (3.9) show that (z, τ) satisfies

$$\partial_t z(t) + A(t) z(t) = \mathcal{R}(w(t), \dot{\rho}(t)) - \mathcal{R}(w_*(t), \dot{\rho}_*(t))$$

$$\begin{aligned}
& + (A(w_*(t)) - \mathcal{A}(w(t)))u(t) =: f(t), \\
\partial_t \tau(t) + B_0(t)z(t) + C_0(t)\tau(t) & = g_0(t), \\
\widehat{B}(t)z(t) + \widehat{C}(t)\tau(t) & = \widehat{g}(t), \\
(z(0), \tau(0)) & = 0,
\end{aligned}$$

for $t \in [0, T]$. We note that $g(0) = \mathcal{D}(w_0) - \mathcal{D}(w_0) = 0$, as well as $f \in \mathbb{E}([0, T])$ and $g \in \mathbb{F}([0, T])$ by (R). Theorem 2.5 and (R) thus yield

$$\begin{aligned}
\|(z, \tau)\|_{\mathbb{E}_1([0, T])} & \leq c(\|f\|_{\mathbb{E}([0, T])} + \|g\|_{\mathbb{F}([0, T])}) \\
& \leq c \left(\|w - w_*\|_{L_p([0, T]; E_\gamma)} + \|\partial_t(\rho - \rho_*)\|_{L_p([0, T]; Y_{0\gamma})} + \|w - w_*\|_{C([0, T]; E_\gamma)} \right. \\
& \quad \left. \cdot \|u\|_{L_p([0, T]; X_1)} + \varepsilon(\|w - w_*\|_{\mathbb{E}_1([0, T])}) \|w - w_*\|_{\mathbb{E}_1([0, T])} \right).
\end{aligned}$$

We next write $w - w_* = w - \tilde{w} + \tilde{w} - w_*$, estimate the p -norm by the supnorm, use the embeddings (2.5), (2.11), (2.19) and the inequality $\|w - \tilde{w}\|_{\mathbb{E}_1} \leq r$. It then follows

$$\begin{aligned}
\|\Phi(w) - \tilde{w}\|_{\mathbb{E}_1([0, T])} & \leq c T^{\frac{1}{p}} (r + \|\tilde{w} - w_*\|_{C([0, T]; E_\gamma)} + r + \|\partial_t(\tilde{\rho} - \rho_*)\|_{C([0, T]; Z_\gamma^1)}) \\
& \quad + c (r + \|\tilde{w} - w_*\|_{C([0, T]; E_\gamma)})(r + \|\tilde{u}\|_{L_p([0, T]; X_1)}) \\
& \quad + \varepsilon(r + \|\tilde{w} - w_*\|_{\mathbb{E}_1([0, T])})(r + \|\tilde{w} - w_*\|_{\mathbb{E}_1([0, T])}).
\end{aligned}$$

Here the constants in Theorem 2.5 and the embeddings do not depend on $T \in (0, T_1]$ since all relevant functions vanish at $t = 0$. Using once more $\tilde{w}(0) = w_*(0)$ and $\partial_t \tilde{\rho}(0) = \partial_t \rho_*(0)$, we obtain $T_2 \in (0, T_1]$ and $r_2 \in (0, r_1]$ such that $\|\Phi(w) - \tilde{w}\|_{\mathbb{E}_1([0, T])} \leq r$ whenever $T \in (0, T_2]$ and $r \in (0, r_2]$; i.e., Φ leaves $\Sigma(T, r)$ invariant for such T and r . By analogous arguments, we can fix $T \in (0, T_2]$ and $r \in (0, r_2]$ such that Φ has a Lipschitz constant less or equal $1/2$ on $\Sigma(T, \rho)$. The resulting fixed point w is a local solution of (1.1) on $[0, T]$.

Assume there is a different solution \hat{w} of (3.7) on $[0, T]$. Then there are numbers $t_0, t_n \in [0, T]$ such that $t_n \searrow t_0$ as $n \rightarrow \infty$, $w(t) = \hat{w}(t)$ for $t \in [0, t_0]$, and $w(t_n) \neq \hat{w}(t_n)$. We may apply the above argument with some $T', r' > 0$, the initial time t_0 , and the initial value $w(t_0) =: w_1 \in \mathcal{M}$. This leads to a contradiction establishing the uniqueness assertion. \square

We now introduce in a standard way the maximal existence interval for the solution with initial value $w_0 \in \mathcal{M}$. Under the assumptions of Proposition 3.1, let $t^+(w_0)$ be the supremum of those $T > 0$ such that (1.1) has a solution $w \in \mathbb{E}_1([0, T])$. Proposition 3.1 implies that $t^+(w_0) > 0$. Moreover, for two given solutions w_1 on $[0, a]$ and w_2 on $[a, b]$ of (1.1) with $w_1(b) = w_2(b)$, we obtain a solution w of (1.1) on $[0, b]$ by setting $w = w_1$ on $[0, a]$ and $w = w_2$ on $[a, b]$. To see this, note that $\dot{\rho}_1(a) = -\mathcal{D}_0(w_1(a)) = \dot{\rho}_2(b)$, where $w_i = (u_i, \rho_i)$, so that $w \in \mathbb{E}_1([0, b])$ by Lemma 2.8.

To state our main well-posedness result, we need some more notation and results related to the solution manifold \mathcal{M} , recalling the definitions (2.15) and

(3.2). For $w_* \in \mathcal{M}$ and $\psi = (v_0, \sigma_0) \in \tilde{E}_\gamma$ (see (2.20)), we define

$$\begin{aligned}\langle \psi \rangle_\gamma &= |\psi|_{E_\gamma} + [\psi]_\gamma, \quad [\psi]_\gamma = |\mathcal{D}_0(\psi + w_*) - \mathcal{D}_0(w_*)|_{Z_\gamma^1} = |(B_0, C_0)\psi - G_0(\psi)|_{Z_\gamma^1}, \\ \langle \psi \rangle_1 &= |\psi|_{E_1} + [\psi]_1, \quad [\psi]_1 = |\mathcal{D}_0(\psi + w_*) - \mathcal{D}_0(w_*)|_{Z_\gamma} = |(B_0, C_0)\psi - G_0(\psi)|_{Z_\gamma}\end{aligned}\tag{3.10}$$

For a solution $\psi(t) = (v(t), \sigma(t))$ of (3.7), the above quantities simplify to

$$\langle \psi(t) \rangle_\gamma = |\psi(t)|_{E_\gamma} + |\dot{\sigma}(t)|_{Z_\gamma^1}, \quad \langle \psi(t) \rangle_1 = |\psi(t)|_{E_1} + |\dot{\sigma}(t)|_{Z_\gamma}. \tag{3.11}$$

We note that $[\psi]_\gamma \leq c|\psi|_{E_\gamma}$ if $\ell \leq 2m$ since then $Z_\gamma^1 = Y_{0\gamma}$ as observed in Section 2, and thus $|\psi|_{E_\gamma}$ and $\langle \psi \rangle_\gamma$ are locally equivalent in this case. Given $r > 0$, we further introduce

$$\mathcal{M}^*(r) := \{\psi \in \mathcal{M}^* \mid \langle \psi \rangle_\gamma < r\}.$$

The next lemma gives a local chart for such restrictions of the solution manifold.

Lemma 3.2. *In the setting of Corollary 2.7, we define G by (3.2) for some $w_* = (u_*, \rho_*) \in \mathcal{M}$. Then the map $\mathcal{Q} = I - \mathcal{N}_\gamma G$ belongs to $C^1(W_\gamma - w_*; E_\gamma)$ with a locally bounded derivative, $\mathcal{Q}(0) = 0$ and $\mathcal{Q}'(0) = I$. It maps \mathcal{M}^* into E_γ^0 (see (2.20)) with $|\psi - \mathcal{N}_\gamma G(\psi)|_{E_\gamma^0} \leq c\langle \psi \rangle_\gamma$ for $\psi \in \mathcal{M}^*$. We can invert $I - \mathcal{N}_\gamma G$ on some ball $B_{E_\gamma}(0, r_0) \subseteq W_\gamma - w_*$ and set $h = \mathcal{N}_\gamma G(I - \mathcal{N}_\gamma G)^{-1}$. There is a radius $r > 0$ such that $\mathcal{M}^*(r)$ is the graph of h , i.e.,*

$$\mathcal{M}^*(r) = \{\psi = \xi + h(\xi) \mid \xi \in B_{E_\gamma^0}(0, r_0), \langle \psi \rangle_\gamma < r\}.$$

In particular, $w_* + E_\gamma^0$ is the tangent plane of \mathcal{M} at w_* and \mathcal{Q} is a local chart.

Proof. Corollary 2.7 and (3.4) imply that the first sentence about \mathcal{Q} holds. The inverse mapping theorem then shows that \mathcal{Q} is invertible in E_γ near 0, so that \mathcal{Q}^{-1} is defined on $B_{E_\gamma}(0, r_0) \subseteq W_\gamma - w_*$ for some $r_0 > 0$. For $\psi = (v, \sigma) \in \mathcal{M}^*$, we obtain $\mathcal{Q}(\psi) \in E_\gamma^0$ since

$$\begin{aligned}(\widehat{B}, \widehat{C})(\psi - \mathcal{N}_\gamma G(\psi)) &= (\widehat{B}, \widehat{C})\psi - \widehat{G}(\psi) = 0, \\ |\mathcal{Q}(\psi)|_{E_\gamma} + |(B_0, C_0)\mathcal{Q}(\psi)|_{Z_\gamma^1} &\leq |\psi|_{E_\gamma} + |\mathcal{N}_\gamma \mathcal{Q}(\psi)|_{E_\gamma} + |(B_0, C_0)\psi - G_0(\psi)|_{Z_\gamma^1} \\ &\quad + |G_0(\psi) - (B_0, C_0)\mathcal{N}_\gamma G(\psi)|_{Z_\gamma^1} \\ &\leq c\langle \psi \rangle_\gamma + c|G(\psi)|_{Y_\gamma} \leq c\langle \psi \rangle_\gamma,\end{aligned}\tag{3.12}$$

where we use $w_* \in \mathcal{M}$, $\psi \in \mathcal{M}^*$ and again Corollary 2.7 and (3.4).

Let $\xi \in E_\gamma^0$ with $|\xi|_{E_\gamma} < r_0$. Define $h = \mathcal{N}_\gamma G(I - \mathcal{N}_\gamma G)^{-1}$ on $B_{E_\gamma}(0, r_0)$ and set $\psi = \xi + h(\xi)$. Then, $\psi = (I - \mathcal{N}_\gamma G)^{-1}(\xi) \in W_\gamma - w_*$ and $\xi = \psi - \mathcal{N}_\gamma G(\psi)$. Corollary 2.7 thus yields

$$\begin{aligned}(\widehat{B}, \widehat{C})\psi &= (\widehat{B}, \widehat{C})(\mathcal{N}_\gamma G(\psi) + \xi) = \widehat{G}(\psi), \\ (B_0, C_0)\psi - G_0(\psi) &= (B_0, C_0)\mathcal{N}_\gamma G(\psi) - G_0(\psi) + (B_0, C_0)\xi \in Z_\gamma^1.\end{aligned}$$

As in (3.12), we obtain $\langle \psi \rangle_\gamma \leq \widehat{c}|\xi|_{E_\gamma^0}$, so that $\psi \in \mathcal{M}^*(r)$ for all $r \in (0, \widehat{c}r_0)$.

Conversely, take $\psi \in \mathcal{M}^*(r)$ for some $r \in (0, \widehat{c}r_0)$. Set $\xi = \mathcal{Q}(\psi) \in E_\gamma^0$. Fixing a sufficiently small $r > 0$, estimate (3.12) yields $|\xi|_{E_\gamma} < r_0$ if $\langle \psi \rangle_\gamma < r$. It follows that

$$\xi + h(\xi) = \psi - \mathcal{N}_\gamma G(\psi) + \mathcal{N}_\gamma G(I - \mathcal{N}_\gamma G)^{-1}(\psi - \mathcal{N}_\gamma \mathcal{Q}(\psi)) = \psi.$$

Thus $\mathcal{M}^*(r)$ is the graph of h , and the other assertions follow. \square

Our main result on wellposedness shows that (1.1) generates a local semiflow on the nonlinear phase space \mathcal{M} . Moreover, the problem possesses a smoothing effect. We write tw for the function $t \mapsto tw(t)$.

Theorem 3.3. *Let (R) and (S) be true. Assume that (E) and (LS), as well as (LS_∞^\pm) if $\ell \geq 2m$, hold for all functions $(u_0, \rho_0) \in W_\gamma$. Let $w_{0*} = (u_{0*}, \rho_{0*}) \in \mathcal{M}$. Proposition 3.1 yields a solution $w_* = w(\cdot; w_{0*})$ of (1.1) with $w_*(0) = w_{0*}$. Take $T \in (0, t^+(w_{0*}))$ and set $J = [0, T]$ and $J^+ = [0, t^+(w_{0*}))$. Then the following assertions are true.*

(a) *If $t^+(w_{0*}) < \infty$, then $\|w_*\|_{\mathbb{E}_1(J^+)} = \infty$ or there are $J^+ \ni t_n \rightarrow t^+(w_{0*})$ such that $w_*(t_n)$ tends to ∂W_γ in E_γ as $n \rightarrow \infty$. Moreover, then $(w_*(t), \dot{\sigma}_*(t))$ does not converge in $W_\gamma \times Z_\gamma^1$ as $t \rightarrow t^+(w_{0*})$.*

(b) *There is a radius $r > 0$ such that for each $\varphi_0 = (v_0, \sigma_0) \in \mathcal{M}^*(r)$ there exists a solution $w = (u, \rho) \in \mathbb{W}_1(J)$ of (1.1) with $w(0) = w_0 = (u_0, \rho_0) = w_{0*} + \varphi_0$. The map $\varphi_0 \mapsto w - w_*$ from $\mathcal{M}^*(r)$ to $\mathbb{W}_1(J)$ is C_b^1 . It holds*

$$\|w - w_*\|_{\mathbb{E}_1(J)} \leq c \langle w_0 - w_{0*} \rangle_\gamma = c |w_0 - w_{0*}|_{E_\gamma} + c |\mathcal{D}_0(w_0 + w_{0*}) - \mathcal{D}_0(w_{0*})|_{Z_\gamma^1}$$

(c) *We have $t\partial_t w \in \mathbb{E}_1(J)$, and thus*

$$\begin{aligned} tu &\in W_p^2(J; X) \cap W_p^1(J; X_1) \cap C^1(J; X_\gamma), \\ t\rho &\in W_p^{2+\kappa_0}(J; L_p) \cap W_p^2(J; Z) \cap W_p^1(J; Z_1) \cap C^1(J; Z_\gamma) \cap \bigcap_{j \in \tilde{\mathcal{J}}} W_p^{1+\kappa_j}(J; W_p^{k_j}). \end{aligned}$$

Proof. (a) Suppose that $t^+(w_{0*}) < \infty$, $w_* \in \mathbb{E}_1(J^+)$ and $d_{E_\gamma}(w_*(t), \partial W_\gamma) \geq \delta > 0$ for all $t \in J^+$. By the embeddings (2.5), (2.11) and (2.19), $w_*(t)$ converges in E_γ to some $w_1 \in W_\gamma$ as $t \rightarrow t^+(w_{0*})$. Due to (1.1) and (R), we obtain $\widehat{\mathcal{D}}(w_1) = 0$ in the limit, and $\dot{\rho}_*(t) = -\mathcal{D}_0(w_*(t))$ converges to $\mathcal{D}_0(w_1)$ in Z_γ^1 . As a result, w_1 belongs to \mathcal{M} and we can extend the solution across $t^+(w_{0*})$ by the remarks before Lemma 3.2, contradicting the definition of $t^+(w_{0*})$.

(b) We linearize the problem along $w_*(t)$, $t \in J$, and obtain the operators given by (2.15) and (3.2). Let $\mathcal{S} : \mathbb{D}(J) \rightarrow \mathbb{E}_1(J)$ be the solution operator for (2.25). For $\xi \in E_\gamma^0$ (see (2.20)) and $\psi = (v, \sigma) \in \mathbb{W}_1^*(J) = \mathbb{W}_1^*$, we define

$$\Phi(\xi, \psi) = \psi - \mathcal{S}(\xi + \mathcal{N}_\gamma G(\psi(0)), F(\psi), G(\psi)).$$

Due to Corollary 2.7, (2.7) and $\xi \in E_\gamma^0$, we have $(\widehat{B}, \widehat{C})(\xi + \mathcal{N}_\gamma G(\psi(0))) = \widehat{G}(\psi(0))$ and $(B_0, C_0)(\xi + \mathcal{N}_\gamma G(\psi(0))) - G_0(\psi(0)) \in Z_\gamma^1$. Corollary 2.6, (3.1) and (3.4) thus yield that that $\Phi(0, 0) = 0$, $\Phi \in C^1(E_\gamma^0 \times \mathbb{W}_1^*; \mathbb{E}_1(J))$, and $\partial_2 \Phi(0, 0) = I$. Therefore the implicit function theorem gives a ball $B(r_0) = B(0, r_0)$ in E_γ^0 and a map $\phi \in C_b^1(B(r_0); \mathbb{W}_1^*)$ such that $\phi(0) = 0$ and $\Phi(\xi, \phi(\xi)) = 0$ for all $\xi \in B(r_0)$. (One obtains the boundedness of the derivative by decreasing r_0 if necessary.) In particular, $\phi(\xi)$ solves (3.7) with the initial value $\xi + \mathcal{N}_\gamma G(\phi(\xi)) \in \mathcal{M}^*$. If we start with a given function $w_0 = (u_0, \rho_0) \in \mathcal{M}$, we set $\varphi_0 = (v_0, \sigma_0) = w_0 - w_{0*} \in \mathcal{M}^*$. Lemma 3.2 yields that $\xi := \mathcal{Q}(\varphi_0)$ belongs to E_γ^0 with $|\xi|_{E_\gamma^0} \leq c \langle \varphi_0 \rangle_\gamma$. We can thus find an $r > 0$ such that $|\xi|_{E_\gamma^0} < r_0$ if $\langle \varphi_0 \rangle_\gamma < r$. Then $\varphi := \phi(\xi) \in \mathbb{W}_1^*$ solves (3.7) with the

initial value $\varphi(0) = \varphi_0 + \mathcal{N}_\gamma(G(\varphi(0)) - G(\varphi_0)) \in W_\gamma$. We further conclude that

$$|\varphi(0)|_{E_\gamma} \leq c \|\varphi\|_{\mathbb{E}_1(J)} = c \|\phi(\xi) - \phi(0)\|_{\mathbb{E}_1(J)} \leq c |\xi|_{E_\gamma^0} \leq c \langle \varphi_0 \rangle_\gamma \leq cr.$$

Corollary 2.7, (3.2) and (R) now yield

$$\begin{aligned} |\varphi(0) - \varphi_0|_{E_\gamma} &\leq c |G(\varphi(0)) - G(\varphi_0)|_{Y_\gamma} \\ &\leq c |(\mathcal{D}'(w_*) - \mathcal{D}'(\varphi_0 + w_*))(\varphi(0) - \varphi_0)|_{Y_\gamma} \\ &\quad + c |\mathcal{D}'(\varphi_0 + w_*)(\varphi(0) - \varphi_0) + \mathcal{D}(\varphi_0 + w_*) - \mathcal{D}(\varphi(0) + w_*)|_{Y_\gamma} \\ &\leq (\varepsilon(|\varphi_0|_{E_\gamma}) + \varepsilon(|\varphi(0) - \varphi_0|_{E_\gamma})) |\varphi(0) - \varphi_0|_{E_\gamma} \\ &\leq \varepsilon(r) |\varphi(0) - \varphi_0|_{E_\gamma}. \end{aligned}$$

Choosing a smaller $r > 0$ if necessary, we see that $\varphi(0) = \varphi_0$, and thus $w := \varphi + w_*$ solves (1.1) with the initial value w_0 . The asserted estimate and differentiability now follow from the above results and Lemma 3.2.

(c) Let $w = (u, \rho)$ solve (1.1) on $[0, T']$, where $w_0 \in \mathcal{M}$ and $T' = (1 + \epsilon)T < t^+(w_0)$ for some $\epsilon \in (0, 1)$ and $T > 0$. Let $J = [0, T]$. For $\lambda \in (1 - \epsilon, 1 + \epsilon)$ and $t \in J$, we put $w_\lambda(t) = w(\lambda t)$. We define the operators $A_*(t)$, $B(t)$ and $C(t)$ as in (2.15) with w_* replaced by w . For $\psi = (v, \sigma) \in \mathbb{W}_1(J) = \mathbb{W}_1$, we then set

$$\begin{aligned} F(\lambda, \psi(t)) &= A_*(t)\psi(t) - \lambda \mathcal{A}(\psi(t))v(t) + \lambda \mathcal{R}(\psi(t), \lambda^{-1}\dot{\sigma}(t)), \\ G_0(\lambda, \psi(t)) &= B_0(t)v(t) + C_0(t)\sigma(t) - \lambda \mathcal{D}_0(\psi(t)), \\ \widehat{G}(\lambda, \psi(t)) &= \widehat{B}(t)v(t) + \widehat{C}(t)\sigma(t) - \widehat{\mathcal{D}}(\psi(t)). \end{aligned}$$

Then $\psi = (v, \sigma) = w_\lambda$ is the unique solution of the problem

$$\begin{aligned} \partial_t v(t) + A_*(t)\psi(t) &= F(\lambda, \psi(t)), & \text{on } \Omega, \text{ a.e. } t > 0, \\ \partial_t \sigma(t) + B_0(t)v(t) + C_0(t)\sigma(t) &= G_0(\lambda, \psi(t)), & \text{on } \Sigma, \text{ } t \geq 0, \\ \widehat{B}(t)v(t) + \widehat{C}(t)\sigma(t) &= \widehat{G}(\lambda, \psi(t)), & \text{on } \Sigma, \text{ } t \geq 0, \\ (v(0), \sigma(0)) &= w_0, & \text{on } \Omega \times \Sigma. \end{aligned} \tag{3.13}$$

Using (R) and (2.15), we infer $F \in C^1((1 - \epsilon, 1 + \epsilon) \times \mathbb{W}_1; \mathbb{E})$ with $\partial_2 F(1, w) = 0$ and $G \in C^1((1 - \epsilon, 1 + \epsilon) \times \mathbb{W}_1; \mathbb{F})$ with $\partial_2 G(1, w) = 0$. As in Lemma 3.2 one sees that $\xi(\lambda) = w_0 - \mathcal{N}_\gamma G(\lambda, w_0) \in E_\gamma^0$. We then define the map

$$\Phi_0(\lambda, \psi) = \psi - \mathcal{S}(\xi(\lambda) + \mathcal{N}_\gamma G(\lambda, \psi(0)), F(\lambda, \psi), G(\lambda, \psi)),$$

where \mathcal{S} is the solution operator of (2.25) for the above introduced operators $A_*(t)$, $B(t)$ and $C(t)$. Since w solves (1.1), we have $\Phi_0(1, w) = 0$. As in part (b), we deduce that $\Phi_0 \in C^1((1 - \epsilon, 1 + \epsilon) \times \mathbb{W}_1; \mathbb{E}_1)$ and $\partial_2 \Phi_0(1, w) = I$. The implicit function theorem thus yields an $\epsilon' \in (0, \epsilon)$ and a map $\phi_0 \in C_b^1((1 - \epsilon', 1 + \epsilon'); \mathbb{W}_1)$ such that $\phi_0(1) = w$ and $\Phi_0(\lambda, \phi_0(\lambda)) = 0$ for all $\lambda \in (1 - \epsilon', 1 + \epsilon')$. Hence, $\phi_0(\lambda)$ solves (3.13) with w_0 replaced by $w_0(\lambda) := [\phi_0(\lambda)](0) \in \mathcal{M}$. Note that

$$\begin{aligned} w_0(\lambda) &= \xi(\lambda) + \mathcal{N}_\gamma G(\lambda, w_0(\lambda)) = w_0 + \mathcal{N}_\gamma(G(\lambda, w_0(\lambda)) - G(\lambda, w_0)), \\ w_0(\lambda) - w_0 &= \mathcal{N}_\gamma(\mathcal{D}'(w_0)(w_0(\lambda) - w_0) + \mathcal{D}(w_0) - \mathcal{D}(w_0(\lambda))) \\ &\quad + [(\lambda - 1)(\mathcal{D}_0(w_0) - \mathcal{D}_0(w_0(\lambda))), 0]. \end{aligned}$$

Therefore, Corollary 2.7, (R), (2.5) and (2.11) yield

$$\begin{aligned} |w_0(\lambda) - w_0|_{E_\gamma} &\leq c(\varepsilon(|w_0(\lambda) - w_0|_{E_\gamma}) + |\lambda - 1|) |w_0(\lambda) - w_0|_{E_\gamma} \\ &\leq c(\varepsilon(\|\phi_0(\lambda) - \phi_0(1)\|_{\mathbb{E}_1}) + |\lambda - 1|) |w_0(\lambda) - w_0|_{E_\gamma} \end{aligned}$$

for constants c and functions ε with $\varepsilon(r) \rightarrow 0$ as $r \rightarrow 0$ which do not depend on λ . Decreasing $\varepsilon' > 0$, we deduce that $w_0(\lambda) = w_0$, and thus $\phi_0(\lambda)$ solves (3.13) provided $|\lambda - 1|$ is sufficiently small.

The uniqueness of (3.13) now yields $w_\lambda = \phi_0(\lambda)$, and hence $\lambda \mapsto w_\lambda \in \mathbb{E}_1(J)$ is continuously differentiable with derivative $(\frac{d}{d\lambda}w_\lambda)(t) = t\dot{w}(\lambda t)$. Taking $\lambda = 1$, we deduce that $t\partial_t w \in \mathbb{E}_1(J)$. Using also (2.5) and (2.11), we conclude that $\partial_t(tw) = t\partial_t w + w \in \mathbb{E}_1(J) \hookrightarrow C(J; E_\gamma)$; i.e., (c) is true (cf. (2.10)). \square

We add a quantitative version of Theorem 3.3(c) which will allow us to improve convergence from E_γ to E_1 in Theorem 5.1 and in [37].

Proposition 3.4. *In the setting of Theorem 3.3 we assume that $w_* \in E_1$ is an equilibrium of (1.1). For $T \in (0, t^+(w_0))$, let $r > 0$ be given by Theorem 3.3. Then there is an $r' \in (0, r]$ such that for $w_0 \in w_* + \mathcal{M}^*(r')$ and $T_0 \in (0, T)$ the solution $w = (u, \rho) = w(\cdot; w_0) \in \mathbb{W}_1([0, T])$ satisfies*

$$|w(t) - w_*|_{E_1} + |\dot{w}(t)|_{E_\gamma} \leq c \langle w_0 - w_* \rangle_\gamma, \quad \|t \partial_t(w - w_*)\|_{\mathbb{E}_1([0, T])} \leq c \langle w_0 - w_* \rangle_\gamma,$$

for $t \in [T_0, T]$ and constants independent of t and w_0 , where $\langle w_0 - w_* \rangle_\gamma = |w_0 - w_*|_{E_\gamma} + |\mathcal{D}_0(w_0 + w_*) - \mathcal{D}_0(w_*)|_{Z_\gamma^1}$, see (3.10).

Proof. In contrast to the proof of Theorem 3.3(c) we now use (3.7) instead of (1.1). We thus set $v(t) = u(t) - u_*$, $\sigma(t) = \rho(t) - \rho_*$, $\varphi_0 = (v_0, \sigma_0) = w_0 - w_*$, and linearize (1.1) at the equilibrium w_* , employing the operators from (2.15) and (3.2) which now do not depend on time explicitly. Due to (3.7), the functions $v_\lambda(t) = v(\lambda t)$ and $\sigma_\lambda(t) = \sigma(\lambda t)$, $t \in J = [0, T]$, uniquely solve the problem

$$\begin{aligned} \partial_t z(t) + A_*(\psi(t), \dot{\tau}(t)) &= A_*(\psi(t), \dot{\tau}(t)) - \lambda A_*(\psi(t), \lambda^{-1}\dot{\tau}(t)) \\ &\quad + \lambda F(\psi(t), \lambda^{-1}\dot{\tau}(t)), \end{aligned} \quad (3.14)$$

$$\begin{aligned} \partial_t \tau(t) + B_0 z(t) + C_0 \tau(t) &= (1 - \lambda) B_0 z(t) + (1 - \lambda) C_0 \tau(t) + \lambda G_0(\psi(t)), \\ \widehat{B} z(t) + \widehat{C} \tau(t) &= \widehat{G}(\psi(t)), \\ (z(0), \tau(0)) &= (v_0, \sigma_0), \end{aligned} \quad (3.15)$$

for $t \in J$, where we write $\psi = (z, \tau)$ and take $\lambda \in (1 - \epsilon, 1 + \epsilon)$ and $\epsilon \in (0, 1)$ such that $(1 + \epsilon)T < t^+(w_0)$. We denote the right hand sides of (3.15) by $F(\lambda, \psi)$, $G_0(\lambda, \psi)$ and $\widehat{G}(\lambda, \psi)$, respectively. We now proceed as in the proof of Theorem 3.3(c) using the operator

$$\Phi(\lambda, \psi) = \psi - \mathcal{S}(\xi(\lambda) + \mathcal{N}_\gamma G(\lambda, \psi(0)), F(\lambda, \psi), G(\lambda, \psi))$$

for $\lambda \in (1 - \epsilon, 1 + \epsilon)$, $\psi \in \mathbb{W}_1^*(J)$, and $\xi(\lambda) = \varphi_0 - \mathcal{N}_\gamma G(\lambda, \varphi_0)$. Here \mathcal{S} is the solution operator of (2.25) for the operators A_* , B and C . Since $\varphi_0 \in \mathcal{M}^*$, we have $\xi(\lambda) \in E_\gamma^0$ by Corollary 2.7. As in Theorem 3.3, we then see that $\Phi \in C^1((1 - \epsilon, 1 + \epsilon) \times \mathbb{W}_1^*(J); \mathbb{E}_1(J))$,

$$\Phi(1, (v, \sigma)) = 0, \quad \text{and} \quad \partial_2 \Phi(1, (v, \sigma)) = I - \mathcal{S}(\mathcal{N}_\gamma G'(\varphi_0), F'(v, \sigma), G'(v, \sigma)),$$

where F and G are given by (3.2). In view of Corollaries 2.6 and 2.7, (3.1), (3.4) and the estimate in Theorem 3.3(b), we obtain the invertibility of $\partial_2\Phi(1, v, \sigma)$ in $\mathbb{E}_1(J)$ if $r > \langle \varphi_0 \rangle_\gamma$ is chosen small enough. Moreover, the inverse is uniformly bounded. So the implicit function theorem provides us with a map $\phi \in C^1((1 - \hat{\epsilon}, 1 + \hat{\epsilon}); \mathbb{W}_1^*(J))$ such that $\phi(1) = (v, \sigma)$ and $\Phi(\lambda, \phi(\lambda)) = 0$ for $|1 - \lambda| \leq \hat{\epsilon}$ and some $\hat{\epsilon} \in (0, 1)$. We set $\varphi_0(\lambda) = [\phi(\lambda)](0)$ and note that $\phi(\lambda)$ solves (3.15) with initial value $\varphi_0(\lambda) \in \mathcal{M}^*$. We then compute

$$\begin{aligned} \varphi_0(\lambda) - \varphi_0 &= \mathcal{N}_\gamma[G(\lambda, \varphi(0)) - G(\lambda, \varphi_0)] \\ &= \mathcal{N}_\gamma \left[G(\varphi_0(\lambda)) - G(\varphi_0) + (1 - \lambda)(\mathcal{D}_0(w_* + \varphi_0(\lambda)) - \mathcal{D}_0(w_* + \varphi_0)) \right] \\ &= \mathcal{N}_\gamma \left[(\mathcal{D}'(w_*) - \mathcal{D}'(w_* + \varphi_0))(\varphi_0(\lambda) - \varphi_0) \right. \\ &\quad \left. + \mathcal{D}'(w_* + \varphi_0)(\varphi_0(\lambda) - \varphi_0) + \mathcal{D}(w_* + \varphi_0) - \mathcal{D}(w_* + \varphi_0(\lambda)) \right. \\ &\quad \left. + (1 - \lambda)(\mathcal{D}_0(w_* + \varphi_0(\lambda)) - \mathcal{D}_0(w_* + \varphi_0))) \right]. \end{aligned}$$

Combined with Corollary 2.7, (2.5), (2.11) and (R), this identity leads to

$$|\varphi_0(\lambda) - \varphi_0|_{E_\gamma} \leq c(\varepsilon(|\varphi_0|_{E_\gamma}) + \varepsilon(\|\phi(\lambda) - \phi(1)\|_{\mathbb{E}_1}) + |\lambda - 1|)|\varphi_0(\lambda) - \varphi_0|_{E_\gamma}.$$

We conclude that $\varphi_0(\lambda) = \varphi_0$, and hence $\phi(\lambda) = (v_\lambda, \sigma_\lambda)$, if $r > 0$ and $\hat{\epsilon} > 0$ are small enough. We put $\varphi = (v, \sigma) = w - w_*$, and further compute

$$\begin{aligned} \partial_1 F(1, \varphi) &= F(\varphi, \dot{\sigma}) - A_*(\varphi, \dot{\sigma}) + (A_{*\dot{\rho}} - \partial_3 F(\varphi, \dot{\sigma}))\dot{\sigma} \\ &= \mathcal{A}(w_*)u_* \pm \mathcal{A}(w_*)u - \mathcal{A}(w)u + \mathcal{R}(w, \dot{\rho}) - \mathcal{R}(w_*, \dot{\rho}_*) + (A_{*\dot{\rho}} - \partial_3 F(w, \dot{\rho}))\dot{\sigma}, \\ \partial_1 G_0(1, \varphi) &= G_0(\varphi) - B_0v - C_0\sigma = \mathcal{D}_0(w_*) - \mathcal{D}_0(w), \\ \phi'(1) &= -[\partial_2\Phi(1, \varphi)]^{-1}\partial_1\Phi(1, \varphi) = [\partial_2\Phi(1, \varphi)]^{-1}\mathcal{S}(0, \partial_1 F(1, \varphi), (\partial_1 G_0(1, \varphi), 0)). \end{aligned}$$

Corollary 2.6, (R), (2.5), (2.11) and (2.19) thus yield

$$\begin{aligned} \|\phi'(1)\|_{\mathbb{E}_1} &\leq c(\|v\|_{\mathbb{E}_u} + \|\varphi\|_{C(J; E_\gamma)} \|u\|_{\mathbb{E}_u} + \|\varphi\|_{L_p(J; E_\gamma)} + \|\partial_t\sigma\|_{L_p(J; Y_{0, \gamma})} + \|\varphi\|_{\mathbb{E}_1}) \\ &\leq c\|\varphi\|_{\mathbb{E}_1}, \end{aligned}$$

where we also use that $\|u\|_{\mathbb{E}_u} \leq \|u_*\|_{\mathbb{E}_u} + \|v\|_{\mathbb{E}_u} \leq c(1 + r)$ by Theorem 3.3(b). Since $t\partial_t\varphi = \phi'(1)$, we arrive at

$$\|t\partial_t(w - w_*)\|_{\mathbb{E}_1(J)} \leq c\|w - w_*\|_{\mathbb{E}_1(J)} \leq c\langle w_0 - w_* \rangle_\gamma,$$

employing also Theorem 3.3(b). The remaining estimate then follows from Sobolev's embedding, (2.5) and (2.11). \square

Finally, we replace in the above proposition the equilibrium w_* by a general solution w_* of (1.1), under the somewhat stronger regularity assumption (RR). This result will imply that certain invariant manifolds are Lipschitz in E_1 (instead of E_γ), see e.g. Theorem 5.1.

Proposition 3.5. *In the setting of Theorem 3.3 we assume that (RR) holds and that $w_* = (u_*, \rho_*) \in \mathbb{W}_1([0, T_*])$ solves (1.1) with $w_*(0) = w_{*0} \in \mathcal{M}$. Take $T \in (0, T_*)$ and $T_0 \in (0, T)$ and let $r > 0$ be given by Theorem 3.3(b).*

Then there is an $r' \in (0, r]$ such that for every $w_0 \in w_{*0} + \mathcal{M}^*(r')$ the solution $w = (u, \rho) \in \mathbb{W}_1([0, T])$ of (1.1) satisfies

$$\langle w(t) - w_*(t) \rangle_1 \leq c \langle w_0 - w_{*0} \rangle_\gamma, \quad \|t \partial_t(w - w_*)\|_{\mathbb{E}_1([0, T])} \leq c \langle w_0 - w_{*0} \rangle_\gamma,$$

for $t \in [T_0, T]$ and constants independent of t and w_0 .

Proof. Theorem 3.3 gives the solution $w = (u, \rho) \in \mathbb{W}_1([0, T])$ of (1.1) on $[0, T]$. Take $\epsilon > 0$ with $(1 + \epsilon)T < T_*$. We set $\varphi = (v, \sigma) = w - w_*$ and $\varphi_0 = (v_0, \sigma_0) = w_0 - w_{*0}$, and linearize (1.1) at the function $w_* \in \mathbb{W}_1([0, T_*])$, employing the operators from (2.15) and (3.2). In view of (3.7), the functions $v_\lambda(t) = v(\lambda t)$ and $\sigma_\lambda(t) = \sigma(\lambda t)$, $t \in J = [0, T]$, uniquely solve the problem

$$\begin{aligned} \partial_t z(t) + A_*(t)(\psi(t), \dot{\tau}(t)) &= A_*(t)(\psi(t), \dot{\tau}(t)) - \lambda A_*(\lambda t)(\psi(t), \lambda^{-1}\dot{\tau}(t)) \\ &\quad + \lambda F(\lambda t, \psi(t), \lambda^{-1}\dot{\tau}(t)), \\ \partial_t \tau(t) + B_0(t)z(t) + C_0(t)\tau(t) &= (B_0(t) - \lambda B_0(\lambda t))z(t) + (C_0(t) - \lambda C_0(\lambda t))\tau(t) \\ &\quad + \lambda G_0(\lambda t, \psi(t)), \\ \widehat{B}(t)z(t) + \widehat{C}(t)\tau(t) &= (\widehat{B}(t) - \widehat{B}(\lambda t))z(t) + (\widehat{C}(t) - \widehat{C}(\lambda t))\tau(t) \\ &\quad + \widehat{G}(\lambda t, \psi(t)), \\ (z(0), \tau(0)) &= (v_0, \sigma_0), \end{aligned} \tag{3.16}$$

for $t \in J$, where we write $\psi = (z, \tau)$ and take $\lambda \in (1 - \epsilon, 1 + \epsilon)$. We denote the right hand sides of (3.16) by $F(\lambda, \psi)$, $G_0(\lambda, \psi)$ and $\widehat{G}(\lambda, \psi)$, respectively. Due to (2.15) and (3.2), these maps can be written as

$$\begin{aligned} F(\lambda, \psi)(t) &= A_*(t)(\psi(t), \dot{\tau}(t)) - \lambda \mathcal{A}(w_*(\lambda t) + \psi(t))(u_*(\lambda t) + z(t)) \\ &\quad + \lambda \mathcal{A}(w_*(\lambda t))u_*(\lambda t) + \lambda \mathcal{R}(w_*(\lambda t) + \psi(t), \dot{\rho}_*(\lambda t) + \lambda^{-1}\dot{\tau}(t)) \\ &\quad - \lambda \mathcal{R}(w_*(\lambda t), \dot{\rho}_*(\lambda t)) \\ G_0(\lambda, \psi)(t) &= B_0(t)z(t) + C_0(t)\tau(t) + \lambda \mathcal{D}_0(w_*(\lambda t)) - \lambda \mathcal{D}_0(w_*(\lambda t) + \psi(t)), \\ \widehat{G}(\lambda, \psi)(t) &= \widehat{B}(t)z(t) + \widehat{C}(t)\tau(t) + \widehat{\mathcal{D}}(w_*(\lambda t)) - \widehat{\mathcal{D}}(w_*(\lambda t) + \psi(t)), \end{aligned}$$

where $(\lambda, \psi) \in (1 - \epsilon, 1 + \epsilon) \times \mathbb{W}_1^*(J)$. As in the proof of Theorem A.1 in [20], one deduces from Theorem 3.3(c) that the map $\lambda \mapsto w_*(\lambda \cdot)$ belongs to $C^1((1 - \epsilon, 1 + \epsilon); \mathbb{E}_1(J))$ having the derivative $\partial_\lambda w_*(\lambda \cdot) = t\dot{w}_*(\lambda \cdot)$. Proceeding as in Proposition 3.4, we then see that $F \in C^1((1 - \epsilon, 1 + \epsilon) \times \mathbb{W}_1^*(J); \mathbb{E}(J))$ and $G \in C^1((1 - \epsilon, 1 + \epsilon) \times \mathbb{W}_1^*(J); \mathbb{F}(J))$ with $F(1, \psi) = F(\psi)$, $G(1, \psi) = G(\psi)$, $\partial_2 F(1, \psi) = F'(\psi)$ and $\partial_2 G(1, \psi) = G'(\psi)$ for $\psi \in \mathbb{W}_1^*(J)$. We further obtain

$$\begin{aligned} \partial_1 F(1, \psi) &= (\mathcal{A}(w_*)u_* - \mathcal{A}(w_* + \psi)(u_* + z)) + (\mathcal{A}(w_*)u_* - \mathcal{A}(w_* + \psi))t\dot{u}_* \\ &\quad + (\mathcal{A}'(w_*)u_* - \mathcal{A}'(w_* + \psi))[t\dot{w}_*, u_*] - \mathcal{A}'(w_* + \psi)[t\dot{w}_*, z] \\ &\quad + (\mathcal{R}(w_* + \psi, \dot{\rho}_* + \dot{\tau}) - \mathcal{R}(w_*, \dot{\rho}_*)) - \partial_3 \mathcal{R}(w_* + \psi, \dot{\rho}_* + \dot{\tau})[\dot{\tau}] \\ &\quad + (\mathcal{R}(w_* + \psi, \dot{\rho}_* + \dot{\tau}) - \mathcal{R}(w_*, \dot{\rho}_*))[t\dot{u}_*, t\dot{\rho}_*, t\ddot{\rho}_*] \\ \partial_1 G_0(1, \psi) &= (D_0(w_*) - D_0(w_* + \psi)) + (D'_0(w_*) - D'_0(w_* + \psi))[t\dot{w}_*], \\ \partial_1 \widehat{G}(1, \psi) &= (\widehat{D}'(w_*) - \widehat{D}'(w_* + \psi))[t\dot{w}_*]. \end{aligned}$$

for $\psi = (z, \tau) \in \mathbb{W}_1^*(J)$. Let \mathcal{S} be the solution operator of (2.25). We now proceed as in the previous proposition, using the operator

$$\Phi(\lambda, \psi) = \psi - \mathcal{S}(\xi + \mathcal{N}_\gamma \gamma_0 G(\lambda, \psi), F(\lambda, \psi), G(\lambda, \psi))$$

for $\lambda \in (1 - \epsilon, 1 + \epsilon)$, $\psi \in \mathbb{W}_1^*(J)$, and $\xi := \varphi_0 - \mathcal{N}_\gamma G(\lambda, \varphi_0) \in E_\gamma^0$, see Corollary 2.7. So Corollaries 2.6 and 2.7, (2.7) and the above stated properties of $F(\lambda, \cdot)$ and $G(\lambda, \cdot)$ yield that $\Phi \in C^1((1 - \epsilon, 1 + \epsilon) \times \mathbb{W}_1^*(J); \mathbb{E}_1(J))$,

$$\Phi(1, \varphi) = 0, \quad \text{and} \quad \partial_2 \Phi(1, \varphi) = I - S(\mathcal{N}_\gamma G'(\varphi_0), F'(\varphi), G'(\varphi)).$$

Employing also (3.1), (3.4) and the estimate in Theorem 3.3(b), we obtain the invertibility of $\partial_2 \Phi(1, \varphi)$ in $\mathbb{E}_1(J)$ if $r' > \langle \varphi_0 \rangle_\gamma$ is chosen small enough, and the inverses are uniformly bounded. The implicit function theorem thus gives a map $\phi \in C^1((1 - \hat{\epsilon}, 1 + \hat{\epsilon}); \mathbb{W}_1^*(J))$ such that $\phi(1) = \varphi = (v, \sigma)$ and $\Phi(\lambda, \phi(\lambda)) = 0$ for $|1 - \lambda| \leq \hat{\epsilon}$ and some $\hat{\epsilon} \in (0, 1)$. We set $\varphi_0(\lambda) = [\phi(\lambda)](0)$ and note that $\phi(\lambda)$ solves (3.16) with the initial value $\varphi_0(\lambda) \in \mathcal{M}^*$. As in the proof of Proposition 3.4, we then compute

$$\begin{aligned} \varphi_0(\lambda) - \varphi_0 &= \mathcal{N}_\gamma [G(\lambda, \varphi_0(\lambda)) - G(\lambda, \varphi_0)] \\ &= \mathcal{N}_\gamma \left[(\mathcal{D}'(w_{*0}) - \mathcal{D}'(w_{*0} + \varphi_0))(\varphi_0(\lambda) - \varphi_0) \right. \\ &\quad + \mathcal{D}'(w_{*0} + \varphi_0)(\varphi_0(\lambda) - \varphi_0) + \mathcal{D}(w_{*0} + \varphi_0) - \mathcal{D}(w_{*0} + \varphi_0(\lambda)) \\ &\quad \left. + (\lambda - 1)(\mathcal{D}_0(w_{*0} + \varphi_0)) - \mathcal{D}_0(w_{*0} + \varphi_0(\lambda)) \right]. \end{aligned}$$

This identity again leads to the estimate

$$|\varphi_0(\lambda) - \varphi_0|_{E_\gamma} \leq c \left(\varepsilon (|\varphi_0|_{E_\gamma}) + \varepsilon (\|\phi(\lambda) - \phi(1)\|_{\mathbb{E}_1}) + |\lambda - 1| \right) |\varphi_0(\lambda) - \varphi_0|_{E_\gamma}.$$

We conclude that $\varphi_0(\lambda) = \varphi_0$, and hence $\phi(\lambda) = (v_\lambda, \sigma_\lambda)$, if $r > 0$ and $\hat{\epsilon} > 0$ are small enough. Observe that

$$\begin{aligned} \phi'(1) &= -[\partial_2 \Phi(1, \varphi)]^{-1} \partial_1 \Phi(1, \varphi) \\ &= -[\partial_2 \Phi(1, \varphi)]^{-1} \mathcal{S}(\mathcal{N}_\gamma \gamma_0 \partial_1 G(1, \varphi), \partial_1 F(1, \varphi), \partial_1 G(1, \varphi)). \end{aligned}$$

The above formulas for $\partial_1 F(1, \cdot)$ and $\partial_1 G(1, \cdot)$, Corollaries 2.6 and 2.7, (RR), Theorem 3.3 and the embeddings (2.5), (2.7), (2.11) and (2.19) thus yield

$$\begin{aligned} \|\phi'(1)\|_{\mathbb{E}_1} &\leq c \left(\|\varphi\|_{C(J; E_\gamma)} (1 + \|v\|_{L_p(J; X_1)} + \|\partial_t \sigma\|_{L_p(J; Y_{0\gamma})}) + \|\varphi\|_{L_p(J; E_\gamma)} \right. \\ &\quad \left. + \|\varphi\|_{C(J; E_\gamma)} + \|\partial_t \sigma\|_{C(J; Y_{0\gamma})} + \|\varphi\|_{\mathbb{E}_1} \right) \\ &\leq c \|\varphi\|_{\mathbb{E}_1} \leq c \langle \varphi_0 \rangle_\gamma. \end{aligned}$$

Taking also into account $\partial_t(t\varphi) = \varphi + t\partial_t \varphi = \varphi + \phi'(1)$ and Sobolev's embedding, we arrive at the assertion as before. \square

Example 3.6. We consider the Stefan problem with surface tension from Examples 2.2 and 2.4, taking as initial values interfaces Γ_0 which are parametrized by a function $\rho_0 \in W_p^{4-3/p}(\Sigma)$ over a sphere Σ as in Example 2.2 (i.e., $\Gamma_0 \in W_p^{4-3/p}$) together with initial temperatures $u_0 \in W_p^{2-2/p}(D \setminus \Gamma_0)$. We further assume the compatibility conditions $u_{0i} = \sigma H(\Gamma_0)$ on Γ_0 , $d_2 \partial_\nu u_2 - d_1 \partial_\nu u_1 \in W_p^{2-6/p}(\Gamma_0)$ and $\partial_\nu u_0 = 0$ on ∂D . The above results then give local solutions of

(2.14) still parametrized over Σ . In Theorem 3.1 of [34] a more realistic version of (2.14) was solved for more general initial data (which would be also possible here). Our results give additional differentiability with respect to initial data and smoothing estimates, but see [12], [32]. \diamond

4. PREPARATIONS FOR THE ASYMPTOTIC THEORY

We first establish several additional results about the linear problem (2.25) and the operator Λ_* from (2.16). We work under the following hypothesis.

Hypothesis 4.1. *Let $(w_*, y_*) \in W_1 \times Y_{0\gamma}$. Assume that (S), (E), (LS) and (if $\ell \gtrless 2m$) (LS $_{\infty}^{\pm}$) hold at w_* . Define A_* for (w_*, y_*) .*

Recall that the restriction $-\Lambda_0$ of $-\Lambda_*$ to $D(\Lambda_0) = E_1^0$ generates an analytic C_0 -semigroup $T(\cdot)$ on E_0 , see Corollary 2.6, (2.17) and (2.20). We need the *extrapolation space* E_{-1} which is the completion of E_0 with respect to the norm $|(\mu + \Lambda_0)^{-1}w|_{E_0}$ for any $\mu \geq \mu_0$ (where μ_0 is given by Corollary 2.6). There is a bounded extension $-\Lambda_{-1} : E_0 \rightarrow E_{-1}$ of $-\Lambda_0$ which is similar to $-\Lambda_0$ and generates the extension $T_{-1}(\cdot)$ of $T(\cdot)$ on E_{-1} . It further holds $T_{-1}(t) \in \mathcal{L}(E_{-1}, E_1^0)$ for $t > 0$. (See e.g. [3] or [10].)

A *solution* of the problem (1.1), (2.25) or (3.7) (or of some equations of them) on an (unbounded) interval J is a function $w \in \mathbb{E}_1^{\text{loc}}(J)$ satisfying the respective problem. Let $\alpha, \beta \in \mathbb{R}$. To study our equations on unbounded time intervals we set $e_{\alpha}(t) = e^{\alpha t}$ for $t \in \mathbb{R}$, denoting restrictions of this function by the same symbol. Moreover, on $J = \mathbb{R}$ we fix a smooth, strictly positive function $e_{\alpha, \beta}$ satisfying $e_{\alpha, \beta}(t) = e_{\alpha}(t)$ for $t \leq -1$ and $e_{\alpha, \beta}(t) = e_{\beta}(t)$ for $t \geq 1$. We then introduce the weighted spaces

$$\mathbb{E}_1(\mathbb{R}_{\pm}, \alpha) = \{w \mid e_{\alpha}w \in \mathbb{E}_1(\mathbb{R}_{\pm})\}, \quad \mathbb{E}_1(\alpha, \beta) = \{w \mid e_{\alpha, \beta}w \in \mathbb{E}_1(\mathbb{R})\}, \quad (4.1)$$

and their analogues for \mathbb{E} , \mathbb{F} and \mathbb{D} , which are complete if endowed with the canonical norms $\|w\|_{\mathbb{E}_1(\mathbb{R}_{+}, \alpha)} = \|e_{\alpha}w\|_{\mathbb{E}_1(\mathbb{R}_{+})}$ etc. We also use the corresponding norms on compact intervals J . We start with a version of the second part of Corollary 2.6 for $J \in \{\mathbb{R}_{-}, \mathbb{R}\}$.

Lemma 4.2. *Assume that Hypothesis 4.1 holds. Let $J \in \{\mathbb{R}_{-}, \mathbb{R}\}$, $f \in \mathbb{E}(J)$, $g \in \mathbb{F}_0(J)$. Then there is a unique $w = (u, \rho) \in \mathbb{E}_1(J)$ satisfying the first three equations of (2.26) on J for any $\mu \geq \mu_0$, where μ_0 is given by Corollary 2.6. Moreover, $\|w\|_{\mathbb{E}_1(J)} \leq c(\|f\|_{\mathbb{E}(J)} + \|g\|_{\mathbb{F}(J)})$.*

Proof. For $n \in \mathbb{N}$, we put $J_n = J \cap [-n, n]$ and take functions $\chi_n \in C^2(J_n)$ with uniformly bounded derivatives, $\chi_n(-n) = 0$ and $\chi_n = 1$ on $J \cap [-n+1, n]$. We set $f_n = \chi_n f \in \mathbb{E}(J_n)$ and $g_n = \chi_n g \in \mathbb{F}(J_n)$. Due to Corollary 2.6 (and an obvious time shift), there is a solution $w_n = (u_n, \rho_n)$ of (2.26) on J_n with data $(0, 0, f_n, g_n) \in \mathbb{D}(J_n)$ and some $\mu \geq \mu_0$. Extending the data to $[-n, \infty)$ and restricting the solution on $[-n, \infty)$ to J_n , we deduce from Corollary 2.6 that the solution operator on J_n is bounded uniformly in n , and so

$$\|w_n\|_{\mathbb{E}_1(J_n)} \leq c(\|f_n\|_{\mathbb{E}(J_n)} + \|g_n\|_{\mathbb{F}(J_n)}) \leq c(\|f\|_{\mathbb{E}(J)} + \|g\|_{\mathbb{F}(J)}) =: c(f, g)$$

for all $n \in \mathbb{N}$. We fix $m \in \mathbb{N}$. There is a subsequence such that $(u_{n_k}, \rho_{n_k}) \rightharpoonup (u^m, \rho^m)$ weakly in $\mathbb{E}_1(J_m)$. The limit functions also satisfy the first three

equations of (2.26) with μ on J_m , due to the mapping properties of the linear operators described in (2.15), see also (R), (2.5) and (2.11). It still holds $\|(u^m, \rho^m)\|_{\mathbb{E}_1(J_m)} \leq c(f, g)$ for all $m \in \mathbb{N}$. We can then take a subsubsequence of (u_n, ρ_n) converging weakly in $\mathbb{E}_1(J_{m+1})$ to a solution (u^{m+1}, ρ^{m+1}) on J_{m+1} . The functions (u^{m+1}, ρ^{m+1}) extend (u^m, ρ^m) since the subsubsequence still converges weakly in $\mathbb{E}_1(J_m)$ to (u^m, ρ^m) . By induction, we thus obtain a solution (u, ρ) of the first three equations of (2.26) on J fulfilling $\|(u, \rho)\|_{\mathbb{E}_1(J)} \leq c(f, g)$.

Let $(v, \sigma) \in \mathbb{E}_1(J)$ satisfy the first three equations of (2.26) on J with $f = 0$ and $g = 0$. Due to the embedding (2.18), the function (v, σ) belongs to $W_p^1(\mathbb{R}_-; E_0) \cap L_p(\mathbb{R}_-; E_1)$. Equations (2.26) thus imply $(v, \sigma) \in L_p(\mathbb{R}_-; E_1^0)$. So Corollary 2.6 and (2.16) yield that $(v(t), \sigma(t)) = e^{-\mu(t-\tau)} T(t-\tau)(v(\tau), \sigma(\tau))$ for all $t \geq \tau$ in J . Since this semigroup is exponentially stable by Corollary 2.6, we derive $(v, \sigma) = 0$. \square

The above lemma allows to solve the stationary problem related to (2.25).

Lemma 4.3. *Assume that Hypothesis 4.1 holds. Let $\mu_0 \geq 0$ be given by Corollary 2.6 and set $\mu = \mu_0 + 1$. Then there is an operator $\mathcal{S}_{st} \in \mathcal{L}(X \times Y_1, E_1)$ such that, for any $(x, y) \in X \times Y_1$, the function $\mathcal{S}_{st}(x, y)$ is the unique solution $w \in E_1 = X_1 \times Z_1$ of the boundary value problem*

$$(\mu + \Lambda_*)w = (x, y_0) \quad \text{and} \quad (\widehat{B}, \widehat{C})w = \widehat{y}.$$

Proof. Let $(x, y) \in X \times Y_1$. We set $f = e_1 x \in \mathbb{E}(\mathbb{R}_-)$ and $g = e_1 y \in \mathbb{F}(\mathbb{R}_-)$. Lemma 4.2 gives a unique solution $(\tilde{v}, \tilde{\sigma}) \in \mathbb{E}_1(\mathbb{R}_-)$ of

$$\begin{aligned} (\partial_t + \mu - 1)u(t) + (A_{*u} - A_{*\rho}B_0)u(t) + (A_{*\rho} - A_{*\rho}C_0)\rho(t) &= f(t), \\ (\partial_t + \mu - 1)\rho(t) + B_0u(t) + C_0\rho(t) &= g_0(t), \\ \widehat{B}u(t) + \widehat{C}\rho(t) &= \widehat{g}(t), \end{aligned} \quad (4.2)$$

for $t \in \mathbb{R}_-$. It further holds

$$\|(\tilde{v}, \tilde{\sigma})\|_{\mathbb{E}_1(\mathbb{R}_-)} \leq c(\|f\|_{\mathbb{E}(\mathbb{R}_-)} + \|g\|_{\mathbb{F}(\mathbb{R}_-)}) \leq c(|x|_X + |y|_{Y_\gamma}).$$

We set $v = e_{-1}\tilde{v}$ and $\sigma = e_{-1}\tilde{\sigma}$. Then (v, σ) solves the first three equations of (2.26) with the constant inhomogeneities x and y . On the other hand, for any $r \geq 0$ the functions $\tilde{v}_r = e^r \tilde{v}(\cdot - r) \in \mathbb{E}_u(\mathbb{R}_-)$ and $\tilde{\sigma}_r = e^r \tilde{\sigma}(\cdot - r) \in \mathbb{E}_\rho(\mathbb{R}_-)$ also satisfy (4.2). Hence, $\tilde{v}_r = \tilde{v}$ and $\tilde{\sigma}_r = \tilde{\sigma}$ by the uniqueness. Since $r \geq 0$ is arbitrary and $(\tilde{v}, \tilde{\sigma}) = e_1(v, \sigma)$, it follows that $(v, \sigma) =: \mathcal{S}_{st}(x, y)$ does not depend on time, and thus $(\mu + \Lambda_*)w = (x, y_0)$, $(\widehat{B}, \widehat{C})w = \widehat{y}$, and

$$\begin{aligned} |(v, \sigma)|_{E_1} &\leq \|(v, \sigma)\|_{\mathbb{E}_1([-1, 0])} = \|(e_{-1}\tilde{v}, e_{-1}\tilde{\sigma})\|_{\mathbb{E}_1([-1, 0])} \leq c \|(\tilde{v}, \tilde{\sigma})\|_{\mathbb{E}_1([-1, 0])} \\ &\leq c \|(\tilde{v}, \tilde{\sigma})\|_{\mathbb{E}_1(\mathbb{R}_-)} \leq c(|\varphi|_X + |\psi|_{Y_\gamma}). \end{aligned}$$

Let $w = (v, \sigma) \in E_1$ satisfy $(\mu + \Lambda_*)w = 0$ and $(\widehat{B}, \widehat{C})w = 0$. Then $(B_0, C_0)w = -\mu\sigma \in Z_1 \hookrightarrow Z_0$ by (2.19) so that $w \in E_0^1$ belongs to the kernel of $\mu + \Lambda_0$. Since $\mu \in \rho(-\Lambda_0)$ by Corollary 2.6, the operator \mathcal{S}_{st} is injective. \square

We fix the number $\mu = \mu_0 + 1$ obtained in the above lemma for the remainder of the paper. The next result allows to use the asymptotic behavior of $T(\cdot)$ (determined by $\sigma(\Lambda_0)$) in the investigation of the longterm behavior of the

nonlinear problem (1.1), by means of the ‘mild formula’ in (d). Part (b) and (2.19) give the embeddings

$$E_1^0 \subset E_1 \hookrightarrow E_\gamma \hookrightarrow E_0 \hookrightarrow E \hookrightarrow E_{-1}.$$

Observe that part (c) describes the difference between Λ_{-1} and Λ_* which expresses the impact of the boundary conditions. We define

$$\Pi = (\mu + \Lambda_{-1})\mathcal{N}_1.$$

Proposition 4.4. *Under Hypothesis 4.1, the following assertions hold.*

- (a) *There are operators $\mathcal{N}_1 \in \mathcal{L}(\widehat{Y}_1, E_1)$ and $R \in \mathcal{L}(E, E_1)$ such that $(\mu + \Lambda_*)\mathcal{N}_1 = 0$ and $(\widehat{B}, \widehat{C})\mathcal{N}_1 = I_{\widehat{Y}_1}$, as well as $(\mu + \Lambda_*)R = I_E$ and $(\widehat{B}, \widehat{C})R = 0$.*
- (b) *We have $E \hookrightarrow E_{-1}$ and $\Lambda_{-1}w = \Lambda_*w$ for all $w \in E_1$ with $(\widehat{B}, \widehat{C})w = 0$.*
- (c) *It holds $\Pi \in \mathcal{L}(\widehat{Y}_1, E_{-1})$ and $\Lambda_*w = \Lambda_{-1}w - \Pi(\widehat{B}, \widehat{C})w$ for all $w \in E_1$.*
- (d) *Let $J = [0, T]$, $(w_0, f, g) \in \mathbb{D}(J)$, and put $\tilde{f} := f - A_{*\dot{\rho}}g_0 \in \mathbb{E}(J)$. Then the solution $w \in \mathbb{E}_1(J)$ of (2.25) is given by*

$$w(t) = T(t)w_0 + \int_0^t T_{-1}(t-\tau)[(\tilde{f}(\tau), g_0(\tau)) + \Pi\widehat{g}(\tau)] \, d\tau, \quad t \in J. \quad (4.3)$$

Moreover, w is the solution of (2.26) with data (w_0, \tilde{f}, g) and $\mu = 0$, where we have $\|\tilde{f}\|_{\mathbb{E}(J)} \leq c(\|f\|_{\mathbb{E}(J)} + \|g_0\|_{L_p(J; Y_{0\gamma})}) \leq c(\|f\|_{\mathbb{E}(J)} + \|g_0\|_{\mathbb{F}(J)})$.

Proof. (a) For $(x, y_0, \widehat{y}) \in E \times Y_1 = X \times Y_{01} \times \widehat{Y}_1$, we set $\mathcal{N}_1\widehat{y} = \mathcal{S}_{\text{st}}(0, 0, \widehat{y})$ and $R(x, y_0) = \mathcal{S}_{\text{st}}(x, y_0, 0)$. Assertion (a) then follows from Lemma 4.3.

(b) Let $(x, y_0) \in E_0$. We then have $(B_0, C_0)R(x, y_0) = y_0 - \mu[R(x, y_0)]_1 \in Z_0$ by (2.19), so that R maps E_0 into $E_1^0 = D(\Lambda_0)$ and

$$(\mu + \Lambda_0)R(x, y_0) = (\mu + \Lambda_*)R(x, y_0) = (x, y_0). \quad (4.4)$$

As a result, $R : E \rightarrow E_1$ is a continuous extension of $(\mu + \Lambda_0)^{-1}$. Since R is injective by Lemma 4.3, one can see as in the proof of Lemma 3.3 of [24] that $E \hookrightarrow E_{-1}$. We can then extend (4.4) to the equation $(\mu + \Lambda_{-1})R = (\mu + \Lambda_*)R$ on E , using $R \in \mathcal{L}(E, E_1)$ and the density of E_0 in E . Lemma 4.3 implies that R is an isomorphism from E to $\{w \in E_1 \mid (\widehat{B}, \widehat{C})w = 0\}$; i.e., (b) holds.

(c) For $w \in E_1$, parts (a) and (b) imply that $(\widehat{B}, \widehat{C})(w - \mathcal{N}_1(\widehat{B}, \widehat{C})w) = 0$ and

$$(\mu + \Lambda_*)w = (\mu + \Lambda_*)(w - \mathcal{N}_1(\widehat{B}, \widehat{C})w) = (\mu + \Lambda_{-1})(w - \mathcal{N}_1(\widehat{B}, \widehat{C})w),$$

as asserted. The mapping property of Π is clear.

(d) Let $w = (u, \rho)$ be the solution of (2.25) and $\tilde{f} = f - A_{*\dot{\rho}}g_0$. We insert $\dot{\rho} = g_0 - B_0u - C_0\rho$ into the term $A_{*\dot{\rho}}\dot{\rho}$ in (2.25), obtaining

$$\partial_t w(t) + \Lambda_*w(t) = (f(t) - A_{*\dot{\rho}}g_0(t), g_0(t)) = (\tilde{f}(t), g_0(t)), \quad t \in [0, T]. \quad (4.5)$$

Moreover, w satisfies (2.26) with data (w_0, \tilde{f}, g) and $\mu = 0$. From (2.15), we deduce that $\tilde{f} \in \mathbb{E}([0, T])$ and the asserted estimate for \tilde{f} . Part (c) and the boundary condition $(\widehat{B}, \widehat{C})w(t) = \widehat{g}(t)$ then lead to

$$\partial_t w + \Lambda_{-1}w(t) = (\tilde{f}(t), g_0(t)) + \Pi\widehat{g}(t), \quad t \in [0, T].$$

Since $E \hookrightarrow E_{-1}$, this is an evolution equation in E_{-1} so that (d) follows. \square

In the following we rewrite the solutions of (2.25) on unbounded time intervals $J \in \{\mathbb{R}_\pm, \mathbb{R}\}$ as in (4.3). To treat the case $J = \mathbb{R}_\pm$, we assume that the (rescaled) semigroup $\{e^{\delta t} T(t)\}_{t \geq 0}$ has an *exponential dichotomy* for $\delta \in [\delta_1, \delta_2]$ and some segment $[\delta_1, \delta_2] \subset \mathbb{R}$ (i.e., $\sigma(-\Lambda_0 + \delta) \cap i\mathbb{R} = \emptyset$). Let $P \in \mathcal{L}(E_0)$ be the (stable) spectral projection for $-\Lambda_0 + \delta$ corresponding to the part of $\sigma(-\Lambda_0 + \delta)$ in the open left halfplane, and set $Q = I - P$. Then, $P \in \mathcal{L}(E_1^0)$, P commutes with $T(t)$ and Λ_0 , $Q \in \mathcal{L}(E_0, E_1^0)$, $T(t)$ is invertible on QE_0 with the inverse $T_Q(-t)Q$, and $\|e^{t\delta} T(t)P\|_{\mathcal{L}(E_0)}, \|e^{-t\delta} T_Q(-t)Q\|_{\mathcal{L}(E_0)} \leq ce^{-\epsilon t}$ for $t \geq 0$ and some $\epsilon > 0$. Further, there are extensions $P_{-1} \in \mathcal{L}(E_{-1})$ of P and $Q_{-1} \in \mathcal{L}(E_{-1}, E_1^0)$ of Q such that $T_{-1}(t)$ has an exponential dichotomy on E_{-1} with the same constants. Since $Q = Q^2$, we throughout write Q instead of Q_{-1} . From $P = I - Q$, we deduce

$$P \in \mathcal{L}(E_1) \cap \mathcal{L}(E_\gamma) \cap \mathcal{L}(E_\gamma^0) \quad \text{and} \quad P_{-1} \in \mathcal{L}(E). \quad (4.6)$$

Occasionally, we omit the subscript -1 . (Compare e.g. §2 of [19] for these facts.) It further holds:

$$\text{If } (w_0, f, g) \in \mathbb{D}(J), \quad \text{then } (Pw_0, f, g) \in \mathbb{D}(J). \quad (4.7)$$

In fact, we have $(B, C)Pw_0 = (B, C)w_0 - (B, C)Qw_0$ and $Qw_0 \in E_1^0$ so that $(\widehat{B}, \widehat{C})Pw_0 = \widehat{g}(0)$ and $(B_0, C_0)Pw_0 - g_0(0) \in Z_\gamma^1$.

Let $e_\delta T(\cdot)$ have an exponential dichotomy. Given $(\varphi_0, f, g) \in E_\gamma \times \mathbb{E}(\mathbb{R}_+, \delta) \times \mathbb{F}(\mathbb{R}_+, \delta)$, resp. $(\varphi_0, f, g) \in E_{-1} \times \mathbb{E}(\mathbb{R}_-, \delta) \times \mathbb{F}(\mathbb{R}_-, \delta)$, we can then define

$$\begin{aligned} L_{P, \Lambda_0}^+(\varphi_0, f, g)(t) &= T(t)\varphi_0 + \int_0^t T_{-1}(t-\tau)P_{-1}[(\tilde{f}(\tau), g_0(\tau)) + \Pi\widehat{g}(\tau)] d\tau \quad (4.8) \\ &\quad - \int_t^\infty T_Q(t-\tau)Q[(\tilde{f}(\tau), g_0(\tau)) + \Pi\widehat{g}(\tau)] d\tau, \quad t \geq 0, \end{aligned}$$

$$\phi_0^+ = - \int_0^\infty T_Q(-\tau)Q[(\tilde{f}(\tau), g_0(\tau)) + \Pi\widehat{g}(\tau)] d\tau, \quad \text{resp.,} \quad (4.9)$$

$$\begin{aligned} L_{P, \Lambda_0}^-(\varphi_0, f, g)(t) &= T_Q(t)Q\varphi_0 + \int_{-\infty}^t T_{-1}(t-\tau)P_{-1}[(\tilde{f}(\tau), g_0(\tau)) + \Pi\widehat{g}(\tau)] d\tau \\ &\quad - \int_t^0 T_Q(t-\tau)Q[(\tilde{f}(\tau), g_0(\tau)) + \Pi\widehat{g}(\tau)] d\tau, \quad t \leq 0, \quad (4.10) \end{aligned}$$

$$\phi_0^- = \int_{-\infty}^0 T_{-1}(-\tau)P_{-1}[(\tilde{f}(\tau), g_0(\tau)) + \Pi\widehat{g}(\tau)] d\tau. \quad (4.11)$$

Here we set again $\tilde{f} := f - A_{*\dot{\rho}}g_0 \in \mathbb{E}(J, \delta)$. Using the properties of $T(\cdot)$ and Proposition 4.4(c), it is easy to verify the existence of these integrals in E_{-1} .

We now take $(w_0, f, g) \in \mathbb{D}(J)$. Clearly, a function $w = (u, \rho) \in \mathbb{E}_1^{\text{loc}}(J)$ solves (2.25) if and only if $\tilde{w} = e_\delta w \in \mathbb{E}_1^{\text{loc}}(J)$ is a solution of the rescaled problem

$$\begin{aligned} \partial_t v(t) + (A_{*u} - \delta - A_{*\dot{\rho}}B_0)v(t) + (A_{*\rho} - A_{*\dot{\rho}}C_0)\sigma &= e_\delta(f(t) - A_{*\dot{\rho}}g_0), \\ \partial_t \sigma(t) + B_0 v(t) + (C_0 - \delta)\sigma(t) &= e_\delta g_0(t), \\ \widehat{B}(t)v + \widehat{C}(t)\sigma &= e_\delta \widehat{g}(t), \\ (v(0), \sigma(0)) &= (u_0, \rho_0), \end{aligned}$$

for $t \in J$. The solution of this system is denoted by $(v, \sigma) =: \mathcal{S}_{\Lambda_0 - \delta}(w_0, e_\delta f, e_\delta g)$, cf. (2.26) in Corollary 2.6. Note that $-\Lambda_0 + \delta$ generates the analytic semigroup $e_\delta T(\cdot)$ on E_0 . We start with the case $J = \mathbb{R}_+$. Using (4.3), (4.8) and (4.9), in a standard way we compute

$$\begin{aligned} e_\delta \mathcal{S}_{\Lambda_0}(w_0, f, g) &= \mathcal{S}_{\Lambda_0 - \delta}(w_0, e_\delta f, e_\delta g) \\ &= e_\delta T(\cdot)[Qw_0 - \phi_0^+] + L_{P, \Lambda_0 - \delta}^+(Pw_0, e_\delta f, e_\delta g) \\ &= L_{P, \Lambda_0 - \delta}^+(w_0 - \phi_0^+, e_\delta f, e_\delta g) = e_\delta L_{P, \Lambda_0}^+(w_0 - \phi_0^+, f, g). \end{aligned} \quad (4.12)$$

The next result is similar to Proposition 8 of [19] in the case of static boundary conditions, but we cannot follow the proof given there. The main problem is that we do not know whether the spectral projections P and Q leave invariant $\mathbb{E}_1(J)$ because of the extra time regularity in the ρ component.

Proposition 4.5. *Assume that Hypothesis 4.1 holds and that for $\delta \in [\delta_1, \delta_2] \subset \mathbb{R}$ the semigroup $e_\delta T(\cdot)$ has an exponential dichotomy with the stable projection P , and let $Q = I - P$. Given $(w_0, f, g) \in \mathbb{D}(\mathbb{R}_+, \delta)$, the following assertions are equivalent.*

- (a) $\mathcal{S}_{\Lambda_0}(w_0, f, g) \in \mathbb{E}(\mathbb{R}_+, \delta)$.
- (b) $L_{P, \Lambda_0}^+(w_0 - \phi_0^+, f, g) \in \mathbb{E}(\mathbb{R}_+, \delta)$.
- (c) $\phi_0^+ = Qw_0$.

If these assertions hold, then $(u, \rho) := \mathcal{S}_{\Lambda_0}(w_0, f, g) = L_{P, \Lambda_0}^+(Pw_0, f, g)$ belongs to $\mathbb{E}_1(\mathbb{R}_+, \delta)$ and solves (2.25), and we have

$$\begin{aligned} \|\mathcal{S}_{\Lambda_0}(w_0, f, g)\|_{\mathbb{E}_1(\mathbb{R}_+, \delta)} &\leq c(|w_0|_{E_\gamma} + |(B_0, C_0)w_0 - g_0(0)|_{Z_\gamma^1} \\ &\quad + \|f\|_{\mathbb{E}(\mathbb{R}_+, \delta)} + \|g\|_{\mathbb{F}(\mathbb{R}_+, \delta)}), \end{aligned}$$

where c does not depend on w_0 , f , g or δ . (Note that $\dot{\rho}(0) = g_0(0) - (B_0, C_0)w_0$.)

Proof. In view of (4.12), we only have to consider the case $\delta = 0$. Moreover, (4.12) implies that (a) and (b) are equivalent and that the equality $\mathcal{S}_{\Lambda_0}(w_0, f, g) = L_{P, \Lambda_0}^+(Pw_0, f, g)$ follows from (c). We check below that the integrals in (4.8) belong to $\mathbb{E}(\mathbb{R}_+)$. Hence, assertions (b) and (c) are equivalent.

We now assume that $(w_0, f, g) \in \mathbb{D}(\mathbb{R}_+)$ and (c) holds, and estimate the solution in $\mathbb{E}_1(\mathbb{R}_+)$. Due to Corollary 2.6 and (4.7), there is a function $\varphi = (u_\mu, \rho_\mu) \in \mathbb{E}_1(\mathbb{R}_+)$ solving (2.26) on \mathbb{R}_+ with μ from Lemma 4.3, the initial value Pw_0 and the inhomogeneities $\tilde{f} = f - A_{*\dot{\rho}}g_0$ and g . Moreover,

$$\begin{aligned} \|\varphi\|_{\mathbb{E}_1(\mathbb{R}_+)} &\leq c(\|\tilde{f}\|_{\mathbb{E}(\mathbb{R}_+)} + \|g\|_{\mathbb{F}(\mathbb{R}_+)} + |w_0|_{E_\gamma} + |(B_0, C_0)w_0 - g_0(0)|_{Z_\gamma^1}) \\ &\leq c\|(w_0, f, g)\|_{\mathbb{D}(\mathbb{R}_+)}. \end{aligned}$$

Using (2.26), integration by parts, $Q\varphi(0) = QPw_0 = 0$, Proposition 4.4, (4.9) and (c), we further compute

$$\begin{aligned} \int_0^\infty T_Q(-\tau)Q\mu\varphi(\tau) d\tau &= \int_0^\infty T_Q(-\tau)Q[(\tilde{f}(\tau), g_0(\tau)) - \dot{\varphi}(\tau) - \Lambda_*\varphi(\tau)] d\tau \\ &= \int_0^\infty T_Q(-\tau)Q[(\tilde{f}(\tau), g_0(\tau)) + (\Lambda_{-1} - \Lambda_*)\varphi(\tau)] d\tau \end{aligned}$$

$$= \int_0^\infty T_Q(-\tau)Q[(\tilde{f}(\tau), g_0(\tau)) + \Pi\hat{g}(\tau)] \, d\tau = -Qw_0.$$

Theorem 2.2 of [8] shows that the operator (Λ, E_1^0) has maximal L_p regularity in E_0 , so that the lower order perturbation Λ_0 also has maximal L_p regularity in E_0 . (See [3], [7] or [18] for this concept and Proposition III.1.6.3 in [3] for the relevant perturbation result.) Since $\mu\varphi \in L_p(\mathbb{R}_+; E_0)$ and Qw_0 satisfy the above equation, Theorem 2.4 in [9] now gives a unique function $\psi = (v, \sigma) \in L_p(\mathbb{R}_+; E_1^0) \cap W_p^1(\mathbb{R}_+; E_0) =: \mathbb{E}_1^0(\mathbb{R}_+)$ such that

$$\partial_t \psi(t) + \Lambda_0 \psi(t) = \mu\varphi, \quad t \geq 0, \quad \psi(0) = Qw_0, \quad (4.13)$$

and we have

$$\|\psi\|_{\mathbb{E}_1^0(\mathbb{R}_+)} \leq c (|Qw_0|_{E_\gamma} + \|\varphi\|_{L_p(\mathbb{R}_+; E_0)}) \leq c \|(w_0, f, g)\|_{\mathbb{D}(\mathbb{R}_+)}.$$

Therefore the function $w = (u, \rho) := \varphi + \psi \in L_p(\mathbb{R}_+; E_1) \cap W_p^1(\mathbb{R}_+; E)$ solves (2.26) with $\mu = 0$, the initial value w_0 and the inhomogeneities \tilde{f} and g ; i.e., w satisfies (2.25). We have further shown that

$$\|w\|_{L_p(\mathbb{R}_+; E_1)} + \|w\|_{W_p^1(\mathbb{R}_+; E)} \leq c \|(w_0, f, g)\|_{\mathbb{D}(\mathbb{R}_+)}.$$

It remains to check that $\|(w_0, f, g)\|_{\mathbb{D}(\mathbb{R}_+)}$ also dominates the norm of σ in the other spaces forming $\mathbb{E}_\rho(\mathbb{R}_+)$. We start with $W_p^{\kappa_j}(\mathbb{R}_+; W_p^{k_j}(\Sigma; V_\rho))$ for $j \in \tilde{\mathcal{J}}$.

We first note that $(\widehat{B}, \widehat{C})Qw_0 = 0$ and $(B_0, C_0)Qw_0 - \mu\rho_\mu(0) \in Z_\gamma^1$ because of $Qw_0 \in E_1^0$, $\rho_\mu \in \mathbb{E}_\rho(\mathbb{R}_+)$, and (2.11). Corollary 2.6 thus gives a solution $\tilde{\psi} \in \mathbb{E}_1([0, 2])$ of (4.13). The embedding (2.18) yields $\tilde{\psi} \in W_p^1([0, 2]; E_0)$ and thus $\tilde{\psi} \in L_p([0, 2]; E_0^1)$ by (4.13) and $\mu\varphi \in L^p([0, 2]; E_0)$. As a consequence, $\tilde{\sigma} \in E_0^1([0, 2]$ and $\psi = \tilde{\psi}$ on $[0, 2]$. The properties of Q and (2.19) then imply

$$\begin{aligned} \|\psi\|_{\mathbb{E}_1([0, 2])} &\leq c (|Qw_0|_{E_\gamma} + |(B_0, C_0)Qw_0 - \mu\rho_\mu(0)|_{Z_\gamma^1} + \|\mu u_\mu\|_{\mathbb{E}([0, 2])} \\ &\quad + \|\mu\rho_\mu\|_{\mathbb{F}_0([0, 2])}) \\ &\leq c \|(w_0, f, g)\|_{\mathbb{D}([0, 2])}. \end{aligned}$$

To proceed, we recall from Theorem 2.4 in [9] that ψ is given by

$$\psi(t) = \int_0^t T(t-\tau)P\mu\varphi(\tau) \, d\tau - \int_t^\infty T_Q(t-\tau)Q\mu\varphi(\tau) \, d\tau =: \psi^1(t) + \psi^2(t)$$

for $t \geq 0$. Let $J_n = [n-1, n+1]$ for $n \in \mathbb{N}$. As in the proof of Proposition 8 of [19], we now use smooth functions $\chi_n : J_n \rightarrow \mathbb{R}$ such that χ_n , χ'_n and χ''_n are uniformly bounded, $\chi_n(n-1) = 1$ and $\chi_n = 0$ on $[n-1/2, n+1]$, for every $n \in \mathbb{N}$. For $t \in [n, n+1]$, we then write

$$\begin{aligned} \psi^1(t) &= P \int_{n-1}^t T(t-\tau)(1 - \chi_n(\tau))\mu\varphi(\tau) \, d\tau \\ &\quad + T(t-n)T(\frac{1}{2}) \int_{n-1}^{n-\frac{1}{2}} T(n-\frac{1}{2}-\tau)P\chi_n(\tau)\mu\varphi(\tau) \, d\tau \\ &\quad + T(t-n)T(1) \int_0^{n-1} T(n-1-\tau)P\mu\varphi(\tau) \, d\tau. \end{aligned} \quad (4.14)$$

We denote the first integral by $\psi_n^{11}(t)$ and the sum of the other two summands by ψ_n^{12} . In view of (4.3), ψ_n^{11} is the solution of (2.26) on J_n with data $(0, \mu(1 - \chi_n)u_\mu, [\mu(1 - \chi_n)\rho_\mu, 0]) \in \mathbb{D}(J_n)$. Corollary 2.6 thus yields $\|\psi_n^{11}\|_{\mathbb{E}_1(J_n)} \leq c(\|u_\mu\|_{\mathbb{E}(J_n)} + \|\rho_\mu\|_{\mathbb{F}_0(J_n)}) =: c_n$. Because $P = I - Q$, $QE_{-1} \subset E_1^0$ and (2.9), we can estimate the norm of $[P\psi_n^{11}]_2$ in $W_p^{\kappa_j}(J_n; W_p^{\kappa_j}(\Sigma; Y_\rho))$ by $c\|\psi_n^{11}\|_{\mathbb{E}_1(J_n)}$ and thus by cc_n . Standard semigroup theory further yields

$$|\partial_t^i \psi_n^{12}(t)|_{E_1} \leq c \int_0^t e^{-\epsilon(t-\tau)} |\varphi(\tau)|_{E_0} d\tau, \quad |\partial_t^i \psi^2(t)|_{E_1} \leq c \int_t^\infty e^{\epsilon(t-\tau)} |\varphi(\tau)|_{E_0} d\tau$$

for some $\epsilon > 0$ and $i = 0, 1$. Using Lemma 2.8 and also a slight variant of Lemma 4.7 below for ψ^1 , we now conclude (writing $\mathbb{G}_j(J) = W_p^{\kappa_j}(J; W_p^{\kappa_j}(\Sigma; V_\rho))$)

$$\begin{aligned} \|\sigma\|_{\mathbb{G}_j(\mathbb{R}_+)}^p &\leq c\|(w_0, f, g)\|_{\mathbb{D}([0, 2])}^p + c\|\psi^2\|_{W_p^1(\mathbb{R}_+; E_1)}^p \\ &\quad + c \sum_{n \geq 1} (\|[P\psi_n^{11}]_2\|_{\mathbb{G}_j(J_n)}^p + \|[\psi_n^{12}]_2\|_{W_p^1(J_n; Z_1)}^p) \\ &\leq c\|(w_0, f, g)\|_{\mathbb{D}(\mathbb{R}_+)}^p + c\|\varphi\|_{L_p(\mathbb{R}_+; E_0)}^p + c \sum_{n \geq 1} (\|u_\mu\|_{\mathbb{E}(J_n)}^p + \|\rho_\mu\|_{\mathbb{F}_0(J_n)}^p) \\ &\leq c\|(w_0, f, g)\|_{\mathbb{D}(\mathbb{R}_+)}^p. \end{aligned}$$

This inequality, $\dot{\sigma} = -B_0 v - C_0 \sigma + \mu \rho_\mu$, (R) and (S) with time-independent coefficients then imply

$$\begin{aligned} \|\dot{\sigma}\|_{W_p^{\kappa_0}(\mathbb{R}_+; Y_\rho)} &\leq c(\|v\|_{\mathbb{E}_u(\mathbb{R}_+)} + \|\sigma\|_{W_p^{\kappa_0}(\mathbb{R}_+; W_p^{\kappa_0}(\Sigma; V_\rho))} + \|\rho_\mu\|_{W_p^{\kappa_0}(\mathbb{R}_+; Y_\rho)}) \\ &\leq c\|(w_0, f, g)\|_{\mathbb{D}(\mathbb{R}_+)}^p. \end{aligned}$$

(We remark that we cannot use (R) to estimate $C_0 \sigma$ since we do not yet know that $\sigma \in \mathbb{E}_\rho$.) Summing up, it holds $\|w\|_{\mathbb{E}_1(\mathbb{R}_+)} \leq c\|(w_0, f, g)\|_{\mathbb{D}(\mathbb{R}_+)}$. \square

The corresponding result for $J = \mathbb{R}_-$ looks a bit different since in (4.10) we have to write $T(t)Qw_0$ rather than $T(t)w_0$ for negative t . Moreover, Proposition 4.6 does not require a compatibility condition since it deals with a final value problem on $J = \mathbb{R}_-$. The proof of this proposition is similar to the previous one: The asserted equivalence and the representation of the solution by $L_{P, \Lambda}^-$ can be shown as in, e.g., Proposition 9 in [19]. As above, it suffices to consider $\delta = 0$. Lemma 4.2 gives a solution $\varphi \in \mathbb{E}_1(\mathbb{R}_-)$ of the first three equations of (2.26) on \mathbb{R}_- with inhomogeneities \tilde{f} and g . Using Theorem 2.5 in [9], one again obtains a solution $\psi \in \mathbb{E}_1^0(\mathbb{R}_-)$ of (4.13) on \mathbb{R}_- with final value $w_0 - \varphi(0)$. The sum $w = \varphi + \psi$ then solves (2.25) on \mathbb{R}_- , and it can be estimated as in the proof of Proposition 4.5. (It is easy to see that the resulting new term $T_Q(\cdot)Q(w_0 - \varphi(0))$ even belongs to $W_p^2(\mathbb{R}_-; E_1^0)$, with norm less than $c|Q(w_0 - \varphi(0))|_E$.) We thus omit further details.

Proposition 4.6. *Assume that Hypothesis 4.1 holds and that for $\delta \in [\delta_1, \delta_2] \subset \mathbb{R}$ the semigroup $e_\delta T(\cdot)$ has an exponential dichotomy with the stable projection P , and let $Q = I - P$. Given $(w_0, f, g) \in E_{-1} \times \mathbb{E}(\mathbb{R}_-, \delta) \times \mathbb{F}(\mathbb{R}_-, \delta)$, there is a*

solution $w = \mathcal{S}_{\Lambda_0}(w_0, f, g)$ of (2.25) in $\mathbb{E}(\mathbb{R}_-, \delta)$ if and only if $P_{-1}w_0 = \phi_0^-$. In this case, this solution is unique, $w = L_{P, \Lambda_0}^-(w_0, f, g) \in \mathbb{E}_1(\mathbb{R}_-, \delta)$, and

$$\|\mathcal{S}_{\Lambda_0}(w_0, f, g)\|_{\mathbb{E}_1(\mathbb{R}_-, \delta)} \leq c (|Qw_0|_E + \|f\|_{\mathbb{E}(\mathbb{R}_-, \delta)} + \|g\|_{\mathbb{F}(\mathbb{R}_-, \delta)}),$$

where c does not depend on w_0 , f , g or δ .

We continue with the discussion of nonlinear maps F and G acting on exponentially weighted function spaces on unbounded time intervals, cf. (4.1). We start with an elementary, but crucial lemma. The straightforward proof is omitted. (It also uses Lemma 11 of [19] when treating the Slobodeckii spaces.) The notation $a \simeq b$ means that $a \leq c_1 b \leq c_2 a$ for some constants $c_1, c_2 > 0$. We put $\mathbb{Z}_+ = \mathbb{N}_0$ and $\mathbb{Z}_- = \{-1, -2, \dots\}$.

Lemma 4.7. *Let V be a Banach space, $J = \mathbb{R}_\pm$, $\kappa \in (0, 1)$, $a > 0$, $d \geq 0$, $|\delta| \leq d$, $J_n = [n, n+1]$, and $J'_n = [n-a, n+1+a] \cap J$ for $n \in \mathbb{Z}$. Then the following assertions hold with constants only depending on a and d .*

- (a) $\|\mathbf{e}_\delta h\|_{L_p(\mathbb{R}_\pm; V)}^p \simeq \sum_{n \in \mathbb{Z}_\pm} e^{\delta np} \|h\|_{L_p(J_n; V)}^p$.
- (b) $\|\mathbf{e}_\delta h\|_{W_p^1(\mathbb{R}_\pm; V)}^p \simeq \sum_{n \in \mathbb{Z}_\pm} e^{\delta np} \|h\|_{W_p^1(J_n; V)}^p$.
- (c) $\|\mathbf{e}_\delta h\|_{W_p^\kappa(\mathbb{R}_\pm; V)}^p \simeq \sum_{n \in \mathbb{Z}_\pm} e^{\delta np} \|h\|_{W_p^\kappa(J'_n; V)}^p$.
- (d) $\|\mathbf{e}_\delta h\|_{W_p^{1+\kappa}(\mathbb{R}_\pm; V)}^p \simeq \sum_{n \in \mathbb{Z}_\pm} e^{\delta np} \|h\|_{W_p^{1+\kappa}(J'_n; V)}^p$,
 $\|\mathbf{e}_\delta h\|_{W_p^{1+\kappa}(\mathbb{R}_\pm; V)}^p \leq c \sum_{n \in \mathbb{Z}_\pm} e^{\delta np} (\|h'\|_{W_p^\kappa(J'_n; V)}^p + \|h\|_{W_p^1(J_n; V)}^p)$.
- (e) $\|h\|_{\mathbb{E}_u(\mathbb{R}_\pm, \delta)}^p \simeq \sum_{n \in \mathbb{Z}_\pm} e^{\delta np} \|h\|_{\mathbb{E}_u(J_n)}^p$.
- (f) $\|h\|_{\mathbb{E}_\rho(\mathbb{R}_\pm, \delta)}^p \simeq \sum_{n \in \mathbb{Z}_\pm} e^{\delta np} \|h\|_{\mathbb{E}_\rho(J'_n)}^p$.
- (g) $\|h\|_{\mathbb{F}(\mathbb{R}_\pm, \delta)}^p \simeq \sum_{n \in \mathbb{Z}_\pm} e^{\delta np} \|h\|_{\mathbb{F}(J'_n)}^p$.

The same results hold on $J = \mathbb{R}$ if we use the function $\mathbf{e}(\alpha, \beta)$ for $|\alpha|, |\beta| \leq d$ instead of \mathbf{e}_δ and replace \mathbb{Z}_\pm by \mathbb{Z} .

We now collect the basic assumptions (and some of the notations) for the rest of paper, where we strengthen Hypothesis 4.1.

Hypothesis 4.8. *Let (R) be true, and (S), (E), (LS) and (if $\ell \geq 2m$) (LS $^\pm_\infty$) hold for any $(w_0, \rho_0) \in W_\gamma$. Let $w_* = (u_*, \rho_*) \in W_1$ be an equilibrium of (1.1) and define the maps A_* , B , C , F , G , Λ_* and $\Lambda_0 = \Lambda_*|E_1^0$ as well as the expressions $\langle \psi \rangle_\gamma$ and $\langle \psi \rangle_1$ for this w_* as in (2.15), (3.2), (2.16), (2.20), (3.10).*

Note that $\mathcal{D}(w_*) = 0$ and $\dot{\rho}_* = 0$ if Hypothesis 4.8 holds, and that $\langle \psi \rangle_\gamma$ is locally equivalent to $|\psi|_{E_\gamma}$ if $\ell \leq 2m$. The next result describes the properties of F and G on \mathbb{R}_\pm with weights larger or equal than 1. For $\delta \geq 0$, we set

$$\mathbb{W}_*^1(\mathbb{R}_\pm, \pm\delta) = \{w \in \mathbb{E}_1(\mathbb{R}_\pm, \pm\delta) \mid w(t) \in W_\gamma - w_* \text{ for all } t \in \mathbb{R}_\pm\}.$$

It is straightforward to check that this set is open in $\mathbb{E}_1(\mathbb{R}_\pm, \pm\delta)$ if $\delta > 0$ using (5.1) below. Moreover, 0 belongs to the interior of $\mathbb{W}_*^1(\mathbb{R}_\pm) := \mathbb{W}_1^*(\mathbb{R}_\pm, 0)$.

Proposition 4.9. *Let (R) hold, $\delta \in (0, d]$ and define F and G as in (3.2) for an equilibrium $w_* = (u_*, \rho_*) \in W_1$. We then have*

$$F \in C^1(\mathbb{W}_1^*(\mathbb{R}_\pm, \pm\delta), \mathbb{E}(\mathbb{R}_\pm, \pm\delta)) \quad \text{and} \quad G \in C^1(\mathbb{W}_1^*(\mathbb{R}_\pm, \pm\delta), \mathbb{F}(\mathbb{R}_\pm, \pm\delta))$$

and $F(0) = 0$, $G(0) = 0$, $F'(0) = 0$, $G'(0) = 0$. Moreover, the derivatives are bounded and uniformly continuous on closed balls. If $\delta = 0$, the above results hold on sufficiently small balls in $\mathbb{E}_1(\mathbb{R}_\pm)$ with center 0.

Proof. We only consider the map G on $J = \mathbb{R}_+$, the other cases can be treated in the same way. Let $w = (u, \rho) \in \mathbb{W}_1^*(\mathbb{R}_\pm, \pm\delta)$ and $\varphi = (v, \sigma) \in \mathbb{E}_1(\mathbb{R}_\pm, \pm\delta)$. Set $J_n = [n, n+1]$ and $J'_n = [n-1, n+2] \cap \mathbb{R}_+$. Lemma 4.7 and (3.1) yield

$$\|G(w)\|_{\mathbb{F}(\mathbb{R}_+, \delta)}^p \leq c \sum_{n \in \mathbb{N}_0} e^{\delta np} \|G(w)\|_{\mathbb{F}(J'_n)}^p \leq c \sum_{n \in \mathbb{N}_0} e^{\delta np} \|w\|_{\mathbb{E}_1(J'_n)}^p \leq c \|w\|_{\mathbb{E}_1(\mathbb{R}_+, \delta)}^p$$

so that G maps properly. Since $\delta n \geq 0$, it is straightforward to check that

$$\|\varphi\|_{\mathbb{E}_1(J'_n)}^p \leq e^{\delta np} \|\varphi\|_{\mathbb{E}_1(J'_n)}^p \leq c \|\varphi\|_{\mathbb{E}_1(J'_n, \delta)}^p \leq c \|\varphi\|_{\mathbb{E}_1(\mathbb{R}_+, \delta)}^p. \quad (4.15)$$

Using also this estimate, as above we obtain for $g := G(w+\varphi) - G(w) - G'(w)\varphi$

$$\begin{aligned} \|g\|_{\mathbb{F}(\mathbb{R}_+, \delta)}^p &\leq c \sum_{n \in \mathbb{N}_0} e^{\delta np} \|g\|_{\mathbb{F}(J'_n)}^p \leq c \sum_{n \in \mathbb{N}_0} e^{\delta np} \varepsilon (\|\varphi\|_{\mathbb{E}_1(J'_n)})^p \|\varphi\|_{\mathbb{E}_1(J'_n)}^p \\ &\leq c \sum_{n \in \mathbb{N}_0} e^{\delta np} \varepsilon (\|\varphi\|_{\mathbb{E}_1(\mathbb{R}_+, \delta)})^p \|\varphi\|_{\mathbb{E}_1(J'_n)} \leq \varepsilon (\|\varphi\|_{\mathbb{E}_1(\mathbb{R}_+, \delta)})^p \|\varphi\|_{\mathbb{E}_1(\mathbb{R}_+, \delta)}^p. \end{aligned}$$

Hence, $G : \mathbb{W}_1^*(\mathbb{R}_+, \delta) \rightarrow \mathbb{F}(\mathbb{R}_+, \delta)$ is differentiable. If $\|w\|_{\mathbb{E}_1(\mathbb{R}_+, \delta)} \leq r$ for some r , then $\|w\|_{\mathbb{E}_1(J'_n)} \leq \bar{c}r$ for a constant \bar{c} by (4.15). Let J' be any interval with length 3 and $c_r < \infty$ be the supremum of the norms of $G'(w) : \mathbb{E}_1(J') \rightarrow \mathbb{F}(J')$ for $w \in \mathbb{W}_1^*(J')$ with $\|w\|_{\mathbb{E}_1(J')} \leq \bar{c}r$, see (3.1). Lemma 4.7 then implies

$$\begin{aligned} \|G'(w)\varphi\|_{\mathbb{F}(\mathbb{R}_+, \delta)}^p &\leq c \sum_{n \in \mathbb{N}_0} e^{\delta np} \|G'(w)\varphi\|_{\mathbb{F}_1(J'_n)}^p \leq c c_r^p \sum_{n \in \mathbb{N}_0} e^{\delta np} \|\varphi\|_{\mathbb{E}_1(J'_n)}^p \\ &\leq c \|\varphi\|_{\mathbb{E}_1(\mathbb{R}_+, \delta)}^p. \end{aligned}$$

The equalities $G(0) = 0$ and $G'(0) = 0$ follow from (3.1). The continuity of G' can be checked by the same methods as above. \square

5. THE SADDLE POINT PROPERTY

In this section we construct the stable and unstable manifolds for (1.1) near the equilibrium w_* assuming Hypothesis 4.8 and that $\sigma(-\Lambda_0) \cap i\mathbb{R} = \emptyset$. (Recall our notation stated in Hypothesis 4.8.) The next theorem shows in particular that these manifolds are uniquely given as sets of initial values of solutions starting near w_* and staying in certain neighborhoods of w_* for all $t \geq 0$, resp. all $t \leq 0$. These solutions then converge exponentially to w_* as $t \rightarrow \infty$, resp. as $t \rightarrow -\infty$. If $\sigma(-\Lambda_0)$ is contained in the open left half plane, then the theorem gives a principle of linearized stability (which could be proved much easier, see e.g. Proposition 16 in [19]).

The following observations are used below several times. Fix $r_0 > 0$ such that $B_{E_\gamma}(0, r_0) \subset W_\gamma^*$. Take $\varphi_0 \in W_\gamma^*$ with $|\varphi_0|_{E_\gamma} \leq r < r_0$. Let $\varphi = (v, \sigma) \in \mathbb{E}_1(\mathbb{R}_+, \delta)$ with $\|\varphi\|_{\mathbb{E}_1(\mathbb{R}_+, \delta)} \leq R$ satisfy $\varphi(0) = \varphi_0$ for some $R, \delta \geq 0$. The embeddings (2.5) and (2.11) imply that

$$|\varphi(t)|_{E_\gamma} + |\dot{\sigma}(t)|_{Z_\gamma^1} \leq |e^{\delta t} \varphi(t)|_{E_\gamma} + |e^{\delta t} \dot{\sigma}(t)|_{Z_\gamma^1} \leq c \|\varphi\|_{\mathbb{E}_1(\mathbb{R}_+, \delta)} \leq cR \quad (5.1)$$

for all $t \geq 0$. Hence, for sufficiently small $R > 0$ we deduce $cR < r_0$, and thus $\varphi(t) \in W_\gamma^*$ for all $t \geq 0$; i.e., $\varphi \in \mathbb{W}_1^*(\mathbb{R}_+, \delta)$. Such φ actually exist if r is small enough. In fact, (2.5) and (2.11) give a function $\psi \in \mathbb{E}_1([0, 1])$ with $\psi(0) = \varphi_0$ and $\|\psi\|_{\mathbb{E}_1([0, 1])} \leq c|\varphi_0|_{E_\gamma} \leq cr$. We extend ψ to a compactly supported function $\varphi \in \mathbb{E}_1(\mathbb{R}_+)$ with norm less or equal $c\|\psi\|_{\mathbb{E}_1([0, 1])} \leq cr$, so that we can control R by cr . Analogous results hold for $J = \mathbb{R}_-$ if $\delta \leq 0$.

Recall the definition of the space E_γ^0 in (2.20) and of the map \mathcal{Q} in Lemma 3.2. By (3.11), for solutions $(v, \sigma) \in \mathbb{E}_1(J)$ of (3.7) and $t \in J$, the expression $\langle(v(t), \sigma(t))\rangle_\gamma = |(v(t), \sigma(t))|_{E_\gamma} + |\dot{\sigma}(t)|_{Z_\gamma^1}$ is the norm on the trace space $E_\gamma \times Z_\gamma^1$.

Theorem 5.1. *Assume that Hypothesis 4.8 and $i\mathbb{R} \subset \rho(-\Lambda_0)$ hold. Let P be the stable projection for $T(\cdot)$, $Q = I - P$, $\delta_0 := \text{dist}(i\mathbb{R}, \rho(-\Lambda_0)) > 0$, and $\delta \in (0, \delta_0)$. Then the following assertions hold.*

(a) *There are numbers $r'_k \geq r_k > 0$ and $r_0^k > 0$ with $k \in \{s, u\}$, and C_b^1 maps*

$$\phi_s : D_s := \{\xi \in PE_\gamma^0 \mid |\xi|_{E_\gamma^0} < r_0^s\} \longrightarrow QE, \quad \vartheta_s : D_s \longrightarrow PE_\gamma,$$

$$\phi_u : D_u := \{\xi \in QE \mid |\xi|_E < r_0^u\} \longrightarrow PE_\gamma$$

such that $\phi_s(0) = \vartheta_s(0) = \phi_u(0) = 0$, $\phi'_s(0) = \vartheta'_s(0) = \phi'_u(0) = 0$ and

$$\mathcal{M}_s := \{w_0 = w_* + \xi + \vartheta_s(\xi) + \phi_s(\xi) \mid \xi \in D_s, \langle w_0 - w_* \rangle_\gamma < r_s\} \quad (5.2)$$

$$= \{w_0 \in \mathcal{M} \mid \langle w_0 - w_* \rangle_\gamma < r_s, \exists \text{ solution } w = (u, \rho) \text{ of (1.1) on } \mathbb{R}_+ \text{ with} \\ \langle w(t) - w_* \rangle_\gamma = |w(t) - w_*|_{E_\gamma} + |\dot{\rho}(t)|_{Z_\gamma^1} \leq r'_s \ (\forall t \geq 0)\}, \quad (5.3)$$

$$\mathcal{M}_u := \{w_0 = w_* + \xi + \phi_u(\xi) \mid \xi \in D_u, \langle w_0 - w_* \rangle_\gamma < r_u\} \quad (5.4)$$

$$= \{w_0 \in \mathcal{M} \mid \langle w_0 - w_* \rangle_\gamma < r_u, \exists \text{ solution } w = (u, \rho) \text{ of (1.1) on } \mathbb{R}_- \text{ with} \\ \langle w(t) - w_* \rangle_\gamma = |w(t) - w_*|_{E_\gamma} + |\dot{\rho}(t)|_{Z_\gamma^1} \leq r'_u \ (\forall t \leq 0)\} \quad (5.5)$$

The above solutions w are given by $w = w_* + \Phi_s(P\mathcal{Q}(w_0 - w_*))$ if $w_0 \in \mathcal{M}_s$ and $w = w_* + \Phi_u(Q(w_0 - w_*))$ if $w_0 \in \mathcal{M}_u$, where $\Phi_s \in C_b^1(D_s; \mathbb{E}_1(\mathbb{R}_+, \delta))$ and $\Phi_u \in C_b^1(D_u; \mathbb{E}_1(\mathbb{R}_-, -\delta))$ with $\Phi_s(0) = 0$ and $\Phi_u(0) = 0$. It further holds

$$|w(t) - w_*|_{E_1} + |\dot{\rho}(t)|_{Z_\gamma} \leq ce^{-\delta t} \langle w_0 - w_* \rangle_\gamma \quad (\forall t \geq 1) \quad \text{if } w_0 \in \mathcal{M}_s,$$

$$|w(t) - w_*|_{E_1} + |\dot{\rho}(t)|_{Z_\gamma} \leq ce^{\delta t} |w_0 - w_*|_E \quad (\forall t \leq 0) \quad \text{if } w_0 \in \mathcal{M}_u.$$

(b) *If $w_0 \in \mathcal{M}_s$ and the forward (resp., a backward) solution w of (1.1) satisfies $\langle w - w_* \rangle_\gamma < r_s$ on $[0, t]$ for some $t > 0$ (resp., on $[t, 0]$ for some $t < 0$), then $w(t) \in \mathcal{M}_s$. If $w_0 \in \mathcal{M}_u$ and the forward solution w of (1.1) fulfills $\langle w - w_* \rangle_\gamma < r_u$ on $[0, t]$ for some $t > 0$, then $w(t) \in \mathcal{M}_u$. If $w_0 \in \mathcal{M}_u$ and the solution w from (5.5) fulfills $\langle w - w_* \rangle_\gamma < r_u$ on $[t, 0]$ for some $t < 0$, then $w(t) \in \mathcal{M}_u$.*

(c) *We have $\mathcal{M}_s \cap \mathcal{M}_u = \{w_*\}$.*

(d) *The dimension of \mathcal{M}_u is equal to the dimension of QE . If $\sigma(-\Lambda_0) \cap \mathbb{C}_+ \neq \emptyset$, then w_* is (Lyapunov) unstable in $E_\gamma \times Z_\gamma^1$ for (1.1).*

(e) *If (RR) holds, then there is a $\hat{r}_0^u \in (0, r_0^u)$ such that the map ϕ_u is Lipschitz from $\hat{D}_u := \{\xi \in QE \mid |\xi|_E < \hat{r}_0^u\}$ to PE_1 , and the operators $\phi'_u(\xi)$ are uniformly bounded in $\mathcal{L}(QE, PE_1)$ for $\xi \in \hat{D}_u$.*

Proof. (1) *Construction of the stable manifold \mathcal{M}_s .* Let $\delta \in [0, \delta_0)$. Observe that $e_\delta T(\cdot)$ has an exponential dichotomy by the assumptions. Let $\xi \in PE_\gamma^0 \subset E_\gamma^0$ (see (4.6)) and $\varphi = (v, \sigma) \in \mathbb{W}_1^*(\mathbb{R}_+, \delta)$. Using Proposition 4.9, (3.4) and Corollary 2.7, we see that $(\xi + \mathcal{N}_\gamma G(\varphi(0)), \tilde{F}(\varphi), G(\varphi)) \in \mathbb{D}(\mathbb{R}_+, \delta)$, cf. the proof of Theorem 3.3(b). Here, we set $\tilde{F} = F - A_{*,\rho} G_0$ as before. It follows that $(\xi + P\mathcal{N}_\gamma G(\varphi(0)), \tilde{F}(\varphi), G(\varphi)) \in \mathbb{D}(\mathbb{R}_+, \delta)$ by (4.7). To apply Proposition 4.5, we put $\zeta = \xi + P\mathcal{N}_\gamma G(\varphi(0))$ and $w_0 := \zeta + \phi_0^+ = Pw_0 + Qw_0$, where ϕ_0^+ is given by (4.9) with $\tilde{f} = \tilde{F}(\varphi)$ and $g = G(\varphi)$. We then define

$$\begin{aligned}\mathcal{T}_s : PE_\gamma^0 \times \mathbb{W}_1^*(\mathbb{R}_+, \delta) &\rightarrow \mathbb{E}_1(\mathbb{R}_+, \delta); \\ \mathcal{T}_s(\xi, \varphi) &= \varphi - L_{\Lambda_0, P}^+(\xi + P\mathcal{N}_\gamma G(\varphi(0)), F(\varphi), G(\varphi)).\end{aligned}$$

The above mentioned results imply that $\mathcal{T}_s \in C^1(PE_\gamma^0 \times \mathbb{W}_1^*(\mathbb{R}_+, \delta); \mathbb{E}_1(\mathbb{R}_+, \delta))$, $\mathcal{T}_s(0, 0) = 0$ and $\partial_2 \mathcal{T}_s(0, 0) = I$. The implicit function theorem thus gives a radius $r_0^s > 0$ and a map $\Phi_s \in C_b^1(B(r_0^s); \mathbb{W}_1^*(\mathbb{R}_+, \delta))$ such that $\Phi_s(0) = 0$ and $\mathcal{T}_s(\xi, \Phi_s(\xi)) = 0$ for $\xi \in B(r_0^s) := B_{PE_\gamma^0}(0, r_0^s)$. Due to Proposition 4.5, the function $\varphi = (v, \sigma) = \Phi_s(\xi)$ solves (3.7) with the initial value

$$\begin{aligned}\varphi(0) &= \xi + P\mathcal{N}_\gamma \gamma_0 G(\Phi_s(\xi)) \\ &\quad - \int_0^\infty T_Q(-\tau) Q[(\tilde{F}(\Phi_s(\xi)(\tau)), G_0(\Phi_s(\xi)(\tau))) + \Pi \tilde{G}(\Phi_s(\xi)(\tau))] d\tau \\ &=: \xi + \vartheta_s(\xi) + \phi_s(\xi).\end{aligned}$$

Combining the above results with Corollary 2.7, the embedding (2.7), $Q \in \mathcal{L}(E_{-1}, E_1^0)$, Propositions 4.4 and 4.9, we conclude that $\phi_s \in C_b^1(B(r_0^s); QE)$, $\vartheta_s \in C_b^1(B(r_0^s); PE_\gamma)$, $\phi_s(0) = \vartheta_s(0) = 0$, and $\phi'_s(0) = \vartheta'_s(0) = 0$.

We now define \mathcal{M}_s as in (5.2), where we choose a sufficiently small $r_s > 0$ below. Our construction yields that $\mathcal{M}_s \subset \mathcal{M}$. Take $w_0 = \varphi_0 + w_* \in \mathcal{M}_s$, where $w_0 = (u_0, \rho_0)$ and $\varphi_0 = (v_0, \sigma_0)$. It follows that $w_0 = w_* + \xi + \vartheta_s(\xi) + \phi_s(\xi)$ for some $\xi \in B(r_0^s)$, and that $\varphi = (v, \sigma) = \Phi_s(\xi) \in \mathbb{W}_1^*(\mathbb{R}_+, \delta)$ solves (3.7). We set $w = (u, \rho) = w_* + \varphi$. It further holds

$$\|\varphi\|_{\mathbb{E}_1(\mathbb{R}_+, \delta)} = \|\Phi_s(\xi) - \Phi_s(0)\|_{\mathbb{E}_1(\mathbb{R}_+, \delta)} \leq c |\xi|_{E_\gamma^0}. \quad (5.6)$$

To control ξ by φ_0 , we compute

$$\begin{aligned}\xi &= P(w_0 - w_* - \mathcal{N}_\gamma \gamma_0 G(\Phi_s(\xi))) = P(w_0 - w_* - \mathcal{N}_\gamma \gamma_0 G(w - w_*)), \\ |\xi|_{E_\gamma^0} &\leq c \left(|\varphi_0 - \mathcal{N}_\gamma G(\varphi_0)|_{E_\gamma} + |(B_0, C_0)[w_0 - w_* - \mathcal{N}_\gamma G(w_0 - w_*)]|_{Z_\gamma^1} \right) \\ &\leq c \left(|w_0 - w_*|_{E_\gamma} + |(B_0, C_0)(w_0 - w_*) - G_0(w_0 - w_*)|_{Z_\gamma^1} \right. \\ &\quad \left. + |G_0(w_0 - w_*) - (B_0, C_0)\mathcal{N}_\gamma G(w_0 - w_*)|_{Z_\gamma^1} \right) \\ &\leq c (\langle w_0 - w_* \rangle_\gamma + |G(w_0 - w_*)|_{Y_\gamma}) \leq \bar{c} \langle w_0 - w_* \rangle_\gamma,\end{aligned}$$

where we used (4.6), (3.4) and Corollary 2.7. We now choose $r_1 > 0$ such that $\bar{c}r_1 < r_0^s$. Hence, $|\xi|_{E_\gamma^0} < r_0^s$ if $\langle w_0 - w_* \rangle_\gamma \leq r_s$ and $r_s \in (0, r_1]$. In view of (5.1), we then obtain

$$e^{\delta t} |w(t) - w_*|_{E_\gamma} + e^{\delta t} |\dot{w}(t)|_{Z_\gamma^1} \leq c \|\varphi\|_{\mathbb{E}_1(\mathbb{R}_+, \delta)} \leq c \langle w_0 - w_* \rangle_\gamma \quad (5.7)$$

for all $t \geq 0$. Possibly after decreasing $r_1 > 0$, Proposition 3.4 finally implies the exponential estimate on \mathcal{M}_s in assertion (a).

(2) *Description of the stable manifold \mathcal{M}_s .* Take an initial value $w_0 \in \mathcal{M}$ with a solution $w = w(\cdot; w_0)$ of (1.1) on \mathbb{R}_+ and assume that

$$\langle w(t) - w_* \rangle_\gamma \leq r' \quad \text{for all } t \geq 0 \quad (5.8)$$

for some number $r' > 0$ with $\overline{B}_{E_\gamma}(w_*, r') \subset W_\gamma$. We want to find a sufficiently small $r'_1 > 0$ such that

$$\|w - w_*\|_{\mathbb{E}_1(\mathbb{R}_+)} \leq cr' \quad (5.9)$$

is true whenever (5.8) holds with $r' \in (0, r'_1]$. Again we put $\varphi = (v, \sigma) = w - w_*$.

To prove this claim, let w satisfy (5.8) for some $r' > 0$ which is small enough to allow the application of Theorem 3.3 with initial values $w(t) \in \mathcal{M}$. This theorem then shows that $\|\varphi\|_{\mathbb{E}_1([n, n+3])} \leq c \langle \varphi(n) \rangle_\gamma \leq cr'$ for all $n \in \mathbb{N}_0$. Let $J_n = [n, n+1]$, $J'_n = [n-1, n+2] \cap \mathbb{R}_+$ and $\delta \in (0, \delta_0/2]$. We then deduce

$$\|\varphi\|_{\mathbb{E}_1(\mathbb{R}_+, -\delta)}^p \simeq \sum_{n=0}^{\infty} e^{-\delta np} \|\varphi\|_{\mathbb{E}_1(J'_n)}^p \leq c(\delta) (r')^p,$$

using Lemma 4.7. Since also $e_{-\delta}T(\cdot)$ has an exponential dichotomy, Proposition 4.5, Lemma 4.7 and (3.1) imply that

$$\begin{aligned} \|\varphi\|_{\mathbb{E}_1(\mathbb{R}_+, -\delta)}^p &\leq c(\langle \varphi(0) \rangle_\gamma^p + \|F(\varphi)\|_{\mathbb{E}(\mathbb{R}_+, -\delta)}^p + \|G(\varphi)\|_{\mathbb{F}(\mathbb{R}_+, -\delta)}^p) \\ &\leq c \langle \varphi(0) \rangle_\gamma^p + c \sum_{n=0}^{\infty} e^{-\delta np} (\|F(\varphi)\|_{\mathbb{E}(J_n)}^p + \|G(\varphi)\|_{\mathbb{F}(J'_n)}^p) \\ &\leq c \langle \varphi(0) \rangle_\gamma^p + \varepsilon(r')^p \sum_{n=0}^{\infty} e^{-\delta np} \|\varphi\|_{\mathbb{E}_1(J'_n)}^p \\ &\leq c \langle \varphi(0) \rangle_\gamma^p + \varepsilon(r')^p \|\varphi\|_{\mathbb{E}_1(\mathbb{R}_+, -\delta)}^p. \end{aligned}$$

Fixing a small a sufficiently small $r'_1 > 0$ and choosing $0 < r' \leq r'_1$, we conclude

$$\|\varphi\|_{\mathbb{E}_1(\mathbb{R}_+, -\delta)} \leq c \langle \varphi(0) \rangle_\gamma \leq cr' \quad (5.10)$$

for constants independent of δ . Observe that $\partial_t(e_{-\delta}\sigma)$ tends to $\partial_t\sigma$ pointwise as $\delta \rightarrow 0$ and that the integrands in the Slobodeckii parts of $\|\varphi\|_{\mathbb{E}_1(\mathbb{R}_+, -\delta)}$ converge pointwise to those of $\|\varphi\|_{\mathbb{E}_1(\mathbb{R}_+)}$. Letting $\delta \rightarrow 0$, we thus deduce (5.9) for $0 < r' \leq r'_1$ from (5.10) and Fatou's lemma.

Let again $w = \varphi + w_*$ be the solution of (1.1) on \mathbb{R}_+ satisfying (5.8). Put $\varphi_0 = w_0 - w_*$. Note that $\varphi \in \mathbb{W}_1^*(\mathbb{R}_+)$ by (5.8) and (5.9). Proposition 4.5 then shows that $\varphi = L_{\Lambda_0, P}^+(P\varphi_0, F(\varphi), G(\varphi))$. The function $\xi := P(\varphi_0 - \mathcal{N}_\gamma G(\varphi_0))$ belongs to E_γ^0 and satisfies $|\xi|_{E_\gamma^0} \leq c \langle \varphi_0 \rangle_\gamma \leq \hat{c}r'$ due to Lemma 3.2 and (4.6). Choosing a sufficiently small $r' \in (0, r'_1]$ such that $\hat{c}r' < r_0^s$, we can now apply Step (1) of the proof with this ξ and obtain a solution $\bar{\varphi} = \Phi_s(\xi) \in \mathbb{W}_1^*(\mathbb{R}_+, \delta) \subset \mathbb{W}_1^*(\mathbb{R}_+)$ of (3.7), where $\|\bar{\varphi}\|_{\mathbb{E}_1(\mathbb{R}_+, \delta)} \leq cr'$ in view of (5.6). By construction, it holds $\bar{\varphi} = L_{\Lambda_0, P}^+(\xi + P\mathcal{N}_\gamma G(\bar{\varphi}(0)), F(\bar{\varphi}), G(\bar{\varphi}))$, and hence

$$\begin{aligned} \varphi - \bar{\varphi} &= L_{\Lambda_0, P}^+(P\varphi_0 - \xi - P\mathcal{N}_\gamma G(\bar{\varphi}(0)), F(\varphi) - F(\bar{\varphi}), G(\varphi) - G(\bar{\varphi})) \\ &= L_{\Lambda_0, P}^+(P\mathcal{N}_\gamma[G(\varphi(0)) - G(\bar{\varphi}(0))], F(\varphi) - F(\bar{\varphi}), G(\varphi) - G(\bar{\varphi})). \end{aligned}$$

We set $\zeta = P\mathcal{N}_\gamma(G(\varphi(0)) - G(\bar{\varphi}(0)))$. Corollary 2.7 and (4.7) yield $(\zeta, F(\varphi) - F(\bar{\varphi}), G(\varphi) - G(\bar{\varphi})) \in \mathbb{D}(\mathbb{R}_+)$. Since φ and $\bar{\varphi}$ have norms less than cr' in $\mathbb{E}_1(\mathbb{R}_+)$, Propositions 4.5 and 4.9, Corollary 2.7 and (2.7) imply that

$$\begin{aligned}\|\varphi - \bar{\varphi}\|_{\mathbb{E}_1(\mathbb{R}_+)} &\leq c(\langle \zeta \rangle_\gamma + \|F(\varphi) - F(\bar{\varphi})\|_{\mathbb{E}(\mathbb{R}_+)} + \|G(\varphi) - G(\bar{\varphi})\|_{\mathbb{F}(\mathbb{R}_+)}) \\ &\leq \varepsilon(r') \|\varphi - \bar{\varphi}\|_{\mathbb{E}_1(\mathbb{R}_+)}.\end{aligned}$$

Fixing a small $r'_s \in (0, r'_1]$ in (5.8), we deduce $\varphi = \bar{\varphi}$. On the other hand, due to (5.7), in the definition of \mathcal{M}_s we can now choose an $r_s \in (0, r_1]$ with $r_s \leq r'_s$ such that $|\tilde{w}(t) - w_*|_{E_\gamma} + |\partial_t \tilde{w}(t)|_{Z_\gamma^1} \leq r'_s$ for all $t \geq 0$ and any solution \tilde{w} with initial value $\tilde{w}_0 \in \mathcal{M}_s$. If we finally assume that $\langle \varphi_0 \rangle_\gamma < r_s$ we arrive at $\varphi_0 \in w_* + \mathcal{M}_s$ and the equality in (5.3).

(3) *Construction of the unstable manifold \mathcal{M}_u .* We proceed as in Step (1) and thus focus on the necessary changes. In view of Proposition 4.6, we define

$$\mathcal{T}_u : QE \times \mathbb{W}_1^*(\mathbb{R}_-, -\delta) \rightarrow \mathbb{E}_1(\mathbb{R}_-, -\delta); \quad \mathcal{T}_u(\xi, \varphi) = \varphi - L_{\Lambda_0, P}^-(\xi, F(\varphi), G(\varphi)).$$

As before we see that $\mathcal{T}_u \in C^1(QE \times \mathbb{W}_1^*(\mathbb{R}_-, -\delta); \mathbb{E}_1(\mathbb{R}_-, -\delta))$, $\mathcal{T}_u(0, 0) = 0$ and $\partial_2 \mathcal{T}_u(0, 0) = I$. The implicit function theorem thus gives a radius $r_0^u > 0$ and a map $\Phi_u \in C_b^1(B(r_0^u); \mathbb{W}_1^*(\mathbb{R}_-, -\delta))$ such that $\Phi_u(0) = 0$ and $\mathcal{T}_u(\xi, \Phi_u(\xi)) = 0$ for $\xi \in B(r_0^u) := B_{QE}(0, r_0)$. Due to Proposition 4.6, the function $\varphi = (v, \sigma) = \Phi_u(\xi)$ solves (3.7) on \mathbb{R}_- with the final value

$$\begin{aligned}\varphi(0) &= \xi + \int_{-\infty}^0 T(-\tau) P_{-1}[(\tilde{F}(\Phi_s(\xi)(\tau)), G_0(\Phi_s(\xi)(\tau))) + \Pi \tilde{G}(\Phi_s(\xi)(\tau))] d\tau \\ &= \xi + P\gamma_0 \Phi_u(\xi) =: \xi + \phi_u(\xi).\end{aligned}\tag{5.11}$$

Combining the above results with the embeddings (2.5) and (2.11) we conclude that $\phi_u \in C_b^1(B(r_0); PE_\gamma)$ and $\phi_u(0) = 0$. Propositions 4.4 and 4.9 further yield $\phi'_u(0) = 0$ if we differentiate the above integral representation in E_{-1} .

We now define \mathcal{M}_u as in (5.4), where we choose $r_u > 0$ below. Our construction yields that $\mathcal{M}_u \subset \mathcal{M}$. Take $w_0 = \varphi_0 + w_* \in \mathcal{M}_u$, where $w_0 = (u_0, \rho_0)$ and $\varphi_0 = (v_0, \sigma_0)$. It follows $w_0 = w_* + \xi + \phi_u(\xi)$ for some $\xi \in B(r_0^u)$, and that $(v, \sigma) = \Phi_s(\xi) \in \mathbb{W}_1^*(\mathbb{R}_-, -\delta)$ solves (3.7). Since $\xi = Q\varphi_0$, we have the estimate

$$\|\varphi\|_{\mathbb{E}_1(\mathbb{R}_-, -\delta)} = \|\Phi_u(\xi) - \Phi_u(0)\|_{\mathbb{E}_1(\mathbb{R}_-, -\delta)} \leq c |\xi|_E \leq c |\varphi_0|_E.\tag{5.12}$$

We then choose $r_1 > 0$ such that $|Q|_{\mathcal{L}(E)} r_1 < r_0^u$ and take any $r_u \in (0, r_1]$. In view of (5.1) we thus obtain

$$e^{-\delta t} |w(t) - w_*|_{E_\gamma} + e^{-\delta t} |\dot{w}(t)|_{Z_\gamma^1} \leq c \|\varphi\|_{\mathbb{E}_1(\mathbb{R}_-, -\delta)} \leq c |w_0 - w_*|_E\tag{5.13}$$

for all $t \leq 0$. Possibly after decreasing $r_1 > 0$, Proposition 3.4 implies the exponential estimate on \mathcal{M}_u in assertion (a).

(4) *Description of the unstable manifold \mathcal{M}_u .* Again we argue similarly as in Step (2). Take a final value $w_0 \in \mathcal{M}$ and a solution $w = \varphi + w_*$ of (1.1) on \mathbb{R}_- satisfying

$$\langle w(t) - w_* \rangle_\gamma \leq r' \quad \text{for all } t \leq 0\tag{5.14}$$

and some $r' > 0$ with $\overline{B}_{E_\gamma}(w_*, r') \subset W_\gamma$. Put $\varphi_0 = w_0 - w_*$. As in Step (2), we find a sufficiently small $r'_1 > 0$ such that

$$\|w - w_*\|_{\mathbb{E}_1(\mathbb{R}_-)} \leq cr'\tag{5.15}$$

is true whenever (5.14) holds with $r' \in (0, r'_1]$. Note that $\varphi \in \mathbb{W}_1^*(\mathbb{R}_-)$ by (5.14) and (5.15). Proposition 4.6 then shows that $\varphi = L_{\Lambda_0, P}^-(Q\varphi_0, F(\varphi), G(\varphi))$. Choosing sufficiently small $r' \in (0, r'_1]$ such that $|Q|_{\mathcal{L}(E)} r' < r_0^u$, we can now apply Step (3) of the proof with $\xi = Q\varphi_0$ and obtain a solution $\bar{\varphi} \in \mathbb{W}_1^*(\mathbb{R}_-, -\delta) \subset \mathbb{W}_1^*(\mathbb{R}_-)$ of (3.7), where $\|\bar{\varphi}\|_{\mathbb{E}_1(\mathbb{R}_-, -\delta)} \leq cr'$ in view of (5.12). By construction, it holds $\bar{\varphi} = L_{\Lambda_0, P}^-(Q\varphi_0, F(\bar{\varphi}), G(\bar{\varphi}))$, and hence

$$\varphi - \bar{\varphi} = L_{\Lambda_0, P}^-(0, F(\varphi) - F(\bar{\varphi}), G(\varphi) - G(\bar{\varphi})).$$

Since φ and $\bar{\varphi}$ have norms less than cr' in $\mathbb{E}_1(\mathbb{R}_-)$, Propositions 4.6 and 4.9 imply that

$$\begin{aligned} \|\varphi - \bar{\varphi}\|_{\mathbb{E}_1(\mathbb{R}_-)} &\leq c (\|\tilde{F}(\varphi) - \tilde{F}(\bar{\varphi})\|_{\mathbb{E}(\mathbb{R}_-)} + \|G(\varphi) - G(\bar{\varphi})\|_{\mathbb{F}(\mathbb{R}_-)}) \\ &\leq \varepsilon(r') \|\varphi - \bar{\varphi}\|_{\mathbb{E}_1(\mathbb{R}_-)}. \end{aligned}$$

Fixing a small $r'_u \in (0, r'_1]$ in (5.14), we deduce $\varphi = \bar{\varphi}$. On the other hand, due to (5.13), in the definition of \mathcal{M}_u we can now choose a radius $r_u \in (0, r_1]$ such that $|\tilde{w}(t) - w_*|_{E_\gamma} + |\partial_t \tilde{w}(t)|_{Z_\gamma^1} \leq r'_u$ for all $t \leq 0$ and any solution \tilde{w} with final value $\tilde{w}_0 \in \mathcal{M}_u$. If we finally assume that $\langle \varphi_0 \rangle_\gamma < r_u$, we arrive at $\varphi_0 \in w_* + \mathcal{M}_u$ and the equality in (5.5). We have now shown assertion (a).

(5) *Remaining properties.* The local forward invariance of \mathcal{M}_s and the local backward invariance of \mathcal{M}_u follow directly from (5.3) and (5.5), respectively, and the time invariance of (1.1). To show the local backward invariance of \mathcal{M}_s and the local forward invariance of \mathcal{M}_u , we need in addition that we can glue solutions as described before Lemma 3.2. Hence, (b) holds.

Let $w_0 \in \mathcal{M}_s \cap \mathcal{M}_u$. Assertion (a) shows that $\langle w(t) - w_* \rangle_\gamma \leq \hat{c} e^{\delta t} r_u$ for all $t \leq 0$ and some $\delta \in (0, \delta_0)$ and $\hat{c} > 0$. Since we fixed $r_u > 0$ only at the end of Step (4), we can decrease it further, obtaining $\hat{c} r_u < r_s$. The invariance thus implies $w(t) \in \mathcal{M}_s$. Assertion (a) now yields

$$\langle w(t) - w_* \rangle_\gamma \leq c e^{\delta t} \langle w(0) - w_* \rangle_\gamma \leq c e^{2\delta t} \langle w(t) - w_* \rangle_\gamma$$

for all $t \leq 0$, so that $w_0 = w_*$ and (c) is true.

The first part of assertion (d) is clear. If $\sigma(-\Lambda_0) \cap \mathbb{C}_+ \neq \emptyset$, then there is a $w_0 \in \mathcal{M}_u \setminus \{w_*\}$. The corresponding solution tends to 0 in the sense that $\langle w(t) - w_* \rangle_\gamma \rightarrow 0$ as $t \rightarrow -\infty$. Since $w_0 = w(-t; w(t))$, the instability follows.

Let (RR) hold. We decrease $r_u > 0$ once more so that Proposition 3.5 can be applied on \mathcal{M}_u . Take $\hat{r}_0^u > 0$ such that $\langle \xi + \phi_u(\xi) \rangle_\gamma \leq c |\xi|_E < r_u$ if $|\xi|_E < \hat{r}_0^u$, cf. (5.12) and (5.13). For $\xi, \bar{\xi} \in B_{QE}(0, \hat{r}_0^u)$ we have solutions $w = \Phi_u(\xi)$ and $\bar{w} = \Phi_u(\bar{\xi})$ of (1.1) on \mathbb{R}_- with $\phi_u(\xi) = P(w(0) - w_*)$ and $\phi_u(\bar{\xi}) = P(\bar{w}(0) - w_*)$. Proposition 3.5, (4.6), (5.1) and assertion (a) then imply

$$\begin{aligned} |\phi_u(\xi) - \phi_u(\bar{\xi})|_{E_1} &\leq c |w(0) - \bar{w}(0)|_{E_1} \leq c \langle w(-1) - \bar{w}(-1) \rangle_\gamma \\ &\leq c \|w - \bar{w}\|_{\mathbb{E}_1(\mathbb{R}_-, -\delta)} = c \|\Phi_u(\xi) - \Phi_u(\bar{\xi})\|_{\mathbb{E}_1(\mathbb{R}_-, -\delta)} \\ &\leq \tilde{c} |\xi - \bar{\xi}|_E; \end{aligned}$$

i.e., the first part of (e) has been verified. Let $\zeta \in QE$. We know that the limit

$$\phi'_u(\xi) \zeta = \lim_{h \rightarrow 0} \frac{1}{h} (\phi_u(\xi + h\zeta) - \phi_u(\xi)) =: \lim_{h \rightarrow 0} D(h)$$

exists in E_γ . The above estimate further yields $|D_h|_{E_1} \leq \tilde{c}|\xi|_E$ for all sufficiently small $|h|$. After passing to a subsequence, the vectors D_h converge weakly in E_1 to $\phi'_u(\xi)\zeta$ so that $|\phi'_u(\xi)\zeta|_{E_1} \leq \tilde{c}|\xi|_E$, as asserted. \square

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