A sufficient condition for exponential dichotomy of parabolic evolution equations

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1 INTRODUCTION

In this note we establish the exponential dichotomy of the abstract Cauchy problem

$$(CP) \quad \begin{cases} \frac{d}{dt}u(t) &= A(t)u(t), \quad t \geq s, \\ u(s) &= x, \end{cases}$$

assuming that $A(t)$, $t \in \mathbb{R}$, generates an analytic semigroup $(e^{\tau A(t)})_{\tau \geq 0}$ and $A(\cdot)^{-1}$ satisfies a certain Hölder condition (see (P) below). Moreover, each semigroup $(e^{\tau A(t)})_{\tau \geq 0}$ is supposed to have exponential dichotomy and the Hölder constant of $A(\cdot)^{-1}$ must be sufficiently small. Already simple matrix examples, [6, p.3], [15, Ex.3.4], show that one cannot omit this smallness condition.

Similar sufficient conditions for exponential dichotomy are known for matrices $A(t)$, [6], bounded operators $A(t)$, [4], and delay equations, [12]. In [15] we have shown the main Theorem 5 of the present paper for the special case of time independent domains $D(A(t)) \equiv D(A(0))$. This seems to be the first result of this kind for parabolic evolution equations; but the hypothesis of constant domains is rather restrictive because it excludes, for instance, Neumann type boundary conditions in the context of parabolic partial differential equations, see [17, §4]. The results in [4] and [15] are further improved by Proposition 6 which determines the rank of the splitting projections for (CP).

Our approach is based on a characterization of the exponential dichotomy of (CP) by means of the associated *evolution semigroup* defined in (10), cf. [5] or [7, §VI.9]. In order to prove our main result in the third section we use ideas from [4] and [15], but we supplement them by the theory of inter- and extrapolation spaces as studied in, e.g., [3], [7], [13]. The necessary tools are introduced in the next section.

2 PREPARATIONS

An operator A on a Banach space X is called *sectorial of type* (ϕ, K, w) if its domain $D(A)$ is dense, $\Sigma \subseteq \rho(A)$, and $\|R(\lambda, A(t))\| \leq \frac{K}{1+|\lambda - w|}$ for $\lambda \in \Sigma := \{w\} \cup \{\lambda : |\arg(\lambda - w)| < \phi\}$ and constants $\phi \in (\frac{\pi}{2}, \pi]$ and $K, w \geq 0$. Recall that a sectorial operator generates an analytic semigroup. Given an invertible sectorial operator A and $0 < \alpha < 1$, we define the interpolation space

$$X_\alpha^A := \overline{X}^{\|\cdot\|_\alpha^A} \quad \text{with norm} \quad \|x\|_\alpha^A := \sup_{r > w} \|r^\alpha A R(r, A)x\|.$$

By X_1^A we denote $D(A)$ endowed with the norm $\|Ax\|$, and $X_0^A := X$. We further introduce the completion X_{-1}^A of X with respect to the norm $\|x\|_{-1}^A := \|A^{-1}x\|$. Then A has a unique continuous extension $A_{-1} : X \to X_{-1}^A$ which is sectorial of type (ϕ, K, w). So we can define $X_{\alpha-1}^A := (X_{-1}^A)_\alpha^{A_{-1}}$, $0 < \alpha < 1$, equipped with the norm

$$\|x\|_{\alpha-1}^A := \sup_{r>w} \|r^\alpha R(r, A_{-1})x\|.$$

The continuous and dense embeddings

$$X_1 \hookrightarrow X_\beta^A \hookrightarrow D((w-A)^\alpha) \hookrightarrow X_\alpha^A \hookrightarrow X \hookrightarrow X_{\alpha-1}^A \hookrightarrow X_{-1}^A$$

hold for $0 \le \alpha < \beta \le 1$, where the domain of the fractional power is endowed with the graph norm and the norms of the embeddings depend only on the type of A and $\|A^{-1}\|$, α, β. Moreover, the operator A can be extended to an isomorphism $A_{\alpha-1} : X_\alpha^A \to X_{\alpha-1}^A$ for $0 \le \alpha \le 1$ which coincides with the part of A_{-1} in $X_{\alpha-1}^A$. The semigroup e^{tA} restricts to an analytic semigroup on X_α^A and extends to an analytic semigroup on $X_{\alpha-1}^A$ generated by $A_{\alpha-1}$, $0 \le \alpha \le 1$, and it satisfies

$$\|A^k e^{tA_{\alpha-1}}x\|_X \le C\, t^{\alpha-k-1}\, \|x\|_{\alpha-1}^A \tag{1}$$

for $k = 0, 1$, $0 \le \alpha \le 1$, $x \in X_{\alpha-1}^A$, $0 < t \le 1$, and a constant C depending only on the type of A and $\|A^{-1}\|$. These facts follow from, e.g., [3, §V.1, V.2.1], [7, §II.5], [13, §1, 2.2].

The next assumption was introduced by P. Acquistapace and B. Terreni, [2], in a slightly different form.

(P) $A(t)$, $t \in \mathbb{R}$, is a sectorial operator of type (ϕ, K, w), with a uniformly bounded inverse, and $\|A(t)^{-1} - A(s)^{-1}\|_{\mathcal{L}(X, X_\nu^t)} \le L\,|t-s|^\mu$ holds for $t, s \in \mathbb{R}$ and constants $L \ge 0$ and $\mu, \nu \in (0, 1]$ with $\mu + \nu > 1$.

Here we set $X_\alpha^t := X_\alpha^{A(t)}$ and $\|\cdot\|_\alpha^t := \|\cdot\|_\alpha^{A(t)}$. Assumption (P) implies that there is a unique *evolution family* $(U(t,s))_{t\ge s}$ in the space $\mathcal{L}(X)$ of bounded linear operators, i.e.,

$$U(t,s) = U(t,r)U(r,s),\ U(s,s) = I, \quad (t,s) \mapsto U(t,s) \text{ is strongly continuous}$$

for $t \ge r \ge s$, $t, r, s \in \mathbb{R}$, such that $U(\cdot, s)x \in C^1([s,\infty), X)$ solves (CP) if $x \in D(A(s))$, see [1, Thm.2.3, Prop.3.2], and also [2], [3], [13], [16], [17]. Moreover, $U(t,s)X \subset D(A(t))$ and

$$\|(w-A(t))^\alpha\, U(t,s)\| \le C'\,(t-s)^{-\alpha}, \tag{2}$$

$$\|A(t)U(t,s)(w-A(s))^{-\alpha}\| \le C'\,(t-s)^{\alpha-1}, \tag{3}$$

$$\|(w-A(t))^\alpha\, U(t,s)(w-A(s))^{-\alpha}\| \le C', \tag{4}$$

$$\|U(t,s)(w-A(s))^\alpha x\| \le \frac{C'}{\mu-\alpha}\,(t-s)^{-\alpha}\,\|x\|, \tag{5}$$

for $0 \le \alpha \le 1$ in (2)–(4), $0 \le \alpha < \mu$ and $x \in D((w-A(s))^\alpha)$ in (5), $0 < t-s \le 1$, and a constant $C' \ge 0$ only depending on the constants in (P), see [17, Thm.2.1, (2.13)]. Thus $U(\cdot,\cdot)$ is *exponentially bounded*, i.e., there are constants $M \geq 1$ and $\gamma \in \mathbb{R}$ such that

$$\|U(t,s)\| \leq M e^{\gamma(t-s)} \quad \text{for} \quad t \geq s. \tag{6}$$

By [1, Thm.2.3] and [17, Thm.2.3] the mappings

$$\{(\tau,\sigma) \in \mathbb{R}^2 : \tau > \sigma\} \ni (t,s) \mapsto U(t,s) \in \mathcal{L}(X) \tag{7}$$

$$\{(\tau,\sigma) \in \mathbb{R}^2 : \tau \geq \sigma\} \ni (t,s) \mapsto (w - A(t))^{\alpha}\, U(t,s)(w - A(s))^{-\alpha} x \in X, \tag{8}$$

$x \in X$, are continuous for $0 \leq \alpha \leq 1$. Using the estimates (5) and $\|(w - A(s))^{\theta-1} x\| \leq c_{\theta,\beta}\, \|x\|^s_{\beta-1}$ for $0 \leq \theta < \beta \leq 1$, we can uniquely extend $U(t,s)$, $t > s$, to a bounded operator $\tilde{U}(t,s) : X^s_{\beta-1} \to X$ for $1 > \beta > 1-\mu$. Finally we note that (P) implies

$$\|(w - A(t))^{\alpha-1} - (w - A(s))^{\alpha-1}\| \leq L_\alpha\, |t-s|^{\mu} \tag{9}$$

for $t, s \in \mathbb{R}$, $0 \leq \alpha < \nu$, and a constant L_α depending only on α and the constants in (P).

Let $E := C_0(\mathbb{R}, X)$ be endowed with the sup–norm. For a given evolution family $U(\cdot,\cdot)$ on X satisfying (6), we define the *evolution semigroup* $T(\cdot)$ on E by

$$(T(t)f)\,(s) := U(s, s-t) f(s-t), \qquad s \in \mathbb{R},\, t \geq 0,\, f \in E, \tag{10}$$

compare [5], [7, §VI.9], [10], [11], [14], [15], and the references therein. Clearly, $T(\cdot)$ is a strongly continuous semigroup on E. Its generator is denoted by G. It is easy to see that the resolvent of G is given by

$$(R(\lambda, G)f)\,(t) = \int_{-\infty}^{t} e^{-\lambda(t-s)} U(t,s) f(s)\, ds, \quad t \in \mathbb{R}, \tag{11}$$

for $\operatorname{Re}\lambda > \gamma$. To obtain a representation of G, we introduce the space $C^1_0(\mathbb{R}, X) := \{f \in C^1(\mathbb{R}, X) : f, f' \in E\}$ and the multiplication operator $A(\cdot)$ on E with domain $D(A(\cdot)) := \{f \in E : f(s) \in D(A(s)) : A(\cdot)f(\cdot) \in E\}$. Notice that $A(\cdot)$ is sectorial of type (ϕ, K, w) with resolvent $R(\lambda, A(\cdot))$ for $\lambda \in \Sigma \cup \{0\}$.

LEMMA 1. Let (P) hold. Then G extends the operator $-\frac{d}{dt} + A(\cdot)$ defined on $C^1_0(\mathbb{R}, X) \cap D(A(\cdot))$. Moreover, $D(A(\cdot)) \cap D(G) \subset C^1_0(\mathbb{R}, X)$.

Proof: For $f \in C^1_0(\mathbb{R}, X) \cap D(A(\cdot))$, $t > 0$, and $s \in \mathbb{R}$, we compute

$$\begin{aligned}
&\tfrac{1}{t}\,(T(t)f(s) - f(s)) + f'(s) - A(s)f(s) \\
&\quad = \frac{1}{t}\,(U(s,s-t)f(s-t) - f(s)) + \frac{1}{t}\int_{s-t}^{s} U(s,\tau) f'(\tau)\, d\tau - A(s)f(s) \\
&\qquad - \frac{1}{t}\int_{s-t}^{s} (U(s,\tau)f'(\tau) - f'(s))\, d\tau \\
&\quad = \frac{1}{t}\int_{s-t}^{s} \Big(U(s,\tau)A(\tau)f(\tau) - A(s)f(s) - U(s,\tau)f'(\tau) + f'(s)\Big) d\tau,
\end{aligned}$$

where we have integrated by parts and used $\frac{\partial^+}{\partial s}\, U(t,s)x = -U(t,s)A(s)x$ for $t > s$ and $x \in A(s)$, [1, Thm.2.3]. This shows the first assertion. The second one can be established exactly as [15, Lem.3.6] using (4) and (8). $\square$

We further need the inter- and extrapolation spaces $E_\alpha := E_\alpha^{A(\cdot)}$ and $E_{\alpha-1} := E_{\alpha-1}^{A(\cdot)}$, $0 \le \alpha \le 1$, of the operator $A(\cdot)$ on E. By [8, Thm.4.7], the extrapolated operator $A(\cdot)_{-1} : E \to E_{-1}$ is given by the multiplication operator $A_{-1}(\cdot)$ and $f \in E_{-1}$ can be identified with an element of $\prod_{t\in\mathbb{R}} X_{-1}^t$. Note that $E_{\alpha-1}$ is the closure of E_1 in E_{-1} with respect to the norm

$$\|f\|_{\alpha-1} := \sup_{r>w} \sup_{s\in\mathbb{R}} \|r^\alpha R(r, A(s)) f(s)\|,$$

$A_{\alpha-1}(\cdot) : E_\alpha \to E_{\alpha-1}$ is an isomorphism for $0 \le \alpha \le 1$, and $E_\beta \hookrightarrow D((w - A(\cdot))^\alpha) \hookrightarrow E_\alpha$ for $0 \le \alpha < \beta \le 1$. Due to (5) and (11) we can extend $R(\lambda, G)$ to a bounded operator $\tilde{R}_\lambda : E_{\alpha-1} \to E$, $1 - \mu < \alpha < 1$, $\lambda > \gamma$, given by

$$(\tilde{R}_\lambda f)(t) = \int_{-\infty}^t e^{-\lambda(t-s)} \tilde{U}(t,s) f(s)\, ds, \quad t \in \mathbb{R},\ f \in E_{\alpha-1}.$$

The following lemma will be crucial for the proof of our main result.

LEMMA 2. Let (P) hold, $\lambda > \gamma$, and $f \in C_0^1(\mathbb{R}, X) \cap D((w - A(\cdot))^\alpha)$ for $\alpha \in (1-\mu, \nu)$. Then we have $R(\lambda, G) f' = f - \lambda R(\lambda, G) f + \tilde{R}_\lambda A_{\alpha-1}(\cdot) f$.

Proof: By rescaling we may assume that $\lambda = 0 > \gamma$ and $w = 0$. First, for $s \in \mathbb{R}$ and $0 < h \le 1$, we write

$$\begin{aligned} D_{h,f}(s) &:= (-A(s+h))^{\alpha-1} \tfrac{1}{h} [U(s+h,s) - I] f(s) + (-A(s))^\alpha f(s) \\ &= \frac{1}{h} \int_0^h \left[(-A(s+h))^{\alpha-1} - (-A(s+\tau))^{\alpha-1}\right] A(s+\tau) U(s+\tau,s) f(s)\, d\tau \\ &\quad - \frac{1}{h} \int_0^h \left[(-A(s+\tau))^\alpha U(s+\tau,s)(-A(s))^{-\alpha} - I\right] (-A(s))^\alpha f(s)\, d\tau. \end{aligned}$$

Using (9), (3) for the first integral and (4), (8) for the second one, we derive

$$\lim_{h\to 0} \|D_{h,f}\|_E = 0. \tag{12}$$

Next, we compute

$$\begin{aligned} \int_{-\infty}^t U(t,s) f'(s)\, ds &= \lim_{h\to 0} \int_{-\infty}^{t-h} U(t,s+h) \frac{1}{h} (f(s+h) - f(s))\, ds \\ &= \lim_{h\to 0} \frac{1}{h} \int_{-\infty}^{t-h} (U(t,s+h) f(s+h) - U(t,s) f(s))\, ds \\ &\quad + \lim_{h\to 0} \frac{1}{h} \int_{-\infty}^{t-h} U(t,s+h)[U(s+h,s) - I] f(s)\, ds \\ &= f(t) + \lim_{h\to 0} \frac{1}{h} \int_{-\infty}^{t-h} U(t,s+h)[U(s+h,s) - I] f(s)\, ds. \end{aligned}$$

To determine the remaining limit, we note that $U(t,s)(-A(s))^{1-\alpha}$, $t>s$, has a unique bounded extension $V(t,s): X \to X$ satisfying

$$\|V(t,s)\|_{\mathcal{L}(X)} \le C' \, (\mu+\alpha-1)^{-1} \, (t-s)^{\alpha-1} \tag{13}$$

for $0<t-s\le 1$ by (5). Hence,

$$\begin{aligned}
&\frac{1}{h}\int_{-\infty}^{t-h} U(t,s+h)[U(s+h,s)-I]f(s)\,ds - \int_{-\infty}^{t} \tilde{U}(t,s)A_{\alpha-1}(s)f(s)\,ds \\
&\quad = \int_{-\infty}^{t-h} V(t,s+h)\left\{(-A(s+h))^{\alpha-1}\tfrac{1}{h}\,[U(s+h,s)-I]f(s)+(-A(s))^{\alpha}f(s)\right\}ds \\
&\qquad - \int_{-\infty}^{t-h} V(t,s+h)\,[(-A(s))^{\alpha}f(s)-(-A(s+h))^{\alpha}f(s+h)]\,ds.
\end{aligned}$$

As consequence of (13), (12), and $(-A(\cdot))^{\alpha}f(\cdot)\in E$, the right hand side tends to 0 in X, and the assertion follows. □

We say that an evolution family $U(\cdot,\cdot)$ has *exponential dichotomy* if there are projections $P(t)\in\mathcal{L}(X)$, $t\in\mathbb{R}$, and constants $N,\delta>0$ such that $P(\cdot)$ is strongly continuous and

(a) $P(t)U(t,s)=U(t,s)P(s)=: U_P(t,s)$ for $t\ge s$,

(b) the restriction $U_Q(t,s): Q(s)X\to Q(t)X$ is invertible for $t\ge s$ (and we set $U_Q(s,t):=U_Q(t,s)^{-1}$),

(c) $\|U_P(t,s)P(s)\|\le Ne^{-\delta(t-s)}$ and $\|U_Q(s,t)Q(t)\|\le Ne^{-\delta(t-s)}$ for $t\ge s$.

Here and below, we let $Q=I-P$ for a projection P. We remark that the dichotomy projections $P(t)$ are uniquely determined, see [15, Cor.3.3]. The *Green's function* of an exponentially dichotomic evolution family is given by

$$\Gamma(t,s):=\begin{cases} U_P(t,s)P(s), & t\ge s,\\ -U_Q(t,s)Q(s), & t<s.\end{cases}$$

Now we can formulate our second main assumption supposing that (P) is valid.

(ED) Each semigroup $(e^{\tau A(t)})_{\tau\ge 0}$, $t\in\mathbb{R}$, has exponential dichotomy with constants $N,\delta>0$ and projection P_t.

Observe that (ED) implies $\|A(t)^{-1}\|\le\frac{2N}{\delta}$. It is not difficult to show that (ED) holds if $[-\delta,\delta]+i\mathbb{R}\subset\rho(A(t))$ and $R(\lambda,A(t))$ is uniformly bounded for $\lambda\in\pm\delta+i\mathbb{R}$, cf. [7, §V.1.c], [13, §2.3]. Assuming (P) and (ED), we set

$$\Gamma_t(s):=\begin{cases} e^{sA_P(t)}P_t, & s\ge 0,\ t\in\mathbb{R},\\ -e^{sA_Q(t)}Q_t, & s<0,\ t\in\mathbb{R},\end{cases}$$

where $A_P(t)$ and $A_Q(t)$ denote the restrictions of $A(t)$ to P_tX and Q_tX, respectively. One sees as in [15, Lem.3.5] that the mappings

$$t\mapsto P_t\in\mathcal{L}(X)\quad\text{and}\quad (t,s)\mapsto\Gamma_t(s)\in\mathcal{L}(X)\quad\text{are continuous} \tag{14}$$

for $t \in \mathbb{R}$ and $s \in \mathbb{R} \setminus \{0\}$. Clearly, P_t and $\Gamma_t(s)$ leave X_α^t invariant and have unique extensions to $X_{\alpha-1}^t$, $0 \le \alpha \le 1$, $t \in \mathbb{R}$. The following estimate easily follows from (1) (where the second integral in the first line is understood as an upper integral).

LEMMA 3. Assume that (P) and (ED) hold. Then

$$\begin{aligned} q &:= \sup_{t\in\mathbb{R}} \max \left\{ \int_{\mathbb{R}} \|A(t)^2\Gamma_t(\tau)\|_{\mathcal{L}(X_\nu^t,X)} \, |\tau|^\mu \, d\tau, \ \int_{\mathbb{R}} \|A(s)\Gamma_s(t-s)\|_{\mathcal{L}(X_{\nu-1}^s,X)} \, |t-s|^\mu \, ds \right\} \\ &\le NC \left(\int_0^1 \tau^{\nu+\mu-2} \, d\tau + \int_0^\infty \left(|\tau+1|^\mu + e^{-\delta} \, |\tau|^\mu \right) e^{-\delta\tau} \, d\tau \right) < \infty. \end{aligned}$$

3 MAIN RESULTS

We start with an auxiliary result which is an interesting refinement of the following fact: An exponentially bounded evolution family has exponential dichotomy if and only if the generator G of the associated evolution semigroup $T(\cdot)$ on $E = C_0(\mathbb{R}, X)$ is bijective. In this case the unit circle $\mathbb{T}$ belongs to $\rho(T(1))$,

$$P(\cdot) = \frac{1}{2\pi i} \int_{\mathbb{T}} R(\lambda, T(1)) \, d\lambda, \quad \text{and} \quad G^{-1}f = -\int_{\mathbb{R}} \Gamma(\cdot, s) f(s) \, ds, \tag{15}$$

see e.g. [10], [11], [14], and also [5], [7, §VI.9] for further references.

LEMMA 4. Let $U(\cdot,\cdot)$ be an evolution family on a Banach space X with $\|U(t,s)\| \le Me^{\gamma(t-s)}$ and let G be the generator of the associated evolution semigroup on $E = C_0(\mathbb{R}, X)$. If G is bijective and $\|G^{-1}\| \le 1/\eta$, then $U(\cdot,\cdot)$ has exponential dichotomy and the dichotomy constants depend only on $\eta > 0$.

Proof: From $\sigma(G) = \sigma(G) + i\mathbb{R}$, see [10, Thm.3.1], follows that $(-\eta, \eta) + i\mathbb{R} \subset \rho(G)$. Using rescaling and the above cited results, we see that $U(\cdot,\cdot)$ has exponential dichotomy with exponent $\delta' \in (0, \eta)$.

Fix $\delta' \in (0, \eta)$. Suppose the dichotomy constant N' were not uniform in η. Then there would exist evolution families $U_n(\cdot,\cdot)$ on Banach spaces X_n such that $\|U_n(t,s)\| \le Me^{\gamma(t-s)}$ and $\|G_n^{-1}\| \le 1/\eta$ for the generator G_n of the associated evolution semigroup on $C_0(\mathbb{R}, X_n)$, but there are $x_n \in X_n$ and $t_n, s_n \in \mathbb{R}$ with $\|x_n\| = 1$ and

$$\lim_{n\to\infty} e^{\delta'|t_n - s_n|} \, \|\Gamma_n(t_n, s_n)x_n\| = \infty,$$

where $\Gamma_n(\cdot,\cdot)$ is the Green's function of $U_n(\cdot,\cdot)$. Set $f_n(s) := \varphi_n(s)e^{\pm\delta'(s-s_n)} \, U_n(s, s_n)x_n$ for $s \ge s_n$ and $f_n(s) := 0$ for $s < s_n$, where $\varphi_n \in C(\mathbb{R})$ has compact support in $(s_n, s_n + 2)$, $0 \le \varphi_n \le 1$, and $\int \varphi_n(s) \, ds = 1$, and we take $+\delta'$ if $t_n \ge s_n$ and $-\delta'$ if $t_n < s_n$. As a result,

$$R(\mp\delta', G_n)f_n(t_n) = \int_{\mathbb{R}} e^{\pm\delta'(t_n - s)}\Gamma_n(t_n, s)f_n(s) \, ds = e^{\delta'|t_n - s_n|} \, \Gamma_n(t_n, s_n)x_n.$$

This contradicts the uniform estimate of G_n^{-1}. $\square$

THEOREM 5. Assume that (P) and (ED) hold and let q be given by Lemma 3. If $qL < 1$, then the evolution family $(U(t,s))_{t\ge s}$ solving (CP) has exponential dichotomy with constants $N',\delta' > 0$ depending only on q and the constants in (P) and (ED).

Proof. (a) If $Gf = 0$ for some $f \in D(G)$, then $f(t) = (T(1)f)(t) = U(t,t-1)f(t-1)$ for $t \in \mathbb{R}$. Thus (2) and [1, Thm.2.3] yield $f \in D(A(\cdot))$ so that $f \in C_0^1(\mathbb{R}, X)$ by Lemma 1. We define on E a bounded operator L by

$$(Lg)(t) := \int_{-\infty}^{\infty} \Gamma_t(t-s)g(s)\,ds, \quad t \in \mathbb{R}.$$

Using Lemma 1 and integration by parts, we obtain

$$0 = LGf = -Lf' + LA(\cdot)f = LA(\cdot)f - A_{-1}(\cdot)Lf - f =: Vf - f,$$

where we have set

$$\begin{aligned}(Vf)(t) &:= \int_{\mathbb{R}} \Gamma_t(t-s)(A(s) - A_{-1}(t))f(s)\,ds \\ &= \int_{\mathbb{R}} A(t)\Gamma_t(t-s)(A(t)^{-1} - A(s)^{-1})A(s)f(s)\,ds.\end{aligned}$$

Here the integral in the first line is understood in the topology of X_{-1}^t. Then (P) and Lemma 3 imply $\|A(t)\,Vf(t)\| \le qL\,\|A(\cdot)f\|_E$ for each $t \in \mathbb{R}$. Since $Vf = f$ and $A(\cdot)$ is invertible, this estimate gives $f = 0$. Hence G is injective.

(b) Define on E a bounded operator R by

$$(Rf)(t) := \int_{-\infty}^{\infty} \Gamma_s(t-s)f(s)\,ds.$$

Notice that due to (1) R extends to an operator $\tilde{R} : E_{\alpha-1} \to E$ for $\alpha \in (0,1)$ with norm

$$\|\tilde{R}\|_{\mathcal{L}(E_{\alpha-1},E)} \le NC\left(\frac{1}{\alpha} + \frac{1+e^{-\delta}}{\delta}\right) =: \rho_\alpha. \tag{16}$$

For $f \in D(A(\cdot))$, one easily sees that $Rf \in C_0^1(\mathbb{R}, X)$ and $\frac{d}{dt}Rf = f + RA(\cdot)f$. We let

$$\begin{aligned}(Sf)(t) &:= \int_{\mathbb{R}} (A_{-1}(t) - A(s))\Gamma_s(t-s)f(s)\,ds \\ &= A_{-1}(t)\int_{\mathbb{R}} (A(s)^{-1} - A(t)^{-1})\Gamma_s(t-s)A(s)f(s)\,ds,\end{aligned}$$

where first integral is understood in the topology of X_{-1}^t. This leads to

$$\tilde{G}Rf := -\tfrac{d}{dt}\,Rf + A_{-1}(\cdot)Rf = (S-I)f. \tag{17}$$

We further need the Banach space $F_{\alpha-1} := \{f \in E_{-1} : \|f\|_{\alpha-1} < \infty\}$ endowed with the norm $\|\cdot\|_{\alpha-1}$ for $0 < \alpha < 1$. We note that E is in general not dense in $F_{\alpha-1}$ and that $F_{\nu-1} \hookrightarrow E_{\alpha-1}$ for $0 \le \alpha < \nu$ by [7, Prop.II.5.14]. Using (P) and Lemma 3, we can extend S to a bounded operator $\tilde{S} : F_{\nu-1} \to F_{\nu-1}$ with norm less than qL. Fix $\alpha \in (1-\mu, \nu)$. Since (P) holds with ν replaced by α, S is also continuous with respect to $\|\cdot\|_{\alpha-1}$.

For a given $g \in E$, we have $f := (\tilde{S} - I)^{-1} g \in F_{\nu-1} \subset E_{\alpha-1}$. There are $f_n \in D(A(\cdot))$ converging to f in $\|\cdot\|_{\alpha-1}$. From (17) we derive

$$\tilde{G} R f_n = (S-I) f_n \longrightarrow g \qquad \text{in } \|\cdot\|_{\alpha-1} \text{ as } n \to \infty.$$

Moreover, $A_{-1}(\cdot) R f_n \in E_{\alpha-1}$ so that $R f_n \in E_\alpha \subset D((w - A(\cdot))^\beta)$ for $1 - \mu < \beta < \alpha < \nu$. Then Lemma 2 yields

$$\tilde{R}_\lambda \tilde{G} R f_n = \lambda R(\lambda, G) R f_n - R f_n$$

from which we deduce

$$\tilde{R}_\lambda g = R(\lambda, G) g = \lambda R(\lambda, G) \tilde{R} f - \tilde{R} f.$$

Therefore $\tilde{R} f \in D(G)$ and $G \tilde{R} f = g$; i.e., G is also surjective.

Considering $(\tilde{S} - I)^{-1}$ as an operator from E to $F_{\nu-1}$, we obtain $G^{-1} = \tilde{R}(\tilde{S} - I)^{-1}$ and $\|G^{-1}\| \leq \frac{c\rho}{1-qL}$ where $\rho := \rho_\nu$ is given by (16) and c is the norm of the embedding of E in $F_{\nu-1}$. So Lemma 4 yields the assertion. □

We further prove that the dimensions of the stable and unstable subspaces of $U(t,s)$ and $e^{\tau A(s)}$ coincide, cf. [6], [12].

PROPOSITION 6. Under the assumptions of Theorem 5 we have $\dim P_s X = \dim P(t) X$ and $\dim \ker P_s = \dim \ker P(t)$ for $t, s \in \mathbb{R}$.

Proof: (a) By [9, p.298] the dimensions of the kernel and the range of two projections P and P' coincide whenever $\|P - P'\| < 1$. So (14) shows that the dimensions of $\ker P_t$ and $P_t X$ do not depend on t.

(b) We set $A_\varepsilon(t) := A(\varepsilon t)$ for $0 < \varepsilon \leq 1$ and $t \in \mathbb{R}$, and $A_0(t) = A(0)$ for $t \in \mathbb{R}$. Observe that $A_\varepsilon(\cdot)$ satisfies (P) and (ED) with the same constants and the same q. So the corresponding evolution families $U_\varepsilon(\cdot,\cdot)$ have a uniform exponential bound and exponential dichotomy with uniform constants $\delta', N' > 0$ and projections $P_\varepsilon(t)$, where $P_1(t) = P(t)$ and $P_0(t) = P_0$ for $t \in \mathbb{R}$. Due to (a) it remains to show that $\varepsilon \mapsto P_\varepsilon(t) \in \mathcal{L}(X)$ is continuous for each $t \in \mathbb{R}$. Clearly, $U_\varepsilon(t,s) = \hat{U}_\varepsilon(\varepsilon t, \varepsilon s)$, where $\hat{U}_\varepsilon(t,s)$ solves the Cauchy problem corresponding to the operators $\frac{1}{\varepsilon} A(t)$, $0 < \varepsilon \leq 1$. We note that $\frac{1}{\varepsilon} A(t)$ fulfills (P) with the same constants except for w replaced by w/ε. For $0 < \varepsilon, \eta \leq 1$ we compute

$$\begin{aligned} U_\varepsilon(s, s-1) - U_\eta(s, s-1) &= \hat{U}_\varepsilon(\varepsilon s, \varepsilon(s-1)) - \hat{U}_\varepsilon(\eta s, \eta(s-1)) \\ &\quad + \int_{\eta(s-1)}^{\eta s} \hat{U}_\varepsilon(\eta s, r)\left(\tfrac{1}{\varepsilon} - \tfrac{1}{\eta}\right) A(r) \hat{U}_\eta(r, \eta(s-1))\, dr \end{aligned}$$

(where the existence of the integral is verified in the next step). The first summand tends to 0 in $\mathcal{L}(X)$ uniformly for s in compact sets as $\eta \to \varepsilon$ by (7). The second summand is equal to

$$\frac{\varepsilon-\eta}{\varepsilon^{1-\alpha}\eta^{\alpha}}\int_{\eta(s-1)}^{\eta s}\hat{U}_{\varepsilon}(\eta s,r)\,\varepsilon^{-\alpha}(w-A(r))^{\alpha}\,\eta^{\alpha-1}(w-A(r))^{1-\alpha}\,\hat{U}_{\eta}(r,\eta(s-1))\,dr$$
$$+\,w\left(\frac{1}{\varepsilon}-\frac{1}{\eta}\right)\int_{\eta(s-1)}^{\eta s}\hat{U}_{\varepsilon}(\eta s,r)\hat{U}_{\eta}(r,\eta(s-1))\,dr,$$

where $0<\alpha<\mu$. Using (5) and (2), we see that this expression tends to 0 in operator norm uniformly in $s\in\mathbb{R}$ as $\eta\to\varepsilon$. In the case $\eta\to 0$, we write

$$U_{\eta}(s,s-1)-e^{A(0)}=\hat{U}_{\eta}(\eta s,\eta(s-1))-e^{\eta\frac{1}{\eta}A(\eta(s-1))}+e^{A(\eta(s-1))}-e^{A(0)}.$$

The right hand side converges to 0 in $\mathcal{L}(X)$ uniformly for s in compact subsets due to [1, (2.6), Lem.2.2] and (P).

(c) Let $T_{\varepsilon}(\cdot)$ be the evolution semigroup on $E=C_0(\mathbb{R},X)$ associated with $U_{\varepsilon}(\cdot,\cdot)$. Notice that

$$\begin{aligned}R(\lambda,T_{\varepsilon}(1))f&=\sum_{n=0}^{\infty}\lambda^{-(n+1)}T_{\varepsilon}(n)P_{\varepsilon}(\cdot)f-\sum_{n=1}^{\infty}\lambda^{n-1}(T_{\varepsilon}(n)Q_{\varepsilon}(\cdot))^{-1}f\\&=\sum_{n=0}^{\infty}\lambda^{-(n+1)}U_{\varepsilon,P}(\cdot,\cdot-n)f(\cdot-n)-\sum_{n=1}^{\infty}\lambda^{n-1}U_{\varepsilon,Q}(\cdot,\cdot+n)f(\cdot+n)\end{aligned}$$

for $f\in E$. In particular, $\|R(\lambda,T_{\varepsilon}(1))\|\le\frac{2N'}{\delta'}=:c$ for each $|\lambda|=1$, $0\le\varepsilon\le 1$. Given $t\in\mathbb{R}$ and $x\in X$, we set $f=\varphi(\cdot)x$ for a function $\varphi\in C(\mathbb{R})$ with support in $(t-1,t+1)$, $\varphi(t)=1$, and $0\le\varphi\le 1$. Then (15) and (6) yield

$$\begin{aligned}\|P_{\eta}(t)x-P_{\varepsilon}(t)x\|&\le\|P_{\eta}(\cdot)f-P_{\varepsilon}(\cdot)f\|_{\infty}\le c\sup_{|\lambda|=1}\|(T_{\eta}(1)-T_{\varepsilon}(1))R(\lambda,T_{\varepsilon}(1))f\|_{\infty}\\&\le c\,\max\{\sup_{|s|\le r,|\lambda|=1}\|U_{\varepsilon}(s,s-1)-U_{\eta}(s,s-1)\|\,\|R(\lambda,T_{\varepsilon}(1))f(s-1)\|,\\&\qquad\sup_{|s|\ge r,|\lambda|=1}2Me^{\gamma}\|R(\lambda,T_{\varepsilon}(1))f(s-1)\|\}\\&\le\max\{c^2\sup_{|s|\le r}\|U_{\varepsilon}(s,s-1)-U_{\eta}(s,s-1)\|,\ 4cMN'e^{\gamma}\sum_{n>N(r)}e^{-\delta' n}\}\|x\|\end{aligned}$$

for each $r>\max\{0,1-t,t+1\}$, where $N(r)$ is the maximum of the integer parts of $r-t-1$ and $r+t-1$. Now part (b) easily implies the assertion. □

REMARK 7. Theorem 5 and Proposition 6 can be generalized to operators $A(t)$ satisfying (P) and (ED) for $t\ge t_0$ by setting $A(t):=A(t_0)$ for $t<t_0$, cf. [15, §3].

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