SUFFICIENT CONDITIONS FOR EXPONENTIAL STABILITY AND DICHOTOMY OF EVOLUTION EQUATIONS

ROLAND SCHNAUBELT

Abstract. We present several sufficient conditions for exponential stability and dichotomy of solutions of the evolution equation $u'(t) = A(t)u(t)$ $(*)$ on a Banach space X. Our main theorem says that if the operators $A(t)$ generate analytic semigroups on X having exponential dichotomy with uniform constants and $A(\cdot)$ has a sufficiently small Hölder constant, then $(*)$ has exponential dichotomy. We further study robustness of exponential dichotomy under time dependent unbounded Miyadera-type perturbations. Our main tool is a characterization of exponential dichotomy of evolution families by means of the spectra of the so-called evolution semigroup on $C_0(\mathbb{R}, X)$ or $L^1(\mathbb{R}, X)$.

1. Introduction and preliminaries

Exponential dichotomy is one of the fundamental asymptotic properties of solutions of the linear Cauchy problem

$$(CP) \quad \begin{cases} \frac{d}{dt}u(t) &= A(t)u(t), \quad t > s, \\ u(s) &= x \end{cases}$$

in a Banach space X. It also plays an important role in the investigation of qualitative properties of nonlinear evolution equations such as linearized (in-)stability or the existence of center manifolds, see e.g. [8, 12, 19, 32] and the references therein.

In the autonomous case, i.e., $A(t) \equiv A$, exponential dichotomy is characterized by the spectral condition $\sigma(A) \cap i\mathbb{R} = \emptyset$ provided that, for instance, A generates an analytic or, more generally, an eventually norm continuous semigroup. In particular, in this case the semigroup is exponentially stable if and only if $s(A) := \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\} < 0$. For this and related facts and the spectral theory of $\mathcal{C}_0$–semigroups we refer to the monograph [20]. Unfortunately, in the non-autonomous situation such a criterion fails even for $X = \mathbb{C}^2$, [7, p.3]. In an appendix we present a rather drastic example illustrating this phenomenon for infinite dimensional X: There exist generators $A(t)$, $t \geq 0$, of uniformly bounded, commuting, positive $\mathcal{C}_0$–semigroups on an L^1–space such that $A(\cdot)$ is smooth in a certain sense, (CP) is well-posed, and $s(A(t)) = -\infty$ except for a sequence (t_k) where $A(t_k) = 0$. However, the corresponding evolution family $\mathcal{U}$ grows faster than any exponential. A related example for bounded operators $A(t)$ is also constructed.

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Moreover, it seems to be rather difficult to find assumptions on unbounded operators $A(t)$ which imply exponential dichotomy of (CP). In fact, to our knowledge the results in the literature are mostly restricted to exponential stability and to robustness of dichotomy, see the references below.

In our paper we present several sufficient conditions for exponential stability and dichotomy of (mostly parabolic) evolution equations. In Section 2 and 3 it is proved that exponential stability and dichotomy, respectively, of the semigroups $(e^{\tau A(t)})_{\tau\geq 0}$ with uniform constants imply the same properties for the solutions of (CP) provided that $A(\cdot)$ does not oscillate too much. Similar criteria for exponential stability can be found in, e.g., [7, 8] for bounded $A(t)$ and in [2, 10, 12, 35] for parabolic equations, see also [21] and the references therein. Our results in Section 2 are consequences of a general approximation theorem for evolution families, Theorem 2.1. For exponential dichtomy, our main Theorem 3.7 is similar to known results for bounded operators $A(t)$, [3, 7], and for delay equations, [18], but it seems to be the first theorem of this type for parabolic equations. We also refer to [19, §6.3] and the references therein concerning periodic parabolic equations, where the monodromy operator can be used. Further, in Section 4 we study robustness of exponential dichotomy. In view of the vast literature on this topic let us only mention [7, 8] for bounded $A(t)$, [5, 15, 16] for bounded perturbations, and [6, 12, 17] for certain unbounded perturbations. We (partially) extend these results to time dependent *Miyadera* perturbations. In Example 2.4 and 3.8, our results are applied to parabolic partial differential equations.

Our approach is based on the use of evolution semigroups which we introduce now. Throughout let $I \in \{\mathbb{R}, \mathbb{R}_+\}$. Further, $J \subseteq \mathbb{R}$ denotes a closed interval, and X is a Banach space. We set $D_J = D := \{(t,s) \in J^2 : t \geq s\}$. Let $J' := J \setminus \inf J$, where $\inf J$ is the left end point of J if J is bounded to the left and $\inf J = -\infty$ otherwise. A family $\mathcal{U} = (U(t,s))_{(t,s)\in D}$ in the space $\mathcal{L}(X)$ of bounded linear operators on X is called *evolution family* if

(E1) $U(s,s) = Id$, $U(t,s) = U(t,r)U(r,s)$ for $t \geq r \geq s$ in J and
(E2) the mapping $D \ni (t,s) \mapsto U(t,s)$ is strongly continuous.

Notice that a $\mathcal{C}_0$–semigroup $(e^{tA})_{t\geq 0}$ yields an evolution family $\mathcal{U}$ by setting $U(t,s) := e^{(t-s)A}$ for $(t,s) \in D_{\mathbb{R}}$. We say that an evolution family $\mathcal{U}$ on X *solves* the Cauchy problem (CP) on spaces Y_s, $s \in J$, (or that (CP) is *well–posed*) if Y_s is dense in X, $U(t,s)Y_s \subseteq Y_t \subseteq D(A(t))$ for $(t,s) \in D$, and $u = U(\cdot,s)x \in C^1(J \cap (s,\infty), X)$ is the unique solution of (CP) for $x \in Y_s$. We remark that this concept of well–posedness could be formulated in terms of solutions of (CP) only, cf. [23].

The *exponential growth bound* $\omega(\mathcal{U})$ of an evolution family $\mathcal{U}$ is given by

$$\omega(\mathcal{U}) := \inf\{w \in \mathbb{R} : \text{there is } M_w \geq 1 \text{ with } \|U(t,s)\| \leq M_w\, e^{w(t-s)} \text{ for } (t,s) \in D\}.$$

The evolution family $\mathcal{U}$ is called *exponentially bounded* if $\omega(\mathcal{U}) < \infty$, and $\mathcal{U}$ is *exponentially stable* if $\omega(\mathcal{U}) < 0$. Observe that $\omega(\mathcal{U}) = -\infty$ if J is compact. Exponential dichotomy (or hyperbolicity) is introduced in Definition 3.1.

Let $\mathcal{U}$ be an exponentially bounded evolution family on X with index set D_J. Consider the Bochner–Lebesgue spaces $L^p(J,X)$, $1 \leq p < \infty$, and the space $C_{00}(J,X)$ of continuous functions $f : J \to X$ vanishing at $\inf J$ and at infinity (if J is unbounded) endowed with the sup–norm $\|\cdot\|_\infty$. We write $C_0(\mathbb{R},X)$ instead of $C_{00}(\mathbb{R},X)$. On $E = L^p(J,X)$ or $C_{00}(J,X)$ we define by

$$(T(t)f)(s) := \chi_J(s-t)\,U(s,s-t)f(s-t) = \begin{cases} U(s,s-t)f(s-t), & s, s-t \in J, \\ 0, & s \in J,\ s-t \notin J, \end{cases}$$

a bounded operator $T(t)$ for $t \geq 0$, where χ_J is the characteristic function of J. It is easily seen that $\mathcal{T} = (T(t))_{t\geq 0}$ is a strongly continuous semigroup on E, see e.g. [31], [33]. We call $\mathcal{T}$ the *evolution semigroup* induced by $\mathcal{U}$ and denote its generator by $(G, D(G))$. Concerning the theory of evolution semigroups we refer to [14, 15, 16, 22, 23, 24, 28, 29, 30, 31, 33] and the references therein. For $U(t,s) \equiv Id$ the evolution semigroup coincides with the (right-) translation semigroup $\mathcal{T}_0 = (T_0(t))_{t\geq 0}$ with generator $G_0 f = -f'$. Notice that on $C_{00}(J,X)$ we have $D(G_0) = \{f \in C^1(J,X) : f, f' \in C_{00}(J,X)\}$. Finally, for operators $(A(t), D(A(t)))$, $t \in J$, we define the multiplication operator $\mathcal{A}f := A(\cdot)f(\cdot)$ with $D(\mathcal{A}) := \{f \in E : f(t) \in D(A(t)) \text{ (a.e.)}, A(\cdot)f(\cdot) \in E\}$ on $E = C_{00}(J,X), L^p(J,X)$. Using this notation, we state the following representation of G which can be shown as [15, Prop. 2.9] or [33, Prop. 1.13]. For simplicity, we set $U(t,s) := 0$ for $t < s$.

Lemma 1.1. *Let $\mathcal{U}$ be an exponentially bounded evolution family on X and $\mathcal{T}$ be the induced evolution semigroup on $E = C_{00}(J,X)$ or $L^p(J,X)$, $1 \leq p < \infty$. Denote by $\mathcal{D}$ the linear span of functions of the form $\varphi(\cdot)U(\cdot,s)x$, where $s \in J$, $\varphi \in C^1(J)$ with compact support contained in $(s,\infty) \cap J$, and $x \in Y_s$ for dense subspaces Y_s of X. Then $T(t)\mathcal{D} \subseteq \mathcal{D} \subseteq D(G)$ and $\mathcal{D}$ is dense in E. If $\mathcal{U}$ solves (CP) on Y_s, then $\mathcal{D} \subseteq D(G_0) \cap D(\mathcal{A})$ and G is the closure of $(G_0 + \mathcal{A}, \mathcal{D})$.*

2. Exponential stability

In the sequel, we prove an approximation theorem for evolution families which implies several sufficient conditions for exponential stability of the solutions of (CP). We assume the well–posedness of the Cauchy problem (CP) given by linear operators $A(t)$ on X which satisfy the following assumptions.

(A) $(A(t), D(A(t)))$ generates a $\mathcal{C}_0$–semigroup $(e^{\tau A(t)})_{\tau\geq 0}$ on a Banach space X with $\|e^{\tau A(t)}\| \leq Me^{\tau w(t)}$ for $t \in I$, $\tau \geq 0$, a constant $M \geq 1$, and $w \in C(I,\mathbb{R})$. The mapping $J \ni t \mapsto R(\lambda, A(t))$ is strongly continuous for $\lambda > \max_J w$, where $I \in \{\mathbb{R}, \mathbb{R}_+\}$ and $J \subseteq I$ is an arbitrary compact interval.

Notice that by considering $A_w(t) := A(t) - w(t)$ with $D(A_w(t)) := D(A(t))$ we may assume $w = 0$ in (A). Moreover, if $\mathcal{U}$ solves the Cauchy problem associated with $A(\cdot)$, then $U_w(t,s) := \exp(-\int_s^t w(\tau)d\tau)\,U(t,s)$ solves (CP) for $A_w(\cdot)$.

By (A) and the Trotter–Kato theorem, see e.g. [27, Thm. 3.4.2], the mapping $J \times \mathbb{R}_+ \ni (t,\tau) \mapsto e^{\tau A(t)}$ is strongly continuous. Using this fact, it is easy to prove that the operator $(\mathcal{A}, D(\mathcal{A}))$ on $E_J := C_{00}(J,X)$ generates the $\mathcal{C}_0$–semigroup $e^{t\mathcal{A}}f = e^{tA(\cdot)}f(\cdot)$, cf. [11, Thm. 3.4].

A special case of the next result was shown in [24, Prop. 3.2]. Let $\log 0 := -\infty$, $s_k := s + \frac{kt}{n}$, and $\prod_{k=1}^n e^{tA(s_k)} := e^{tA(s_n)} \cdots e^{tA(s_1)}$ for $s \in I$, $t \geq 0$, and $k = 1, \ldots, n$.

Theorem 2.1. *Suppose that the operators $A(t)$, $t \in I$, on X satisfy (A) and that the related Cauchy problem is solved by the evolution family $(U(t,s))_{(t,s)\in D}$. If there is a* *locally bounded function $M : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\left\| \prod_{k=1}^{n} e^{\frac{t}{n} A(s_k)} \right\| \le M(t) \tag{2.1}$$

for $n \in \mathbb{N}$, $t \ge 0$, $s \in I$, and $s_k = s + \frac{kt}{n}$, then

$$U(s+t,s)x = \lim_{n\to\infty} \prod_{k=1}^{n} e^{\frac{t}{n} A(s_k)}\, x, \qquad x \in X, \tag{2.2}$$

uniformly for t and s in compact subsets of $\mathbb{R}_+$ and I', respectively. Moreover, $\|U(s+t,s)\| \le M(t)$ and $\omega(\mathcal{U}) \le \inf_{t>0} \frac{1}{t} \log M(t)$.

Proof. Let $J \subseteq I$ be a compact interval of length t_0. We use the evolution semigroup $\mathcal{T}$ on E_J induced by $\mathcal{U}$ and the translation semigroup $\mathcal{T}_0$ on E_J with generators G and G_0, respectively. Observe that $T(t) = T_0(t) = 0$ for $t \ge t_0$. The generator $(G, D(G))$ is the closure of $(G_0 + \mathcal{A}, \mathcal{D})$ for a dense subspace $\mathcal{D}$ of E_J by Lemma 1.1. In particular, $G\mathcal{D}$ is dense in E_J. Further,

$$\left(e^{\frac{t}{n}\mathcal{A}} T_0\left(\tfrac{t}{n}\right)\right)^n f\,(s+t) = \begin{cases} e^{\frac{t}{n} A(s_n)} \dots e^{\frac{t}{n} A(s_1)} f(s), & \text{if } s, s+t \in J, \\ 0, & \text{if } s+t \in J,\ s \notin J, \end{cases} \tag{2.3}$$

for $f \in E_J$, $n \in \mathbb{N}$, and $t \ge 0$. By (2.1) we obtain $\|(e^{r\mathcal{A}} T_0(r))^n\| \le \sup_{0\le\tau\le t_0} M(\tau) < \infty$ for $r \ge 0$ and $n \in \mathbb{N}$. The Lie–Trotter product formula now yields

$$(e^{\frac{t}{n}\mathcal{A}} T_0(\tfrac{t}{n}))^n f \to T(t) f \tag{2.4}$$

for $f \in E_J$ and for t uniformly in compact intervals, see e.g. [27, Thm. 3.5.3, Cor. 3.5.5].

Fix $x \in X$ and a parameter set $K = \{(s+t, s) \in D_{I'} : 0 \le t \le t_1,\ a \le s \le b\}$. Choose a compact interval $J \subseteq I$ so that $[a,b] \subseteq J'$ and a function $f \in E_J$ being equal to x on $[a,b]$. From (2.3) follows

$$\begin{aligned} \left\| \prod_{k=1}^{n} e^{\frac{t}{n} A(s_k)}\, x - U(s+t,s)x \right\| &= \|(e^{\frac{t}{n}\mathcal{A}} T_0(\tfrac{t}{n}))^n f\,(s+t) - T(t)f\,(s+t)\| \\ &\le \|(e^{\frac{t}{n}\mathcal{A}} T_0(\tfrac{t}{n}))^n f - T(t)f\|_\infty \end{aligned}$$

for $(s+t,s) \in K$. Hence, (2.4) implies (2.2) uniformly on K.

To show the second assertion, first notice that due to (2.2) and (2.1) we have $\|U(s+t,s)\| \le M(t)$ for $s \in I$ and $t \ge 0$. Fix $t > 0$ and write $t' = nt + \tau$ for $t' > 0$, where $n \in \mathbb{N}$ and $0 \le \tau < t$. Set $c := \sup_{0\le\tau\le t} M(\tau)$. We estimate

$$\|U(s+t',s)\| \le M(t)^n\, \|U(s+t', s+nt)\| \le c\, e^{nt\,\frac{1}{t}\log M(t)} \le c\, e^{t|\log M(t)|}\, e^{t'\,\frac{1}{t}\log M(t)}$$

for all $s \in I$. As a result, $\omega(\mathcal{U}) \le \frac{1}{t} \log M(t)$ for all $t > 0$. ☐

Due to the principle of uniform boundedness, the products $\prod_{k=1}^{n} e^{\frac{t}{n} A(s_k)}$ are uniformly bounded on $K \subseteq D_I$ if (2.2) holds uniformly on K. Observe that Theorem 2.1 and its consequences stated below are also true for compact intervals I by the same proof.

Corollary 2.2. *Let $A(t)$ and $B(t)$, $t \in I$, be operators on a Banach lattice X satisfying (A) and $0 \le e^{\tau A(t)} \le e^{\tau B(t)}$ for $t \in I$ and $\tau \ge 0$. Assume that the corresponding Cauchy* *problems (CP) are solved by evolution families $\mathcal{U}_A$ and $\mathcal{U}_B$. Suppose that $B(\cdot)$ satisfies (2.1). Then $A(\cdot)$ satisfies (2.1) and $0 \le U_A(t,s) \le U_B(t,s)$ for $(t,s) \in D$.*

In [2, §II.6.4,IV.2.4] a different approximation method was used to show positivity of an evolution family.

Corollary 2.3. *Assume that the operators $A(t)$, $t \in I$, on a Banach space X satisfy (A) and that the corresponding Cauchy problem is solved by an evolution family $(U(t,s))_{(t,s)\in D}$. Suppose that there exist norms $\|\cdot\|_t$ on X satisfying*

$$\|x\| \le \|x\|_t \le \|x\|_s \le M\,\|x\| \quad \text{and} \quad \|e^{\tau A(t)}x\|_t \le e^{\tau\tilde{w}(t)}\,\|x\|_t \tag{2.5}$$

for $(t,s) \in D$, $\tau \ge 0$, $x \in X$, a constant $M \ge 1$, and $\tilde{w} \in C(I,\mathbb{R})$. Then, (2.2) holds for $(t,s) \in D_{I'}$ and $\|U(t,s)\| \le M\,\exp(\int_s^t \tilde{w}(\tau)d\tau)$ for $(t,s) \in D_I$.

Proof. As seen above, by considering $A(t) - \tilde{w}(t)$ we may assume $\tilde{w} = 0$. Then

$$\begin{aligned} \left\|\prod_{k=1}^{n} e^{\frac{t}{n}A(s_k)}x\right\| &\le \left\|\prod_{k=1}^{n} e^{\frac{t}{n}A(s_k)}x\right\|_{s_n} \le \left\|\prod_{k=1}^{n-1} e^{\frac{t}{n}A(s_k)}x\right\|_{s_n} \le \left\|\prod_{k=1}^{n-1} e^{\frac{t}{n}A(s_k)}x\right\|_{s_{n-1}} \\ &\le \cdots \le \|x\|_{s_1} \le M\,\|x\| \end{aligned}$$

for $x \in X$, $n \in \mathbb{N}$, $s \in I$, and $t \ge 0$. So the assertion follows from Theorem 2.1. $\square$

We remark that condition (2.5) with $\tilde{w}(\cdot) \equiv w_0$ is equivalent to Kato's stability condition, cf. [27, §5.2], which can be expressed by

$$\left\|\prod_{k=1}^{n} e^{\tau_k A(s_k)}\right\| \le M\,e^{w_0\sum_{k=1}^{n}\tau_k}$$

for $n \in \mathbb{N}$, $\tau_k \ge 0$, and $s_k \in I$ with $s_n \ge \cdots \ge s_1$, see [24, Prop 1.3] and the references therein. In the following example we check (2.5) for $\|\cdot\|_t = \|\cdot\|$. A similar result in sup-norm context can be found in [35, pp.14], where the maximum principle is used.

Example 2.4. We consider the Cauchy problem

$$\begin{aligned} \partial_t\,u(t,\xi) &= A(t,\xi,D)u(t,\xi), \qquad t > s \ge 0,\ \xi \in \Omega, \\ u(t,\xi) &= 0, \qquad t > s \ge 0,\ \xi \in \partial\Omega, \\ u(s,\xi) &= x(\xi), \qquad \xi \in \Omega, \end{aligned} \tag{2.6}$$

for a bounded region $\Omega \subseteq \mathbb{R}^n$ and the (formal) differential operator $A(t,\xi,D) = \sum_{k,l=1}^{n} \partial_k\,a_{kl}(t,\xi)\,\partial_l$, where $\partial_t = \frac{\partial}{\partial t}$, $\partial_k = \frac{\partial}{\partial \xi_k}$, and the coefficients a_{kl} are real-valued, symmetric, uniformly elliptic and belong to $C^\alpha(\mathbb{R}_+, L^\infty(\Omega))$ for $\frac{1}{2} < \alpha \le 1$. We define the sesquilinear form

$$a_t(x,y) = \int_\Omega \sum\nolimits_{k,l=1}^{n} a_{kl}(t,\xi)\,\partial_k x(\xi)\,\overline{\partial_l y(\xi)}\,d\xi$$

for $x,y \in W_0^{1,2}(\Omega)$. Denote by $A(t)$ the operator on $X = L^2(\Omega)$ induced by a_t, see e.g. [9, §6.1] or [34, §2.2]. Then $A(t)$ generates an analytic semigroup on X and satisfies (A), see [34, Thm. 3.6.1, p.145]. By [34, Thm. 5.4.2], there exists an evolution family $\mathcal{U}$ solving the (abstract) Cauchy problem (CP) on $Y_t = X$. In particular, $u = U(\cdot,s)x$ is a weak solution of (2.6), cf. [34, §5.5]. Finally, we have $\|e^{\tau A(t)}\| \le e^{\tau s(A(t))}$ and $s(A(t)) \le -c$ for a constant $c > 0$ due to, e.g., [9, Thm. 5.2.2, 6.3.1]. Therefore, Corollary 2.3 shows that $\|U(t,s)\| \le e^{-c(t-s)}$. $\square$

For simplicity, we now let $w = 0$ in (A). The norms $\|x\|_t := \sup_{\tau\ge 0} \|e^{\tau A(t)}x\|$ are natural candidates in order to check condition (2.5). But, in general, these norms do not satisfy $\|x\|_t \le \|x\|_s$ for all $x \in X$ and $t \ge s$ (this can be seen, for instance, by the examples in the appendix). However, if we require that $t \mapsto e^{\tau A(t)}$ is Lipschitz continuous, then we can verify a weakened version of (2.5).

Proposition 2.5. *Let the operators $A(t)$, $t \in I$, satisfy (A) with $w = 0$ and let $\mathcal{U}$ solve the corresponding Cauchy problem (CP). Assume $\|e^{\tau A(t)} - e^{\tau A(s)}\| \le L\,|t-s|$ for $t,s \in I$, $\tau \ge 0$, and a constant $L \ge 0$. Then, (2.1) holds with $M(t) = Me^{Lt}$. In particular, $\|U(t,s)\| \le M\,e^{L(t-s)}$ for $(t,s) \in D$.*

Proof. Define the norms $\|x\|_t := \sup_{\tau\ge 0} \|e^{\tau A(t)}x\|$ for $t \in I$ on X. We will show that

$$\|x\|_t \le (1 + L(t-s))\,\|x\|_s \le e^{L(t-s)}\,\|x\|_s\,, \qquad \text{for } (t,s) \in D,\ x \in X. \tag{2.7}$$

Then the result follows as in the proof of Corollary 2.3. Let $(t,s) \in D$ and $x \in X$. Fix $\varepsilon > 0$ and choose $\tau_0 \ge 0$ such that $\|x\|_t \le \varepsilon + \|e^{\tau_0 A(t)}x\|$. By the assumption,

$$\|x\|_t - \|x\|_s \le \varepsilon + \|e^{\tau_0 A(t)}x\| - \|e^{\tau_0 A(s)}x\| \le \varepsilon + L(t-s)\|x\| \le \varepsilon + L(t-s)\|x\|_s.$$

Thus (2.7) is verified. $\square$

We can replace the Lipschitz condition in Proposition 2.5 by an estimate for the resolvents $R(\lambda, A(\cdot))$ in the so-called *parabolic case*, where one assumes that the operators $(A(t), D(A(t)))$ generate bounded analytic semigroups of type (ϕ, K). This means that $D(A(t))$ is dense, $\Sigma \cup \{0\} \subseteq \rho(A(t))$, and $\|R(\lambda, A(t))\| \le \frac{K}{|\lambda|}$ for $\lambda \in \Sigma = \Sigma_\phi := \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| < \phi\}$ and constants $K \ge 0$ and $\frac{\pi}{2} < \phi \le \pi$. We say that A is of type (ϕ, K, w) if $A_w = A - w$ is of type (ϕ, K) for some $w \in \mathbb{R}$.

Corollary 2.6. *Let $A(t)$, $t \in I$, be generators of bounded analytic semigroups of the same type (ϕ, K). Suppose that $\|R(\lambda, A(t)) - R(\lambda, A(s))\| \le L\,\frac{|t-s|}{|\lambda|}$ for $t,s \in I$, $\lambda \in \Sigma_\phi$, and a constant $L \ge 0$. Let $\mathcal{U}$ solve (CP). Then, (2.1) holds for $M(t) = cKe^{cLt}$ and $\|U(s+t,s)\| \le cKe^{cLt}$ for $t \ge 0$ and $s \in I$, where $c = c_\phi$ only depends on ϕ.*

Proof. Obviously, $A(\cdot)$ satisfies condition (A). For $x \in X$ we have

$$e^{\tau A(t)}x - e^{\tau A(s)}x = \frac{1}{2\pi i}\int_\Gamma e^{\lambda\tau}(R(\lambda, A(t)) - R(\lambda, A(s)))x\,d\lambda\,,$$

where Γ is a path in Σ_ϕ independent of t, s, τ and chosen as in [27, p.30]. Hence,

$$\|e^{\tau A(t)}x - e^{\tau A(s)}x\| \le L\,|t-s|\,\|x\|\,\frac{1}{2\pi}\int_\Gamma \frac{e^{Re\lambda\tau}}{|\lambda|}\,|d\lambda| =: c_\phi L\,|t-s|\,\|x\|.$$

Similarly, $\|e^{\tau A(t)}\| \le c_\phi K$ for $t \in I$ and $\tau \ge 0$. Consequently, Proposition 2.5 implies the assertion. $\square$

In the context of the well–posedness theorems in the parabolic case estimates of the same type were obtained in [2, II.5.1, IV.2.4], [10, Thm. 2.3], and [35, p.6] (see also [7, §1] and [12, §7.4] for special cases). These authors assume (roughly speaking) the conditions

(P1) or (P2) introduced in Section 3 and 4, respectively, which imply an Hölder estimate for the resolvents of $A(\cdot)$. In [33, §4.2] our approach was extended to the situation of condition (P1).

3. Exponential dichotomy

Let us first recall the definition of exponential dichotomy, see e.g. [7, 8, 12]. For a projection P we set $Q := Id - P$. Throughout we consider $I \in \{\mathbb{R}, \mathbb{R}_+\}$.

Definition 3.1. An evolution family $\mathcal{U} = (U(t,s))_{(t,s)\in D}$ on a Banach space X is called *hyperbolic* (or has *exponential dichotomy*) if there are projections $P(t) \in \mathcal{L}(X)$, $t \in I$, and constants $N, \delta > 0$ such that $P(\cdot)$ is uniformly bounded, strongly continuous, and

(a) $U(t,s)P(s) = P(t)U(t,s)$ for $(t,s) \in D$,
(b) the restriction $U_Q(t,s) : Q(s)X \to Q(t)X$ is invertible for $(t,s) \in D$ (and we set $U_Q(s,t) := U_Q(t,s)^{-1}$),
(c) $\|U(t,s)P(s)\| \leq Ne^{-\delta(t-s)}$ and $\|U_Q(s,t)Q(t)\| \leq Ne^{-\delta(t-s)}$ for $(t,s) \in D$.

For a hyperbolic evolution family $\mathcal{U}$ one easily shows that $I^2 \ni (t,s) \mapsto U_Q(t,s)Q(s)$ is strongly continuous and satisfies (E1) with Id replaced by $Q(s)$, [33, Lemma 5.15]. If $\mathcal{U}$ is exponentially bounded, then conditions (a)–(c) imply that $P(\cdot)$ is uniformly bounded and strongly continuous, [22, Lemma 4.2].

Our approach is based on the following characterization of hyperbolic evolution families. Different proofs of (parts of) the theorem can be found in, e.g., [14, 15, 16, 29, 30, 31]. Versions of the result for $I = \mathbb{R}_+$ are shown in [22] and the references therein. We set $\Gamma := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$.

Theorem 3.2. *Let $I = \mathbb{R}$ and $\mathcal{U}$ be an exponentially bounded evolution family on X. Let $\mathcal{T}$ be the induced evolution semigroup on $E = C_0(\mathbb{R}, X)$ or $L^p(\mathbb{R}, X)$, $1 \leq p < \infty$, with generator G. Then, $\sigma(G) = \sigma(G) + i\mathbb{R}$ and $\sigma(T(t)) = \Gamma \cdot \sigma(T(t))$ for $t > 0$. Further, $\mathcal{U}$ is hyperbolic if and only if G or, equivalently, $1 - T(t)$, $t > 0$, is invertible on E. In this case the spectral projection $\mathcal{P} = \frac{1}{2\pi i}\int_\Gamma R(\lambda, T(1))d\lambda$ is given by $\mathcal{P} = P(\cdot)$, where $P(\cdot)$ satisfies (a)–(c) above.*

The theorem implies uniqueness of the dichotomy projections $P(\cdot)$ if $I = \mathbb{R}$. For an invertible evolution family this fact can be proved directly, cf. [8, p.164].

Corollary 3.3. *Let $I = \mathbb{R}$ and $\mathcal{U}$ be an exponentially bounded, hyperbolic evolution family on X. Then the projections in Definition 3.1 are uniquely determined by (a)–(c). If $\mathcal{U}$ is given by a $\mathcal{C}_0$–semigroup $(e^{tA})_{t\geq 0}$, i.e., $U(s+t,s) = e^{tA}$ for $s \in \mathbb{R}$ and $t \geq 0$, then $P(\cdot) \equiv P$.*

Proof. Given an exponentially bounded, hyperbolic evolution family $\mathcal{U}$ on X with projections $P(t)$, we define $\mathcal{P} = P(\cdot)$ on $E = C_0(\mathbb{R}, X)$. By condition (a), $\mathcal{P}$ commutes with the hyperbolic evolution semigroup $T(t)$ on E, and hence with the spectral projection $\mathcal{P}_1 = P_1(\cdot)$ of $\mathcal{T}$. Further, (c) yields

$$N^{-1}e^{\delta(t-s)}\|Q(s)x\| \leq \|U(t,s)Q(s)x\| \leq \|U(t,s)x\| + Ne^{-\delta(t-s)}\|P(s)x\|.$$

This estimate implies $P(s)X = \{x \in X : \lim_{t\to\infty} U(t,s)x = 0\}$. So $P_1(s)$ and $P(s)$ commute and have the same range. As a consequence, $P(s) = P_1(s)$ for $s \in \mathbb{R}$.

If additionally $U(s+t,s) = e^{tA}$, then we have $\sigma(e^{tA}) \cap \Gamma = \emptyset$ by Theorem 3.2 and [14, Thm. 2.5]. By standard spectral theory, [20, A-III.3], the semigroup $(e^{tA})_{t\geq 0}$ is hyperbolic with a projection P satisfying (a)–(c). Thus, $P \equiv P(t)$ due to the first assertion. $\square$

Recall that in the case $U(s+t,s) = e^{tA}$ the restriction A_Q of A to $D(A)\cap QX$ generates the $\mathcal{C}_0$–group $(e^{tA_Q})_{t\in\mathbb{R}} := (e^{tA}_{|QX})_{t\in\mathbb{R}}$ on QX, [20, A-III.3].

Assume now that $\mathcal{U}$ solves the Cauchy problem (CP) given by generators $A(t)$. We have seen in Corollary 2.3 that an estimate $\|e^{\tau A(t)}\| \leq e^{-\varepsilon\tau}$ for $\varepsilon > 0$ implies exponential stability of $\mathcal{U}$. In contrast, the following simple matrix example (a variant of [7, p.3]) shows that hyperbolicity of $(e^{\tau A(t)})_{\tau\geq 0}$ with uniform constants $M = 1$ and $\varepsilon > 0$ and projections $P_t \neq Id$ does not imply exponential dichotomy of $\mathcal{U}$.

Example 3.4. On $X = \mathbb{C}^2$ endowed with the Euclidean norm let $A(t) = D(-t)A_0D(t)$ for $A_0 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $D(t) = \left(\begin{smallmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{smallmatrix}\right)$ for $t \in \mathbb{R}$. Then the evolution family

$$U(t,s) = D(-t)\begin{pmatrix} 1+t & t \\ -t & 1-t \end{pmatrix} D(s) = D(-t)\ \exp\left[t\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\right] D(s)$$

solves (CP). Hence, $\mathcal{U}$ is not hyperbolic. However, $e^{\tau A(t)} = D(-t)e^{\tau A_0}D(t)$ is hyperbolic with constants $M = \varepsilon = 1$.

Using Hölder estimates for $A(\cdot)$, we will be able to derive the hyperbolicity of $\mathcal{U}$ from the hyperbolicity of $(e^{\tau A(t)})_{\tau\geq 0}$. To that purpose, we introduce the following hypotheses.

- (P1) Let $(A(t), D(A(t)))$, $t \in I$, be generators of analytic semigroups on X of the same type (ϕ, K, w) for some $w \geq 0$. Suppose that $D(A(t)) \equiv D(A(0))$, $A(t)$ is invertible, $\sup_{t,s\in I} \|A(t)A(s)^{-1}\| < \infty$, and $\|A(t)A(s)^{-1} - Id\| \leq L|t-s|^\alpha$ for $t,s \in I$ and constants $L \geq 0$ and $0 < \alpha \leq 1$.
- (ED) Assume that the semigroups $(e^{\tau A(t)})_{\tau\geq 0}$, $t \in I$, are hyperbolic with projections P_t and constants $N, \delta > 0$. Moreover, let $\|A(t)e^{\tau A(t)}P_t\| \leq \psi(\tau)$ and $\|A(t)e^{-\tau A_Q(t)}Q_t\| \leq \psi(-\tau)$ for $\tau > 0$ and a function ψ such that $\mathbb{R} \ni s \mapsto \varphi(s) := |s|^\alpha\psi(s)$ is integrable.

Condition (P1) implies $\|(A_w(t) - A_w(s))A_w(r)^{-1}\| \leq L'|t-s|^\alpha$ for $t,s,r \in I$, a constant $L' \geq 0$, and $A_w(t) = A(t) - w$. Consequently, there exists a unique exponentially bounded evolution family $\mathcal{U}$ solving the Cauchy problem (CP) corresponding to $A(\cdot)$ on X, see e.g. [2, Thm. II.4.4.1] or [27, Thm. 5.6.1]. In the case $I = \mathbb{R}_+$ we extend $A(\cdot)$ to $I = \mathbb{R}$ by setting $\hat{A}(t) := A(0)$ and $D(\hat{A}(t)) := D(A(0))$ for $t < 0$. Then, $\hat{A}(\cdot)$ also satisfies (P1) and (ED) with the same constants and the corresponding Cauchy problem is solved by the evolution family

$$\hat{U}(t,s) = \begin{cases} U(t,s), & t \geq s \geq 0, \\ U(t,0)e^{-sA(0)}, & t \geq 0 > s, \\ e^{(t-s)A(0)}, & 0 > t \geq s. \end{cases} \tag{3.1}$$

In the sequel, we will assume $I = \mathbb{R}$ and omit the "hats".

We remark that (P1) and the first sentence of (ED) already imply the existence of a function $\psi : \mathbb{R} \to \mathbb{R}_+$ with the properties required in (ED). Let us indicate one way to determine ψ, see also Example 3.8. It follows from, e.g., [27, (2.5.13)] that an analytic semigroup $(e^{tA})_{t\geq 0}$ of type (ϕ, K, w) satisfies $\|Ae^{tA}\| \leq \frac{C}{t}$ for $0 < t \leq 1$ and a constant C only depending on K, ϕ, and w. Hence, if $(e^{tA})_{t\geq 0}$ is hyperbolic with a projection P and constants $N, \delta > 0$, then $\|Ae^{tA}P\| \leq \frac{CN}{t}$ for $0 < t \leq 1$ and $\|Ae^{tA}P\| \leq \|Ae^{A}\| \, \|e^{(t-1)A}P\| \leq CNe^{-\delta(t-1)}$ for $t \geq 1$. Further,

$$Ae^{-tA_Q}Q = Ae^{A}\, e^{-(t+1)A_Q}Q \tag{3.2}$$

and so $\|Ae^{-tA_Q}Q\| \leq CNe^{-\delta}e^{-\delta t}$. For $\alpha = 1$, this gives $\|\varphi\|_1 = CN(1+\delta^{-2}(1+\delta+e^{-\delta}))$.

To prove our main theorem, we need two prelimary results. Assuming (ED), we set

$$\Gamma_s(\tau) := \begin{cases} e^{\tau A(s)}P_s, & \tau \geq 0,\ s \in \mathbb{R} \\ -e^{\tau A_Q(s)}Q_s, & \tau < 0,\ s \in \mathbb{R}. \end{cases}$$

Lemma 3.5. *Let (P1) and (ED) hold. Then the mappings* $(s,\tau) \mapsto \Gamma_s(\tau) \in \mathcal{L}(X)$ *and* $(t,s,\tau) \mapsto A(t)\Gamma_s(\tau) \in \mathcal{L}(X)$ *are continuous for* $\tau \neq 0$ *and* $t, s \in \mathbb{R}$.

Proof. One verifies that (P1) yields Hölder continuity of $t \mapsto e^{\tau A(t)} \in \mathcal{L}(X)$ uniformly for $0 \leq \tau \leq d$ as in the proof of Corollary 2.6. The resolvents

$$R(\lambda, e^{A(t)}) = \sum_{n=0}^{\infty} \lambda^{-(n+1)}\, e^{nA(t)}P_t - \sum_{n=1}^{\infty} \lambda^{n-1}\, e^{-nA_Q(t)}Q_t$$

are uniformly bounded for $t \in \mathbb{R}$ and $\lambda \in \Gamma$ due to (ED). Thus, $R(\lambda, e^{A(t)})$ and, hence, the spectral projections P_t are Hölder continuous with respect to t. As a result, $(s,\tau) \mapsto e^{\tau A(s)}P_s \in \mathcal{L}(X)$ is continuous for $\tau > 0$. Further,

$$\begin{aligned} e^{-\tau A_Q(t)}Q_t - e^{-\sigma A_Q(s)}Q_s =& e^{-\tau A_Q(t)}Q_t\,(Q_t - Q_s) + e^{-\tau A_Q(t)}Q_t\,(e^{\sigma A(s)} - e^{\tau A(t)})\, e^{-\sigma A_Q(s)}Q_s \\ &+ (Q_t - Q_s)\, e^{-\sigma A_Q(s)}Q_s \end{aligned}$$

for $t, s \in I$ and $\tau, \sigma > 0$. So $(s,\tau) \mapsto \Gamma_s(\tau) \in \mathcal{L}(X)$ is continuous for $\tau \neq 0$. Then, from [27, Lemma 5.6.2] and (3.2) follows that $(t,s,\tau) \mapsto A(t)\Gamma_s(\tau) \in \mathcal{L}(X)$ is continuous for $\tau \neq 0$. ☐

The next lemma is essentially contained in [23]. For completeness we give the details. Recall the definition of the operators $\mathcal{A}$ and G_0 on $E = C_0(\mathbb{R}, X)$ given in the first section.

Lemma 3.6. *Assume (P1). Then the generator* $(G, D(G))$ *of the evolution semigroup on* $E = C_0(\mathbb{R}, X)$ *is the closure of* $(G_0 + \mathcal{A}, D(G_0) \cap D(\mathcal{A}))$*. Further, if* $f \in D(G) \cap D(\mathcal{A})$*, then* $f \in D(G_0)$.

Proof. By rescaling, we may assume that $w = 0$ in (P1). The first assertion follows from $D(G_0) + \mathcal{A} \subseteq G$ due to Lemma 1.1. We estimate

$$\begin{aligned} &\|\tfrac{1}{t}(T(t)f - f) - (-f' + \mathcal{A}f)\|_\infty \\ &\leq\ \sup_{s\in\mathbb{R}} \Big(\|\tfrac{1}{t}(U(s+t,s)f(s) - f(s)) - A(s)f(s)\| + \|A(s)f(s) - A(s+t)f(s+t)\| \\ &\qquad + \|\tfrac{1}{t}(f(s) - f(s+t)) + f'(s)\| + \|f'(s+t) - f'(s)\| \Big) \end{aligned}$$

for $f \in D(G_0) \cap D(\mathcal{A})$ and $t > 0$. Clearly, the second, third, and fourth term on the right hand side tend to 0 as $t \to 0$. Further,

$$\begin{aligned}\|\tfrac{1}{t}(U(s+t,s)f(s) - f(s)) - A(s)f(s)\| &= \left\|\tfrac{1}{t}\int_0^t (A(\tau)U(\tau,s) - A(s))f(s)\,d\tau\right\| \\ &\le \sup_{s\in\mathbb{R}, s\le\tau\le s+t} \|(A(\tau)U(\tau,s) - A(s))f(s)\|\,.\end{aligned}$$

Since $(\tau, s) \mapsto A(\tau)U(\tau, s)A(s)^{-1}$ is strongly continuous and uniformly bounded for $s \le \tau \le s+1$ by [2, Thm. II.4.4.1], the first summand also converges to 0 as $t \to 0$. Now let $f \in D(G) \cap D(\mathcal{A})$. Then,

$$\tfrac{1}{t}(f(s-t) - f(s)) = \tfrac{1}{t}(U(s,s-t)f(s-t) - f(s)) + \tfrac{1}{t}(1 - U(s,s-t))f(s-t)$$

The first term on the right hand side converges in E as $t \to 0$. The convergence of the second one follows as above if one observes that

$$\|\tfrac{1}{t}(1-U(\cdot,\cdot-t))f(\cdot-t)+\mathcal{A}f\|_\infty \le \|\tfrac{1}{t}(1-U(\cdot+t,\cdot))f(\cdot)+\mathcal{A}f\|_\infty + \|\mathcal{A}f(\cdot+t)-\mathcal{A}f\|_\infty.$$

□

We now come to our main theorem. A similar approach has been sketched by A.G. Baskakov, [3], for bounded operators $A(t)$. Analogous results can be found in [7, Prop. 6.1] for $X = \mathbb{C}^n$ and in [18, Thm. 4] for delay equations. Moreover, Proposition 6.2 in [7] shows that, roughly speaking, (ED) is necessary for exponential dichotomy of $\mathcal{U}$ if $A(\cdot)$ has a small Lipschitz constant.

Theorem 3.7. *Assume (P1) and (ED). Let $q := L\|\varphi\|_1 < 1$. Then $\mathcal{U}$ is hyperbolic with an exponent $0 < \delta' < \frac{\delta(1-q)}{2N}$.*

Proof. (1) As observed above, we may assume $I = \mathbb{R}$. We define

$$\begin{aligned}(Rf)(t) &:= \int_{-\infty}^{\infty} \Gamma_s(t-s)f(s)\,ds \\ &= \int_{-\infty}^{t} e^{(t-s)A(s)}P_s\,f(s)\,ds - \int_t^{\infty} e^{(t-s)A_Q(s)}Q_s\,f(s)\,ds, \\ (Lf)(t) &:= \int_{-\infty}^{\infty} \Gamma_t(t-s)f(s)\,ds \\ &= \int_{-\infty}^{t} e^{(t-s)A(t)}P_t\,f(s)\,ds - \int_t^{\infty} e^{(t-s)A_Q(t)}Q_t\,f(s)\,ds\end{aligned}$$

for $f \in E = C_0(\mathbb{R}, X)$ and $t \in \mathbb{R}$. Using (ED) and Lemma 3.5, it is easy to verify that $R, L \in \mathcal{L}(E)$ and $\|R\|, \|L\| \le \frac{2N}{\delta}$. Let G be the generator of the evolution semigroup $\mathcal{T}$ on E induced by $\mathcal{U}$. We show in two steps that G is bijective.

(2) For $f \in D(\mathcal{A})$ we have

$$\int_{-\infty}^{\infty} A(t)\Gamma_s(t-s)f(s)\,ds = \int_{-\infty}^{\infty} A(t)A(s)^{-1}\,\Gamma_s(t-s)A(s)f(s)\,ds.$$

So we infer from (ED) and Lemma 3.5 that $Rf \in D(\mathcal{A})$. Moreover,

$$\left(\tfrac{d}{dt}Rf\right)(t) = P_tf(t) + Q_tf(t) + \int_{-\infty}^{\infty} \Gamma_s(t-s)A(s)f(s)\,ds$$

so that $\frac{d}{dt}Rf \in E$. As a consequence, Lemma 3.6 implies $RD(\mathcal{A}) \subseteq D(\mathcal{A}) \cap D(G_0) \subseteq D(G)$ and $(GR + Id)f = Sf$ for $f \in D(\mathcal{A})$, where

$$(Sf)(t) := \int_{-\infty}^{\infty} (A(t) - A(s))\, \Gamma_s(t-s) f(s)\, ds.$$

Due to (P1), (ED), and Lemma 3.5, we obtain $Sf \in E$ and

$$\begin{aligned} \|Sf(t)\| &\leq \int_{-\infty}^{\infty} \|(A(t) - A(s))A(s)^{-1}\|\, \|A(s)\Gamma_s(t-s)f(s)\|\, ds \\ &\leq L\, \|f\|_\infty \int_{-\infty}^{\infty} \varphi(t-s)\, ds \;=\; q\, \|f\|_\infty \end{aligned}$$

for $f \in E$ and $t \in \mathbb{R}$. Since $q < 1$, the operator $S - Id$ is invertible on E.

Fix $g \in E$. Let $f := (S - Id)^{-1} g$ and choose $D(\mathcal{A}) \ni f_n \to f$. Then, $Rf_n \to Rf$ and $GRf_n = (S - Id)f_n \to g$ in E. The closedness of G gives $Rf \in D(G)$ and $GRf = g$. As a result, G is surjective and has the right inverse $R(S - Id)^{-1}$.

(3) Let $f \in D(\mathcal{A}) \cap D(G_0) \subseteq D(G)$. Integrating by parts we obtain

$$\begin{aligned} (LGf)(t) &= \int_{-\infty}^{\infty} \Gamma_t(t-s) A(s) f(s)\, ds - \int_{-\infty}^{\infty} \Gamma_t(t-s) f'(s)\, ds \\ &= \int_{-\infty}^{\infty} \Gamma_t(t-s) A(s) f(s)\, ds - (P_t + Q_t) f(t) - \int_{-\infty}^{\infty} \Gamma_t(t-s) A(t) f(s)\, ds \\ &= -f(t) + (Vf)(t) \end{aligned} \tag{3.3}$$

for $t \in \mathbb{R}$, where

$$(Vf)(t) := \int_{-\infty}^{\infty} \Gamma_t(t-s)(A(s) - A(t)) f(s)\, ds.$$

Again, for $f \in D(\mathcal{A})$ conditions (P1) and (ED) and Lemma 3.5 yield $Vf \in D(\mathcal{A})$ and

$$\begin{aligned} \|A(t)\,(Vf)(t)\| &\leq \int_{-\infty}^{\infty} \|A(t)\Gamma_t(t-s)\|\, \|(A(t) - A(s))A(s)^{-1}\|\, \|A(s)f(s)\|\, ds \\ &\leq L\, \|\mathcal{A}f\|_\infty \int_{-\infty}^{\infty} \varphi(t-s)\, ds \;=\; q\, \|\mathcal{A}f\|_\infty \end{aligned}$$

for $t \in \mathbb{R}$. Therefore V is a strict contraction on $D(\mathcal{A})$ endowed with the norm $\|f\|_{\mathcal{A}} := \|\mathcal{A}f\|_\infty$ (notice that $\mathcal{A}^{-1} = A(\cdot)^{-1} \in \mathcal{L}(E)$ by (P1)).

Assume $Gf = 0$ for some $f \in D(G)$. Then, $T(1)f = f$ and so $f(s) = U(s, s-1)f(s-1)$ for $s \in \mathbb{R}$. By [2, Thm. II.4.4.1], the function $s \mapsto A(s)U(s, s-1) \in \mathcal{L}(X)$ is strongly continuous and uniformly bounded; that is, $f \in D(\mathcal{A})$. Consequently, $f \in D(\mathcal{A}) \cap D(G_0)$ due to Lemma 3.6. Now, (3.3) implies $Vf = f + LGf = f$, and hence $f = 0$.

(4) By part (2) and (3), we have $G^{-1} = R(S - Id)^{-1} \in \mathcal{L}(E)$ and $\|G^{-1}\| \leq \frac{2N}{\delta(1-q)} =: \frac{1}{\eta}$. By the symmetry of $\sigma(G)$, see Theorem 3.2, this implies $(-\eta, \eta) + i\mathbb{R} \subseteq \rho(G)$. Now, using rescaling and Theorem 3.2, one sees that the evolution families $(e^{\pm\delta'(t-s)}U(t,s))_{t \geq s}$ are hyperbolic with exponent $0 \leq \delta' < \eta$ and the same projections $P(s)$. □

Our result can be applied to the following situation.

Example 3.8. We consider the same situation as in Example 2.4. Additionally, we assume that $\partial\Omega$ is compact and of class C^2 and let $a_{kl} \in C_b^1(\mathbb{R}_+, C^1(\overline{\Omega}))$. Then, the operator $A_0(t)x = A(t, \xi, D)x$ with domain $D(A_0(t)) \equiv Y = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ in $X = L^2(\Omega)$

is self adjoint and generates an analytic semigroup, see e.g. [34, §3.7]. By Example 2.4 these semigroups have a common growth bound $-c < 0$. Further, let $V \in C_b^1(\mathbb{R}_+, L^\infty(\Omega))$ be real valued. Then, the operators $A(t) = A_0(t) + V(t)$ with domain Y are self adjoint and generate analytic semigroups of the same type. Assume that there is $\delta > 0$ such that $(-\delta, \delta) \subseteq \rho(A(t))$. Hence, $A(t)$ generates a hyperbolic semigroup with constants $\delta > 0$ and $N = 1$. By virtue of the Agmon–Douglis–Nirenberg estimate, see [34, §3.7], the operators $A(t)$ satisfy (P1) with constants $\alpha = 1$ and L proportional to $\max_{k,l}\{\|a_{kl}\|_{C_b^1}, \|V\|_{C_b^1}\} =: L_0$. On the other hand, a straightforward application of the spectral theorem shows that (ED) holds with $\psi(t) = (e|t|)^{-1}$ for $0 < |t| \le 1$ and $\psi(t) = \frac{1}{e} e^{-(|t|-1)\delta}$ for $|t| \ge 1$. Thus, $\|\varphi\|_1 = \frac{2}{e}(1 + \delta^{-1} + \delta^{-2})$. As a result, for sufficiently small Lipschitz constant L_0 the evolution family $\mathcal{U}$ solving (CP) has exponential dichotomy.

4. Robustness of exponential dichotomy

In this section we show robustness of exponential dichotomy of an evolution family under time dependent *Miyadera* type perturbations by means of the spectral characterization provided by Theorem 3.2. Given an exponentially bounded evolution family $(U(t,s))_{(t,s)\in D}$ on X with $I \in \{\mathbb{R}, \mathbb{R}_+\}$, we make the following assumption on linear operators $B(t)$, $t \in I$, on X.

(M) $(B(t), D(B(t)))$ is closed for a.e. $t \in I$. For all $s \in I$ and $x \in X$ we have $U(t,s)x \in D(B(t))$ for a.e. $t \in [s, \infty)$, $B(\cdot)U(\cdot, s)x$ is measurable, and
$$\int_0^{\alpha'} \|B(s+t)U(s+t,s)x\| \, dt \le \beta' \, \|x\| \tag{4.1}$$
for constants $\alpha' > 0$ and $\beta' \in [0, 1)$,

see [28] for a somewhat weaker condition. It is proved in [28, Thm. 3.4, Cor. 3.5] that there is a unique exponentially bounded evolution family $\mathcal{U}_B = (U_B(t,s))_{(t,s)\in D}$ on X satisfying
$$U_B(t,s)x = U(t,s)x + \int_s^t U_B(t,\tau)B(\tau)U(\tau,s)x \, d\tau \tag{4.2}$$
for all $x \in X$ and $(t,s) \in D$. Moreover, for $x \in X$ and $s \in I$, we have $U_B(t,s)x \in D(B(t))$ for a.e. $t \in [s, \infty)$, $B(\cdot)U_B(\cdot, s)x$ is locally integrable, and
$$U_B(t,s)x = U(t,s)x + \int_s^t U(t,\tau)B(\tau)U_B(\tau,s)x \, d\tau \tag{4.3}$$
for all $x \in X$ and $(t,s) \in D$. Observe that if $\mathcal{U}$ solves the Cauchy problem (CP), then by (4.3) the function $u(\cdot) = U_B(\cdot, s)x$ can be interpreted as a *mild solution* of the perturbed problem
$$\tfrac{d}{dt}u(t) = (A(t) + B(t))\, u(t), \quad t > s, \qquad u(s) = x.$$

We now assume that $\mathcal{U}$ has exponential dichotomy with projections $P(s)$ and constants $N, \delta > 0$. If $I = \mathbb{R}_+$, we extend $\mathcal{U}$ to the time interval $I = \mathbb{R}$. To that purpose set

$R := Q(0) - P(0) = Id - 2P(0)$ and

$$\tilde{U}(t,s) := \begin{cases} U(t,s), & t \geq s \geq 0, \\ U(t,0)e^{-s\delta R}, & t \geq 0 > s, \\ e^{(t-s)\delta R}, & 0 > t \geq s, \end{cases}$$

cf. [4, p.109]. This defines an hyperbolic evolution family $\tilde{\mathcal{U}}$ on X with the same constants N, δ and projections $\tilde{P}(t) := P(t)$ for $t \geq 0$ and $\tilde{P}(t) := P(0)$ for $t \leq 0$. Further, suppose that (M) holds for $\mathcal{U}$ and operators $B(t)$, $t \geq 0$. Let $\tilde{B}(t) := B(t)$ for $t \geq 0$ and $\tilde{B}(t) := 0$ for $t < 0$. Then (M) is satisfied for $\tilde{\mathcal{U}}$ and $\tilde{B}(\cdot)$.

Since $B(t)U_Q(t,s)Q(s)x = B(t)U_Q(t,r)Q(r)y$ for $y = U_Q(r,s)Q(s)x$ and $r \leq s$, we see that $B(\cdot)U_Q(\cdot,s)Q(s)x \in L^1_{loc}(\mathbb{R}, X)$ for $x \in X$ and $s \in \mathbb{R}$. Using this fact, we can now formulate our robustness theorem. For bounded perturbations our approach was already used in [15] and [16]. For a smaller class of perturbations and $U(s+t,s) = e^{tA}$ a similar result was shown in [6, Thm. 4.1] in the context of linear skew product flows.

Theorem 4.1. *Assume that (M) holds for an exponentially bounded evolution family $\mathcal{U}$ on X and operators $(B(t), D(B(t)))$, $t \in I$. Let $\mathcal{U}$ be hyperbolic with projections $P(t)$ and constants $N, \delta > 0$. Suppose that there are constants $\alpha > 0$ and $\beta_P, \beta_Q \geq 0$ such that*

$$\int_0^\alpha \|B(s+t)U_P(s+t,s)P(s)x\|\, dt \;\leq\; \beta_P \|x\|, \tag{4.4}$$

$$\int_0^\alpha \chi_I(s-t)\, \|B(s-t)U_Q(s-t,s)Q(s)x\|\, dt \;\leq\; \beta_Q \|x\| \tag{4.5}$$

for $s \in I$ and $x \in X$, where $\beta_Q = 0$ if $P(\cdot) \equiv Id$. Let $b := \frac{N(\beta_P+\beta_Q)}{1-e^{-\alpha\delta}}$ for $\alpha < \infty$ and $b := \beta_P + \beta_Q$ if $\alpha = \infty$. If $b < 1$, then the perturbed evolution family $\mathcal{U}_B$ is hyperbolic.

Proof. Using the extension introduced above, we may assume $I = \mathbb{R}$. Let G be the generator of the evolution semigroup $\mathcal{T}$ on $E = L^1(\mathbb{R}, X)$ induced by $\mathcal{U}$ and $\mathcal{B} = B(\cdot)$ be multiplication operator on E with maximal domain. Notice that $\mathcal{B}$ is closed. The operator G is invertible by Theorem 3.2 with corresponding spectral projection $\mathcal{P} = P(\cdot)$. Set $\mathcal{Q} = Id - \mathcal{P}$ and denote by $T_P(t)$ and $T_Q(t)$ the restrictions of $T(t)$ to $\mathcal{P}E$ and $\mathcal{Q}E$, respectively. It is easy to see that

$$\begin{aligned} -G^{-1}f &= \int_0^\infty T_P(t)\mathcal{P}f\, dt - \int_0^\infty T_Q^{-1}(t)\mathcal{Q}f\, dt \quad \text{and} \\ T_Q^{-1}(t)\mathcal{Q}f &= U_Q(\cdot, \cdot + t)Q(\cdot + t)f(\cdot + t) \qquad \text{for } f \in E. \end{aligned}$$

Due to [28, Thm. 3.4], there exists a perturbed evolution family $\mathcal{U}_B$ on X and the induced evolution semigroup $(T_B(t))_{t\geq 0}$ on E is generated by $G_B = G + \mathcal{B}$ with $D(G_B) = D(G) \subseteq D(\mathcal{B})$. For $Y_t = X$, we define as in Lemma 1.1 a dense subspace $\mathcal{D} \subseteq D(G)$. We claim that

$$\|\mathcal{B}G^{-1}f\| \leq b\|f\| \qquad \text{for} \quad f \in \mathcal{D}. \tag{4.6}$$

Since $\mathcal{B}G^{-1}$ is bounded, (4.6) yields $\|\mathcal{B}g\| \leq b\|Gg\|$ for $g \in D(G)$. Due to [13, IV.1.16], the condition $b < 1$ implies that G_B is invertible. Then the assertion follows from Theorem 3.2.

We now show (4.6). Let $f \in \mathcal{D}$ be given by $f(s) = \sum_{k=1}^n \varphi_k(s)\, U(s,s_k)x_k$. Observe that $\mathcal{B}T_P(t)\mathcal{P}f = \mathcal{B}G^{-1}\, T_P(t)\mathcal{P}Gf$ and $\mathcal{B}T_Q(t)^{-1}\mathcal{Q}f = \mathcal{B}G^{-1}\, T_Q(t)^{-1}\mathcal{Q}Gf$. Therefore the functions $\mathcal{B}T_P(\cdot)\mathcal{P}f$ and $\mathcal{B}T_Q(\cdot)^{-1}\mathcal{Q}f$ are continuous. First, we obtain

$$\begin{aligned}
\int_0^\alpha \|\mathcal{B}T(t)\mathcal{P}f\|\,dt &= \int_0^\alpha \int_{\mathbb{R}} \left\| \sum_{k=1}^n \varphi_k(s-t)B(s)U(s,s_k)P(s_k)x_k \right\| ds\,dt \\
&= \int_{\mathbb{R}} \int_0^\alpha \left\| B(s+t)U(s+t,s) \sum_{k=1}^n \varphi_k(s)\,U(s,s_k)P(s_k)x_k \right\| dt\,ds \\
&= \int_{\mathbb{R}} \int_0^\alpha \|B(s+t)U(s+t,s)P(s)f(s)\|\,dt\,ds \\
&\leq \beta_P\,\|f\|
\end{aligned}$$

by Fubini's theorem and (4.4). Second, notice that

$$(T_Q^{-1}(t)\mathcal{Q}f)(s) = \sum_{k=1}^n \varphi_k(s+t)\,U_Q(s,s_k)Q(s_k)x_k$$

for $t \geq 0$. Thus, (4.5) implies

$$\begin{aligned}
&\int_0^\alpha \|\mathcal{B}T_Q^{-1}(t)T_Q^{-1}(r)\mathcal{Q}f\|\,dt \\
&= \int_0^\alpha \int_{\mathbb{R}} \left\| B(s) \sum_{k=1}^n \varphi_k(s+r+t)\,U_Q(s,s_k)Q(s_k)x_k \right\| ds\,dt \\
&= \int_{\mathbb{R}} \int_0^\alpha \left\| B(s-t)U_Q(s-t,s)Q(s) \sum_{k=1}^n \varphi_k(s+r)\,U_Q(s,s_k)Q(s_k)x_k \right\| dt\,ds \\
&\leq \beta_Q\,\|T_Q^{-1}(r)\mathcal{Q}f\|
\end{aligned}$$

for $t, r \geq 0$. So we conclude for finite $\alpha > 0$ that

$$\begin{aligned}
\|\mathcal{B}G^{-1}f\| &\leq \int_0^\infty \|\mathcal{B}T(t)\mathcal{P}f\|\,dt + \int_0^\infty \|\mathcal{B}T_Q^{-1}(t)\mathcal{Q}f\|\,dt \\
&\leq \sum_{k=0}^\infty \int_0^\alpha \Big(\|\mathcal{B}T_P(t)(T_P(k\alpha)\mathcal{P}f)\| + \|\mathcal{B}T_Q^{-1}(t)(T_Q^{-1}(k\alpha)\mathcal{Q}f)\| \Big) dt \\
&\leq \sum_{k=0}^\infty \Big(\beta_P\,\|T_P(k\alpha)\mathcal{P}f\| + \beta_Q \|T_Q^{-1}(k\alpha)\mathcal{Q}f\| \Big) \\
&\leq N\,(\beta_P + \beta_Q) \sum_{k=0}^\infty e^{-\alpha\delta k}\,\|f\| = b\,\|f\|,
\end{aligned}$$

which is (4.6). For $\alpha = \infty$ the assertion follows in a similar way. ☐

We give some applications of Theorem 4.1. First we consider bounded perturbations, see e.g. [5, Thm. 2.5], [7, §4], [8, Thm. IV.5.1, Remark 5.1], [15, §5], and [16, Thm. 3.9] for similar results. We denote by $L^p_{loc,u}(I)$, $1 \leq p < \infty$, the space of uniformly locally p–integrable functions endowed with the norm $\|\psi\|_{p,loc,u} := \sup_{s\in I} \|\chi_{[s,s+1]}\psi\|_p$. Let $\frac{1}{p}+\frac{1}{q} = 1$.

Corollary 4.2. *Let $(U(t,s))_{(t,s)\in D}$ be an exponentially bounded evolution family on X and let $B(t) \in \mathcal{L}(X)$ such that $B(\cdot)x$ is measurable and $\|B(t)\| \leq \psi(t)$ for a.e. $t \in I$,* *where $\psi \in L^p_{loc,u}(I)$ for $1 < p \leq \infty$. Assume that $\mathcal{U}$ is exponentially dichotomic (resp. stable) with constants $N, \delta > 0$. If*

$$\|\psi\|_{p,loc,u} < \frac{1-e^{-\delta}}{2N} \qquad \left(\textit{resp. } \|\psi\|_{p,loc,u} < \frac{1-e^{-\delta}}{N}\right),$$

then the perturbed evolution family $\mathcal{U}_B$ is exponentially dichotomic (resp. stable). In the case $p = \infty$ we can replace $1 - e^{-\delta}$ by δ in the above estimates.

Proof. From Hölder's inequality and the exponential boundedness of $\mathcal{U}$ we derive

$$\int_0^\alpha \|B(s+t)U(s+t,s)x\|\,dt \leq C\,\|x\|\,\|\psi\|_{p,loc,u}\,\alpha^{\frac{1}{q}}$$

for $\alpha \in (0,1]$, a constant C, $s \in I$, and $x \in X$. Letting $\alpha \to 0$ we obtain (M). Further,

$$\begin{aligned}\int_0^\infty \|B(s+t)U(s+t,s)P(s)x\|\,dt &\leq \sum_{k=0}^\infty N\,\|x\| \int_k^{k+1} \psi(s+t)e^{-\delta t}\,dt \\ &\leq N\,\|\psi\|_{p,loc,u}\,(1-e^{-\delta})^{-1}\,\|x\|\end{aligned}$$

for $x \in X$. Analogously one determines β_Q. The case $p = \infty$ is treated in the same way. □

If X is finite dimensional and the operators $A(t)$ and $B(t)$ are uniformly bounded, refinements of the above estimates are known to be optimal in certain situations, see [26] and the references therein.

Next we apply Theorem 4.1 to parabolic equations, where we use a condition introduced by P. Acquistapace and B. Terreni.

(P2) Let $(A(t), D(A(t)))$, $t \in I$, be generators of analytic semigroups on X of the same type (ϕ, K, w). For $A_w(t) = A(t) - w$, we assume

$$\|A_w(t)R(\lambda, A_w(t))\,(A_w(t)^{-1} - A_w(s)^{-1})\| \leq L\,|t-s|^\mu\,|\lambda|^{-\nu}$$

for $\lambda \in \Sigma$, $t, s \in I$, and constants $L \geq 0$ and $\mu, \nu \in (0,1]$ with $\mu + \nu > 1$.

Due to [1, Thm. 2.3] there is a unique exponentially bounded evolution family $\mathcal{U}$ solving the Cauchy problem corresponding to $A(\cdot)$ on X which is exponentially bounded by [10, Thm. 2.3]. For $0 \leq \theta \leq 1$ and $t \in I$, let Z_t be the domain of the fractional power $(-A_w(t))^\theta$ endowed with the norm $\|x\|_{Z_t} := \|(-A_w(t))^\theta x\|$. It is shown in [10, Thm. 2.3] that $U(t,s)Z_s \subseteq X$, $t > s$, and

$$\|(-A_w(t))^\theta U(t,s)\| \leq C(t-s)^{-\theta} \tag{4.7}$$

for $0 < t - s \leq 1$ and a constant C. Condition (P2) holds for a large class of elliptic operators with mixed boundary conditions, see e.g. [1], [2, §IV.2.6, IV.3.1], [10], and the references therein. On the perturbations $B(\cdot)$ we impose the following condition.

(B) Let $(B(t), D(B(t)))$, $t \in I$, be closed operators in X and $Z_t \subseteq D(B(t))$. Assume that there are $0 \leq \theta < 1$ and $1 < p \leq \infty$ such that $p(1-\theta) > 1$, $B(t) \in \mathcal{L}(Z_t, X)$, $\|B(t)\|_{\mathcal{L}(Z_t,X)} \leq \psi(t)$ for a.e. $t \in I$, and $\psi \in L^p_{loc,u}(I)$. Finally, let $B(\cdot)U(\cdot,s)x$ be measurable for $s \in I$ and $x \in X$.

By [28, Lemma 4.1] conditions (P2) and (B) imply (M). In [28, §5] condition (B) was verified for first order perturbations $B(t)$ with singular coefficients of an elliptic second order operator $A(t)$.

Corollary 4.3. *Assume (P2) and (B). Let $\mathcal{U}$ be hyperbolic with constants $M, \varepsilon > 0$ and projections $P(s)$. If*

$$\|\psi\|_{p,loc,u} < \frac{1}{NC}\left((1-\theta q)^{-\frac{1}{q}} + \frac{1+e^{-\delta}}{1-e^{-\delta}}\right)^{-1},$$

then the perturbed evolution family $\mathcal{U}_B$ has exponential dichotomy. If $P(s) \equiv Id$ and

$$\|\psi\|_{p,loc,u} < \frac{1}{NC}\left((1-\theta q)^{-\frac{1}{q}} + \frac{1}{1-e^{-\delta}}\right)^{-1},$$

then $\mathcal{U}_B$ is exponentially stable. In the case $p = \infty$ we can replace $1 - e^{-\delta}$ by δ in the above estimates.

Proof. Using (B), (4.7), exponential dichotomy of $\mathcal{U}$, and Hölder's inequality, we calculate

$$\begin{aligned}
&\int_0^\infty \|B(s+t)U(s+t,s)P(s)x\|\,dt \\
&\le \int_0^1 \psi(s+t)Ct^{-\theta}\|P(\cdot)\|_\infty\,\|x\|\,dt + \int_1^\infty \psi(s+t)C\,\|U(s+t-1,s)P(s)x\|\,dt \\
&\le C\,\|\psi\|_{p,loc,u}\,\|x\|\left(\|P(\cdot)\|_\infty\left(\int_0^1 t^{-\theta q}\,dt\right)^{\frac{1}{q}} + Ne^{\delta}\sum_{k=1}^{\infty}\left(\int_k^{k+1} e^{-\delta q t}\,dt\right)^{\frac{1}{q}}\right) \\
&\le NC\,\|\psi\|_{p,loc,u}\,\|x\|\left((1-\theta q)^{-\frac{1}{q}} + (1-e^{-\delta})^{-1}\right)
\end{aligned}$$

for $x \in X$. Similarly,

$$\begin{aligned}
&\int_0^\infty \chi_I(s-t)\,\|B(s-t)U_Q(s-t,s)Q(s)x\|\,dt \\
&\le \int_0^\infty \chi_I(s-t)\,\psi(s-t)\,\|(-A_w(s-t))^\theta U(s-t,s-t-1)\|\,\|U_Q(s-t-1,s)Q(s)x\|\,dt \\
&\le CNe^{-\delta}\|x\|\int_0^\infty \chi_I(s-t)\,\psi(s-t)e^{-\delta t}\,dt \\
&\le CNe^{-\delta}(1-e^{-\delta})^{-1}\|\psi\|_{p,loc,u}\,\|x\|.
\end{aligned}$$

Now the corollary follows from Theorem 4.1. □

For a more specific class of parabolic equations a variety of results concerning robustness of dichotomy can be found in [12, §7.6]. In [17] perturbations $B(t)$ defined on $D(A(t))$ are considered, where it is assumed that $D(A(t))$ and certain interpolation spaces are time invariant.

5. Appendix: The spectral bounds of $A(t)$ do not determine the growth bound of $\mathcal{U}$

The following examples were jointly developed with G. Nickel and F. Räbiger.

Fix $M > 1$ and let $d := \frac{1}{2} + \frac{1}{2M}$. Set $J := [0, d]$ and let $h : J \to \mathbb{R}$ be defined by $h(\xi) = 1$ for $0 \le \xi < \frac{1}{2}$ and $h(\xi) = M$ for $\frac{1}{2} \le \xi \le d$. Set $\mu := h\lambda$, where λ is the

Lebesgue measure on J. Then (J, μ) is a probability space and there exists the product space $(\Omega, \nu) := \bigotimes_{n\in\mathbb{N}}(J, \mu)$ and the space $X := L^1(\Omega, \nu)$. On X we define for each $k \in \mathbb{N}$ a nilpotent right–translation semigroup by

$$\left(e^{\tau A_k} f\right)(\xi_1, \cdots) := \begin{cases} f(\xi_1, \cdots, \xi_{k-1}, \xi_k - \tau, \xi_{k+1}, \cdots), & \xi_k - \tau \in J, \\ 0, & \xi_k - \tau \notin J. \end{cases}$$

The following lemma shows that these semigroups are strongly continuous. A proof can be found in [25, Lemma 3.1] or [33, Lemma 3.9]. We denote the generator of $(e^{\tau A_k})_{\tau\geq 0}$ by $(A_k, D(A_k))$. Formally, we have $A_k f = -\frac{\partial}{\partial \xi_k} f$.

Lemma 5.1. *Let X and $(e^{\tau A_k})_{\tau\geq 0}$, $k \in \mathbb{N}$, be defined as above. Then we have:*

(a) *The semigroups $(e^{\tau A_k})_{\tau\geq 0}$ are strongly continuous and positive, and commute pairwise. Moreover, $\|e^{\tau A_k}\| \leq M$ for $\tau \geq 0$ and $e^{\tau A_k} = 0$ for $\tau \geq d$.*
(b) *We have $\|e^{\tau_n A_n} \cdots e^{\tau_m A_m}\| = M^{n-m+1}$ for $0 < \tau_k \leq \frac{1}{2M}$ and $m \leq k \leq n$.*
(c) *The space $Y := \{f \in \cap_{k\in\mathbb{N}} D(A_k) : \sup_{k\in\mathbb{N}} \|A_k f\| < \infty\}$ endowed with the norm $\|f\|_Y := \max\{\|f\|, \sup_k \|A_k f\|\}$ is continuously and densely embedded in X.*

We remark that the first example can easily be changed into $\sup_{0\leq t<t_0} \|U(t, 0)\| = \infty$ for some $0 < t_0 < \infty$, see [25]. Further, observe that the proofs of Example 5.2 and 5.3 yield the following result: For each given sequence $(c_n) \subseteq \mathbb{R}_+$ there are operators $A(t)$, $t \geq 0$, fulfilling the assumptions stated below such that the corresponding evolution family satisfies $\|U(n + 1, n)\| \geq c_n$ for $n \in \mathbb{N}$.

Example 5.2. *There is a family $(A(t))_{t\geq 0}$ of generators of commuting, positive $\mathcal{C}_0$–semigroups $(e^{\tau A(t)})_{\tau\geq 0}$ on X such that $\|e^{\tau A(t)}\| \leq M$ and $Y \subseteq D(A(t))$ for $t, \tau \geq 0$, $A(\cdot) \in C_b^1(\mathbb{R}_+, \mathcal{L}(Y, X))$, the corresponding Cauchy problem (CP) is well–posed on Y, and $s(A(t)) = -\infty$, $t \geq 0$, except for a sequence (t_k) where $A(t_k) = 0$. However, the corresponding evolution family $\mathcal{U}$ satisfies $\omega(\mathcal{U}) = \infty$.*

Proof. Let $(t_k)_{k\geq 0} := (0, 1, 2, \frac{5}{2}, 3, \frac{10}{3}, \frac{11}{3}, 4, \ldots)$. Take functions $\alpha_k \in C^1(\mathbb{R}_+)$ such that $\alpha_k(t) = \alpha_k'(t) = 0$ for $t \notin (t_{k-1}, t_k)$, $\alpha_k > 0$ on (t_{k-1}, t_k), and $\|\alpha_k\|_\infty$, $\|\alpha_k'\|_\infty \leq C$ for $k \in \mathbb{N}$. We set $a_k := \int_{t_{k-1}}^{t_k} \alpha_k(r)\, dr$. By choosing $C > 0$ sufficiently small, we can also assume that $a_k \leq \frac{1}{2M}$ for $k \in \mathbb{N}$. Now define $A(t) := \alpha_k(t)A_k$ and $D(A(t)) := D(A_k)$ for $t_{k-1} < t < t_k$ and $A(t_k) := 0$ for $k \in \mathbb{N}$. Due to Lemma 5.1, we only have to check well-posedness and the last assertion. For $t_{k-1} < s \leq t_k \leq t_l \leq t < t_{l+1}$ we set

$$U(t, s) := \left(e^{\int_{t_l}^t \alpha_{l+1}(r)dr\, A_{l+1}}\right)\left(e^{a_l A_l}\right) \cdots \left(e^{a_{k+1}A_{k+1}}\right)\left(e^{\int_s^{t_k} \alpha_k(r)dr\, A_k}\right). \tag{5.1}$$

This defines an evolution family $(U(t, s))_{t\geq s\geq 0}$ on X. Since the operators $e^{\tau A_k}$ commute, $\mathcal{U}$ leaves Y invariant and solves the Cauchy problem (CP) on Y. Finally, $\|U(n+1, n)\| \geq M^n$ for $n \in \mathbb{N}$ by Lemma 5.1(b). □

Example 5.3. *There is a family $(A(t))_{t\geq 0}$ of bounded, commuting operators on X such that $\|e^{\tau A(t)}\| \leq M$ for $t, \tau \geq 0$, $A(\cdot) \in C^1(\mathbb{R}_+, \mathcal{L}(X))$, and $\omega(\mathcal{U}) = \infty$, where $\mathcal{U}$ is the evolution family corresponding to $A(\cdot)$.*

Proof. Define t_k, α_k, and a_k for $k \in \mathbb{N}$ as in Example 5.2 . For $\lambda > 0$ we consider the Yosida approximations $A_{\lambda,k} := \lambda\, A_k\, R(\lambda, A_k)$. Note that we have $\|e^{\tau A_{\lambda,k}}\| \leq M$ for $\tau \geq 0$.

Moreover, finite products $\prod e^{a_k A_{\lambda,k}}$ converge strongly to $\prod e^{a_k A_k}$ as $\lambda \to \infty$. Therefore, by Lemma 5.1(b), there are positive numbers λ_n such that

$$\left\| \prod_{k=l+1}^{m} e^{a_k A_{\lambda_n,k}} \right\| \geq N^n \tag{5.2}$$

for a constant $N \in (1, M)$, where l and m are chosen such that $t_l = n$ and $t_m = n+1$. Now set $A(t) := \alpha_k(t)\, A_{\lambda_n,k}$ for $t_{k-1} \leq t < t_k$ and $n < t_k \leq n+1$. Clearly, the corresponding Cauchy problem (CP) is well-posed, and the solving evolution family $(U(t,s))_{t \geq s \geq 0}$ is given as in (5.1). However, from (5.2) follows that $\|U(n+1,n)\| \geq N^n$. □

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Mathematisches Institut, Universität Tübingen, Auf der Morgenstelle 10, 72076 Tübingen, Germany.

E-mail address: `rosc@michelangelo.mathematik.uni-tuebingen.de`