THE NON–AUTONOMOUS KATO CLASS

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Abstract. We discuss singular time dependent absorption–excitation rates for second order parabolic equations. A suitable class, the non–autonomous Kato class, is defined for the heat equation in connection with the non–autonomous Miyadera perturbation theorem. We give sufficient conditions in terms of integrability for a potential to belong to the non–autonomous Kato class.

1. Introduction

The Kato class $\mathbf{K}$ of potentials $V_0 : \mathbb{R}^n \to \mathbb{R}$ was introduced in 1982 by M. Aizenman and B. Simon, [1], [10], to study Schrödinger operators $-(\Delta + V_0)$ for singular V_0. It is a special case of a Stummel class defined in terms of the fundamental solution of the Laplacian Δ whose importance for second order elliptic operators was realized by T. Kato in [5]. Aizenman and Simon proved that $\mathbf{K}$ coincides with the set of potentials having relative bound 0 with respect to Δ on $L^1(\mathbb{R}^n)$, [1, Thm. 4.5]. Moreover, $\mathbf{K}$ is closely related to the set of potentials which are Miyadera perturbations of the semigroup $(U(t))_{t\geq 0}$ on $L^1(\mathbb{R}^n)$ generated by Δ, see [6, 11, 12, 13]. More precisely, the *enlarged Kato class* is defined by

$$\hat{\mathbf{K}} := \{V_0 : \mathbb{R}^n \to \mathbb{C} : V_0 \text{ measurable}, \|V_0\|_U < \infty\}, \quad \text{where}$$

$$\|V_0\|_U := \sup_{f \in D(\Delta),\, \|f\|_1 \leq 1} \int_0^1 \|V_0 U(t) f\|_1 \, dt.$$

(To be precise, in [6, 12, 13] the class $\hat{\mathbf{K}}$ was introduced for real potentials V_0. However, the results of these papers which are used below carry over to complex-valued V_0 by considering $|V_0|$.) We remark that $(\hat{\mathbf{K}}, \|\cdot\|_U)$ is a Banach space, see e.g. [13]. For $V_0 \in \hat{\mathbf{K}}$ and $\alpha > 0$, let

$$c'_\alpha(V_0) := \sup_{f \in D(\Delta),\, \|f\|_1 \leq 1} \int_0^\alpha \|V_0 U(t) f\|_1 \, dt \quad \text{and}$$

$$c(V_0) := \lim_{\alpha \to 0} c'_\alpha(V_0) = \inf_{\alpha > 0} c'_\alpha(V_0),$$

cf. [6, 13]. Then the *Kato class* is given by

$$\mathbf{K} = \{V_0 \in \hat{\mathbf{K}} : c(V_0) = 0\},$$

[12, Prop. 4.7, 5.1]. In particular, if $c(V_0) < 1$ then $\Delta + V_0$ with domain $D(\Delta)$ generates a strongly continuous semigroup on $X = L^1(\mathbb{R}^n)$ by virtue of the Miyadera perturbation theorem, see e.g. [11].

*Support by the Deutsche Forschungsgemeinschaft is gratefully acknowledged.

†Partly supported by the Deutsche Forschungsgemeinschaft.

1991 *Mathematics Subject Classification.* 35K10, 47A55, 47D06.

In the present note, we investigate perturbations of the Laplacian by time dependent potentials

$$V : [0,\infty) \times \mathbb{R}^n \to \mathbb{C}.$$

By $V(t)$ we denote the multiplication operator with maximal domain induced by $V(t,\cdot)$ on $L^1(\mathbb{R}^n)$. As in the autonomous case, we set

$$\|V(\cdot)\|_U := \sup_{s\geq 0} \sup_{f\in D(\Delta),\, \|f\|_1\leq 1} \int_0^1 \|V(s+t)U(t)f\|_1 \, dt,$$

and define the *non–autonomous enlarged Kato class* by

$$\widehat{\mathbf{NK}} = \{V : [0,\infty) \times \mathbb{R}^n \to \mathbb{C} : V \text{ measurable}, \|V(\cdot)\|_U < \infty\}.$$

Observe that $\|V(\cdot)\|_U = \|V_0\|_U$ for a time independent potential $V(\cdot) = V_0$. We further introduce

$$\begin{aligned} c'_\alpha(V(\cdot)) &:= \sup_{s\geq 0} \sup_{f\in D(\Delta),\, \|f\|_1\leq 1} \int_0^\alpha \|V(s+t)U(t)f\|_1 \, dt \quad \text{and} \\ c(V(\cdot)) &:= \lim_{\alpha\to 0} c'_\alpha(V(\cdot)) = \inf_{\alpha>0} c'_\alpha(V(\cdot)), \end{aligned}$$

for $V \in \widehat{\mathbf{NK}}$ and $\alpha > 0$. We remark that, for $V \in \widehat{\mathbf{NK}}$, the quantity $c'_\alpha(V(\cdot))$ does not change if we replace "$f \in D(\Delta)$" by "$f \in L^1(\mathbb{R}^n)$" in the second supremum, see [8, Thm. 3.4].

Recently, the Miyadera perturbation theorem has been extended to the non–autonomous situation in [7] and [8]. This result is applied in Theorem 1 to obtain mild solutions of the parabolic problem

$$\begin{aligned} \frac{\partial}{\partial t} u(t,x) &= \sum_{k,l=1}^n \frac{\partial}{\partial x_k} a_{kl}(t,x) \frac{\partial}{\partial x_l} u(t,x) + V(t,x)u(t,x), \\ u(s,x) &= f(x), \quad t\geq s,\ x\in\mathbb{R}^n, \end{aligned} \tag{1}$$

provided that V satisfies an estimate similar to $c(V(\cdot)) < 1$. In the remainder of the paper we investigate the class $\widehat{\mathbf{NK}}$. In Theorem 2 we see that the estimate $c(V(\cdot)) < q$ is equivalent to an integrability condition on V which involves the heat kernel

$$K_t(x) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}} \qquad (t>0,\ x\in\mathbb{R}^n).$$

Recall that $U(t)f = K_t * f$. Further, it is shown in Theorem 6 that Hölder continuity and boundedness of $t \mapsto V(t) \in \hat{\mathbf{K}}$ imply $V \in \widehat{\mathbf{NK}}$. By means of examples, we first see that continuity is not sufficient for this conclusion and that, second, boundedness in $\hat{\mathbf{K}}$ is not necessary for $V \in \widehat{\mathbf{NK}}$.

2. Results

First, we study the problem (1) in the space $L^1(\mathbb{R}^n)$, where we assume that

(A) $a_{kl} \in L^\infty([0,\infty)\times\mathbb{R}^n,\mathbb{R})$ and $\sum_{k,l=1}^n a_{kl}(t,x)\eta_k\eta_l \geq \mu\,|\eta|^2$ for $x,\eta\in\mathbb{R}^n$, $t\geq 0$, $k,l = 1,\cdots,n$, and a constant $\mu > 0$.

It is known that there exists a unique weak fundamental solution

$$\Gamma : \{(t,x,s,y) \in [0,\infty)\times\mathbb{R}^n\times[0,\infty)\times\mathbb{R}^n : t\geq s\} \to [0,\infty)$$

of (1) with $V = 0$ which satisfies a *Gaussian estimate*

$$0 \le \Gamma(t,x,s,y) \le M\, K_{a(t-s)}(x-y) \qquad (x,y \in \mathbb{R}^n,\ t > s \ge 0) \tag{2}$$

for some constants $M, a > 0$, see [2, Thm. 10], [3, Thm. 5.3], and [4, Thm. 1.6]. In [9, Lemma 5.1] it is shown that

$$(U(t,s)f)(x) := \int_{\mathbb{R}^n} \Gamma(t,x,s,y) f(y)\, dy \;\text{ for } t > s \ge 0 \;\text{ and }\; U(s,s) = Id \;\text{ for } s \ge 0$$

defines an *evolution family* on $L^1(\mathbb{R}^n)$. This means that

$$(t,s) \mapsto U(t,s) \in \mathcal{L}(L^1(\mathbb{R}^n)) \quad \text{is strongly continuous for } t \ge s \ge 0 \quad \text{and}$$
$$U(t,s) = U(t,r)U(r,s) \text{ and } U(s,s) = Id \;\text{ for } t \ge r \ge s \ge 0.$$

Notice that the estimate (2) yields $0 \le U(t,s) \le M\, U(a(t-s))$. We call a function $u \in C([s,\infty), L^1(\mathbb{R}^n))$ a *mild solution* of the problem (1) if $V(\cdot)u(\cdot) \in L^1_{loc}([s,\infty), L^1(\mathbb{R}^n))$ and

$$u(t) = U(t,s)f + \int_s^t U(t,r)V(r)u(r)\, dr$$

for all $t \ge s$. Let $V_a(t) = V(t/a)$ for $a > 0$.

Theorem 1. *Assume that (A) holds and that $V : [0,\infty) \times \mathbb{R}^n \to \mathbb{C}$ satisfies $c(V_a(\cdot)) < a/M$, where $M, a > 0$ are given by (2). Then there is a unique evolution family $W(t,s)$, $t \ge s \ge 0$, on $L^1(\mathbb{R}^n)$ satisfying*

$$W(t,s)f = U(t,s)f + \int_s^t W(t,\tau)V(\tau)U(\tau,s)f\, d\tau$$

for $f \in L^1(\mathbb{R}^n)$ and $t \ge s \ge 0$. Further, $W(\cdot,s)f$ is the unique mild solution of (1) for all $f \in L^1(\mathbb{R}^n)$.

Proof. Let $c(V_a(\cdot))\, \frac{M}{a} < q < 1$. Then we derive

$$\begin{aligned}\int_0^\alpha \|V(s+t)U(s+t,s)f\|_1\, dt &\le \frac{M}{a} \int_0^{a\alpha} \|V_a(as+\tau)U(\tau)\,|f|\,\|_1\, d\tau \\ &\le \frac{M}{a}\, c'_{a\alpha}(V_a(\cdot))\, \|f\|_1 \le q\, \|f\|_1\end{aligned}$$

for $s \ge 0$, $f \in L^1(\mathbb{R}^n)$, and sufficiently small $\alpha > 0$. Except for uniqueness, the assertion now follows from the non–autonomous Miyadera perturbation theorem [8, Thm. 3.4].

In order to show the uniqueness of mild solutions, let u be a mild solution of (1) with $u(s) = 0$. Take $\alpha > 0$ such that $c'_\alpha(V_a(\cdot))\, \frac{M}{a} \le q < 1$. The function $v = V(\cdot)u(\cdot) \in L^1_{loc}([s,\infty), L^1(\mathbb{R}^n))$ satisfies

$$u(t) = \int_s^t U(t,r)v(r)\, dr \quad \text{and} \quad v(t) = V(t) \int_s^t U(t,r)v(r)\, dr \tag{3}$$

for $t \ge s$. The second identity implies

$$\int_s^{s+\alpha} \|v(r)\|_1\, dr \le \int_s^{s+\alpha} \int_\tau^{s+\alpha} \|V(r)U(r,\tau)v(\tau)\|_1\, dr\, d\tau \le q \int_s^{s+\alpha} \|v(r)\|_1\, dr.$$

Therefore, we have $v(t) = 0$ for a.e. $t \in [s, s+\alpha]$, and so $u(t) = 0$ for $s \le t \le s+\alpha$ due to (3). Continuing to the right, one obtains $u(t) = 0$ for $t \ge s$. ☐

We refer to [2] or [3] concerning related results for a smaller class of potentials V (see Corollary 3) and to [9] for negative V. In Theorem 2 we reformulate the condition $c(V(\cdot)) < 1$; a stronger condition was introduced in a similar context in [14], [15], [16].

We now investigate the quantity $c(V(\cdot))$ in more detail. Compare also [1, 5, 10, 12] concerning the autonomous case.

Theorem 2. *Let $V : [0,\infty) \times \mathbb{R}^n \to \mathbb{C}$ be measurable,*

$$M(\alpha,\beta) := \sup_{s\geq 0} \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} \int_s^{s+\alpha} \int_{|x-y|\leq\beta} |V(t,y)|\, K_{t-s}(x-y)\, dy\, dt$$

for $\alpha,\beta > 0$. Then $c(V(\cdot)) = \inf_{\alpha,\beta>0} M(\alpha,\beta)$.

Proof. Notice that we have

$$\int_0^\alpha \|V(s+t)U(t)f\|_1\, dt = \int_{\mathbb{R}^n} f(x) \int_s^{s+\alpha} \int_{\mathbb{R}^n} |V(t,y)|\, K_{t-s}(x-y)\, dy\, dt\, dx$$

for $0 \leq f \in L^1(\mathbb{R}^n)$, and that therefore

$$c'_\alpha(V(\cdot)) = \sup_{s\geq 0} \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} \int_s^{s+\alpha} \int_{\mathbb{R}^n} |V(t,y)|\, K_{t-s}(x-y)\, dy\, dt. \tag{4}$$

This implies $M(\alpha,\beta) \leq c'_\alpha(V(\cdot))$ for all $\beta > 0$ and $\inf_{\alpha,\beta>0} M(\alpha,\beta) \leq c(V(\cdot))$.

In order to prove the converse implication, we choose $q > \inf_{\alpha,\beta>0} M(\alpha,\beta)$ and $\alpha',\beta > 0$ such that

$$M(\alpha',\beta) < q; \tag{5}$$

without restriction $\alpha' \leq 1$. For $0 < \alpha \leq \alpha'$, $s \geq 0$, and $x \in \mathbb{R}^n$, we estimate

$$\begin{aligned} q &\geq \sup_{s\geq 0} \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} \int_s^{s+\alpha} \int_{|y-x|\leq\frac{\beta}{2}} |V(t,y)|\, (4\pi(t-s))^{-\frac{n}{2}}\, e^{-\frac{|y-x|^2}{4(t-s)}}\, dy\, dt \\ &\geq \sup_{s\geq 0} \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} \int_0^\alpha (4\pi t)^{-\frac{n}{2}}\, e^{-\frac{\beta^2}{16t}} \int_{|y|\leq\frac{\beta}{2}} |V(s+t,x+y)|\, dy\, dt, \end{aligned} \tag{6}$$

$$\begin{aligned} &\int_s^{s+\alpha} \int_{|y-x|\geq\beta} |V(t,y)|\, K_{t-s}(y-x)\, dy\, dt \\ &\quad\leq \int_0^\alpha (4\pi t)^{-\frac{n}{2}}\, e^{-\frac{\beta^2}{8t}} \int_{|y|\geq\beta} |V(s+t,x+y)|\, e^{-\frac{|y|^2}{8}}\, dy\, dt \\ &\quad= C_1 \int_0^\alpha (4\pi t)^{-\frac{n}{2}}\, e^{-\frac{\beta^2}{8t}} \int_{|y|\geq\beta} \int_{|z-y|\leq\frac{\beta}{3}} |V(s+t,x+y)|\, e^{-\frac{|y|^2}{8}}\, dz\, dy\, dt \\ &\quad\leq C_1 \int_0^\alpha (4\pi t)^{-\frac{n}{2}}\, e^{-\frac{\beta^2}{8t}} \int_{|z|\geq\frac{2}{3}\beta} \int_{|y-z|\leq\frac{\beta}{3}} |V(s+t,x+y)|\, e^{-\frac{|y|^2}{8}}\, dy\, dz\, dt \\ &\quad\leq C_1 \int_0^\alpha (4\pi t)^{-\frac{n}{2}}\, e^{-\frac{\beta^2}{8t}} \int_{|z|\geq\frac{2}{3}\beta} e^{-\frac{|z|^2}{32}} \int_{|y|\leq\frac{\beta}{3}} |V(s+t,x+z+y)|\, dy\, dz\, dt \\ &\quad\leq C_1\, e^{-\frac{\beta^2}{16\alpha}} \int_{\mathbb{R}^n} e^{-\frac{|z|^2}{32}} \int_0^\alpha (4\pi t)^{-\frac{n}{2}}\, e^{-\frac{\beta^2}{16t}} \int_{|y|\leq\frac{\beta}{3}} |V(s+t,x+z+y)|\, dy\, dt\, dz \\ &\quad\leq C_2\, q\, e^{-\frac{\beta^2}{16\alpha}} \end{aligned} \tag{7}$$

with constants C_1 and C_2 (depending on β), where we have used (6) in the last step. Now (4), (5), and (7) imply

$$c'_\alpha(V(\cdot)) \le M(\alpha,\beta) + C_2\, q\, e^{-\frac{\beta^2}{16\alpha}} < q$$

for small $\alpha > 0$. □

As an easy consequence of the above characterization, we obtain a slight generalization of a well-known integrability condition for the existence of weak solutions of (1), see e.g. [2]. We use the space $L^p_{loc,u}(\Omega)$ of locally integrable functions f having a finite norm

$$\|f\|_{L^p_{loc,u}(\Omega)} = \sup_{x\in\Omega} \|\chi_{\Omega\cap B(x,1)} f\|_p\,,$$

where $\Omega \subseteq \mathbb{R}^n$, $B(x,1) = \{y \in \mathbb{R}^n : |x-y| \le 1\}$, and $1 \le p \le \infty$.

Corollary 3. *Let* $1 \le p, q \le \infty$, $\frac{1}{q} + \frac{n}{2p} < 1$, *and* $V \in L^q_{loc,u}([0,\infty), L^p_{loc,u}(\mathbb{R}^n))$. *Then* $c(V(\cdot)) = 0$.

Proof. Notice that $\|K_t\|_{p'} = C\, t^{-\frac{n}{2p}}$ for $\frac{1}{p} + \frac{1}{p'} = 1$ and a constant C depending on n and p. Therefore Hölder's inequality yields

$$\begin{aligned}\int_0^\alpha \int_{|y|\le 1} |V(s+t, x-y)|\, K_t(y)\, dy\, dt &\le C \int_0^\alpha t^{-\frac{n}{2p}}\, \|V(s+t)\|_{L^p_{loc,u}(\mathbb{R}^n)}\, dt \\ &\le C_1\, \|V\|\, \alpha^{\frac{1}{q'} - \frac{n}{2p}},\end{aligned}$$

where $\|V\|$ denotes the norm of V in $L^q_{loc,u}([0,\infty), L^p_{loc,u}(\mathbb{R}^n))$. □

Remark 4. For $q = \infty$, Corollary 3 yields the condition $p > \frac{n}{2}$ which is the usual integrability condition in the t–independent case; cf. [1, Prop. 4.3], [10]. In the other extreme case, $p = \infty$, we get the condition $q > 1$. It is also easy to show that

$$L^1([0,\infty), L^\infty(\mathbb{R}^n)) \subseteq \mathbf{NK} := \{V \in \widehat{\mathbf{NK}} : c(V(\cdot)) = 0\}.$$

We now discuss the relationship between the classes $\widehat{\mathbf{NK}}$ and $\hat{\mathbf{K}}$. Our first observation indicates that $\widehat{\mathbf{NK}}$ is an appropriate extension of $\hat{\mathbf{K}}$ to the non–autonomous situation.

Remark 5. Let $E = L^1([0,\infty), L^1(\mathbb{R}^n)) \cong L^1([0,\infty) \times \mathbb{R}^n)$. It is easy to see that the operator $L = -\frac{d}{dt} + \Delta$ defined on

$$\{g \in W^{1,1}([0,\infty), L^1(\mathbb{R}^n)) : g(0) = 0,\ g(t) \in D(\Delta) \text{ for a.e. } t \ge 0,\ \Delta g \in E\}$$

has a closure G in E which generates a positive semigroup $T(\cdot)$; namely the evolution semigroup on E corresponding to the semigroup $U(\cdot)$, compare [7, pp. 525]. Let $V : [0,\infty) \times \mathbb{R}^n \to \mathbb{C}$ be measurable. By the proof of [8, Thm. 3.4] the potential V belongs to $\widehat{\mathbf{NK}}$ if and only if

$$\sup_{g\in E,\, \|g\|\le 1} \int_0^1 \|VT(t)g\|_E\, dt < \infty.$$

The latter condition is equivalent to the G–boundedness of V due to [12, Prop. 4.7]. In other words, $\widehat{\mathbf{NK}}$ is the class of potentials which are relatively bounded with respect to the closure of the parabolic operator L in $L^1([0,\infty) \times \mathbb{R}^n)$, whereas $\hat{\mathbf{K}}$ is the space of Δ–bounded potentials $V_0 : \mathbb{R}^n \to \mathbb{C}$ by [12, Prop. 5.1].

Theorem 6. *Assume that* $[0,\infty) \ni s \mapsto V(s) \in \hat{\mathbf{K}}$ *is Hölder continuous and that* $c'_\alpha(V(s)) \le q$ *for all* $s \ge 0$ *and some* $\alpha > 0$. *Then* $c(V(\cdot)) \le q$.

Proof. Due to [12, Prop. 4.7, 5.1], the assumption yields

$$\|(V(s+t)-V(s))(1-\Delta)^{-1}\| \le C\, t^{\beta}$$

for $t,s \ge 0$ and some constants $C \ge 0$ and $0 < \beta \le 1$. Recall that $\|\Delta U(t)\| \le \frac{c}{t}$ for a suitable constant c. Thus,

$$\begin{aligned}\int_0^{\alpha} \|V(s+t)U(t)f\|_1\,dt &\le \int_0^{\alpha} t^{\beta}\,\|(1-\Delta)U(t)f\|_1\,dt + \int_0^{\alpha}\|V(s)U(t)f\|_1\,dt \\ &\le (C_1\,(\alpha^{\beta}+\alpha^{\beta+1})+q)\,\|f\|_1\end{aligned}$$

for a constant C_1. This implies the assertion. □

The next two examples show that neither of the spaces $\widehat{\mathbf{NK}}$ and $C_b([0,\infty),\hat{\mathbf{K}})$ is included in the other.

Example 7. Set $V(t,x) = \varphi(t)V_0(x)$ for $\varphi \in L^1([0,\infty))$ and $V_0 \in L^{\infty}(\mathbb{R}^n)$. Then $c(V(\cdot)) = 0$, but $V \in C_b([0,\infty),\hat{\mathbf{K}})$ if and only if $\varphi \in C_b([0,\infty))$.

Example 8. Let $n \ge 3$. The potential

$$V(t,x) = \begin{cases} \|K_t\|_U^{-1}\,K_t(x), & t \ge 1/e,\ x\in\mathbb{R}^n, \\ (|\ln t|\ \|K_t\|_U)^{-1}\,K_t(x), & 0 < t \le 1/e,\ x \in \mathbb{R}^n, \\ 0, & t = 0,\ x\in\mathbb{R}^n, \end{cases}$$

belongs to $C_b([0,\infty),\hat{\mathbf{K}})$, but not to $\widehat{\mathbf{NK}}$.

Proof. The first assertion is clear. From [12, Prop. 4.7] we derive by a change of variables that

$$\begin{aligned}\|K_t\|_U &= \left\| K_t \int_0^1 U(s)\,ds \right\|_{\mathcal{L}(L^1(\mathbb{R}^n))} \\ &= \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} \tfrac{1}{4}\,\pi^{-\frac{n}{2}} \int_{\mathbb{R}^n} K_t(x-y)\,|y|^{2-n} \int_{|y|^2/4}^{\infty} \tau^{\frac{n}{2}-2}\,e^{-\tau}\,d\tau\,dy \\ &\le C_1 \int_{\mathbb{R}^n} K_t(y)\,|y|^{2-n}\,dy \\ &= C_2\,t^{-\frac{n}{2}} \int_0^{\infty} r^{n-1}\,r^{2-n}\,e^{-\frac{r^2}{4t}}\,dr \\ &= 2C_2\,t^{1-\frac{n}{2}} \int_0^{\infty} e^{-r}\,dr = C_3\,t^{1-\frac{n}{2}}\,.\end{aligned}$$

As a consequence,

$$\begin{aligned}\int_0^{1/e}\int_{\mathbb{R}^n} V(t,y)\,K_t(y)\,dy\,dt &\ge \int_0^{1/e} \frac{t^{\frac{n}{2}}}{C_3\,t\,|\ln t|} \int_{\mathbb{R}^n} K_t(y)^2\,dy\,dt \\ &= C_4 \int_0^{1/e} \frac{dt}{t\,|\ln t|} = \infty.\end{aligned}$$

Using (4), this easily implies $V \notin \widehat{\mathbf{NK}}$. □

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