Local wellposedness of Maxwell systems with retarded material laws on domains with interfaces

Christopher Bresch, Roland Schnaubelt

Department of Mathematics, Karlsruhe Institute of Technology
P.O. Box 6980, 76049 Karlsruhe, Germany
Email: christopher.bresch@kit.edu, schnaubelt@kit.edu

We study a Maxwell system on a bounded domain consisting of two subdomains with different materials, separated by an interface. The nonlinear constitutive relations are nonlocal in time and of a form commonly used in nonlinear optics. We develop a comprehensive local wellposedness theory for solutions belonging to the Sobolev space $\mathcal{H}^k$ on each subdomain, where $k \geq 2$ is an integer. The blow-up condition only involves the sup-norm. A main tool are div-curl estimates.

1 Introduction

The Maxwell equations form the mathematical basis for the theory of electromagnetism. In matter, they have to be supplemented by constitutive relations modeling the response of the material to electric and magnetic fields. We treat polynomial nonlinear material laws with retardation, which are frequently used in the field of nonlinear optics, see [6] or [9], for example. Assuming time differentiablity of the coefficients in the nonlinearity, the Maxwell equations can be regarded as a retarded semilinear evolution equation. In such problems one has to prescribe the history of the fields for $t \leq 0$ as the prehistory.

In the present work, we study the the equations on a bounded domain G consisting of two subdomains G_1 and G_2 with different materials, separated by an interface Σ, having a positive distance to the boundary ∂G. The retardations may affect both the electric and magnetic fields, also cross effects between them are admitted. We establish

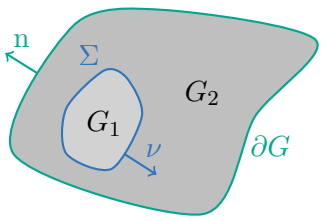

Figure 1: Schematic depiction of the domain.

a comprehensive local wellposedness theory for solutions with $\mathcal{H}^k$-regularity on each

Date: October 20, 2026
2020 *Mathematics Subject Classification*: 35Q61; 35B65, 35L60, 35R09, 78A48.
Keywords: Nonlinear Maxwell system, nonlocal material laws, interfaces, nonlinear optics.
Funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – Project-ID 258734477 – SFB 1173

subdomain, where $k \geq 2$ is an integer. The blow-up condition only depends on the sup-norm of solutions so that the maximal existence time is independent of k. The coefficients are required to be piecewise C^k on G_1 and G_2.

Instantaneous material laws assume that there is no time delay in the response. A local wellposedness theory for instantaneous and nonlinear material laws has been established in [22] and [19]. For retarded nonlinear problems on $\mathbb{R}^3$, existence and uniqueness of solutions in the Sobolev space $\mathcal{H}^k$ with $k \geq 2$ was shown in [5] using energy methods and the Banach algebra property of $\mathcal{H}^k$. This approach can be extended to $\mathcal{H}^s$ with $s > 3/2$. In [8] the regularity level has been lowered to $s \in \left(1, \frac{3}{2}\right]$ for scalar coefficients with the help of Strichartz estimates from [18]. Such estimates are not available on domains with interfaces. Also on domains, the wellposedness of linear Maxwell systems with retarded material laws is well known, see [17] and the references therein. In the recent work [11] the Maxwell interface problem with purely electric nonlinear retardations has been treated within the framework of evolutionary equations as introduced by Picard. (See also [16] for previous results with Lipschitz nonlinearities of different form.) Corollary 2.30 in [11] shows existence and uniqueness of piecewise $\mathcal{H}^2$ solutions on $\mathbb{R}$ in exponentially weighted L^2-spaces in time in a more abstract setting. As explained in Remark 2.14 therein and the text before it, this setting does not directly cover our problem. Examples 2.20 and 2.31 indicate the necessary modifications to obtain existence and uniqueness of local solutions. Continuous dependence on data, maximally defined solutions, or blow-up conditions are not treated in [11]. As in [11], we use div-curl estimates in our regularity analysis. In [11] also global existence for small data is shown under certain coercivity conditions on the material laws. We plan to treat such results in a further paper, cf. Chapter 10 of [7].

Using a unit system in which the physical constants ε_0 and μ_0 have the value 1, the macroscopic Maxwell equations are given by

$$\begin{aligned} \partial_t D &= \operatorname{curl} H - J, \quad \operatorname{div} D = \rho, \\ \partial_t B &= -\operatorname{curl} E, \qquad \operatorname{div} B = 0, \end{aligned} \qquad t \geq 0,\ x \in G. \tag{1.1}$$

Here E and H are the electric and magnetic fields, D is the electric displacement and B the magnetic induction. These fields are related by

$$D = E + P, \quad B = H + M,$$

via the polarization P and magnetization M which describe the density of electric, respectively magnetic dipoles in the medium. We refer to Section 6.6 of [13] for a discussion of the macroscopic Maxwell equations.

The source terms ρ and J are the density of the free charges, respectively the free current. On the boundary of G we impose perfectly conducting boundary conditions,

$$\mathrm{n} \times E = 0, \quad \mathrm{n} \cdot B = 0, \qquad t \geq 0,\ x \in \partial G, \tag{1.2}$$

where n denotes the outer unit normal vector on ∂G. We assume that there are no free surface currents on the interface Σ. However there may be a free surface charge density ρ_Σ on Σ. The Maxwell equations then imply the interface conditions

$$\begin{aligned} [\nu \cdot D]_\Sigma &= -\rho_\Sigma, \qquad [\nu \cdot B]_\Sigma = 0, \\ [\nu \times E]_\Sigma &= 0, \qquad\quad [\nu \times H]_\Sigma = 0, \end{aligned} \qquad t \geq 0,\ x \in \Sigma, \tag{1.3}$$

see Section I.5 in [13]. Here, ν stands for the unit normal vector on Σ, pointing outwards from G_1, and $[f]_\Sigma = (f|_{G_1} - f|_{G_2})|_\Sigma$ denotes the jump of a quantity f across the interface

Σ. Both (1.2) and (1.3) are understood in the sense of traces, as described in Lemma 2.3. It is known that the two divergence equations in (1.1) and the boundary condition for B in (1.2) as well as the interface conditions for D and B in (1.3) are fulfilled if they hold at the initial time, the remaining equations in (1.1), (1.2) and (1.3) are satisfied, and if the free bulk and surface charge densities are given by

$$\rho(t) = \rho_0 - \int_0^t \operatorname{div} J(s)\,\mathrm{d}s, \quad \rho_\Sigma(t) = \rho_{\Sigma,0} + \int_0^t [\nu \cdot J(s)]_\Sigma\,\mathrm{d}s,$$

where ρ_0 and $\rho_{\Sigma,0}$ denote the corresponding initial values, see Section 1 of [22] and Lemma 8.1 in [19]. This motivates the study of the reduced system

$$\begin{aligned} \partial_t D &= \operatorname{curl} H - J, && t \geq 0,\ x \in G, \\ \partial_t B &= -\operatorname{curl} E, && t \geq 0,\ x \in G, \\ \mathrm{n} \times E &= 0, && t \geq 0,\ x \in \partial G, \\ [\nu \times E]_\Sigma = [\nu \times H]_\Sigma &= 0, && t \geq 0,\ x \in \Sigma. \end{aligned} \tag{1.4}$$

We split $J = J_0 + \sigma E$ into an induced free current density σE with the conductivity σ of the medium, and an external free current density J_0 considered as a given inhomogeneity.

In this paper, $u = (E, H)$ is used as the variable describing the state of the system. We treat constitutive relations of the form

$$\begin{aligned} P(E,H)(t,x) &= \chi_\mathrm{e}(x)E(t,x) + P_\mathrm{ret}(E,H)(t,x), \\ M(E,H)(t,x) &= \chi_\mathrm{m}(x)H(t,x) + M_\mathrm{ret}(E,H)(t,x), \end{aligned}$$

The terms $\chi_\mathrm{e}E$ and $\chi_\mathrm{m}H$ describe a linear and instantaneous response characterized by the matrix valued electric and magnetic susceptibilities χ_e and χ_m. The permittivity ε and permeability μ of the medium are then given by $\varepsilon = I + \chi_\mathrm{e}$ and $\mu = I + \chi_\mathrm{m}$. In contrast, the contributions $P_\mathrm{ret}(E,H)$ and $M_\mathrm{ret}(E,H)$ contain both linear and nonlinear terms and include time retardations between the fields and the materials response. The model for P_ret and M_ret is basically the standard model from nonlinear optics [9], with the difference that we include a possible coupling between electric and magnetic effects. In optics, one often neglects magnetic effects and uses $P_\mathrm{ret} = P_\mathrm{ret}(E)$ and $M_\mathrm{ret} = 0$. We describe the model in Section 4. An example is the Kerr effect given by

$$P(E)(t,x) = \int_{-\infty}^t \int_{-\infty}^t \int_{-\infty}^t \kappa(t-s_1, t-s_2, t-s_3, x)\big(E(s_1,x)\cdot E(s_2,x)\big)E(s_3,x)\,\mathrm{d}s_1\,\mathrm{d}s_2\,\mathrm{d}s_3$$

in isotropic materials, where κ is a scalar function, cf. Section 4.1 of [6]. Because of this third-order effect the index of refraction to change depends on the intensity of a light beam travelling through the medium, roughly speaking. It gives rise to several phenomena with applications, for example in fiber optics [1].

Our approach is based on linearization and a fixed-point argument, in which the nonlinearity is treated as an inhomogeneity. In Section 2, we state the assumptions on the domain and the coefficients ε, μ and σ and introduce the needed function spaces. We also discuss their relevant properties and state div-curl estimates for functions with inhomogeneous interface conditions, which were proven in [11] for the electric boundary conditions, see [24] for the homogenous case. In addition, the semigroup governing the time evolution of the instantaneous part of the Maxwell equations is introduced.

After these preparations, we study the linear problem in Section 3. Here the main result is Theorem 3.2, which yields existence and uniqueness of solutions with piecewise $\mathcal{H}^k$-regularity, where $k \geq 2$, provided the initial data and the inhomogeneity are piecewise $\mathcal{H}^k$-regular and satisfy certain compatibility conditions at time $t = 0$. These are invariant under the evolution and are necessary due to the interface and boundary conditions. For the application to the nonlienar problem it is crucial that we do not require interface or boundary conditions for inhomogeneity alone. The proof of Theorem 3.2 consists of two parts: We first use semigroup theory to establish a solution in L^2 possessing additional time regularity. Then div-curl estimates allow us to show higher spatial regularity in an intricate iteration argument, taking advantage of the structure of the Maxwell system.

Section 4 contains a description of the constitutive relations and a study of their relevant properties. By demanding and technical calculations, we derive estimates needed to treat the nonlinearity. In Section 5, we finally combine these estimates with the result for the linear problem in a fixed-point argument. We first obtain a solution on a small interval depending on the size of the initial data and inhomogeneity. This is then extended to a unique solution with piecewise $\mathcal{H}^k$ regularity, where $k \geq 2$, on a maximal existence interval using standard methods. We also show a blow-up condition involving only the L^∞-norm of the solution, which implies that the maximal existence time is independent of the regularity level k. With some effort, here we have to refine the analysis of Section 4 by means of Moser-type inequalities. In addition, continuous dependence of the solution on the initial data and inhomogeneity is proved. These findings are collected in Theorem 5.8, which is the main result of this paper.

2 Function spaces and the semigroup

We write $A \lesssim_\alpha B$ if $A \leq cB$ for some constant $c = c(\alpha) > 0$, $A \eqsim_\alpha B$ if $A \lesssim_\alpha B$ and $B \lesssim_\alpha A$, $|\alpha| := \sum_{i=1}^n \alpha_i$ for a multi-index $\alpha \in \mathbb{N}_0^n$, as well as $C_T^m Y := C^m\big([0,T], Y\big)$ for $T > 0$, $m \in \mathbb{N}_0$ and a Banach space Y, and analogously for $L_T^p Y$. The empty sum is considerded to be zero, and the term *domain* means an open and connected set. We state the assumptions on the domain and the material parameters used throughout. For $\ell \in \mathbb{N}_0$ we define the space

$$\mathcal{C}_\Sigma^\ell(G) := \big\{\alpha \colon G \to \mathbb{R}^{3\times 3} \;\big|\; \alpha|_{G_i} \text{ has an extension in } C^\ell(\overline{G_i})^{3\times 3},\ i \in \{1,2\}\big\}$$

with its canonical norm, as well as the subspace

$$\mathcal{C}_{\Sigma,\mathrm{sym}}^\ell(G) := \big\{\alpha \in \mathcal{C}_\Sigma^\ell(G) \;\big|\; \alpha \text{ is symmetric } \big\}.$$

Assumption 2.1. *Let $k \in \mathbb{N}$ with $k \geq 2$.*

(i) The domain $G \subseteq \mathbb{R}^3$ is bounded and can be decomposed into disjoint subdomains G_1 and G_2 and an interface Σ such that $G = G_1 \cup G_2 \cup \Sigma$, where $\Sigma := \partial G_1$ and ∂G are both of class C^{k+1} with $\operatorname{dist}(\Sigma, \partial G) > 0$.

(ii) The matrices ε and μ are contained in $\mathcal{C}_{\Sigma,\mathrm{sym}}^k(G)$ and σ belongs to $\mathcal{C}_\Sigma^k(G)$. Furthermore there exists a constant $\eta > 0$ such that $\xi \cdot \big(\varepsilon(x)\,\xi\big) \geq \eta\,|\xi|^2$ and $\xi \cdot \big(\mu(x)\,\xi\big) \geq \eta\,|\xi|^2$ hold for all $\xi \in \mathbb{R}^3$ and $x \in G \setminus \Sigma$.

We note that the above assumption on ε and μ implies that the inverse matrices exist and have the same $\mathcal{C}_\Sigma^k$-regularity.

For open sets $M \subseteq \mathbb{R}^3$ and the distributional curl and div we define the spaces

$$\mathcal{H}(\operatorname{curl}, M) := \big\{u \in L^2(M)^3 \;\big|\; \operatorname{curl} u \in L^2(M)^3\big\},$$
$$\mathcal{H}(\operatorname{div}, M) := \big\{u \in L^2(M)^3 \;\big|\; \operatorname{div} u \in L^2(M)\big\},$$

equipped with their (Hilbertian) graph norms, and the closed subpaces

$$\mathcal{H}_0(\operatorname{curl}, M) := \overline{C_c^\infty(M)^3}^{\mathcal{H}(\operatorname{curl},M)}, \quad \mathcal{H}_0(\operatorname{div}, M) := \overline{C_c^\infty(M)^3}^{\mathcal{H}(\operatorname{div},M)}.$$

The spaces

$$X := L^2(G)^6, \quad \mathcal{H}^\ell_\Sigma(G)^d := \big\{w \in L^2(G)^d \;\big|\; w\big|_{G_i} \in \mathcal{H}^\ell(G_i)^d, i = 1, 2\big\},$$

are endowed with the scalar products

$$\big((u, v) \;\big|\; (f, g)\big)_X := (\varepsilon u|v)_{L^2(G)^3} + (\mu f|g)_{L^2(G)^3}\,,$$
$$(u|v)_{\mathcal{H}^\ell_\Sigma(G)^d} := (u|v)_{\mathcal{H}^\ell(G_1)^d} + (u|v)_{\mathcal{H}^\ell(G_2)^d}\,,$$

where $\ell, d \in \mathbb{N}$. Due to the assumptions on ε and μ, the norm on X is equivalent to the usual L^2-norm. We set $\mathcal{H}^0_\Sigma(G)^d := L^2(G)^d$ and often omit the domain G and the superscript d when referring to function spaces. The space $\mathcal{H}^2_\Sigma(G)$ is a Banach algebra and we have the embeddings $\mathcal{H}^2_\Sigma(G) \hookrightarrow L^\infty(G)$ and $\mathcal{H}^1_\Sigma(G) \hookrightarrow L^6(G) \hookrightarrow L^4(G)$.

Note that the restrictions u_i of $u \in \mathcal{H}(\operatorname{curl}, G)$ to G_i belong to $\mathcal{H}(\operatorname{curl}, G_i)$ and satisfy $\operatorname{curl} u\big|_{G_i} = \operatorname{curl} u_i$. Moreover by Assumption 2.1, the product and inversion rules imply that multiplication with $\alpha \in \{\varepsilon, \mu\}$ and α^{-1} leaves invariant $\mathcal{H}^\ell_\Sigma(G)^3$ for $\ell \le k$ and

$$\|\alpha u\|_{\mathcal{H}^\ell_\Sigma} \lesssim \|\alpha\|_{\mathcal{C}^k_\Sigma} \|u\|_{\mathcal{H}^\ell_\Sigma}\,, \quad \big\|\alpha^{-1} u\big\|_{\mathcal{H}^\ell_\Sigma} \le C \|u\|_{\mathcal{H}^\ell_\Sigma}\,, \tag{2.1}$$

where $C = C\big(\|\alpha\|_{\mathcal{C}^k_\Sigma}, \eta^{-1}\big) > 0$ is nondecreasing in $\|\alpha\|_{\mathcal{C}^k_\Sigma}$ and η^{-1}.

To deal with products of field components appearing in the constitutive relations for P_{ret} and M_{ret}, we use several product rules stated in the next lemma (for which the assumptions on the regularity of the boundary and the interface could be relaxed). The last two statements are variants of a Moser-type inequality proved in Lemma 7.19 of [21].

Lemma 2.2. *Let Assumption 2.1 be true, $\ell, j, m \in \mathbb{N}_0$ satisfy $j, \ell \le m$, $m \ge 2$, and $I \subseteq \mathbb{R}$ be an interval.*

(i) Let $v \in \mathcal{H}^j_\Sigma(G)$ and $w \in \mathcal{H}^m_\Sigma(G)$. Then $vw \in \mathcal{H}^j_\Sigma(G)$ and we have

$$\|vw\|_{\mathcal{H}^j_\Sigma} \lesssim_{m,j,G} \|v\|_{\mathcal{H}^j_\Sigma} \|w\|_{\mathcal{H}^m_\Sigma}\,. \tag{2.2}$$

(ii) Let $v, w \in \mathcal{H}^1_\Sigma(G)$. Then $vw \in L^2(G)$ and we have

$$\|vw\|_{L^2} \lesssim_G \|v\|_{\mathcal{H}^1_\Sigma} \|w\|_{\mathcal{H}^1_\Sigma}\,.$$

(iii) Let $v \in C^1_b\big(I, \mathcal{H}^j_\Sigma(G)\big)$ and $w \in C^1_b\big(I, \mathcal{H}^m_\Sigma(G)\big)$. Then $vw \in C^1_b\big(I, \mathcal{H}^j_\Sigma(G)\big)$ with $\partial_t(vw) = \partial_t vw + v\partial_t w$.

(iv) Let $v, w \in C^1_b\big(I, \mathcal{H}^1_\Sigma(G)\big)$. Then $vw \in C^1_b\big(I, L^2(G)\big)$ with $\partial_t(vw) = \partial_t vw + v\partial_t w$.

(v) Let $n \in \mathbb{N}$ and $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le \ell$ and $u_i \in \mathcal{H}^{m-\alpha_i}_\Sigma(G)$ for all $i \in \{1, \dots, n\}$. Then $\prod_{i=1}^n u_i \in \mathcal{H}^{m-\ell}_\Sigma(G)$ and we have

$$\Big\|\prod_{i=1}^n u_i\Big\|_{\mathcal{H}^{m-\ell}_\Sigma} \lesssim_{m,n,\ell,G} \prod_{i=1}^n \|u_i\|_{\mathcal{H}^{m-\alpha_i}_\Sigma}\,. \tag{2.3}$$

(vi) *Let* $m \in \mathbb{N}_0$, $n \in \mathbb{N}$ *and* $u_1, \dots, u_n \in L^\infty(G) \cap \mathcal{H}^m_\Sigma(G)$. *Then* $\prod_{i=1}^n u_i \in L^\infty(G) \cap \mathcal{H}^m_\Sigma(G)$ *and we have*

$$\Big\|\prod_{i=1}^n u_i\Big\|_{\mathcal{H}^m_\Sigma} \lesssim_{m,n,G} \sum_{i=1}^n \|u_i\|_{\mathcal{H}^m_\Sigma} \prod_{j=1, j\neq i}^n \|u_j\|_{L^\infty}. \tag{2.4}$$

(vii) *Let* $n \in \mathbb{N}$ *with* $n \geq 2$, $u_1, \dots, u_n \in \mathcal{H}^2_\Sigma(G)$ *and* $\alpha \in \mathbb{N}_0^3$ *with* $|\alpha| = 1$. *Then* $\partial^\alpha u_1 \prod_{j=2}^n u_j$ *is contained in* $\mathcal{H}^1_\Sigma(G)$ *and we have*

$$\Big\|\partial^\alpha u_1 \prod_{j=2}^n u_j\Big\|_{\mathcal{H}^1_\Sigma} \lesssim_G \sum_{i=1}^n \|u_i\|_{\mathcal{H}^2_\Sigma} \prod_{j=1, j\neq i}^n \|u_j\|_{L^\infty}. \tag{2.5}$$

Proof. Claims (i)–(v) are straightforward consequences of Sobolev's and Hölder's inequalities. In (vi), the case $n = 2$ is an easy consequence of the corresponding result on $\mathbb{R}^3$, see e.g. Lemma 7.19 in [21], and the Stein extension theorem. The general case is then shown by iteration. The proof of the last statement follows that of Lemma 7.19 in [21]. The Gagliardo–Nirenberg estimates and Stein's extension theorem lead to

$$\|\partial^\alpha w\|_{L^4(G_k)} \lesssim_G \|w\|^{\frac{1}{2}}_{L^\infty(G_k)} \|w\|^{\frac{1}{2}}_{\mathcal{H}^2(G_i)}, \quad k = 1, 2, \tag{2.6}$$

for all $w \in \mathcal{H}^2_\Sigma(G)$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| = 1$. Take $u_1, \dots, u_n \in \mathcal{H}^2_\Sigma(G)$ and $\alpha, \beta \in \mathbb{N}_0^3$ with $|\alpha| = |\beta| = 1$. The estimate

$$\Big\|\partial^\alpha u_1 \prod_{j=2}^n u_j\Big\|_{L^2} + \Big\|\partial^{\alpha+\beta} u_1 \prod_{j=2}^n u_j\Big\|_{L^2} \leq 2\, \|u_1\|_{\mathcal{H}^2_\Sigma} \prod_{j=2}^n \|u_j\|_{L^\infty} \tag{2.7}$$

is clear. For the remaining term, we first apply Hölder's inequality to obtain

$$\begin{aligned} \Big\|\partial^\alpha u_1 \partial^\beta \Big(\prod_{j=2}^n u_j\Big)\Big\|_{L^2} &\leq \sum_{i=2}^n \Big\|\partial^\alpha u_1 \partial^\beta u_i\Big\|_{L^2} \prod_{j=2,\, j\neq i}^n \|u_j\|_{L^\infty} \\ &\leq \sum_{i=2}^n \|\partial^\alpha u_1\|_{L^4} \Big\|\partial^\beta u_i\Big\|_{L^4} \prod_{j=2,\, j\neq i}^n \|u_j\|_{L^\infty}. \end{aligned} \tag{2.8}$$

Formula (2.6) and Young's inequality then imply

$$\begin{aligned} \|\partial^\alpha u_1\|_{L^4} \Big\|\partial^\beta u_i\Big\|_{L^4} &\lesssim_G \Big(\|u_1\|_{\mathcal{H}^2_\Sigma} \|u_i\|_{L^\infty}\Big)^{\frac{1}{2}} \Big(\|u_1\|_{L^\infty} \|u_i\|_{\mathcal{H}^2_\Sigma}\Big)^{\frac{1}{2}} \\ &\lesssim \|u_1\|_{\mathcal{H}^2_\Sigma} \|u_i\|_{L^\infty} + \|u_1\|_{L^\infty} \|u_i\|_{\mathcal{H}^2_\Sigma}. \end{aligned} \tag{2.9}$$

Equations (2.7), (2.8) and (2.9) yield (2.5). □

The boundary and interface conditions are made precise by means of trace operators. We write $C_c^\infty\left(\overline{M}\right) := \left\{u|_M \;\middle|\; u \in C_c^\infty(\mathbb{R}^3)\right\}$ for open sets $M \subseteq \mathbb{R}^3$. The nextresult follows by localization from Theorems 2.1.62, 2.2.18 and 2.2.24 in [3].

Lemma 2.3. *Let Assumption 2.1 hold and* $i = 1, 2$. *There exist linear bounded operators*

$$\mathrm{tr}_{\partial G}\colon \mathcal{H}^1(G) \to \mathcal{H}^{1/2}(\partial G), \quad \mathrm{tr}_\Sigma\colon \mathcal{H}^1(G) \to \mathcal{H}^{1/2}(\Sigma), \quad \mathrm{tr}_{\Sigma,i}\colon \mathcal{H}^1(G_i) \to \mathcal{H}^{1/2}(\Sigma),$$

$$\begin{aligned}
&\operatorname{tr}_{\Sigma,\mathrm{n},i}\colon \mathcal{H}(\operatorname{div},G_i)\to\mathcal{H}^{-1/2}(\Sigma), \quad \operatorname{tr}_{\Sigma,\mathrm{t},i}\colon \mathcal{H}(\operatorname{curl},G_i)\to\mathcal{H}^{-1/2}(\Sigma)^3,\\
&\operatorname{tr}_{\partial G,\mathrm{n}}\colon \mathcal{H}(\operatorname{div},G)\to\mathcal{H}^{-1/2}(\partial G), \quad \operatorname{tr}_{\partial G,\mathrm{t}}\colon \mathcal{H}(\operatorname{curl},G)\to\mathcal{H}^{-1/2}(\partial G)^3,\\
&\operatorname{tr}_{\partial G,\mathrm{t},2}\colon \mathcal{H}(\operatorname{curl},G_2)\to\mathcal{H}^{-1/2}(\partial G)^3,
\end{aligned}$$

which satisfy $\operatorname{tr}_{\partial G}\mathcal{H}^1(G)=\mathcal{H}^{1/2}(\partial G)$, $\operatorname{tr}_{\Sigma}\mathcal{H}^1_0(G)=\mathcal{H}^{1/2}(\Sigma)$ *and are given by*

$$\begin{aligned}
\operatorname{tr}_{\partial G} u = u|_{\partial G}, \quad \operatorname{tr}_{\Sigma} u = u|_{\Sigma} \quad &\text{for all } u\in C_c^\infty\left(\overline{G}\right),\\
\operatorname{tr}_{\Sigma,i} u = u|_{\Sigma} \quad &\text{for all } u\in C_c^\infty\left(\overline{G_i}\right),\\
\operatorname{tr}_{\Sigma,\mathrm{n},i} u = \nu\cdot u|_{\Sigma}, \quad \operatorname{tr}_{\Sigma,\mathrm{t},i} u = \nu\times u|_{\Sigma} \quad &\text{for all } u\in C_c^\infty\left(\overline{G_i}\right)^3,\\
\operatorname{tr}_{\partial G,\mathrm{n}} u = \mathrm{n}\cdot u|_{\partial G} \quad \operatorname{tr}_{\partial G,\mathrm{t}} u = \mathrm{n}\times u|_{\partial G} \quad &\text{for all } u\in C_c^\infty\left(\overline{G}\right)^3,\\
\operatorname{tr}_{\partial G,\mathrm{t},2} u = \mathrm{n}\times u|_{\partial G} \quad &\text{for all } u\in C_c^\infty\left(\overline{G_2}\right)^3.
\end{aligned}$$

To simplify the notation, we set $[u]_\Sigma := \operatorname{tr}_{\Sigma,1} u|_{G_1} - \operatorname{tr}_{\Sigma,2} u|_{G_2}$ for $u\in\mathcal{H}^1_\Sigma(G)$ as well as $[\nu\cdot u]_\Sigma := \operatorname{tr}_{\Sigma,\mathrm{n},1} u - \operatorname{tr}_{\Sigma,\mathrm{n},2} u$ for vector fields $u\in L^2(G)^3$ with $u|_{G_i}\in\mathcal{H}(\operatorname{div},G_i)$ and analogously $[\nu\times u]_\Sigma := \operatorname{tr}_{\Sigma,\mathrm{t},1} u - \operatorname{tr}_{\Sigma,\mathrm{t},2} u$ for $u\in L^2(G)^3$ with $u|_{G_i}\in\mathcal{H}(\operatorname{curl},G_i)$, where $i=1,2$. We also write $\mathrm{n}\cdot u|_{\partial G}$ and $\mathrm{n}\times u|_{\partial G}$ instead of $\operatorname{tr}_{\partial G,\mathrm{n}} u$ and $\operatorname{tr}_{\partial G,\mathrm{t}} u$ or $\operatorname{tr}_{\partial G,\mathrm{t},2} u$ for $u\in\mathcal{H}(\operatorname{div},G)$ respectively $u\in\mathcal{H}(\operatorname{curl},G)$ or $u\in\mathcal{H}(\operatorname{curl},G_2)$. The following statements are needed for the treatment of the interface and boundary conditions. See Propositions 2.2.30 and 2.2.32 as well as Theorems 2.2.18 and 2.2.24 in [3].

Lemma 2.4. *Let Assumption 2.1 be true. Then we have the characterizations*

$$\begin{aligned}
\mathcal{H}(\operatorname{div},G) &= \left\{u\in L^2(G)^3 \;\middle|\; u|_{G_i}\in\mathcal{H}(\operatorname{div},G_i),\ i=1,2,\ [\nu\cdot u]_\Sigma=0\right\},\\
\mathcal{H}_0(\operatorname{div},G) &= \left\{u\in\mathcal{H}(\operatorname{div},G) \;\middle|\; \mathrm{n}\cdot u|_{\partial G}=0\right\},\\
\mathcal{H}(\operatorname{curl},G) &= \left\{u\in L^2(G)^3 \;\middle|\; u|_{G_i}\in\mathcal{H}(\operatorname{curl},G_i),\ i=1,2,\ [\nu\times u]_\Sigma=0\right\},\\
\mathcal{H}_0(\operatorname{curl},G) &= \left\{u\in\mathcal{H}(\operatorname{curl},G) \;\middle|\; \mathrm{n}\times u|_{\partial G}=0\right\}.
\end{aligned}$$

A crucial tool in the treatment of the linear problem in Section 3 are div-curl estimates due to Weber [24]. We use the following extensions of these results. The first case (corresponding to the electric boundary condition $\mathrm{n}\times u=0$ on ∂G) is proved as Proposition 2.24 in [11] (with slightly different assumptions on the regularity of Σ, ∂G and ε). The second case (with the magnetic boundary condition $\mathrm{n}\cdot u=0$ on ∂G) can be shown analogously. To this aim, by means of a Helmholtz decomposition in a $L^2(G)$-space with weight ε (see Lemmas 3.8 and 3.9 in [23]) one reduces the problem a regularity theorem for solutions of elliptic transmission problems. The regularity theory for elliptic transmission problems is known and has been studied by several authors, see e.g. [4], [10], [14], but detailed proofs of the results are difficult to find, so we provide a proof of the needed results in Appendix A.2 of [7]. We remark that Proposition 2.5 also holds if one allows the case $k=1$ in Assumption 2.1.

Proposition 2.5. *Let Assumption 2.1 be true and let $\ell\in\mathbb{N}_0$ with $\ell<k$. There exists a constant $C=C(\ell,\|\varepsilon\|_{\mathcal{C}^{\ell+1}_\Sigma},\eta^{-1},G)>0$ nondecreasing in $\|\varepsilon\|_{\mathcal{C}^{\ell+1}_\Sigma}$ and η^{-1} such that the following holds. Let $v\in\mathcal{H}^{\ell+1}_\Sigma(G)^3$ and let $u\in L^2(G)^3$ satisfy either of the cases*

(i) $u\in\mathcal{H}_0(\operatorname{curl},G)$ and $\varepsilon u+v\in\mathcal{H}(\operatorname{div},G)$, or

(ii) $u \in \mathcal{H}(\operatorname{curl}, G)$ *and* $\varepsilon u + v \in \mathcal{H}_0(\operatorname{div}, G)$.

Furthermore, let $\operatorname{curl} u \in \mathcal{H}^\ell_\Sigma(G)^3$ *and* $\operatorname{div}\big(\varepsilon u|_{G_i}\big) \in \mathcal{H}^\ell(G_i)$ *for all* $i \in \{1,2\}$. *Then* u *is contained in* $\mathcal{H}^{\ell+1}_\Sigma(G)^3$ *and fulfills*

$$\|u\|_{\mathcal{H}^{\ell+1}_\Sigma(G)} \leq C\Big(\|u\|_{L^2(G)} + \|\operatorname{curl} u\|_{\mathcal{H}^\ell_\Sigma(G)} + \sum\nolimits_{i=1}^2 \big\|\operatorname{div}\big(\varepsilon u|_{G_i}\big)\big\|_{\mathcal{H}^\ell(G_i)} + \|v\|_{\mathcal{H}^{\ell+1}_\Sigma(G)}\Big).$$

We define the *Maxwell operator* by

$$A := \begin{pmatrix} 0 & \varepsilon^{-1}\operatorname{curl} \\ -\mu^{-1}\operatorname{curl} & 0 \end{pmatrix} \tag{2.10}$$

on $X = L^2(G)^6$ with dense domain

$$\mathrm{D}(A) := \mathcal{H}_0(\operatorname{curl}, G) \times \mathcal{H}(\operatorname{curl}, G).$$

We write X_1 for $\mathrm{D}(A)$ endowed with the graph norm

$$\|(u,v)\|_{X_1} := \|(u,v)\|_X + \|A(u,v)\|_X \eqsim_{\varepsilon,\mu,\eta} \|u\|_{\mathcal{H}(\operatorname{curl},G)} + \|v\|_{\mathcal{H}(\operatorname{curl},G)}.$$

Furthermore, the extrapolation space of A is denoted by X^A_{-1}, cf. Section II.5 in [12] Note that $X \hookrightarrow X^A_{-1}$ and A has an extension to $\mathcal{B}\left(X, X^A_{-1}\right)$ also denoted by A.

The boundary and interface conditions in (1.4) can be written compactly in the form $K_{\partial G,\Sigma}(E,H) = 0$ with the operator

$$\begin{aligned} K_{\partial G,\Sigma} : &\Big\{(E,H) \in L^2(G)^6 \;\Big|\; E|_{G_i}, H|_{G_i} \in \mathcal{H}(\operatorname{curl}, G_i),\ i = 1,2\Big\} \\ &\longrightarrow \mathcal{H}^{-1/2}(\partial G)^3 \times \mathcal{H}^{-1/2}(\Sigma)^3 \times \mathcal{H}^{-1/2}(\Sigma)^3, \\ K_{\partial G,\Sigma}(E,H) &= \Big(\mathrm{n} \times E|_{\partial G}, [\nu \times E]_\Sigma, [\nu \times H]_\Sigma\Big) \end{aligned} \tag{2.11}$$

in the sense of traces. We note that Lemma 2.4 implies $K_{\partial G,\Sigma}\, u = 0$ for all $u \in \mathrm{D}(A)$.

It can be shown that the Maxwell operator is skew-adjoint on X, see Chapter 8 of [15]. By Stone's theorem, it therefore generates a unitary strongly continuous group. This implies the following result.

Lemma 2.6. *Let Assumption 2.1 be true. The operator* A *generates a strongly continuous group* $S(\cdot)$ *on* $L^2(G)^6$. *It satisfies* $\|S(t)\|_{\mathcal{B}(L^2)} \leq c(\|\varepsilon\|_{L^\infty} + \|\mu\|_{L^\infty})^{1/2}\eta^{-1/2}$ *for all* $t \in \mathbb{R}$, *where* $c > 0$ *is a constant.*

We require E and H to have a spatial regularity of $\mathcal{H}^k_\Sigma(G)$ for some $k \geq 2$. This ensures that the expressions in the nonlinearity are well-defined, see Section 4.

Our treatment of the linear unretarded problem consists of two steps: The first one is based on semigroup theory in $L^2(G)^6$ to obtain a solution in $C^k_T L^2(G)^6$, under appropriate conditions on the initial data and inhomogeneity. In the second step we use the equation and the div-curl estimates from Proposition 2.5 to trade time derivatives for increased spatial regularity. Therefore our approach yields solutions having the same amount of regularity in time as in space, namely $u \in \bigcap_{\ell=0}^k C^\ell_T \mathcal{H}^{k-\ell}_\Sigma(G)^6$.

In order to include retardation effects, we need to work in spaces incorporating the history of the fields, which means using $(-\infty, T]$ instead of $[0,T]$ as the time interval. To avoid extra conditions on the initial data, we allow discontinuities of the highest time

derivative of the solution at $t = 0$. We therefore define the following function spaces, which also allow for jumps at $t = \vartheta$ to deal with time translations in e.g. Lemma 5.3.

Let $k \in \mathbb{N}$ with $k \geq 2$, $I \subseteq \mathbb{R}$ be an interval containing more than one point, and ϑ belong to the interior I^0 of I. In the case $\sup I \in I$, we set

$$\begin{aligned}
Z^k(I) &:= \bigcap_{\ell=0}^{k} C_b^\ell\big(I, \mathcal{H}_\Sigma^{k-\ell}(G)^6\big),\\
Z_\vartheta^k(I) &:= \Big\{u \in \bigcap_{\ell=0}^{k-1} C_b^\ell\big(I, \mathcal{H}_\Sigma^{k-\ell}(G)^6\big) \;\Big|\; u|_{I\cap(-\infty,\vartheta]} \in C_b^k\big(I \cap (-\infty, \vartheta], L^2(G)^6\big),\\
&\qquad\qquad u|_{I\cap[\vartheta,\infty)} \in C_b^k\big(I \cap [\vartheta, \infty), L^2(G)^6\big)\Big\}.
\end{aligned}$$

These spaces are complete with respect to the norms

$$\begin{aligned}
\|u\|_{Z^k(I)} &:= \sum_{\ell=0}^{k} \sup_{t\in I} \Big\|\partial_t^\ell u(t)\Big\|_{\mathcal{H}_\Sigma^{k-\ell}},\\
\|u\|_{Z_\vartheta^k(I)} &:= \sum_{\ell=0}^{k-1} \sup_{t\in I} \Big\|\partial_t^\ell u(t)\Big\|_{\mathcal{H}_\Sigma^{k-\ell}} + \sup_{t\in I\cap(-\infty,\vartheta]} \Big\|\partial_t^k u|_{I\cap(-\infty,\vartheta]}(t)\Big\|_{L^2}\\
&\quad + \sup_{t\in I\cap[\vartheta,\infty)} \Big\|\partial_t^k u|_{I\cap[\vartheta,\infty)}(t)\Big\|_{L^2}.
\end{aligned}$$

If $\sup I \notin I$, we define

$$\begin{aligned}
Z^k(I) &:= \Big\{u \mid u|_{I\cap(-\infty,T]} \in Z^k\big(I \cap (-\infty, T]\big) \text{ for all } T \in I\Big\},\\
Z_\vartheta^k(I) &:= \Big\{u \mid u|_{I\cap(-\infty,T]} \in Z_\vartheta^k\big(I \cap (-\infty, T]\big) \text{ for all } T \in (\vartheta, \sup I)\Big\}.
\end{aligned}$$

In this case, we use $Z^k(I)$ and $Z_\vartheta^k(I)$ as vector spaces (without norm). We also set $Z_\vartheta^k(I) := Z^k(I)$ if $\vartheta \notin I^0$ and use the abbreviations

$$Z_\vartheta^k(T) := Z_\vartheta^k\big((-\infty, T]\big), \quad Z^k(T) := Z^k\big((-\infty, T]\big) \qquad \text{for } T \in \mathbb{R}.$$

3 Linear problem

In this section we study the linear unretarded part of the problem (1.4), that is,

$$\begin{aligned}
\partial_t u(t) &= Au(t) + f(t), && t \geq 0,\\
K_{\partial G,\Sigma}\, u(t) &= 0, && t \geq 0,\\
u(0) &= u_0
\end{aligned} \tag{3.1}$$

with initial value $u_0 = (E_0, H_0)$ and inhomogeneity $f = (f_\mathrm{e}, f_\mathrm{m})$. Since P_ret and M_ret are omitted, it suffices to prescribe $u(0)$ as initial condition. In (1.4), the magnetic part f_m of f vanishes, but we include it here since the nonlinearity M_ret leads to such a term in the fixed-point argument below.

In the following, for $k \geq 2$ and $T > 0$ we obtain a solution $u \in Z^k\big([0, T]\big)$ of (3.1) under appropriate assumptions on the initial value and inhomogeneity. The latter does not need to satisfy boundary or interface conditions itself, but has the same regularity

in time and space as the solution. The initial value and the inhomogeneity have to fulfill so-called compatibility conditions. In view of Lemma 2.4, the boundary and interface conditions lead to the requirement $u(t) \in \mathrm{D}(A)$. If the solution is contained in $Z^k\big([0,T]\big)$, the traces exist and we can take the first time derivative and insert the equation for $\partial_t u$. So we get the condition $Au(t) + f(t) \in \mathrm{D}(A)$. Another differentiation then yields $A(Au(t) + f(t)) + \partial_t f(t) \in \mathrm{D}(A)$ etc. In total we obtain k conditions on u and f. As shown in Theorem 3.2, it suffices that these conditions are satisfied at the initial time.

Definition 3.1. *Let Assumption 2.1 be true and $T > 0$. For a pair*

$$(u_0, f) \in \mathcal{H}^k_\Sigma(G)^6 \times Z^k\big([0,T]\big),$$

we define

$$w_0 := u_0, \quad w_{\ell+1} := Aw_\ell + \partial_t^\ell f(0) \tag{3.2}$$

for $\ell \in \{0, \dots, k-1\}$. We say that (u_0, f) satisfies the compatibility conditions *of order k if w_ℓ is contained in $\mathrm{D}(A)$ for all $\ell \in \{0, \dots, k-1\}$.*

Note that inductively one sees that w_ℓ belongs to $\mathcal{H}^{k-\ell}_\Sigma(G)^6$ for all $\ell \in \{0, \dots, k\}$. We can now formulate the main result for the linear problem.

Theorem 3.2. *Let Assumption 2.1 be true. Then there exists a constant*

$$C_1 = C_1(k, \|\varepsilon\|_{\mathcal{C}^k_\Sigma}, \|\mu\|_{\mathcal{C}^k_\Sigma}, \eta^{-1}, G) \geq 1$$

(nondecreasing in $\|\varepsilon\|_{\mathcal{C}^k_\Sigma}$,$\|\mu\|_{\mathcal{C}^k_\Sigma}$ and η^{-1}) such that for any $T > 0$ and

$$(u_0, f) \in \mathcal{H}^k_\Sigma(G)^6 \times Z^k\big([0,T]\big)$$

satisfying the compatibility conditions of order k, the problem (3.1) has a unique solution u in $Z^k\big([0,T]\big) \cap C^{k-1}_T X_1$. It holds

$$u(t) = S(t)u_0 + \int_0^t S(t-\tau) f(\tau)\,\mathrm{d}\tau, \tag{3.3}$$

$$\partial_t^{\ell+1} u(t) = A\partial_t^\ell u(t) + \partial_t^\ell f(t),$$

$$\|u\|_{Z^k([0,T])} \leq C_1 \Big(\|u_0\|_{\mathcal{H}^k_\Sigma} + \sum_{i=0}^{k-1} \big\|\partial_t^i f(0)\big\|_{\mathcal{H}^{k-i-1}_\Sigma} + \sum_{i=0}^{k} \big\|\partial_t^i f\big\|_{L^1_T \mathcal{H}^{k-i}_\Sigma} \Big) \tag{3.4}$$

for all $t \in [0,T]$ and $\ell \in \{0, \dots, k-1\}$.

Proof. The quantities u_0, u and f are split into electric and magnetic parts via $u_0 = (E_0, H_0)$, $u = (E, H)$ and $f = (f_\mathrm{e}, f_\mathrm{m})$. The constants can depend on k, $\|\varepsilon\|_{\mathcal{C}^k_\Sigma}$, $\|\mu\|_{\mathcal{C}^k_\Sigma}$, η^{-1} and G and are nondecreasing in $\|\varepsilon\|_{\mathcal{C}^k_\Sigma}$, $\|\mu\|_{\mathcal{C}^k_\Sigma}$ and η^{-1}. Wo do not specify this in this proof, and just write $\lesssim$. Let $t \in [0,T]$ throughout.

1) In a first step, we show inductively

$$u \in C^{\ell+1}_T X \cap C^\ell_T X_1, \tag{3.5}$$

$$\partial_t^{\ell+1} u(t) = A\partial_t^\ell u(t) + \partial_t^\ell f(t), \tag{3.6}$$

$$\partial_t^\ell u(0) = w_\ell \in \mathrm{D}(A) \cap \mathcal{H}^{k-\ell}_\Sigma(G)^6 \tag{3.7}$$

for all $\ell \in \{0, \dots, k-1\}$, as well as

$$\partial_t^\ell u(t) = S(t)w_\ell + \int_0^t S(t-\tau)\partial_t^\ell f(\tau)\,\mathrm{d}\tau, \tag{3.8}$$

for all $\ell \in \{0, \dots, k\}$, where w_ℓ is defined as in Definition 3.1.

For $\ell = 0$, property (3.7) is true by the assumptions on u_0. Since $u_0 \in \mathrm{D}(A)$ and f is contained in $C_T^1 X$, Corollary 3.1.17 in [2] yields that (3.1) has a unique classical solution $u \in C_T^1 X \cap C_T X_1$ given by (3.3). So we have (3.5)-(3.8) for $\ell = 0$.

Now we assume that (3.5)-(3.8) hold for some $\ell \in \{0, \dots, k-2\}$. Then $A\partial_t^\ell u$ belongs to $C_T^1 X_{-1}^A$. Therefore (3.6) implies $\partial_t^{\ell+1} u \in C_T^1 X_{-1}^A$ and

$$\begin{aligned}
\partial_t^{\ell+2} u(t) &= A\partial_t^{\ell+1} u(t) + \partial_t^{\ell+1} f(t),\\
\partial_t^{\ell+1} u(0) &= A\partial_t^\ell u(0) + \partial_t^\ell f(0) = Aw_\ell + \partial_t^\ell f(0) = w_{\ell+1}
\end{aligned}$$

in the space X_{-1}^A. So $v := \partial_t^{\ell+1} u$ solves

$$\partial_t v = Av(t) + \partial_t^{\ell+1} f(t),\ t \in [0,T], \quad v(0) = w_{\ell+1}.$$

Due to the compatibility conditions, $w_{\ell+1}$ is contained in $\mathrm{D}(A) \cap \mathcal{H}_\Sigma^{k-\ell-1}(G)^6$. Since $\partial_t^{\ell+1} f \in C_T^1 X$, the function v belongs to $C_T^1 X \cap C_T X_1$ with

$$v(t) = S(t)w_{\ell+1} + \int_0^t S(t-\tau)\partial_t^{\ell+1} f(\tau)\,\mathrm{d}\tau,$$

and thus u to $C_T^{\ell+2} X \cap C_T^{\ell+1} X_1$. We therefore have shown (3.5)-(3.8) for $\ell \le k-1$. In particular, $u \in C_T^k X$ satisfies

$$\partial_t^{k-1} u(t) = S(t)w_{k-1} + \int_0^t S(t-\tau)\partial_t^{k-1} f(\tau)\,\mathrm{d}\tau. \tag{3.9}$$

As $w_{k-1} \in \mathrm{D}(A)$ and $\partial_t^{k-1} f \in C_T^1 X$, we can differentiate (3.9) to obtain

$$\partial_t^k u(t) = S(t)\big(Aw_{k-1} + \partial_t^{k-1} f(0)\big) + \int_0^t S(t-\tau)\partial_t^k f(\tau)\,\mathrm{d}\tau = S(t)w_k + \int_0^t S(t-\tau)\partial_t^k f(\tau)\,\mathrm{d}\tau$$

in X, proving (3.8) for $\ell = k$.

2) Let $\ell \in \{0, \dots, k\}$ and $j \in \{0, \dots, k-\ell\}$. The inequality

$$\|w_\ell\|_{\mathcal{H}_\Sigma^j} \lesssim \|u_0\|_{\mathcal{H}_\Sigma^{\ell+j}} + \sum_{i=0}^{\ell-1} \|\partial_t^i f(0)\|_{\mathcal{H}_\Sigma^{\ell+j-i-1}} \tag{3.10}$$

can be easily shown by an induction over ℓ, using (3.2) and (2.1).

Formulas (3.8), (3.10) and the boundedness of the group $S(\cdot)$ on $L^2(G)^6$ then yield

$$\left\|\partial_t^\ell u(t)\right\|_{L^2} \lesssim \|u_0\|_{\mathcal{H}_\Sigma^\ell} + \sum_{i=0}^{\ell-1} \|\partial_t^i f(0)\|_{\mathcal{H}_\Sigma^{\ell-i-1}} + \left\|\partial_t^\ell f\right\|_{L^1((0,t),L^2)}. \tag{3.11}$$

3) We now establish higher spatial regularity for the solution using the div-curl estimates from Proposition 2.5. We claim

$$u \in C_T^{k-m} \mathcal{H}_\Sigma^m(G)^6, \tag{3.12}$$

$$\left\|\partial_t^{k-m}u(t)\right\|_{\mathcal{H}_\Sigma^m} \lesssim \|u_0\|_{\mathcal{H}_\Sigma^k} + \sum_{i=0}^{k-1}\left\|\partial_t^i f(0)\right\|_{\mathcal{H}_\Sigma^{k-i-1}} + \sum_{i=0}^{m}\left\|\partial_t^{k-i} f\right\|_{L^1((0,t),\mathcal{H}_\Sigma^i)} \tag{3.13}$$

for all $m \in \{0,\dots,k-1\}$. For $m=0$, this follows from (3.5) and (3.11). So assume that (3.12) and (3.13) hold for some $m \in \{0,\dots,k-2\}$. By (3.5), the function $\partial_t^{k-m-1}u$ is contained in $C_T X_1$ which implies $\partial_t^{k-m-1}E \in C_T\mathcal{H}_0(\operatorname{curl},G)$. From the electric component of (3.6) and $\operatorname{div}\operatorname{curl}=0$, we obtain

$$\begin{aligned}\varepsilon\partial_t^{k-m-1}E - \varepsilon\partial_t^{k-m-2}f_{\mathrm{e}} &= \operatorname{curl}\partial_t^{k-m-2}H \in C_T\mathcal{H}(\operatorname{div},G),\\ \operatorname{div}\big(\varepsilon\partial_t^{k-m-1}E_{|G_i}\big) &= \operatorname{div}\big(\varepsilon\partial_t^{k-m-2}f_{\mathrm{e}|G_i}\big) \in C_T\mathcal{H}^{m+1}(G_i)\end{aligned} \tag{3.14}$$

for $i \in \{1,2\}$. By (3.12), the function $\partial_t^{k-m}H$ is contained in $C_T\mathcal{H}_\Sigma^m(G)^3$. The magnetic component of (3.6) and the assumptions on μ and f therefore yield

$$\operatorname{curl}\partial_t^{k-m-1}E = -\mu\partial_t^{k-m}H + \mu\partial_t^{k-m-1}f_{\mathrm{m}} \in C_T\mathcal{H}_\Sigma^m(G)^3.$$

As $\varepsilon\partial_t^{k-m-2}f_{\mathrm{e}} \in C_T\mathcal{H}_\Sigma^{m+2}(G)^3$, we can apply the first part of Proposition 2.5 to infer that $\partial_t^{k-m-1}E \in C_T\mathcal{H}_\Sigma^{m+1}(G)^3$ satisfies

$$\begin{aligned}\left\|\partial_t^{k-m-1}E(t)\right\|_{\mathcal{H}_\Sigma^{m+1}} &\lesssim \left\|\partial_t^{k-m-1}E(t)\right\|_{L^2} + \left\|\operatorname{curl}\partial_t^{k-m-1}E(t)\right\|_{\mathcal{H}_\Sigma^m}\\ &\quad + \sum_{i=1}^{2}\left\|\operatorname{div}\big(\varepsilon\partial_t^{k-m-1}E_{|G_i}\big)\right\|_{H^m(G_i)} + \left\|\varepsilon\partial_t^{k-m-2}f_{\mathrm{e}}(t)\right\|_{\mathcal{H}_\Sigma^{m+1}}\\ &\lesssim \left\|\partial_t^{k-m-1}u(t)\right\|_{L^2} + \left\|A\partial_t^{k-m-1}u(t)\right\|_{\mathcal{H}_\Sigma^m} + \left\|\partial_t^{k-m-2}f(t)\right\|_{\mathcal{H}_\Sigma^{m+1}},\end{aligned}$$

where we have used (3.14) in the last step. The first term on the right hand side can be treated using (3.11). For the second one, (3.6) yields

$$\left\|A\partial_t^{k-m-1}u(t)\right\|_{\mathcal{H}_\Sigma^m} \le \left\|\partial_t^{k-m}u(t)\right\|_{\mathcal{H}_\Sigma^m} + \left\|\partial_t^{k-m-1}f(t)\right\|_{\mathcal{H}_\Sigma^m}.$$

Here we can insert (3.13). To estimate the terms involving $f(t)$, we write

$$\partial_t^j f(t) = \partial_t^j f(0) + \int_0^t \partial_t^{j+1} f(s)\,\mathrm{d}s$$

for $j \in \{k-m-2, k-m-1\}$. This finally results in the inequality

$$\left\|\partial_t^{k-m-1}E(t)\right\|_{\mathcal{H}_\Sigma^{m+1}} \lesssim \|u_0\|_{\mathcal{H}_\Sigma^k} + \sum_{i=0}^{k-1}\left\|\partial_t^i f(0)\right\|_{\mathcal{H}_\Sigma^{k-i-1}} + \sum_{i=0}^{m+1}\left\|\partial_t^{k-i} f\right\|_{L^1((0,t),\mathcal{H}_\Sigma^i)}. \tag{3.15}$$

The magnetic field is treated analogously. Due to (3.5), the field $\partial_t^{k-m-1}H$ is contained in $C_T\mathcal{H}(\operatorname{curl},G)$. Furthermore, the trace $\mathrm{n}\times\partial_t^{k-m-2}E_{|\partial G}$ vanishes since $\partial_t^{k-m-2}E$ belongs to $C_T\mathcal{H}_0(\operatorname{curl},G)$, see Lemma 2.4. Using also the magnetic component of (3.6) and Corollary 3.1.16 in [3], we conclude

$$\mu\partial_t^{k-m-1}H - \mu\partial_t^{k-m-2}f_{\mathrm{m}} = -\operatorname{curl}\partial_t^{k-m-2}E \in C_T\mathcal{H}_0(\operatorname{div},G). \tag{3.16}$$

Additionally, it holds

$$\operatorname{div}\big(\mu\partial_t^{k-m-1}H_{|_{G_i}}\big) = \operatorname{div}\big(\mu\partial_t^{k-m-2}f_{\mathrm{m}|_{G_i}}\big) \in C_T\mathcal{H}^{m+1}(G_i)$$

for $i \in \{1,2\}$. By (3.12), the function $\partial_t^{k-m}E$ belongs to $C_T\mathcal{H}_\Sigma^m(G)^3$. The electric component of (3.6) and the assumptions on ε and f now lead to

$$\operatorname{curl}\partial_t^{k-m-1}H = \varepsilon\partial_t^{k-m}E - \varepsilon\partial_t^{k-m-1}f_{\mathrm{e}} \in C_T\mathcal{H}_\Sigma^m(G)^3.$$

As in the previous step we then derive $\partial_t^{k-m-1}H \in C_T\mathcal{H}_\Sigma^{m+1}(G)^3$ and

$$\left\|\partial_t^{k-m-1}H(t)\right\|_{\mathcal{H}_\Sigma^{m+1}} \lesssim \left\|\partial_t^{k-m-1}u(t)\right\|_{L^2} + \left\|A\partial_t^{k-m-1}u(t)\right\|_{\mathcal{H}_\Sigma^m} + \left\|\partial_t^{k-m-2}f(t)\right\|_{\mathcal{H}_\Sigma^{m+1}},$$

using Proposition 2.5 ii). The same calculations as above lead to (3.15) with E replaced by H (and a with different constant) and we conclude

$$\left\|\partial_t^{k-m-1}u(t)\right\|_{\mathcal{H}_\Sigma^{m+1}} \lesssim \|u_0\|_{\mathcal{H}_\Sigma^k} + \sum_{i=0}^{k-1}\left\|\partial_t^i f(0)\right\|_{\mathcal{H}_\Sigma^{k-i-1}} + \sum_{i=0}^{m+1}\left\|\partial_t^{k-i}f\right\|_{L^1((0,t),\mathcal{H}_\Sigma^i)}.$$

By induction, (3.12) and (3.12) are proven.

5) To complete the proof, we still need to show $u \in C_T\mathcal{H}_\Sigma^k(G)^6$ and

$$\|u(t)\|_{\mathcal{H}_\Sigma^k} \lesssim \|u_0\|_{\mathcal{H}_\Sigma^k} + \sum_{i=0}^{k-1}\left\|\partial_t^i f(0)\right\|_{\mathcal{H}_\Sigma^{k-i-1}} + \sum_{i=0}^{k}\left\|\partial_t^{k-i}f\right\|_{L^1((0,t),\mathcal{H}_\Sigma^i)}.$$

We again start with the electric field, for which we already know $E \in C_T\mathcal{H}_0(\operatorname{curl}, G)$. The electric part of (3.1) can be rewritten as

$$\varepsilon E(t) + V(t) = \int_0^t \operatorname{curl} H(s)\,\mathrm{d}s \quad \text{with} \quad V(t) := -\varepsilon E(0) - \int_0^t \varepsilon f_{\mathrm{e}}(s)\,\mathrm{d}s.$$

The assumptions on u_0 and f as well as the identity $\operatorname{div}\operatorname{curl} = 0$ imply $V \in C_T\mathcal{H}_\Sigma^k(G)^3$, $\varepsilon E + V \in C_T\mathcal{H}(\operatorname{div}, G)$, and $\operatorname{div}(\varepsilon E_{|_{G_i}}) = -\operatorname{div}(V_{|_{G_i}}) \in C_T\mathcal{H}^{k-1}(G_i)$ for $i \in \{1,2\}$. By the equation for $\partial_t H$, the field $\operatorname{curl} E = -\mu\partial_t H + \mu f_{\mathrm{m}}$ belongs to $C_T\mathcal{H}_\Sigma^{k-1}(G)^3$. Therefore Proposition 2.5 again yields $E \in C_T\mathcal{H}_\Sigma^k(G)^3$ and

$$\begin{aligned}\|E(t)\|_{\mathcal{H}_\Sigma^k} &\lesssim \|E(t)\|_{L^2} + \|\operatorname{curl} E(t)\|_{\mathcal{H}_\Sigma^{k-1}} + \sum_{i=1}^{2}\left\|\operatorname{div}\big(\varepsilon E(t)_{|_{G_i}}\big)\right\|_{\mathcal{H}^{k-1}(G_i)} + \|V(t)\|_{\mathcal{H}_\Sigma^k}\\ &\lesssim \|u(t)\|_{L^2} + \|Au(t)\|_{\mathcal{H}_\Sigma^{k-1}} + \|u_0\|_{\mathcal{H}_\Sigma^k} + \|f\|_{L^1((0,t),\mathcal{H}_\Sigma^k)}.\end{aligned}$$

Since

$$Au(t) = \partial_t u(t) + f(0) + \int_0^t \partial_t f(s)\,\mathrm{d}s,$$

estimates (3.11) and (3.13) lead to

$$\|E(t)\|_{\mathcal{H}_\Sigma^k} \lesssim \|u_0\|_{\mathcal{H}_\Sigma^k} + \sum_{i=0}^{k-1}\left\|\partial_t^i f(0)\right\|_{\mathcal{H}_\Sigma^{k-i-1}} + \sum_{i=0}^{k}\left\|\partial_t^{k-i}f\right\|_{L^1((0,t),\mathcal{H}_\Sigma^i)}. \tag{3.17}$$

The magnetic field analogously satisfies $H \in C_T\mathcal{H}(\mathrm{curl}, G)$ and

$$\mu H(t) + W(t) = -\int_0^t \mathrm{curl}\, E(s)\,\mathrm{d}s \quad \text{with} \quad W(t) := -\mu H(0) - \int_0^t \mu f_{\mathrm{m}}(s)\,\mathrm{d}s.$$

Moreover, W belongs to $C_T\mathcal{H}^k_\Sigma(G)^3$, $\mathrm{div}(\mu H_{|_{G_i}}) = -\mathrm{div}(W_{|_{G_i}})$ to $C_T\mathcal{H}^{k-1}(G_i)$, and $\mathrm{curl}\, H = \varepsilon\partial_t E - \varepsilon f_{\mathrm{e}}$ to $C_T\mathcal{H}^{k-1}_\Sigma(G)^3$. for $i \in \{1,2\}$. Arguing as in the derivation of (3.16), we see that $\mu H + W$ is contained in $C_T\mathcal{H}_0(\mathrm{div}, G)$. We can then estimate H as in (3.17). □

4 Constitutive relations

We now describe the model for the main parts $P_{\mathrm{ret}}(E,H)$ and $M_{\mathrm{ret}}(E,H)$ of the material law and study its properties. We combine the variables as $u = (E,H)$ and $\Gamma(u) = \big(P_{\mathrm{ret}}(E,H), M_{\mathrm{ret}}(E,H)\big)$ and assume Γ to be given by

$$\Gamma(u) = \sum\nolimits_{n=1}^{N} \Gamma^{(n)}(u), \tag{4.1}$$

where $N \in \mathbb{N}$ is an arbitrary but fixed number and $\Gamma^{(n)}(u)$ is n-linear in u. The components of the individual contributions are modeled by the expressions

$$\Gamma^{(n)}_{j_0}(u)(t,x) = \int_{-\infty}^t \cdots \int_{-\infty}^t R^{(n)}_{j_0 j_1 \dots j_n}(t-s_1,\dots,t-s_n,x) u_{j_1}(s_1,x)\dots u_{j_n}(s_n,x)\,\mathrm{d}s_1\dots\,\mathrm{d}s_n \tag{4.2}$$

for $n \in \{1,\dots,N\}$ and $j_0 \in \{1,\dots,6\}$. Here and below we use the convention of summing over repeated indices and from now on we mostly omit the spatial variable x.

The model (4.2) is local in space, but nonlocal in time: The polarization at a given time depends in general on all past values of u. The kernels $R^{(n)}_{j_0 j_1 \dots j_n}$ are called response functions of the material and are tensors with components $j_0,\dots,j_n \in \{1,\dots,6\}$. This accounts for a possible anisotropy of the material. The response functions weight the influence of the field components $u_{j_i}(s_i)$ on the value of the mediums response $\Gamma_{j_0}(u)(t)$. Expressions for the response functions can be obtained from models for the material. For example in the Lorentz oscillator the motion of bound electrons in the material under the influence of an external electric field is modelled by a damped harmonic oscillator. This yields an expression for the linear response function ($n = 1$). Generalizations to anharmonic oscillators lead to nonlinearities ($n > 1$), see for example Section 1.4 of [6].

The expression (4.2) is causal, since only the values of $u(s)$ for $s \le t$ can influence $\Gamma(u)(t)$. In addition, it displays time invariance in the sense that $R^{(n)}_{j_0 j_1 \dots j_n}$ only depend on time differences $t - s_i$. The response functions possess an intrinsic permutation symmetry: $R^{(n)}_{j_0 j_1 \dots j_n}(\tau_1 \dots, \tau_n)$ is unchanged under any interchange $(j_\ell, \tau_\ell) \leftrightarrow (j_k, \tau_k)$ for $\ell, k \in \{1,\dots,n\}$, see Chapter 2 of [9]. More precisely, we have

$$R^{(n)}_{j_0 j_1 \dots j_n}(\tau_1,\dots,\tau_n) = R^{(n)}_{j_0 j_{\pi(1)} \dots j_{\pi(n)}}\big(\tau_{\pi(1)},\dots,\tau_{\pi(n)}\big)$$

for all $\tau \in \mathbb{R}^n_{\ge 0}$, $j_0 \dots, j_n \in \{1,\dots,6\}$ and $\pi \in S_n$, where S_n denotes the group of permutations on $\{1,\dots,n\}$. For derivatives and $\alpha \in \mathbb{N}^n_0$, this fact implies

$$\partial^\alpha_\tau R^{(n)}_{j_0 j_1 \dots j_n}(\tau) = \partial_\tau^{\left(\alpha_{\pi(1)},\dots,\alpha_{\pi(n)}\right)} R^{(n)}_{j_0 j_{\pi(1)} \dots j_{\pi(n)}}(\tau_{\pi(1)},\dots,\tau_{\pi(n)}).$$

Recall that $\mathcal{H}^k_\Sigma(G)$ is a Banach algebra for $k \geq 2$. The Maxwell equations contain the time derivatives of P as well as M and our fixed-point argument treats $\partial_t \Gamma$ as a fixed inhomogeneity for the linear problem (see the proof of Lemma 5.2). In order to be able to use the estimate (3.4), we therefore need appropriate estimates for $\partial_t^\ell \Gamma$ with $\ell \in \{1, \dots, k+1\}$. The next lemma is a first step. We omit the proof, see Lemma 6.4 in [7]. The estimates easily follow from Lemma 2.2, but the elementary argument for continuity and differentiability in time is tedious.

Lemma 4.1. *Let Assumption 2.1 be true, $m \in \mathbb{N}$, $T \in \mathbb{R}$ and $\ell \in \mathbb{N}_0$ with $\ell \leq k$. Let $u_1, \dots, u_m \in C_b\big((-\infty, T], \mathcal{H}^k_\Sigma(G)\big)$ and $\phi \in L^1\big((0,\infty)\big) \cap C_b\big([0,\infty)\big)$.*

Let $\zeta \in C\left(\mathbb{R}^m_{\geq 0}, \mathcal{H}^\ell_\Sigma(G)\right)$ satisfy

$$\|\zeta(\tau)\|_{\mathcal{H}^\ell_\Sigma} \leq \prod\nolimits_{i=1}^m \phi(\tau_i)$$

for all $\tau \in \mathbb{R}^m_{\geq 0}$. Let ψ be defined by

$$\psi(t) := \int_{-\infty}^t \cdots \int_{-\infty}^t \zeta(t-s_1, \dots, t-s_m) u_1(s_1) \dots u_m(s_m) \,\mathrm{d}s_1 \dots \,\mathrm{d}s_m$$

for all $t \leq T$. Then ψ is contained in $C_b\big((-\infty, T], \mathcal{H}^\ell_\Sigma(G)\big)$ with

$$\sup_{t \leq T} \|\psi(t)\|_{\mathcal{H}^\ell_\Sigma} \leq C \, \|\phi\|^m_{L^1((0,\infty))} \prod_{i=1}^m \sup_{s \leq T} \|u_i(s)\|_{\mathcal{H}^k_\Sigma} \,.$$

If $\zeta \in C^1\left(\mathbb{R}^m_{\geq 0}, \mathcal{H}^\ell_\Sigma(G)\right)$ satisfies

$$\sum_{\alpha \in \mathbb{N}_0^m,\, |\alpha| \leq 1} \|\partial_\tau^\alpha \zeta(\tau)\|_{\mathcal{H}^\ell_\Sigma} \leq \prod_{i=1}^m \phi(\tau_i)$$

for all $\tau \in \mathbb{R}^m_{\geq 0}$, then ψ is contained in $C^1_b\big((-\infty, T], \mathcal{H}^\ell_\Sigma(G)\big)$ with

$$\begin{aligned} \partial_t \psi(t) = \sum_{i=1}^m \Bigg[& \int_{-\infty}^t \cdots \int_{-\infty}^t \partial_{\tau_i} \zeta(t-s_1, \dots, t-s_m) u_1(s_1) \dots u_m(s_m) \,\mathrm{d}s_1 \dots \,\mathrm{d}s_m \\ & + u_i(t) \int_{-\infty}^t \cdots \int_{-\infty}^t \zeta(t-s_1, \dots, t-s_{i-1}, 0, t-s_{i+1}, \dots, t-s_m) \\ & \cdot u_1(s_1) \dots u_{i-1}(s_{i-1}) u_{i+1}(s_{i+1}) \dots u_m(s_m) \,\mathrm{d}s_1 \dots \,\mathrm{d}s_{i-1} \,\mathrm{d}s_{i+1} \dots \,\mathrm{d}s_m \Bigg] \end{aligned} \tag{4.3}$$

(for $m = 1$ the $(m-1)$-fold integral in (4.3) is replaced by $\zeta(0)$) and

$$\sup_{t \leq T} \|\partial_t \psi(t)\|_{\mathcal{H}^\ell_\Sigma} \leq C \left(\|\phi\|^m_{L^1((0,\infty))} + \phi(0) \, \|\phi\|^{m-1}_{L^1((0,\infty))} \right) \prod_{i=1}^m \sup_{s \leq T} \|u_i(s)\|_{\mathcal{H}^k_\Sigma} \,,$$

Here $C = C\big(k, \ell, m, G\big) \geq 0$ are constants independent of T.

We now state the assumptions on the response functions. In light of the the function spaces in which we seek solutions, the regularity conditions below are a natural choice. Examples from physics often include an exponential decay of the response functions in time caused by damping effects, see Chapter 11 of [20]. Here Assumption 4.2 is satisfied with $\phi(t) = c\mathrm{e}^{-\gamma t}$ with positive constants c and γ. In the following, we often use the assumptions on the response functions without further comment.

Assumption 4.2. *Let Assumption 2.1 be true. There exists a function* $\phi \in L^1\big((0,\infty)\big) \cap C_b\big([0,\infty)\big)$ *such that* $R^{(n)}_{j_0\dots j_n} \in \bigcap_{\ell=0}^k C^{\ell+1}\big(\mathbb{R}^n_{\geq 0}, \mathcal{H}^{k-\ell}_\Sigma(G)\big)$ *with*

$$\Big\| R^{(n)}_{j_0\dots j_n}(\tau)\Big\|_{\mathcal{H}^k_\Sigma} + \sum_{\ell=0}^k \sum_{\alpha\in\mathbb{N}_0^n,\,|\alpha|=\ell+1} \Big\|\partial_\tau^\alpha R^{(n)}_{j_0\dots j_n}(\tau)\Big\|_{\mathcal{H}^{k-\ell}_\Sigma} \leq \prod_{i=1}^n \phi(\tau_i)$$

for all $n \in \{1,\dots,N\}$, $j_0,\dots,j_n \in \{1,\dots,6\}$, *and* $\tau \in \mathbb{R}^n_{\geq 0}$.

We now prove expressions for the time derivatives of $\Gamma(u)$. The time variable appears in (4.2) in the limits of the integrals and in the response functions, In formula (4.4) below, the index q counts the number of derivatives in $\partial_t^m \Gamma(u)$ applied to the integral limits. The remaining $m-q$ derivatives fall on the response functions or the field components.

Lemma 4.3. *Let Assumptions 2.1 and 4.2 be true,* $n \in \{1,\dots,N\}$, *and* $T \in \mathbb{R}$.

(i) Let $u \in Z^k(T)$. *Then we have*

$$\Gamma^{(n)}(u) \in C_b^1\big((-\infty,T], \mathcal{H}^k_\Sigma(G)^6\big), \quad \partial_t \Gamma^{(n)}(u) \in Z^k(T).$$

(ii) Let $\vartheta < T$ *and* $u \in Z^k_\vartheta(T)$. *Then we have*

$$\Gamma^{(n)}(u) \in C_b^1\big((-\infty,T], \mathcal{H}^k_\Sigma(G)^6\big), \quad \partial_t \Gamma^{(n)}(u) \in Z^k_\vartheta(T).$$

In both cases, there exist constants $a^{(n)}_{m,q,\alpha,\beta}, b^{(n)}_{\alpha,\beta} \in \mathbb{N}_0$ *such that*

$$\begin{aligned}
\partial_t^m \Gamma^{(n)}_{j_0}(u)|_I(t) &= \sum_{\alpha\in\mathbb{N}_0^n,\,|\alpha|=m} \int_{-\infty}^t \dots \int_{-\infty}^t \partial_\tau^\alpha R^{(n)}_{j_0\dots j_n}(t-s_1,\dots,t-s_n) \prod_{i=1}^n u_{j_i}(s_i)\,\mathrm{d}s_1\dots\mathrm{d}s_n \\
&+ \sum_{q=1}^{\min\{m,n-1\}} \sum_{\substack{\alpha\in\mathbb{N}_0^n,\,\beta\in\mathbb{N}_0^q,\\ |\alpha|+|\beta|=m-q}} a^{(n)}_{m,q,\alpha,\beta} \left(\prod_{i=1}^q \partial_t^{\beta_i} u_{j_i}|_I(t)\right) \\
&\quad \cdot \int_{-\infty}^t \cdots \int_{-\infty}^t \partial_\tau^\alpha R^{(n)}_{j_0\dots j_n}(0,\dots,0,t-s_{q+1},\dots,t-s_n) \prod_{i=q+1}^n u_{j_i}(s_i)\,\mathrm{d}s_{q+1}\dots\mathrm{d}s_n \\
&+ \sum_{\substack{\alpha,\beta\in\mathbb{N}_0^n,\\ |\alpha|+|\beta|=m-n}} b^{(n)}_{\alpha,\beta} \partial_\tau^\alpha R^{(n)}_{j_0\dots j_n}(0,\dots,0) \prod_{i=1}^n \partial_t^{\beta_i} u_{j_i}|_I(t)
\end{aligned} \tag{4.4}$$

holds for all $j_0 \in \{1,\dots,6\}$, $m \in \{1,\dots,k+1\}$ *and all* $t \in I$, *where* $I = (-\infty,T]$ *in case (i) or if* $m \leq k$, *and* $I \in \{(-\infty,\vartheta],[\vartheta,T]\}$ *in case (ii) with* $m = k+1$.

Proof. 1) We only prove the second case. The first case is treated by the same calculations without requiring restrictions to the subintervals for the highest time derivative. Lemma 4.1 implies $\Gamma^{(n)}_{j_0}(u) \in C_b^1\big((-\infty,T], \mathcal{H}^k_\Sigma(G)\big)$ and (4.4) for $m=1$ in the form

$$\partial_t \Gamma^{(1)}_{j_0}(u)(t) = \int_{-\infty}^t \partial_{\tau_1} R^{(1)}_{j_0 j_1}(t-s_1) u_{j_1}(s_1)\,\mathrm{d}s_1 + R^{(1)}_{j_0 j_1}(0) u_{j_1}(t), \tag{4.5}$$

$$\partial_t \Gamma^{(n)}_{j_0}(u)(t) = \sum_{\alpha\in\mathbb{N}_0^n,\,|\alpha|=1} \int_{-\infty}^t \cdots \int_{-\infty}^t \partial_\tau^\alpha R^{(n)}_{j_0\dots j_n}(t-s_1,\dots,t-s_n) \prod_{i=1}^n u_{j_i}(s_i)\,\mathrm{d}s_1\dots\mathrm{d}s_n$$

$$+ nu_{j_1}(t)\int_{-\infty}^{t}\dots\int_{-\infty}^{t} R^{(n)}_{j_0\dots j_n}(0,t-s_2\dots,t-s_n)\prod_{i=2}^{n}u_{j_i}(s_i)\,\mathrm{d}s_2\dots\mathrm{d}s_n \quad (4.6)$$

if $n>1$, for all $t\le T$. In the last term we have used the intrinsic permutation symmetry of the response functions to avoid a summation. This is done below multiple times. From these expressions and Lemmas 2.2 and 4.1, we see that $\partial_t\Gamma^{(n)}_{j_0}(u)$ is contained in $C^1_b\big((-\infty,T],\mathcal{H}^{k-1}_\Sigma(G)\big)$ with

$$\partial_t^2\Gamma^{(1)}_{j_0}(u)(t) = \int_{-\infty}^{t}\partial^2_{\tau_1}R^{(1)}_{j_0j_1}(t-s_1)u_{j_1}(s_1)\,\mathrm{d}s_1 + \partial_{\tau_1}R^{(1)}_{j_0j_1}(0)u_{j_1}(t) + R^{(1)}_{j_0j_1}(0)\partial_t u_{j_1}(t), \quad (4.7)$$

$$\begin{aligned}\partial_t^2\Gamma^{(2)}_{j_0}(u)(t) = &\sum_{\alpha\in\mathbb{N}_0^2,\,|\alpha|=2}\int_{-\infty}^{t}\cdots\int_{-\infty}^{t}\partial^\alpha_\tau R^{(2)}_{j_0j_1j_2}(t-s_1,t-s_2)u_{j_1}(s_1)u_{j_2}(s_2)\,\mathrm{d}s_1\,\mathrm{d}s_2\\ &+ a^{(2)}_{2,1,(1,0),0}u_{j_1}(t)\int_{-\infty}^{t}\partial_{\tau_1}R^{(2)}_{j_0j_1j_2}(0,t-s_2)u_{j_2}(s_2)\,\mathrm{d}s_2\\ &+ a^{(2)}_{2,1,(0,1),0}u_{j_1}(t)\int_{-\infty}^{t}\partial_{\tau_2}R^{(2)}_{j_0j_1j_2}(0,t-s_2)u_{j_2}(s_2)\,\mathrm{d}s_2\\ &+ a^{(2)}_{2,1,(0,0),1}\partial_t u_{j_1}(t)\int_{-\infty}^{t}R^{(2)}_{j_0j_1j_2}(0,t-s_2)u_{j_2}(s_2)\,\mathrm{d}s_2\\ &+ b^{(2)}_{(0,0),(0,0)}R^{(2)}_{j_0j_1j_2}(0,0)u_{j_1}(t)u_{j_2}(t)\end{aligned}$$

for $t\le T$ with constants $a^{(2)}_{2,1,(1,0),0}, a^{(2)}_{2,1,(0,1),0}, a^{(2)}_{2,1,(0,0),1}, b^{(2)}_{(0,0),(0,0)}\in\mathbb{N}$. If $n\ge 3$ we have

$$\begin{aligned}\partial_t^2\Gamma^{(n)}_{j_0}(u)(t) = &\sum_{\alpha\in\mathbb{N}_0^n,\,|\alpha|=2}\int_{-\infty}^{t}\cdots\int_{-\infty}^{t}\partial^\alpha_\tau R^{(n)}_{j_0\dots j_n}(t-s_1,\dots,t-s_n)\prod_{i=1}^{n}u_{j_i}(s_i)\,\mathrm{d}s_1\dots\mathrm{d}s_n\\ &+ a^{(n)}_{2,1,(1,0),0}u_{j_1}(t)\int_{-\infty}^{t}\cdots\int_{-\infty}^{t}\partial_{\tau_1}R^{(n)}_{j_0\dots j_n}(0,t-s_2,\dots,t-s_n)\prod_{i=2}^{n}u_{j_i}(s_i)\,\mathrm{d}s_2\dots\mathrm{d}s_n\\ &+ a^{(n)}_{2,1,(0,1),0}u_{j_1}(t)\int_{-\infty}^{t}\cdots\int_{-\infty}^{t}\partial_{\tau_2}R^{(n)}_{j_0\dots j_n}(0,t-s_2,\dots,t-s_n)\prod_{i=2}^{n}u_{j_i}(s_i)\,\mathrm{d}s_2\dots\mathrm{d}s_n\\ &+ a^{(n)}_{2,1,(0,0),1}\partial_t u_{j_1}(t)\int_{-\infty}^{t}\cdots\int_{-\infty}^{t}R^{(n)}_{j_0\dots j_n}(0,t-s_2,\dots,t-s_n)\prod_{i=2}^{n}u_{j_i}(s_i)\,\mathrm{d}s_2\dots\mathrm{d}s_n\\ &+ b^{(n)}_{(0,0),(0,0)}u_{j_1}(t)u_{j_2}(t)\int_{-\infty}^{t}\dots\int_{-\infty}^{t}R^{(n)}_{j_0\dots j_n}(0,0,t-s_3,\dots,t-s_n)\prod_{i=3}^{n}u_{j_i}(s_i)\,\mathrm{d}s_3\dots\mathrm{d}s_n\end{aligned}$$

for all $t\le T$ with constants $a^{(n)}_{2,1,(1,0),0}, a^{(n)}_{2,1,(0,1),0}, a^{(n)}_{2,1,(0,0),1}, b^{(n)}_{(0,0),(0,0)}\in\mathbb{N}$.

2) We now establish by induction that

$$\partial_t\Gamma^{(n)}_{j_0}(u)\in\bigcap_{\ell=0}^{\nu+1}C^\ell_b\big((-\infty,T],\mathcal{H}^{k-\ell}_\Sigma(G)\big) \quad\text{and}\quad \partial_t^{\nu+2}\Gamma^{(n)}_{j_0}(u)\text{ is of the form (4.4)} \quad (4.8)$$

for all $\nu\in\{0,\dots,k-2\}$. The case $\nu=0$ was shown above. Let $k>2$. We assume that (4.8) holds for some $\nu\in\{0,\dots,k-3\}$. Setting $m:=\nu+2$, we prove that $\partial_t^m\Gamma^{(n)}_{j_0}(u)\in C^1_b\big((-\infty,T],\mathcal{H}^{k-m}_\Sigma(G)\big)$ has a derivative as in (4.4), treating the terms in (4.4) separately.

2.1) Let $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = m$. We define

$$\psi_{j_0}^{\alpha,n}(t) := \int_{-\infty}^{t} \cdots \int_{-\infty}^{t} \partial_\tau^\alpha R_{j_0\dots j_n}^{(n)}(t-s_1,\dots,t-s_n) \prod_{i=1}^{n} u_{j_i}(s_i)\,\mathrm{d}s_1\dots\mathrm{d}s_n \tag{4.9}$$

for $t \le T$. The integrand satisfies the assumptions of the second part of Lemma 4.1 because of $u \in C_b\big((-\infty,T],\mathcal{H}_\Sigma^k(G)^6\big)$, Assumption 4.2 and Lemma 2.2. Lemma 4.1 then shows that $\psi_{j_0}^{\alpha,n}$ is contained in $C_b^1\big((-\infty,T],\mathcal{H}_\Sigma^{k-m}(G)\big)$ with derivative

$$\begin{aligned}\partial_t \psi_{j_0}^{\alpha,n}(t) = {}& \sum_{p=1}^{n} \int_{-\infty}^{t} \dots \int_{-\infty}^{t} \partial_\tau^{\alpha+e_p} R_{j_0\dots j_n}^{(n)}(t-s_1,\dots,t-s_n) \prod_{i=1}^{n} u_{j_i}(s_i)\,\mathrm{d}s_1\dots\mathrm{d}s_n \\ &+ n u_{j_1}(t) \int_{-\infty}^{t} \cdots \int_{-\infty}^{t} \partial_\tau^\alpha R_{j_0\dots j_n}^{(n)}(0,t-s_2,\dots,t-s_n) \prod_{i=2}^{n} u_{j_i}(s_i)\,\mathrm{d}s_2\dots\mathrm{d}s_n\end{aligned} \tag{4.10}$$

for all $t \le T$, where e_p denotes the pth unit vector (for $n=1$, the $(n-1)$-fold integral is replaced by $\partial_\tau^\alpha R^{(1)}(0)$).

2.2) Let $q \in \{1,\dots,\min\{m,n-1\}\}$, $\alpha \in \mathbb{N}_0^n$ and $\beta \in \mathbb{N}_0^q$ with $|\alpha|+|\beta| = m-q$. We set

$$\chi_{j_0}^{\alpha,n,q}(t) := \int_{-\infty}^{t} \cdots \int_{-\infty}^{t} \partial_\tau^\alpha R_{j_0\dots j_n}^{(n)}(0,\dots,0,t-s_{q+1},\dots,t-s_n) \prod_{i=q+1}^{n} u_{j_i}(s_i)\,\mathrm{d}s_{q+1}\dots\mathrm{d}s_n \tag{4.11}$$

for all $t \le T$. As above, we see that $\chi_{j_0}^{\alpha,n,q}$ is contained in $C_b^1\big((-\infty,T],\mathcal{H}_\Sigma^{k-|\alpha|}(G)\big)$. Since $|\alpha| \le m-q \le m-1$, we have $\chi_{j_0}^{\alpha,n,q} \in C_b^1\big((-\infty,T],\mathcal{H}_\Sigma^{k-m+1}(G)\big)$. To treat the terms in the second sum in (4.4), we also define

$$\gamma_{j_0}^{\beta,q}(t) := \prod_{i=1}^{q} \partial_t^{\beta_i} u_{j_i}(t) \tag{4.12}$$

for all $t \le T$. Note that $\partial_t^{\beta_i} u_{j_i}$ belongs to $C_b^1\big((-\infty,T],\mathcal{H}_\Sigma^{k-\beta_i-1}(G)\big)$ for all $i \in \{1,\dots,q\}$. The property $\gamma_{j_0}^{\beta,q}\chi_{j_0}^{\alpha,n,q} \in C_b^1\big(I,\mathcal{H}_\Sigma^{k-m}(G)\big)$ is a consequence of (iii) and (iv) of Lemma 2.2 and the following case distinctions.

- $m \le k-2$: We have $k-m+1 \ge 3$ and $k-\beta_i-1 \ge k-m \ge 2$ since $\beta_i \le m-1$ for all $i \in \{1,\dots,q\}$.
- $k=2$, $m=1$: Here it holds $q=1$, $\alpha=0$ and $\beta=0$.
- $k>2$, $m=k-1$, $q=1$: We have $k-m+1=2$ and $k-\beta_1-1 \ge 1$.
- $k>2$, $m=k-1$, $q>1$: In this case, $k-m+1=2$ and $k-\beta_i-1 \ge 2$ because of $\beta_i \le m-q \le k-3$ for all $i \in \{1,\dots,q\}$.

In addition, we obtain

$$\begin{aligned}\partial_t\Big(\gamma_{j_0}^{\beta,q}\chi_{j_0}^{\alpha,n,q}\Big)(t) = {}& \sum_{r=1}^{q}\left(\prod_{i=1}^{q}\partial_t^{(\beta+e_r)_i}u_{j_i}(t)\right)\chi_{j_0}^{\alpha,n,q}(t) + \left(\prod_{i=1}^{q}\partial_t^{\beta_i}u_{j_i}(t)\right)\sum_{r=q+1}^{n}\chi_{j_0}^{\alpha+e_r,n,q}(t) \\ &+ (n-q)\left(\prod_{i=1}^{q}\partial_t^{\beta_i}u_{j_i}(t)\right)u_{j_{q+1}}(t)\cdot \\ &\cdot \int_{-\infty}^{t}\dots\int_{-\infty}^{t}\partial_\tau^\alpha R_{j_0\dots j_n}^{(n)}(0,\dots,0,t-s_{q+2},\dots,t-s_n)\prod_{i=q+2}^{n}u_{j_i}(s_i)\,\mathrm{d}s_{q+2}\dots\mathrm{d}s_n\end{aligned} \tag{4.13}$$

for all $t \in I$, if $q \le n-2$. In the case $q = n-1$, the last term in (4.13) is replaced by

$$\partial_\tau^\alpha R^{(n)}_{j_0\dots j_n}(0,\dots,0) u_{j_n}(t) \prod_{i=1}^{n-1} \partial_t^{\beta_i} u_{j_i}(t).$$

2.3) Let $n \le m$ and $\alpha, \beta \in \mathbb{N}_0^n$ with $|\alpha| + |\beta| = m - n$. We set

$$\omega_{j_0}^{\alpha,\beta,n}(t) := \partial_\tau^\alpha R^{(n)}_{j_0\dots j_n}(0,\dots,0) \prod_{i=1}^{n} \partial_t^{\beta_i} u_{j_i}(t) \tag{4.14}$$

for all $t \le T$. The term $\partial_\tau^\alpha R^{(n)}_{j_0\dots j_n}(0,\dots,0)$ is contained in $\mathcal{H}^k_\Sigma(G)$ if $\alpha = 0$ and in $\mathcal{H}^{k-|\alpha|+1}_\Sigma(G) \hookrightarrow \mathcal{H}^{k-m}_\Sigma(G)$ otherwise. Additionally, $\partial_t^{\beta_i} u_{j_i}$ is contained in the space $C^1_b\big((-\infty,T], \mathcal{H}^{k-\beta_i-1}_\Sigma(G)\big) \hookrightarrow C^1_b\big((-\infty,T], \mathcal{H}^{k-m}_\Sigma(G)\big)$. We argue similarly as above.

- $m \le k-2$: We use $k - m \ge 2$.
- $m = k-1$: Here it holds $k - |\alpha| + 1 \ge k - (m-n) + 1 = n+2$ as well as $k - \beta_i - 1 \ge k - (m-n) - 1 = n$ for all $i \in \{1,\dots,n\}$. If $n \ge 2$, we can again use the Banach algebra property, while for $n = 1$, we have $\partial_\tau^{\alpha_1} R^{(1)}(0) \in \mathcal{H}^2_\Sigma(G)$ and $\partial_t^{\beta_1} u_{j_1} \in C^1_b\big((-\infty,T], \mathcal{H}^1_\Sigma(G)\big)$, so the product is contained in $C^1_b\big((-\infty,T], \mathcal{H}^1_\Sigma(G)\big)$.

In all these cases, we derive $\omega_{j_0}^{\alpha,\beta,n} \in C^1_b\big((-\infty,T], \mathcal{H}^{k-m}_\Sigma(G)\big)$ and

$$\partial_t \omega_{j_0}^{\alpha,\beta,n}(t) = \partial_\tau^\alpha R^{(n)}_{j_0\dots j_n}(0,\dots,0) \sum_{p=1}^{n} \prod_{i=1}^{n} \partial_t^{(\beta+e_p)_i} u_{j_i}(t) \tag{4.15}$$

for $t \le T$. Cases 2.1)-2.3) yield $\partial_t^m \Gamma^{(n)}_{j_0}(u) \in C^1_b\big((-\infty,T], \mathcal{H}^{k-m}_\Sigma(G)\big)$ and that $\partial_t^{m+1} \Gamma^{(n)}_{j_0}(u)$ is of the form (4.4) with m replaced by $m+1$. This shows (4.8) for $\nu+1$ and we conclude that $\partial_t \Gamma^{(n)}_{j_0}(u) \in \bigcap_{\ell=0}^{k-1} C^\ell_b\big((-\infty,T], \mathcal{H}^{k-\ell}_\Sigma(G)\big)$ and $\partial_t^m \Gamma^{(n)}_{j_0}(u)$ is of the form (4.4) for all $m \in \{1,\dots,k\}$.

3) It remains to show that $\partial_t^k \Gamma^{(n)}_{j_0}(u)|_I$ is contained in $C^1_b\big(I, L^2(G)\big)$ for $I \in \{(-\infty,\vartheta], [\vartheta,T]\}$ and (4.4) for $m = k+1$.

3.1) Let $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = k$ and $\psi_{j_0}^{\alpha,n}$ be defined by (4.9). As in step 2.1), we see $\psi_{j_0}^{\alpha,n}$ belongs to $C^1_b\big((-\infty,T], L^2(G)\big)$ with time derivative given by (4.10).

3.2) Let $q \in \big\{1,\dots,\min\{k, n-1\}\big\}$, $\alpha \in \mathbb{N}_0^n$ and $\beta \in \mathbb{N}_0^q$ with $|\alpha|+|\beta| = k-q$. We define $\chi_{j_0}^{\alpha,n,q}$ and $\gamma_{j_0}^{\beta,q}$ as in (4.11) and (4.12). We again have $\chi_{j_0}^{\alpha,n,q} \in C^1_b\big((-\infty,T], \mathcal{H}^{k-|\alpha|}_\Sigma(G)\big)$, and $\gamma_{j_0}^{\beta,q} \chi_{j_0}^{\alpha,n,q}|_I$ is contained in $C^1_b\big(I, L^2(G)\big)$ is because of the following observations.

- $q = 1$: Here $\partial_t^{\beta_1} u_{j_1} \chi_{j_0}^{\alpha,n,q}|_I$ is contained in $C^1_b\big(I, L^2(G)\big)$ due to $k - |\alpha| + k - \beta_1 - 1 = k \ge 2$. (We use $fg \in L^2(G)$ if $f \in \mathcal{H}^a_\Sigma(G)$ and $g \in \mathcal{H}^b_\Sigma(G)$ with $a + b \ge 2$.)
- $k = q = 2$: We have $\alpha = 0$, $\beta = 0$, so $u_{j_i} \in C^1_b\big(I, \mathcal{H}^1_\Sigma(G)\big)$ for all $i \in \{1,2\}$ and $\chi_{j_0}^{\alpha,n,q} \in C^1_b\big(I, \mathcal{H}^2_\Sigma(G)\big)$.
- $k > 2$, $q > 1$: Here it holds $k - |\alpha| \ge 2$ and $|\beta| \le k-2$. If $\beta_i < k-2$ for all $i \in \{1,\dots,q\}$, we have $k - \beta_i - 1 \ge 2$ for all $i \in \{1,\dots,q\}$. If on the other hand there exists an index $i_0 \in \{1,\dots,q\}$ with $\beta_{i_0} = k-2$, then $\beta_i < k-2$ for all $i \ne i_0$. In this case $k - \beta_{i_0} - 1 = 1$ and $k - \beta_i - 1 \ge 2$ for $i \ne i_0$.

For the time derivative of $\gamma_{j_0}^{\beta,q} \chi_{j_0}^{\alpha,n,q}$ on I we obtain the same expression as in step 2.2).

3.3) Let $n \le k$ and $\alpha, \beta \in \mathbb{N}_0^n$ with $|\alpha| + |\beta| = k - n$ and $\omega_{j_0}^{\alpha,\beta,n}$ be given by (4.14). Since $k - |\alpha| + 1 \ge n+1$, we have $\partial_\tau^\alpha R^{(n)}_{j_0\dots j_n}(0,\dots,0) \in \mathcal{H}^2_\Sigma(G)$. It holds $\partial_t^{\beta_i} u_{j_i} \in$

$C_b^1\big((-\infty,T],\mathcal{H}_\Sigma^{k-\beta_i-1}(G)\big)$ with $k-\beta_i-1\geq n-1$ for all $i\in\{1,\dots,n\}$. If $n=1$, it directly follows $\omega_{j_0}^{\alpha,\beta,1}|_I\in C_b^1\big(I,L^2(G)\big)$, while for $n=2$ we can use Lemma 2.2 to obtain $\omega_{j_0}^{\alpha,\beta,n}|_I\in C_b^1\big(I,L^2(G)\big)$. The time derivative is given by (4.15). □

Lemma 4.3 leads to the core estimates for the nonlinearity of the Maxwell system. Since the polarization and magnetization only enter the Maxwell equations in the form of $\partial_t P$ and $\partial_t M$, it suffices to estimate $\partial_t\Gamma(u)$.

Corollary 4.4. *Let Assumptions 2.1 and 4.2 be true and $n\in\{1,\dots,N\}$. There exists a constant $C=C(k,n,\phi(0),\|\phi\|_{L^1((0,\infty))},G)>0$ (nondecreasing in $\phi(0)$ and $\|\phi\|_{L^1((0,\infty))}$) such that the following statements hold for any $r>0$, $T\in\mathbb{R}$ and $\vartheta<T$.*

(i) Let $u\in Z_\vartheta^k(T)$. Then we have $\partial_t\Gamma^{(n)}(u)\in Z_\vartheta^k(T)$ and

$$\left\|\partial_t\Gamma^{(n)}(u)\right\|_{Z_\vartheta^k(T)}\leq C\,\|u\|_{Z_\vartheta^k(T)}^n\,. \tag{4.16}$$

(ii) Let $u,\widetilde{u}\in Z_\vartheta^k(T)$ satisfy $\max\big\{\|u\|_{Z_\vartheta^k(T)},\|\widetilde{u}\|_{Z_\vartheta^k(T)}\big\}\leq r$. Then we have

$$\left\|\partial_t\Gamma^{(n)}(u)-\partial_t\Gamma^{(n)}(\widetilde{u})\right\|_{Z_\vartheta^k(T)}\leq Cr^{n-1}\,\|u-\widetilde{u}\|_{Z_\vartheta^k(T)}\,. \tag{4.17}$$

The above statements are also true if $Z_\vartheta^k(T)$ is replaced by $Z^k(T)$ everywhere.

Proof. We only prove the claims for $u\in Z_\vartheta^k(T)$ and omit stating the dependence of the constants on G and the various parameters k, m, n and q below.

(i) We have $\partial_t\Gamma^{(n)}(u)\in Z_\vartheta^k(T)$ by Lemma 4.3 (ii). Let $I\in\{(-\infty,\vartheta],[\vartheta,T]\}$. To prove (4.16), we need to estimate $\partial_t^m\Gamma^{(n)}(u)|_I(t)$ in $\mathcal{H}_\Sigma^{k-m+1}(G)$ for all $m\in\{1,\dots,k+1\}$. We treat the different terms in (4.4) separately.

1) Let $\alpha\in\mathbb{N}_0^n$ with $|\alpha|=m$ and define $\psi_{j_0}^{\alpha,n}$ as in (4.9). In view of the assumptions on u and R, Lemma 4.1 implies

$$\sup_{t\in I}\left\|\psi_{j_0}^{\alpha,n}(t)\right\|_{\mathcal{H}_\Sigma^{k-m+1}}\lesssim\|\phi\|_{L^1((0,\infty))}^n\sup_{t\leq T}\|u(t)\|_{\mathcal{H}_\Sigma^k}^n\lesssim\|\phi\|_{L^1((0,\infty))}^n\,\|u\|_{Z_\vartheta^k(T)}^n\,.$$

2) Let $q\in\big\{1,\dots,\min\{m,n-1\}\big\}$, $\alpha\in\mathbb{N}_0^n$ and $\beta\in\mathbb{N}_0^q$ with $|\alpha|+|\beta|=m-q$. In particular, it holds $|\alpha|<m$. Let $\chi_{j_0}^{\alpha,n,q}$ and $\gamma_{j_0}^{\beta,q}$ be defined as in (4.11) and (4.12). As in step 1), we bound

$$\sup_{t\in I}\left\|\chi_{j_0}^{\alpha,n,q}(t)\right\|_{\mathcal{H}_\Sigma^{k-|\alpha|+1}}\lesssim\phi(0)^q\,\|\phi\|_{L^1((0,\infty))}^{n-q}\,\|u\|_{Z_\vartheta^k(T)}^{n-q}\qquad\text{if }\ \alpha\neq0,$$
$$\sup_{t\in I}\left\|\chi_{j_0}^{0,n,q}(t)\right\|_{\mathcal{H}_\Sigma^k}\lesssim\phi(0)^q\,\|\phi\|_{L^1((0,\infty))}^{n-q}\,\|u\|_{Z_\vartheta^k(T)}^{n-q}\,.$$

Since $\partial_t^{\beta_i}u\in C_b\big(I,\mathcal{H}_\Sigma^{k-\beta_i}(G)\big)$ and $|\beta|=m-q-|\alpha|$, inequality (2.3) yields

$$\sup_{t\in I}\left\|\gamma_{j_0}^{\beta,q}(t)\right\|_{\mathcal{H}_\Sigma^{k-m+q+|\alpha|}}\lesssim\prod_{i=1}^q\sup_{t\in I}\left\|\partial_t^{\beta_i}u(t)\right\|_{\mathcal{H}_\Sigma^{k-\beta_i}}\lesssim\|u\|_{Z_\vartheta^k(T)}^q\,.$$

The estimate

$$\sup_{t\in I}\left\|\gamma_{j_0}^{\beta,q}(t)\chi_{j_0}^{\alpha,n}(t)\right\|_{\mathcal{H}_\Sigma^{k-m+1}}\lesssim\phi(0)^q\,\|\phi\|_{L^1((0,\infty))}^{n-q}\,\|u\|_{Z_\vartheta^k(T)}^n$$

is now a consequence of (2.2) and the following observations.

Let $v\in\mathcal{H}_\Sigma^{k-m+q+|\alpha|}(G)$. Let $w\in\mathcal{H}_\Sigma^k(G)$ if $\alpha=0$ and $w\in\mathcal{H}_\Sigma^{k-|\alpha|+1}(G)$ if $\alpha\neq0$.

- $\alpha = 0$: Then $\|vw\|_{\mathcal{H}^{k-m+1}_\Sigma} \leq \|vw\|_{\mathcal{H}^{k-m+q}_\Sigma} \lesssim \|v\|_{\mathcal{H}^{k-m+q}_\Sigma} \|w\|_{\mathcal{H}^k_\Sigma}$.
- $0 < |\alpha| < \min\{m, k\}$: Here it holds $k - |\alpha| + 1 \geq 2$ and

$$\min\{k - m + q + |\alpha|\,, k - |\alpha| + 1\} \geq k - m + 1$$

  and thus $\|vw\|_{\mathcal{H}^{k-m+1}_\Sigma} \lesssim \|v\|_{\mathcal{H}^{k-m+q+|\alpha|}_\Sigma} \|w\|_{\mathcal{H}^{k-|\alpha|+1}_\Sigma}$.
- $0 < |\alpha| < m$, $|\alpha| = k$: In this case we have $m = k+1$, $k-m+q+|\alpha| = k+q-1 \geq 2$ and $k - |\alpha| + 1 = 1$ and we deduce $\|vw\|_{L^2} \lesssim \|v\|_{\mathcal{H}^{k-m+q+|\alpha|}_\Sigma} \|w\|_{\mathcal{H}^1_\Sigma}$.

3) Let $n \leq m$, $\alpha, \beta \in \mathbb{N}_0^n$ with $|\alpha| + |\beta| = m - n$, and $\omega_{j_0}^{\alpha,\beta,n}$ be defined as in (4.14). The factor $\partial_\tau^\alpha R^{(n)}_{j_0\dots j_n}(0, \dots, 0)$ is contained in $\mathcal{H}^k_\Sigma(G)$ if $\alpha = 0$ and in $\mathcal{H}^{k-|\alpha|+1}_\Sigma(G)$ otherwise. Arguing as in step 2) with q replaced by n, we first obtain

$$\sup_{t\in I} \Big\| \prod_{i=1}^n \partial_t^{\beta_i} u_{j_i}(t) \Big\|_{\mathcal{H}^{k-m+n+|\alpha|}_\Sigma} \lesssim \|u\|^n_{Z^k_\vartheta(T)}$$

and finally

$$\sup_{t\in I} \Big\| \omega_{j_0}^{\alpha,\beta,n}(t) \Big\|_{\mathcal{H}^{k-m+1}_\Sigma} \lesssim \big(\phi(0)\big)^n \|u\|^n_{Z^k_\vartheta(T)}\,.$$

This concludes the proof of (4.16).

(ii) We show (4.17) by the same computations as above, combined with the formula

$$\prod_{m=1}^n v_m - \prod_{m=1}^n w_m = \sum_{\ell=1}^n \left(\prod_{m=1}^{\ell-1} v_m \right) (v_\ell - w_\ell) \prod_{m=\ell+1}^n w_m$$

to handle the differences of products. □

5 Local wellposedness

The polarization and magnetization depend in general on all past values of the electric and magnetic fields. To describe this, the following notation is used.

For $\vartheta, T \in \mathbb{R}$, $u = (E, H) \in Z^k_\vartheta(T)$ and $t \leq T$ we introduce the *history function* $u_t \in Z^k_{\vartheta-t}(0)$ by $u_t(s) := u(t + s)$ with $s \leq 0$. Thus u_t determines the past of u up to time t. We now define the map F representing the retarded part of the material law. Let

$$\nu(x) := \begin{pmatrix} \varepsilon(x)^{-1} & 0 \\ 0 & \mu(x)^{-1} \end{pmatrix} \in \mathbb{R}^{6\times 6} \quad \text{and} \quad F : Z^k_\vartheta(0)^6 \to \mathcal{H}^k_\Sigma(G)^6,\ F(v) := -\nu\partial_t\Gamma(v)(0).$$

By (4.5) and (4.6), in components this reads

$$F_i(v) = -\nu_{ij_0} \sum_{n=1}^N \left[\sum_{\ell=1}^n \int_{\mathbb{R}^n_{>0}} \partial_{\tau_\ell} R^{(n)}_{j_0\dots j_n}(\tau) \prod_{k=1}^n v_{j_k}(-\tau_k)\,\mathrm{d}\tau + \int_{\partial\mathbb{R}^n_{>0}} R^{(n)}_{j_0\dots j_n}(\tau) \prod_{k=1}^n v_{j_k}(-\tau_k)\,\mathrm{d}\tau \right]$$

for $i \in \{1, \dots, 6\}$. Here we have used the intrinsic permutation symmetry of the response functions to write the second terms on the right-hand sides of (4.5) and (4.6) as a boundary integral. Inserting u_t, the nonlinearities are written as

$$F(u_t) = -\nu\partial_t\Gamma(u)(t) = -\big(\varepsilon^{-1}\partial_t P_{\mathrm{ret}}(E, H)(t), \mu^{-1}\partial_t M_{\mathrm{ret}}(E, H)(t)\big)$$

for all $t \leq T$. Since the compatibility conditions of Definition 3.1 involve time derivatives of the inhomogeneity, we furthermore need time derivatives of the map $t \mapsto F(u_t)$. To this end we define

$$F^{\langle \ell \rangle}(u) := -\nu \partial_t^{\ell+1} \Gamma(u)(0).$$

for $\ell \in \{0, \dots, k\}$, and have

$$F^{\langle \ell \rangle}(u_t) = -\nu \partial_t^{\ell+1} \Gamma(u)(t) = -\left(\varepsilon^{-1} \partial_t^{\ell+1} P_{\mathrm{ret}}(E,H)(t), \mu^{-1} \partial_t^{\ell+1} M_{\mathrm{ret}}(E,H)(t)\right)$$

for all $t \leq T$. Note that in the part $\Gamma^{(n)}(u)$ of the material law $\Gamma(u) = \sum_{n=1}^{N} \Gamma^{(n)}(u)$, the superscript (n) refers to the order of dependence on u, while the superscript $\langle \ell \rangle$ in $F^{\langle \ell \rangle}$ denotes the number $\ell + 1$ of time derivatives of $\Gamma(u)$.

Inequalities (2.1) and Corollary 4.4 imply the following two estimates which play a crucial role in the fixed-point argument of Lemma 5.2 below.

Corollary 5.1. *Let Assumptions 2.1 and 4.2 be true. There exists a constant*

$$C_2 = C_2(k, N, \|\varepsilon\|_{\mathcal{C}^k_\Sigma}, \|\mu\|_{\mathcal{C}^k_\Sigma}, \eta^{-1}, \phi(0), \|\phi\|_{L^1((0,\infty))}, G) > 0$$

(nondecreasing in $\|\varepsilon\|_{\mathcal{C}^k_\Sigma}$, $\|\mu\|_{\mathcal{C}^k_\Sigma}$, η^{-1}, $\phi(0)$ and $\|\phi\|_{L^1((0,\infty))}$) such that for any $r > 0$, $T \in \mathbb{R}$, $\vartheta < T$ and $u, \widetilde{u} \in Z^k_\vartheta(T)$ satisfying $\max\{\|u\|_{Z^k_\vartheta(T)}, \|\widetilde{u}\|_{Z^k_\vartheta(T)}\} \leq r$, we have $F(u_\bullet)$, $F(\widetilde{u}_\bullet) \in Z^k_\vartheta(T)$ and it holds

$$\begin{aligned} \|F(u_\bullet)\|_{Z^k_\vartheta(T)} &\leq C_2\left(1 + r^N\right), \\ \|F(u_\bullet) - F(\widetilde{u}_\bullet)\|_{Z^k_\vartheta(T)} &\leq C_2\left(1 + r^{N-1}\right) \|u - \widetilde{u}\|_{Z^k_\vartheta(T)}. \end{aligned}$$

These statements are also true if $Z^k_\vartheta(T)$ is replaced by $Z^k(T)$ everywhere.

We further rewrite the external free current density as the inhomogeneity

$$g(t) := -\left(\varepsilon^{-1} J_0(t), 0\right) \in \mathbb{R}^6$$

for all $t \geq 0$. In order to include currents induced by a conductivity, on $\mathcal{H}^k_\Sigma(G)^6$ we define the bounded linear operator

$$Lu = \begin{pmatrix} -\varepsilon^{-1}\sigma & 0 \\ 0 & 0 \end{pmatrix} u \qquad \text{and} \qquad C_3 := \max_{m \in \{0,\dots,k\}} \|L\|_{\mathcal{B}(\mathcal{H}^m_\Sigma)}, \tag{5.1}$$

where $C_3 = C_3\left(k, \|\varepsilon\|_{\mathcal{C}^k_\Sigma}, \|\sigma\|_{\mathcal{C}^k_\Sigma}, \eta^{-1}\right) \geq 0$ is nondecreasing in $\|\varepsilon\|_{\mathcal{C}^k_\Sigma}$, $\|\sigma\|_{\mathcal{C}^k_\Sigma}$ and η^{-1}. The Maxwell system (1.4) thus becomes the retarded evolution equation

$$\begin{aligned} \partial_t u(t) &= (A + L)u(t) + F(u_t) + g(t), && t \geq 0, \\ K_{\partial G, \Sigma}\, u(t) &= 0, && t \geq 0, \\ u(t) &= u_{\mathrm{h}}(t), && t \leq 0, \end{aligned} \tag{5.2}$$

with the prescribed initial history u_{h} of the system. The operators A and $K_{\partial G, \Sigma}$ were defined in (2.10), respectively (2.11).

Lemma 5.2 below is the basic step to establish local wellposedness of (5.2). It relies on a fixed-point argument, in which we consider $f(t) = Lu(t) + F(u_t) + g(t)$ as a fixed inhomogeneity and use the results for the linear problem from Theorem 3.2. This result

requires $u_{\mathrm{h}}(0)$ and f to satisfy the compatibility conditions (5.4) involving $\partial_t^\ell f(0)$ for $\ell \in \{0,\dots,k-2\}$. Furthermore, there are conditions on u_{h} which ensure that the solution has the necessary time regularity at $t=0$. Corollary 5.1 and the product rules in Lemma 2.2 are used to obtain the needed estimates for f.

In the case $k=2$, by Lemma 2.4 the conditions in (5.4) can be equivalently written in terms of the initial fields $u_{\mathrm{h}} = (E_{\mathrm{h}}, H_{\mathrm{h}})$ and the external current density J_0 as

$$\begin{aligned}
\operatorname{curl} H_{\mathrm{h}}(0) - \sigma E_{\mathrm{h}}(0) - \partial_t P_{\mathrm{ret}}(E_{\mathrm{h}}, H_{\mathrm{h}})(0) - J_0(0) &= \varepsilon \partial_t E^{\mathrm{h}}(0),\\
-\operatorname{curl} E_{\mathrm{h}}(0) - \partial_t M_{\mathrm{ret}}(E_{\mathrm{h}}, H_{\mathrm{h}})(0) &= \mu \partial_t H^{\mathrm{h}}(0),\\
\mathrm{n} \times E_{\mathrm{h}}(0)|_{\partial G} &= 0,\\
[\nu \times E_{\mathrm{h}}(0)]_\Sigma = [\nu \times H_{\mathrm{h}}(0)]_\Sigma &= 0,\\
\mathrm{n} \times \partial_t E_{\mathrm{h}}(0)|_{\partial G} &= 0,\\
[\nu \times \partial_t E_{\mathrm{h}}(0)]_\Sigma = [\nu \times \partial_t H_{\mathrm{h}}(0)]_\Sigma &= 0.
\end{aligned}$$

Lemma 5.2 has two parts: In the first one, the initial function u_{h} lies in $Z^k(0)$ and we obtain a solution u in the space $Z_0^k(b_0)$ for some small positive time b_0. So $\partial_t^k u$ possibly has a jump at time zero. To extend u to a maximal time interval, we need to be able to use the shifted function $u(\bullet + b_0)$ as initial datum u_{h}. In the second part we therefore allow for the kth time derivative of u_{h} to have a jump at a negative time. Here we impose additionally a condition on $\partial_t^k u_{\mathrm{h}}(0)$ which ensures the continuity of the kth time derivative of the solution at time zero, so no second jump occurs. The condition on $\partial_t^k u_{\mathrm{h}}(0)$ is automatically fulfilled if u_{h} solves $\partial_t v = (A+L)v(t) + F(v_t) + g(t)$ on an interval around zero. This is the case for $u_{\mathrm{h}} = u(\bullet + b_0)$.

The time variable T below serves as a cut-off for the inhomogeneity g. This plays a role in the proof of Theorem 5.8: It is used to ensure that in the result for the continuous dependence of the solution on u_{h} and g on an interval $[0,b]$ no information about g at times $t > b$ enters.

In the proof of Lemma 5.2, we use the fact that the matrix $V \in \mathbb{R}^{(k+1)\times(k+1)}$ with entries $v_{ij} = \big(-\frac{j}{k+1}\big)^{i-1}$ is invertible (k is the parameter from Assumption 2.1) since it is the transpose of the Vandermonde-matrix corresponding to $\big(-\frac{1}{k+1},\dots,-\frac{k+1}{k+1}\big)$. We define the constant

$$C_4(k) = C_4 := \sum\nolimits_{j=1}^{k+1} |a_j| \geq 0, \tag{5.3}$$

where $a \in \mathbb{R}^{k+1}$ is the solution of $Va = \mathbb{1}$ with $\mathbb{1} := (1,\dots,1) \in \mathbb{R}^{k+1}$.

Lemma 5.2. *Let Assumptions 2.1 and 4.2 be true, $T > 0$ and $r_0 > 0$. We set*

$$r := 1 + (2C_1 + C_1C_3 + C_4)r_0 + C_1C_2(1 + r_0^N) > 2r_0$$

with the constants $C_1 \geq 1$, C_2, C_3 and C_4 from Theorem 3.2, Corollary 5.1 and equations (5.1) and (5.3). Furthermore, the radius

$$r = r(r_0, k, N, \|\varepsilon\|_{\mathcal{C}_\Sigma^k}, \|\mu\|_{\mathcal{C}_\Sigma^k}, \|\sigma\|_{\mathcal{C}_\Sigma^k}, \eta^{-1}, \phi(0), \|\phi\|_{L^1((0,\infty))}, G)$$

is nondecreasing in r_0, $\|\varepsilon\|_{\mathcal{C}_\Sigma^k}$, $\|\mu\|_{\mathcal{C}_\Sigma^k}$, $\|\sigma\|_{\mathcal{C}_\Sigma^k}$, η^{-1}, $\phi(0)$, and $\|\phi\|_{L^1((0,\infty))}$.

(i) There exists a time $b_0 = b_0(r,T) \in (0,T]$ (given by (5.11) below) such that for every $u_{\mathrm{h}} \in Z^k(0)$ and $g \in Z^k\big([0,\infty)\big)$ satisfying

$$\begin{aligned}
&\partial_t^{\ell+1} u_{\mathrm{h}}(0) = (A+L)\partial_t^\ell u_{\mathrm{h}}(0) + F^{\langle \ell \rangle}(u_{\mathrm{h}}) + \partial_t^\ell g(0), \quad \ell = 0,\dots,k-2,\\
&\partial_t^\ell u_{\mathrm{h}}(0) \in \mathrm{D}(A), \quad \ell = 0,\dots,k-1,
\end{aligned} \tag{5.4}$$

as well as

$$\|u_{\mathrm{h}}\|_{Z^k(0)} + \|g\|_{Z^k([0,T])} \le r_0, \tag{5.5}$$

there is a solution $u \in Z_0^k(b_0) \cap C^{k-1}\big([0,b_0], X_1\big)$ *of* (5.2) *on* $(-\infty, b_0]$. *It fulfills*

$$u(t) = S(t)\,(u_{\mathrm{h}}(0)) + \int_0^t S(t-\tau)\big(Lu(\tau) + F(u_\tau) + g(\tau)\big)\,\mathrm{d}\tau$$

for all $t \in [0, b_0]$. *For each* $b \in (0, b_0]$, *the restriction* $u|_{(-\infty,b]}$ *is the only solution in* $Z_0^k(b)$ *with* $\|u\|_{Z_0^k(b)} \le r$.

(ii) Let $\vartheta < 0$. *There exists a time* $\widetilde{b}_0 = \widetilde{b}_0(r, \vartheta, T) \in \big(0, \min\{-\vartheta, T\}\big]$ *(given by* (5.12) *below) such that for each* $u_{\mathrm{h}} \in Z_\vartheta^k(0)$ *and* $g \in Z^k\big([0,\infty)\big)$ *satisfying*

$$\begin{aligned} &\partial_t^{\ell+1} u_{\mathrm{h}}(0) = (A+L)\partial_t^\ell u_{\mathrm{h}}(0) + F^{\langle \ell \rangle}(u_{\mathrm{h}}) + \partial_t^\ell g(0), \quad \ell = 0, \dots, k-1, \\ &\partial_t^\ell u_{\mathrm{h}}(0) \in \mathrm{D}(A), \quad \ell = 0, \dots, k-1, \end{aligned} \tag{5.6}$$

as well as

$$\|u_{\mathrm{h}}\|_{Z_\vartheta^k(0)} + \|g\|_{Z^k([0,T])} \le r_0 \tag{5.7}$$

there is a solution $u \in Z_\vartheta^k(\widetilde{b}_0) \cap C^{k-1}\big([0,\widetilde{b}_0], X_1\big)$ *of* (5.2) *on* $(-\infty, \widetilde{b}_0]$. *It fulfills*

$$u(t) = S(t)\,(u_{\mathrm{h}}(0)) + \int_0^t S(t-\tau)\big(Lu(\tau) + F(u_\tau) + g(\tau)\big)\,\mathrm{d}\tau$$

for all $t \in [0, \widetilde{b}_0]$. *For each* $b \in (0, \widetilde{b}_0]$, *the restriction* $u|_{(-\infty,b]}$ *is the only solution in* $Z_\vartheta^k(b)$ *with* $\|u\|_{Z_\vartheta^k(b)} \le r$.

In both cases, the conditions (5.4) *respectively* (5.6) *are preserved in time; i.e.,*

$$\begin{aligned} &\partial_t^{\ell+1} u(t) = (A+L)\partial_t^\ell u(t) + F^{\langle \ell \rangle}(u_t) + \partial_t^\ell g(t), \quad \ell = 0, \dots, k-1, \\ &\partial_t^\ell u(t) \in \mathrm{D}(A), \quad \ell = 0, \dots, k-1, \end{aligned} \tag{5.8}$$

hold for all $t \in (0, b_0]$ *respectively* $t \in (0, \widetilde{b}_0]$. *The times* b_0 *and* $\widetilde{b}_0$ *are nonincreasing in* r_0, $\|\varepsilon\|_{\mathcal{C}_\Sigma^k}$, $\|\mu\|_{\mathcal{C}_\Sigma^k}$, $\|\sigma\|_{\mathcal{C}_\Sigma^k}$, η^{-1}, $\phi(0)$, *and* $\|\phi\|_{L^1((0,\infty))}$.

Proof. (i) Let $u_{\mathrm{h}} \in Z^k(0)$ and $g \in Z^k([0,\infty))$ satisfy (5.4) and (5.5). For a time $b \in (0, T]$ specified later, we define the space

$$\begin{aligned} Z_0^k(b,r) := \big\{ u \in Z_0^k(b) \;\big|\; u|_{(-\infty,0]} = u_{\mathrm{h}},\ \|u\|_{Z_0^k(b)} \le r, \\ \partial_t^{\ell+1} u(0) = (A+L)\partial_t^\ell u(0) + F^{\langle \ell \rangle}(u_{\mathrm{h}}) + \partial_t^\ell g(0),\ \ell = 0, \dots, k-2 \big\} \end{aligned}$$

with $r > 2r_0$ from the statement.

1) By means of Corollary 5.1 one easily checks that $Z_0^k(b,r)$ is closed in $Z_0^k(b)$ and therefore the metric induced by $\|\cdot\|_{Z_0^k(b)}$ is complete on $Z_0^k(b,r)$. To see that it is nonempty, we set

$$\widehat{u}(t) := \begin{cases} u_{\mathrm{h}}(t), & t \le 0, \\ \sum_{j=1}^{k+1} a_j u_{\mathrm{h}}\left(-\frac{j}{k+1} t\right), & 0 < t \le b, \end{cases} \tag{5.9}$$

for constants $a = (a_1, \dots, a_{k+1}) \in \mathbb{R}^{k+1}$ satisfying $Va = \mathbb{1}$ with the Vandermonde matrix V from above. Then $\partial_t^\ell \widehat{u}$ is continuous in $\mathcal{H}_\Sigma^{k-\ell}(G)^6$ at time zero for all $\ell \in \{0, \dots, k\}$,

so that $\widehat{u}$ is contained in $Z^k(b) \subseteq Z_0^k(b)$. Furthermore, the time derivatives of $\widehat{u}$ at $t=0$ satisfy the conditions of $Z_0^k(b,r)$ due to (5.4). Finally, it holds

$$\begin{aligned}\|\widehat{u}(t)\|_{Z_0^k(b)} &\leq \sum_{\ell=0}^{k} \sup_{t\leq 0} \left\|\partial_t^\ell u_{\mathrm{h}}(t)\right\|_{\mathcal{H}_\Sigma^{k-\ell}(G)} + \sum_{\ell=0}^{k}\sum_{j=1}^{k+1} |a_j| \left(\frac{j}{k+1}\right)^\ell \sup_{0\leq t\leq b} \left\|\partial_t^\ell u_{\mathrm{h}}(-jt)\right\|_{\mathcal{H}_\Sigma^{k-\ell}} \\ &\leq (1+C_4)r_0 \leq r.\end{aligned}$$

2) On $Z_0^k(b,r)$ we define the map $\Phi_{u_{\mathrm{h}},g}$ by

$$\Phi_{u_{\mathrm{h}},g}(u)(t) = \begin{cases} S(t)u_{\mathrm{h}}(0) + \int_0^t S(t-\tau)\big(Lu(\tau)+F(u_\tau)+g(\tau)\big)\,\mathrm{d}\tau, & 0<t\leq b, \\ u_{\mathrm{h}}(t), & t\leq 0. \end{cases} \tag{5.10}$$

and prove that it is a strict contraction for sufficiently small b. Let u and $\widetilde{u}$ be contained in $Z_0^k(b,r)$ and let $f:[0,b]\to\mathcal{H}_\Sigma^k(G)^6$ be given by $f(t) := Lu(t)+F(u_t)+g(t)$. Due to Corollary 5.1 and the assumptions, f is contained in $Z^k\big([0,b]\big)$. Set $w_0 := u_{\mathrm{h}}(0)$ and $w_{\ell+1} := Aw_\ell + \partial_t^\ell f(0)$ for $\ell\in\{0,\dots,k-2\}$. We prove that $w_\ell = \partial_t^\ell u_{\mathrm{h}}(0)$ belongs to $\mathrm{D}(A)$ for all $\ell\in\{0,\dots,k-1\}$. By assumptions, this is true for $\ell=0$. Let $w_\ell = \partial_t^\ell u_{\mathrm{h}}(0)\in\mathrm{D}(A)$ for some $\ell\in\{0,\dots,k-2\}$. Since $u\in Z_0^k(b,r)$, the function $\partial_t^{\ell+1}u$ is continuous at time zero in $\mathcal{H}_\Sigma^{k-\ell}(G)^6$ and we have

$$w_{\ell+1} = (A+L)\partial_t^\ell u_{\mathrm{h}}(0) + F^{\langle\ell\rangle}(u_{\mathrm{h}}) + \partial_t^\ell g(0) = \partial_t^{\ell+1}u_{\mathrm{h}}(0)\in\mathrm{D}(A)$$

by (5.4). Therefore the pair $\big(u_{\mathrm{h}}(0), f\big)$ satisfies the compatibility conditions of order k. A comparison of (5.10) with (3.3) shows that $v := \Phi_{u_{\mathrm{h}},g}(u)|_{[0,b]}$ is the unique solution of

$$\partial_t v(t) = Av(t)+f(t),\ t\geq 0, \qquad K_{\partial G,\Sigma}\,v(t)=0,\ t\geq 0, \qquad v(0)=u_{\mathrm{h}}(0)$$

on $[0,b]$. So Theorem 3.2 yields $v\in Z^k\big([0,b]\big)\cap C^{k-1}\big([0,b],X_1\big)$ and

$$\partial_t^{\ell+1}v(t) = A\partial_t^\ell v(t) + \partial_t^\ell f(t)$$

for all $t\in[0,b]$ and $\ell\in\{0,\dots,k-1\}$ as well as

$$\|v\|_{Z^k([0,b])} \leq C_1\Big(\|u_{\mathrm{h}}(0)\|_{\mathcal{H}_\Sigma^k} + \sum_{\ell=0}^{k-1}\left\|\partial_t^\ell f(0)\right\|_{\mathcal{H}_\Sigma^{k-\ell-1}} + \sum_{\ell=0}^{k}\left\|\partial_t^\ell f\right\|_{L_b^1\mathcal{H}_\Sigma^{k-\ell}}\Big).$$

From Corollary 5.1 and 1) we then deduce

$$\|v\|_{Z^k([0,b])} \leq C_1\Big((1+C_3)r_0 + C_2(1+r_0^N) + b\big((1+C_3)r + C_2(1+r^N)\big)\Big)$$

using that $\partial_t^\ell u(0) = \partial_t^\ell u_{\mathrm{h}}(0)$ for all $\ell\in\{0,\dots,k-1\}$. By (5.4), the function $\Phi_{u_{\mathrm{h}},g}(u)$ is contained in $Z_0^k(b)$ and satisfies

$$\|\Phi_{u_{\mathrm{h}},g}(u)\|_{Z_0^k(b)} \leq C_1\Big((2+C_3)r_0 + C_2(1+r_0^N) + b\big((1+C_3)r + C_2(1+r^N)\big)\Big).$$

3) In the next step we estimate the difference $d := \Phi_{u_{\mathrm{h}},g}(u) - \Phi_{u_{\mathrm{h}},g}(\widetilde{u})$. It is given by

$$d(t) = \begin{cases} \int_0^t S(t-\tau)h(\tau)\,\mathrm{d}\tau, & 0<t\leq b, \\ 0, & t\leq 0, \end{cases}$$

where $h(t) := L\big(u(t) - \widetilde{u}(t)\big) + F(u_t) - F(\widetilde{u}_t)$ for $t \in [0,b]$. As in step 2) we see that h is contained in $Z^k\big([0,b]\big)$. The functions u and $\widetilde{u}$ coincide on $(-\infty,0]$ and it holds $\partial_t^\ell u(0) = \partial_t^\ell \widetilde{u}(0)$ for all $\ell \in \{0,\dots,k-1\}$. Equation (4.4) therefore implies that $\partial_t^\ell h(0)$ vanishes for all $\ell \in \{0,\dots,k-1\}$. So the pair $(0,h)$ satisfies the compatibility conditions of order k. Since d solves

$$\partial_t d(t) = Ad(t) + h(t),\ t \geq 0, \qquad K_{\partial G,\Sigma}\, d(t) = 0,\ t \geq 0, \qquad d(0) = 0,$$

on $[0,b]$, Theorem 3.2 now yields

$$\big\|d|_{[0,b]}\big\|_{Z^k([0,b])} \leq C_1 \sum\nolimits_{\ell=0}^{k} \Big\|\partial_t^\ell h\Big\|_{L^1_b \mathcal{H}^{k-\ell}_\Sigma}.$$

Corollary 5.1 and the fact that $d(t)$ vanishes for $t \leq 0$ lead to

$$\|d\|_{Z^k_0(b)} \leq bC_1\big(C_3 + C_2\left(1 + r^{N-1}\right)\big)\, \|u - \widetilde{u}\|_{Z^k_0(b)}.$$

We set

$$b_0(r,T) := \min\left\{T, \frac{1}{C_1\big((1+C_3)\,r + C_2\,(1+r^N)\big)}, \frac{1}{2C_1\big(C_3 + C_2\,(1+r^{N-1})\big)}\right\}. \tag{5.11}$$

Then for any $b \in \big(0, b_0(r,T)\big]$, the map Φ_{u_h} is a strict contraction on $Z^k_0(b,r)$ and Banachs fixed-point theorem yields a solution $u^{(b)} = \Phi_{u_\mathrm{h},g}(u^{(b)})$ of (5.2) on $(-\infty,b]$ which is unique in the space $Z^k_0(b,r)$. We set $u := u^{(b_0(r,T))}$. The restriction of u to $(-\infty,b]$ is also a solution whose norm in $Z^k_0(b)$ is bounded by r. Uniqueness thus implies that it coincides with $u^{(b)}$ on the interval $(-\infty,b]$.

(ii) We prove the second part of the lemma analogously. Let $u_\mathrm{h} \in Z^k_\vartheta(0)$ and $g \in Z^k\big([0,\infty)\big)$ satisfy (5.6) and (5.7). For $b \in \big(0, \min\{-\vartheta, T\}\big]$, we set

$$\begin{aligned} Z^k_\vartheta(b,r) := \big\{u \in Z^k_\vartheta(b) \;\big|\; u|_{(-\infty,0]} = u_\mathrm{h},\ \|u\|_{Z^k_\vartheta(b)} \leq r, \\ \partial_t^{\ell+1} u(0) = (A+L)\partial_t^\ell u(0) + F^{\langle \ell \rangle}(u_\mathrm{h}) + \partial_t^\ell g(0),\ \ell = 0,\dots,k-1\big\}. \end{aligned}$$

1) Using $\widehat{u}$ from (5.9) and $b \leq -\vartheta$, we again see that $Z^k_\vartheta(b,r)$ is a nonempty.

2) We show that $\Phi_{u_\mathrm{h},g}$ from part (i) is a strict contraction on $Z^k_\vartheta(b,r)$ for any $b \in \big(0, \widetilde{b}_0(r_0,\vartheta,T)\big]$, where

$$\widetilde{b}_0(r,\vartheta,T) := \min\{-\vartheta, b_0(r,T)\}, \tag{5.12}$$

by the same reasoning. The assumption on $\partial_t^k u_\mathrm{h}(0)$ from (5.6) ensures that $\partial_t^k \Phi_{u_\mathrm{h},g}(u)$ is continuous in $L^2(G)^6$ at time zero for all $u \in Z^k_\vartheta(b,r)$.

We finally note that in both cases, $\partial_t^\ell u(t)$ is contained in $\mathrm{D}(A)$ for all $\ell \in \{0,\dots,k-1\}$ as a consequence of Theorem 3.2 and therefore the boundary and interface conditions hold for all $t \in [0,b_0]$ respectively $t \in [0,\widetilde{b}_0]$. In case (i), we can allow for $\ell = k-1$ in the first line of (5.8), since $\partial_t^k u$ can only have a discontinuity at $t = 0$. □

We now show that concatenating and shifting solutions lead again to solutions. The straightforward proof is omitted, see Lemma 8.3 in [7].

Lemma 5.3. *Let Assumptions 2.1 and 4.2 be true and $b_2, b_1 > 0$. Let $u_\mathrm{h} \in Z^k(0)$ and $g \in Z^k\big([0,\infty)\big)$ satisfy (5.4) and $u \in Z^k_0(b_1) \cap C^{k-1}\big([0,b_1], X_1\big)$ solve (5.2) on $(-\infty,b_1]$.*

(i) We define $\widetilde{u}_{\mathrm{h}} : (-\infty, 0] \to \mathcal{H}^k_\Sigma(G)^6$ *and* $\widetilde{g} : [0,\infty) \to \mathcal{H}^k_\Sigma(G)^6$ *by* $\widetilde{u}_{\mathrm{h}}(t) = u(t+b_1)$ *for* $t \le 0$ *and* $\widetilde{g}(t) = g(t+b_1)$ *for* $t \ge 0$*. Then* $\widetilde{u}_{\mathrm{h}}$ *is contained in* $Z^k_{-b_1}(0)$ *and satisfies* (5.6) *with* u_{h} *and* g *replaced by* $\widetilde{u}_{\mathrm{h}}$ *and* $\widetilde{g}$*. Let* $\widetilde{u} \in Z^k_{-b_1}(b_2) \cap C^{k-1}\big([0,b_2], X_1\big)$ *be a solution of*

$$\begin{aligned} \partial_t \widetilde{u}(t) &= (A+L)\widetilde{u}(t) + F(\widetilde{u}_t) + \widetilde{g}(t), && t \ge 0,\\ K_{\partial G,\Sigma}\, \widetilde{u}(t) &= 0, && t \ge 0,\\ \widetilde{u}(t) &= \widetilde{u}_{\mathrm{h}}(t), && t \le 0, \end{aligned}$$

on $(-\infty, b_2]$*. Then the function* v *given by*

$$v(t) = \begin{cases} u(t), & t \le b_1,\\ \widetilde{u}(t-b_1), & b_1 < t \le b_1 + b_2, \end{cases}$$

is contained in $Z^k_0(b_1+b_2)$ *and solves* (5.2) *on* $(-\infty, b_1+b_2]$*.*

(ii) Let $\beta \in (0, b_1)$*. We define* $\widehat{u}_{\mathrm{h}} : (-\infty, 0] \to \mathcal{H}^k_\Sigma(G)^6$ *and* $\widehat{g} : [0,\infty) \to \mathcal{H}^k_\Sigma(G)^6$ *by* $\widehat{u}_{\mathrm{h}}(t) = u(t+\beta)$ *for* $t \le 0$ *and* $\widehat{g}(t) = g(t+\beta)$ *for* $t \ge 0$*. Then* $\widehat{u}_{\mathrm{h}}$ *is contained in* $Z^k_{-\beta}(0)$ *and* $\widehat{u} \in Z^k_{-\beta}(b_1-\beta)$ *given by* $\widehat{u}(t) = u(t+\beta)$ *for* $t \le b_1 - \beta$ *is a solution of*

$$\begin{aligned} \partial_t \widehat{u}(t) &= (A+L)\widehat{u}(t) + F(\widehat{u}_t) + \widehat{g}(t), && t \ge 0,\\ K_{\partial G,\Sigma}\, \widehat{u}(t) &= 0, && t \ge 0,\\ \widehat{u}(t) &= \widehat{u}_{\mathrm{h}}(t), && t \le 0, \end{aligned}$$

on $(-\infty, b_1 - \beta]$*.*

Lemma 5.2 yields the existence of a local solution of (5.2), but the uniqueness statement contains a condition on the size of the initial data and inhomogeneity. We now show that solutions are in fact unique unconditionally.

Lemma 5.4. *Let Assumptions 2.1 and 4.2 be true. Let* $0 < T_1 \le T_2$*,* $u_{\mathrm{h}} \in Z^k(0)$*,* $g \in Z^k\big([0,\infty)\big)$ *satisfy* (5.4)*, and let* $u \in Z^k_0(T_1)$*,* $v \in Z^k_0(T_2)$ *be solutions of* (5.2) *on* $(-\infty, T_1]$*, respectively* $(-\infty, T_2]$*. Then* $u(t) = v(t)$ *for all* $t \le T_1$*.*

Proof. We set

$$\hat{t} := \sup\big\{ t \le T_1 \;\big|\; u(\tau) = v(\tau) \text{ for all } \tau \le t \big\}.$$

Then we have $\hat{t} \ge 0$ and by continuity of u and v in time also $u(\hat{t}) = v(\hat{t})$. We assume $\hat{t} < T_1$ and define the shifted functions $\widehat{u}_{\mathrm{h}} : (-\infty, 0] \to \mathcal{H}^k_\Sigma(G)^6$ and $\widehat{g} : [0,\infty) \to \mathcal{H}^k_\Sigma(G)^6$ by $\widehat{u}_{\mathrm{h}}(t) = u(t+\hat{t})$ and $\widehat{g}(t) := g(t+\hat{t})$. According to Lemma 5.3, the functions $\widehat{u} := u(\,\cdot + \hat{t}) \in Z^k_{-\hat{t}}(T_1 - \hat{t})$ and $\widehat{v} := v(\,\cdot + \hat{t}) \in Z^k_{-\hat{t}}(T_2 - \hat{t})$ are solutions of

$$\begin{aligned} \partial_t w(t) &= (A+L)w(t) + F(w_t) + \widehat{g}(t), && t \ge 0,\\ K_{\partial G,\Sigma}\, w(t) &= 0, && t \ge 0,\\ w(t) &= \widehat{u}_{\mathrm{h}}(t), && t \le 0, \end{aligned} \tag{5.13}$$

on $(-\infty, T_1 - \hat{t}]$. We set $\widehat{r}_0 := \|\widehat{u}_{\mathrm{h}}\|_{Z^k_{-\hat{t}}(0)} + \|\widehat{g}\|_{Z^k([0,T])}$ for some fixed $T > 0$. Lemma 5.2 yields a time $\widehat{b}_0 = b_0(\widehat{r}_0, \hat{t}, T) \in (0, T]$ and a solution $w \in Z^k_{-\hat{t}}(\widehat{b}_0)$ of (5.13) on $(-\infty, \widehat{b}_0]$. It is the only solution in $Z^k_{-\hat{t}}(\widehat{b}_0)$ which satisfies

$$\|w\|_{Z^k_{-\hat{t}}(b_0)} \le 1 + (2C_1 + C_1C_3 + C_4)\widehat{r}_0 + C_1C_2(1 + \widehat{r}_0^N).$$

Using continuity, we can now find a time $b_1 \in (0, b_0]$ with $\hat{t} + b_1 \le T_1$ and

$$\max\{\|\widehat{u}\|_{Z^k_{-\hat{t}}(b_1)}, \|\widehat{v}\|_{Z^k_{-\hat{t}}(b_1)}\} \le 1 + (2C_1 + C_1C_3 + C_4)\widehat{r}_0 + C_1C_2(1 + \widehat{r}_0^N).$$

The uniqueness statement in Lemma 5.2 then implies $\widehat{u}(t) = w(t) = \widehat{v}(t)$ for all $t \le b_1$. Shifting back leads to $u(t) = v(t)$ for all $t \le \hat{t} + b_1$, contradicting the definition of $\hat{t}$. □

By Lemmas 5.3 and 5.4, we extend the solution from Lemma 5.2 to a maximal interval.

Definition 5.5. *Let Assumptions 2.1 and 4.2 be true. For each $u_\mathrm{h} \in Z^k(0)$ and $g \in Z^k\big([0,\infty)\big)$ satisfying (5.4), we define the* maximal existence time

$$t^+(u_\mathrm{h}, g, k) := \sup\big\{b \;\big|\; \textit{there exists a solution } u \in Z^k_0(b) \textit{ of (5.2) on } (-\infty, b]\big\}.$$

The interval $I^+(u_\mathrm{h}, g, k) := \big(-\infty, t^+(u_\mathrm{h}, g, k)\big)$ is called the maximal existence interval. A map $u \in Z^k_0\big(I^+(u_\mathrm{h}, g, k)\big)$ which satisfies (5.2) on $I^+(u_\mathrm{h}, g, k)$ is called maximal solution.

We note that a maximal solution necessarily satisfies $u \in C^{k-1}\big([0, t^+(u_\mathrm{h}, g, k)), X_1\big)$, as can be seen by differentiating the boundary and interface conditions $K_{\partial G,\Sigma}\, u = 0$ and using Lemma 2.4. As defined above, the maximal existence time depends on the regularity index $k \ge 2$. Since for $\widetilde{k} > k$, a $\mathcal{H}^{\widetilde{k}}_\Sigma$-solution is also a $\mathcal{H}^k_\Sigma$-solution, t^+ is nonincreasing in k. We prove in Lemma 5.7 that it is even independent of k and then just write $t^+(u_\mathrm{h}, g)$ and $I^+(u_\mathrm{h}, g)$. We omit the dependence of t^+ on ε, μ, σ, η, ϕ and G.

With the help of Lemmas 5.2, 5.3 and 5.4, we can prove the existence of a unique maximal solution and state a first blow-up condition (which is refined in Lemma 5.7).

Lemma 5.6. *Let Assumptions 2.1 and 4.2 hold, and let $u_\mathrm{h} \in Z^k(0)$ and $g \in Z^k\big([0,\infty)\big)$ satisfy (5.4). Then $t^+(u_\mathrm{h}, g, k) \in (b_0, \infty]$ with b_0 as in Lemma 5.2 and $T = 1$, say. There exists a unique maximal solution $u \in Z^k_0\big(I^+(u_\mathrm{h}, g, k)\big)$ of (5.2). If $t^+(u_\mathrm{h}, g, k) < \infty$, there exists a sequence (t_m) in $\big(0, t^+(u_\mathrm{h}, g, k)\big)$ with $t_m \to t^+(u_\mathrm{h}, g, k)$ and*

$$\sum_{\ell=0}^{k} \Big\|\partial_t^\ell u(t_m)\Big\|_{\mathcal{H}^{k-\ell}_\Sigma} \to \infty, \qquad m \to \infty.$$

Proof. 1) The first part of Lemma 5.2 yields a solution $u \in Z^k_0(b_0)$ with b_0 given by (5.11). By the first part of Lemma 5.3, the shifted function $\widetilde{u}_\mathrm{h} = u(\bullet + b_0)$ satisfies the conditions on the initial data in the second part of Lemma 5.2 and we obtain a solution $\widetilde{u} \in Z^k_{-b_0}(\widetilde{b}_0)$ for a positive time $\widetilde{b}_0$. Again using Lemma 5.3, this yields a solution of (5.2) in the space $Z^k_0(b_0 + \widetilde{b}_0)$. Hence $t^+(u_\mathrm{h}, g, k)$ belongs to $(b_0, \infty]$.

Let $b \in \big(0, t^+(u_\mathrm{h}, g, k)\big)$. Definition 5.5 implies the existence a solution $u^{(b)}$ of (5.2) on $(-\infty, b]$. By Lemma 5.4 we have $u^{(b')} = u^{(b)}$ on $(-\infty, b']$ for $0 < b' < b < t^+(u_\mathrm{h}, g, k)$. So we can define a maximal solution u on $\big(-\infty, t^+(u_\mathrm{h}, g, k)\big)$ by choosing for any $t \in \big(0, t^+(u_\mathrm{h}, g, k)\big)$ a time $b \in \big[t, t^+(u_\mathrm{h}, g, k)\big)$ and setting $u(t) := u^{(b)}(t)$. Due to Lemma 5.4, this procedure gives functions solving (5.2) uniquely in $Z^k_0\big(I^+(u_\mathrm{h}, g, k)\big)$.

2) Let $t^+ := t^+(u_\mathrm{h}, g, k) < \infty$. We argue by contradiction. Suppose

$$C := \sup_{0<\tau<t^+} \|u\|_{Z^k([0,\tau])} < \infty$$

and choose some time $T > 2t^+$. By assumption, $\widetilde{C} := \|g\|_{Z^k([0,T])}$ is finite. Let (t_m) be a sequence in $(0, t^+)$ with $t_m \to t^+$ as $m \to \infty$. We define sequences of functions $(g^{(m)})$

and $(w_{\mathrm{h}}^{(m)})$ by $g^{(m)}(t) := g(t+t_m)$ for all $t \geq 0$ and $w_{\mathrm{h}}^{(m)}(t) := u(t+t_m)$ for all $t \leq 0$. They satisfy $w_{\mathrm{h}}^{(m)} \in Z^k_{-t_m}(0)$,

$$\begin{aligned}
\|w_{\mathrm{h}}^{(m)}\|_{Z^k_{-t_m}(0)} + \left\|g^{(m)}\right\|_{Z^k([0,T-t^+])} &\leq C + \widetilde{C} =: r_0, \\
\partial_t^{\ell+1} w_{\mathrm{h}}^{(m)}(0) = \partial_t^{\ell+1} u(t_m) &= (A+L)\partial_t^\ell u(t_m) + F^{\langle \ell \rangle}(u_{t_m}) + \partial_t^\ell g(t_m) \\
&= (A+L)\partial_t^\ell w_{\mathrm{h}}^{(m)}(0) + F^{\langle \ell \rangle}(w_{\mathrm{h}}^{(m)}) + \partial_t^\ell g^{(m)}(0), \\
\partial_t^\ell w_{\mathrm{h}}^{(m)}(0) &= \partial_t^\ell u(t_m) \in \mathrm{D}(A)
\end{aligned}$$

for all $\ell \in \{0, \dots, k-1\}$ and $m \in \mathbb{N}$. The second part of Lemma 5.2 yields a time $\widetilde{b}_0^{(m)} = \widetilde{b}_0^{(m)}(r_0, t_m, T-t^+) \in (0, T-t^+)$ such that the problem

$$\begin{aligned}
\partial_t v(t) &= (A+L)v(t) + F(v_t) + g^{(m)}(t), && t \geq 0, \\
K_{\partial G,\Sigma}\, v(t) &= 0, && t \geq 0, \\
v(t) &= w_{\mathrm{h}}^{(m)}(t), && t \leq 0,
\end{aligned}$$

has a solution $v_m \in Z^k_{-t_m}(\widetilde{b}_0^{(m)})$ on $(-\infty, \widetilde{b}_0^{(m)}]$ for each $m \in \mathbb{N}$. Since $t_m \to t^+$, there exists an index $m_0 \in \mathbb{N}$ with $t_m \geq t^+/2 > 0$ for all $m \geq m_0$. From (5.12) we see that

$$\widetilde{b}_0^{(m)} \geq \min\left\{\frac{t^+}{2}, T, \frac{1}{C_1\big((1+C_3)\,r + C_2\,(1+r^N)\big)}, \frac{1}{2C_1\big(C_3 + C_2\,(1+r^{N-1})\big)}\right\} =: \widetilde{b}$$

for all $m \geq m_0$, where we have used $T - t^+ > t^+$. We can now find an index $m_1 \geq m_0$ with $t_{m_1} + \widetilde{b} > t^+$. Using Lemma 5.3, we can combine u and v_{m_1} to construct a solution on $(-\infty, t_{m_1} + \widetilde{b}]$, which contradicts the definition of t^+. □

We can improve the blow-up condition to involve only the L^∞-norm of the solution. The proof uses ideas from Proposition 4.4 in [22], where the $W^{1,\infty}$-norm is used to control the L^2-norms of spatial derivatives of the solution. Compared to the instantaneous material laws treated in [22], the retarded material laws considered here allow us to work in L^∞ instead of $W^{1,\infty}$.

Lemma 5.7. *Let Assumptions 2.1 and 4.2 be true. Furthermore, let $u_{\mathrm{h}} \in Z^k(0)$ and $g \in Z^k\big([0,\infty)\big)$ satisfy (5.4). Let u be the unique maximal solution of (5.2) on $(-\infty, t^+(u_{\mathrm{h}}, g, k))$. If $t^+(u_{\mathrm{h}}, g, k) < \infty$, there exists a sequence (t_i) in $(0, t^+(u_{\mathrm{h}}, g, k))$ with $t_i \to t^+(u_{\mathrm{h}}, g, k)$ and*

$$\|u(t_i)\|_{L^\infty} \to \infty$$

as $i \to \infty$. In particular, t^+ is independent of k.

Proof. The last statement follows from the embedding $\mathcal{H}^2_\Sigma(G) \hookrightarrow L^\infty(G)$. We omit the restriction to subintervals and the dependence of the constants on the parameters.

We assume $t^+ := t^+(u_{\mathrm{h}}, g, k) < \infty$ and $\sup_{0 \leq t < t^+} \|u(t)\|_{L^\infty(G)} < \infty$. Let $T \in (0, t^+)$. For $m \in \{2, \dots, k\}$, we define the function $\varphi_m : [0, t^+) \to [0, \infty)$ by

$$\varphi_m(t) := \|u\|_{Z^m_0(t)} = \sum_{\ell=0}^{m-1} \sup_{\tau \leq t} \left\|\partial_t^\ell u(\tau)\right\|_{\mathcal{H}^{m-\ell}_\Sigma} + \sup_{\tau \leq 0} \left\|\partial_t^m u(\tau)\right\|_{L^2} + \sup_{0 \leq \tau \leq t} \left\|\partial_t^m u(\tau)\right\|_{L^2}.$$

The strategy of the proof is to show by induction that φ_m is bounded for all $m \in \{2,\dots,k\}$. The case $m=k$ is then a contradiction to Lemma 5.6. We set

$$f(t) := Lu(t) + F(u_t) + g(t),\ t \in [0,T].$$

As in the proof of Lemma 5.2, we see that $\big(u_{\mathrm{h}}(0), f\big)$ satisfies the compatibility conditions of order k and u solves

$$\partial_t u(t) = Au(t) + f(t),\ t \geq 0, \qquad K_{\partial G,\Sigma}\, u(t) = 0,\ t \geq 0, \qquad u(0) = u_{\mathrm{h}}(0)$$

on $[0,T]$. We now apply Theorem 3.2 with k replaced by m to obtain the estimate

$$\begin{aligned}\varphi_m(t) &\lesssim \|u_{\mathrm{h}}\|_{Z^m(0)} + \sum_{\ell=0}^{m-1} \left\|\partial_t^\ell f(0)\right\|_{\mathcal{H}_\Sigma^{m-1-\ell}} + \sum_{\ell=0}^{m} \left\|\partial_t^\ell f\right\|_{L^1((0,t),\mathcal{H}_\Sigma^{m-\ell})} \\ &\lesssim 1 + \sum_{\ell=0}^{m} \left\|\partial_t^\ell f\right\|_{L^1((0,t),\mathcal{H}_\Sigma^{m-\ell})} \end{aligned} \tag{5.14}$$

for all $m \in \{2,\dots,k\}$ and $t \in [0,T]$. This and the following estimates are uniform in $T < t^+$. We furthermore have

$$\left\|\partial_t^\ell f(t)\right\|_{\mathcal{H}_\Sigma^{m-\ell}} \lesssim 1 + \varphi_m(t) + \sum_{n=1}^{N} \left\|\partial_t^{\ell+1}\Gamma^{(n)}(u)(t)\right\|_{\mathcal{H}_\Sigma^{m-\ell}} \tag{5.15}$$

for all $\ell \in \{0,\dots,m\}$ and $t \in [0,T]$.

1) In the first step we prove the boundedness of φ_2. Let $n \in \{1,\dots,N\}$ and $t \in [0,T]$. Throughout, we use $\mathcal{H}_\Sigma^2(G) \hookrightarrow L^\infty(G)$ and the assumptions on the response functions.

a) In the case $\ell = 0$, inequality (2.4) and formulas (4.5) and (4.6) then lead to

$$\left\|\partial_t\Gamma^{(n)}(u)(t)\right\|_{\mathcal{H}_\Sigma^2} \lesssim \sup_{\tau\leq t} \|u(\tau)\|_{\mathcal{H}_\Sigma^2} \sup_{\tau\leq t} \|u(\tau)\|_{L^\infty}^{n-1} \lesssim \varphi_2(t).$$

In addition, it holds

$$\left\|\partial_t\Gamma^{(n)}(u)(t)\right\|_{L^\infty} \lesssim \sup_{\tau\leq t} \|u(\tau)\|_{L^\infty}^{n} \lesssim 1,$$

and from (5.15) we conclude

$$\|f(t)\|_{\mathcal{H}_\Sigma^2} \lesssim 1 + \varphi_2(t), \quad \|f(t)\|_{L^\infty} \lesssim 1. \tag{5.16}$$

b) Now let $\ell = 1$. If $n = 1$, equation (4.7) implies $\left\|\partial_t^2\Gamma^{(1)}(u)(t)\right\|_{\mathcal{H}_\Sigma^1} \lesssim \varphi_2(t)$. In the case $n \geq 2$, we again use (2.4) to estimate the $\mathcal{H}_\Sigma^1(G)$-norms of the terms

$$\int_{-\infty}^t \cdots \int_{-\infty}^t \partial_\tau^\alpha R_{j_0\dots j_n}^{(n)}(t-s_1,\dots,t-s_n) \prod_{i=1}^n u_{j_i}(s_i)\,\mathrm{d}s_1\dots\mathrm{d}s_n,$$

$$u_{j_1}(t) \int_{-\infty}^t \cdots \int_{-\infty}^t \partial_\tau^\beta R_{j_0\dots j_n}^{(n)}(0,t-s_2,\dots,t-s_n) \prod_{i=2}^n u_{j_i}(s_i)\,\mathrm{d}s_2\dots\mathrm{d}s_n,$$

$$u_{j_1}(t)u_{j_2}(t) \int_{-\infty}^t \cdots \int_{-\infty}^t R_{j_0\dots j_n}^{(n)}(0,0,t-s_3,\dots,t-s_n) \prod_{i=3}^n u_{j_i}(s_i)\,\mathrm{d}s_3\dots\mathrm{d}s_n,$$

where $\alpha, \beta \in \mathbb{N}_0^n$ with $|\alpha| = 2$ and $|\beta| = 1$, by a constant times $\varphi_2(t)$. (For $n = 2$, the last term is replaced by $R^{(2)}_{j_0 j_1 j_2}(0,0)u_{j_1}(t)u_{j_2}(t)$.) It remains to treat the contribution

$$I(t) := \partial_t u_{j_1}(t) \int_{-\infty}^t \cdots \int_{-\infty}^t R^{(n)}_{j_0 \dots j_n}(0, t-s_2, \dots, t-s_n) \prod_{i=2}^n u_{j_i}(s_i)\, \mathrm{d}s_1 \dots \mathrm{d}s_n.$$

Here we insert $\partial_t u(t) = Au(t) + f(t)$ and derive from (2.2), (2.4), (2.5) and (5.16) the inequality

$$\begin{aligned}
\|I(t)\|_{\mathcal{H}^1_\Sigma} \lesssim & \sum_{\alpha \in \mathbb{N}_0^3,\, |\alpha|=1} \int_{-\infty}^t \cdots \int_{-\infty}^t \left\| R^{(n)}_{j_0 \dots j_n}(0, t-s_2, \dots, t-s_n) \right\|_{\mathcal{H}^2_\Sigma} \\
& \cdot \left\| \partial^\alpha u_{j_1}(t) \prod_{i=2}^n u_{j_i}(s_i) \right\|_{\mathcal{H}^1_\Sigma} \mathrm{d}s_2 \dots \mathrm{d}s_n \\
& + \|f(t)\|_{\mathcal{H}^2_\Sigma} \int_{-\infty}^t \cdots \int_{-\infty}^t \left\| R^{(n)}_{j_0 \dots j_n}(0, t-s_2, \dots, t-s_n) \right\|_{L^\infty} \\
& \cdot \left\| \prod_{i=2}^n u_{j_i}(s_i) \right\|_{L^\infty} \mathrm{d}s_2 \dots \mathrm{d}s_n \\
& + \|f(t)\|_{L^\infty} \int_{-\infty}^t \cdots \int_{-\infty}^t \left\| R^{(n)}_{j_0 \dots j_n}(0, t-s_2, \dots, t-s_n) \right\|_{\mathcal{H}^2_\Sigma} \\
& \cdot \sum_{i=2}^n \|u_{j_i}(s_i)\|_{\mathcal{H}^2_\Sigma} \prod_{m=1,\, m \neq i}^n \|u_{j_m}(s_m)\|_{L^\infty} \mathrm{d}s_2 \dots \mathrm{d}s_n \\
\lesssim & \sup_{\tau \le t} \|u(\tau)\|_{\mathcal{H}^2_\Sigma} \sup_{\tau \le t} \|u(\tau)\|_{L^\infty}^{n-1} + \|f(t)\|_{\mathcal{H}^2_\Sigma} \sup_{\tau \le t} \|u(\tau)\|_{L^\infty}^{n-1} \\
& + \|f(t)\|_{L^\infty} \sup_{\tau \le t} \|u(\tau)\|_{\mathcal{H}^2_\Sigma} \|u(\tau)\|_{L^\infty}^{n-2} \\
\lesssim & \, 1 + \varphi_2(t).
\end{aligned}$$

So (5.15) leads to $\|\partial_t f(t)\|_{\mathcal{H}^1_\Sigma(G)} \lesssim 1 + \varphi_2(t)$ for all $t \in [0, T]$.

c) We turn to the case $\ell = 2$. If $n \ge 4$, we see from (4.4) that $\partial_t^3 \Gamma^{(n)}(u)(t)$ is a linear combination of the terms

$$\int_{-\infty}^t \cdots \int_{-\infty}^t \partial_\tau^\alpha R^{(n)}_{j_0 \dots j_n}(t-s_1, \dots, t-s_n) \prod_{i=1}^n u_{j_i}(s_i)\, \mathrm{d}s_1 \dots \mathrm{d}s_n,$$

$$u_{j_1}(t) \int_{-\infty}^t \cdots \int_{-\infty}^t \partial_\tau^\beta R^{(n)}_{j_0 \dots j_n}(0, t-s_2, \dots, t-s_n) \prod_{i=2}^n u_{j_i}(s_i)\, \mathrm{d}s_2 \dots \mathrm{d}s_n,$$

$$\partial_t u_{j_1}(t) \int_{-\infty}^t \cdots \int_{-\infty}^t \partial_\tau^\gamma R^{(n)}_{j_0 \dots j_n}(0, t-s_2, \dots, t-s_n) \prod_{i=2}^n u_{j_i}(s_i)\, \mathrm{d}s_2 \dots \mathrm{d}s_n,$$

$$\partial_t^2 u_{j_1}(t) \int_{-\infty}^t \cdots \int_{-\infty}^t R^{(n)}_{j_0 \dots j_n}(0, t-s_2, \dots, t-s_n) \prod_{i=2}^n u_{j_i}(s_i)\, \mathrm{d}s_2 \dots \mathrm{d}s_n,$$

$$u_{j_1}(t) u_{j_2}(t) \int_{-\infty}^t \cdots \int_{-\infty}^t \partial_\tau^\delta R^{(n)}_{j_0 \dots j_n}(0, 0, t-s_3, \dots, t-s_n) \prod_{i=3}^n u_{j_i}(s_i)\, \mathrm{d}s_3 \dots \mathrm{d}s_n,$$

$$\partial_t u_{j_1}(t) u_{j_2}(t) \int_{-\infty}^t \cdots \int_{-\infty}^t R^{(n)}_{j_0\dots j_n}(0,0,t-s_3\dots,t-s_n) \prod_{i=3}^n u_{j_i}(s_i)\,\mathrm{d}s_3\dots\,\mathrm{d}s_n,$$

$$u_{j_1}(t) u_{j_2}(t) u_{j_3}(t) \int_{-\infty}^t \cdots \int_{-\infty}^t R^{(n)}_{j_0\dots j_n}(0,0,0,t-s_4\dots,t-s_n) \prod_{i=4}^n u_{j_i}(s_i)\,\mathrm{d}s_4\dots\,\mathrm{d}s_n,$$

where $\alpha,\beta,\gamma,\delta\in\mathbb{N}_0^n$ with $|\alpha|=3$, $|\beta|=2$, $|\gamma|=1$, $|\delta|=1$. Using $\left\|\partial_t^j u(t)\right\|_{L^2}\le\varphi_2(t)$ for all $j\in\{0,1,2\}$, as above we obtain $\left\|\partial_t^3\Gamma^{(n)}(u)(t)\right\|_{L^2}\lesssim 1+\varphi_2(t)$. The contributions for $n\in\{1,2,3\}$ are handled analogously. We conclude $\sum_{\ell=0}^2\left\|\partial_t^\ell f(t)\right\|_{\mathcal{H}_\Sigma^{2-\ell}}\lesssim 1+\varphi_2(t)$ with a constant independent of T. As $T\in(0,t^+)$ is arbitrary, (5.14) implies that there exist constants $a,b\ge 0$ such that $\varphi_2(t)\le a+b\int_0^t\varphi_2(\tau)\,\mathrm{d}\tau$ holds for all $t\in(0,t^+)$. Gronwall's inequality finally implies $\sup_{0\le\tau<t^+}\varphi_2(\tau)<\infty$. If $k=2$, the proof is finished.

2) Let $k\ge 3$ and $m\in\{2,\dots,k-1\}$. We assume $\sup_{0\le\tau<t^+}\varphi_j(\tau)<\infty$ for all $j\in\{2,\dots,m\}$. The boundedness of φ_{m+1} on $[0,t^+)$ follows by the same arguments as above, if we can show

$$\left\|\partial_t^{\ell+1}\Gamma^{(n)}(u)(t)\right\|_{\mathcal{H}_\Sigma^{m+1-\ell}}\lesssim 1+\varphi_{m+1}(t) \tag{5.17}$$

for all $\ell\in\{0,\dots,m+1\}$, $n\in\{1,\dots,N\}$ and $t\in[0,T]$. So let $\ell\in\{0,\dots,m+1\}$, $n\in\{1,\dots,N\}$ and $t\in[0,T]$. By (4.4), we have

$$\partial_t^{\ell+1}\Gamma^{(n)}_{j_0}(u)(t)=\sum_{\substack{\alpha\in\mathbb{N}_0^n,\\|\alpha|=\ell+1}} I^{(n)}_{1,\alpha}(t)+\sum_{q=1}^{\min\{\ell+1,n-1\}}\sum_{\substack{\alpha\in\mathbb{N}_0^n,\,\beta\in\mathbb{N}_0^q,\\|\alpha|+|\beta|=\ell+1-q}} a^{(n)}_{\ell,q,\alpha,\beta} I^{(n)}_{2,q,\alpha,\beta}(t)+\sum_{\substack{\alpha,\beta\in\mathbb{N}_0^n,\\|\alpha|+|\beta|=\ell+1-n}} b^{(n)}_{\alpha,\beta} I^{(n)}_{3,\alpha,\beta}(t)$$

with

$$I^{(n)}_{1,\alpha}(t):=\int_{-\infty}^t\cdots\int_{-\infty}^t \partial_\tau^\alpha R^{(n)}_{j_0\dots j_n}(t-s_1,\dots,t-s_n)\prod_{i=1}^n u_{j_i}(s_i)\,\mathrm{d}s_1\dots\,\mathrm{d}s_n,$$

$$I^{(n)}_{2,q,\alpha,\beta}(t):=I^{(n)}_{2,q,\alpha}(t)\prod_{i=1}^q\partial_t^{\beta_i}u_{j_i}(t),$$

$$I^{(n)}_{2,q,\alpha}(t):=\int_{-\infty}^t\dots\int_{-\infty}^t \partial_\tau^\alpha R^{(n)}_{j_0\dots j_n}(0,\dots,0,t-s_{q+1},\dots,t-s_n)\prod_{i=q+1}^n u_{j_i}(s_i)\,\mathrm{d}s_{q+1}\dots\mathrm{d}s_n,$$

$$I^{(n)}_{3,\alpha,\beta}(t):=\partial_\tau^\alpha R^{(n)}_{j_0\dots j_n}(0,\dots,0)\prod_{i=1}^n\partial_t^{\beta_i}u_{j_i}(t)$$

and constants $a^{(n)}_{\ell,q,\alpha,\beta}, b^{(n)}_{\alpha,\beta}\in\mathbb{N}_0$. We note that $\partial_\tau^\alpha R^{(n)}_{j_0\dots j_n}$ belongs to $C\big(\mathbb{R}^n_{\ge 0},\mathcal{H}_\Sigma^{k+1-|\alpha|}(G)\big)$ for all $\alpha\in\mathbb{N}_0^n$ with $|\alpha|\le k+1$.

2.1) We first treat $I^{(n)}_{1,\alpha}(t)$. Let $\alpha\in\mathbb{N}_0^n$ with $|\alpha|=\ell+1$. We distinguish four cases.

a) $\ell\in\{0,\dots,m-2\}$: Then $m+1-\ell\ge 3$ and $\partial_\tau^\alpha R^{(n)}_{j_0\dots j_n}$ is contained in $C\big(\mathbb{R}^n_{\ge 0},\mathcal{H}_\Sigma^{m+1-\ell}(G)\big)$ due to $k+1-|\alpha|=k-\ell\ge m+1-\ell$. We use (2.4) and the embedding $\mathcal{H}^2_\Sigma(G)\hookrightarrow L^\infty(G)$ to obtain

$$\left\|I^{(n)}_{1,\alpha}(t)\right\|_{\mathcal{H}_\Sigma^{m+1-\ell}}\lesssim\sup_{\tau\le t}\|u(\tau)\|_{\mathcal{H}_\Sigma^{m+1-\ell}}\sup_{\tau\le t}\|u(\tau)\|_{L^\infty}^{n-1}\lesssim\varphi_{m+1}(t).$$

b) $\ell = m-1$: Here we have $m+1-\ell = 2$ and $k+1-|\alpha| = k-\ell = k-m+1 \geq 2$. Thus $\partial_\tau^\alpha R^{(n)}_{j_0\dots j_n}$ belongs to $C\big(\mathbb{R}^n_{\geq 0}, \mathcal{H}^2_\Sigma(G)\big)$ and we infer

$$\big\|I^{(n)}_{1,\alpha}(t)\big\|_{\mathcal{H}^{m+1-\ell}_\Sigma} \lesssim \sup_{\tau\leq t} \|u(\tau)\|^n_{\mathcal{H}^2_\Sigma} \lesssim \varphi_2(t)^n \lesssim 1.$$

c) $\ell = m$: It holds $m+1-\ell = 1$ and $k+1-|\alpha| \geq 1$. The product rule (2.2) yields

$$\big\|I^{(n)}_{1,\alpha}(t)\big\|_{\mathcal{H}^1_\Sigma} \lesssim \sup_{\tau\leq t} \|u(\tau)\|^n_{\mathcal{H}^2_\Sigma} \lesssim \varphi_2(t)^n \lesssim 1.$$

d) $\ell = m+1$: In this case, $m+1-\ell = 0$, $k+1-|\alpha| \geq 0$ and again the embedding $\mathcal{H}^2_\Sigma(G) \hookrightarrow L^\infty(G)$ lead to

$$\big\|I^{(n)}_{1,\alpha}(t)\big\|_{L^2} \lesssim \sup_{\tau\leq t} \|u(\tau)\|^n_{\mathcal{H}^2_\Sigma} \lesssim \varphi_2(t)^n \lesssim 1.$$

2.2) We turn to $I^{(n)}_{2,q,\alpha,\beta}(t)$. These terms appear only if $n \geq q+1$. Let $q \in \mathbb{N}$, $\alpha \in \mathbb{N}_0^n$ and $\beta \in \mathbb{N}_0^q$ satisfy

$$q \leq \min\{\ell+1, n-1\}, \quad |\alpha| + |\beta| = \ell + 1 - q.$$

Without loss of generality we can assume $\beta_1 \geq \dots \geq \beta_q$.

i) We begin with the case $m = 2$, using again product rules from Lemma 2.2.

a) $m = 2$, $\ell = 0$: Here we have $q = 1$, $\alpha = 0$, $\beta = 0$ and

$$\begin{aligned}\big\|I^{(n)}_{2,1,0,0}(t)\big\|_{\mathcal{H}^3_\Sigma} &\lesssim \|u(t)\|_{\mathcal{H}^3_\Sigma} \big\|I^{(n)}_{2,1,0}(t)\big\|_{L^\infty} + \|u(t)\|_{L^\infty} \big\|I^{(n)}_{2,1,0}(t)\big\|_{\mathcal{H}^3_\Sigma} \\ &\lesssim \|u(t)\|_{\mathcal{H}^3_\Sigma} \sup_{\tau\leq t} \|u(\tau)\|^{n-1}_{L^\infty} + \|u(t)\|_{L^\infty} \sup_{\tau\leq t} \|u(\tau)\|_{\mathcal{H}^3_\Sigma} \sup_{\tau\leq t} \|u(\tau)\|^{n-2}_{L^\infty} \\ &\lesssim \varphi_3(t).\end{aligned}$$

b) $m = 2$, $\ell = 1$: If $q = 1$, it holds $|\alpha| \leq 1$ and $\beta_1 \in \{0,1\}$. Since $\big\|\partial_t^{\beta_1} u(t)\big\|_{\mathcal{H}^2_\Sigma(G)} \leq \varphi_3(t)$ and

$$\big\|I^{(n)}_{2,1,\alpha}(t)\big\|_{\mathcal{H}^2_\Sigma} \lesssim \|u(t)\|_{\mathcal{H}^2_\Sigma} \sup_{\tau\leq t} \|u(\tau)\|^{n-1}_{L^\infty} \lesssim \varphi_2(t) \lesssim 1$$

by assumption, we have

$$\big\|I^{(n)}_{2,1,\alpha,\beta}(t)\big\|_{\mathcal{H}^2_\Sigma} \lesssim \big\|\partial_t^{\beta_1} u(t)\big\|_{\mathcal{H}^2_\Sigma} \big\|I^{(n)}_{2,1,\alpha}(t)\big\|_{\mathcal{H}^2_\Sigma} \lesssim \varphi_3(t).$$

For $q = 2$ it holds $\alpha = 0$, $\beta = 0$ and

$$\big\|I^{(n)}_{2,2,0,0}(t)\big\|_{\mathcal{H}^2_\Sigma} \lesssim \|u(t)\|^2_{\mathcal{H}^2_\Sigma} \big\|I^{(n)}_{2,2,0}(t)\big\|_{\mathcal{H}^2_\Sigma} \lesssim 1.$$

c) $m = 2$, $\ell = 2$: In the case $q = 1$, it follows $|\alpha| \leq 2$ and $\beta_1 \in \{0,1,2\}$. Furthermore, $k+1-|\alpha| \geq 2$, so $\partial_\tau^\alpha R^{(n)}_{j_0\dots j_n}$ is contained in $C\big(\mathbb{R}^n_{\geq 0}, \mathcal{H}^2_\Sigma(G)\big)$ and we obtain

$$\big\|I^{(n)}_{2,1,\alpha,\beta}(t)\big\|_{\mathcal{H}^1_\Sigma} \lesssim \big\|\partial_t^{\beta_1} u(t)\big\|_{\mathcal{H}^1_\Sigma} \big\|I^{(n)}_{2,1,\alpha}(t)\big\|_{\mathcal{H}^2_\Sigma} \lesssim \varphi_3(t).$$

If $q = 2$, we have $|\alpha| \leq 1$, $\beta_1 \in \{0,1\}$, $\beta_2 = 0$ and

$$\big\|I^{(n)}_{2,2,\alpha,\beta}(t)\big\|_{\mathcal{H}^1_\Sigma} \lesssim \big\|\partial_t^{\beta_1} u(t)\big\|_{\mathcal{H}^1_\Sigma} \|u(t)\|_{\mathcal{H}^2_\Sigma} \big\|I^{(n)}_{2,2,\alpha}(t)\big\|_{\mathcal{H}^2_\Sigma} \lesssim 1.$$

For $q=3$, it holds $\alpha=0$, $\beta=0$, and

$$\big\|I^{(n)}_{2,3,0,0}(t)\big\|_{\mathcal{H}^1_\Sigma} \lesssim \|u(t)\|_{\mathcal{H}^1_\Sigma}\,\|u(t)\|^2_{\mathcal{H}^2_\Sigma}\,\big\|I^{(n)}_{2,3,0}(t)\big\|_{\mathcal{H}^2_\Sigma} \lesssim 1.$$

d) $m=2$, $\ell=3$: If $q=1$, it holds $|\alpha|+|\beta|=3$. We first consider the case $|\alpha|=3$. Then we have $\beta=0$, $\|u(t)\|_{\mathcal{H}^1_\Sigma}\le\varphi_1(t)\lesssim 1$ and $\partial^\alpha_\tau R^{(n)}_{j_0\dots j_n}C\big(\mathbb{R}^n_{\ge 0},\mathcal{H}^1_\Sigma(G)\big)$, which implies

$$\big\|I^{(n)}_{2,1,\alpha}(t)\big\|_{\mathcal{H}^1_\Sigma} \lesssim \sup_{\tau\le t}\|u(\tau)\|^{n-1}_{\mathcal{H}^2_\Sigma} \lesssim 1.$$

So we deduce

$$\big\|I^{(n)}_{2,1,\alpha,0}(t)\big\|_{L^2} \lesssim \big\|u(t)\big\|_{\mathcal{H}^1_\Sigma}\big\|I^{(n)}_{2,1,\alpha}(t)\big\|_{\mathcal{H}^1_\Sigma} \lesssim 1.$$

In the case $|\alpha|<3$, we have $\beta_1\in\{0,1,2,3\}$, $\big\|\partial_t^{\beta_1}u(t)\big\|_{L^2}\le\varphi_3(t)$ and

$$\partial^\alpha_\tau R^{(n)}_{j_0\dots j_n}C\big(\mathbb{R}^n_{\ge 0},\mathcal{H}^2_\Sigma(G)\big)\hookrightarrow C\big(\mathbb{R}^n_{\ge 0},L^\infty(G)\big).$$

It follows

$$\big\|I^{(n)}_{2,1,\alpha,\beta}(t)\big\|_{L^2} \lesssim \big\|\partial_t^{\beta_1}u(t)\big\|_{L^2}\big\|I^{(n)}_{2,1,\alpha}(t)\big\|_{L^\infty} \lesssim \varphi_3(t).$$

Let $q=2$. Then $|\alpha|\le 2$. First let $\beta_1\in\{0,1,2\}$ and $\beta_2=0$. This implies

$$\big\|I^{(n)}_{2,2,\alpha,\beta}(t)\big\|_{L^2} \lesssim \big\|\partial_t^{\beta_1}u(t)\big\|_{L^2}\,\|u(t)\|_{L^\infty}\,\big\|I^{(n)}_{2,2,\alpha}(t)\big\|_{L^\infty} \lesssim \varphi_2(t)\lesssim 1.$$

The second case is $\beta_1=\beta_2=1$. Here it holds

$$\big\|I^{(n)}_{2,2,\alpha,\beta}(t)\big\|_{L^2} \lesssim \big\|\partial_t^{\beta_1}u(t)\big\|_{\mathcal{H}^1_\Sigma}\big\|\partial_t^{\beta_2}u(t)\big\|_{\mathcal{H}^1_\Sigma}\big\|I^{(n)}_{2,2,\alpha}(t)\big\|_{L^\infty} \lesssim \varphi_2(t)^2\lesssim 1.$$

If $q=3$, we have $|\alpha|\le 1$, $\beta_1\in\{0,1\}$ and $\beta_2=\beta_3=0$, while $q=4$ implies $\alpha=0$ and $\beta=0$. In both cases we obtain $\big\|I^{(n)}_{2,2,\alpha,\beta}(t)\big\|_{L^2}\lesssim 1$.

For $m=2$ and all $\ell\in\{0,\dots,3\}$, we conclude

$$\Big\|\partial_t^{\ell+1}\Gamma^{(n)}(u)(t)\Big\|_{\mathcal{H}^{3-\ell}_\Sigma} \lesssim 1+\varphi_3(t).$$

ii) It remains to consider $m\in\{3,\dots,k-1\}$ with $k\ge 4$. Again there are four subcases.

a) $\ell=m+1$: We argue similarly as in the case $m=2$, $\ell=3$ above and first consider $q=1$: It holds $|\alpha|+|\beta|=m+1$. If $|\alpha|=m+1$, we have $\beta=0$, $\|u(t)\|_{\mathcal{H}^1_\Sigma}\lesssim 1$, $\partial^\alpha_\tau R^{(n)}_{j_0\dots j_n}C\big(\mathbb{R}^n_{\ge 0},\mathcal{H}^1_\Sigma(G)\big)$ and therefore

$$\big\|I^{(n)}_{2,1,\alpha,0}(t)\big\|_{L^2} \lesssim \big\|u(t)\big\|_{\mathcal{H}^1_\Sigma}\big\|I^{(n)}_{2,1,\alpha}(t)\big\|_{\mathcal{H}^1_\Sigma} \lesssim 1.$$

If $|\alpha|\le m$, it follows $\beta_1\in\{0,\dots,m+1\}$, $\big\|\partial_t^{\beta_1}u(t)\big\|_{L^2}\le\varphi_{m+1}(t)$ as well as $\partial^\alpha_\tau R^{(n)}_{j_0\dots j_n}\in C\big(\mathbb{R}^n_{\ge 0},\mathcal{H}^2_\Sigma(G)\big)$, implying

$$\big\|I^{(n)}_{2,1,\alpha,\beta}(t)\big\|_{L^2} \lesssim \big\|\partial_t^{\beta_1}u(t)\big\|_{L^2}\big\|I^{(n)}_{2,1,\alpha}(t)\big\|_{L^\infty} \lesssim \varphi_3(t).$$

If $q\ge 2$, it must hold $\beta_i\le m-2$ for all $i\in\{2,\dots,q\}$, since otherwise $2m-2\le 2\beta_i\le\beta_1+\beta_i\le|\beta|\le m+2-q\le m$, which would imply $m\le 2$. This yields the estimates

$$\big\|\partial_t^{\beta_1}u(t)\big\|_{L^2}\le\varphi_{\beta_1}(t)\le\varphi_{m+1}(t),$$

$$\big\|\partial_t^{\beta_i} u(t)\big\|_{L^\infty} \lesssim \big\|\partial_t^{\beta_i} u(t)\big\|_{\mathcal{H}^2_\Sigma} \leq \varphi_{2+\beta_i}(t) \leq \varphi_m(t) \lesssim 1,\ i = 2, \dots, q,$$

We again have $\partial_\tau^\alpha R^{(n)}_{j_0 \dots j_n} \in C\big(\mathbb{R}^n_{\geq 0}, \mathcal{H}^2_\Sigma(G)\big)$ and deduce

$$\big\|I^{(n)}_{2,q,\alpha,\beta}(t)\big\|_{L^2} \lesssim \big\|I^{(n)}_{2,q,\alpha}(t)\big\|_{L^\infty} \big\|\partial_t^{\beta_1} u(t)\big\|_{L^2} \prod_{i=2}^q \big\|\partial_t^{\beta_i} u(t)\big\|_{L^\infty} \lesssim \varphi_{m+1}(t).$$

b) $\ell = m$: We have $\beta_1 \leq |\beta| \leq \ell + 1 - q \leq m$ and in case $q \geq 2$ additionally $\beta_i + 2 \leq m$ for all $i \in \{2, \dots, q\}$ by the same reasoning as in the step above. We obtain

$$\big\|I^{(n)}_{2,q,\alpha,\beta}(t)\big\|_{\mathcal{H}^1_\Sigma} \lesssim \big\|I^{(n)}_{2,q,\alpha}(t)\big\|_{\mathcal{H}^2_\Sigma} \big\|\partial_t^{\beta_1} u(t)\big\|_{\mathcal{H}^1_\Sigma} \prod_{i=2}^q \big\|\partial_t^{\beta_i} u(t)\big\|_{\mathcal{H}^2_\Sigma} \lesssim \varphi_{m+1}(t).$$

c) $\ell \in \{1, \dots, m-1\}$: It holds $m + 1 - \ell \geq 2$, and $\partial_\tau^\alpha R^{(n)}_{j_0 \dots j_n}$ is contained in the space $C\big(\mathbb{R}^n_{\geq 0}, \mathcal{H}^{m+1-\ell}_\Sigma(G)\big)$ due to $k + 1 - |\alpha| \geq k + 1 - \ell \geq m + 2 - \ell$ (recall that $|\alpha| \leq \ell + 1 - q \leq \ell$). This yields

$$\big\|I^{(n)}_{2,q,\alpha}(t)\big\|_{\mathcal{H}^{m+1-\ell}_\Sigma} \lesssim \sup_{\tau \leq t} \|u(\tau)\|_{\mathcal{H}^{m+1-\ell}_\Sigma} \sup_{\tau \leq t} \|u(\tau)\|_{L^\infty}^{n-q-1} \lesssim \varphi_m(t) \lesssim 1.$$

In addition, $\big\|\partial_t^{\beta_i} u(t)\big\|_{\mathcal{H}^{m+1-\ell}_\Sigma} \leq \varphi_{m+1}(t)$ for all $i \in \{1, \dots, q\}$, since $\beta_i + m + 1 - \ell \leq m + 1$ by $|\beta| \leq \ell$. If $q = 1$, we use (2.2) to obtain

$$\big\|I^{(n)}_{2,q,\alpha,\beta}(t)\big\|_{\mathcal{H}^{m+1-\ell}_\Sigma} \lesssim \Big\|\partial_t^{\beta_1} u(t)\Big\|_{\mathcal{H}^{m+1-\ell}_\Sigma} \big\|I^{(n)}_{2,q,\alpha}(t)\big\|_{\mathcal{H}^{m+1-\ell}_\Sigma} \lesssim \varphi_{m+1}(t).$$

If $q \geq 2$, for each $i \in \{1, \dots, q\}$ we have $\beta_i \leq |\beta| \leq \ell + 1 - q \leq m - 2$, which implies $\big\|\partial_t^{\beta_i} u(t)\big\|_{L^\infty(G)} \leq \varphi_m(t) \lesssim 1$. We can now use (2.4) to derive

$$\begin{aligned}
\big\|I^{(n)}_{2,q,\alpha,\beta}(t)\big\|_{\mathcal{H}^{m+1-\ell}_\Sigma} &\lesssim \sum_{i=1}^q \Big\|\partial_t^{\beta_i} u(t)\Big\|_{\mathcal{H}^{m+1-\ell}_\Sigma} \prod_{j=1,\, j \neq i}^q \Big\|\partial_t^{\beta_j} u(t)\Big\|_{L^\infty} \big\|I^{(n)}_{2,q,\alpha}(t)\big\|_{L^\infty} \\
&\quad + \prod_{j=1}^q \Big\|\partial_t^{\beta_j} u(t)\Big\|_{L^\infty} \big\|I^{(n)}_{2,q,\alpha}(t)\big\|_{\mathcal{H}^{m+1-\ell}_\Sigma} \\
&\lesssim 1 + \varphi_{m+1}(t).
\end{aligned}$$

d) $\ell = 0$: Here it holds $q = 1$, $\alpha = 0$, $\beta = 0$ and

$$\big\|I^{(n)}_{2,1,0,0}(t)\big\|_{\mathcal{H}^{m+1}_\Sigma} \lesssim \|u(t)\|_{\mathcal{H}^{m+1}_\Sigma} \big\|I^{(n)}_{2,1,0}(t)\big\|_{L^\infty} + \|u(t)\|_{L^\infty} \big\|I^{(n)}_{2,1,0}(t)\big\|_{\mathcal{H}^{m+1}_\Sigma}.$$

With $\|u(t)\|_{\mathcal{H}^{m+1}_\Sigma} \lesssim \varphi_{m+1}(t)$, $\big\|I^{(n)}_{2,1,0}(t)\big\|_{L^\infty} \lesssim 1$ and

$$\big\|I^{(n)}_{2,1,0}(t)\big\|_{\mathcal{H}^{m+1}_\Sigma} \lesssim \sup_{\tau \leq t} \|u(\tau)\|_{\mathcal{H}^{m+1}_\Sigma} \sup_{\tau \leq t} \|u(\tau)\|_{L^\infty}^{n-2} \lesssim \varphi_{m+1}(t),$$

it follows $\big\|I^{(n)}_{2,1,0,0}(t)\big\|_{\mathcal{H}^{m+1}_\Sigma} \lesssim \varphi_{m+1}(t)$.

Combining the above results, we deduce

$$\big\|I^{(n)}_{2,q,\alpha,\beta}(t)\big\|_{\mathcal{H}^{m+1}_\Sigma} \lesssim 1 + \varphi_{m+1}(t).$$

2.3) The term $I^{(n)}_{3,\alpha,\beta}(t)$ can be estimated in the same way as $I^{(n)}_{2,q,\alpha,\beta}(t)$ since apart from the missing time integrals, it is of the same structure, and contains fewer time derivatives. So we have shown (5.17), which implies $\varphi_{m+1}(t) \le a + b\int_0^t \varphi_{m+1}(\tau)\,\mathrm{d}\tau$ for all $t \in (0,t^+)$ and some constants $a, b \ge 0$. Gronwall's inequality yields the boundedness of φ_{m+1}, concluding the proof. □

The solution of the Maxwell system (5.2) depends continuously on the initial data and the inhomogeneity, as shown below. We collect the obtained local wellposedness properties of (5.2) in the following theorem which is the main result of this paper.

Theorem 5.8. *Let Assumptions 2.1 and 4.2 be true and let $u_\mathrm{h} \in Z^k(0)$ and $g \in Z^k\big([0,\infty)\big)$ satisfy (5.4). Then the following statements hold.*

(i) There exists a unique maximal solution $u \in Z^k_0\big((-\infty, t^+(u_\mathrm{h}, g))\big)$ of (5.2), where $t^+(u_\mathrm{h}, g) \in (b_0, \infty]$ with b_0 as in Lemma 5.2 and $T = 1$, say.

(ii) If $t^+(u_\mathrm{h}, g) < \infty$, there exists a sequence (t_m) in $(0, t^+(u_\mathrm{h}, g))$ with $t_m \to t^+(u_\mathrm{h}, g)$ and

$$\|u(t_m)\|_{L^\infty} \to \infty, \quad m \to \infty.$$

(iii) Let $b \in \big(0, t^+(u_\mathrm{h}, g)\big)$. There exist constants $\delta > 0$ and $C > 0$ such that for any initial data $\widetilde{u}_\mathrm{h}, \widehat{u}_\mathrm{h} \in Z^k(0)$ and inhomogeneities $\widetilde{g}, \widehat{g} \in Z^k\big([0,\infty)\big)$ satisfying

$$\begin{aligned}
\partial_t^{\ell+1}\widetilde{u}_\mathrm{h}(0) &= (A+L)\partial_t^\ell \widetilde{u}_\mathrm{h}(0) + F^{\langle \ell\rangle}(\widetilde{u}_\mathrm{h}) + \partial_t^\ell \widetilde{g}(0), \quad \ell = 0,\dots,k-2,\\
\partial_t^\ell \widetilde{u}_\mathrm{h}(0) &\in \mathrm{D}(A), \quad \ell = 0,\dots,k-1,\\
\partial_t^{\ell+1}\widehat{u}_\mathrm{h}(0) &= (A+L)\partial_t^\ell \widehat{u}_\mathrm{h}(0) + F^{\langle \ell\rangle}(\widehat{u}_\mathrm{h}) + \partial_t^\ell \widehat{g}(0), \quad \ell = 0,\dots,k-2,\\
\partial_t^\ell \widehat{u}_\mathrm{h}(0) &\in \mathrm{D}(A), \quad \ell = 0,\dots,k-1,
\end{aligned}$$

as well as

$$\|\widetilde{u}_\mathrm{h} - u_\mathrm{h}\|_{Z^k(0)} + \|\widetilde{g} - g\|_{Z^k([0,b])} \le \delta, \quad \|\widehat{u}_\mathrm{h} - u_\mathrm{h}\|_{Z^k(0)} + \|\widehat{g} - g\|_{Z^k([0,b])} \le \delta,$$

we have $\min\big\{t^+(\widetilde{u}_\mathrm{h}, \widetilde{g}), t^+(\widehat{u}_\mathrm{h}, \widehat{g})\big\} > b$ and the estimate

$$\|\widetilde{u} - \widehat{u}\|_{Z^k_0(b)} \le C\left(\|\widetilde{u}_\mathrm{h} - \widehat{u}_\mathrm{h}\|_{Z^k(0)} + \|\widetilde{g} - \widehat{g}\|_{Z^k([0,b])}\right),$$

where $\widetilde{u}$ and $\widehat{u}$ denote the solutions of (5.2) for $\widetilde{u}_\mathrm{h}$ and $\widetilde{g}$, respectively $\widehat{u}_\mathrm{h}$ and $\widehat{g}$. The constants δ and C depend on k, N, b, $\|u\|_{Z^k_0(b)}$, $\|g\|_{Z^k([0,b])}$, $\|\varepsilon\|_{\mathcal{C}^k_\Sigma}$, $\|\mu\|_{\mathcal{C}^k_\Sigma}$, $\|\sigma\|_{\mathcal{C}^k_\Sigma}$, η^{-1}, $\phi(0)$, $\|\phi\|_{L^1((0,\infty))}$ and G. While δ is nonincreasing in b, $\|u\|_{Z^k_0(b)}$, $\|g\|_{Z^k([0,b])}$, $\|\varepsilon\|_{\mathcal{C}^k_\Sigma}$, $\|\mu\|_{\mathcal{C}^k_\Sigma}$, $\|\sigma\|_{\mathcal{C}^k_\Sigma}$, η^{-1}, $\phi(0)$ and $\|\phi\|_{L^1((0,\infty))}$, the constant C is nondecreasing in these quantities.

Proof. It only remains to prove the last statement concerning the continuous dependence of u on u_h and g. We set

$$\overline{r}_0 := \overline{r}_0(k, b, u, g) := 1 + \|u\|_{Z^k_0(b)} + \|g\|_{Z^k([0,b])}.$$

Let $\delta_1 := \frac{1}{2}$ and $\widetilde{u}_\mathrm{h}, \widehat{u}_\mathrm{h}, \widetilde{g}, \widehat{g}$ be as in the claim with δ_1 instead of δ. In particular, we have

$$\|\widetilde{u}_\mathrm{h}\|_{Z^k(0)} + \|\widetilde{g}\|_{Z^k([0,b])} \le \overline{r}_0, \quad \|\widehat{u}_\mathrm{h}\|_{Z^k(0)} + \|\widehat{g}\|_{Z^k([0,b])} \le \overline{r}_0.$$

Lemma 5.2 thus yields a time $b_1 := b_0(\overline{r}_0, b) \in (0, b]$ smaller than $t^+(\widetilde{u}_\mathrm{h}, \widetilde{g})$ and $t^+(\widehat{u}_\mathrm{h}, \widehat{g})$ such that

$$\max\left\{\|\widetilde{u}\|_{Z_0^k(b_1)}, \|\widehat{u}\|_{Z_0^k(b_1)}\right\} \leq 1 + (2C_1 + C_1C_3 + C_4)\overline{r}_0 + C_1C_2(1 + \overline{r}_0^N) =: \overline{r} \tag{5.18}$$

holds. By equation (5.11), we can choose

$$b_1 = \min\left\{b, \frac{1}{C_1\big((1+C_3)\,\overline{r} + C_2\left(1+\overline{r}^N\right)\big)}, \frac{1}{2C_1\big(C_3 + C_2\left(1+\overline{r}^{N-1}\right)\big)}\right\}.$$

We set $v := \widetilde{u} - \widehat{u}$, $v_0 := \widetilde{u}_\mathrm{h}(0) - \widehat{u}_\mathrm{h}(0)$, and

$$f(t) := L\big(\widetilde{u}(t) - \widehat{u}(t)\big) + F\left(\widetilde{u}_t\right) - F\left(\widehat{u}_t\right) + \widetilde{g}(t) - \widehat{g}(t)$$

for $t \in [0, b_1]$. The function $v|_{[0,b_1]}$ solves the problem

$$\partial_t v(t) = Av(t) + f(t),\ t \geq 0, \qquad K_{\partial G,\Sigma}\, v(t) = 0,\ t \geq 0, \qquad v(0) = v_0$$

on $[0, b_1]$. Since (v_0, f) satisfies the compatibility conditions of order k (see the proof of Lemma 5.2), Theorem 3.2 yields the estimate

$$\left\|v|_{[0,b_1]}\right\|_{Z^k([0,b_1])} \leq C_1\left(\|v_0\|_{\mathcal{H}_\Sigma^k} + \sum_{\ell=0}^{k-1}\left\|\partial_t^\ell f(0)\right\|_{\mathcal{H}_\Sigma^{k-\ell-1}} + \sum_{\ell=0}^{k}\left\|\partial_t^\ell f\right\|_{L^1((0,b_1),\mathcal{H}_\Sigma^{k-\ell})}\right) \tag{5.19}$$

We use (5.1), (5.18), and Corollary 5.1 to bound the terms on the right and obtain

$$\begin{aligned}
\|v_0\|_{\mathcal{H}_\Sigma^k} &\leq \|\widetilde{u}_\mathrm{h} - \widehat{u}_\mathrm{h}\|_{Z^k(0)}\,,\\
\sum_{\ell=0}^{k-1}\left\|\partial_t^\ell f(0)\right\|_{\mathcal{H}_\Sigma^{k-\ell-1}} &\leq C(R)\,\|\widetilde{u}_\mathrm{h} - \widehat{u}_\mathrm{h}\|_{Z^k(0)} + \sum_{\ell=0}^{k-1}\left\|\partial_t^\ell \widetilde{g}(0) - \partial_t^\ell \widehat{g}(0)\right\|_{\mathcal{H}_\Sigma^{k-\ell-1}}\,,\\
\sum_{\ell=0}^{k}\left\|\partial_t^\ell f\right\|_{L^1((0,b_1),\mathcal{H}_\Sigma^{k-\ell})} &\leq b_1\big(C(R)\,\|\widetilde{u} - \widehat{u}\|_{Z_0^k(b_1)} + \|\widehat{g} - \widetilde{g}\|_{Z^k([0,b_1])}\big),
\end{aligned}$$

with $C(R) := C_3 + C_2(1 + \overline{r}^{N-1})$. Observe that $\left\|v|_{(-\infty,0]}\right\|_{Z^k(0)} = \left\|\widetilde{u}_\mathrm{h} - \widehat{u}_\mathrm{h}\right\|_{Z^k(0)}$ and recall $C_1 \geq 1$. These facts and (5.19) lead to

$$\begin{aligned}
\|v\|_{Z_0^k(b_1)} &\leq \left\|v|_{(-\infty,0]}\right\|_{Z^k(0)} + \left\|v|_{[0,b_1]}\right\|_{Z^k([0,b_1])}\\
&\leq \big(1 + C_1 + C_1C_2(1+\overline{r}^{N-1}) + C_1C_3\big)\,\|\widetilde{u}_\mathrm{h} - \widehat{u}_\mathrm{h}\|_{Z^k(0)}\\
&\qquad + C_1(1+b_1)\,\|\widehat{g} - \widetilde{g}\|_{Z^k([0,b_1])} + b_1C_1\big(C_3 + C_2(1+\overline{r}^{N-1})\big)\,\|v\|_{Z_0^k(b_1)}
\end{aligned}$$

We note that $b_1C_1\Big(C_3 + C_2\left(1+\overline{r}^{N-1}\right)\Big) \leq \frac{1}{2}$, set

$$K_1 := 2\max\left\{1 + C_1 + C_1C_2\left(1+\overline{r}^{N-1}\right) + C_1C_3, C_1\left(1+b_1\right)\right\} > 4,$$

and conclude

$$\|\widetilde{u} - \widehat{u}\|_{Z_0^k(b_1)} \leq K_1\big(\,\|\widetilde{u}_\mathrm{h} - \widehat{u}_\mathrm{h}\|_{Z^k(0)} + \|\widehat{g} - \widetilde{g}\|_{Z^k([0,b_1])}\,\big). \tag{5.20}$$

If $b_1 = b$, we have $\min\{t^+(\widetilde{u}_\mathrm{h}, \widetilde{g}), t^+(\widehat{u}_\mathrm{h}, \widehat{g})\} > b$ and the proof is complete with $\delta := \delta_1$ and $C := K_1$.

In the case $b_1 < b$, we shift the solutions $\widetilde{u}$ and $\widehat{u}$ as well as the inhomogeneities $\widetilde{g}$ and $\widehat{g}$ by b_1 and argue similarly (possibly reducing δ_1) to extend the estimate for $\widetilde{u} - \widehat{u}$ by another step of size b_1 if $2b_1 \le b$. If $2b_1 > b$, we choose a smaller step size in order to not exceed the time b. This yields a procedure that terminates at b after a finite number of steps, since the step size b_1 depends only on u_h, g, b, C_1, C_2, C_3 and C_4.

To make this precise, we formulate a step of the iteration. Let $n \in \mathbb{N}$ with $nb_1 < b$ and let $\delta_n \in (0, \frac{1}{2})$ and $K_n > 1$ be constants such that for any initial data $\widetilde{u}_\mathrm{h}$, $\widehat{u}_\mathrm{h}$ and inhomogeneities $\widetilde{g}$, $\widehat{g}$ fulfilling the conditions in the claim with δ replaced by δ_n, we have that the corresponding solutions $\widetilde{u}$ and $\widehat{u}$ exist on $(-\infty, nb_1]$ and satisfy

$$\|\widetilde{u} - \widehat{u}\|_{Z_0^k(nb_1)} \le K_n\big(\|\widetilde{u}_\mathrm{h} - \widehat{u}_\mathrm{h}\|_{Z^k(0)} + \|\widehat{g} - \widetilde{g}\|_{Z^k([0,b])}\big). \tag{5.21}$$

We furthermore assume that δ_n and K_n have the same dependence on the parameters as δ and C from the claim.

The case $n = 1$ was shown above. If $(n+1)b_1 \le b$, we can make a full step of size b_1. Otherwise, a smaller step of size $b - nb_1 > 0$ is chosen. We set

$$T := \min\{b_1, b - nb_1\}, \quad \delta_{n+1} := \min\big\{\delta_n, (2K_n)^{-1}\big\}.$$

Let $\widetilde{u}_\mathrm{h}$, $\widehat{u}_\mathrm{h}$, $\widetilde{g}$ and $\widehat{g}$ be as in the claim with δ replaced by δ_{n+1} and set

$$\widetilde{u}_\mathrm{h}^\mathrm{s} := \widetilde{u}(\bullet + nb_1), \quad \widehat{u}_\mathrm{h}^\mathrm{s} := \widehat{u}(\bullet + nb_1), \quad \widetilde{g}^\mathrm{s} := \widetilde{g}(\bullet + nb_1), \quad \widehat{g}^\mathrm{s} := \widehat{g}(\bullet + nb_1). \tag{5.22}$$

Using the special case $\widehat{u}_\mathrm{h} = u_\mathrm{h}$, $\widehat{g} = g$ in (5.21) and noting that $nb_1 + T \le b$, we obtain the estimate

$$\begin{aligned}
\|\widetilde{u}_\mathrm{h}^\mathrm{s}\|_{Z^k_{-nb_1}(0)} + \|\widetilde{g}^\mathrm{s}\|_{Z^k([0,T])} &\le \|\widetilde{u}\|_{Z_0^k(nb_1)} + \|\widetilde{g}\|_{Z^k([0,b])} \\
&\le \|\widetilde{u} - u\|_{Z_0^k(nb_1)} + \|\widetilde{g} - g\|_{Z^k([0,b])} + \|u\|_{Z_0^k(nb_1)} + \|g\|_{Z^k([0,b])} \\
&\le K_n\delta_{n+1} + \delta_{n+1} + \|u\|_{Z_0^k(nb_1)} + \|g\|_{Z^k([0,b])} \le \overline{r}_0
\end{aligned}$$

and analogously $\|\widehat{u}_\mathrm{h}^\mathrm{s}\|_{Z^k_{-nb_1}(0)} + \|\widehat{g}^\mathrm{s}(t)\|_{Z^k([0,T])} \le \overline{r}_0$. According to the first part of Lemma 5.3, the shifted functions (5.22) satisfy the requirements of the second part of Lemma 5.2. The latter therefore yields solutions $\widetilde{u}^\mathrm{s}$ and $\widehat{u}^\mathrm{s}$ to the shifted problems in the space $Z^k_{-nb_1}(\overline{b}_0)$ satisfying $\max\big\{\|\widetilde{u}^\mathrm{s}\|_{Z^k_{-nb_1}(\overline{b}_0)}, \|\widehat{u}^\mathrm{s}\|_{Z^k_{-nb_1}(\overline{b}_0)}\big\} \le \overline{r}$, where $\overline{b}_0$ is given by

$$\overline{b}_0 = \min\left\{T, \frac{1}{C_1\big((1 + C_3)\,\overline{r} + C_2(1 + \overline{r}^N)\big)}, \frac{1}{2C_1\big(C_3 + C_2(1 + \overline{r}^{N-1})\big)}\right\},$$

see (5.12) and note $T \le b_1$. We have $\overline{b}_0 = T$ since $\overline{b}_0 < T$ would imply the contradiction $\overline{b}_0 = b_1 \ge T$ where we have used $b_1 < b$.

We set $v^\mathrm{s} := \widetilde{u}^\mathrm{s} - \widehat{u}^\mathrm{s}$, $v_0^\mathrm{s} := \widetilde{u}_\mathrm{h}^\mathrm{s}(0) - \widehat{u}_\mathrm{h}^\mathrm{s}(0)$ and

$$f^\mathrm{s}(t) := L\big(\widetilde{u}^\mathrm{s}(t) - \widehat{u}^\mathrm{s}(t)\big) + F\left(\widetilde{u}_t^\mathrm{s}\right) - F\left(\widehat{u}_t^\mathrm{s}\right) + \widetilde{g}^\mathrm{s}(t) - \widehat{g}^\mathrm{s}(t)$$

for $t \in [0, T]$, and proceed as in the derivation of (5.20). It holds

$$\|v^\mathrm{s}\|_{Z^k_{-nb_1}(T)} \le \|v^\mathrm{s}\|_{Z^k_{-nb_1}(0)} + C_1\Big(\|v_0^\mathrm{s}\|_{\mathcal{H}^k_\Sigma} + \sum_{\ell=0}^{k-1}\Big\|\partial_t^\ell f^\mathrm{s}(0)\Big\|_{\mathcal{H}^{k-\ell-1}_\Sigma} + \sum_{\ell=0}^{k}\Big\|\partial_t^\ell f^\mathrm{s}\Big\|_{L^1_T\mathcal{H}^{k-\ell}_\Sigma}\Big)$$

$$
\begin{aligned}
&\leq \big(1 + C_1 + C_1 C_2 (1 + \overline{r}^{N-1}) + C_1 C_3\big) \, \|\widetilde{u}^{\mathrm{s}}_{\mathrm{h}} - \widehat{u}^{\mathrm{s}}_{\mathrm{h}}\|_{Z^k_{-nb_1}(0)} \\
&\quad + C_1 (1 + T) \, \|\widetilde{g}^{\mathrm{s}} - \widehat{g}^{\mathrm{s}}\|_{Z^k([0,T])} + T C_1 \big(C_3 + C_2 (1 + \overline{r}^{N-1})\big) \, \|v^{\mathrm{s}}\|_{Z^k_{-nb_1}(T)} \, .
\end{aligned}
$$

Since $T \leq b_1$, we arrive at

$$
\|\widetilde{u}^{\mathrm{s}} - \widehat{u}^{\mathrm{s}}\|_{Z^k_{-nb_1}(T)} \leq K_1 \big(\|\widetilde{u}^{\mathrm{s}}_{\mathrm{h}} - \widehat{u}^{\mathrm{s}}_{\mathrm{h}}\|_{Z^k_{-nb_1}(0)} + \|\widetilde{g}^{\mathrm{s}} - \widehat{g}^{\mathrm{s}}\|_{Z^k([0,T])} \big).
$$

With the aid of Lemma 5.3 we get

$$
\|\widetilde{u} - \widehat{u}\|_{Z^k_0(nb_1+T)} \leq K_1 \big(\|\widetilde{u} - \widehat{u}\|_{Z^k_0(nb_1)} + \|\widetilde{g} - \widehat{g}\|_{Z^k([nb_1, nb_1+T])} \big).
$$

Inserting (5.21) results in

$$
\begin{aligned}
\|\widetilde{u} - \widehat{u}\|_{Z^k_0(nb_1+T)} &\leq K_1 K_n \big(\|\widetilde{u} - \widehat{u}\|_{Z^k(0)} + \|\widetilde{g} - \widehat{g}\|_{Z^k([b_1, nb_1])} \big) + K_1 \, \|\widetilde{g} - \widehat{g}\|_{Z^k([nb_1, nb_1+T])} \\
&\leq K_{n+1} \big(\|\widetilde{u} - \widehat{u}\|_{Z^k(0)} + \|\widetilde{g} - \widehat{g}\|_{Z^k([0,b])} \big)
\end{aligned}
$$

with $K_{n+1} := 2K_1 K_n$. This completes the induction step. The constants δ and C from the claim are then given by δ_N respectively K_N, where N is the number of steps needed to reach b. □

Bibliography

[1] G. Agrarwal. *Nonlinear Fiber Optics*. Fifth edition. Elsevier, 2013.

[2] W. Arendt, C. J. K. Batty, M. Hieber, and F. Neubrander. *Vector-valued Laplace Transforms and Cauchy Problems*. Second edition. Birkhäuser, 2011.

[3] F. Assous, P. Ciarlet, and S. Labrunie. *Mathematical Foundations of Computational Electromagnetism*. Springer, 2018.

[4] C. Athanasiadis and I. G. Stratis. "On some elliptic transmission problems". In: *Ann. Polon. Math.* 63.2 (1996), pp. 137–154.

[5] A. Babin and A. Figotin. "Nonlinear Maxwell equations in inhomogeneous media". In: *Comm. Math. Phys.* 241 (2003), pp. 519–581.

[6] R. W. Boyd. *Nonlinear Optics*. Third edition. Elsevier, Academic Press, 2008.

[7] C. Bresch. *Wellposedness of Maxwell systems with nonlinear retarded material laws*. PhD thesis (submitted). Karlsruher Institut für Technologie.

[8] C. Bresch and R. Schnaubelt. "Local wellposedness of Maxwell systems with retarded material laws in low regularity". In: *NoDEA Nonlinear Differential Equations Appl.* 33.1 (2026), Paper No. 13, 33 pp.

[9] P. N. Butcher and D. Cotter. *The Elements of Nonlinear Optics*. First paperback edition, reprinted. Cambridge University Press, 1998.

[10] M. Costabel, M. Dauge, and S. Nicaise. *Corner Singularities and Analytic Regularity for Linear Elliptic Systems. Part I: Smooth domains*. Unpublished book, 2010. URL: https://hal.science/hal-00453934v2.

[11] T. Dohnal, M. Ionescu-Tira, and M. Waurick. "Well-posedness and exponential stability of nonlinear Maxwell equations for dispersive materials with interface". In: *J. Differential Equations* 383 (2024), pp. 24–77.

[12] K.-J. Engel and R. Nagel. *One-Parameter Semigroups for Linear Evolution Equations*. Springer, 2000.

[13] J. D. Jackson. *Classical Electrodynamics*. Third edition. Wiley, 1999.

[14] O. A. Ladyzhenskaya and N. N. Ural'tseva. *Linear and Quasilinear Elliptic Equations*. Academic Press, 1968.

[15] R. Leis. *Initial Boundary Value Problems in Mathematical Physics*. Teubner, 1986.

[16] A. Milani and R. Picard. "Evolution equations with nonlinear constitutive laws and memory effects". In: *Differential Integral Equations* 15.3 (2002), pp. 327–344.

[17] S. Nicaise and C. Pignotti. "Asymptotic behavior of dispersive electromagnetic waves in bounded domains". In: *Z. Angew. Math. Phys.* 71.3 (2020), Paper No. 76.

[18] R. Schippa. "Strichartz estimates for Maxwell equations in media: the partially anisotropic case". In: *Comm. Partial Differential Equations* 49.4 (2024), pp. 279–332.

[19] R. Schnaubelt and M. Spitz. "Local wellposedness of quasilinear Maxwell equations with conservative interface conditions". In: *Comm. Math. Sci.* 20.8 (2022), pp. 2265–2313.

[20] A. Sihvola. *Electromagnetic Mixing Formulas and Applications*. Reprint of the first edition. The Institution of Engineering and Technology, 2008.

[21] M. Spitz. "Local Wellposedness of Nonlinear Maxwell Equations". PhD thesis. Karlsruher Institut für Technologie, 2017.

[22] M. Spitz. "Local wellposedness of nonlinear Maxwell equations with perfectly conducting boundary conditions". In: *J. Differential Equations* 266.8 (2019), pp. 5012–5063.

[23] C. Weber. "A local compactness theorem for Maxwell's equations". In: *Math. Methods Appl. Sci.* 2.1 (1980), pp. 12–25.

[24] C. Weber. "Regularity theorems for Maxwell's equations". In: *Math. Methods Appl. Sci.* 3.4 (1981), pp. 523–536.