LOCAL WELLPOSEDNESS OF QUASILINEAR MAXWELL EQUATIONS WITH ABSORBING BOUNDARY CONDITIONS

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Abstract. In this article we provide a local wellposedness theory for quasilinear Maxwell equations with absorbing boundary conditions in $\mathcal{H}^m$ for $m \geq 3$. The Maxwell equations are equipped with instantaneous nonlinear material laws leading to a quasilinear symmetric hyperbolic first order system. We consider both linear and nonlinear absorbing boundary conditions. We show existence and uniqueness of a local solution, provide a blow-up criterion in the Lipschitz norm, and prove the continuous dependence on the data. In the case of nonlinear boundary conditions we need a smallness assumption on the tangential trace of the solution. The proof is based on detailed apriori estimates and the regularity theory for the corresponding linear problem which we also develop here.

1. Introduction

The Maxwell system is the foundation of electromagnetism and thus one of the core partial differential equations in physics. For nonlinear instantaneous material laws, it can be written as a symmetric hyperbolic system under natural assumptions. On the full space $\mathbb{R}^d$, for such systems a satisfactory local wellposedness theory in $\mathcal{H}^s(\mathbb{R}^d)$ for $s > 1 + \frac{d}{2}$ is provided by Kato's work [12]. On domains $G \subsetneq \mathbb{R}^3$ the Maxwell system is characteristic, and with its standard boundary conditions it does not fit into the classes of hyperbolic problems for which one has a local wellposedness theory in $\mathcal{H}^3$. The available results are stated in Sobolev spaces of much higher order and with weights encoding a loss of derivatives in normal direction, see [10] or [20]. In the very recent papers [22] and [23] by one of the authors, an encompassing local wellposedness theory in $\mathcal{H}^m$ with $m \geq 3$ was derived for the Maxwell system endowed with perfectly conducting boundaries, and it has been extended to interface problems in [19].

In this paper we treat the quasilinear Maxwell system (1.1) with absorbing boundary conditions which occur if one has a finite, strictly positive conductivity at the boundary, see [9]. We establish a comprehensive local wellposedness theory in $\mathcal{H}^m$ with $m \geq 3$ for linear boundary conditions and also treat nonlinear ones under a smallness condition (which automatically holds in the linear

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case). Our result provides the framework to show global existence and exponential decay of the solutions if the initial data are small, see the companion paper [16] co-authored by one of us.

For such boundary conditions, local solutions in $\mathcal{H}^3$ were already constructed in [15] under a similar smallness assumption. However, neither uniqueness, nor blow-up criteria, nor the continuous dependence on data were addressed in this paper. These results (and ours below) rely on a regularity theory for the linearized non-autonomous problem. It seems to us that the corresponding estimates in [15] were not precise enough to show uniqueness of the nonlinear problem and to treat its wellposedness theory. The crucial problem in this respect is to derive differentiability in normal direction to the boundary, whereas tangential regularity can be shown in much greater generality, see [18]. In [3] and [14] this difficulty was solved on the linear level by transforming the system in a non-characteristic one, but the resulting estimates do not fit to the fixed point argument for the nonlinear system, as already observed in [15] concerning [14]. We note that [14] deals with a far more general situation.

In this work we study the Maxwell system

$$\begin{aligned} \partial_t \theta_e(E,H) &= \operatorname{curl} H - \sigma_e(E,H)E - J_0, && x \in G, \quad t \geq t_0, \\ \partial_t \theta_m(E,H) &= -\operatorname{curl} E, && x \in G, \quad t \geq t_0, \\ H \times \nu &= \nu \times (\zeta(E \times \nu)E \times \nu) + g, && x \in \Sigma, \quad t \geq t_0, \\ E(t_0) &= E_0, \qquad H(t_0) = H_0, && x \in G, \end{aligned} \tag{1.1}$$

for an initial time $t_0 \in \mathbb{R}$, an open subset G of $\mathbb{R}^3$ with a smooth compact boundary Σ, and the unit outward normal ν. We look for the electric and magnetic fields $(E(t,x), H(t,x)) \in \mathcal{U}$, where $\mathcal{U} \subseteq \mathbb{R}^6$ is a fixed open convex set.

The material laws $\theta = (\theta_e, \theta_m) : G \times \mathcal{U} \to \mathbb{R}^6$ are differentiable and their derivative $\partial_{(E,H)}\theta$ is C^m for some $m \in \mathbb{N}$ with $m \geq 3$, and it is assumed to be symmetric and positive definite. The latter is a standard assumption already in the linear case and was also imposed in [15], for instance. It is true for isotropic nonlinearities and large classes of constitutive relations arising in optics, see e.g. Example 2.1 in [13]. We refer to [2] and [9] for further background. The conductivities σ_e and ζ are also of class C^m, and ζ is symmetric and positive definite. The given current densities J_0 and g and the initial fields E_0 and H_0 are supposed to belong to $\mathcal{H}^m$.

Guided by the basic energy estimate (3.1), we look for solutions

$$u = (E,H) \in \bigcap\nolimits_{j=0}^{m} C^j(\overline{J}, \mathcal{H}^{m-j}(G)^6)$$

having tangential traces in $\mathcal{H}^m(J \times \Sigma)$, where $J = (t_0, T)$ for some $T > t_0$. The space of these functions is called G^m_Σ. For such a solution the data and coefficients have to satisfy the compatibility conditions (2.21). Assuming them, in our main Theorem 6.4 we show that

(1) the system (1.1) has a unique maximal solution in G^m_Σ with $m \geq 3$,
(2) blow-up can be characterized in the Lipschitz-norm,
(3) the solution depends continuously on the data in $\mathcal{H}^m$.

In the case of nonlinear boundary conditions we have to add a smallness assumption on the product $\overline{\kappa}\,|\partial_E\zeta(\cdot, E\times\nu)|$ for fields with $|E\times\nu|\le\overline{\kappa}$. So, either the boundary condition is close to be linear or the solution has to be uniformly small (as in [15]). We also deal with non-autonomous linear boundary conditions, see (2.4), without assuming smallness. The smallness condition is enforced by the basic energy estimate (3.1) which allows us to bound the tangential traces of the solution in $L^2(J\times\Sigma)$ by the boundary data in the same norm, but with a constant which cannot be made small. This behavior reappears on higher regularity levels and spoils the fixed point argument if the boundary condition is nonlinear. Still the situation is much better than for perfectly conducting boundaries where one may lose a derivative at the boundary, cf. [7].

For linear material laws θ one can treat nonlinear boundary conditions with bounded ζ even on an L^2–level without a smallness condition, see e.g. [8]. These results are based on the theory of monotone operators and semigroups. In our setting it seems to be impossible to argue in this way, also in view of blow-up examples in $\mathcal{H}(\mathrm{curl})$, see [5].

We follow the strategy of [22] and [23]. One freezes the solutions in the nonlinearities of the system and solves the resulting non-autonomous linear problems via localization, duality and regularization with precise apriori estimates. Then local solutions are constructed via a contraction argument. Similar ideas had been used in [15], though there core parts (like the regularization procedure) were not worked out. The improved blow-up condition and the continuous dependence on data require additional significant efforts.

We first rewrite the system (1.1) in the equivalent form of a standard hyperbolic system in Section 2, where we also collect our notation. Moreover, we describe the localization procedure. It is crucial for our arguments that in various steps we can partly decouple the normal direction from the time and tangential ones. To achieve this, we perform an additional transformation after the localization in order to keep the form of the boundary condition and the constant coefficient in front of ∂_3. For perfectly conducting boundaries and interfaces this has been discussed in [21] respectively [19], so that we can focus below on the new boundary conditions. The compatibility conditions for the linear and nonlinear problems are derived in (2.20) and (2.21). They differ from those in [22] and [23] in several respects.

In Section 3 we first solve the non-autonomous linear problem in G^0_Σ with L^2–data and Lipschitz coefficients and derive the basic L^2–estimate in Proposition 3.1. This result is known, see e.g. [3], but it is hard to find complete proofs and we need more precise information about the constants than given in e.g. [3]. So we sketch the proof and also obtain a rather general uniqueness statement. The apriori estimates in G^m_Σ are then proven inductively by combining bounds for normal derivatives and for those in tangential and time directions. Here and later on we can use the results in normal direction from [22] and [23] since they do not involve boundary conditions. (And so we can omit a few very lengthy and intricate proofs.) However, the absorbing boundary conditions lead to new terms in the tangential bounds which have to be estimated carefully, since the nonlinear boundary conditions lead to coefficients at the boundary with less integrability (only L^2 and not L^∞ in time) than those in the interior. This fact causes the smallness condition mentioned above.

In Section 4 we then derive the main linear regularity Theorem 4.7 needed for the nonlinear theory. We again follow the procedure from [22] and employ different regularization procedures in normal, tangential and time directions which have to be intertwined in a careful induction. We again have to deal with new terms at the boundaries, but also various additional difficulties arise because of the more complicated compatibility conditions due to the time-dependent boundary coefficient on the linear level.

Based on the linear theory, we can then perform the core fixed point argument in Theorem 5.3. Relying on the reasoning from [23], we can focus on the smallness conditions needed to deal with the semilinear boundary conditions. They allow us to absorb error terms in the crucial estimates. In the last section we derive our main local wellposedness Theorem 6.4. It is based on auxiliary results preparing the improved blow-up condition and the continuous dependence on data. Compared to [23], we have to deal again with additional boundary terms, the new compatibility conditions and the needed smallness assumptions.

2. Auxiliary results and notation

Let $m \in \mathbb{N}$, $\tilde{m} = \max\{m, 3\}$, and $G \subseteq \mathbb{R}^3$ be either an open set with a compact boundary $\Sigma := \partial G$ of class $C^{\tilde{m}+2}$ or $G = \mathbb{R}^3_+ = \{x \in \mathbb{R}^3 \,|\, x_3 > 0\}$. Its outer unit normal is ν. We write c and C for generic positive constants, as well as $c(a, b, \dots)$ or $C(a, b, \dots)$ if they depend on $a, b, \dots$. The range of a map v is denoted by $\operatorname{ran}(v)$, and $B(x, r)$ or $B_M(x, r)$ is the closed ball in a metric space M with center x and radius r. Let $t_0 \in \mathbb{R}$ be the initial time, where we often take $t_0 = 0$ in view of time invariance. To control constants, we partly fix a time $T' > t_0$ and let $T \in (t_0, T')$. For $J = (t_0, T)$, we set $\Omega = J \times \mathbb{R}^3_+$, and $\Gamma = J \times (\mathbb{R}^2 \times \{0\}) \cong J \times \mathbb{R}^2$. Sometimes J also denotes other open intervals.

We often use the same symbols for spaces of scalar, vector or matrix valued functions. Sobolev spaces are designated by $W^{s,p}$ with $W^{s,2} = \mathcal{H}^s$. Spaces on Σ are equipped with the surface measure, written as $\mathrm{d}x$. For $\gamma \geq 0$ and $t \in \mathbb{R}$ we set $e_{-\gamma}(t) = \mathrm{e}^{-\gamma t}$. We employ time-weighted norms such as

$$\|f\|^2_{\mathcal{H}^m_\gamma(\Omega)} = \sum_{|\alpha| \leq m} \|e_{-\gamma} \partial^\alpha f\|^2_{L^2(\Omega)},$$

and denote the respective spaces by the subscript γ.

Let $v : G \to \mathbb{R}^3$ be sufficiently regular. We write $\operatorname{tr}_n v$ for the trace of the normal component $v \cdot \nu$ on Γ, and $\operatorname{tr}_t v$ for the tangential trace $v \times \nu$ on Γ. We further employ its rotated variant $\operatorname{tr}_\tau v = \nu \times (\operatorname{tr}_t v)$ which is the tangential component $\operatorname{tr} v - (\operatorname{tr}_n v)\nu$ of the full trace $\operatorname{tr} v$. Note that the Euclidean norms of $\operatorname{tr}_t v$ and $\operatorname{tr}_\tau v$ coincide. For vector fields $u = (u^1, u^2) : G \to \mathbb{R}^6$ and $k \in \{t, \tau, n\}$ we further set $\operatorname{tr}_k u = (\operatorname{tr}_k u^1, \operatorname{tr}_k u^2)$. It is well known that the mappings

$$\operatorname{tr}_n : \mathcal{H}(\operatorname{div}) \to \mathcal{H}^{-1/2}(\Sigma) \qquad \text{and} \qquad \operatorname{tr}_t : \mathcal{H}(\operatorname{curl}) \to \mathcal{H}^{-1/2}(\Sigma)^3$$

are continuous, where their maximal domains $\mathcal{H}(\operatorname{curl})$ and $\mathcal{H}(\operatorname{div})$ in $L^2(G)^3$ are endowed with the respective graph norm, see Theorems IX.1.1+2 in [6]. For sufficiently regular functions $w : J \times G \to \mathbb{R}^l$ with $l \in \{3,6\}$ we set $(\operatorname{tr}_k w)(t) = \operatorname{tr}_k(w(t))$, see also the remarks at the beginning of Section 3. We introduce

$$G^m = G^m(J \times G) = \bigcap_{j=0}^{m} C^j(\overline{J}, \mathcal{H}^{m-j}(G)^6) \qquad \text{and}$$
$$\tilde{G}^m = \tilde{G}^m(J \times G) = \bigcap_{j=0}^{m} W^{j,\infty}(J, \mathcal{H}^{m-j}(G)^6).$$

Our solutions have extra trace regularity expressed by the space

$$G^m_\Sigma = G^m_\Sigma(J \times G) = \{u \in G^m(J \times G) \mid \operatorname{tr}_\tau u \in \mathcal{H}^m(J \times \Sigma)^6\},$$

which is equipped with its canonical norm. In the fixed point argument we also need the slightly larger one

$$\tilde{G}^m_\Sigma = \tilde{G}^m_\Sigma(J \times G) = \{u \in \tilde{G}^m(J \times G) \mid \operatorname{tr}_\tau u \in \mathcal{H}^m(J \times \Sigma)^6\}.$$

We also use the subscript Σ if $G = \mathbb{R}^3_+$.

To reformulate (1.1) as a standard first order system, we write $u = (E,H) = (u^1, u^2)$ for the unknowns and introduce the matrices and maps

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} -J_0 \\ 0 \end{pmatrix},$$
$$A_j^{\mathrm{co}} = \begin{pmatrix} 0 & -J_j \\ J_j & 0 \end{pmatrix}, \quad \theta = (\theta_e, \theta_m), \quad \chi = \partial_u \theta, \quad \sigma = \begin{pmatrix} \sigma_e & 0 \\ 0 & 0 \end{pmatrix}, \tag{2.1}$$
$$B_1 = \begin{pmatrix} \operatorname{tr}_t & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & \operatorname{tr}_t \end{pmatrix}, \quad B(u) = B_2 - \nu \times \zeta(B_1 u) B_1$$

for $j \in \{1,2,3\}$. Observe that $\operatorname{curl} = J_1\partial_1 + J_2\partial_2 + J_3\partial_3$. With this notation the Maxwell system (1.1) becomes

$$\begin{aligned} \chi(u(t))\partial_t u(t) + \sum_{j=1}^{3} A_j^{\mathrm{co}} \partial_j u(t) + \sigma(u(t))u(t) &= f(t), \qquad && x \in G, \quad t \geq t_0, \\ B(u(t))u(t) &= g(t), && x \in \Sigma, \quad t \geq t_0, \\ u(t_0) &= u_0 && x \in G, \end{aligned} \tag{2.2}$$

where $u_0 := (E_0, H_0)$. A *solution* u on an interval J (with $t_0 \in J$) to this system belongs to $G^m_\Sigma(J' \times G)$ for every compact interval $J' \subseteq J$, $u(t)$ takes values in a closed subset of $\mathcal{U}$ for each $t \in J$, and u satisfies (2.2) for $t \in J$.

Let $\mathcal{U} \subseteq \mathbb{R}^6$ be an open convex set. We write $V \Subset \mathcal{U}$ for open subsets V of $\mathcal{U}$ with a compact closure $\overline{V} \subseteq \mathcal{U}$. We assume that the coefficient θ belongs to $C^{\tilde{m}}(G \times \mathcal{U}, \mathbb{R}^{6\times 6})$ and that

$$\begin{aligned} &\chi = \partial_u \theta \in C^{\tilde{m}}(G \times \mathcal{U}, \mathbb{R}^{6\times6}_{\mathrm{sym}}), \quad \sigma \in C^{\tilde{m}}(G \times \mathcal{U}, \mathbb{R}^{6\times 6}), \quad \zeta \in C^{\tilde{m}}_\tau(T_{\mathcal{U}}\Sigma, \mathbb{R}^{3\times3}_{\mathrm{sym}}), \\ &\forall\, V \Subset \mathcal{U},\ \alpha \in \mathbb{N}_0^9,\ |\alpha| \leq \tilde{m} : \sup_{G\times V} (|\partial^\alpha \chi| + |\partial^\alpha \sigma|) < \infty,\ \sup_{\Sigma \times V} |\partial^\alpha \zeta| < \infty, \\ &\chi, \zeta \geq \eta I > 0, \end{aligned} \tag{2.3}$$

for some number $\eta > 0$, where $T_{\mathcal{U}}\Sigma = \{(x,v) \mid x \in \Sigma, v \in B_1(x)\mathcal{U}\}$. Mainly to unify notation, here we allow for a larger class of 'conductivities' than in (1.1). The subscript τ means that ζ is tangential in the sense that $\zeta\nu^\perp \subseteq \nu^\perp$. Actually we only need uniform positive definiteness on each set $G \times \overline{V}$, respectively $\Sigma \times \overline{V}$, but we impose the above condition to simplify the presentation a bit. Below we also state a variant of these assumptions for linear boundary conditions.

Let $\hat{u} \in \tilde{G}^m_\Sigma$ with $m \geq 3$. We freeze the coefficients in the nonlinearities setting $b = \zeta(B_1\hat{u})$, $A_0 = \chi(\hat{u})$, $D = \sigma(\hat{u})$, and

$$B(t) = B_2 - \nu \times b(t)B_1$$

for $t \in J$. Such coefficients belong to the function spaces

$$\begin{aligned} F^m = F^m(J \times G) = \{A \in W^{1,\infty}(J \times G)^{k\times k} \,|\, \forall\, \alpha \in \mathbb{N}_0^4 \text{ with } 1 \leq |\alpha| \leq m : \\ \partial^\alpha A \in L^\infty(J, L^2(G))\}, \\ F^{m,0} = F^{m,0}(G) = \{A \in L^\infty(G)^{k\times k} \,|\, \forall \alpha \in \mathbb{N}_0^3 \text{ with } 1 \leq |\alpha| \leq m : \partial^\alpha A \in L^2(G)\}, \end{aligned}$$

which are endowed with their natural norms. The corresponding spaces for the domains Ω, $\mathbb{R}^3_+$, $J \times \Sigma$, Σ, $\partial\mathbb{R}^3_+$, and Γ are denoted analogously. For b we need the space

$$F^m_{\mathcal{H}} = F^m_{\mathcal{H}}(J \times \Sigma) = F^{m-1}_\tau(J \times \Sigma) \cap \mathcal{H}^m(J \times \Sigma)^k$$

whose norm is given by

$$\begin{aligned} \|b\|^2_{F^m_{\mathcal{H}}(J\times\Sigma)} &= \|b\|^2_{F^{m-1}(J\times\Sigma)} + [b]^2_{\mathcal{H}^m(J\times\Sigma)} \qquad \text{with} \\ [b]^2_{\mathcal{H}^m(J\times\Sigma)} &= \sum_{\alpha\in\mathbb{N}_0^4, |\alpha|=m} \|\partial^\alpha b\|^2_{L^2(J\times\Sigma)}, \end{aligned}$$

Here, $k \in \mathbb{N}$ usually is 1, 3, or 6. If we drop J in F^m, we refer to the subspace of maps being constant in time. The subscript $\eta > 0$ in any of these or other spaces means that the functions take values in symmetric matrices with lower bound $\eta > 0$; whereas cp indicates that the maps are constant outside a compact subset of $\overline{J \times G}$, respectively $\overline{G}$. (For bounded G the latter subspace is equal to F^m or $F^{m,0}$.) In Lemma 2.1 and Corollary 2.2 of [23] one can find detailed results concerning the mapping properties of χ and σ in these spaces. Several variants of the product rules in $\tilde{G}^m$, F^m and $\mathcal{H}^m$ are shown in Lemma 2.1 of [22] for G (or in Lemma 2.22 of [21]), which easily extend to the boundary. In the context of $F^m_{\mathcal{H}}(J \times \Sigma)$ new issues arise for the terms of highest order, which are discussed in the relevant parts of the proofs.

We also look at the case of linear non-autonomous boundary conditions assuming that

$$\begin{aligned} &\chi \in C^{\tilde{m}}_\eta(G \times \mathcal{U}, \mathbb{R}^{6\times 6}_{\mathrm{sym}}), \quad \sigma \in C^{\tilde{m}}(G \times \mathcal{U}, \mathbb{R}^{6\times 6}), \quad \zeta = b \in F^{\tilde{m}}_{\eta,\tau}(\mathbb{R}_+ \times \Sigma), \\ &\forall\, V \Subset \mathcal{U},\ \alpha \in \mathbb{N}_0^9,\ |\alpha| \leq \tilde{m} : \sup_{G\times V}(|\partial^\alpha\chi| + |\partial^\alpha\sigma|) < \infty, \end{aligned} \tag{2.4}$$

for some number $\eta > 0$. For time-independent b the above assumption is (almost) a special case of (2.3), except that one requires $\mathcal{H}^{\tilde{m}}$ instead of $C^{\tilde{m}}$–regularity of b.

Let $A_0 \in F^{\tilde{m}}_\eta(J \times G)$, $D \in F^{\tilde{m}}(J \times G)$, $b \in F^{\tilde{m}}_{\mathcal{H},\eta}(J \times \Sigma)$, $f \in \mathcal{H}^m(J \times G)$, $g \in \mathcal{H}^m(J \times \Sigma)$ with $g \cdot \nu = 0$, and $v_0 \in \mathcal{H}^m(G)$. We then look for a solution

$v \in G^m_\Sigma(J \times G)$ of the linearized problem

$$
\begin{aligned}
A_0(t)\partial_t v(t) + \sum_{j=1}^{3} A_j^{\mathrm{co}} \partial_j v(t) + D(t)v(t) &= f(t), \qquad && x \in G, \qquad && t \geq t_0, \\
B(t)v(t) &= g(t), && x \in \Sigma, && t \geq t_0, \\
v(t_0) &= v_0, && x \in G.
\end{aligned} \tag{2.5}
$$

We solve the system (2.5) via localization, proceeding as in [19], [21] and [22]. To this aim, we cover $\overline{G}$ with connected open sets $U_0, U_1, \dots, U_N$ where $U_0 \subseteq G$ and $\Sigma \subseteq U_1 \cup \dots \cup U_N$. For each $i \in \{1, \dots, N\}$ we fix a $C^{\tilde{m}+2}$–diffeomorphism φ_i from U_i onto an open subset V_i of $B(0,1)$ which maps $\Sigma \cap U_i$ onto $\{y \in V_i \,|\, y_3 = 0\}$ and $G \cap U_i$ onto $\{y \in V_i \,|\, y_3 > 0\} \subseteq \mathbb{R}^3_+$ such that all partial derivatives of φ_i and φ_i^{-1} up to order $\tilde{m} + 2$ are bounded. The resulting composition operators are denoted by

$$\Phi_i : L^2(U_i) \to L^2(V_i); \;\; v \mapsto v \circ \varphi_i^{-1}, \qquad \Phi_i^{-1} : L^2(V_i) \to L^2(U_i); \;\; v \mapsto v \circ \varphi_i.$$

We use the same notation for the induced maps on Sobolev spaces, and also for the spaces on the domains $J \times U_i$ or $J \times V_i$. The extension by 0 or restrictions of a function v are also denoted by v.

Let $\{\theta_i \,|\, i \in \{0, 1, \dots, N\}\}$ be a smooth partition of unity subordinate to the sets U_i, and σ_i (resp. ω_i) be test functions in U_i (resp. V_i) which are equal to 1 on $\operatorname{supp} \theta_i$ (resp. on $K_i := \varphi_i(\operatorname{supp} \sigma_i)$). We can find a constant $\tau \in (0,1)$ and an index $z(i) \in \{1,2,3\}$ with $|\partial_{z(i)} \varphi_{i,3}| \geq \tau$ for each i, see e.g. Lemma 5.1 of [21], where $\varphi_{i,3}$ is the third component of φ_i. We assume that $z(i) = 3$. The other cases are treated analogously, cf. Section 5 of [21] and Section 3 of [19]. Let $i \in \{1, \dots, N\}$.

The usual localization procedure leads to a first order system on $\mathbb{R}^3_+$ with variable coefficient matrices, see e.g. (2.5) in [22]. For our analysis it is important to keep the constant matrix A_3^{co} and the form of the boundary condition. To this aim, we multiply the coefficients by x-depending matrices R_i defined below. We first introduce the auxiliary function

$$\beta_i = \omega_i \Phi_i(\partial_3 \varphi_{i,3}) + (1 - \omega_i) \frac{\partial_3 \varphi_{i,3}}{|\partial_3 \varphi_{i,3}|}(\varphi_i^{-1}(y_i))$$

for a fixed point $y_i \in V_i$. The second summand is not important since we mostly work on K_i where we have $\beta_i = \Phi_i \partial_3 \varphi_{i,3}$. It is easy to check the lower bound $|\beta_i| \geq \tau$. We assume that $\beta_i \geq \tau$ as the other sign is handled in the same way, cf. Section 5 of [21] and Section 3 of [19]. We then set

$$\hat{R}_i = \beta_i^{-1/2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \omega_i \Phi_i(\partial_1 \varphi_{i,3}) & \omega_i \Phi_i(\partial_2 \varphi_{i,3}) & -\beta_i \end{pmatrix}, \qquad R_i = \begin{pmatrix} \hat{R}_i & 0 \\ 0 & \hat{R}_i \end{pmatrix},$$

and define the 'localized' coefficients

$$
\begin{aligned}
A_0^i &= R_i \big(\omega_i \Phi_i A_0 + (1 - \omega_i) \eta I \big) R_i^T, \\
A_j^i &= R_i \Big(\omega_i \Phi_i \Big(\textstyle\sum_{k=1}^{3} A_k^{\mathrm{co}} \partial_k \varphi_{i,j} \Big) + (1 - \omega_i) \dfrac{\partial_3 \varphi_{i,3}}{|\partial_3 \varphi_{i,3}|} (\varphi_i^{-1}(y_i)) A_3^{\mathrm{co}} \Big) R_i^T,
\end{aligned}
$$

$$
\begin{aligned}
A_3^i &= A_3^{\mathrm{co}}, \qquad (2.6)\\
D^i &= \omega_i R_i \Phi_i D R_i^T - \sum\nolimits_{j=1}^3 A_j^i \partial_j (R_i^T)^{-1} R_i^T
\end{aligned}
$$

for $j \in \{1,2\}$ on $\mathbb{R}^3_+$ as in Section 3 of [19] or Section 2 of [22], where partly a different notation was used. We note that A_0^i belongs to $F^{\tilde{m}}_{\eta,\mathrm{cp}}(J \times \mathbb{R}^3_+)$ and D^i to $F^{\tilde{m}}_{\mathrm{cp}}(J \times \mathbb{R}^3_+)$, and that their norms in these spaces are bounded by a constant times the analogous norms of A_0 and D. The maps A_1^i and A_2^i are contained in the space

$$
F^{\tilde{m}}_{\mathrm{cf}}(\mathbb{R}^3_+) = \Big\{ A \in F^{\tilde{m}}_{\mathrm{cp}}(\mathbb{R}^3_+) \,\Big|\, \exists\, \mu_j \in F^{\tilde{m}}_{\mathrm{cp}}(\mathbb{R}^3_+, \mathbb{R}) : A = \sum\nolimits_{j=1}^3 A_j^{\mathrm{co}} \mu_j \Big\},
$$

and they are dominated in the norm of $F^{\tilde{m}}$ by a constant only depending on G, the charts, and ω_i. We stress that the functions μ_j are scalar.

To deal with the boundary condition, we set $\kappa_i = -\nabla \varphi_{i,3} \cdot \nu \in C^{\tilde{m}+1}(\Sigma \cap U_i)$. Observe that $\nabla \varphi_{i,3} = -\kappa_i \nu$ since $\nabla \varphi_{i,3}$ is normal to Σ and that $\kappa_i > 0$ on $\Sigma \cap U_i$ by the properties of $\varphi_{i,3}$. We further define

$$
\begin{aligned}
B_0 &= \begin{pmatrix} 0 & \nu_3 & -\nu_2 \\ -\nu_3 & 0 & \nu_1 \\ \nu_2 & -\nu_1 & 0 \end{pmatrix}, \qquad B_0^{\mathrm{co}} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = J_3,\\
B_1^{\mathrm{co}} &= \begin{pmatrix} B_0^{\mathrm{co}} & 0 \end{pmatrix}, \qquad B_2^{\mathrm{co}} = \begin{pmatrix} 0 & B_0^{\mathrm{co}} \end{pmatrix},
\end{aligned}
$$

cf. (2.1). Abusing notation, we identify B_0 with tr_t at Σ and B_0^{co} with tr_t at $\partial\mathbb{R}^3_+$ (where $\nu = -e_3$). As in (3.12) of [19] or Section 2 of [22], we then compute

$$
B_0^{\mathrm{co}} = \hat{R}_i \Big(\omega_i \Phi_i(\kappa_i B_0) + (1-\omega_i) \frac{\partial_3 \varphi_{i,3}}{|\partial_3 \varphi_{i,3}|} (\varphi_i^{-1}(y_i)) B_0^{\mathrm{co}} \Big) \hat{R}_i^T
$$

on $\mathbb{R}^2 \times \{0\}$. For the transformed coefficients, we take further cut-offs $\tilde{\omega}_i \in C_c^\infty(V_i)$ which are equal to 1 on $\operatorname{supp} \omega_i$ and define the auxiliary maps

$$
\begin{aligned}
\tilde{b}_i(t) &= \tilde{\omega}_i \Phi_i(\kappa_i^{-1} b(t)) + (1-\tilde{\omega}_i)\eta I,\\
C^i(t) &= B_0^{\mathrm{co}} \tilde{b}_i(t) \Phi_i(\kappa_i B_1) + \Phi_i(\kappa_i B_0) \tilde{b}_i(t) B_1^{\mathrm{co}},\\
\Omega_i &= \begin{pmatrix} \omega_i I & 0 \\ 0 & I \end{pmatrix}.
\end{aligned}
$$

We now introduce the localized boundary operators and coefficients

$$
\begin{aligned}
B^i(t) = \hat{R}_i \Big[& \omega_i \Phi_i(\kappa_i B(t)) \Omega_i + (1-\omega_i) \frac{\partial_3 \varphi_{i,3}}{|\partial_3 \varphi_{i,3}|} (\varphi_i^{-1}(y_i)) B_2^{\mathrm{co}} \qquad (2.7)\\
& + \omega_i (1-\omega_i) \frac{\partial_3 \varphi_{i,3}}{|\partial_3 \varphi_{i,3}|} (\varphi_i^{-1}(y_i)) C_i(t) + (1-\omega_i)^2 B_0^{\mathrm{co}} \tilde{b}_i(t) B_1^{\mathrm{co}} \Big] R_i^T,\\
b_i(t) = (\hat{R}_i^T&)^{-1} \tilde{b}_i(t) \hat{R}_i^{-1},
\end{aligned}
$$

on $\mathbb{R}^2 \times \{0\}$. One can then derive the identity

$$
B^i(t) = B_2^{\mathrm{co}} + B_0^{\mathrm{co}} b_i(t) B_1^{\mathrm{co}}. \qquad (2.8)
$$

For the reconstruction of the original boundary condition on Σ from the localized ones it is crucial to note that $B^i(t) = \hat{R}_i \Phi_i(\kappa_i B(t)) R_i^T$ on $\omega_i^{-1}(\{1\})$. Redefining η if necessary, we obtain that b_i is contained in $F^{\tilde{m}}_{\mathcal{H},\eta,\mathrm{cp}}(\Gamma)$ and bounded in this norm by a constant times the norm of b in $F^{\tilde{m}}_{\mathcal{H},\eta}(J \times \Sigma)$.

Let $h \in \mathcal{H}^m(J \times G)$ and $v \in G^m(J \times G)$. We introduce the transformed data

$$v_0^i = (R_i^T)^{-1} \Phi_i(\theta_i v_0), \qquad g^i = \hat{R}_i \Phi_i(\theta_i \kappa_i g), \tag{2.9}$$
$$f^i = f^i(h, v) = R_i \Big(\Phi_i(\theta_i h) + \Phi_i \Big(\sum\nolimits_{j=1}^3 A_j^{\mathrm{co}} \partial_j \theta_i v \Big) \Big).$$

These functions belong to $\mathcal{H}^m(\mathbb{R}^3_+)$, $\mathcal{H}^m(\Gamma)$, and $\mathcal{H}^m(\Omega)$, with norms bounded by a constant times the corresponding norms of v_0, g, h, and v, respectively. (In the existence proof one has to construct a suitable map h for a given f.) Instead of (2.5), we are now looking at the linear system

$$\begin{aligned} A_0^i(t)\partial_t v(t) + \sum_{j=1}^3 A_j^i \partial_j v(t) + D^i(t) v(t) &= f^i(t), \qquad & x \in \mathbb{R}^3_+ \quad & t \geq t_0, \\ B^i(t) v(t) &= g^i(t), & x \in \partial\mathbb{R}^3_+ \quad & t \geq t_0, \\ v(t_0) &= v_0^i & x \in \mathbb{R}^3_+, & \end{aligned} \tag{2.10}$$

for $i \in \{1, \ldots, N\}$ with the operators and maps from (2.6), (2.8), and (2.9). Once we have established apriori estimates and the regularity theory for (2.10), we obtain the corresponding assertions on G in Theorem 4.7. To that purpose, we also need the case $i = 0$ which leads to a much simpler full space problem already treated in [22], for instance. We put $\partial_0 = \partial_t$ and define the hyperbolic operators

$$L(w) = \chi(w)\partial_t + \sum\nolimits_{j=1}^3 A_j^{\mathrm{co}} \partial_j + \sigma(w) \qquad \text{on } J \times G,$$
$$L^{\mathrm{co}}(A_0, D) = L^{\mathrm{co}} = A_0 \partial_t + \sum\nolimits_{j=1}^3 A_j^{\mathrm{co}} \partial_j + D \qquad \text{on } J \times G,$$
$$L^i(A_0, A_1, A_2, A_3^{\mathrm{co}}, D) = L^i = \sum\nolimits_{j=0}^3 A_j^i \partial_j + D^i \qquad \text{on } \Omega. \tag{2.11}$$

In the last operator and in (2.10) we often omit the superscript i.

If u or v in G^m solves one of the above linear or nonlinear Maxwell systems, we can differentiate the evolution equation and the boundary condition $m - 1$ times and then take the time trace at $t = t_0 \in \overline{J}$. The resulting compatibility conditions on $\{t_0\} \times \Sigma$ are thus a necessary property for any sufficiently regular solution. ¿From (2.1) of [22] and (2.9) of [23] (see also Lemma 2.31 of [21]) we first recall several important formulas relating time derivatives of a solution v with its value at the same time t_0, where we assume conditions (2.4), respectively (2.3) for the nonlinear boundary condition.

Take a time $t_0 \in \overline{J}$, an inhomogeneity $f \in \mathcal{H}^m(\Omega)$, and initial values $u_0, v_0 \in \mathcal{H}^m(\mathbb{R}^3_+)$. Let $p \in \{0, \ldots, m\}$. Assume that $v \in G^m(\Omega)$ solves (2.10) without the boundary condition. Differentiating the evolution equation in time and dropping the superscript i, we deduce that this function satisfies

$$\partial_t^p v(t_0) = S_{m,p,A_j,D}(t_0, v(t_0), f), \tag{2.12}$$

for all $p \in \{0, \dots, m\}$, where the term $S^{\mathrm{lin}}_{m,p} = S_{m,p,A_j,D}(t_0, u_0, f)$ is defined by

$$
\begin{aligned}
S^{\mathrm{lin}}_{m,0} &= u_0, \\
S^{\mathrm{lin}}_{m,p} &= A_0(t_0)^{-1}\Big[\partial_t^{p-1} f(t_0) - \sum_{j=1}^{3} A_j \partial_j S^{\mathrm{lin}}_{m,p-1} - \sum_{l=1}^{p-1} \binom{p-1}{l} \partial_t^l A_0(t_0) S^{\mathrm{lin}}_{m,p-l} \\
&\qquad - \sum_{l=0}^{p-1} \binom{p-1}{l} \partial_t^l D(t_0) S^{\mathrm{lin}}_{m,p-1-l}\Big], \quad p \geq 1. \qquad (2.13)
\end{aligned}
$$

An analogous formula is true on G if v fulfills (2.5) and we replace A_j by A_j^{co} for $j \in \{1, 2, 3\}$.

Next, let $u \in G^m(J \times G)$ satisfy (2.2). We then obtain

$$
\partial_t^p u(t_0) = S_{m,p,\chi,\sigma}(t_0, u(t_0), f) \qquad (2.14)
$$

for all $p \in \{0, \dots, m\}$. Here we inductively define the maps $S^{\mathrm{nl}}_{m,p} = S_{m,p,\chi,\sigma}(t_0, u_0, f)$ by

$$
\begin{aligned}
S^{\mathrm{nl}}_{m,0} &= u_0, \\
S^{\mathrm{nl}}_{m,p} &= \chi(u_0)^{-1}\Big[\partial_t^{p-1} f(t_0) - \sum_{j=1}^{3} A_j^{\mathrm{co}} \partial_j S^{\mathrm{nl}}_{m,p-1} - \sum_{l=1}^{p-1} \binom{p-1}{l} M_1^l S^{\mathrm{nl}}_{m,p-l} \\
&\qquad - \sum_{l=0}^{p-1} \binom{p-1}{l} M_2^l S^{\mathrm{nl}}_{m,p-1-l}\Big], \qquad (2.15) \\
M_k^p &:= \sum_{1 \leq j \leq p} \sum_{\substack{\gamma_1, \dots, \gamma_j \in \mathbb{N} \\ \sum \gamma_i = p}} \sum_{l_1, \dots, l_j = 1}^{6} C(p, \gamma)(\partial_{y_{l_j}} \dots \partial_{y_{l_1}} \theta_k)(u_0) \prod_{i=1}^{j} (S^{\mathrm{nl}}_{m,\gamma_i})_{l_i},
\end{aligned}
$$

where $p \geq 1$, $k \in \{1, 2\}$, $\theta_1 = \chi$, $\theta_2 = \sigma$, $M_2^0 = \sigma(u_0)$, and $C(p, \gamma) = C((p, 0, 0, 0), \gamma_1, \dots, \gamma_j)$ is the constant from Lemma 2.1 of [23].

We have to estimate these maps. Lemma 2.3 of [22] (or Lemma 2.33 of [21]) shows the inequality

$$
\begin{aligned}
&\|S_{m,p,A_j,D}(t_0, u_0, f)\|_{\mathcal{H}^{m-p}(G)} \qquad (2.16) \\
&\qquad \leq c(r_0, \eta, m, p)\Big(\sum_{j=0}^{p-1} \|\partial_t^j f(t_0)\|_{\mathcal{H}^{m-1-j}(G)} + \|u_0\|_{\mathcal{H}^m(G)}\Big),
\end{aligned}
$$

provided that $A_0 \in F^{\tilde{m}}_{\eta}(J \times G)$, $A_1, A_2, A_3 \in F^{\tilde{m}}(G)$ and $D \in F^{\tilde{m}}(J \times G)$ and that the quantities

$$
\|A_k(t_0)\|_{F^{\tilde{m}-1,0}(G)}, \|D(t_0)\|_{F^{\tilde{m}-1,0}(G)}, \|\partial_t^j A_0(t_0)\|_{\mathcal{H}^{\tilde{m}-1-j}(G)}, \|\partial_t^j D(t_0)\|_{\mathcal{H}^{\tilde{m}-1-j}(G)}
$$

are bounded by r_0 for $k \in \{0, 1, 2, 3\}$ and $j \in \{1, \dots, m-1\}$. Here one can replace G by $\mathbb{R}^3_+$. Similarly, Lemma 2.4 of [23] says that

$$
\|S_{m,p,\chi,\sigma}(t_0, u_0, f)\|_{\mathcal{H}^{m-p}(G)} \qquad (2.17)
$$

$$\leq c(r_0,\eta,V,m,p)\Big(\sum_{j=0}^{p-1}\|\partial_t^j f(t_0)\|_{\mathcal{H}^{m-1-j}(G)}+\|u_0\|_{\mathcal{H}^m(G)}\Big),$$

if χ and σ fulfill (2.3), the range of u_0 is contained in $V \Subset \mathcal{U}$ and the number in parentheses is less or equal r_0. Lemma 2.4 of [23] also provides an analogous Lipschitz estimate for arguments (u_0, f) and $(\tilde{u}_0, \tilde{f})$.

On the other hand, for $g \in \mathcal{H}^m(J \times \Sigma)$ and $v \in G^m_\Sigma(J \times G)$ solving (2.5) we can differentiate the boundary condition in (2.5) up to $m-1$ times in time arriving at

$$B(t_0)\partial_t^p v(t_0) = \partial_t^p g(t_0) + \nu \times \sum_{k=1}^{p}\binom{p}{k}\partial_t^k b(t_0) B_1 \partial_t^{p-k} v(t_0) \tag{2.18}$$

on Σ for all $p \in \{1, \dots, m-1\}$. Replacing B_1 by B_1^{co} the same equation is true on $\partial\mathbb{R}^3_+$ and a function $v \in G^m_\Sigma(\Omega)$ fulfilling (2.10). Analogously, each solution $u \in G^m_\Sigma(J \times G)$ of (2.2) satisfies

$$B(u(t_0))\partial_t^p u(t_0) = \partial_t^p g(t_0) + \nu \times \sum_{k=1}^{p}\binom{p}{k}\partial_t^k \zeta(B_1 u)(t_0) B_1 \partial_t^{p-k} u(t_0) \tag{2.19}$$

on Σ for all $p \in \{1, \dots, m-1\}$. To express the factors $\partial_t^k \zeta(B_1 u)(t_0)$, as in (2.15) we set

$$M_3^p = \sum_{1\leq j\leq p}\ \sum_{\substack{\gamma_1,\dots,\gamma_j\in\mathbb{N}\\ \sum\gamma_i=p}}\ \sum_{l_1,\dots,l_j=1}^{2} C(p,\gamma)(\partial_{y_{l_j}}\dots\partial_{y_{l_1}}\zeta)(B_1 u_0)\prod_{i=1}^{j}(B_1 S^{\mathrm{nl}}_{m,\gamma_i})_{l_i}$$

for $p \geq 1$. Taking into account (2.12) and (2.14), formulas (2.18) and (2.19) lead to the equations

$$B(t_0)S^{\mathrm{lin}}_{m,p} = \partial_t^p g(t_0) + \nu \times \sum_{k=1}^{p}\binom{p}{k}\partial_t^k b(t_0) B_1 S^{\mathrm{lin}}_{m,p-k}, \tag{2.20}$$

$$B(u(t_0))S^{\mathrm{nl}}_{m,p} = \partial_t^p g(t_0) + \nu \times \sum_{k=1}^{p}\binom{p}{k} M_3^k B_1 S^{\mathrm{nl}}_{m,p-k}, \tag{2.21}$$

on Σ for all $p \in \{0, \dots, m-1\}$ respectively, which are called the *compatibility conditions* of order m for the systems (2.5), (2.10), respectively (2.2). (For $p = 0$ the sums are omitted.)

Remark 2.1. In the following two sections we develop the regularity theory for the linear problem (2.10) with coefficients $A_0 \in F^{\tilde{m}}_\eta(\Omega)$, $A_1, A_2 \in F^{\tilde{m}}_{\mathrm{cf}}(\mathbb{R}^3_+)$, $A_3 = A_3^{\mathrm{co}}$, $D \in F^{\tilde{m}}(\Omega)$, and $b \in F^{\tilde{m}}_{\mathcal{H},\eta}(\Gamma)$, where we partly add further smoothness and decay conditions (as $|x| \to \infty$) on the derivatives. In our reasoning, we use results from Section 3 and 4 of [22] or [21] which only involve the evolution equation $L^i u = f$ but not the boundary condition.

We observe that there it is also required that the coefficients A_0 and D are constant outside a compact subset of $\mathbb{R}^3_+$. For unbounded domains G, in the main Theorem 1.1 of [22] it is thus assumed that A_0 and D converge at infinity. However, these restrictions only enter because of one result cited from another work. They are needed in the basic linear wellposedness result in L^2 from [7] to derive deeper trace results in the context of boundary conditions which are different from ours. This result is replaced by Proposition 3.1 below, which does not need that A_0 and D are constant outside a compact subset. For this reason we can use the results of Section 3 and 4 of [22], *provided that* they do not involve the boundary condition.

3. Linear apriori estimates

We first state the basic well-posedness result of the localized linear problem (2.10) on the regularity level $m = 0$. In particular, the data f, g and u_0 belong to L^2. If we have a solution $u \in L^2(\Omega)$, the evolution equation implies that u is contained in $\mathcal{H}^1(J, \mathcal{H}^{-1}(\mathbb{R}^3_+))$. The initial condition thus makes sense in $\mathcal{H}^{-1}(\mathbb{R}^3_+)$. Moreover, the tangential trace can be extended from regular functions on Ω to those $u \in L^2(\Omega)$ with $Lu \in L^2(\Omega)$ yielding a distribution $\operatorname{tr}_\tau u$ in $\mathcal{H}^{-1/2}(\Gamma)$, see e.g. Remark 2.14 of [21]. The boundary condition can thus be understood as an equation in $\mathcal{H}^{-1/2}(\Gamma)$. We put $\operatorname{Div} A = \sum_{j=0}^3 \partial_j A_j$.

Proposition 3.1. *Let $t_0 = 0$, $A_j = A_j^T \in W^{1,\infty}(\Omega)^{6\times 6}$, $A_0 \geq \eta I$, $A_3 = A_3^{\mathrm{co}}$, $D \in L^\infty(\Omega)^{6\times 6}$, $b = b^T \in L^\infty_\tau(\Gamma)^{3\times 3}$ with $A_0, b \geq \eta I$, $u_0 \in L^2(\mathbb{R}^3_+)^6$, $f \in L^2(\Omega)^6$, and $g \in L^2(\Gamma)^3$ with $g \cdot \nu = 0$. Then there is a unique solution $u \in L^2(\Omega)^6$ of* (2.10)*. Moreover, u belongs to $C(\overline{J}, L^2(\mathbb{R}^3_+)^6)$, $\operatorname{tr}_\tau u$ to $L^2(\Gamma)^6$, and they satisfy equation* (3.2) *and the estimate*

$$\begin{aligned} &\mathrm{e}^{-2\gamma T} \|u(T)\|^2_{L^2(\mathbb{R}^3_+)} + \gamma \|u\|^2_{L^2_\gamma(\Omega)} + \|\operatorname{tr}_\tau u\|^2_{L^2_\gamma(\Gamma)} \\ &\qquad \leq c \left(\|A_0(0)\|_{L^\infty(\mathbb{R}^3_+)} \|u_0\|^2_{L^2(\mathbb{R}^3_+)} + \tfrac{1}{\gamma} \|f\|^2_{L^2_\gamma(\Omega)} + \|g\|^2_{L^2_\gamma(\Gamma)} \right) \end{aligned} \tag{3.1}$$

for constants $c = c(\eta, \|b\|_\infty) \geq 0$ and $\gamma_0(\eta, r) \geq 1$ and all $\gamma \geq \gamma_0(\eta, r)$ with $r := \|D - \operatorname{Div} A/2\|_\infty$.

This result is essentially known, see e.g. Proposition 2.1 of [3], so that we only indicate the main steps of the proof. Since the precise form of the constants is crucial for us, we fully show the estimate (3.1) for a solution $u \in G^0$ such that also $\partial_j u$ for $j \in \{0, 1, 2\}$ and $A_3^{\mathrm{co}} \partial_3 u$ belong to $L^2(\Omega)$. Set $v = e_{-\gamma} u$. The equation $Lu = f$ yields

$$\int_\Omega Lv \cdot v \,\mathrm{d}x\,\mathrm{d}t = \int_\Omega (e_{-\gamma} f \cdot v - \gamma A_0 v \cdot v) \,\mathrm{d}x\,\mathrm{d}t.$$

Using the symmetry of A_j, we next compute

$$\int_\Omega Lv \cdot v \,\mathrm{d}x\,\mathrm{d}t = \frac{1}{2} \sum_{j=0}^3 \int_\Omega \partial_j (A_j v \cdot v) \,\mathrm{d}x\,\mathrm{d}t + \int_\Omega (D - \tfrac{1}{2} \operatorname{Div} A) v \cdot v \,\mathrm{d}x\,\mathrm{d}t.$$

The first term on the right-hand side is equal to

$$\frac{1}{2} \int_{\mathbb{R}^3_+} (A_0(T) v(T) \cdot v(T) - A_0(0) u_0 \cdot u_0) \,\mathrm{d}x - \frac{1}{2} \int_\Gamma A_3^{\mathrm{co}} v \cdot v \,\mathrm{d}x\,\mathrm{d}t.$$

Basic vector algebra and the boundary condition in (2.10) then lead to

$$\begin{aligned}
-\frac{1}{2}\int_\Gamma A_3^{\mathrm{co}} v\cdot v\,\mathrm{d}x\,\mathrm{d}t &= \frac{1}{2}\int_\Gamma e_{-2\gamma}\,(u^1\cdot \operatorname{tr}_t u^2 - u^2\cdot \operatorname{tr}_t u^1)\,\mathrm{d}x\,\mathrm{d}t\\
&= \int_\Gamma e_{-2\gamma}\,u^1\cdot \operatorname{tr}_t u^2\,\mathrm{d}x\,\mathrm{d}t\\
&= \int_\Gamma e_{-2\gamma}\,u^1\cdot (g-(b\operatorname{tr}_t u^1)\times\nu)\,\mathrm{d}x\,\mathrm{d}t\\
&= \int_\Gamma e_{-2\gamma}\,(g\cdot \operatorname{tr}_\tau u^1 + (b\operatorname{tr}_t u^1)\cdot \operatorname{tr}_t u^1)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

The assumptions now imply the basic estimate

$$\begin{aligned}
&\mathrm{e}^{-2\gamma T}\tfrac{\eta}{2}\,\|u(T)\|^2_{L^2(\mathbb{R}^3_+)} + \gamma\eta\,\|u\|^2_{L^2_\gamma(\Omega)} + \eta\,\|\operatorname{tr}_\tau u^1\|^2_{L^2_\gamma(\Gamma)}\\
&\quad\le \tfrac{1}{2}\|A_0(0)\|_{L^\infty(\mathbb{R}^3_+)}\,\|u_0\|^2_{L^2(\mathbb{R}^3_+)} + r\,\|u\|^2_{L^2_\gamma(\Omega)} + \|f\cdot u\|_{L^1_{2\gamma}(\Omega)} + \|g\cdot \operatorname{tr}_\tau u^1\|_{L^1_{2\gamma}(\Gamma)}.
\end{aligned}$$

By means of the Cauchy-Schwarz inequality and choosing $\gamma\ge 4r/\eta$, one easily deduces (3.1) with $\operatorname{tr}_\tau u^1$ instead of $\operatorname{tr}_\tau u$. The remaining summand can be recovered from the boundary condition $\operatorname{tr}_t u^2 = g-(b\operatorname{tr}_t u^1)\times\nu$. We note that for $\gamma=0$ we also obtain the equality

$$\begin{aligned}
&\int_{\mathbb{R}^3_+} \tfrac{1}{2}A_0(T)u(T)\cdot u(T)\,\mathrm{d}x + \int_\Gamma (b\operatorname{tr}_t u^1)\cdot \operatorname{tr}_t u^1\,\mathrm{d}(t,x) \qquad (3.2)\\
&\quad= \int_{\mathbb{R}^3_+}\tfrac{1}{2}A_0(0)u_0\cdot u_0\,\mathrm{d}x + \int_\Omega \big((\tfrac{1}{2}\operatorname{Div}A - D)u\cdot u + u\cdot f\big)\,\mathrm{d}(t,x)\\
&\qquad - \int_\Gamma g\cdot \operatorname{tr}_\tau u^1\,\mathrm{d}(t,x).
\end{aligned}$$

The other steps follow a standard procedure going back (at least) to [17], see also [1] or [4]. Estimate (3.1) and a duality argument yield a solution u of (2.10) in $L^2(\Omega)$. One can also show a variant of (3.1) for the interval $J=\mathbb{R}$ without the terms at times $t=T$ and $t=0$, assuming analogous regularity assumptions. One thus obtains a solution u in $L^2_\gamma(\Omega)$ of (2.10) on $J=\mathbb{R}$ without an initial condition. Mollifiers in (t,x_1,x_2) yield aproximate solutions u_n, where $A_3^{\mathrm{co}}\partial_3 u_n$ belongs to $L^2_\gamma(\Omega)$ because of the evolution equation. Using the variant of (3.1) on $J=\mathbb{R}$ for u_n, we see that this estimate is also valid for solutions in $L^2_\gamma(\mathbb{R}\times\mathbb{R}^3_+)$ and that $\operatorname{tr}_\tau u$ is an element of $L^2_\gamma(\Gamma)$.

As in Theorem 6.11 of [4] one next shows that the solution vanishes on $(-\infty,0)$ if the same is true for f and g. For $u_0=0$, again by mollification one can now construct a unique solution $u\in L^2_\gamma(\Omega)$ of (2.10) satisfying $u\in C(\overline{J},L^2(\mathbb{R}^3_+)^6)$, $\operatorname{tr}_\tau u\in L^2(\Gamma)^6$, (3.1), and (3.2). This fact also leads to the uniqueness statement in Proposition 3.1. It thus remains to show that for $f=g=0$ the solution u is contained in $C(\overline{J},L^2(\mathbb{R}^3_+)^6)$ and fullfills $\operatorname{tr}_\tau u\in L^2(\Gamma)^6$, (3.1), and (3.2). In view of the estimate one only has to consider compactly supported u_0. In this case, the available full space result and the finite speed of propagation imply that u is continuous in $L^2(\mathbb{R}^3_+)$ for small times $t\ge 0$. The result then follows by mollification.

We next establish higher order apriori estimates for solutions $u \in G^m$ of (3.1), extending the approach of [22]. In the first step we treat the 'tangential' derivatives $\partial^\alpha u$ with $\alpha \in \mathbb{N}_0^4$ and $\alpha_3 = 0$. We use the space $\mathcal{H}^m_{\mathrm{ta}}(\Omega)$ containing those functions $u \in L^2(\Omega)$ such that all such derivatives with $|\alpha| \le m$ belong to $L^2(\Omega)$, which is equipped with its natural norm. The space $\mathcal{H}^m_{\mathrm{ta}}(\mathbb{R}^3_+)$ is defined analogously. The number γ_0 is taken from Proposition 3.1, whereas $\delta_{m>2}$ is equal to 1 if $m \in \{3, 4, \dots\}$ and zero if $m \in \{1, 2\}$.

Lemma 3.2. *Let $t_0 = 0$, $T \in (0, T')$, $\rho, \eta > 0$, $r \ge r_0 > 0$, $m \in \mathbb{N}$, and $\tilde m = \max\{m, 3\}$. Take coefficients $A_0 \in F^{\tilde m}_\eta(\Omega)$, $A_1, A_2 \in F^{\tilde m}_{\mathrm{cf}}(\mathbb{R}^3_+)$, $A_3 = A_3^{\mathrm{co}}$, $D \in F^{\tilde m}(\Omega)$, and $b \in F^{\tilde m}_{\mathcal{H},\eta}(\Gamma)$ satisfying*

$$\begin{aligned}
&\|A_i\|_{F^{\tilde m}(\Omega)} \le r, \quad \|D\|_{F^{\tilde m}(\Omega)} \le r, \quad \|b\|_{F^{\tilde m-1}(\Gamma)} \le r, \quad [b]_{\mathcal{H}^{\tilde m}(\Gamma)} \le \rho,\\
&\max\{\|A_i(0)\|_{F^{\tilde m-1,0}(\mathbb{R}^3_+)}, \max_{1\le j\le m-1} \|\partial_t^j A_0(0)\|_{\mathcal{H}^{\tilde m-1-j}(\mathbb{R}^3_+)}\} \le r_0,\\
&\max\{\|D(0)\|_{F^{\tilde m-1,0}(\mathbb{R}^3_+)}, \max_{1\le j\le m-1} \|\partial_t^j D(0)\|_{\mathcal{H}^{\tilde m-1-j}(\mathbb{R}^3_+)}\} \le r_0
\end{aligned}$$

for all $i \in \{0, 1, 2\}$. Choose data $u_0 \in \mathcal{H}^m(\mathbb{R}^3_+)$, $f \in \mathcal{H}^m_{\mathrm{ta}}(\Omega)$, and $g \in \mathcal{H}^m(\Gamma)$ with $g \cdot \nu = 0$. Assume that the solution u of (2.10) belongs to $G^m(\Omega)$. Then it is also contained in $G^m_\Sigma(\Omega)$ and we have

$$\begin{aligned}
&\sum_{\substack{|\alpha|\le m\\ \alpha_3=0}} \|\partial^\alpha u\|^2_{G^0_\gamma(\Omega)} + \gamma\, \|u\|^2_{\mathcal{H}^m_{\mathrm{ta},\gamma}(\Omega)} + \|\operatorname{tr}_\tau u\|^2_{\mathcal{H}^m_\gamma(\Gamma)} \qquad (3.3)\\
&\quad \le C^{\mathrm{ta}}_{m,0} \Big[\sum_{j=0}^{m-1} \|\partial_t^j f(0)\|^2_{\mathcal{H}^{m-1-j}(\mathbb{R}^3_+)} + \|u_0\|^2_{\mathcal{H}^m(\mathbb{R}^3_+)} + \delta_{m>2}\rho^2\, \|B_1 u\|^2_{L^\infty_\gamma(\Gamma)} \Big]\\
&\qquad + c\, \|g\|^2_{\mathcal{H}^m_\gamma(\Gamma)} + \frac{C^{\mathrm{ta}}_m}{\gamma} \Big(\|f\|^2_{\mathcal{H}^m_{\mathrm{ta},\gamma}(\mathbb{R}^3_+)} + \|u\|^2_{G^m_\gamma(\Omega)} \Big)
\end{aligned}$$

for all $\gamma \ge \gamma_0$, where $C^{\mathrm{ta}}_m = C^{\mathrm{ta}}_m(\eta, r, T')$, $C^{\mathrm{ta}}_{m,0} = C^{\mathrm{ta}}_{m,0}(\eta, r_0, \|b\|_\infty)$, and $c = c(\eta, m, \|b\|_\infty)$. If b even belongs to $b \in F^{\tilde m}_\eta(\Gamma)^{3\times 3}$ with norm less or equal r, then one can set $\rho = 0$ in the above inequality.

Proof. Let $\alpha \in \mathbb{N}_0^4$ with $|\alpha| \le m$ and $\alpha_3 = 0$. We use the differential operator $L = L^i(A_0, A_1, A_2, A_3^{\mathrm{co}}, D)$ from (2.11). Let $j \in \{0, 1, 2\}$. Exactly as in Lemma 3.2 of [22], elementary calculations yield the equations

$$\begin{aligned}
L\partial^\alpha u &= f_\alpha, \qquad && x \in \mathbb{R}^3_+, \qquad t \in J,\\
\partial^\alpha u(0) &= u_{0,\alpha}, && x \in \mathbb{R}^3_+,
\end{aligned}$$

for the functions

$$\begin{aligned}
f_\alpha &= \partial^\alpha f - \sum_{j=0}^{2} \sum_{0<\beta\le\alpha} \binom{\alpha}{\beta} \partial^\beta A_j \partial^{\alpha-\beta} \partial_j u - \sum_{0<\beta\le\alpha} \binom{\alpha}{\beta} \partial^\beta D \partial^{\alpha-\beta} u,\\
u_{0,\alpha} &= \partial^\alpha u(0) = \partial^{(0,\alpha_1,\alpha_2,0)} S_{m,\alpha_0,A_k,D}(0, u_0, f),
\end{aligned}$$

where we used $A_3 = A_3^{\mathrm{co}}$. Moreover, f_α is an element of $\mathcal{H}^{m-|\alpha|}(\Omega)$ and $u_{0,\alpha}$ of $\mathcal{H}^{m-|\alpha|}(\mathbb{R}^3_+)$ satisfying

$$\|f_\alpha\|_{L^2_\gamma(\Omega)} \le \|f\|_{\mathcal{H}^m_{\mathrm{ta},\gamma}(\Omega)} + c(m,r,T')\,\|u\|_{G^m(\Omega)}, \tag{3.4}$$

$$\|u_{0,\alpha}\|_{\mathcal{H}^{m-|\alpha|}(\mathbb{R}^3_+)} \le c(\eta,m,r_0)\Big(\sum_{k=0}^{m-1}\|\partial_t^k f(0)\|_{\mathcal{H}^{m-1-k}(\mathbb{R}^3_+)} + \|u_0\|_{\mathcal{H}^m(\mathbb{R}^3_+)}\Big).$$

In particular, $\partial^\alpha u$ has a tangential trace in $\mathcal{H}^{-1/2}(\Gamma)$. On the other hand, we can apply ∂_j to $\mathrm{tr}_t\,\partial^{\alpha-e_j}u = (-\partial^{\alpha-e_j}u_2, \partial^{\alpha-e_j}u_1, 0, -\partial^{\alpha-e_j}u_5, \partial^{\alpha-e_j}u_4, 0)$ in $\mathcal{H}^{-1/2}(\Gamma)$ and obtain $\partial_j\,\mathrm{tr}_t\,\partial^{\alpha-e_j}u = \mathrm{tr}_t\,\partial^\alpha u$ for $j\in\{0,1,2\}$. The boundary condition in (2.10) thus leads to the equation

$$B\partial^\alpha u = g_\alpha := \partial^\alpha g + \nu \times \sum_{0<\beta\le\alpha}\binom{\alpha}{\beta}\partial^\beta b\,\mathrm{tr}_t\,\partial^{\alpha-\beta}u^1$$

on Γ. We bound the terms of this sum in $L^2(\Gamma)$, at first for $b \in F^{\tilde m}_{\mathcal{H},\eta}(\Gamma)$. We use the trace theorem in the form

$$\|\,\mathrm{tr}\,v\|^2_{H^{1/2}(\partial\mathbb{R}^3_+)} \le \kappa\,\|v\|^2_{\mathcal{H}^1_{\mathrm{ta}}(\mathbb{R}^3_+)} + c\kappa^{-1}\,\|\partial_3 v\|^2_{L^2(\mathbb{R}^3_+)} \tag{3.5}$$

for any $\kappa > 0$, which can easily be derived using the Fourier transform.

Let $2 \le |\beta| \le \tilde m - 1$. Then the norm of $\partial^\beta b$ in $L^\infty(J, L^2(\partial\mathbb{R}^3_+))$ is less or equal $\|b\|_{F^{\tilde m-1}}$, whereas $v := e_{-\gamma}\,\mathrm{tr}_t\,\partial^{\alpha-\beta}u^1$ can be estimated in $L^2(J,\mathcal{H}^{3/2}(\partial\mathbb{R}^3_+)) \hookrightarrow L^2(J, L^\infty(\partial\mathbb{R}^3_+))$ and thus, after taking squares, by

$$\varepsilon\gamma\,\|u\|^2_{\mathcal{H}^m_{\mathrm{ta},\gamma}(\Omega)} + c(\varepsilon\gamma)^{-1}\,\|u\|^2_{\mathcal{H}^m_\gamma(\Omega)} \le \varepsilon\gamma\,\|u\|^2_{\mathcal{H}^m_{\mathrm{ta},\gamma}(\Omega)} + cT'(\varepsilon\gamma)^{-1}\,\|u\|^2_{G^m_\gamma(\Omega)},$$

using (3.5) with $\kappa = \varepsilon\gamma$ and any $\varepsilon > 0$. If $|\beta| = 1$ we argue similarly, invoking the spaces $L^\infty(J,\mathcal{H}^1(\partial\mathbb{R}^3_+)) \hookrightarrow L^\infty(J, L^4(\partial\mathbb{R}^3_+))$ for b and $L^2(J,\mathcal{H}^1(\mathbb{R}^3_+))$ for v.

The cases $m\in\{1,2\}$ are thus settled. It remains to consider the case $m \ge 3$ and $\alpha = \beta$. We can now only use the L^2-norm of $\partial^\beta b$ and bound v by its sup-norm which is dominated by $\|u\|_{L^\infty_\gamma(J,\mathcal{H}^2(\mathbb{R}^3_+))}$. We conclude

$$\begin{aligned}&\|g_\alpha\|^2_{L^2_\gamma(\Gamma)} \\ &\le \|g\|^2_{\mathcal{H}^m_\gamma(\Gamma)} + c(m,r)\big[\varepsilon\gamma\|u\|^2_{\mathcal{H}^m_{\mathrm{ta},\gamma}(\Omega)} + T'(\varepsilon\gamma)^{-1}\|u\|^2_{G^m_\gamma(\Omega)}\big] + c(m)\rho^2\|\,\mathrm{tr}_t u^1\|^2_{L^\infty_\gamma(\Gamma)}.\end{aligned} \tag{3.6}$$

The last term disappears if $m\in\{1,2\}$.

Since $g_\alpha\cdot\nu = 0$, Proposition 3.1 shows that $\mathrm{tr}_\tau\,\partial^\alpha u \in L^2(\Gamma)$ and that

$$\|\partial^\alpha u\|^2_{G^0_\Sigma(\Omega)} + \gamma\,\|\partial^\alpha u\|^2_{L^2(\Omega)} \le c\,\big(c(r_0)\,\|u_{0,\alpha}\|^2_{L^2(\mathbb{R}^3_+)} + \tfrac{1}{\gamma}\,\|f_\alpha\|^2_{L^2_\gamma(\Omega)} + \|g_\alpha\|^2_{L^2_\gamma(\Gamma)}\big)$$

for a constant $c = c(\eta,\|b\|_\infty)$ and all $\gamma \ge \gamma_0(\eta,r)$. We now insert estimates (3.4) and (3.6). Fixing a small number $\varepsilon = \varepsilon(\eta,m,r) > 0$, one can absorb the second term in the right-hand side in (3.6) by the above left-hand side. The assertion follows.

If b belongs to $F^{\tilde m}_{\eta,\mathrm{cp}}(\Gamma)$, then we can always estimate $\partial^\beta b$ in $L^\infty(J, L^2(\partial\mathbb{R}^3_+))$, so that (3.6) is true without the last term and we can proceed as above. □

The normal derivatives can be treated by means of Proposition 3.3 of [22] which is independent of the boundary condition. We recall this result, see also Lemma 3.11 of [21] and Remark 2.1. Let the coefficients and data satisfy the assumptions of Theorem 3.3 below with $m = 1$ and $\tilde m$ replaced by 1, and let $u \in G^1(\Omega)$ solve $L(A_0, A_1, A_2, A_3, D)u = f$ and $u(0) = u_0$. Then there are constants $C'_{1,0} = C'_{1,0}(\eta, r_0) \geq 1$ and $C'_1 = C'_1(\eta, r, T') \geq 1$ with

$$\|\nabla u\|^2_{G^0_\gamma(\Omega)} \leq e^{C'_1 T}\Big((C'_{1,0} + TC'_1)\Big[\sum_{j=0}^{2} \|\partial_j u\|^2_{G^0_\gamma(\Omega)} + \|f(0)\|^2_{L^2(\mathbb{R}^3_+)} + \|u_0\|^2_{\mathcal{H}^1(\mathbb{R}^3_+)}\Big] + \frac{C'_1}{\gamma}\|f\|^2_{\mathcal{H}^1_\gamma(\Omega)}\Big) \tag{3.7}$$

for all $\gamma \geq 1$.

The full apriori estimate now follows by an induction argument. We note that for $m \in \{1, 2\}$ one can probably reduce the regularity conditions of the coefficients here. For $m = 1$ it should be enough to require that they belong to $W^{1,\infty} \subseteq F^1$, and for $m = 2$ one seems to need $W^{1,\infty}$ and that the second order derivatives of the coefficients are contained in $L^\infty(J, L^3(\mathbb{R}^3_+))$. Since we are mostly interested in the case $m \geq 3$ and for the ease of the presentation, we do not elaborate on these points. For $m = 1$, Theorem 1.3 of [3] provides such a result in a setting including more boundary regularity. For different boundary conditions and $m = 1$, one finds related partial results in Corollary 3.12 in [21] and in the proof of Lemma 4.2 of [13].

Theorem 3.3. *Let $t_0 = 0$, $T \in (0, T')$, $\rho, \eta > 0$, $r \geq r_0 > 0$, and $m \in \mathbb{N}$. Take coefficients $A_0 \in F^{\tilde m}_\eta(\Omega)$, $A_1, A_2 \in F^{\tilde m}_{\mathrm{cf}}(\mathbb{R}^3_+)$, $A_3 = A_3^{\mathrm{co}}$, $D \in F^{\tilde m}(\Omega)$, and $b \in F^{\tilde m}_{\mathcal{H},\eta}(\Gamma)$ satisfying*

$$\|A_i\|_{F^{\tilde m}(\Omega)} \leq r, \quad \|D\|_{F^{\tilde m}(\Omega)} \leq r, \quad \|b\|_{F^{\tilde m-1}(\Gamma)} \leq r, \quad [b]_{\mathcal{H}^{\tilde m}(\Gamma)} \leq \rho,$$
$$\max\{\|A_i(0)\|_{F^{\tilde m-1,0}(\mathbb{R}^3_+)}, \max_{1\leq j\leq m-1} \|\partial_t^j A_0(0)\|_{\mathcal{H}^{\tilde m-1-j}(\mathbb{R}^3_+)}\} \leq r_0,$$
$$\max\{\|D(0)\|_{F^{\tilde m-1,0}(\mathbb{R}^3_+)}, \max_{1\leq j\leq m-1} \|\partial_t^j D(0)\|_{\mathcal{H}^{\tilde m-1-j}(\mathbb{R}^3_+)}\} \leq r_0$$

for all $i \in \{0, 1, 2\}$. Choose data $u_0 \in \mathcal{H}^m(\mathbb{R}^3_+)^6$, $f \in \mathcal{H}^m(\Omega)^6$, and $g \in \mathcal{H}^m(\Gamma)^3$ with $g \cdot \nu = 0$. Assume that the solution u of (2.10) belongs to $G^m(\Omega)$. Then it is also contained in $G^m_\Sigma(\Omega)$ and there is a number $\gamma_m = \gamma_m(\eta, r, T') \geq \gamma_0$ with

$$\|u\|^2_{G^m_\gamma(\Omega)} + \gamma\,\|u\|^2_{\mathcal{H}^m_{\mathrm{ta},\gamma}(\Omega)} + \|\operatorname{tr}_\tau u\|^2_{\mathcal{H}^m(\Gamma)} \tag{3.8}$$
$$\leq (C_{m,0} + TC_m)\mathrm{e}^{mC_1T}\Big(\sum_{j=0}^{m-1} \|\partial_t^j f(0)\|^2_{\mathcal{H}^{m-1-j}(\mathbb{R}^3_+)} + \|u_0\|^2_{\mathcal{H}^m(\mathbb{R}^3_+)} + \|g\|^2_{\mathcal{H}^m_\gamma(\Gamma)} + \delta_{m>2}\rho^2\,\|B_1 u\|^2_{L^\infty_\gamma(\Gamma)}\Big) + \frac{C_m}{\gamma}\|f\|^2_{\mathcal{H}^m_\gamma(\Omega)}$$

for all $\gamma \geq \gamma_m$, where $C_m = C_m(\eta, r, T') \geq 1$, $C_{m,0} = C_{m,0}(\eta, r_0, \|b\|_\infty) \geq 1$, and $C_1 = C_1(\eta, r, T')$ does not depend on m. If b even belongs to $F^{\tilde m}_\eta(\Gamma)^{3\times 3}$ with norm less or equal r, then one can set $\rho = 0$ in the above inequality.

Proof. For $m = 1$ the result follows from (3.7), Proposition 3.1 and Lemma 3.2, after choosing $\gamma_1(\eta, r, T') \geq \gamma_0$ large enough. We now assume that $m \geq 2$ and that the assertion has been shown for $m - 1$, keeping the assumptions on the coefficients.

We take the derivative ∂_p of (2.10) for $p \in \{0,1,2\}$. The function $\partial_p u$ then satisfies

$$\begin{aligned} L(A_j, D)\partial_p u = f_p &:= \partial_p f - \sum_{i=0}^{2} \partial_p A_i \partial_i u - \partial_p D u \quad \text{on } \Omega, \\ B\partial_p u = g_p &:= \partial_p g + \nu \times (\partial_p b \operatorname{tr}_t u^1) \quad \text{on } \Gamma, \\ \partial_p u(0) = u_{0,p} &:= \partial_p u_0 \quad \text{on } \mathbb{R}^3_+, \end{aligned} \tag{3.9}$$

where $\partial_0 u_0 = S_{m,1,A_j,D}(0, u_0, f)$. By straightforward estimates, the functions f_p and $u_{0,p}$ belong to $\mathcal{H}^{m-1}$ and are bounded by

$$\begin{aligned} \|f_p\|_{\mathcal{H}^{m-1}_\gamma(\Omega)} &\le \|f\|_{\mathcal{H}^m_\gamma(\Omega)} + c(m,r)\|u\|_{G^m_\gamma(\Omega)}, \\ \|u_{0,p}\|_{\mathcal{H}^{m-1}(\mathbb{R}^3_+)} + \|\partial_t^j f_p(0)\|^2_{\mathcal{H}^{m-1-j}(\mathbb{R}^3_+)} &\le c(\eta, r_0)\Big(\sum_{k=0}^{m-1} \|\partial_t^k f(0)\|^2_{\mathcal{H}^{m-1-k}(\mathbb{R}^3_+)} \\ &\qquad + \|u_0\|^2_{\mathcal{H}^m(\mathbb{R}^3_+)}\Big) \end{aligned} \tag{3.10}$$

for $j \in \{0, \dots, m-2\}$. See the inequalities before (3.38) in the proof of Theorem 3.4 of [22], or Theorem 3.13 of [21]. As in (3.6) we infer

$$\begin{aligned} &\|g_p\|^2_{\mathcal{H}^{m-1}_\gamma(\Gamma)} \\ &\le \|g\|^2_{\mathcal{H}^m_\gamma(\Gamma)} + c(m,r)\big[\varepsilon\gamma\|u\|^2_{\mathcal{H}^m_{\mathrm{ta},\gamma}(\Omega)} + T'(\varepsilon\gamma)^{-1}\|u\|^2_{G^m_\gamma(\Omega)}\big] + c(m)\rho^2\|\operatorname{tr}_t u^1\|^2_{L^\infty_\gamma(\Gamma)}, \end{aligned} \tag{3.11}$$

where the last term vanishes if $m = 2$ or if $b \in F^{\tilde m}_\eta(\Gamma)$. We apply the induction hypothesis to (3.9) and insert (3.10) and (3.11). It follows

$$\begin{aligned} &\|u\|^2_{G^{m-1}_\gamma(\Omega)} + \gamma\,\|u\|^2_{\mathcal{H}^m_{\mathrm{ta},\gamma}(\Omega)} + \|\operatorname{tr}_\tau u\|^2_{\mathcal{H}^m(\Gamma)} + \sum_{p=0}^{2} \|\partial_p u\|^2_{G^{m-1}_\gamma(\Omega)} \\ &\le (C'_{m,0} + TC'_m)\mathrm{e}^{(m-1)C_1 T}\Big[\sum_{j=0}^{m-1} \|\partial_t^j f(0)\|^2_{\mathcal{H}^{m-1-j}(\mathbb{R}^3_+)} + \|u_0\|^2_{\mathcal{H}^m(\mathbb{R}^3_+)} + \|g\|^2_{\mathcal{H}^m_\gamma(\Gamma)} \\ &\qquad + \delta_{m>2}\rho^2\,\|B_1 u\|^2_{L^\infty_\gamma(\Gamma)}\Big] + \frac{C'_m}{\gamma}\,(\|f\|^2_{\mathcal{H}^m_\gamma(\Omega)} + \|u\|^2_{G^m_\gamma(\Omega)}), \end{aligned} \tag{3.12}$$

where we already have absorbed the term $c(m,r)\varepsilon\gamma\,\|u\|^2_{\mathcal{H}^m_{\mathrm{ta},\gamma}(\Omega)}$ by the left-hand side fixing a small $\varepsilon = \varepsilon(m,\eta,r) > 0$. To control the remaining normal derivative $\partial_3^m u$, we apply (3.7) to $\partial_3^{m-1} u$. Since the boundary condition does not enter, we can directly use (3.42) of [22] (or (3.57) of [21]) yielding

$$\begin{aligned} \|\partial_3^m u\|^2_{G^0_\gamma(\Omega)} &\le (C'_{m,0} + TC'_m)\mathrm{e}^{C_1 T}\Big(\sum_{p=0}^{2} \|\partial_p u\|^2_{G^{m-1}_\gamma(\Omega)} + \|f(0)\|^2_{\mathcal{H}^{m-1}(\mathbb{R}^3_+)} \\ &\qquad + \|u_0\|^2_{\mathcal{H}^m(\mathbb{R}^3_+)}\Big) + \frac{C'_m}{\gamma}\,(\|f\|^2_{\mathcal{H}^m_\gamma(\Omega)} + \|u\|^2_{G^m_\gamma(\Omega)}). \end{aligned} \tag{3.13}$$

For $\gamma \ge \gamma_m$ and a sufficiently large $\gamma_m = \gamma_m(\eta, r, T')$, the inequalities (3.12) and (3.13) imply (3.8). $\square$

Using the trace theorem and the estimate (3.8) for $m = 2$, we can get rid of the extra term in this inequality.

Corollary 3.4. *Let the assumptions of Theorem 3.3 be true with $m \geq 3$ and $r \geq \rho$. Using the notation of this theorem, we obtain*

$$
\begin{aligned}
&\|u\|^2_{G^m_\gamma(\Omega)} + \gamma\, \|u\|^2_{\mathcal{H}^m_{\mathrm{ta},\gamma}(\Omega)} + \|\operatorname{tr}_\tau u\|^2_{\mathcal{H}^m(\Gamma)} \qquad (3.14)\\
&\leq (C_{m,0} + TC_m)\mathrm{e}^{mC_1T}\Big(\sum_{j=0}^{m-1} \|\partial_t^j f(0)\|^2_{\mathcal{H}^{m-1-j}(\mathbb{R}^3_+)} + \|u_0\|^2_{\mathcal{H}^m(\mathbb{R}^3_+)} + \|g\|^2_{\mathcal{H}^m_\gamma(\Gamma)}\Big)\\
&\quad + C_m \mathrm{e}^{(m+2)C_1T}\big[\|u_0\|^2_{\mathcal{H}^2(\mathbb{R}^3_+)} + \|f(0)\|^2_{\mathcal{H}^1(\mathbb{R}^3_+)} + \|\partial_t f(0)\|^2_{L^2(\mathbb{R}^3_+)} + \|g\|^2_{\mathcal{H}^2_\gamma(\Gamma)}\big]\\
&\quad + \frac{C_m}{\gamma}\|f\|^2_{\mathcal{H}^m_\gamma(\Omega)}.
\end{aligned}
$$

Unfortunately, the estimate (3.14) does not fit to the nonlinear fixed point argument since the constants in front of the data in [...] depend on r and thus on the size of functions inserted in the fixed point operator, see (5.21).

4. Linear regularity results

We still have to construct solutions of the linear problem (2.5) in the class G^m_Σ. In view of the localization procedure, we can focus on the halfspace case (2.10). We start with the L^2-solution from Proposition 3.1. Different regularization techniques in normal, tangential, and time direction have to be applied. The apriori estimates from the previous section then allow us to pass to the limit and derive the required smoothness. We can directly use the results of [22] concerning normal regularity since they do not involve boundary conditions. The relevant Lemma 4.1 and Corollary 4.2 of [22] are stated below as Lemma 4.3. To regularize in spatial tangential variables, we employ the norms

$$
\begin{aligned}
\|v\|^2_{\mathcal{H}^k_{\mathrm{ta}}(\mathbb{R}^3_+)} &= \int_{\mathbb{R}_+}\int_{\mathbb{R}^2} (1+|\xi|^2)^k |(\mathcal{F}_2 v)(\xi, x_3)|^2 \,\mathrm{d}\xi\,\mathrm{d}x_3,\\
\|v\|^2_{\mathcal{H}^k_{\mathrm{ta},\delta}(\mathbb{R}^3_+)} &= \int_{\mathbb{R}_+}\int_{\mathbb{R}^2} (1+|\xi|^2)^{k+1}(1+|\delta\xi|^2)^{-1} |(\mathcal{F}_2 v)(\xi, x_3)|^2 \,\mathrm{d}\xi\,\mathrm{d}x_3
\end{aligned}
$$

for $k \in \mathbb{Z}$, $\delta > 0$, and functions $v \in \mathcal{S}'(\overline{\mathbb{R}^3_+})$ whose Fourier transform $\mathcal{F}_2 v$ in (x_1, x_2) belongs to $L^2_{\mathrm{loc}}(\mathbb{R}^3_+)$. The space $\mathcal{H}^k_{\mathrm{ta}}(\mathbb{R}^3_+)$ consists of those v with finite norm $\|v\|_{\mathcal{H}^k_{\mathrm{ta}}(\mathbb{R}^3_+)}$. For $k \in \mathbb{N}_0$ we obtain the standard tangential Sobolev spaces as defined before. (See Sections 1.7 and 2.4 in [11].) The norm of $\mathcal{H}^{k+1}_{\mathrm{ta}}(\mathbb{R}^3_+)$ dominates that of $\mathcal{H}^k_{\mathrm{ta},\delta}(\mathbb{R}^3_+)$. Conversely, if $\|v\|_{\mathcal{H}^k_{\mathrm{ta},\delta}(\mathbb{R}^3_+)}$ is bounded as $\delta \to 0$, then v is contained in $\mathcal{H}^{k+1}_{\mathrm{ta}}(\mathbb{R}^3_+)$, see (2.4.4) in [11].

To construct mollifiers, we take a map $\chi \in C^\infty_c(\mathbb{R}^2)$ such that $\mathcal{F}_2\chi(\xi) = O(|\xi|^{m+1})$ as $\xi \to 0$ and $\mathcal{F}_2\chi(t\xi) = 0$ for all $t \in \mathbb{R}$ implies $\xi = 0$. Set $\chi_\varepsilon(x) = \varepsilon^{-2}\chi(\varepsilon^{-1}x)$ for all $x \in \mathbb{R}^2$ and $\varepsilon > 0$. The convolution in spatial tangential variables (x_1, x_2) by χ_ε is called J_ε. We collect the properties of J_ε in the above norms which follow from Theorems 2.4.5 and 2.4.6 in [11]. There it was assumed that the coefficient A belongs to Schwartz' class. An inspection of the proofs in [11] shows that it suffices to require the regularity stated below.

Lemma 4.1. *Let $k \in \{0, \dots, m\}$, $\delta \in (0,1)$, $v \in \mathcal{H}^{k-1}_{\mathrm{ta}}(\mathbb{R}^3_+)$, and $A \in F^m(\mathbb{R}^3_+)$ with $\partial^\alpha A \in L^2(\mathbb{R}^3_+)$ for all $\alpha \in \mathbb{N}_0^3$. Then there are constants $c, C > 0$ not depending on δ and v such that*

$$c\,\|v\|^2_{\mathcal{H}^{k-1}_{\mathrm{ta},\delta}(\mathbb{R}^3_+)} \leq \|v\|^2_{\mathcal{H}^{k-1}_{\mathrm{ta}}(\mathbb{R}^3_+)} + \int_0^1 \|J_\varepsilon v\|^2_{L^2(\mathbb{R}^3_+)}\,\varepsilon^{-2k-1}\Big(1 + \frac{\delta^2}{\varepsilon^2}\Big)^{-1}\,\mathrm{d}\varepsilon$$
$$\leq C\,\|v\|^2_{\mathcal{H}^{k-1}_{\mathrm{ta},\delta}(\mathbb{R}^3_+)},$$
$$\int_0^1 \|AJ_\varepsilon v - J_\varepsilon(Av)\|^2_{L^2(\mathbb{R}^3_+)}\,\varepsilon^{-2k-1}\Big(1 + \frac{\delta^2}{\varepsilon^2}\Big)^{-1}\,\mathrm{d}\varepsilon \leq C\,\|v\|^2_{\mathcal{H}^{k-2}_{\mathrm{ta},\delta}(\mathbb{R}^3_+)}.$$

We also use the analogous results on $\mathbb{R}^2 \cong \partial\mathbb{R}^3_+$ (dropping the subscript ta) which are taken from Theorems 2.4.1 and 2.4.2 of [11]. Because of the above lemma, for some time we have to work with smooth coefficients whose derivatives of arbitrary order belong to L^2. An approximation argument will bring us back to limited regularity of the coefficients later. The next result provides tangential regularity.

Lemma 4.2. *Let $\eta > 0$ and $m \in \mathbb{N}$. Take $A_0 \in F^{\tilde{m}}_\eta(\Omega)$, $A_1, A_2 \in F^{\tilde{m}}_{\mathrm{cf}}(\mathbb{R}^3_+)$, $A_3 = A_3^{\mathrm{co}}$, $D \in F^{\tilde{m}}(\Omega)$, and $b \in F^{\tilde{m}}_{\mathcal{H},\eta}(\Gamma)$. We further assume that $\partial^\alpha A_i, \partial^\alpha D \in L^2(\Omega)$ and $\partial^\beta b \in L^2(\Gamma)$ for all $\alpha \in \mathbb{N}_0^4$, $\beta \in \mathbb{N}_0^3$, and $i \in \{0,1,2\}$. Choose data $u_0 \in \mathcal{H}^m(\mathbb{R}^3_+)$, $f \in \mathcal{H}^m_{\mathrm{ta}}(\Omega)$, and $g \in \mathcal{H}^m(\Gamma)$ with $g \cdot \nu = 0$. Let u be the solution of* (2.10) *from Proposition 3.1. Suppose that u belongs to $\bigcap_{j=1}^m C^j(\overline{J}, \mathcal{H}^{m-j}(\mathbb{R}^3_+))$ and $\operatorname{tr}_\tau u$ to $\bigcap_{j=1}^m \mathcal{H}^j(J, \mathcal{H}^{m-j}(\partial\mathbb{R}^3_+))$. Then u is an element of $C(\overline{J}, \mathcal{H}^m_{\mathrm{ta}}(\mathbb{R}^3_+))$ and $\operatorname{tr}_\tau u$ of $L^2(J, \mathcal{H}^m(\partial\mathbb{R}^3_+))$.*

Proof. 1) We first show that u belongs to $L^\infty(J, \mathcal{H}^m_{\mathrm{ta}}(\mathbb{R}^3_+))$ and $\operatorname{tr}_\tau u$ to $L^2(J, \mathcal{H}^m(\partial\mathbb{R}^3_+))$. Let $\varepsilon, \delta \in (0,1)$ and $\gamma > 0$. The generic constants below do not depend on δ or γ. We let $t_0 = 0$ for simplicity.

Applying the operators $L = L(A_0, A_1, A_2, A_3^{\mathrm{co}}, D)$ and $B = B_2^{\mathrm{co}} - \nu \times bB_1^{\mathrm{co}}$ to $J_\varepsilon u$, we obtain

$$\begin{aligned} LJ_\varepsilon u &= J_\varepsilon f + \sum_{j=0}^{2} [A_j, J_\varepsilon]\partial_j u + [D, J_\varepsilon]u \qquad \text{on } \Omega, \\ BJ_\varepsilon u &= J_\varepsilon g - \nu \times [b, J_\varepsilon] B_1^{\mathrm{co}} u \qquad \text{on } \Gamma, \end{aligned} \tag{4.1}$$

where $[A_j, J_\varepsilon] = A_j J_\varepsilon - J_\varepsilon A_j$ etc. Lemma 4.1 implies the commutator estimate

$$\begin{aligned} \int_J e^{-2\gamma t} \int_0^1 \|[A_j, J_\varepsilon]\partial_j u(t)\|^2_{L^2(\mathbb{R}^3_+)}\,\varepsilon^{-2m-1}\Big(1 + \frac{\delta^2}{\varepsilon^2}\Big)^{-1}\,\mathrm{d}\varepsilon\,\mathrm{d}t \\ \leq c\,\big(\|u\|^2_{L^2_\gamma(J, \mathcal{H}^{m-1}_{\mathrm{ta},\delta}(\mathbb{R}^3_+))} + \|\partial_t u\|^2_{\mathcal{H}^{m-1}_\gamma(\Omega)}\big) \end{aligned} \tag{4.2}$$

for all $j \in \{0,1,2\}$. The other commutators are treated analogously. In particular, $LJ_\varepsilon u$ is an element of $L^2(\Omega)$, $J_\varepsilon u_0$ of $L^2(\mathbb{R}^3_+)$, and $BJ_\varepsilon u$ of $L^2(\Gamma)$. Hence, the apriori estimate from Proposition 3.1 can be applied to $J_\varepsilon u$. We first use Lemma 4.1 to derive

$$S_\delta := \sup_{t \in J} \mathrm{e}^{-2\gamma t}\|u(t)\|^2_{\mathcal{H}^{m-1}_{\mathrm{ta},\delta}(\mathbb{R}^3_+)} + \int_0^T \mathrm{e}^{-2\gamma t}\|\operatorname{tr}_\tau u(t)\|^2_{\mathcal{H}^{m-1}_\delta(\mathbb{R}^2)}\,\mathrm{d}t \tag{4.3}$$

$$
\begin{aligned}
&\le c \sup_{t\in J} e^{-2\gamma t}\Big(\|u(t)\|^2_{\mathcal{H}^{m-1}_{\mathrm{ta}}(\mathbb{R}^3_+)} + \int_0^1 \|J_\varepsilon u(t)\|^2_{L^2(\mathbb{R}^3_+)}\,\varepsilon^{-2m-1}\Big(1+\frac{\delta^2}{\varepsilon^2}\Big)^{-1}\mathrm{d}\varepsilon\Big)\\
&\quad + c\int_0^T \mathrm{e}^{-2\gamma t}\|\operatorname{tr}_\tau u(t)\|^2_{\mathcal{H}^{m-1}(\mathbb{R}^2)}\,\mathrm{d}t\\
&\quad + c\int_0^1\int_0^T \mathrm{e}^{-2\gamma t}\|\operatorname{tr}_\tau J_\varepsilon u(t)\|^2_{L^2(\mathbb{R}^2)}\,\mathrm{d}t\,\varepsilon^{-2m-1}\Big(1+\frac{\delta^2}{\varepsilon^2}\Big)^{-1}\mathrm{d}\varepsilon\\
&\le c\,\Big(\|u\|^2_{G^{m-1}_\gamma(\Omega)} + \|\operatorname{tr}_\tau u\|^2_{\mathcal{H}^{m-1}_\gamma(\Gamma)} + \int_0^1 [\|J_\varepsilon u\|^2_{G^0_\gamma(\Omega)} + \|J_\varepsilon \operatorname{tr}_\tau u\|^2_{L^2_\gamma(\Gamma)}]\\
&\qquad\quad \cdot \varepsilon^{-2m-1}\Big(1+\frac{\delta^2}{\varepsilon^2}\Big)^{-1}\mathrm{d}\varepsilon\Big).
\end{aligned}
$$

By Proposition 3.1 and (4.1) there are constants $C_0, \gamma_0 > 0$ such that the term in brackets $[\dots]$ is bounded by C_0 times

$$
\begin{aligned}
&\|J_\varepsilon u_0\|^2_{L^2(\mathbb{R}^3_+)} + \int_0^T \mathrm{e}^{-2\gamma t}\big(\|J_\varepsilon g(t)\|^2_{L^2(\mathbb{R}^2)} + \|[b, J_\varepsilon]B_1^{\mathrm{co}} u(t)\|^2_{L^2(\mathbb{R}^2)}\big)\,\mathrm{d}t\\
&+\frac{1}{\gamma}\int_0^T \mathrm{e}^{-2\gamma t}\Big[\|J_\varepsilon f(t)\|^2_{L^2(\mathbb{R}^3_+)} + \|[D, J_\varepsilon]u(t)\|^2_{L^2(\mathbb{R}^3_+)} + \sum_{j=0}^2 \|[A_j, J_\varepsilon]\partial_j u(t)\|^2_{L^2(\mathbb{R}^3_+)}\Big]\mathrm{d}t
\end{aligned}
$$

for all $\gamma \ge \gamma_0$. We insert these quantities in (4.3) and interchange $\mathrm{d}t$ and $\mathrm{d}\varepsilon$. Combined with Lemma 4.1 and (4.2), it follows

$$
\begin{aligned}
S_\delta \le c\,\Big[&\|u\|^2_{G^{m-1}_\gamma(\Omega)} + \|\operatorname{tr}_\tau u\|^2_{\mathcal{H}^{m-1}_\gamma(\Gamma)} + \|u_0\|^2_{\mathcal{H}^m(\mathbb{R}^3_+)} + \gamma^{-1}\big(\|f\|^2_{\mathcal{H}^m_\gamma(\Omega)}\\
&+ \|u\|^2_{L^2_\gamma(J,\mathcal{H}^{m-1}_{\mathrm{ta},\delta}(\mathbb{R}^3_+))} + \|\partial_t u\|^2_{\mathcal{H}^{m-1}_\gamma(\Omega)}\big) + \|g\|^2_{\mathcal{H}^m_\gamma(\Gamma)} + \|\operatorname{tr}_\tau u\|^2_{L^2_\gamma(J,\mathcal{H}^{m-2}_\delta(\mathbb{R}^2))}\Big].
\end{aligned}
\tag{4.4}
$$

The last summand is bounded by $\|\operatorname{tr}_\tau u\|^2_{\mathcal{H}^{m-1}_\gamma(\Gamma)}$. We can absorb the term with u in the regularized norm by the left-hand side choosing a sufficiently large γ, depending on T'. As a result, the quantity S_δ is bounded uniformly in $\delta \in (0,1)$. We conclude that u belongs to $L^\infty(J, \mathcal{H}^m_{\mathrm{ta}}(\mathbb{R}^3_+))$. Fatou's lemma further yields

$$
\begin{aligned}
\|\operatorname{tr}_\tau u\|^2_{L^2_\gamma(J,\mathcal{H}^m(\mathbb{R}^2))} &= \int_0^T \mathrm{e}^{-2\gamma t}\int_{\mathbb{R}^2}(1+|\xi|^2)^m\,|\mathcal{F}_2(\operatorname{tr}_\tau u)(t,\xi)|^2\,\mathrm{d}\xi\,\mathrm{d}t\\
&\le \liminf_{\delta\to 0}\int_0^T \mathrm{e}^{-2\gamma t}\int_{\mathbb{R}^2}(1+|\xi|^2)^m(1+|\delta\xi|^2)^{-1}\,|\mathcal{F}_2(\operatorname{tr}_\tau u)(t,\xi)|^2\,\mathrm{d}\xi\,\mathrm{d}t\\
&= \liminf_{\delta\to 0}\|\operatorname{tr}_\tau u\|^2_{L^2_\gamma(J,\mathcal{H}^{m-1}_\delta(\mathbb{R}^2))}.
\end{aligned}
$$

The right-hand side is finite because of estimate (4.4), and so $\operatorname{tr}_\tau u$ belongs to $L^2_\gamma(J, \mathcal{H}^m(\partial\mathbb{R}^3_+))$.

2) Step 1) and Lemma 4.3 below imply that u is an element of $\mathcal{H}^m(\Omega)$. To show $u \in C(\overline{J}, \mathcal{H}^m_{\mathrm{ta}}(\mathbb{R}^3_+))$, we fix $\alpha \in \mathbb{N}_0^4$ with $|\alpha| = m$ and $\alpha_0 = \alpha_3 = 0$. Using that $u \in \mathcal{H}^m(\Omega)$ solves (2.10) and that $\operatorname{tr}_\tau u \in \mathcal{H}^m(\Gamma)$, we derive

$$
L\partial^\alpha u = \partial^\alpha f - \sum_{j=0}^2 \sum_{0<\beta\le\alpha}\binom{\alpha}{\beta}\partial^\beta A_j\,\partial^{\alpha-\beta}\partial_j u - \sum_{0<\beta\le\alpha}\binom{\alpha}{\beta}\partial^\beta D\partial^{\alpha-\beta}u =: f_\alpha,
$$

$$B\partial^\alpha u = \partial^\alpha g + \nu \times \sum_{0<\beta\le\alpha} \binom{\alpha}{\beta} \partial^\beta b \,\mathrm{tr}_t\, \partial^{\alpha-\beta} u^1 =: g_\alpha,$$
$$\partial^\alpha u(0) = \partial^\alpha u_0.$$

The function f_α belongs to $L^2(\Omega)$, g_α to $L^2(\Gamma)$, and $\partial^\alpha u_0$ to $L^2(\mathbb{R}^3_+)$, cf. (3.4) and (3.6). Proposition 3.1 then shows that $\partial^\alpha u$ is contained in $G^0(\Omega)$, as required. $\square$

The following result summarizes Lemma 4.1 and Corollary 4.2 of [22] (or Lemma 4.1 and Corollary 4.2 of [21]). See also Remark 2.1.

Lemma 4.3. *Let the coefficients satisfy the assumptions of the previous lemma, $u_0 \in \mathcal{H}^m(\mathbb{R}^3_+)$, $f \in \mathcal{H}^m(\Omega)$, and let $u \in \bigcap_{j=1}^m C^j(\overline{J}, \mathcal{H}^{m-j}(\mathbb{R}^3_+))$ fulfill $L(A_0, A_1, A_2, A_3, D)u = f$ and $u(0) = u_0$. Take $k \in \{1, \ldots, m\}$ and a multi-index $\alpha \in \mathbb{N}_0^4$ with $|\alpha| = m$, $\alpha_0 = 0$, and $\alpha_3 = k$. Suppose that $\partial^\beta u$ is contained in $G^0(\Omega)$, respectively $L^2(\Omega)$, for all $\beta \in \mathbb{N}_0^4$ with $|\beta| = m$ and $\beta_3 \le k-1$. Then $\partial^\alpha u$ is an element of $G^0(\Omega)$, respectively $L^2(\Omega)$.*

The next lemma allows us to gain one derivative in time. In the proof one constructs a solution v to the initial boundary value problem which $\partial_t u$ formally satisfies. One then checks that the time integral of v coincides with u. Here and in the next proposition the compatibility conditions enter in a crucial way.

Lemma 4.4. *Let $\eta > 0$ and $J = (t_0, T)$. Take coefficients $A_0 \in F^3_\eta(\Omega)$, $A_1, A_2 \in F^3_{\mathrm{cf}}(\mathbb{R}^3_+)$, $A_3 = A_3^{\mathrm{co}}$, $D \in F^3(\Omega)$, and $b \in F^3_{\mathcal{H},\eta}(\Gamma)$. Choose data $u_0 \in \mathcal{H}^1(\mathbb{R}^3_+)$, $f \in \mathcal{H}^1(\Omega)$, and $g \in \mathcal{H}^1(\Gamma)$ with $g \cdot \nu = 0$. Assume that the tuple $(t_0, A_0, A_1, A_2, A_3, D, b, u_0, f, g)$ fulfills the compatibility conditions* (2.20) *of order* 1. *Let u be the solution of* (2.10) *from Proposition 3.1. Assume that u belongs to $G^1_\Sigma(J' \times \mathbb{R}^3_+)$ whenever $u \in C^1(\overline{J'}, L^2(\mathbb{R}^3_+))$ and $\mathrm{tr}_\tau\, u \in \mathcal{H}^1(J', L^2(\partial\mathbb{R}^3_+))$ for every open interval $J' \subseteq J$. Then u is contained in $G^1_\Sigma(\Omega)$.*

Proof. 1) Without loss of generality we assume $J = (0, T)$. Take $r > 0$ with

$$\begin{aligned}
&\|A_i\|_{F^3(\Omega)} \le r, \quad \|D\|_{F^3(\Omega)} \le r, \quad \|b\|_{F^3_{\mathcal{H}}(\Omega)} \le r,\\
&\max\{\|A_i(t)\|_{F^{2,0}(\mathbb{R}^3_+)}, \max_{j\in\{1,2\}} \|\partial_t^j A_0(t)\|_{\mathcal{H}^{2-j}(\mathbb{R}^3_+)}\} \le r,\\
&\max\{\|D(t)\|_{F^{2,0}(\mathbb{R}^3_+)}, \max_{j\in\{1,2\}} \|\partial_t^j D(t)\|_{\mathcal{H}^{2-j}(\mathbb{R}^3_+)}\} \le r
\end{aligned}$$

for all $i \in \{0, 1, 2\}$ and $t \in \overline{J}$. Let $\gamma = \gamma_1(\eta, r, T) \ge 1$ be defined by Theorem 3.3. We further choose a number $C_0 = C_0(\eta, r, T) \ge 1$ dominating the constants in (2.16), Proposition 3.1, and Theorem 3.3. We finally set

$$R_1 := \mathrm{e}^{2\gamma T} C_0 (\|f\|^2_{G^0_\gamma(\Omega)} + \|f\|^2_{\mathcal{H}^1_\gamma(\Omega)} + \|g\|^2_{\mathcal{H}^1_\gamma(\Gamma)} + \|u_0\|^2_{\mathcal{H}^1(\mathbb{R}^3_+)}). \tag{4.5}$$

2) Take an initial time $t_0 \in \overline{J}$ and assume that $u(t_0)$ belongs to $\mathcal{H}^1(\mathbb{R}^3_+)$ with $\|u(t_0)\|^2_{\mathcal{H}^1(\mathbb{R}^3_+)} \le R_1$. Choose a time step

$$0 < T_s \le \min\{1, (6C_0 r^2)^{-1}\}.$$

Following the proof of Lemma 4.5 of [22], we want to construct a function v in $C([t_0, T_s'], L^2(\mathbb{R}^3_+))$ satisfying

$$\begin{aligned}\tilde{L}v &= \partial_t f - \partial_t D\Big(\int_{t_0}^t v(s)\,\mathrm{d}s + u(t_0)\Big) \qquad \text{on } \Omega',\\ Bv &= \partial_t g + \nu \times \partial_t b\Big(\int_{t_0}^t B_1^{\mathrm{co}} v(s)\,\mathrm{d}s + B_1^{\mathrm{co}} u(t_0)\Big) \qquad \text{on } \Gamma',\\ v(t_0) &= S_{1,1,A_j,D}(t_0, f, u(t_0)) =: v_0 \qquad \text{on } \mathbb{R}^3_+.\end{aligned} \tag{4.6}$$

Here we define $T_s' = \min\{t_0 + T_s, T\}$, $J' = (t_0, T_s')$, $\Omega' = J' \times \mathbb{R}^3_+$, $\Gamma' = J' \times \partial\mathbb{R}^3_+$, and $\tilde{L} = L(A_0, A_1, A_2, A_3, \partial_t A_0 + D)$. To solve (4.6), we set

$$R := (4C_0^2(1 + C_0) + 6 + 6C_0(1 + c_{\mathrm{tr}}^2)r^2)R_1 \geq R_1,$$

where c_{tr} is the norm of the trace operator from $\mathcal{H}^1(\mathbb{R}^3_+)$ to $L^2(\partial\mathbb{R}^3_+)$. Let E be the closed ball in $G^0_{\Sigma,\gamma}(\Omega')$ with radius $R^{1/2}$ and center 0. Take $w \in E$. We look at the problem

$$\begin{aligned}\tilde{L}v &= \partial_t f - \partial_t D\Big(\int_{t_0}^t w(s)\,\mathrm{d}s + u(t_0)\Big) =: f_w \qquad \text{on } \Omega',\\ Bv &= \partial_t g + \nu \times \partial_t b\Big(\int_{t_0}^t B_1^{\mathrm{co}} w(s)\,\mathrm{d}s + B_1^{\mathrm{co}} u(t_0)\Big) =: g_w \qquad \text{on } \Gamma',\\ v(t_0) &= v_0 \qquad \text{on } \mathbb{R}^3_+.\end{aligned} \tag{4.7}$$

Note that the data in the above problem fulfill the assumptions of Proposition 3.1. (Use (2.16) for v_0.) This proposition thus provides a unique solution $\Phi(w) \in G^0_{\Sigma,\gamma}(\Omega')$ of (4.7). Proposition 3.1 and (2.16) imply that

$$\begin{aligned}&\|\Phi(w)\|^2_{G^0_{\Sigma,\gamma}(\Omega')} \leq C_0\left(\|v_0\|^2_{L^2(\mathbb{R}^3_+)} + \|f_w\|^2_{L^2_\gamma(\Omega')} + \|g_w\|^2_{L^2_\gamma(\Gamma')}\right)\\ &\leq 2C_0^3\left(\|f(0)\|^2_{L^2(\mathbb{R}^3_+)} + \|u(t_0)\|^2_{\mathcal{H}^1(\mathbb{R}^3_+)}\right) + 3C_0\left(\|\partial_t f\|^2_{L^2_\gamma(\Omega')} + \|\partial_t g\|^2_{L^2_\gamma(\Gamma')}\right)\\ &\quad + 3C_0 r^2 \int_{t_0}^{T_s'} \mathrm{e}^{-2\gamma t}\Big[T_s \int_{t_0}^t \left(\|w(s)\|^2_{L^2(\mathbb{R}^3_+)} + \|\operatorname{tr}_\tau w(s)\|^2_{L^2(\partial\mathbb{R}^3_+)}\right)\mathrm{d}s\\ &\qquad\qquad + \|u(t_0)\|^2_{L^2(\mathbb{R}^3_+)} + \|\operatorname{tr}_\tau u(t_0)\|^2_{L^2(\partial\mathbb{R}^3_+)}\Big]\,\mathrm{d}t\\ &\leq 2C_0^2(1 + C_0)R_1 + 3R_1 + 3C_0(1 + c_{\mathrm{tr}}^2)r^2 R_1 T_s + 3C_0 r^2 T_s\,\|w\|^2_{G^0_{\Sigma,\gamma}(\Omega')} \leq R.\end{aligned}$$

Hence, Φ maps E into itself. In a similar way we estimate

$$\begin{aligned}\|\Phi(w) - \Phi(\hat{w})\|^2_{G^0_{\Sigma,\gamma}(\Omega')} &\leq C_0\left(\|f_w - f_{\hat{w}}\|^2_{L^2_\gamma(\Omega')} + \|g_w - g_{\hat{w}}\|^2_{L^2_\gamma(\Gamma')}\right)\\ &\leq r^2 C_0 T_s\,\|w - \hat{w}\|^2_{G^0_{\Sigma,\gamma}(\Omega')} \leq \tfrac{1}{6}\,\|w - \hat{w}\|^2_{G^0_{\Sigma,\gamma}(\Omega')}\end{aligned}$$

for all $w, \hat{w} \in E$. The contraction mapping principle thus gives a unique function $v \in E$ solving (4.6).

3) In this step we assume that $u(t_0)$ belongs to $\mathcal{H}^1(\mathbb{R}^3_+)$ with $\|u(t_0)\|^2_{\mathcal{H}^1(\mathbb{R}^3_+)} \leq R_1$ and that the tuple $(t_0, A_0, A_1, A_2, A_3, D, b, f, g, u(t_0))$ fulfills the compatibility conditions (2.20) of order one; i.e., $B(t_0)u(t_0) = g(t_0)$ on $\partial\mathbb{R}^3_+$. We use the

solution $v \in G^0_\Sigma(\Omega')$ of (4.6) on the time interval J' from part 2) and define

$$\check{v}(t) = u(t_0) + \int_{t_0}^t v(s)\,\mathrm{d}s$$

for $t \in \overline{J'}$. An elementary calculation shows that $L(A_j, D)\check{v} = f$ on Ω' and $\check{v}(t_0) = u(t_0)$. (See Step II) of the proof of Lemma 4.5 of [22] or Lemma 4.7 of [21].) Using also (4.6), we compute

$$\begin{aligned}
B(t)\check{v}(t) &= B(t_0)u(t_0) + (B(t) - B(t_0))u(t_0) + \int_{t_0}^t B_2^{\mathrm{co}} v(s)\,\mathrm{d}s \\
&\qquad + (b(t)B_1^{\mathrm{co}}(\check{v}(t) - u(t_0)) \times \nu \\
&= g(t_0) + (b(t) - b(t_0))B_1^{\mathrm{co}} u(t_0) \times \nu + (b(t)B_1^{\mathrm{co}}(\check{v}(t) - u(t_0)) \times \nu \\
&\qquad + \int_{t_0}^t \partial_t g(s)\,\mathrm{d}s - \int_{t_0}^t (\partial_t b(s) B_1^{\mathrm{co}}\check{v}(s) + b(s)B_1^{\mathrm{co}}\partial_t \check{v}(s)) \times \nu\,\mathrm{d}s \\
&= g(t)
\end{aligned}$$

for $t \in \overline{J'}$. The uniqueness statement in Proposition 3.1 thus yields that $u = \check{v}$ on J' (where we use the obvious variant of this result with initial time t_0). We conclude that $u \in C^1(\overline{J'}, L^2(\mathbb{R}^3_+))$ and $\mathrm{tr}_\tau\, u \in \mathcal{H}^1(J', L^2(\partial\mathbb{R}^3_+))$ as tr_τ commutes with integration in time, and hence u belongs to $G^1_\Sigma(\Omega')$ by the assumption.

4) Let $t_0 = 0$. By the assumptions and $\|u_0\|^2_{\mathcal{H}^1(\mathbb{R}^3_+)} \le R_1$, step 3) shows that u belongs to $G^1_\Sigma((0, T_0) \times \mathbb{R}^3_+)$ with $T_0 = \min\{T_s, T\}$. If $T_0 = T$, the assertion is shown. Otherwise, Theorem 3.3 and the definition (4.5) yield the bound

$$\begin{aligned}
\|u\|^2_{G^1_\gamma(\Omega)} &\le C_0\big(\|f\|^2_{G^0_\gamma(\Omega)} + \|u_0\|^2_{\mathcal{H}^1(\mathbb{R}^3_+)} + \|g\|^2_{\mathcal{H}^1_\gamma(\Gamma)} + \frac{1}{\gamma}\|f\|^2_{\mathcal{H}^1_\gamma(\Omega)}\big) \\
&\le e^{-2\gamma T} R_1,
\end{aligned}$$

so that $\|u(T_0)\|^2_{\mathcal{H}^1(\mathbb{R}^3_+)} \le R_1$. Because $u \in G^1_\Sigma$ solves (2.10), the compatibility conditions of first order are also valid for the initial value $u(T_0)$ at time T_0, cf. (2.20). We conclude the proof in finitely many iterations. $\square$

For smooth coefficients we now obtain the desired regularity properties on $\mathbb{R}^3_+$.

Proposition 4.5. *Let $\eta > 0$ and $m \in \mathbb{N}$. Take $A_0 \in F^{\tilde{m}}_\eta(\Omega)$ with $\partial_t A_0 \in F^{\max\{m-1,3\}}_\eta(\Omega)$, $A_1, A_2 \in F^{\tilde{m}}_{\mathrm{cf}}(\mathbb{R}^3_+)$, $A_3 = A_3^{\mathrm{co}}$, $D \in F^{\tilde{m}}(\Omega)$, and $b \in F^{\tilde{m}}_{\mathcal{H},\eta}(\Gamma)$. We assume that $\partial^\alpha A_i, \partial^\alpha D \in L^2(\Omega)$ and $\partial^\beta b \in L^2(\Gamma)$ for all $\alpha \in \mathbb{N}_0^4$, $\beta \in \mathbb{N}_0^3$, and $i \in \{0,1,2\}$. Choose data $u_0 \in \mathcal{H}^m(\mathbb{R}^3_+)^6$, $f \in \mathcal{H}^m(\Omega)^6$, and $g \in \mathcal{H}^m(\Gamma)^3$ with $g \cdot \nu = 0$ such that the tuple $(t_0, A_j, D, b, u_0, f, g)$ fulfills the compatibility conditions* (2.20) *of order m. Then the solution u of* (2.10) *from Proposition 3.1 belongs to $G^m_\Sigma(\Omega)$.*

Proof. We proceed as in the proof of Proposition 4.7 of [22], letting $t_0 = 0$ for simplicity. The result for $m = 1$ is a consequence of the three previous lemmas. We assume that we have shown the assertion for some $m \in \mathbb{N}$ and that the assumptions are satisfied for $m + 1$. Hence the solution u of (2.10) belongs to

$G^m_\Sigma(\Omega)$. With the notation of the proof of Lemma 4.4, we see that the function $v = \partial_t u$ fulfills

$$\begin{aligned} \tilde{L}v &= \partial_t f - \partial_t D u =: f_u \qquad \text{on } \Omega', \\ Bv &= \partial_t g + \nu \times \partial_t b\, B_1^{\mathrm{co}} u =: g_u \qquad \text{on } \Gamma', \\ v(0) &= S_{m+1,1,A_j,D}(0,u_0,f) =: v_0 \qquad \text{on } \mathbb{R}^3_+. \end{aligned} \tag{4.8}$$

Observe that the coefficients and data in (4.8) satisfy the regularity assumptions of the induction hypothesis. For the compatibility conditions, we note that Lemma 4.6 of [22] (or Lemma 4.8 of [21]) yields

$$S_{m,p,A_j,\tilde{D}}(0,v_0,f_u) = S_{m+1,p+1,A_j,D}(0,u_0,f)$$

for all $p \in \{0,1,\dots,m-1\}$ and $\tilde{D} := \partial_t A_0 + D$. Equations (2.12) and (2.20) thus imply

$$\begin{aligned} B(0)S_{m,p,A_j,\tilde{D}}(0,v_0,f_u) &= B(0)S_{m+1,p+1,A_j,D}(0,u_0,f) \\ &= \partial_t^{p+1} g(0) + \nu \times \sum_{k=1}^{p+1} \binom{p+1}{k} \partial_t^k b(0) B_1^{\mathrm{co}} \partial_t^{p+1-k} u(0). \end{aligned} \tag{4.9}$$

On the other hand, by means of $v = \partial_t u$ we calculate

$$\begin{aligned} \partial_t^p g_u(0) + \nu \times \sum_{k=1}^{p} \binom{p}{k} \partial_t^k b(0) B_1^{\mathrm{co}} \partial_t^{p-k} v(0) \\ = \partial_t^{p+1} g(0) + \nu \times \sum_{k=0}^{p} \binom{p}{k} \partial_t^{k+1} b(0) B_1^{\mathrm{co}} \partial_t^{p-k} u(0) \\ + \nu \times \sum_{k=1}^{p} \binom{p}{k} \partial_t^{k} b(0) B_1^{\mathrm{co}} \partial_t^{p-k+1} u(0) \\ = \partial_t^{p+1} g(0) + \nu \times \sum_{k=1}^{p+1} \binom{p+1}{k} \partial_t^{k} b(0) B_1^{\mathrm{co}} \partial_t^{p+1-k} u(0). \end{aligned}$$

Combined with (4.9), we have established the compatibility condition (2.20) of order m for (4.8).

The induction hypothesis now shows that $\partial_t u$ belongs to $G^m_\Sigma(\Omega)$. By Lemma 4.3, Lemma 4.2, and the fact that tr_τ commutes with differentiation in time, the map u is thus contained in $G^{m+1}_\Sigma(\Omega)$. □

Above we have assumed extra smoothness of the coefficients. This assumption can be removed by an approximation argument. Take $A_0 \in F^{\tilde{m}}_\eta(\Omega)$, $A_1, A_2 \in F^{\tilde{m}}_{\mathrm{cf}}(\mathbb{R}^3_+)$, $A_3 = A_3^{\mathrm{co}}$, $D \in F^{\tilde{m}}(\Omega)$, and $b \in F^{\tilde{m}}_{\mathcal{H},\eta}(\Gamma)$. Using standard methods, one constructs functions $A_{0,\varepsilon} \in F^{\tilde{m}}_\eta(\Omega)$, $A_{1,\varepsilon}, A_{2,\varepsilon} \in F^{\tilde{m}}_{\mathrm{cf}}(\mathbb{R}^3_+)$, $D_\varepsilon \in F^{\tilde{m}}(\Omega)$, and $b_\varepsilon \in F^{\tilde{m}}_{\mathcal{H},\eta}(\Gamma)$ for $\varepsilon > 0$, which are uniformly bounded in the respective F–space and tend uniformly to A_0, A_1, A_2, D and b, respectively, as $\varepsilon \to 0$. Moreover, all their partial derivatives are contained in the respective F–space and thus in L^2. The analogous results are true for $A_{0,\varepsilon}(0)$ and $D_\varepsilon(0)$ with $\tilde{m}$ replaced by $\tilde{m}-1$. (Compare Lemma 2.2 of [22] or Lemma 2.21 of [21].)

We again choose data $u_0 \in \mathcal{H}^m(\mathbb{R}^3_+)$, $f \in \mathcal{H}^m(\Omega)$, and $g \in \mathcal{H}^m(\Gamma)$ with $g \cdot \nu = 0$ such that the tuple $(0, A_0, A_1, A_2, A_3, D, b, u_0, f, g)$ fulfills the compatibility conditions (2.20) of order m. To use the approximating coefficients, one has to modify the initial value in such a way that (2.20) is still satisfied.

Lemma 4.6. *Under the above assumptions, there is a number $\varepsilon_0 > 0$ and functions $u_{0,\varepsilon}$ in $\mathcal{H}^m(\mathbb{R}^3_+)$ for $\varepsilon \in (0, \varepsilon_0]$ such that the compatibility conditions* (2.20) *of order m are satisfied by the tuple $(0, A_{0,\varepsilon}, A_{1,\varepsilon}, A_{2,\varepsilon}, A_3, D_\varepsilon, b_\varepsilon, u_{0,\varepsilon}, f, g)$. Moreover, the new initial values $u_{0,\varepsilon}$ tend to u_0 in $\mathcal{H}^m(\mathbb{R}^3_+)$ as $\varepsilon \to 0$.*

Proof. Slightly modifiying the notation in (2.12), (2.13) and (2.20), we set

$$S^{\mathrm{lin}}_{m,p}(u_0) = S_{m,p,A_j,D}(0, u_0, f) = \partial_t^p u(0)$$

for $p \in \{0, \dots, m-1\}$ as $t_0 = 0$ and f remain fixed. We further define

$$\mathcal{B} := \begin{pmatrix} -I & B_0^{\mathrm{co}} b(0) \end{pmatrix} \qquad \text{and} \qquad \mathcal{B}_\varepsilon := \begin{pmatrix} -I & B_0^{\mathrm{co}} b_\varepsilon(0) \end{pmatrix}.$$

The compatibility conditions (2.20) can be rewritten as

$$\mathcal{B} A_3^{\mathrm{co}} S^{\mathrm{lin}}_{m,p}(u_0) = \partial_t^p g(0) + \nu \times \sum_{k=1}^{p} \binom{p}{k} \partial_t^k b(0) B_1^{\mathrm{co}} S^{\mathrm{lin}}_{m,p-k}(u_0), \tag{4.10}$$

$$\mathcal{B} A_3^{\mathrm{co}} \Big[S^{\mathrm{lin}}_{m,p}(u_0) + \Big(0, \sum_{k=1}^{p} \binom{p}{k} \partial_t^k b(0) B_1^{\mathrm{co}} S^{\mathrm{lin}}_{m,p-k}(u_0)\Big)^T \Big] = \partial_t^p g(0)$$

on $\partial\mathbb{R}^3_+$. Here and below, sums from 1 to 0 or from 0 to -1 are defined as zero. Here we understand B_1^{co} just as matrix and not as a trace operator. Since $\partial_t^k b(0) \in \mathcal{H}^{\tilde{m}-k-1/2}(\mathbb{R}^2)$ for all $k \in \{0, \dots, \tilde{m}-1\}$, the extension result Theorem 2.5.7 in [11] yields a function $\tilde{b} \in \mathcal{H}^{\tilde{m}}(\mathbb{R}^3_+)$ such that $\mathrm{tr}_{\partial\mathbb{R}^3_+} \partial_3^k \tilde{b} = \partial_t^k b(0)$ for all $k \in \{0, \dots, \tilde{m}-1\}$. In particular, we can extend $\partial_t^k b(0)$ by $\partial_3^k \tilde{b}$ to a function in $\mathcal{H}^{\tilde{m}-k}(\mathbb{R}^3_+)$. We write $\tilde{S}^{\mathrm{lin}}_{m,p}(u_0)$ for the term $[\dots]$ in (4.10), where we extend it to a function on $\mathbb{R}^3_+$ as described above. This term is expanded as

$$\tilde{S}^{\mathrm{lin}}_{m,p}(u_0) = \mathcal{A}^p \partial_3^p u_0 + \sum_{j=0}^{p-1} \tilde{C}_{p,p-j} \partial_3^j u_0 + \tilde{B}_p f, \tag{4.11}$$

where $\mathcal{A} := -A_0(0)^{-1} A_3$, the tangential differential operators $\tilde{C}_{p,p-j}$ belong to $\mathcal{L}(\mathcal{H}^{m-j}(\mathbb{R}^3_+), \mathcal{H}^{m-p}(\mathbb{R}^3_+))$ and $\tilde{B}_p$ is an element of $\mathcal{L}(\mathcal{H}^m(\Omega), \mathcal{H}^{m-p}(\mathbb{R}^3_+))$. These mapping properties can be shown using the product rules Lemma 2.1 of [22] (or Lemma 2.22 of [21]) and the regularity of $\partial_3^k \tilde{b}$.

By $\mathcal{B}_\varepsilon$, $\tilde{S}^{\mathrm{lin}}_{m,p,\varepsilon}$, $\mathcal{A}_\varepsilon$ and $\tilde{C}_{p,p-j,\varepsilon}$ we denote the variants of the above operators for $A_{j,\varepsilon}$, D_ε and $\tilde{b}_\varepsilon$, where we obtain $\tilde{b}_\varepsilon$ as $\tilde{b}$ above. As in the previous paragraph, one sees that the functions $\tilde{C}_{p,p-j,\varepsilon}$ are bounded in $\mathcal{L}(\mathcal{H}^{m-j}(\mathbb{R}^3_+), \mathcal{H}^{m-p}(\mathbb{R}^3_+))$ uniformly in $\varepsilon > 0$ by the properties of the coefficients. We further note that $\partial_t^k b_\varepsilon(0)$ converges to $\partial_t^k b(0)$ as $\varepsilon \to 0$, since b_ε tends to b in $\mathcal{H}^{\tilde{m}}(\Gamma)$. Since the extension map from Theorem 2.5.7 in [11] is continuous, we infer that $\tilde{b}_\varepsilon$ converges to $\tilde{b}$ in $\mathcal{H}^{\tilde{m}}(\mathbb{R}^3_+)$ as $\varepsilon \to 0$. One can thus show that in $\mathcal{H}^{m-p}(\mathbb{R}^3_+)$ the maps $\tilde{S}^{\mathrm{lin}}_{m,p,\varepsilon}(u_0)$ tend to $\tilde{S}^{\mathrm{lin}}_{m,p}(u_0)$ and $(\tilde{b} - \tilde{b}_\varepsilon) B_1 \tilde{S}^{\mathrm{lin}}_{m,p}(u_0)$ to 0 as $\varepsilon \to 0$.

We are looking for functions $u_{0,\varepsilon} \in \mathcal{H}^m(\mathbb{R}^3_+)$ satisfying

$$\mathcal{B}_\varepsilon A_3^{\mathrm{co}} \tilde{S}^{\mathrm{lin}}_{m,p,\varepsilon}(u_{0,\varepsilon}) = \partial_t^p g(0) = \mathcal{B} A_3^{\mathrm{co}} \tilde{S}^{\mathrm{lin}}_{m,p}(u_0) \quad \text{on } \partial\mathbb{R}^3_+.$$

Let $u_{0,\varepsilon} = u_0 + h$ for some $h \in \mathcal{H}^m(\mathbb{R}^3_+)$. The ε–variant of (4.11) yields

$$\tilde{S}^{\mathrm{lin}}_{m,p,\varepsilon}(u_0 + h) = \tilde{S}^{\mathrm{lin}}_{m,p,\varepsilon}(u_0) + \mathcal{A}^p_\varepsilon \partial_3^p h + \sum_{j=0}^{p-1} \tilde{C}_{p,p-j,\varepsilon} \partial_3^j h.$$

We thus have to find a map $h \in \mathcal{H}^m(\mathbb{R}^3_+)$ solving

$$\begin{aligned}
\mathcal{B}_\varepsilon A_3^{\mathrm{co}} \mathcal{A}^p_\varepsilon \partial_3^p h &= \mathcal{B}_\varepsilon A_3^{\mathrm{co}} \Big[\tilde{S}^{\mathrm{lin}}_{m,p}(u_0) - \tilde{S}^{\mathrm{lin}}_{m,p,\varepsilon}(u_0) - \sum_{j=0}^{p-1} \tilde{C}_{p,p-j,\varepsilon} \partial_3^j h \Big] \\
&\qquad + (\mathcal{B} - \mathcal{B}_\varepsilon) A_3^{\mathrm{co}} \tilde{S}^{\mathrm{lin}}_{m,p}(u_0) \\
&= \mathcal{B}_\varepsilon A_3^{\mathrm{co}} \Big[\tilde{S}^{\mathrm{lin}}_{m,p}(u_0) - \tilde{S}^{\mathrm{lin}}_{m,p,\varepsilon}(u_0) - \sum_{j=0}^{p-1} \tilde{C}_{p,p-j,\varepsilon} \partial_3^j h \qquad (4.12) \\
&\qquad + \big(0, (\tilde{b} - \tilde{b}_\varepsilon) B_1^{\mathrm{co}} \tilde{S}^{\mathrm{lin}}_{m,p}(u_0)\big)^T \Big].
\end{aligned}$$

We first construct functions $a^p_\varepsilon \in \mathcal{H}^{m-p}(\mathbb{R}^3_+)$ for $p \in \{0, \dots, m-1\}$, $\varepsilon \in (0, \varepsilon_0)$ and some $\varepsilon_0 > 0$, which satisfy the variant of (4.12) where we drop $\mathcal{B}_\varepsilon$ and replace $\partial_3^j h$ by $a^j_\varepsilon \in \mathcal{H}^{m-j}(\mathbb{R}^3_+)$. Moreover, the functions a^p_ε tend to 0 in $\mathcal{H}^{m-p}(\mathbb{R}^3_+)$ as $\varepsilon \to 0$.

This is done via induction over p. For $p = 0$, set $a^0_\varepsilon = (0, (\tilde{b} - \tilde{b}_\varepsilon) B_1^{\mathrm{co}} u_0)^T$ in $\mathcal{H}^m(\mathbb{R}^3_+)$. Let the functions a^k_ε be constructed for $0 \le k \le p-1 < m-1$. The right-hand side of (4.12) without $\mathcal{B}_\varepsilon$ is equal to $A_3^{\mathrm{co}} v_\varepsilon$ for maps v_ε, which tend to 0 in $\mathcal{H}^{m-p}(\mathbb{R}^3_+)$ as $\varepsilon \to 0$ by the above observations. We now use Lemma 4.9 of [22] (or Lemma 4.11 of [21]) which does not depend on boundary conditions. It yields functions a^p_ε for $\varepsilon \in (0, \varepsilon_0]$ and some $\varepsilon_0 > 0$ such that $a^p_\varepsilon \to 0$ in $\mathcal{H}^{m-p}(\mathbb{R}^3_+)$ as $\varepsilon \to 0$ and $A_3^{\mathrm{co}} \mathcal{A}^p_\varepsilon a^p_\varepsilon = A_3^{\mathrm{co}} v_\varepsilon$. So the maps a^p_ε exist.

Again by Theorem 2.5.7 in [11], we can find functions $h_\varepsilon \in \mathcal{H}^m(\mathbb{R}^3_+)$ satisfying $\mathrm{tr}_{\partial\mathbb{R}^3_+} \partial_3^p h_\varepsilon = \mathrm{tr}_{\partial\mathbb{R}^3_+} a^p_\varepsilon$ for $p \in \{0, \dots, m-1\}$. Moreover, h_ε converges to 0 in $\mathcal{H}^m(\mathbb{R}^3_+)$ as $\varepsilon \to 0$. The maps h_ε thus satisfy (4.12) and $u_{0,\varepsilon} = u_0 + h_\varepsilon$ fulfills the variant of (4.10) for the approximating coefficients. $\square$

Based on the results established so far, we can now derive the desired regularity result. Recall that we allow for $G = \mathbb{R}^3_+$.

Theorem 4.7. *Let $T \in (0, T')$, $\rho, \eta > 0$, $r \ge r_0 > 0$, and $m \in \mathbb{N}$. Take coefficients $A_0 \in F^{\tilde{m}}_\eta(J \times G)$, $D \in F^{\tilde{m}}(J \times G)$, $A_3 = A_3^{\mathrm{co}}$, and $b \in F^{\tilde{m}}_{\mathcal{H},\eta}(J \times \Sigma)$. If $G = \mathbb{R}^3_+$, pick $A_1, A_2 \in F^{\tilde{m}}_{\mathrm{cf}}(\mathbb{R}^3_+)$. Otherwise, let $A_1 = A_1^{\mathrm{co}}$ and $A_2 = A_2^{\mathrm{co}}$. Assume that the coefficients satisfy*

$$\|A_i\|_{F^{\tilde{m}}(J\times G)} \le r, \quad \|D\|_{F^{\tilde{m}}(J\times G)} \le r, \quad \|b\|_{F^{\tilde{m}-1}(J\times \Sigma)} \le r, \quad [b]_{\mathcal{H}^{\tilde{m}}(J\times\Sigma)} \le \rho,$$

$$\max\{\|A_i(0)\|_{F^{\tilde{m}-1,0}(J\times G)}, \max_{j\in\{1,\dots,m-1\}} \|\partial_t^j A_0(0)\|_{\mathcal{H}^{\tilde{m}-1-j}(G)}\} \le r_0,$$

$$\max\{\|D(0)\|_{F^{\tilde{m}-1,0}(J\times G)}, \max_{j\in\{1,\dots,m-1\}} \|\partial_t^j D(0)\|_{\mathcal{H}^{\tilde{m}-1-j}(G)}\} \le r_0,$$

for $i \in \{0,1,2\}$*. Choose data* $u_0 \in \mathcal{H}^m(G)^6$*,* $f \in \mathcal{H}^m(J \times G)^6$*, and* $g \in \mathcal{H}^m(J \times \Sigma)^3$ *with* $g \cdot \nu = 0$ *such that the tuple* $(0, A_0, A_1, A_2, A_3, D, b, u_0, f, g)$ *fulfills the compatibility conditions* (2.20) *of order* m*.*

Then there is a unique solution $u \in G^m_\Sigma(J \times G)$ *of* (2.5) *and there is a number* $\gamma_m = \gamma_m(\eta, r, T') \geq 1$ *such that*

$$\begin{aligned}
&\|u\|^2_{G^m_\gamma(J\times G)} + \|\operatorname{tr}_\tau u\|^2_{\mathcal{H}^m(J\times\Sigma)} \qquad (4.13)\\
&\quad \leq (C_{m,0} + TC_m)\mathrm{e}^{mC_1T}\Big(\sum_{j=0}^{m-1} \|\partial_t^j f(0)\|^2_{\mathcal{H}^{m-1-j}(G)} + \|u_0\|^2_{\mathcal{H}^m(G)} + \|g\|^2_{\mathcal{H}^m_\gamma(J\times\Sigma)}\\
&\qquad\qquad + \delta_{m>2}\rho^2 \, \|B_1 u\|^2_{L^\infty_\gamma(J\times\Sigma)}\Big) + \frac{C_m}{\gamma}\|f\|^2_{\mathcal{H}^m_\gamma(J\times G)}
\end{aligned}$$

for all $\gamma \geq \gamma_m$ *and with constants* $C_i = C_i(\eta, r, T') \geq 1$ *for* $i \in \{1, m\}$ *and* $C_{m,0} = C_{m,0}(\eta, r_0, \|b\|_\infty) \geq 1$*. If* b *even belongs to* $F^{\tilde{m}}_{\eta,\tau}(J \times \Sigma)^{3\times 3}$ *with norm less or equal* r*, then one can set* $\rho = 0$ *in the above inequality. For* $m \geq 3$ *and* $r \geq \rho$ *we further have the estimate*

$$\begin{aligned}
&\|u\|^2_{G^m_\gamma(J\times G)} + \|\operatorname{tr}_\tau u\|^2_{\mathcal{H}^m(J\times\Sigma)} \qquad (4.14)\\
&\quad \leq (C_{m,0} + TC_m)\mathrm{e}^{mC_1T}\Big(\sum_{j=0}^{m-1} \|\partial_t^j f(0)\|^2_{\mathcal{H}^{m-1-j}(G)} + \|u_0\|^2_{\mathcal{H}^m(G)} + \|g\|^2_{\mathcal{H}^m_\gamma(J\times\Sigma)}\Big)\\
&\qquad + C_m\mathrm{e}^{(m+2)C_1T}\big(\|u_0\|^2_{\mathcal{H}^2(G)} + \|f(0)\|^2_{\mathcal{H}^1(G)} + \|\partial_t f(0)\|^2_{L^2(G)} + \|g\|^2_{\mathcal{H}^2_\gamma(J\times\Sigma)}\big)\\
&\qquad + \frac{C_m}{\gamma}\|f\|^2_{\mathcal{H}^m_\gamma(J\times G)}.
\end{aligned}$$

Proof. We only sketch the proof since it is very similar to those of Theorems 4.10 and 1.1 of [22] (or Theorem 4.12 of [21]). We first treat the localized problem (2.10) on $\mathbb{R}^3_+$. We take approximating data as in Lemma 4.6 for $\varepsilon = \frac{1}{n}$. Proposition 4.5 then provides solutions $u_n \in G^m_\Sigma(\Omega)$ which are uniformly bounded in this space due to (3.8) and (3.14). From Banach-Alaoglu we thus obtain a weak* accumulation point u of (u_n) which belongs to $\tilde{G}^m_\Sigma(\Omega)$. We apply (3.1) with the given coefficients to the difference $u_n - u$. By means of the uniform convergence of the coefficients, it follows that the maps u_n tend to u in $G^0_\Sigma(\Omega)$. Using this fact, one sees that u satisfies (2.10).

To show that $u \in G^m_\Sigma(\Omega)$, one first applies ∂_t^{m-1} to the system (2.10). The resulting data satisfy the compatibility conditions (2.20) of order 1 as the given data fulfill them up to order m. Since $\partial_t^{m-1}u \in \tilde{G}^1_\Sigma(\Omega)$, as in step 2) of the proof of Lemma 4.2 we can deduce that $\partial_t^{m-1}u$ belongs to $C(\overline{J}, \mathcal{H}^1_{\mathrm{ta}}(\mathbb{R}^3_+))$. By Lemma 4.3, the function $\partial_t^{m-1}u$ is an element of $G^1(J' \times \mathbb{R}^3_+)$ provided that $\partial_t^{m-1}u \in C^1(\overline{J'}, L^2(\mathbb{R}^3_+))$ for any open interval $J' \subseteq J$. Lemma 4.4 then shows that $\partial_t^{m-1}u$ is contained in $G^1_\Sigma(\Omega)$. As in the proof of Theorem 4.10 in [22], one now inductively infers that u belongs to $G^m_\Sigma(\Omega)$.

Finally, one passes to the domain $G \neq \mathbb{R}^3_+$ by a localization argument. See steps IV–VI of the proof of Theorem 5.6 in [21] or of Theorem 3.1 in [19]. □

If G is unbounded, the above result imposes decay of the derivatives of A_0 and D as $|x| \to \infty$. Actually, if these derivatives are bounded one obtains the same results much easier. As in [22], we have thus focused on the case treated in the theorem and describe the easy extension in the next result.

Remark 4.8. Let G be unbounded. As in Remark 1.2 of [22] we can weaken the regularity assumptions in Theorem 4.7 to $A_0, D \in W^{1,\infty}(J \times G)$ and

$$\forall\, \alpha \in \mathbb{N}_0^4 \text{ with } 1 \le |\alpha| \le m: \ \partial^\alpha A_0, \partial^\alpha D \in L^\infty(J, L^2(G)) + L^\infty(J \times G). \quad (4.15)$$

One further has to assume that the corresponding norms of A_0 and D are less or equal r, and $\partial^\alpha A_0(0)$ and $\partial^\alpha D(0)$ are bounded in $L^2(G) + L^\infty(G)$ by r_0 for all $|\alpha| \le m-1$. Here one can also replace G by $\mathbb{R}^3_+$. The conditions on b remain unchanged. Throughout, in new terms involving bounded parts of $\partial^\alpha A_0$ and $\partial^\alpha D$ these derivatives can easily be estimated by their sup-norms. $\Diamond$

5. Local existence and uniqueness

The apriori estimate of Theorem 4.7 does not allow us to treat the nonlinear absorbing boundary conditions as described in (2.2) and (2.3) in full generality. The problem arises in the terms with highest derivatives of $b = \zeta(B_1 \hat{u})$ for a function $\hat{u} \in \tilde{G}^m_\Sigma(J \times G)$ with range in $\mathcal{U}$. For simplicity we first look at the problem on Γ, the case $J \times \Sigma$ then follows by the localization procedure described in Section 2 and in [21]. We often use Lemma 2.1 and Corollary 2.2 in [23] which deal with mapping properties of substitution operators. This lemma yields the formula

$$\partial^\alpha \zeta(B_1\hat{u}) = \sum_{\substack{\beta,\gamma \in \mathbb{N}_0^3, \beta_0 = 0\\ \beta+\gamma=\alpha}} \sum_{1 \le j \le |\gamma|} \sum_{\substack{\gamma_1,\dots,\gamma_j \in \mathbb{N}_0^3 \setminus \{0\}\\ \sum \gamma_i = \gamma}} \sum_{l_1,\dots,l_j=1}^{2} C(\alpha, \gamma_1, \dots, \gamma_j) \quad (5.1)$$

$$\cdot\, (\partial_{y_{l_j}} \cdots \partial_{y_{l_1}} \partial_x^{(\beta_1,\beta_2)} \zeta)(B_1\hat{u}) \prod_{i=1}^{j} \partial^{\gamma_i} (B_1 \hat{u})_{l_i},$$

for $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \le m$. We take $m \ge 3$. This expression can be written as a sum S_{main} of the terms with $\alpha = \gamma$ and $j = 1$ plus the sum S_{rem} of the other terms. The summands in S_{rem} can be estimated by the norm in $\mathcal{H}^1(\mathbb{R}^3_+)$ of the product $\prod_{i=1}^j \partial^{\gamma_i}(B_1\hat{u})_{l_i}$ using the trace theorem. The product rules in Lemma 2.1 in [22] or Lemma 2.22 of [21] (and localization) lead to the inequality

$$\|S_{\text{rem}}\|^2_{L^2(J\times\Sigma)} \le T\, \|S_{\text{rem}}\|^2_{L^\infty(J, L^2(\Sigma))} \le cT\,(1 + \|\hat{u}\|^{2m}_{G^m(J\times G)}) = \tilde{C}(R)T, \quad (5.2)$$

where $\|\hat{u}\|_{G^m_\Sigma(J\times G)} \le R$. If $|\alpha| < m$, one can estimate the full function $\partial^\alpha \zeta(B_1\hat{u})$ in this way.

To treat the term $S_{\text{main}} = \partial_\xi \zeta(B_1\hat{u}) \partial^\alpha (B_1\hat{u})$ in the case $|\alpha| = m$, we define the quantities

$$z_0(\overline{\kappa}) = \max_{x \in \Sigma, |\xi| \le \overline{\kappa}} |\partial_\xi \zeta(x,\xi)|, \qquad z(\overline{\kappa}) = z_0(\overline{\kappa})\, \overline{\kappa} \quad (5.3)$$

for $\overline{\kappa} \geq 0$. Let $\|B_1\hat{u}(0)\|_{L^\infty(J\times\Sigma)} \leq \overline{\kappa}$. Assuming also that $\|\hat{u}\|_{G^m_\Sigma(J\times G)} \leq R$, we estimate

$$\begin{aligned}\|\partial_\xi\zeta(B_1\hat{u})\|_{L^\infty(J\times\Sigma)} &\leq \|\partial_\xi\zeta(B_1\hat{u}(0))\|_{L^\infty(\Sigma)} + T\|\partial_\xi^2\zeta(B_1\hat{u})B_1\partial_t\hat{u}\|_{L^\infty(J\times\Sigma)}\\ &\leq z_0(\overline{\kappa}) + \tilde{C}(R)T\|\hat{u}\|_{G^3(J\times G)} \leq z_0(\overline{\kappa}) + \tilde{C}(R)T.\end{aligned}$$

We derive

$$\|S_{\mathrm{main}}\|^2_{L^2(J\times\Sigma)} \leq c(z_0(\overline{\kappa})^2 + \tilde{C}(R)T^2)\,\|B_1\hat{u}\|^2_{\mathcal{H}^m(J\times\Sigma)}.$$

It thus follows

$$\sum_{|\alpha|=m}\|\partial^\alpha\zeta(B_1\hat{u})\|^2_{L^2(J\times\Sigma)} \leq \overline{C}(T + z_0(\overline{\kappa})^2). \tag{5.4}$$

We further take functions $\hat{v}$ and v with ranges in $\mathcal{U}$, $\|B_1\hat{v}(0)\|_{L^\infty(J\times\Sigma)} \leq \overline{\kappa}$, $\|\hat{v}\|_{G^m_\Sigma(J\times G)} \leq R$, and analogously for v. In a similar way we estimate

$$\|(\zeta(B_1\hat{u}) - \zeta(B_1\hat{v}))B_1v\|^2_{\mathcal{H}^{m-1}_\gamma(J\times\Sigma)} \leq \overline{C}(T + z(\overline{\kappa})^2)\,\|\hat{u}-\hat{v}\|^2_{G^{m-1}_{\Sigma,\gamma}}. \tag{5.5}$$

The constant $\overline{C}$ depends on R, ζ, and a time $T' > T$. In the fixed point argument, one part of the resulting right-hand sides can be made small choosing a small time interval $(0,T)$ depending on the radius R. For the other one we will have to assume that $z(\overline{\kappa})$ is small, which either means that we are close to a linear boundary condition or that we deal with electric fields having uniformly small tangential traces initially. In the linear case, where ζ does not depend on the state u, the number $z(\overline{\kappa})$ is even 0. Here we actually allow for time depending coefficients $b = \zeta$ in $F^{\tilde{m}}_{\eta,\tau}(J\times\Sigma)$, see (2.4). This case is much easier to treat since for linear boundary conditions with such coefficients we can use our main linear estimate (4.13) with $\rho = 0$, i.e., without an additional boundary term involving u on the right-hand side. Inequalities like (5.4) and (5.5) are used several times below.

Exactly as in Remark 2.5 of [23], for unbounded G in our proofs we will make a simplifying assumption on the coefficients χ and σ in order to avoid certain easier terms in the calculations.

Remark 5.1. Let G be unbounded, $m \geq 3$, $\hat{u} \in \tilde{G}^m(J\times G)$, and χ and σ be given by (2.3) or (2.4). As noted in Remark 4.8, for our linear results we can admit coefficients $A_0 = \chi(\hat{u})$ and $D = \sigma(\hat{u})$ belonging to the space described in (4.15). The additional bounded terms can easily be estimated in each computation. Without loss of generality, in the proofs we will therefore exclude such terms by imposing extra decay on the space derivatives of χ and σ as $|x| \to \infty$. More precisely, for all multiindices $\alpha \in \mathbb{N}_0^9$ with $\alpha_4 = \ldots = \alpha_9 = 0$ and $1 \leq |\alpha| \leq m$, $R > 0$, $V \Subset \mathcal{U}$, and $v \in L^\infty(J, L^2(G))$ with range in V and $\|v\|_{L^\infty(J,L^2(G))} \leq R$ we require that

$$\begin{aligned}&(\partial^\alpha\chi)(v), (\partial^\alpha\sigma)(v) \in L^\infty(J, L^2(G)),\\ &\|(\partial^\alpha\chi)(v)\|_{L^\infty(J,L^2(G))} + \|(\partial^\alpha\sigma)(v)\|_{L^\infty(J,L^2)(G))} \leq C,\end{aligned} \tag{5.6}$$

where $C = C(\chi,\sigma,m,R,V)$. With this assumption, Lemma 2.1 of [23] yields that $\chi(\hat{u})$ and $\sigma(\hat{u})$ are contained in $F^m(J\times G)$. $\Diamond$

We start with the uniqueness of solutions to (2.2).

Lemma 5.2. *Let $t_1 > t_0$ in $\mathbb{R}$ and $J = (t_0, t_1)$. Assume that either (2.3) or (2.4) is valid. Let $u_1, u_2 \in G^3_\Sigma(J \times G)$ solve (2.2) with inhomogeneity f, boundary value g, and initial value u_0 at initial time t_0. If assumption (2.3) is satisfied, we require that $z_0(\overline{\kappa}_1)^2\overline{\kappa}_2^2 \leq (2C_0)^{-1}$ where $\overline{\kappa}_1 \geq \max_{j\in\{1,2\}} \|B_1 u_j\|_{L^\infty(J\times\Sigma)}$, $\overline{\kappa}_2 \geq \min_{j\in\{1,2\}} \|B_1 u_j\|_{L^\infty(J\times\Sigma)}$, and C_0 is taken from (5.7) and depends on the norm of u_1 and u_2 in $W^{1,\infty}(J \times G)$ and on the lower bound η of χ and ζ. Then $u_1 = u_2$.*

Proof. We focus on the assumption (2.3) of a nonlinear boundary condition, and indicate the necessary changes for the linear one from (2.4) at the end of the proof. Let $T_0 \in \overline{J}$ be the supremum of all $t \in \overline{J}$ such that $u_1 = u_2$ on $[t_0, t]$. The two functions coincide on $[t_0, T_0]$ by their continuity.

We suppose that $T_0 < t_1$. We take a time $T' \in (T_0, t_1)$ and set $J' = (T_0, T')$. We fix a compact set $V \subset \mathcal{U}$ containing the ranges of u_1 and u_2 on J'. The maps u_1 and u_2 in $G^3_\Sigma(J' \times G)$ both solve (2.2) on $\overline{J'}$ with the same initial value $u_1(T_0)$, inhomogeneities f and g, and the operators $L_j = L(\chi(u_j), A_1^{\mathrm{co}}, A_2^{\mathrm{co}}, A_3^{\mathrm{co}}, \sigma(u_j))$ and $B^j = B(u_j)$ for $j = 1$, respectively $j = 2$. Without loss of generality we assume that $\|B_1 u_2\|_{L^\infty(J\times\Sigma)} = \min_{j\in\{1,2\}} \|B_1 u_j\|_{L^\infty(J\times\Sigma)}$. The difference $u_1 - u_2$ fulfills

$$
\begin{aligned}
L_1(u_1 - u_2) &= (\chi(u_2) - \chi(u_1))\partial_t u_2 + (\sigma(u_2) - \sigma(u_1))u_2 =: f_1 \qquad \text{on } J' \times G,\\
B_1(u_1 - u_2) &= ((\zeta(B_1 u_2) - \zeta(B_1 u_1))B_1 u_2) \times \nu =: g_1 \qquad \text{on } J' \times \Sigma,\\
(u_1 - u_2)(T_0) &= 0 \qquad \text{on } G.
\end{aligned}
$$

Lemma 2.1 of [23] and Sobolev's embedding theorem yield that $\chi(u_j)$ and $\sigma(u_j)$ are elements of $F^3(J \times G)$, and $\zeta(u_j)$ of $F^3_{\mathcal{H}}(J \times \Sigma)$. Moreover, $\chi(u_1)$ and $\zeta(u_1)$ are symmetric and bounded from below by $\eta > 0$. Let r (resp. R) be the maximum of the norms of u_1 and u_2 in $W^{1,\infty}(J \times G)$ (resp. in $G^3_\Sigma(J \times G)$). Then the Lipschitz norms of $\chi(u_1)$ and $\sigma(u_1)$ and the sup-norm of $\zeta(u_1)$ are bounded by a constant depending on r. Proposition 3.1 now provides constants $C_0 = C_0(\eta, r)$ and $\gamma = \gamma(\eta, r)$ such that

$$\|u_1 - u_2\|^2_{G^0_{\Sigma,\gamma}(J'\times G)} \leq C_0\,(\|f_1\|^2_{L^2_\gamma(J'\times G)} + \|g_1\|^2_{L^2_\gamma(J'\times\Sigma)}). \tag{5.7}$$

The Lipschitz estimates on χ and σ from Corollary 2.22 in [23] imply

$$\|f_1\|^2_{L^2_\gamma(J'\times G)} \leq c(\eta, R)(T' - T_0)\,\|u_1 - u_2\|^2_{G^0_\gamma(J'\times G)}\,. \tag{5.8}$$

Recalling the definition of $\overline{\kappa}_j$ in the statement and of z in (5.3), we next derive

$$\|g_1\|^2_{L^2_\gamma(J'\times\Sigma)} \leq z_0(\overline{\kappa}_1)^2\overline{\kappa}_2^2\,\|\operatorname{tr}_\tau(u_1 - u_2)\|^2_{L^2_\gamma(J'\times G)}\,. \tag{5.9}$$

By the assumption on $\overline{\kappa}_j$, we can choose $T' > T_0$ so small that (5.7), (5.8), and (5.9) imply that $u_1 = u_2$ on $[T_0, T']$ and thus on $[t_0, T']$. This result contradicts the definition of T_0, and hence $u_1 = u_2$ on J.

Under assumption (2.4) we have the boundary conditions

$$Bu_j = (B_2 - \nu \times bB_1)u_j = g$$

and hence $B(u_1 - u_2) = 0$. So the above argument can be used with $g_1 := 0$ and thus without any assumption on $\overline{\kappa}_j$. $\square$

We next construct local in time solutions of (2.2) using Banach's fixed point theorem and our linear result Theorem 4.7. Special care in the treatment of the constants is required to close the argument, and we need the structure of the estimate (3.8) here. For the data we define the quantity

$$d_k(J) := \|u_0\|^2_{\mathcal{H}^k(G)} + \sum_{j=0}^{k-1} \|\partial_t^j f(t_0)\|^2_{\mathcal{H}^{k-1-j}(G)} + \|f\|^2_{\mathcal{H}^k(J\times G)} + \|g\|^2_{\mathcal{H}^k(J\times\Sigma)}. \quad (5.10)$$

Moreover, C_S is the norm of the Sobolev embedding $\mathcal{H}^2(G) \hookrightarrow C_b(\overline{G})$. We note that below the number C_0 only depends on a radius $r_3 \geq d_3(J)^{1/2}$ instead of r, as an inspection of the proof shows.

Theorem 5.3. *Let $t_0 \in \mathbb{R}$, $T > 0$, $J = (t_0, t_0 + T)$, and $m \in \mathbb{N}$ with $m \geq 3$. Assume that either* (2.3) *or* (2.4) *is valid. Choose data $u_0 \in \mathcal{H}^m(G)^6$, $f \in \mathcal{H}^m(J \times G)^6$, and $g \in \mathcal{H}^m(J \times \Sigma)^3$ with $g \cdot \nu = 0$ such that the tuple $(t_0, \chi, \sigma, \zeta, u_0, f, g)$ fulfills the compatibility conditions* (2.21) *of order m. Pick a radius $r > 0$ satisfying*

$$d_m(J) \leq r^2.$$

Take a number $\kappa > 0$ with

$$\operatorname{dist}(\{u_0(x) \mid x \in G\}, \partial\mathcal{U}) > \kappa.$$

If (2.3) *is valid, we take $\widetilde{\kappa} > 0$ with*

$$z(\widetilde{\kappa}) \leq \min\left\{\tfrac{1}{8}(C_{m,0}\overline{C})^{-1/2},\, (2C_0)^{-1/2}\right\}. \quad (5.11)$$

and assume that $\|B_1 u_0\|_{L^\infty(\Sigma)} < \widetilde{\kappa}$. The constants C_0, $C_{m,0}$, and $\overline{C}$ depending on χ, σ, ζ, m, r, κ, and T are given by Lemma 5.2, (5.17), (5.4), *and* (5.5).

Then there is a time $\tau = \tau(\chi, \sigma, \zeta, m, T, r, \kappa, \widetilde{\kappa}) > 0$ such that the nonlinear initial boundary value problem (2.2) *with data f, g, and u_0 has a solution u on $[t_0, t_0 + \tau]$ which belongs to $G^m_\Sigma((t_0, t_0 + \tau) \times G)$. It is unique among those solutions with $\|B_1 u\|_{L^\infty((t_0,t_0+\tau)\times\Sigma)} < \widetilde{\kappa}$.*

Proof. 1) We focus on the assumption (2.3) of a nonlinear boundary condition, and indicate the necessary changes for the linear one under hypothesis (2.4) at the appropriate points. Without loss of generality we assume $t_0 = 0$ and, if G is unbounded, that χ and σ satisfy (5.6), cf. Remark 5.1. Moreover, the quantities $d_k(J)$ can be chosen to be positive since $u = 0$ is the unique solution of (2.2) for $(u_0, f, g) = 0$ by Lemma 5.2.

Let $\tau \in (0, T]$ and $R > 0$. We introduce $J_\tau = (0, \tau)$ and

$$V_\kappa = \{y \in \mathcal{U} \mid \operatorname{dist}(y, \partial\mathcal{U}) \geq \kappa\} \cap \overline{B}(0, C_S r).$$

Note that $\operatorname{ran}(u_0)$ is contained in the compact set V_κ. Our fixed point space is

$$\begin{aligned} E(R,\tau) = \{v \in \tilde{G}^m_\Sigma(J_\tau \times G) \mid \|v\|_{G^m_\Sigma(J_\tau\times G)} \leq R,\ \|v - u_0\|_{L^\infty(J_\tau\times\overline{G})} \leq \kappa/2, \\ \partial_t^j v(0) = S_{m,j,\chi,\sigma}(0, u_0, f) \text{ for } 0 \leq j \leq m-1\} \end{aligned}$$

endowed with the metric induced by the norm of $\tilde{G}^{m-1}_\Sigma(J_\tau \times G)$. We have

$$\operatorname{ran}(v) \subseteq \tilde{V}_\kappa := V_\kappa + \overline{B}(0, \kappa/2) \subseteq \mathcal{U} \quad (5.12)$$

for $v \in E(R,\tau)$. Lemma 5.4 and a localization provide a function $w \in \tilde{G}^m_\Sigma(J_\tau \times G)$ satisfying the initial conditions in $E(R,\tau)$. Using Lemma 5.4 and the estimates on $S_{m,j,\chi,\sigma}$ from Lemma 2.4 in [23], one obtains a constant $C_{5.4} = C_{5.4}(\chi,\sigma,m,T,r,\kappa)$ such that $\|w\|_{\tilde{G}^m_\Sigma} \leq C_{5.4} r$. Take $R > C_{5.4} r$. Since

$$w(t) - u_0 = \int_0^t \partial_s w(s)\,\mathrm{d}s,$$

we can bound $\|w - u_0\|_{L^\infty(J_\tau \times \overline{G})} \leq C_S R\tau$. As a result, $E(R,\tau)$ is non-empty, if we choose $R > C_{5.4} r$ and $\tau \in (0, \kappa/(2C_S R)]$. To show completeness of $E(R,\tau)$, we take a Cauchy sequence (u_n) in $E(R,\tau)$. It then has a limit u in $\tilde{G}^{m-1}_\Sigma(J_\tau \times G) \hookrightarrow L^\infty(J_\tau \times G)$, which satisfies the initial condition and the L^∞–bound of $E(R,\tau)$. Moreover, the m-th order derivatives of u_n have weak* limits in $L^\infty(J_\tau, L^2(G))$, respectively weak limits in $L^2(J_\tau, L^2(\partial G))$. These limits are the corresponding derivatives of u which thus belongs to $\tilde{G}^m_\Sigma(J_\tau \times G)$ with norm less or equal R.

2) Let $\hat{u} \in E(R,\tau)$. Take $\eta > 0$ from (2.3). Then $A_0 := \chi(\hat{u})$ is contained in $F^m_\eta(J_\tau \times G)$, $b := \zeta(B_1 \hat{u})$ in $F^m_{\mathcal{H},\eta}(J_\tau \times \Sigma)$, and $D := \sigma(\hat{u})$ in $F^m(J_\tau \times G)$ by Lemma 2.1 of [23], Remark 5.1, Sobolev's embedding, and the remarks before (5.4). The tuple $(0, \chi(\hat{u}), A_1^{\mathrm{co}}, A_2^{\mathrm{co}}, A_3^{\mathrm{co}}, \sigma(\hat{u}), \zeta(\hat{u}), u_0, f, g)$ satisfies the linear compatibility conditions (2.20) due to the formula

$$S_{m,p,\chi(\hat{u}),A_j^{\mathrm{co}},\sigma(\hat{u})}(0,u_0,f) = S_{m,p,\chi,\sigma}(0,u_0,f) \tag{5.13}$$

from Lemma 2.6 of [23], the initial conditions in $E(R,\tau)$, and formula (5.1). Theorem 4.7 yields a solution $u \in G^m_\Sigma(J_\tau \times G)$ of the system (2.5) with the coefficients A_0, D, b and the data u_0, f, g. In this way one defines a mapping $\Phi\colon \hat{u} \mapsto u$ from $E(R,\tau)$ to $G^m_\Sigma(J_\tau \times G)$. We want to prove that Φ is a strict contraction on $E(R,\tau)$ for a suitable radius R and a sufficiently small time step τ.

To this aim, take numbers $\tau \in (0,T]$ with $\tau \leq \kappa/(2C_S R)$ and $R > C_{5.4} r$ which will be fixed below. Let $\hat{u} \in E(R,\tau)$. Because of (5.12), the map $\zeta(\hat{u})$ is bounded by a constant $c(\tilde{V}_\kappa)$. Using Lemma 2.1 of [23], one finds radii $r_0 = r_0(\chi,\sigma,\zeta,m,r,\kappa)$ and $R_1 = R_1(\chi,\sigma,\zeta,m,R,\kappa,T)$ such that

$$\begin{aligned}
&\max\{\|\chi(\hat{u})(0)\|_{F^{m-1,0}(G)},\ \max_{1\leq l\leq m-1} \|\partial_t^l \chi(\hat{u})(0)\|_{\mathcal{H}^{m-l-1}(G)}\} \leq r_0,\\
&\max\{\|\sigma(\hat{u})(0)\|_{F^{m-1,0}(G)},\ \max_{1\leq l\leq m-1} \|\partial_t^l \sigma(\hat{u})(0)\|_{\mathcal{H}^{m-l-1}(G)}\} \leq r_0,
\end{aligned} \tag{5.14}$$

$$\|\chi(\hat{u})\|_{F^m(J\times G)},\ \|\sigma(\hat{u})\|_{F^m(J\times G)},\ \|\zeta(\hat{u})\|_{F^{m-1}(J\times G)} \leq R_1. \tag{5.15}$$

(See step II) of the proof of Theorem 3.3 in [23].) Moreover, the relations (5.4) and (5.12) imply the bound

$$\sum_{|\alpha|=m} \|\partial^\alpha \zeta(B_1\hat{u})\|^2_{L^2(J\times\Sigma)} \leq \overline{C}(\zeta,R,T,\kappa)(\tau + z_0(\widetilde{\kappa})^2). \tag{5.16}$$

Let the constant

$$C_{m,0} = C_{m,0}(\chi,\sigma,\zeta,r,\kappa) = C_{m,0}(\eta(\chi,\zeta), r_0(\chi,\sigma,\zeta,m,r,\kappa), c(\tilde{V}_\kappa)) \tag{5.17}$$

be given by Theorem 4.7. The radius $R = R(\chi,\sigma,\zeta,m,r,\kappa,T)$ for $E(R,\tau)$ is now defined as

$$R := \max\Big\{\sqrt{32C_{m,0}}\,r,\ C_{5.4}r+1\Big\}. \tag{5.18}$$

Let $\gamma_m = \gamma_m(\chi,\sigma,\zeta,T,r,\kappa)$ and $C_m = C_m(\chi,\sigma,\zeta,T,r,\kappa)$ be the constants from Theorem 4.7 with $\eta(\chi,\zeta)$ and $R_1(\chi,\sigma,\zeta,m,R,\kappa,T)$. Lemma 2.1 in [22] (or Lemma 2.22 in [21]) yields product rules and Corollary 2.2 in [23] Lipschitz bounds of composition operators. We write $C_{2.1,[22]}$ for the maximum of the constants in Lemma 2.1 in [22] and $C_{2.2,[23]}$ for that of Corollary 2.2 in [23] applied to our material laws and with the numbers m and R and the set $\tilde{V}_\kappa$. We finally introduce the parameter $\gamma = \gamma(\chi,\sigma,\zeta,m,T,r,\kappa)$ and the time step $\tau = \tau(\chi,\sigma,\zeta,m,T,r,\kappa,\widetilde{\kappa})$ by

$$\gamma := \max\Big\{\gamma_m,\ C_{m,0}^{-1}C_m\Big\}, \tag{5.19}$$

$$\begin{aligned}\tau := \min\Big\{T, \frac{\kappa}{2C_SR}, \frac{\ln 2}{2\gamma+mC_1}, \frac{C_{m,0}}{C_m}, [16C_{m,0}\overline{C}(4\widetilde{\kappa}^2 + C_S^2T(T+z_0(\widetilde{\kappa})^2))]^{-1},\\ (16C_{m,0}\overline{C})^{-1}, [32R^2C_{m,0}C_{2.1,[22]}^2C_{2.2,[23]}^2]^{-1}\Big\}.\end{aligned} \tag{5.20}$$

3) With the definitions and notations of step 2), Theorem 4.7, (5.16), and Sobolev's embedding yield

$$\begin{aligned}
&\|\Phi(\hat{u})\|^2_{G^m_\Sigma(J_\tau\times G)} \le e^{2\gamma\tau}\,\|\Phi(\hat{u})\|^2_{G^m_{\Sigma,\gamma}(J_\tau\times G)}\\
&\le (C_{m,0}+\tau C_m)e^{(mC_1+2\gamma)\tau}\Big(\sum_{j=0}^{m-1}\|\partial_t^j f(0)\|^2_{\mathcal{H}^{m-1-j}(G)} + \|u_0\|^2_{\mathcal{H}^m(G)}\\
&\qquad + \|g\|^2_{\mathcal{H}^m_\gamma(J_\tau\times\Sigma)} + \overline{C}(\tau+z_0(\widetilde{\kappa}^2))\|B_1\Phi(\hat{u})\|^2_{L^\infty_\gamma(J_\tau\times\Sigma)}\Big) + \frac{C_m}{\gamma}e^{2\gamma\tau}\,\|f\|^2_{\mathcal{H}^m_\gamma(J_\tau\times G)}\\
&\le 2C_{m,0}\cdot 2\Big(r^2+\overline{C}(\tau+z_0(\widetilde{\kappa})^2)(2\|B_1u_0\|^2_{L^\infty(\Sigma)} + 2\tau^2\|\partial_t(B_1\Phi(\hat{u}))\|^2_{L^\infty(J_\tau\times\Sigma)})\Big)\\
&\le 8C_{m,0}\Big(r^2+\overline{C}(\tau+z_0(\widetilde{\kappa})^2)(\widetilde{\kappa}^2+\tau^2C_S^2\|\Phi(\hat{u})\|^2_{G^m_\Sigma(J_\tau\times G)})\Big).
\end{aligned} \tag{5.21}$$

Employing (5.11) and $R\ge 1$, we thus obtain

$$\begin{aligned}\|\Phi(\hat{u})\|^2_{G^m_\Sigma(J_\tau\times G)} &\le 16C_{m,0}\Big(r^2+\overline{C}(\tau+z_0(\widetilde{\kappa})^2)\widetilde{\kappa}^2\Big)\\ &\le \frac{R^2}{2} + 16C_{m,0}\overline{C}\,\widetilde{\kappa}^2\tau + 16C_{m,0}\overline{C}z(\widetilde{\kappa})^2 \le R^2.\end{aligned}$$

Step III) of the proof of Theorem 3.3 in [23] shows that the map $\Phi(\hat{u})$ satisfies the initial and sup-norm conditions in $E(R,\tau)$ which are the same as in the fixed-point space used in [23]. Hence, Φ maps $E(R,\tau)$ into itself.

In the case (2.4) with linear boundary conditions, we can apply Theorem 4.7 with $\rho = 0$, so that in the third line of (5.21) the term with pre-factor $\overline{C}$ disappears. Hence, (5.21) is even true with the smaller right-hand side $8C_{m,0}r^2$ and the invariance is valid without a condition on $\widetilde{\kappa}$.

Take $\hat{u},\hat{v}\in E(R,\tau)$. Set $u = \Phi(\hat{u})$ and $v = \Phi(\hat{v})$. As above, we look at the linear system (2.5) with coefficients $A_0 = \chi(\hat{u})$, $D = \sigma(\hat{u})$, and $b = \zeta(B_1\hat{u})$.

The difference $v-u$ solves this system with initial value 0 and inhomogeneities

$$\tilde{f} = (\chi(\hat{u}) - \chi(\hat{v}))\partial_t v + (\sigma(\hat{u}) - \sigma(\hat{v}))v, \qquad \tilde{g} = B_0(\zeta(B_1\hat{u}) - \zeta(B_1\hat{v}))B_1 v.$$

We can thus use Theorem 4.7 as in (5.21) with $(0, \tilde{f}, \tilde{g})$ instead of (u_0, f, g). Lemma 2.1 in [22] (or Lemma 2.22 in [21]), Corollary 2.2 in [23], (5.5), and (5.20) then yield

$$\begin{aligned}\|\Phi(\hat{v}) - \Phi(\hat{u})\|^2_{G^{m-1}_\Sigma(J_\tau\times G)} &\le e^{2\gamma\tau}\,\|\Phi(\hat{v}) - \Phi(\hat{u})\|^2_{G^{m-1}_{\Sigma,\gamma}(J_\tau\times G)}\\ &\le \frac{1}{4}\,\|\hat{v}-\hat{u}\|^2_{G^{m-1}_\Sigma(J_\tau\times G)} + 4C_{m,0}\overline{C}(\tau + z(\widetilde{\kappa})^2)\|\hat{u}-\hat{v}\|^2_{G^{m-1}_\Sigma(J_\tau\times G)}\\ &\le \frac{3}{4}\,\|\hat{v}-\hat{u}\|^2_{G^{m-1}_\Sigma(J_\tau\times G)}\,. \end{aligned} \tag{5.22}$$

(Compare step IV) of the proof of Theorem 3.3 in [23].) Together with Lemma 5.2, the result is proven under assumption (2.3).

In the case (2.4) with linear boundary conditions, we simply have $\tilde{g}=0$ and we can use Theorem 4.7 with $\rho = 0$. As a result, estimate (5.22) holds with factor $\frac{1}{4}$ instead of $\frac{3}{4}$ without a condition on $\widetilde{\kappa}$. □

We add a lemma used in the proof of Theorem 5.3.

Lemma 5.4. *Let $m\in\mathbb{N}$ and $k\in\{0,\dots,m-1\}$. Take maps h_k in $\mathcal{H}^{m-k}(\mathbb{R}^3_+)$. Then there is a function $u\in G^m(\mathbb{R}\times\mathbb{R}^3_+)$ such that $\partial_t^k u(0) = h_k$ for all k, the trace of u on $\partial\mathbb{R}^3_+$ belongs to $\mathcal{H}^m(\mathbb{R}\times\partial\mathbb{R}^3_+)$, and*

$$\begin{aligned}&\|u\|_{G^m(\mathbb{R}\times\mathbb{R}^3_+)} + \|\partial_t^j u\|_{L^2(\mathbb{R},\mathcal{H}^{m+1/2-j}(\mathbb{R}^3_+))} + \|\operatorname{tr}_{\partial\mathbb{R}^3_+} u\|_{\mathcal{H}^m(\mathbb{R}\times\partial\mathbb{R}^3_+)}\\ &\quad\le c\sum_{k=0}^{m-1}\|h_k\|_{\mathcal{H}^{m-k}(\mathbb{R}^3_+)},\end{aligned} \tag{5.23}$$

for all $j\in\{0,\dots,m\}$ and a constant $c = c(m)$.

Proof. Let $k\in\{0,\dots,m-1\}$. Take $g_k\in\mathcal{S}(\mathbb{R}^3)$. Fix a map $\psi\in C_c^\infty(\mathbb{R})$ which equals 1 in $(-\frac{1}{2},\frac{1}{2})$ and vanishes on $\mathbb{R}\setminus(-2,2)$. We define the function v by

$$v(t,x) = \mathcal{F}^{-1}\Big(\psi((1+|\cdot|^2)^{1/2}t)\sum_{k=0}^{m-1}\hat{g}_k\frac{t^k}{k!}\Big)(x), \qquad (t,x)\in\mathbb{R}^4,$$

where $\mathcal{F}$ and the hat denote the spatial Fourier transform. Observe that we apply $\mathcal{F}^{-1}$ to a function in $\mathcal{S}(\mathbb{R}^4)$. The dominated convergence theorem yields

$$\partial_t^k v(0) = g_k$$

for all k. To show (5.23) for v and g_k, we take $j\in\{0,\dots,m\}$ and compute

$$\begin{aligned}&\|\partial_t^j v\|^2_{L^\infty(\mathbb{R},\mathcal{H}^{m-j}(\mathbb{R}^3))} = \sup_{t\in\mathbb{R}}\int_{\mathbb{R}^3}(1+|\xi|^2)^{m-j}\,|\mathcal{F}(\partial_t^j v)(t,\xi)|^2\,d\xi\\ &\le C\sup_{t\in\mathbb{R}}\int_{\mathbb{R}^3}(1+|\xi|^2)^{m-j}\sum_{k=0}^{m-1}\Big|\partial_t^j\Big(\psi((1+|\xi|^2)^{1/2}t)\hat{g}_k(\xi)\frac{t^k}{k!}\Big)\Big|^2\,d\xi\end{aligned}$$

$$\leq C \sum_{k=0}^{m-1} \sup_{t\in\mathbb{R}} \int_{\mathbb{R}^3} (1+|\xi|^2)^{m-j-k} |\hat{g}_k(\xi)|^2 \left|\partial_t^j [\psi((1+|\xi|^2)^{\frac12} t)((1+|\xi|^2)^{\frac12} t)^k)]\right|^2 \mathrm{d}\xi$$
$$\leq C \sum_{k=0}^{m-1} \int_{\mathbb{R}^3} (1+|\xi|^2)^{m-k} |\hat{g}_k(\xi)|^2 \sup_{s\in\mathbb{R}} |\partial_s^j(\psi(s)s^k)|^2 \,\mathrm{d}\xi$$
$$= C \sum_{k=0}^{m-1} \int_{\mathbb{R}^3} (1+|\xi|^2)^{m-k} |\hat{g}_k(\xi)|^2 \,\mathrm{d}\xi = C \sum_{k=0}^{m-1} \|g_k\|^2_{\mathcal{H}^{m-k}(\mathbb{R}^3)}.$$

So the first estimate in (5.23) has been shown.

For the second one we proceed similarly, now abbreviating $\psi_{l,k} := (s \mapsto \psi(s)s^k)^{(l)}$ for each $l \in \{0, \dots, m\}$. We then derive

$$\|\partial_t^j v\|^2_{L^2(\mathbb{R},\mathcal{H}^{m+1/2-j}(\mathbb{R}^3))} = \int_{\mathbb{R}} \int_{\mathbb{R}^3} (1+|\xi|^2)^{m+1/2-j} \, |\mathcal{F}(\partial_t^j v)(t,\xi)|^2 \,\mathrm{d}\xi \,\mathrm{d}t$$
$$\leq C \sum_{k=0}^{m-1} \int_{\mathbb{R}^3} (1+|\xi|^2)^{m-k} \, |\hat{g}_k(\xi)|^2 \int_{\mathbb{R}} |\psi_{j,k}((1+|\xi|^2)^{1/2}t)|^2 \, (1+|\xi|^2)^{1/2} \,\mathrm{d}t \,\mathrm{d}\xi$$
$$= C \sum_{k=0}^{m-1} \int_{\mathbb{R}} |\psi_{j,k}(s)|^2 \,\mathrm{d}s \int_{\mathbb{R}^3} (1+|\xi|^2)^{m-k} \, |\hat{g}_k(\xi)|^2 \,\mathrm{d}\xi \leq C \sum_{k=0}^{m-1} \|g_k\|^2_{\mathcal{H}^{m-k}(\mathbb{R}^3)}.$$

Denoting (x_1, x_2) by x' and (ξ_1, ξ_2) by ξ', we finally compute

$$\partial_t^m v(t,x',0) = \partial_t^m \mathcal{F}^{-1}\Big(\psi((1+|\cdot|^2)^{1/2}t) \sum_{k=0}^{m-1} \hat{g}_k \frac{t^k}{k!}\Big)(x',0)$$
$$= \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{\mathrm{i}x'\cdot\xi'} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \partial_t^m \Big(\psi((1+|\xi|^2)^{1/2}t) \sum_{k=0}^{m-1} \hat{g}_k(\xi) \frac{t^k}{k!}\Big) \,\mathrm{d}\xi_3 \,\mathrm{d}\xi'.$$

The spatial Fourier transform on $\mathbb{R}^2$ of $\partial_t^m v(t,x',0)$ is thus given by

$$\mathcal{F}(\partial_t^m v(t,\cdot,0))(\xi') = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \partial_t^m \Big(\psi((1+|\xi|^2)^{1/2}t) \sum_{k=0}^{m-1} \hat{g}_k(\xi) \frac{t^k}{k!}\Big) \,\mathrm{d}\xi_3.$$

We next fix a time $t \in \mathbb{R} \setminus \{0\}$. Since ψ vanishes on the complement of $(-2,2)$, the integrand above vanishes if $|\xi_3| > 2/|t|$. This fact yields the estimate

$$|\mathcal{F}(\partial_t^m v(t,\cdot,0))(\xi')|^2 \leq C \sum_{k=0}^{m-1} \Big[\int_{\mathbb{R}} (1+|\xi|^2)^{\frac{m-k}{2}} \, |\hat{g}_k(\xi)| \, \psi_{m,k}((1+|\xi|^2)^{1/2}t) \,\mathrm{d}\xi_3\Big]^2$$
$$\leq C \sum_{k=0}^{m-1} \int_{-2/|t|}^{2/|t|} 1 \,\mathrm{d}\xi_3 \int_{\mathbb{R}} (1+|\xi|^2)^{m-k} \, |\hat{g}_k(\xi)|^2 \, |\psi_{m,k}((1+|\xi|^2)^{1/2}t)|^2 \,\mathrm{d}\xi_3$$
$$= C \sum_{k=0}^{m-1} \int_{\mathbb{R}} (1+|\xi|^2)^{m-k} \, |\hat{g}_k(\xi)|^2 \, \frac{|\psi_{m,k}((1+|\xi|^2)^{1/2}t)|^2}{|(1+|\xi|^2)^{1/2}t|} (1+|\xi|^2)^{1/2} \,\mathrm{d}\xi_3.$$

Since $k < m$, at least one derivative falls onto ψ in $\psi_{m,k} = \partial_t^m(t \mapsto \psi(t)t^k)$. As ψ is constant on $(-1/2, 1/2)$, the function $\psi_{m,k}$ vanishes on this interval, and hence the map $s \mapsto \frac{|\psi_{m,k}(s)|^2}{|s|}$ belongs to $C_c^\infty(\mathbb{R})$. We infer

$$\begin{aligned}\|\partial_t^m v(\cdot,0)\|^2_{L^2(\mathbb{R}\times\mathbb{R}^2)} &= \int_{\mathbb{R}}\int_{\mathbb{R}^2} |\mathcal{F}(\partial_t^m v(t,\cdot,0))(\xi')|^2 \,\mathrm{d}\xi'\,\mathrm{d}t \\ &\le C\sum_{k=0}^{m-1}\int_{\mathbb{R}} \frac{|\psi_{m,k}(s)|^2}{|s|}\,\mathrm{d}s \int_{\mathbb{R}^3}(1+|\xi|^2)^{m-k}|\hat{g}_k(\xi)|^2\,\mathrm{d}\xi \le C\sum_{k=0}^{m-1}\|g_k\|^2_{\mathcal{H}^{m-k}(\mathbb{R}^3)}.\end{aligned}$$

Also employing the trace theorem, we obtain (5.23) for the functions v and g_k. The assertion now follows by approximation. □

We assume that the conditions of Theorem 5.3 concerning the data are valid and that the inhomogeneities f and g belong to the spaces $\mathcal{H}^m((t_0,T)\times G)$ respectively $\mathcal{H}^m((t_0,T)\times\Sigma)$, for all $T>0$. For the assumption (5.11) we take the quantity $d_m((t_0,t_0+1))$ unless something else is specified. We then define the *maximal existence time* by

$$\begin{aligned}T_+ &= T_+(m,t_0,u_0,f,g)\\ &:= \sup\{\tau\ge t_0 \,|\, \exists \text{ unique } \mathcal{G}^m_\Sigma\text{-solution } u_\tau \text{ of (2.2) on } [t_0,\tau]\}.\end{aligned} \tag{5.24}$$

The interval $(t_0,T_+) =: J_{\max}$ is called the *maximal interval of existence*. These notions are modified in a straightforward way if the inhomogeneities are given only on a bounded interval (t_0,T_1). Theorem 5.3 shows that $T_+>t_0$. (If there were two solutions, one can shrink τ to use the uniqueness assertion in Theorem 5.3.) Since we have included uniqueness in (5.24), we can define a *maximal solution* u of (2.2) on $J_{\max}$ by setting $u(t)=u_\tau(t)$ for $t\in J_{\max}$ and any $\tau\in(t,T_+)$. The function u thus belongs to $G^m_\Sigma((t_0,T)\times G)$ for all $T<T_+$. In the next result we also state the basic blow-up condition. To show it, assume that $T_+<\infty$ and conditions (a)–(c) were wrong. Theorem 5.3 would thus allow us to extend u beyond T_+ using Theorem 5.3, which is impossible.

Proposition 5.5. *Let $t_0\in\mathbb{R}$ and $m\in\mathbb{N}$ with $m\ge 3$. Assume that either* (2.3) *or* (2.4) *is valid. Choose data (u_0,f,g) such that $u_0\in\mathcal{H}^m(G)^6$, $f\in\mathcal{H}^m((t_0,T)\times G)^6$, $g\in\mathcal{H}^m((t_0,T)\times\Sigma)^3$ for all $T>t_0$, $g\cdot\nu=0$, and the tuple $(t_0,\chi,\sigma,\zeta,u_0,f,g)$ fulfills the compatibility conditions* (2.21) *of order m. If assumption* (2.3) *is valid, we require condition* (5.11) *and that $\|B_1u_0\|_{L^\infty(\Sigma)}<\widetilde{\kappa}$. Let u be the maximal solution of* (2.2) *on $J_{\max}$ introduced above. If $T_+<\infty$, then one of the following blow-up properties*

(a) $\liminf_{t\nearrow T_+}\operatorname{dist}(\{u(t,x)\,|\,x\in G\},\partial\mathcal{U})=0$,
(b) $\lim_{t\nearrow T_+}\|u(t)\|_{\mathcal{H}^m(G)}=\infty$,
(c) $\limsup_{t\nearrow T_+}\|B_1u(t)\|_{L^\infty(\Sigma)}\ge\widetilde{\kappa}$ *for any $\widetilde{\kappa}$ satisfying* (5.11),

occurs, where the last item is removed if (2.4) *is satisfied. In (c), we assume that (a) and (b) do not occur and define the constants in* (5.11) *for the quantities $\kappa:=\operatorname{dist}(\{u(t,x)\,|\,x\in G, t\in(t_0,T_+)\},\partial\mathcal{U})>0$ and $r^2=d_m(T_+-\delta,T_++\delta)$ for some $\delta\in(0,T_+-t_0)$ and with $\|u_0\|_{\mathcal{H}^m(G)}$ replaced by $\liminf_{t\nearrow T_+}\|u(t)\|_{\mathcal{H}^m(G)}$.*

6. Local wellposedness

In this section we improve the blow-up criterion of Proposition 5.5 and show the continuous dependence on the data. For various quasilinear hyperbolic systems, one has established such criteria in terms of Lipschitz norms. (See Section 4 of [23] for references.) These results rely on Moser-type estimates as stated in Lemma 4.2 of [23]. They will imply in particular that the maximal existence time is independent of $m \geq 3$ in the case of linear boundary conditions. The next proposition is the key step in this direction, where we recall the definition of $d_k(J)$ in (5.10). Since we have to localize again, we recall the abbreviations $\Omega = J \times \mathbb{R}^3_+$ and $\Gamma = J \times \partial\mathbb{R}^3_+$.

Proposition 6.1. *Let $m \in \mathbb{N}$ with $m \geq 3$ and $t_0 \in \mathbb{R}$. Assume that either* (2.3) *or* (2.4) *is valid. Choose data $u_0 \in \mathcal{H}^m(G)^6$, $f \in \mathcal{H}^m((t_0,T) \times G))^6$, and $g \in \mathcal{H}^m((t_0,T) \times \Sigma)^3$ with $g \cdot \nu = 0$ for $T > t_0$ such that the tuple $(t_0, \chi, \sigma, \zeta, u_0, f, g)$ fulfills the compatibility conditions* (2.21) *of order m. Let u be the maximal solution of* (2.2) *provided by Proposition 5.5 on $J_{\max} = (t_0, T_+)$. We introduce the quantity*

$$\omega(T) = \sup_{t \in (t_0,T)} \|u(t)\|_{W^{1,\infty}(G)}$$

for every $T \in (t_0, T_+)$. We further take $r > 0$ with $d_m(J_{\max}) \leq r^2$. We set $T^* = T_+$ if $T_+ < \infty$ and pick any $T^* > t_0$ if $T_+ = \infty$. Take $\omega_0 > 0$ and a compact subset $\mathcal{U}_1$ of $\mathcal{U}$ such that $\omega(T) \leq \omega_0$ and $\operatorname{ran} u(t) \subseteq \mathcal{U}_1$ for all $t \in [t_0, T]$ and some $T \in (t_0, T^*)$. If (2.3) *is true, we also assume that* (5.11) *is valid for $\overline{\kappa}$ and that*

$$z(\overline{\kappa})^2 \leq 1/(2\tilde{C}_m), \tag{6.1}$$

where $\overline{\kappa} = \|B_1 u\|_{L^\infty((t_0,T)\times\Sigma)}$ and $\tilde{C}_m = \tilde{C}_m(\chi, \sigma, \zeta, m, r, \omega_0, \mathcal{U}_1, T^* - t_0)$ is defined as $\max_{1 \leq k \leq m} \sum_{|\alpha|=k} C_{k,\alpha}$ with $C_{k,\alpha}$ appearing in (6.5).

Then there exists a constant $C = C(\chi, \sigma, \zeta, m, r, \omega_0, \mathcal{U}_1, T^* - t_0)$ such that

$$\|u\|^2_{G^m_\Sigma((t_0,T)\times G)} \leq C\, d_m((t_0,T)).$$

Proof. 1) We focus on the assumption (2.3) of a nonlinear boundary condition, and indicate the necessary changes for the linear one under hypothesis (2.4) at the appropriate point. Without loss of generality we assume $t_0 = 0$ and that, if G is unbounded, the nonlinearities χ and σ satisfy (5.6), cf. Remark 5.1. We fix a number $T' \in (0, T^*)$ such that $\omega(T') \leq \omega_0$ and $\operatorname{ran} u(t) \subseteq \mathcal{U}_1$ for all $t \in [0, T']$. Let $T \in (0, T']$ and set $J = (0, T)$. We have to work with the localized nonlinear problem on $G = \mathbb{R}^3_+$ and coefficents $A_1, A_2 \in F^{\tilde{m}}_{\mathrm{cf}}(\mathbb{R}^3_+)$ and $A_3 = A_3^{\mathrm{co}}$. The full space case has already been treated in Proposition 7.20 in [21]. We do not repeat the localization procedure itself, cf. Section 2. Using

$$\partial_t u = \chi(u)^{-1}\Big(f - \sigma(u)u - \sum\nolimits_{j=1}^{3} A_j \partial_j u\Big),$$

we obtain a constant $c = c(\chi, \sigma, r, \omega_0, \mathcal{U}_1, T^*)$ such that

$$\|u\|_{W^{1,\infty}(\Omega)} \leq \|\partial_t u\|_{L^\infty(\Omega)} + \omega(T) \leq c.$$

We put $L(u) = L(\chi(u), A_1, A_2, A_3, \sigma(u))$. Let $\alpha \in \mathbb{N}_0^4$ with $|\alpha| \leq m$. In view of differentiated versions of (2.2), we define

$$f_\alpha = \partial^\alpha f - \sum_{0<\beta\leq\alpha} \binom{\alpha}{\beta} \partial^\beta \chi(u) \partial_t \partial^{\alpha-\beta} u - \sum_{j=1}^{2} \sum_{0<\beta\leq\alpha} \binom{\alpha}{\beta} \partial^\beta A_j \partial_j \partial^{\alpha-\beta} u$$

$$
- \sum_{0<\beta\leq\alpha} \binom{\alpha}{\beta} \partial^\beta \sigma(u) \partial^{\alpha-\beta} u,
$$

$$
g_\alpha = \partial^\alpha g + \nu \times \sum_{0<\beta\leq\alpha} \binom{\alpha}{\beta} \partial^\beta \zeta(B_1^{\mathrm{co}} u) \partial^{\alpha-\beta} B_1^{\mathrm{co}} u
$$

As u solves (2.2), the function $v = \partial^\alpha u$ satisfies the system

$$
\begin{aligned}
L(u)v &= f_\alpha, && x \in \mathbb{R}^3_+, \quad t \in (0,T), \\
v(0) &= \partial^{(0,\alpha_1,\alpha_2,\alpha_3)} S_{m,\alpha_0,\chi,\sigma}(0,u_0,f), && x \in \mathbb{R}^3_+.
\end{aligned} \tag{6.2}
$$

If additionally $\alpha_3 = 0$, it is a solution of the boundary value problem

$$
\begin{aligned}
L(u)v &= f_\alpha, && x \in \mathbb{R}^3_+, \quad t \in (0,T), \\
B(u)v &= g_\alpha, && x \in \partial\mathbb{R}^3_+, \quad t \in (0,T), \\
v(0) &= \partial^{(0,\alpha_1,\alpha_2,\alpha_3)} S_{m,\alpha_0,\chi,\sigma}(0,u_0,f), && x \in \mathbb{R}^3_+.
\end{aligned} \tag{6.3}
$$

Here we used that $\partial_t^j u(0) = S_{m,j,\chi,\sigma}(0,u_0,f)$ for all $j \in \{0,\dots,m\}$ by (2.14).

Let $|\alpha'| \leq m-1$. Step I) of the proof of Proposition 4.4 of [23] shows that

$$
\begin{aligned}
\|f_\alpha\|_{L^2(\Omega)} &\leq \|f\|_{\mathcal{H}^{|\alpha|}(\Omega)} + c\,\|u\|_{\mathcal{H}^{|\alpha|}(\Omega)}, \\
\|f_{\alpha'}\|_{\mathcal{H}^1(\Omega)} + \|f_{\alpha'}(0)\|_{L^2(\mathbb{R}^3_+)} &\leq c d_{|\alpha'|+1}(J) + c\|u\|_{\mathcal{H}^{|\alpha'|+1}(\Omega)}
\end{aligned}
$$

with a constant $c = c(\chi,\sigma,m,r,\omega_0,\mathcal{U}_1)$. These estimates follow from the Moser-type Lemma 4.2 of [23] which is actually true for $u \in \mathcal{H}^m(\Omega)$, cf. Lemma 7.19 of [21].

As in (5.4), we reduce most terms in g_α to those appearing in f_α by means of the trace theorem. The main ones then lead to a summand involving $z(\overline{\kappa})$. So we arrive at

$$
\begin{aligned}
\|g_\alpha\|_{L^2(\Gamma)} &\leq \|g\|_{\mathcal{H}^{|\alpha|}(\Gamma)} + c\,\big(\|u\|_{\mathcal{H}^{|\alpha|}(\Omega)} + z(\overline{\kappa})\,\|B_1^{\mathrm{co}}\partial^\alpha u\|_{L^2(\Gamma)}\big), \\
\|g_{\alpha'}\|_{\mathcal{H}^1(\Gamma)} &\leq \|g\|_{\mathcal{H}^{|\alpha'|+1}(\Gamma)} + c\,\big(\|u\|_{\mathcal{H}^{|\alpha|}(\Omega)} + z(\overline{\kappa}) \sum_{|\beta|=|\alpha'|+1} \|B_1^{\mathrm{co}}\partial^\beta u\|_{L^2(\Gamma)}\big),
\end{aligned}
$$

where $c = c(\chi,\sigma,\zeta,m,r,\omega_0,\mathcal{U}_1)$. Under assumption (2.4), for linear boundary conditions g_α is defined with b instead of $\zeta(B_1^{\mathrm{co}}u)$ so that g_α does not contain highest order derivatives of u. The above estimates are thus true without the summands involving $z(\overline{\kappa})$, which then also do not appear below. Hence, the result is valid without conditions on $\overline{\kappa}$ in this case.

2) We next show that there are constants $C_k = C_k(\chi,\sigma,\zeta,m,r,\omega_0,\mathcal{U}_1,T^*)$ with

$$
\|\partial^\alpha u\|^2_{G^0(\Omega)} + \delta_{\alpha_3 0}\|\partial^{(\alpha_0,\alpha_1,\alpha_2)} \operatorname{tr}_\tau u\|^2_{L^2(\Gamma)} \leq C_k d_k(J) \tag{6.4}
$$

for all $\alpha \in \mathbb{N}_0^4$ with $|\alpha| = k$ and $k \in \{0,\dots,m\}$. Proposition 3.1 yields the case $k=0$ as in the proof of Proposition 4.4 of [23]. So let (6.4) be true for all $j \in \{0,\dots,k-1\}$ and some $k \in \{1,\dots,m\}$. Take $\alpha \in \mathbb{N}_0^4$ with $|\alpha| = k$. We first show that there is a constant $C_{k,\alpha} = C_{k,\alpha}(\chi,\sigma,\zeta,m,r,\omega_0,\mathcal{U}_1,T^*)$ with

$$
\|\partial^\alpha u\|^2_{G^0(\Omega)} + \delta_{\alpha_3 0}\|\partial^{(\alpha_0,\alpha_1,\alpha_2)} \operatorname{tr}_\tau u\|^2_{L^2(\Gamma)} \tag{6.5}
$$

$$\leq C_{k,\alpha}\Big[d_k(J)+\sum_{|\beta|=k}\int_0^T\big[\|\partial^\beta u(s)\|^2_{L^2(\mathbb{R}^3_+)}+\delta_{\beta_3 0}z(\overline{\kappa})^2\|B_1^{\mathrm{co}}\partial^\beta u(s)\|^2_{L^2(\partial\mathbb{R}^3_+)}\big]\mathrm{d}s\Big]$$

for each $\alpha\in\mathbb{N}_0^4$ with $|\alpha|=k$. This claim is shown via induction over α_3.

So let $\alpha_3=0$. Since $\partial^\alpha u$ solves (6.3), Proposition 3.1, the bounds on f_α and g_α, and estimate (2.17) yield a constant $c=c(\chi,\sigma,\zeta,k,r,\omega_0,\mathcal{U}_1,T^*)$ such that

$$\|\partial^\alpha u\|^2_{G^0_\Sigma(\Omega)}\leq c\,\Big(d_k(J)+\|u\|^2_{\mathcal{H}^{|\alpha|}(\Omega)}+z(\overline{\kappa})^2\sum_{|\beta|=k}\|B_1^{\mathrm{co}}\partial^\beta u\|^2_{L^2(\Gamma)}\Big).$$

The derivatives of u of order up to $k-1$ can be bounded by the induction hypothesis (6.4). So we have shown (6.5) for k and $\alpha_3=0$.

In another induction assume that (6.5) is true for k and all $\alpha_3\in\{0,\dots,l-1\}$ and some $l\in\{1,\dots,k\}$. Set $\alpha'=\alpha-e_3$. The function $\partial^{\alpha'}u$ then satisfies the initial value problem (6.2) without a boundary condition, where α is replaced by α'. Estimate (3.7) applied with $\gamma=1$ thus yields

$$\begin{aligned}\|\partial^\alpha u\|^2_{G^0(\Omega)}\leq \mathrm{e}^{2T}\|\nabla\partial^{\alpha'}u\|^2_{G^0_1(\Omega)}\leq c\,\Big(\sum_{j=0}^2\|\partial_j\partial^{\alpha'}u\|^2_{G^0(\Omega)}+\|f_{\alpha'}(0)\|^2_{L^2(\mathbb{R}^3_+)}\\ +\|\partial^{(0,\alpha_1,\alpha_2,\alpha_3-1)}S_{m,\alpha_0,\chi,\sigma}(0,f,u_0)\|^2_{\mathcal{H}^1(\mathbb{R}^3_+)}+\|f_{\alpha'}\|^2_{\mathcal{H}^1(\Omega)}\Big),\end{aligned}$$

where $c=c(\chi,\sigma,\zeta,k,r,\omega_0,\mathcal{U}_1,T^*)$. We then employ the induction hypothesis (6.5) for $\alpha_3=l-1$ to the terms $\|\partial_j\partial^{\alpha'}u\|^2_{G^0(\Omega)}$, the bounds for $f_{\alpha'}$, and estimate (2.17) to infer that

$$\begin{aligned}\|\partial^\alpha u\|^2_{G^0(\Omega)}\leq c\big(d_k(J)+\|u\|^2_{\mathcal{H}^{|\alpha|}(\Omega)}\big)+c\Big(\sum_{j=0}^2 C_{k,\alpha'+e_j}\Big)\Big[d_k(J)\\ +\sum_{|\beta|=k}\int_0^T\big[\|\partial^\beta u(s)\|^2_{L^2(\mathbb{R}^3_+)}+\delta_{\beta_3 0}z(\overline{\kappa})^2\|B_1^{\mathrm{co}}\partial^\beta u(s)\|^2_{L^2(\partial\mathbb{R}^3_+)}\big]\mathrm{d}s\Big].\end{aligned}$$

Using the induction hypothesis (6.4) for $k-1$ once again, we conclude that (6.5) is true for all $\alpha_3=l$, and thus for all $\alpha_3\leq k$.

We now sum in (6.5) over all $\alpha\in\mathbb{N}_0^4$ with $|\alpha|=k$. Assumption (6.1) then allows to absorb the boundary terms in the left-hand side. Afterwards, we use Gronwall's inequality to obtain a constant $C_k=C_k(\chi,\sigma,\zeta,m,r,\omega_0,\mathcal{U}_1,T^*)$ such that

$$\sum_{|\alpha|=k}\big(\|\partial^\alpha u\|^2_{G^0(\Omega)}+\delta_{\alpha_3 0}\|\partial^{(\alpha_0,\alpha_1,\alpha_2)}\operatorname{tr}_\tau u\|^2_{L^2(\Gamma)}\big)\leq C_k d_k(J). \tag{6.6}$$

We have thus shown (6.4). The assertion now follows by induction. □

The blow-up criterion for (2.2) will be established in the local wellposedness Theorem 6.4 below. Before, we provide auxiliary results needed to show the continuous dependence on data, starting with an approximation lemma in lowest order. Its proof is omitted since it is a minor modification of that of Lemma 5.1 in [23].

Lemma 6.2. *Let $J \subset \mathbb{R}$ be an open interval and $t_0 \in \overline{J}$. Take coefficients $A_{0,n}, A_0 \in F^3_\eta(\Omega)$, $A_1, A_2 \in F^3_{\mathrm{cf}}(\mathbb{R}^3_+)$, $A_3 = A_3^{\mathrm{co}}$, $D_n, D \in F^3(\Omega)$, and $b_n, b \in F^3_{\mathcal{H},\eta}(\Gamma)$ for all $n \in \mathbb{N}$ such that $(A_{0,n})_n$, $(D_n)_n$ respectively $(b_n)_n$ are bounded in $W^{1,\infty}(\Omega)$ respectively $W^{1,\infty}(\Gamma)$ and converge to A_0, D respectively b uniformly. Let $B_j = B_j^{\mathrm{co}}$ for $j \in \{1,2\}$ and $G = \mathbb{R}^3_+$. Choose $u_0 \in L^2(\mathbb{R}^3_+)$, $f \in L^2(\Omega)$, and $g \in L^2(\Gamma)$ with $g \cdot \nu = 0$. Let $u_n, u \in G^0_\Sigma(\Omega)$ solve the linear Maxwell system (2.5) with the above coefficients and data. Then $(u_n)_n$ tends to u in $G^0_\Sigma(\Omega)$.*

The next result is the core of the proof of continuous dependence. It improves the norm in which solutions converge by one regularity level, provided one has appropriate apriori information.

Lemma 6.3. *Let $J \subseteq \mathbb{R}$ be an open bounded interval, $t_0 \in \overline{J}$, and $m \in \mathbb{N}$ with $m \geq 3$. Assume that either (2.3) or (2.4) is valid. Choose data $u_0, u_{0,n} \in \mathcal{H}^m(G)$, $f, f_n \in \mathcal{H}^m(J \times G)$, and $g, g_n \in \mathcal{H}^m(J \times \Sigma)$ with $g \cdot \nu = 0$ and $g_n \cdot \nu = 0$ for all $n \in \mathbb{N}$ such that*

$$\|u_{0,n} - u_0\|_{\mathcal{H}^m(G)} \longrightarrow 0, \quad \|f_n - f\|_{\mathcal{H}^m(J\times G)} \longrightarrow 0, \quad \|g_n - g\|_{\mathcal{H}^m(J\times \partial G)} \longrightarrow 0,$$

as $n \to \infty$. We further assume that (2.2) with data $(t_0, u_{0,n}, f_n, g_n)$ and (t_0, u_0, f, g) have $G^m_\Sigma(J \times G)$-solutions u_n and u for all $n \in \mathbb{N}$, that there is a compact subset $\mathcal{U}_1$ of $\mathcal{U}$ with $\operatorname{ran} u(t) \subseteq \mathcal{U}_1$ for all $t \in J$, that $(u_n)_n$ is bounded in $G^m_\Sigma(J \times G)$, and that $(u_n)_n$ converges to u in $G^{m-1}_\Sigma(J \times G)$. If (2.3) is valid, we require that

$$z(\overline{\kappa})^2 \leq 1/(2\hat{C}_m) \tag{6.7}$$

for a fixed number $\overline{\kappa} > \|B_1 u\|_{L^\infty(\Gamma)}$, where $\hat{C}_m = \hat{C}_m(\chi, \sigma, \zeta, r, \mathcal{U}_1, T')$ appears in (6.17), $|J| \leq T'$, and r only depends on $d_m(J)$, $\|u\|_{G^m_\Sigma(J\times G)}$ and Ω. Then the solutions u_n converge to u in $G^m_\Sigma(J \times G)$.

Proof. 1) We focus on the assumption (2.3) of a nonlinear boundary condition, and indicate the necessary changes for the linear one under hypothesis (2.4) at the appropriate point. Without loss of generality we take $t_0 = 0$, $J = (0, T)$ and that, if G is unbounded, the nonlinearities χ and σ satisfy (5.6), cf. Remark 5.1. Moreover, T is less or equal than a fixed time $T' < \infty$. As in Proposition 6.1 we have to work with the localized nonlinear problem on $G = \mathbb{R}^3_+$ and coefficients $A_1, A_2 \in F^{\tilde{m}}_{\mathrm{cf}}(\mathbb{R}^3_+)$ and $A_3 = A_3^{\mathrm{co}}$. We do not repeat the localization procedure itself, cf. Section 2.

Throughout, we let $\alpha \in \mathbb{N}_0^4$ with $|\alpha| \leq m$ and $n \in \mathbb{N} \cup \{\infty\}$, where we put $u_\infty = u$ etc. Due to our assumptions, we can fix a number $r > 0$ that only depends on $d_m(J)$, $\|u\|_{G^m_\Sigma(J\times G)}$, and Ω and that dominates the quantities $d^n_m(J)$ for the data $(u_{0,n}, f_n, g_n)$ and the norms of u_n in $G^m_\Sigma(\Omega)$ and $L^\infty(\Omega)$ as well as A_1 and A_2 in $F^m(\Omega)$. Here and in the next statement we may omit $n \leq n_0$ for some $n_0 \in \mathbb{N}$. Let $\kappa = \operatorname{dist}(\mathcal{U}_1, \partial\mathcal{U}) > 0$ and $\mathcal{U}_1' = \mathcal{U}_1 + \overline{B}(0, \kappa/2)$. Then we obtain $\overline{\kappa} > \|B_1 u_n\|_{L^\infty(\Gamma)}$ and $\operatorname{ran} u_n(t) \subseteq \mathcal{U}_1'$ for all $t \in \overline{J}$ and $n \in \mathbb{N}$. There is another radius $R = R(\chi, \sigma, \zeta, m, r, \mathcal{U}_1)$ dominating the functions $\chi(u_n)$ and $\sigma(u_n)$ in $F^m(\Omega)$ and $\zeta(B_1^{\mathrm{co}} u_n)$ in $F^m_{\mathcal{H}}(\Gamma)$.

Let L_n and B_n be the differential and boundary operator from (2.5) with coefficients $A_0 = \chi(u_n)$ and $D = \sigma(u_n)$ respectively $b = \zeta(B_1^{\mathrm{co}} u_n)$. We use the modified inhomogeneities

$$
\begin{aligned}
f_{\alpha,n} &= \partial^\alpha f - \sum_{0<\beta\le\alpha}\binom{\alpha}{\beta}\partial^\beta\chi(u_n)\partial_t\partial^{\alpha-\beta}u_n - \sum_{j=1}^{2}\sum_{0<\beta\le\alpha}\binom{\alpha}{\beta}\partial^\beta A_j\partial_j\partial^{\alpha-\beta}u_n\\
&\qquad - \sum_{0<\beta\le\alpha}\binom{\alpha}{\beta}\partial^\beta\sigma(u_n)\partial^{\alpha-\beta}u_n,\\
g_{\alpha,n} &= \partial^\alpha g + \nu\times\sum_{0<\beta\le\alpha}\binom{\alpha}{\beta}\partial^\beta\zeta(B_1^{\mathrm{co}}u_n)\partial^{\alpha-\beta}B_1^{\mathrm{co}}u_n,
\end{aligned}
$$

where we assume that $\alpha_3 = 0$ when considering $g_{\alpha,n}$ here and below. Exploiting that A_3 and B_j^{co} are constant, we see that the function $v = \partial^\alpha u_n$ solves the linear initial boundary value problem

$$
\begin{aligned}
L_n v &= f_{\alpha,n}, && x\in\mathbb{R}^3_+, && t\in J,\\
B_n v &= g_{\alpha,n}, && x\in\partial\mathbb{R}^3_+, && t\in J, \qquad (6.8)\\
v(0) &= \partial^{(0,\alpha_1,\alpha_2,\alpha_3)}S_{m,\alpha_0,\chi,\sigma}(0,u_{0,n},f_n), && x\in\mathbb{R}^3_+,
\end{aligned}
$$

if $\alpha_3 = 0$. We further introduce the auxiliary map

$$
\begin{aligned}
h_n(t) = \sum_{i=1}^{3}\sum_{0\le j\le m}\sum_{\substack{0\le\gamma\le\alpha,\gamma_0=0\\|\gamma|=m-j}}\sum_{l_1,\dots,l_j=1}^{6} &\|(\partial_{y_{l_j}}\dots\partial_{y_{l_1}}\partial_x^{(\gamma_1,\gamma_2,\gamma_3)}\theta_i)(u_n(t))\\
&- (\partial_{y_{l_j}}\dots\partial_{y_{l_1}}\partial_x^{(\gamma_1,\gamma_2,\gamma_3)}\theta_i)(u(t))\|_{L^\infty(\mathbb{R}^3_+)}\\
+ \sum_{0\le j\le m}\sum_{\substack{0\le\gamma\le\alpha',\gamma_0=0\\|\gamma|=m-j}}\sum_{l_1,\dots,l_j=1}^{3} &\|(\partial_{y_{l_j}}\dots\partial_{y_{l_1}}\partial_x^{(\gamma_1,\gamma_2)}\zeta)(B_1^{\mathrm{co}}u_n(t))\\
&- (\partial_{y_{l_j}}\dots\partial_{y_{l_1}}\partial_x^{(\gamma_1,\gamma_2)}\zeta)(B_1^{\mathrm{co}}u(t))\|_{L^\infty(\partial\mathbb{R}^3_+)},
\end{aligned}
$$

where $t\in\overline{J}$, $n\in\mathbb{N}$, $\theta_1=\chi$, $\theta_2=\sigma$, $\theta_3=\chi^{-1}$, and $\alpha'\in\mathbb{N}_0^3$ with $|\alpha'|=m$. Observe that the functions h_n tend to 0 uniformly as $n\to\infty$.

Using the calculus results Lemma 2.1 of [22] (or Lemma 2.22 of [21]) and Corollary 2.2 of [23], one can show that all maps $f_{\alpha,n}$ and $g_{\alpha,n}$ are bounded in $L^2(\Omega)$ respectively $L^2(\Gamma)$ by a constant $c = c(\chi,\sigma,\zeta,m,r,\mathcal{U}_1,T')$. If $|\alpha|\le m-1$, then we have analogous bounds in $\mathcal{H}^1(\Omega)$, $G^0(\Omega)$ respectively, $\mathcal{H}^1(\Gamma)$. We further derive the inequalities

$$
\begin{aligned}
\|f_{\alpha,n}-f_{\alpha,\infty}\|^2_{\mathcal{H}^k(\Omega)} &\le c\Big[\|f_n-f\|^2_{\mathcal{H}^m(\Omega)} + \|u_n-u\|^2_{G^{m-1}(\Omega)} + \delta_{|\alpha|(m-k)}\|h_n\|^2_\infty\\
&\qquad + \int_0^T\sum_{\tilde\alpha\in\mathbb{N}_0^4,|\tilde\alpha|=m}\|\partial^{\tilde\alpha}(u_n(s)-u(s))\|^2_{L^2(\mathbb{R}^3_+)}\,\mathrm{d}s\Big],\\
\|f_{\alpha,n}-f_{\alpha,\infty}\|^2_{G^0(\Omega)} &\le c\left(\|f_n-f\|^2_{G^{m-1}(\Omega)} + \|u_n-u\|^2_{G^{m-1}(\Omega)}\right), \qquad (6.9)\\
\|g_{\alpha,n}-g_{\alpha,\infty}\|^2_{\mathcal{H}^k(\Omega)} &\le c\Big[\|g_n-g\|^2_{\mathcal{H}^m(\Gamma)} + \|u_n-u\|^2_{G^{m-1}(\Omega)} + \delta_{|\alpha|(m-k)}\|h_n\|^2_\infty
\end{aligned}
$$

$$+\int_0^T \sum_{\tilde\alpha\in\mathbb{N}_0^4,|\tilde\alpha|=m} \|\partial^{\tilde\alpha}(u_n(s)-u(s))\|^2_{L^2(\mathbb{R}^3_+)}\,\mathrm{d}s$$
$$+\, z(\overline{\kappa})^2\int_0^T \sum_{\tilde\alpha\in\mathbb{N}_0^3,|\tilde\alpha|=m} \|B_1^{\mathrm{co}}\partial^{\tilde\alpha}(u_n(s)-u(s))\|^2_{L^2(\partial\mathbb{R}^3_+)}\,\mathrm{d}s\Big]$$

for $k\in\{0,1\}$ and $|\alpha|\le m-1$, using also (5.4) and (5.5). Here the first and the last estimate are also true for $|\alpha|=m$ in the case $k=0$. For linear boundary conditions under assumption (2.4), in $g_{\alpha,n}$ we replace $\partial^\beta\zeta(B_1^{\mathrm{co}}u_n)$ by b, so that highest order derivatives of u do not appear in $g_{\alpha,n}$. For this reason, the last summand and hence $z(\overline{\kappa})$ disappear in the above estimate and the calculations below. So the conclusions will be true without restriction on $\overline{\kappa}$.

2) We first treat tangential derivatives with $\alpha_3=0$. We set $w_{0,n}=\partial^{(0,\alpha_1,\alpha_2,0)}S_{m,\alpha_0,\chi,\sigma}(0,u_{0,n},f_n)$. To decompose $\partial^\alpha u_n=w_n+z_n$, we use the solution $w_n\in G^0_\Sigma(\Omega)$ of the linear system

$$\begin{aligned} L_n v &= f_{\alpha,\infty}, && x\in\mathbb{R}^3_+, && t\in J,\\ B_n v &= g_{\alpha,\infty}, && x\in\partial\mathbb{R}^3_+, && t\in J,\\ v(0) &= w_{0,\infty}, && x\in\mathbb{R}^3_+, \end{aligned} \tag{6.10}$$

with fixed data, and $z_n\in G^0_\Sigma(\Omega)$ of the linear problem

$$\begin{aligned} L_n v &= f_{\alpha,n}-f_{\alpha,\infty}, && x\in\mathbb{R}^3_+, && t\in J,\\ B_n v &= g_{\alpha,n}-g_{\alpha,\infty}, && x\in\partial\mathbb{R}^3_+, && t\in J,\\ v(0) &= w_{0,n}-w_{0,\infty}, && x\in\mathbb{R}^3_+, \end{aligned} \tag{6.11}$$

with data tending to 0 as we show below. These solutions exist due to Proposition 3.1, and we have $w_\infty=\partial^\alpha u$ by uniqueness and (6.8). By our assumptions, the coefficients $\chi(u_n)$, $\sigma(u_n)$ and $\zeta(B_1^{\mathrm{co}}u_n)$ converge uniformly to $\chi(u)$, $\sigma(u)$ respectively $\zeta(B_1^{\mathrm{co}}u)$. In view of the estimates in step 1), Lemma 6.2 shows

$$\|w_n-\partial^\alpha u\|_{G^0_\Sigma(\Omega)}=\|w_n-w_\infty\|_{G^0_\Sigma(\Omega)}\;\longrightarrow\;0,\qquad n\to\infty. \tag{6.12}$$

Let $\gamma=\gamma(\chi,\sigma,\zeta,m,r,\mathcal{U}_1,T')\ge 1$ be the parameter $\gamma_0(\eta,R)$ from Proposition 3.1. We now apply this result to (6.11). By means of (6.9), we thus obtain

$$\begin{aligned} \|z_n\|^2_{G^0_\Sigma(\Omega)} &\le c\Big[\tilde d^n_m(J)+\|u_n-u\|^2_{G^{m-1}(\Omega)}+\|h_n\|^2_\infty \\ &+\sum_{\tilde\alpha,\alpha'}\int_0^T\big(\|\partial^{\tilde\alpha}(u_n(s)-u(s))\|^2_{L^2(\mathbb{R}^3_+)}+z(\overline{\kappa})^2\|\partial^{\alpha'}B_1^{\mathrm{co}}(u_n(s)-u(s))\|^2_{L^2(\partial\mathbb{R}^3_+)}\big)\,\mathrm{d}s\Big] \end{aligned} \tag{6.13}$$

for a constant $c=c(\chi,\sigma,\zeta,m,r,\mathcal{U}_1,T')$, where we sum over all multi-indices $\tilde\alpha\in\mathbb{N}_0^4$ and $\alpha'\in\mathbb{N}_0^3$ with $|\tilde\alpha|,|\alpha'|=m$ and $\tilde\alpha_3=0$, and the quantity $\tilde d^n_m(J)$ is defined as in (5.10) for $u_{0,n}-u_0$, f_n-f and g_n-g. We write $I_n(T)$ for the above sum of integrals. Since $\partial^\alpha(u_n-u)=w_n-\partial^\alpha u+z_n$ by uniqueness again, estimates (6.12) and (6.13) imply the bound

$$\|\partial^\alpha(u_n(t)-u(t))\|^2_{L^2(\mathbb{R}^3_+)}+\|\partial^\alpha\operatorname{tr}_\tau(u_n-u)\|^2_{L^2(\Gamma)}\le a_{\alpha,n}+cI_n(T) \tag{6.14}$$

for all $t\in\overline{J}$ and numbers $a_{\alpha,n}$ tending to 0 as $n\to\infty$.

We want to extend the above estimate to the general case $\alpha_3 \leq m$; i.e., we have to show the claim

$$\|\partial^\alpha(u_n(t) - u(t))\|^2_{L^2(\mathbb{R}^3_+)} \leq a_{\alpha,n} + \hat{C}_{m,\alpha} \sum_{\substack{\alpha' \in \mathbb{N}_0^4 \\ |\alpha'| = m}} \int_0^t \|\partial^{\alpha'}(u_n(s) - u(s))\|^2_{L^2(\mathbb{R}^3_+)} \, \mathrm{d}s + z(\overline{\kappa})^2 \hat{C}_{m,\alpha} \sum_{\substack{\beta' \in \mathbb{N}_0^3 \\ |\beta'| = m}} \|\partial^{\beta'} B_1^{\mathrm{co}}(u_n - u)\|^2_{L^2(\Gamma)} \tag{6.15}$$

for null sequences $(a_{\alpha,n})_n$, a constant $\hat{C}_{m,\alpha} = \hat{C}_{m,\alpha}(\chi, \sigma, \zeta, r, \mathcal{U}_1, T')$, and all $t \in \overline{J}$ and $\alpha \in \mathbb{N}_0^4$ with $|\alpha| = m$.

We show this claim by an induction over $l = \alpha_3 \in \{0, 1, \ldots, m\}$. Let the inequality

$$\|\partial^\alpha(u_n(t) - u(t))\|^2_{L^2(\mathbb{R}^3_+)} \leq a'_{\alpha,n} + \hat{C}_{m,\alpha} \sum_{|\alpha'| = m} \int_0^t \|\partial^{\alpha'}(u_n(s) - u(s))\|^2_{L^2(\mathbb{R}^3_+)} \, \mathrm{d}s + z(\overline{\kappa})^2 \hat{C}_{m,\alpha} \sum_{|\beta'| = m} \|\partial^{\beta'} B_1^{\mathrm{co}}(u_n - u)\|^2_{L^2(\Gamma)} \tag{6.16}$$

be true for null sequences $(a'_{\alpha,n})_n$ and all $t \in \overline{J}$ and $\alpha \in \mathbb{N}_0^4$ with $\alpha_3 = l - 1$ for some $l \in \{1, \ldots, m\}$, where we sum over $\alpha' \in \mathbb{N}_0^4$ and $\beta' \in \mathbb{N}_0^3$.

The case $l = 0$ was settled in (6.14). To show the claim for $\alpha_3 = l$, one needs a version of Proposition 3.3 in [22] (i.e., our estimate (3.7)) which can be applied to the differences $u - u_n$. Unfortunately, this result does not apply directly since it only involves a fixed operator. However, the arguments in its proof can be extended to the case of data and coefficients that converge as in our situation. This has been carried out in steps III)–V) of the proof of Lemma 7.22 in [21], see also step III) of Lemma 5.2 in [23]. In this reasoning the boundary conditions do not enter. The boundary terms in the right-hand side of (6.15) only appear since they are needed for the pure tangential case $l = 0$. They are then carried on when invoking the induction hypothesis. So (6.16) follows from estimate (7.150) of [21]. We have thus shown (6.15).

Next, we sum in (6.15) over all $\alpha \in \mathbb{N}_0^4$ with $|\alpha| = m$ and $\alpha_3 \geq 1$ and in (6.14) over all $\alpha \in \mathbb{N}_0^4$ with $|\alpha| = m$ and $\alpha_3 = 0$. Combining the resulting inequalities, we arrive at

$$\begin{aligned} &\sum_{\alpha \in \mathbb{N}_0^4, |\alpha| = m} \|\partial^\alpha(u_n(t) - u(t))\|^2_{L^2(\mathbb{R}^3_+)} + \sum_{\beta \in \mathbb{N}_0^3, |\beta| = m} \|\partial^\beta \operatorname{tr}_\tau(u_n - u)\|^2_{L^2(\Gamma)} \\ &\leq a_n + \hat{C}_m \sum_{\alpha' \in \mathbb{N}_0^4, |\alpha'| = m} \int_0^t \|\partial^{\alpha'}(u_n(s) - u(s))\|^2_{L^2(\mathbb{R}^3_+)} \, \mathrm{d}s \\ &\quad + z(\overline{\kappa})^2 \hat{C}_m \sum_{\beta' \in \mathbb{N}_0^3, |\beta'| = m} \|\partial^{\beta'} B_1^{\mathrm{co}}(u_n - u)\|^2_{L^2(\Gamma)}, \end{aligned} \tag{6.17}$$

where $a_n = \sum_{\alpha \in \mathbb{N}_0^4, |\alpha| = m} a_{\alpha,n}$ and $\hat{C}_m = \hat{C}_m(\chi, \sigma, \zeta, r, \mathcal{U}_1, T')$ is given by $\hat{C}_m = \sum_{\alpha \in \mathbb{N}_0^4, |\alpha| = m} \hat{C}_{m,\alpha}$.

One can then use the smallness assumption (6.7) on $z(\overline{\kappa})$ to absorb the boundary terms on the right-hand side by the left-hand side. Finally, Gronwall's inequality implies that $\partial^\alpha(u_n - u)$ and $\partial^\beta \operatorname{tr}_\tau(u_n - u)$ tend to 0 in $G^0(\Omega)$ respectively $L^2(\Gamma)$ as $n \to \infty$, where $|\alpha| = |\beta| = m$. The lower order derivatives converge by assumption. $\square$

We finally establish the full local wellposedness theorem. For times $t_0 < T$ we introduce the data space

$$M_{\chi,\sigma,\zeta,m}(t_0,T) = \{(\tilde{u}_0, \tilde{f}, \tilde{g}) \in \mathcal{H}^m(G) \times \mathcal{H}^m((t_0,T) \times G) \times \mathcal{H}^m((t_0,T) \times \Sigma) \mid (\chi, \sigma, \zeta, t_0, \tilde{f}, \tilde{g}, \tilde{u}_0) \text{ is compatible of order } m,\ \tilde{g} \cdot \nu = 0\}$$

and endow it with its natural norm.

Theorem 6.4. *Let $m \in \mathbb{N}$ with $m \geq 3$ and $t_0 \in \mathbb{R}$. Assume that either* (2.3) *or* (2.4) *is valid. Choose data $u_0 \in \mathcal{H}^m(G)$, $f \in \mathcal{H}^m((t_0,T) \times G))$, and $g \in \mathcal{H}^m(((t_0,T) \times \Sigma)$ with $g \cdot \nu = 0$ for all $T > t_0$ such that $\overline{\operatorname{ran}(u_0)} \subseteq \mathcal{U}$ and the tuple $(\chi, \sigma, \zeta, t_0, u_0, f, g)$ fulfills the compatibility conditions* (2.21) *of order m. If assumption* (2.3) *is true, we pick $\widetilde{\kappa} > 0$ satisfying* (5.11) *and we require $\|B_1 u_0\|_{L^\infty(\Sigma)} < \widetilde{\kappa}/4$.*

Then the maximal existence times $T_+(k, t_0, u_0, f, g)$ from (5.24) *do not depend on $k \in \{3, \ldots, m\}$ if* (2.4) *is true. Moreover, the following assertions hold.*

(1) *There exists a unique maximal solution u of* (2.2) *which belongs to the function space $G^m_\Sigma((t_0,T) \times G)$ for all $T < T_+$.*

(2) *If $T_+ < \infty$, then*

(a) *the solution u leaves every compact subset of $\mathcal{U}$, or*

(b) $\limsup_{t \nearrow T_+} \|\nabla u(t)\|_{L^\infty(G)} = \infty$, *or*

(c) *condition (c) from Proposition 5.5 occurs or* (6.1) *fails as $T \to T_+$.*

If (2.4) *is valid, the last condition can be dropped.*

(3) *Let $T \in (t_0, T_+)$. Fix $T' \in (T, T_+)$. If assumption* (2.3) *is true, let* (6.7) *hold on (t_0, T') and assume that $\|B_1 u\|_{L^\infty((t_0,T') \times \Sigma)} < \widetilde{\kappa}/4$ and that* (6.21) *is valid for $\widetilde{\kappa}$. (The constants in these conditions depend on r from* (6.19), *$(T' - t_0)$, and $\kappa = \frac{1}{2}\operatorname{dist}(\mathcal{U}_1, \partial\mathcal{U})$ for a compact subset $\mathcal{U}_1 \subseteq \mathcal{U}$ with $\operatorname{ran}(u(t)) \subseteq \mathcal{U}_1$ for all $t \in [t_0, T']$.) Then there is a number $\delta > 0$ such that for all data $(\tilde{u}_0, \tilde{f}, \tilde{g}) \in M_{\chi,\sigma,\zeta,m}(t_0,T)$ fulfilling*

$$\|\tilde{u}_0 - u_0\|_{\mathcal{H}^m(G)} < \delta, \quad \|\tilde{f} - f\|_{\mathcal{H}^m((t_0,T') \times G)} < \delta, \quad \|\tilde{g} - g\|_{\mathcal{H}^m((t_0,T') \times \Sigma)} < \delta$$

the maximal existence time satisfies $T_+(m, t_0, \tilde{f}, \tilde{g}, \tilde{u}_0) > T$. Let $u(\cdot; \tilde{u}_0, \tilde{f}, \tilde{g})$ be the corresponding maximal solution of (2.2). *The flow map*

$$\Psi \colon B_{M_{\chi,\sigma,\zeta,m}(t_0,T')}((u_0, f, g), \delta) \to G^m_\Sigma((t_0,T) \times G), \quad (\tilde{u}_0, \tilde{f}, \tilde{g}) \mapsto u(\cdot; \tilde{u}_0, \tilde{f}, \tilde{g}),$$

is continuous. Moreover, there is a constant $C = C(\chi, \sigma, \zeta, m, r, T' - t_0, \kappa)$ with

$$\begin{aligned} &\|\Psi(\tilde{u}_{0,1}, \tilde{f}_1, \tilde{g}_1) - \Psi(\tilde{u}_{0,2}, \tilde{f}_2, \tilde{g}_2)\|_{G^{m-1}_\Sigma((t_0,T) \times G)} \qquad (6.18) \\ &\quad \leq C\|\tilde{u}_{0,1} - \tilde{u}_{0,2}\|_{\mathcal{H}^m(G)} + C \sum_{j=0}^{m-1} \|\partial_t^j \tilde{f}_1(t_0) - \partial_t^j \tilde{f}_2(t_0)\|_{\mathcal{H}^{m-j-1}(G)} \end{aligned}$$

$$+ C\,\|\tilde f_1-\tilde f_2\|_{H^{m-1}((t_0,T)\times G)} + C\,\|\tilde g_1-\tilde g_2\|_{\mathcal{H}^{m-1}((t_0,T)\times\Sigma)}$$

for all $(\tilde u_{0,j},\tilde f_j,\tilde g_j)\in B_{M_{\chi,\sigma,\zeta,m}(t_0,T')}((u_0,f,g),\delta)$.

Proof. Assertions (1) and (2) follow from Propositions 5.5 and 6.1.

1) To show (3), let $t_0<T<T'<T_+$ be as in the statement and $J'=(t_0,T')$. Again, we focus on assumptions (2.3) concerning nonlinear boundary conditions, and indicate the necessary changes for the linear one under hypothesis (2.4) at the appropriate point. Let C_S' be the norm of the embedding of $\mathcal{H}^m(J'\times G)$ in $G^{m-1}(J'\times G)$ and C_S of $\mathcal{H}^2(G)$ into $C_b(\overline{G})$. We take a radius $r>0$ such that

$$\begin{aligned}
&\|u_0\|_{\mathcal{H}^m(G)}+\|f\|_{G^{m-1}(J'\times G)}+\|f\|_{\mathcal{H}^m(J'\times G)}+\|g\|_{\mathcal{H}^m(J'\times\Sigma)}<r/(mC_S'),\\
&\|u\|_{G^m_\Sigma(J'\times G)}<r, \qquad (6.19)
\end{aligned}$$

Let $\mathcal{U}_1\subseteq\mathcal{U}$ and $\widetilde\kappa>4\|B_1u\|_{L^\infty((t_0,T')\times\Sigma)}$ be given as in the statement. Using Lemma 2.1 of [23] and (5.4), one finds a radius $\tilde r=\tilde r(\chi,\sigma,\zeta,m,r,\widetilde\kappa,\mathcal{U}_1)$ larger than the norms of $\theta(u)$ in $F^m((t_0,T')\times G)$, of $\zeta(B_1u)$ in $F^m_{\mathcal{H}}((t_0,T')\times\Sigma)$, of $\theta(u(t_0))$ in $F^{m-1,0}(G)$ and that of $\partial_t^j\theta(u)(t_0)$ in $\mathcal{H}^{m-1-j}(G)$ for $j\in\{1,\dots,m-1\}$ and $\theta\in\{\chi,\sigma\}$. We fix a number $\kappa<\frac12\operatorname{dist}(\mathcal{U}_1,\partial\mathcal{U})$ and set

$$V_\kappa=\{y\in\mathcal{U}\mid \operatorname{dist}(y,\partial\mathcal{U})\ge\kappa\}\cap\overline{B}(0,2C_Sr)\quad\text{and}\quad \tilde V_\kappa=V_\kappa+\overline{B}(0,\kappa/2)\subseteq\mathcal{U}.$$

We take $R=R(\chi,\sigma,\zeta,m,4r,\kappa,T')>4r$ of (5.18) in the proof of Theorem 5.3.

Choose a number $\hat T\in(t_0,T')$ and data $(\hat u_0,\hat f,\hat g)\in M_{\chi,\sigma,\zeta,m}(J')$ such that $\hat u_0$ maps into V_κ and the data satisfy the bounds (6.19) with $2r$ instead of r. Let $\hat J=(t_0,\hat T)$. We assume that a solution $\hat u\in G^m_\Sigma(\hat J\times G)$ of (2.2) exists for these data with norm less or equal R in this space and taking values in $\tilde V_\kappa$. Let $\hat\kappa\ge\overline{\kappa}$ bound the supnorm of $B_1\hat u$ on $(t_0,\hat T)\times\Sigma$. Then there exists a constant $\hat C=\hat C(\chi,\sigma,\zeta,m,2r,R,\tilde V_\kappa,T')$ and a time step $\hat\tau=\hat\tau(\chi,\sigma,\zeta,m,2r,R,\tilde V_\kappa,\hat\kappa,T')$ such that the difference of u and $\hat u$ is controlled by

$$\begin{aligned}
\|\hat u-u\|^2_{G^{m-1}_\Sigma(\hat J\times G)}\le\hat C\Big[&\|\hat u_0-u_0\|^2_{\mathcal{H}^{m-1}(G)}+\sum_{j=0}^{m-1}\|\partial_t^j\hat f(t_0)-\partial_t^jf(t_0)\|^2_{\mathcal{H}^{m-j-1}(G)}\\
&+\|\hat f-f\|^2_{H^{m-1}(\hat J\times G)}+\|\hat g-g\|^2_{\mathcal{H}^{m-1}(\hat J\times\Sigma)}\Big], \qquad (6.20)
\end{aligned}$$

where $\hat J=(t_0,t_0+\hat\tau)$ and we assume that

$$\hat\tau,\,z(\hat\kappa)^2\le(6C\overline{C}(R))^{-1} \qquad (6.21)$$

with $\overline{C}=\overline{C}(R)$ from (5.4) and (5.5) and C introduced below. To show this claim, we note that $\hat u-u$ solves the linear problem (2.5) with coefficients $A_0=\chi(u)$, $D=\sigma(u)$ and $b=\zeta(B_1u)$, as well as data $\hat u_0-u_0$,

$$\begin{aligned}
\varphi&=\hat f-f+(\chi(u)-\chi(\hat u))\partial_t\hat u+(\sigma(u)-\sigma(\hat u))\hat u,\\
\psi&=\hat g-g+B_0(\zeta(B_1u)-\zeta(B_1\hat u))B_1\hat u.
\end{aligned}$$

We now apply the apriori estimate (4.13) in Theorem 4.7 to this problem, using (5.4) and the bounds on the coefficients stated after (6.19). Inequality (4.13) yields constants $c = c(m, \eta, r, \tilde{r}, T')$ and $\gamma_0 = \gamma_0(m, \eta, r, \tilde{r}, T')$ such that

$$\|\hat{u} - u\|^2_{G^{m-1}_{\Sigma,\gamma}(\hat{G})} \leq c\Big[\|\hat{u}_0 - u_0\|^2_{\mathcal{H}^{m-1}(G)} + \sum_{j=0}^{m-1} \|\partial_t^j \varphi(t_0)\|^2_{\mathcal{H}^{m-j-1}(G)} + \|\psi\|^2_{\mathcal{H}^{m-1}_\gamma(\hat{J}\times\Sigma)}$$
$$+ \delta_{m>3}\overline{C}(R)(\hat{\tau} + z_0(\widetilde{\kappa})^2)\|B_1(\hat{u} - u)\|^2_{L^\infty_\gamma(\hat{J}\times\Sigma)} + \frac{1}{\gamma}\|\varphi\|^2_{\mathcal{H}^{m-1}_\gamma(\hat{G})}\Big],$$

where $\hat{G} := \hat{J} \times G$ and $\gamma \geq \gamma_0$. To treat the extra term in the case $m \geq 4$, we first note that $z_0(\widetilde{\kappa})^2 \leq c(r)$ because of (6.19). As in (5.21) we additionally use Sobolev's embedding to estimate

$$\|B_1(\hat{u} - u)\|^2_{L^\infty_\gamma(\hat{J}\times\Sigma)} \leq 2\,\|\hat{u}_0 - u_0\|^2_{L^\infty(\Sigma)} + 2\hat{\tau}^2\|\partial_t(\hat{u} - u)\|^2_{L^\infty_\gamma(\hat{J}\times\Sigma)}$$
$$\leq c\big(\|\hat{u}_0 - u_0\|^2_{\mathcal{H}^{m-1}(G)} + \hat{\tau}\|\hat{u} - u\|^2_{G^{m-1}_\gamma(\hat{J}\times G)}\big).$$

The functions φ and ψ can be controlled by means of (5.5) and the calculus results Lemma 2.1 of [22] (see also Lemma 2.22 of [21]) as well as Lemma 2.1 and Corollary 2.2 of [23]. We then obtain a constant $C = C(\chi, \sigma, \zeta, m, 2r, R, \tilde{V}_\kappa, T')$ such that

$$\|\hat{u} - u\|^2_{G^{m-1}_{\Sigma,\gamma}(\hat{G})} \leq C\Big(\|\hat{u}_0 - u_0\|^2_{\mathcal{H}^{m-1}(G)} + \sum_{j=0}^{m-1} \|\partial_t^j \hat{f}(t_0) - \partial_t^j f(t_0)\|^2_{\mathcal{H}^{m-j-1}(G)}$$
$$+ \|\hat{f} - f\|^2_{\mathcal{H}^{m-1}_\gamma(\hat{G})} + \|\hat{g} - g\|^2_{\mathcal{H}^{m-1}_\gamma(\hat{J}\times\Sigma)} + \frac{1}{\gamma}\|\hat{u} - u\|^2_{G^{m-1}_\gamma(\hat{G})}$$
$$+ \overline{C}(R)(\hat{\tau} + z(\hat{\kappa})^2)\|\hat{u} - u\|^2_{G^{m-1}_{\Sigma,\gamma}(\hat{J}\times G)}\Big),$$

for $\gamma \geq \gamma_0$. Fixing a large γ, condition (6.21) implies (6.20). In a linear boundary condition we only have $\psi = g - \hat{g}$ and thus the term with $\overline{C}(R)$ is not present above, so that we do not need assumption (6.21) in this case. This fact allows to omit the restrictions on $\widetilde{\kappa}$ below if we work under assumption (2.4).

2) We take as a time step τ the minimum of $\hat{\tau}$ in step 1), of $\widetilde{\kappa}/(2C_S R)$, and of $\tau(\chi, \sigma, \zeta, m, T', 2r, \kappa, \widetilde{\kappa})$ from (5.20). There is an index $N \in \mathbb{N}$ with $t_0 + (N-1)\tau < T \leq t_0 + N\tau$. We set $t_k = t_0 + k\tau$ for $k \in \{1, \ldots, N-1\}$. If $t_0 + N\tau < T'$, we put $t_N = t_0 + N\tau$; else we take any t_N from (T, T'). Next, we choose a radius $\delta_0 > 0$ which is less than $r/(4mC'_S)$, $\widetilde{\kappa}/(4C_S)$, and κ/C_S. Let $(\tilde{u}_0, \tilde{f}, \tilde{g}) \in B_{M_{\chi,\sigma,\zeta,m}(t_0,T')}((u_0, f, g), \delta_0) =: B_M(\delta_0)$. These data thus satisfy the bounds (6.19) with a right-hand side $\delta_0 + r/(mC'_S) \leq 2r/(mC'_S)$. We also have $\|B_1\tilde{u}_0\|_{L^\infty(\Sigma)} < C_S\delta_0 + \widetilde{\kappa}/4 \leq \widetilde{\kappa}/2$ as well as $\|u_0 - \tilde{u}_0\|_{L^\infty(G)} \leq C_S\delta_0$ so that the range of $\tilde{u}_0$ is contained in V_κ.

As a result, Theorem 5.3 yields a solution $\tilde{u} \in G^m_\Sigma((t_0, t_1) \times G)$ of (2.2) with data $(\tilde{u}_0, \tilde{f}, \tilde{g})$ instead of (u_0, f, g). The proof of this theorem also shows that $\tilde{u}$ is bounded by R in $G^m_\Sigma((t_0, t_1) \times G)$ and thus Ψ maps $B_M(\delta_0)$ into the ball in $G^m_\Sigma((t_0, t_1) \times G)$ with center 0 and radius R. Moreover, both $\|B_1 u\|_{L^\infty((t_0,t_1)\times\Sigma)}$ and $\|B_1\tilde{u}\|_{L^\infty((t_0,t_1)\times\Sigma)}$ are smaller than $\widetilde{\kappa}$ by the choice of τ. It follows that estimate (6.20) is true for $\tilde{u}$ instead of $\hat{u}$, with time step τ and constant $\hat{C}$, because $\widetilde{\kappa}$ satisfies (6.21).

Next take a sequence $(u_{0,n}, f_n, g_n)_n$ in $B_{M_{\chi,\sigma,\zeta,m}(t_0,T')}((u_0, f, g), \delta_0)$ which converges to (u_0, f, g) in this space. Since

$$\sum_{j=0}^{m-1} \|\partial_t^j f_n(t_0) - \partial_t^j f(t_0)\|^2_{\mathcal{H}^{m-j-1}(G)} \leq mC_S' \|f_n - f\|^2_{\mathcal{H}^m((t_0,T')\times G)} \longrightarrow 0$$

as $n \to \infty$, estimate (6.20) yields the limit

$$\|\Psi(u_{0,n}, f_n, g_n) - \Psi(u_0, f, g)\|_{G_\Sigma^{m-1}((t_0,t_1)\times G)} \longrightarrow 0.$$

Lemma 6.3 thus shows that $(\Psi(u_{0,n}, f_n, g_n))_n$ converges to $\Psi(f, g, u_0)$ in $G_\Sigma^m((t_0, t_1) \times G)$. We conclude that Ψ is continuous in (u_0, f, g). Using (6.19) and the choice of κ, we then find a number $\delta_1 \in (0, \delta_0]$ such that for all data $(\tilde{u}_0, \tilde{f}, \tilde{g}) \in B_{M_{\chi,\sigma,\zeta,m}(t_0,T')}((u_0, f, g), \delta_1)$ the solution $\Psi(\tilde{u}_0, \tilde{f}, \tilde{g})$ exists on $[t_0, t_1]$ and satisfies (6.20) on (t_0, t_1) and

$$\begin{aligned}
&\|\Psi(\tilde{u}_0, \tilde{f}, \tilde{g})\|_{G_\Sigma^m((t_0,t_1)\times G)} \\
&\quad \leq \|\Psi(\tilde{u}_0, \tilde{f}, \tilde{g},) - \Psi(u_0, f, g)\|_{G_\Sigma^m((t_0,t_1)\times G)} + \|\Psi(u_0, f, g)\|_{G_\Sigma^m((t_0,t_1)\times G)} < 2r, \\
&\|B_1 \tilde{u}(t_1)\|_{L^\infty(\Sigma)} < \widetilde{\kappa}/2, \\
&\operatorname{dist}(\operatorname{ran} \Psi(\tilde{u}_0, \tilde{f}, \tilde{g})(t), \partial\mathcal{U}) > \kappa,
\end{aligned}$$

for all $t \in [t_0, t_1]$. In particular, $\Psi(\tilde{u}_0, \tilde{f}, \tilde{g})(t_1)$ satisfies the assumptions of Theorem 5.3 with the same parameters as used before.

3) We can iterate the above argument up to time $t_N \geq T$, arriving at a final radius $\delta := \delta_N$ for the data (cf. the proof of Theorem 5.3 in [23]). In particular, the final existence time $T_+(m, t_0, \tilde{u}_0, \tilde{f}, \tilde{g})$ is larger than T if $(\tilde{u}_0, \tilde{f}, \tilde{g})$ belongs to $B_M(\delta)$. Next fix two tuples $(\tilde{u}_{0,j}, \tilde{f}_j, \tilde{g}_j)$ from this ball. Replacing u by $\Psi(\tilde{u}_{0,2}, \tilde{f}_2, \tilde{g}_2)$ in step I), we deduce from (6.20) that

$$\begin{aligned}
&\|\Psi(\tilde{u}_{0,1}, \tilde{f}_1, \tilde{g}_1) - \Psi(\tilde{u}_{0,2}, \tilde{f}_2, \tilde{g}_2)\|^2_{G_\Sigma^{m-1}((t_0,T)\times G)} \qquad (6.22) \\
&\leq \hat{C}\Big(\|\tilde{u}_{0,1} - \tilde{u}_{0,2}\|^2_{\mathcal{H}^m(G)} + \|\tilde{f}_1 - \tilde{f}_2\|^2_{\mathcal{H}^{m-1}((t_0,T)\times G)} + \|\tilde{g}_1 - \tilde{g}_2\|^2_{\mathcal{H}^{m-1}((t_0,T)\times \Sigma)} \\
&\qquad + \sum_{j=0}^{m-1} \|\partial_t^j \tilde{f}_1(t_0) - \partial_t^j \tilde{f}_2(t_0)\|^2_{\mathcal{H}^{m-j-1}(G)}\Big),
\end{aligned}$$

where $\hat{C} = \hat{C}(\chi, \sigma, \zeta, m, 2r, R, \tilde{V}_\kappa, T')$. This estimate implies (6.18).

Finally, take a sequence $(\tilde{u}_{0,n}, \tilde{f}_n, \tilde{g}_n)_n$ in $B_M(\delta)$ with limit $(\tilde{u}_{0,1}, \tilde{f}_1, \tilde{g}_1)$ in this ball. Inequality (6.22) shows that the solutions $\Psi(\tilde{u}_{0,n}, \tilde{f}_n, \tilde{g}_n)$ tend to $\Psi(\tilde{u}_{0,1}, \tilde{f}_1, \tilde{g}_1)$ in $G_\Sigma^{m-1}((t_0, T) \times G)$ as $n \to \infty$. Lemma 6.3 thus shows that this convergence takes place in $G_\Sigma^m((t_0, T) \times G)$. So also part (3) is established. □

Remark 6.5. Reversing time and adapting coefficients, data and smallness assumptions accordingly, we can transfer the results of Theorem 6.4 to the negative time direction, cf. Remark 3.3 in [21].

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