

Non-autonomous Miyadera perturbations

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Abstract: We consider time dependent perturbations $B(t)$ of a non-autonomous Cauchy problem $\dot{v}(t) = A(t)v(t)$ (CP) on a Banach space X . The existence of a mild solution u of the perturbed problem is proved under Miyadera type conditions on $B(\cdot)$. In the parabolic case and $X = L^d(\Omega)$, $1 < d < \infty$, we show that u is differentiable a.e. and satisfies $\dot{u}(t) = (A(t) + B(t))u(t)$ for a.e. t . Our approach uses perturbation results due to one of the authors, [29], and S. Monniaux and J. Prüß, [14], which are applied to the evolution semigroup induced by the evolution family related to (CP). As an application we obtain solutions of a second order parabolic equation with singular lower order coefficients.

1. Introduction

Consider the non-autonomous linear abstract Cauchy problems

$$(CP) \begin{cases} \frac{d}{dt}u(t) = A(t)u(t), \\ u(s) = x, \quad t \geq s, \end{cases} \quad \text{and} \quad (pCP) \begin{cases} \frac{d}{dt}u(t) = (A(t) + B(t))u(t), \\ u(s) = x, \quad t \geq s, \end{cases}$$

on a Banach space X . In order to solve (pCP) we assume that (CP) is “well-posed” and that the operators $B(t)$ are “small with respect to $A(t)$ ”. In the autonomous case, i.e., $A(t) \equiv A$, $B(t) \equiv B$, and A generates a \mathcal{C}_0 -semigroup $(T(t))_{t \geq 0}$, there are various conditions on A and B ensuring that the operator $A + B$ is also a generator, and hence (pCP) is well-posed. For instance, if B is A -bounded and satisfies the *Miyadera condition*

$$\exists \alpha > 0, \beta \in [0, 1) \quad \text{such that} \quad \int_0^\alpha \|BT(t)x\| dt \leq \beta \|x\|, \quad x \in D(A), \quad (1.1)$$

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then $(A + B, D(A))$ generates a \mathcal{C}_0 -semigroup $(T_B(t))_{t \geq 0}$, [13], [29], where $D(A)$ is the domain of A .

In the present paper we prove a corresponding perturbation theorem in the non-autonomous situation extending a previous result by part of the authors, [24]. Here, instead of a \mathcal{C}_0 -semigroup $(T(t))_{t \geq 0}$, we consider an *evolution family* $\mathcal{U} = (U(t, s))_{(t, s) \in D}$ in the space $\mathcal{L}(X)$ of bounded linear operators on X , that is,

$$(E1) \quad U(s, s) = Id, \quad U(t, s) = U(t, r)U(r, s) \text{ for } t \geq r \geq s \text{ in } I \text{ and}$$

$$(E2) \quad \text{the mapping } D \ni (t, s) \mapsto U(t, s) \text{ is strongly continuous,}$$

where I is an interval in \mathbb{R} and $D = D_I := \{(t, s) \in I^2 : t \geq s\}$. Observe that for a \mathcal{C}_0 -semigroup $(T(t))_{t \geq 0}$ the operators $U(t, s) := T(t - s)$, $t \geq s$, define an evolution family with index set $D_{\mathbb{R}}$. Further, for $I \in \{[a, b], [a, \infty), \mathbb{R}\}$ the Cauchy problem (CP) is called *well-posed* (on spaces Y_s , $s \in I$, is dense in X and for $x \in Y_s$ there is a unique solution $u \in C^1(I \cap [s, \infty), X)$ of (CP) depending continuously on initial data. In this case there exists an evolution family \mathcal{U} such that $u(t) = U(t, s)x$, that is, \mathcal{U} solves (CP) , see [19].

In contrast to the autonomous case, it can happen that the mapping $t \mapsto U(t, s)x$ is differentiable only if $x = 0$ (e.g., take $X = \mathbb{C}$ and $U(t, s) = p(t)/p(s)$, where $p, 1/p \in C_b(\mathbb{R})$ and p is nowhere differentiable). Moreover, regularity properties of \mathcal{U} are not preserved even for bounded perturbations $B(\cdot)$, see [23, Ex. 6.4]. Thus, at first, we will not assume that the evolution family \mathcal{U} is differentiable in any sense and look for an evolution family \mathcal{U}_B such that the *variation of constants formula*

$$U_B(t, s)x = U(t, s)x + \int_s^t U(t, \tau)B(\tau)U_B(\tau, s)x \, d\tau \quad (1.2)$$

holds. If \mathcal{U} is related to (CP) then we say that $t \mapsto u(t) = U_B(t, s)x$ is a *mild solution* of (pCP) . Note that if \mathcal{U} solves (CP) on Y_s and $\tau \mapsto B(\tau)U_B(\tau, s)x$, $x \in Y_s$, is sufficiently smooth, then u satisfies (pCP) , see e.g. [22, §5.5, 5.7].

In Theorem 3.4 we show the existence of \mathcal{U}_B by assuming a non-autonomous version of the Miyadera condition (1.1). Our approach is based on the so-called *evolution semigroup* $\mathcal{T} = (T(t))_{t \geq 0}$ on $E = L^p(I, X)$ given by $(T(t)f)(s) := \chi_I(s - t)U(s, s - t)f(s - t)$, $f \in E$, see Section 2. We perturb \mathcal{T} by the multiplication operator $\mathcal{B} = B(\cdot)$ on E . By applying a Miyadera type perturbation theorem for semigroups, [29], and an abstract characterization of evolution semigroups (see Theorem 2.4) we then obtain a perturbed evolution semigroup \mathcal{T}_B . Finally, the associated evolution family \mathcal{U}_B satisfies (1.2).

In Section 4 we consider evolution families \mathcal{U} solving (CP) where the operators $A(t)$ generate analytic semigroups and perturbations $B(t)$ which are bounded

operators from the domain of a fractional power of $A(t)$ to X . For $X = L^d(\Omega)$, $1 < d < \infty$, we can show that $t \mapsto U_B(t, s)x$, $x \in X$, is differentiable for a.e. $t > s$ and satisfies (pCP) for a.e. $t > s$. Our proof relies on a recent perturbation result of Dore–Venni type by S. Monniaux and J. Pr  b, [14], which we apply to the evolution semigroup \mathcal{T} . Finally, in the last section, we use the above results to solve a second order parabolic equation with singular lower order coefficients.

2. A characterization of evolution semigroups

In this section we consider an evolution family $\mathcal{U} = (U(t, s))_{(t,s) \in D}$ on a Banach space X defined for a right–closed interval $I \subseteq \mathbb{R}$. We call \mathcal{U} *exponentially bounded* if there are constants $M \geq 1$ and $w \in \mathbb{R}$ such that $\|U(t, s)\| \leq Me^{w(t-s)}$ for $(t, s) \in D$. For compact intervals I this estimate always holds with $w = 0$ due to the principle of uniform boundedness. For exponentially bounded \mathcal{U} ,

$$(T(t)f)(s) := \chi_I(s-t)U(s, s-t)f(s-t) = \begin{cases} U(s, s-t)f(s-t), & \text{if } s-t \in I, \\ 0, & \text{if } s-t \notin I, \end{cases}$$

defines a bounded operator $T(t)$ on the Bochner–Lebesgue space $E = L^p(I, X)$, $1 \leq p < \infty$, where χ_I is the characteristic function of I . It is easily seen that $\mathcal{T} = (T(t))_{t \geq 0}$ is a strongly continuous semigroup on E , see e.g. [27], [28]. We call \mathcal{T} the *evolution semigroup* associated with \mathcal{U} and denote its generator by $(G, D(G))$. For further information concerning this approach we refer to [5], [9], [11], [12], [16]–[19], [21], [24]–[28], and the references therein.

The resolvent $R(\lambda, G) := (\lambda - G)^{-1}$, $\lambda \in \rho(G)$, has an important smoothing property which is stated below. For $I = \mathbb{R}$ this result is contained in [25, Prop. 3.3]. The general case can be shown analogously and the proof is therefore omitted, see also [28, Prop. 2.2]. For a similar result in the case of bounded I we refer to [17, Cor. 4.7]. By $C_{00}(I, X)$ we denote the space of continuous functions $f : I \rightarrow X$ vanishing at $a := \inf I$ and at infinity (if I is unbounded) endowed with the sup–norm $\|\cdot\|_\infty$. (Notice that $C_{00}(\mathbb{R}, X)$ is usually denoted by $C_0(\mathbb{R}, X)$.)

Proposition 2.1. *Let I be a right–closed interval in \mathbb{R} and $(U(t, s))_{(t,s) \in D}$ an exponentially bounded evolution family on X . Let $(G, D(G))$ be the generator of the associated evolution semigroup on $L^p(I, X)$, $1 \leq p < \infty$. Consider $\lambda \in \rho(G)$. Then $R(\lambda, G) : L^p(I, X) \rightarrow C_{00}(I, X)$ is continuous with dense image. In particular, $D(G)$ is dense in $C_{00}(I, X)$.*

Following an approach due to J. Howland, [9], we reduce the characterization of evolution semigroups to the characterization of multiplication operators on E .

We consider the space $C_b(I, \mathcal{L}_s(X))$ of bounded, strongly continuous functions $M(\cdot) : I \rightarrow \mathcal{L}(X)$, where $I \subseteq \mathbb{R}$ is an interval. Clearly, $M(\cdot) \in C_b(I, \mathcal{L}_s(X))$ induces a bounded *multiplication operator* \mathcal{M} on $L^p(I, X)$ by setting $\mathcal{M}f := M(\cdot)f(\cdot)$, and $\|\mathcal{M}\| = \|M(\cdot)\|_\infty$. We need the following notion (see also [17, Def. 4.4], [24, Def. 3.1]).

Definition 2.2. *Let $\mathcal{M} \in \mathcal{L}(L^p(I, X))$, $1 \leq p < \infty$. A subspace F of $L^p(I, X)$ is called \mathcal{M} -determining if*

(D1) *F and $\mathcal{M}F$ consist of continuous functions;*

(D2) *$F^s := \{f(s) : f \in F\}$ is dense in X for all $s \in I$;*

(D3) *F is dense in $L^p(I, X)$.*

For $I = \mathbb{R}$ the next result was shown in [24, Prop. 3.3]. The proof carries over to the case $I \neq \mathbb{R}$ with obvious changes. For related characterizations we refer to [5] and [9].

Proposition 2.3. *Let I be an interval in \mathbb{R} and X a Banach space. Consider $\mathcal{M} \in \mathcal{L}(L^p(I, X))$, $1 \leq p < \infty$. Then $\mathcal{M} = M(\cdot) \in C_b(I, \mathcal{L}_s(X))$ if and only if there is an \mathcal{M} -determining subspace F of $L^p(I, X)$ such that $\mathcal{M}(\varphi f) = \varphi \mathcal{M}f$ for $f \in F$ and $\varphi \in L^\infty(I)$.*

The right translations $\mathcal{R}(t)$, $t \in \mathbb{R}$, defined by $(\mathcal{R}(t)f)(s) := \chi_I(s-t)f(s-t)$ for $s \in I$ and $f \in E$, yield a \mathcal{C}_0 -semigroup $(\mathcal{R}(t))_{t \geq 0}$ on $E = L^p(I, X)$. For $X = \mathbb{C}$ we write $R(t)$ instead of $\mathcal{R}(t)$. Let $I_t := I \cap (I+t)$ for $t \in \mathbb{R}$. We identify the space $L^p(I_t, X)$ with a subspace of $E = L^p(I, X)$, $1 \leq p < \infty$, by extending functions by 0. Recall that a *core* of a closed operator $(A, D(A))$ is a subspace of $D(A)$ which is dense in the graph norm $\|x\|_A := \|x\| + \|Ax\|$. Further, $C_c^1(I)$ is the space of continuously differentiable functions with compact support. Finally, we make use of the following condition, cf. Proposition 2.1:

(R) There is $\lambda \in \rho(G)$ such that $R(\lambda, G) : L^p(I, X) \rightarrow C_{00}(I, X)$ is continuous with dense image $D(G)$.

If I contains its left endpoint a and \mathcal{U} is an exponentially bounded evolution family with index set D_I , then the restriction of \mathcal{U} to the index set $D_{I'}$, $I' := I \setminus \{a\}$, induces the same evolution semigroup as \mathcal{U} on $L^p(I, X)$. So, as in [17], in our characterization of evolution semigroups it suffices to consider a left half-open interval I , that is, I is left-open and right-closed. Observe that, in general, an evolution family defined for a left-open interval has no continuous

extension to $D_{\bar{f}}$. For instance, let $X = \mathbb{C}$, $I = (0, \infty)$, and $U(t, s) = p(t)/p(s)$ for $p(t) = 2 + \sin \frac{1}{t}$.

We now formulate the main result of this section. It extends [24, Thm. 3.4] where the case $I = \mathbb{R}$ was considered. A closely related result for bounded intervals I is due to H. Neidhardt, [17, Thm. 4.12]. Evolution semigroups on $C_{00}(I, X)$ were characterized in [5], [12], [17], [21].

Theorem 2.4. *Let I be a left half-open interval in \mathbb{R} . Consider a C_0 -semigroup $\mathcal{T} = (T(t))_{t \geq 0}$ on $E = L^p(I, X)$, $1 \leq p < \infty$, with generator $(G, D(G))$. Let \mathcal{D} be a core of G . Then the following assertions are equivalent:*

- (a) \mathcal{T} is an evolution semigroup induced by an exponentially bounded evolution family $(U(t, s))_{(t,s) \in \mathcal{D}}$.
- (b) Condition (R) holds, and we have $T(t)(\varphi f) = (R(t)\varphi)T(t)f$ for $\varphi \in L^\infty(I)$, $f \in E$, and $t \geq 0$.
- (c) Condition (R) holds, and for $f \in \mathcal{D}$ and $\varphi \in C_c^1(I)$ we have $\varphi f \in D(G)$ and

$$G(\varphi f) = -\varphi' f + \varphi Gf. \quad (2.1)$$

- (d) Condition (R) holds, and there is $\mu \in \rho(G)$ such that $R(\mu, G)(\varphi' R(\mu, G)f) = \varphi R(\mu, G)f - R(\mu, G)(\varphi f)$ for $f \in E$ and $\varphi \in C_c^1(I)$.

Proof. By Proposition 2.1, assertion (a) implies (R). The implications (a) \Rightarrow (c) \Rightarrow (b) and (c) \Leftrightarrow (d) can be shown as in [24, Thm. 3.4], cf. [28, Thm. 2.6].

(b) \Rightarrow (a): Observe that the assumption implies $T(t)f \in L^p(I_t, X)$ for $f \in E$ and $t \geq 0$. Set $\mathcal{M}_t := T(t)\mathcal{R}(-t)$ and $F_t := \mathcal{R}(t)D(G)$ for $t \geq 0$. By (R) the space F_t consists of continuous functions vanishing on $I \setminus I_t$. We have

$$\mathcal{M}_t(\varphi f) = (R(t)R(-t)\varphi)T(t)\mathcal{R}(-t)f = \varphi \mathcal{M}_t f$$

for $f \in E$ and $\varphi \in L^\infty(I)$ with support in I_t . In particular, \mathcal{M}_t leaves $L^p(I_t, X)$ invariant. From (R) we derive that F_t is dense in $L^p(I_t, X)$ and $C_{00}(I_t, X)$. Thus, F_t satisfies (D2) and (D3) for the interval I_t . Further, let $f \in F_t$. Then $f = \mathcal{R}(t)g$ for some $g \in D(G)$, and

$$\mathcal{M}_t f = T(t)\mathcal{R}(-t)\mathcal{R}(t)g = T(t)(\chi_{I_{-t}}g) = \chi_{I_t}T(t)g = T(t)g \in D(G) \subseteq C_{00}(I, X).$$

Hence, F_t is \mathcal{M}_t -determining on $L^p(I_t, X)$, $t \geq 0$. By Proposition 2.3 there are bounded operators $M(t, s)$, $s \in I_t$, such that $\mathcal{M}_t = M(t, \cdot) \in C_b(I_t, \mathcal{L}_s(X))$,

$t \geq 0$. Moreover, $\|M(t, s)\| \leq \|\mathcal{M}_t\| \leq Me^{wt}$ for some constants $M \geq 1$ and $w \in \mathbb{R}$. Notice that $T(t) = M(t, \cdot)\mathcal{R}(t)$. We set $U(t, s) := M(t - s, t)$ for $(t, s) \in D_I$. Then

$$T(t)f(s) = \chi_I(s - t)U(s, s - t)f(s - t)$$

for $f \in E$, $s \in I$, and $t \geq 0$. From the semigroup law for \mathcal{T} one easily derives (E1) for \mathcal{U} , see e.g. [17, p.291] or [24, p.524]. It remains to show the strong continuity of $M(\cdot, \cdot)$. Fix $t \geq 0$ and $s \in I_t$. By (D2) and the exponential bound for $M(t, s)$ it suffices to consider $x \in F_t^s$. So let $f \in D(G)$ with $f(s - t) = x$. Set $g = (\lambda - G)f$. Then, $T(t)f = \mathcal{R}(\lambda, G)T(t)g \in C_{00}(I, X)$ by (R). Thus we obtain

$$\begin{aligned} & \|M(t', s')x - M(t, s)x\| \\ &= \|M(t', s')(f(s - t) - f(s' - t)) + M(t', s')(f(s' - t) - f(s' - t')) \\ &\quad + M(t', s')f(s' - t') - M(t, s')f(s' - t) + M(t, s')f(s' - t) - M(t, s)f(s - t)\| \\ &\leq Me^{wt'} (\|\mathcal{R}(t)f(s) - \mathcal{R}(t)f(s')\| + \|\mathcal{R}(t)f - \mathcal{R}(t')f\|_\infty) \\ &\quad + \|T(t')f - T(t)f\|_\infty + \|T(t)f(s') - T(t)f(s)\| \end{aligned}$$

for $t' \geq 0$ and $s' \in I_{t'} \cap I_t$. The first, second, and fourth summand converge to 0 as $(t', s') \rightarrow (t, s)$ since $f, T(t)f \in C_{00}(I, X)$. For the third summand we use

$$\|T(t')f - T(t)f\|_\infty \leq \|R(\lambda, G)\|_{\mathcal{L}(L^p, C_{00})} \|T(t')g - T(t)g\|_p$$

which yields the assertion. \square

Remark 2.5.

- (a) Implication “(b) \Rightarrow (a)” is false if we drop condition (R). Consider for instance $X = \mathbb{C}$, $E = L^1(\mathbb{R})$, and $U(t, s) = p(t)/p(s)$, where $p, 1/p \in L^\infty(\mathbb{R})$ are discontinuous, but $p(r) \rightarrow p(t)$ as $r \nearrow t$ for a.e. $t \in \mathbb{R}$. See also [5, Thm. 6.4] and [9, Thm. 1].
- (b) The results of this section remain valid if we replace $L^p(I, X)$ by an X -valued Banach function space $E(X)$ such that the norm on E is order continuous and the translations $R(t)$, $t \in \mathbb{R}$, are uniformly bounded on E , see [24] and [28].

3. Miyadera perturbations

In this section we show a perturbation result for evolution families by applying Theorem 2.4 and the following Miyadera type perturbation theorem proved in

[29, Thm. 1]. The proof of part (d) of the theorem uses an idea contained in the proof of [30, Prop. 1.3]. Let $(A, D(A))$ and $(B, D(B))$ be linear operators. Recall that B is A -bounded if $D(A) \subseteq D(B)$ and $\|Bx\| \leq a\|x\| + b\|Ax\|$ for $x \in D(A)$ and constants $a, b \geq 0$. Observe that if there exists $\lambda \in \rho(A)$ then B is A -bounded if and only if $D(A) \subseteq D(B)$ and $BR(\lambda, A)$ is closed (and hence bounded).

Theorem 3.1. *Let $\mathcal{T} = (T(t))_{t \geq 0}$ be a \mathcal{C}_0 -semigroup on a Banach space F with generator $(G, D(G))$. Consider a dense subspace \mathcal{D} of F and a linear operator $(B, D(B))$ such that*

- (i) $T(t)\mathcal{D} \subseteq \mathcal{D} \subseteq D(G) \cap D(B)$ and $[0, \infty) \ni t \mapsto BT(t)f \in F$ is continuous for $f \in \mathcal{D}$;
- (ii) there are constants $\alpha > 0$ and $\gamma \in [0, 1)$ such that $\int_0^\alpha \|BT(t)f\| dt \leq \gamma \|f\|$ for $f \in \mathcal{D}$.

Then the following assertions hold:

- (a) The operator $(G + B, \mathcal{D})$ is closable. Its closure $(G_B, D(G_B))$ generates a \mathcal{C}_0 -semigroup $\mathcal{T}_B = (T_B(t))_{t \geq 0}$ and satisfies $D(G_B) = D(G)$. The operator (B, \mathcal{D}) possesses a unique G -bounded extension \hat{B} to $D(G)$ and we have $G_B = G + \hat{B}$. Further, \mathcal{D} is a core of G and G_B .
- (b) For sufficiently large real $\lambda \in \rho(G) \cap \rho(G_B)$ there exists an invertible operator $C_\lambda \in \mathcal{L}(F)$ such that $R(\lambda, G_B) = R(\lambda, G)C_\lambda$.
- (c) For $f \in D(G)$ and $t \geq 0$ we have

$$T_B(t)f = T(t)f + \int_0^t T_B(t-\tau)\hat{B}T(\tau)f d\tau, \quad (3.1)$$

$$T_B(t)f = T(t)f + \int_0^t T(t-\tau)\hat{B}T_B(\tau)f d\tau. \quad (3.2)$$

Moreover, \mathcal{T}_B is the only \mathcal{C}_0 -semigroup satisfying (3.1) for $f \in \mathcal{D}$.

- (d) In addition, assume that B is closed. Then $\hat{B}f = Bf$ for $f \in D(G)$. Moreover, for all $f \in F$ we have $T(t)f, T_B(t)f \in D(B)$ for a.e. $t \geq 0$, the functions $BT(\cdot)f$ and $BT_B(\cdot)f$ are locally integrable on \mathbb{R}_+ , and (3.1) and (3.2) hold for all $t \geq 0$ and $f \in F$ with \hat{B} replaced by B .

Proof. By [15, A-I.1.9] \mathcal{D} is a core of G . Assertion (a) and (b) and uniqueness follow from [29], Thm. 1, Remark 2, and (1.7). Equation (3.1) is shown in [29, Thm. 1] for $f \in \mathcal{D}$. Since \mathcal{D} is a core of G and \hat{B} is G -bounded, (3.1) holds

for $f \in D(G)$ by the dominated convergence theorem. For $0 \leq \tau \leq t$ and $f \in D(G) = D(G_B)$ we have

$$\begin{aligned} \frac{d}{d\tau} T(t-\tau)T_B(\tau)f &= -T(t-\tau)GT_B(\tau)f + T(t-\tau)G_B T_B(\tau)f \\ &= T(t-\tau)\hat{B}T_B(\tau)f. \end{aligned}$$

Integration from 0 to t yields (3.2).

Now assume that B is closed. Then \hat{B} is the restriction of B to $D(G)$, [29, Cor. 4]. Further, (ii) holds for all $f \in D(G)$ by [30, Thm. 1.1]. Moreover, [20, Lemma 1.1] yields $\int_0^\alpha \|BT_B(t)f\| dt \leq \frac{\gamma}{1-\gamma} \|f\|$ for every $f \in D(G)$. One easily sees that this implies

$$\int_0^t \|BT(\tau)f\| d\tau \leq c \|f\| \quad \text{and} \quad \int_0^t \|BT_B(\tau)f\| d\tau \leq \tilde{c} \|f\| \quad (3.3)$$

for $f \in D(G)$, $t \geq 0$, and constants c, \tilde{c} depending only on t . Next, fix $f \in F$. Choose $f_n \in D(G)$ converging to f . By (3.3) the functions $BT(\cdot)f_n$ and $BT_B(\cdot)f_n$ converge in $L^1([0, t], F)$, and thus (by passing to subsequences) they converge pointwise a.e.. Since B is closed, we derive $T(t)f, T_B(t)f \in D(B)$ for a.e. $t \geq 0$ and $BT(\cdot)f$ and $BT_B(\cdot)f$ are locally integrable on \mathbb{R}_+ . Moreover, the estimates (3.3) hold for all $f \in F$ and $t \geq 0$. Observe that (3.1) and (3.2) are satisfied for f_n with \hat{B} replaced by B . Hence, due to (3.3) the last assertion in (d) follows by approximation. \square

In the sequel we need the following measure theoretical lemma which is essentially shown in [17, Thm. 4.2], see also [26, Lemma 2.2]. Throughout “measurable” means “strongly measurable” and the integrals are Bochner integrals.

Lemma 3.2. *Let X be a Banach space and let I be an interval in \mathbb{R} . Assume that $u : [c, d] \rightarrow L^p(I, X)$, $1 \leq p < \infty$, is integrable. Then there is a measurable function $\Phi : [c, d] \times I \rightarrow X$ such that for a.e. $t \in [c, d]$ we have $u(t)(s) = \Phi(t, s)$ for a.e. $s \in I$. Moreover,*

$$\Phi(\cdot, s) \text{ is integrable and } \left(\int_c^d u(t) dt \right) (s) = \int_c^d \Phi(t, s) dt \quad (3.4)$$

for a.e. $s \in I$. If, in addition, there is a measurable function $\tilde{u} : [c, d] \times I \rightarrow X$ such that for a.e. $t \in [c, d]$ we have $u(t)(s) = \tilde{u}(t, s)$ for a.e. $s \in I$, then (3.4) holds with Φ replaced by \tilde{u} .

We often use the fact that for a measurable function $u : I \times J \rightarrow X$ and intervals $I, J \subseteq \mathbb{R}$ we have $u(t, s) = 0$ for a.e. $(t, s) \in I \times J$ if and only if for a.e. $t \in I$ we have $u(t, s) = 0$ for a.e. $s \in J$. This is an easy consequence of Fubini’s theorem.

Let I be a left half-open interval in \mathbb{R} . Consider an exponentially bounded evolution family $\mathcal{U} = (U(t, s))_{(t, s) \in D}$ on a Banach space X . Let $\mathcal{T} = (T(t))_{t \geq 0}$ be the associated evolution semigroup on $E = L^p(I, X)$, $1 \leq p < \infty$, with generator $(G, D(G))$. In the next lemma we describe a certain class of functions contained in $D(G)$. To simplify notation we set $U(t, s) := 0$ for $t < s$.

Lemma 3.3. *Let $x \in X$, $r \in I$, and $\varphi \in C_c^1(I)$ with $\varphi(s) = 0$ for $s \leq r$. Set $f(\cdot) := \varphi(\cdot)U(\cdot, r)x$. Then $T(t)f = (R(t)\varphi)U(\cdot, r)x$ and $f \in D(G)$.*

For the straightforward proof we refer to [28, Lemma 1.12], see also [25, Lemma 2.1]

We now introduce the assumptions on the operators $(B(t), D(B(t)))$, $t \in I$, on X which are needed for our perturbation result. For $q \geq 1$ we denote by q' the conjugate exponent satisfying $\frac{1}{q} + \frac{1}{q'} = 1$.

- (M) There are dense subspaces Y_t , $t \in I$, of X such that $U(t, s)Y_s \subseteq Y_t$ for $(t, s) \in D$. Moreover, for all $s \in I$ and $y \in Y_s$ we have $U(t, s)y \in D(B(t))$ for a.e. $t \in I \cap [s, \infty)$, $B(\cdot)U(\cdot, s)y$ is measurable, and

$$\int_0^\alpha \chi_I(s+t) \|B(s+t)U(s+t, s)y\|^p dt \leq \beta^p \|y\|^p \quad (3.5)$$

for constants $p \geq 1$, $\alpha > 0$ and $\beta \geq 0$ such that $\gamma := \alpha^{1/p'} \beta < 1$.

It is straightforward to see that (M) implies

$$\int_0^d \chi_I(s+t) \|B(s+t)U(s+t, s)y\|^p dt \leq c \|y\|^p \quad (3.6)$$

for $s \in I$, $d \geq 0$, and $y \in Y_s$, where the constant c is independent of s and y . Observe that if (M) is satisfied for $p > 1$, $\alpha > 0$, and $\beta \geq 0$ then, by Hölder's inequality, (M) is also true for $p = 1$, $\alpha > 0$, and $\gamma \in [0, 1)$. Clearly, (M) holds if $Y_t \equiv X$ and $B(\cdot) \subseteq \mathcal{L}(X)$ such that $\|B(t)\| \leq C$ for a.e. $t \in I$ and $B(\cdot)x$ is measurable for $x \in X$. Other examples will be studied in Section 4 and 5.

On $E = L^p(I, X)$ we define the linear operator \mathcal{B} by setting $\mathcal{B}f := B(\cdot)f(\cdot)$ and $D(\mathcal{B}) := \{f \in E : f(s) \in D(B(s)) \text{ for a.e. } s \in I, B(\cdot)f(\cdot) \in E\}$. Further, we consider the space

$$\mathcal{D} := \text{lin} \{f \in E : f(\cdot) = \varphi(\cdot)U(\cdot, r)y, r \in I, y \in Y_r, \varphi \in C_c^1(I) \text{ with } \varphi(s) = 0, s \leq r\}.$$

We now state the main result of this section. It generalizes [24, Thm. 4.2] where the case $U(t, s) = e^{(t-s)A}$ was considered; cf. [18, Thm 4.3] for the case $U(t, s) = Id$. Bounded perturbations $B(\cdot)$ were treated in [26, Thm. 2.3] with the same approach; see also the references therein.

Theorem 3.4. *Let I be a left half-open interval in \mathbb{R} . Let $\mathcal{U} = (U(t, s))_{(t,s) \in D}$ be an exponentially bounded evolution family on X and $(B(t), D(B(t)))$, $t \in I$, linear operators on X satisfying (M). Denote by $(G, D(G))$ the generator of the associated evolution semigroup \mathcal{T} on $E = L^p(I, X)$. Then the following assertions hold:*

(a) *There is a unique exponentially bounded evolution family $\mathcal{U}_B = (U_B(t, s))_{(t,s) \in D}$ satisfying*

$$U_B(t, s)f(s) = U(t, s)f(s) + \int_s^t U_B(t, \tau)B(\tau)U(\tau, s)f(s) d\tau \quad (3.7)$$

for $f \in \mathcal{D}$ and $(t, s) \in D$.

(b) *Assume, in addition, that $(\mathcal{B}, D(\mathcal{B}))$ is G -bounded and denote by $(G_B, D(G_B))$ the generator of the evolution semigroup \mathcal{T}_B on E associated with \mathcal{U}_B . Then $D(G) = D(G_B) \subseteq D(\mathcal{B})$ and $G_B f = Gf + \mathcal{B}f$ for $f \in D(G)$. Moreover, $(\mathcal{B}, D(\mathcal{B}))$ is G_B -bounded.*

(c) *Suppose now that $(B(t), D(B(t)))$ is closed for a.e. $t \in I$. Then \mathcal{B} is G -bounded, hypothesis (M) is valid with $Y_t = X$ for all $t \in I$, and*

$$U_B(t, s)x = U(t, s)x + \int_s^t U_B(t, \tau)B(\tau)U(\tau, s)x d\tau \quad (3.8)$$

holds for all $x \in X$ and $(t, s) \in D$. Further, for $x \in X$ and $s \in I$ we have $U_B(t, s)x \in D(B(t))$ for a.e. $t \in I \cap [s, \infty)$, the function $B(\cdot)U_B(\cdot, s)x$ is locally integrable and

$$U_B(t, s)x = U(t, s)x + \int_s^t U(t, \tau)B(\tau)U_B(\tau, s)x d\tau \quad (3.9)$$

holds for all $x \in X$ and $(t, s) \in D$.

Proof. (a) At first we verify conditions (i) and (ii) in Theorem 3.1 for \mathcal{T} , G , \mathcal{B} , and \mathcal{D} as defined above. By Lemma 3.3 we have $T(t)\mathcal{D} \subseteq \mathcal{D} \subseteq D(G)$. Using the arguments of the proof of [11, Prop. 2.9] it is easy to show that \mathcal{D} is dense in E , cf. [28, Prop. 1.13]. Moreover, (3.6) yields $\mathcal{D} \subseteq D(\mathcal{B})$. Let $f(\cdot) = \varphi(\cdot)U(\cdot, r)y$ for $r \in I$, $y \in Y_r$, and $\varphi \in C_c^1(I)$ with $\varphi(s) = 0$ for $s \leq r$. Then for $0 \leq t, t' \leq d$ we have

$$\begin{aligned} \|\mathcal{B}T(t')f - \mathcal{B}T(t)f\|_p^p &= \int_I \|(R(t')\varphi(s) - R(t)\varphi(s))B(s)U(s, r)y\|^p ds \\ &\leq \|R(t')\varphi - R(t)\varphi\|_\infty \int_r^c \|B(s)U(s, r)y\|^p ds, \end{aligned}$$

where c only depends on d and φ . By (3.6) this yields (i). In order to check (ii) we consider a function $f \in \mathcal{D}$ given by

$$f(s) = \sum_{k=1}^n \varphi_k(s) U(s, s_k) y_k.$$

Notice that due to (M) the mapping $(t, s) \mapsto \chi_I(s+t) \|B(s+t)U(s+t, s)f(s)\|^p$ is measurable. We compute

$$\begin{aligned} \int_0^\alpha \|\mathcal{B}T(t)f\|_p dt &= \int_0^\alpha \left(\int_I \left\| \sum_{k=1}^n \chi_I(s-t) \varphi_k(s-t) B(s)U(s, s_k) y_k \right\|^p ds \right)^{1/p} dt \\ &= \int_0^\alpha \left(\int_{I-t} \left\| \sum_{k=1}^n \varphi_k(s) B(s+t)U(s+t, s)U(s, s_k) y_k \right\|^p ds \right)^{1/p} dt \\ &= \int_0^\alpha \left(\int_I \chi_I(s+t) \|B(s+t)U(s+t, s)f(s)\|^p ds \right)^{1/p} dt \\ &\leq \alpha^{1/p'} \left(\int_0^\alpha \int_I \chi_I(s+t) \|B(s+t)U(s+t, s)f(s)\|^p ds dt \right)^{1/p} \\ &= \alpha^{1/p'} \left(\int_I \int_0^\alpha \chi_I(s+t) \|B(s+t)U(s+t, s)f(s)\|^p dt ds \right)^{1/p} \\ &\leq \alpha^{1/p'} \beta \left(\int_I \|f(s)\|^p ds \right)^{1/p} \\ &= \alpha^{1/p'} \beta \|f\|_p, \end{aligned}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. (Here we have used Hölder's inequality, Fubini's theorem, and (3.5).) By setting $\gamma := \alpha^{1/p'} \beta < 1$, we obtain (ii).

Hence, by Theorem 3.1, the operator $(\mathcal{B}, \mathcal{D})$ can be extended to a G -bounded operator $\hat{\mathcal{B}}$ on $D(G)$, and $G_B := G + \hat{\mathcal{B}}$ with $D(G_B) := D(G)$ generates a \mathcal{C}_0 -semigroup \mathcal{T}_B on E . Also, \mathcal{D} is a core for G_B . Since G generates an evolution semigroup, G_B satisfies (2.1) in Theorem 2.4 for $f \in \mathcal{D}$. Further, (R) holds for $R(\lambda, G)$, $\lambda \in \rho(G)$, due to Proposition 2.1. So, by Theorem 3.1(b), $R(\lambda, G_B)$ satisfies (R) for $\lambda \in \mathbb{R}$ sufficiently large. Thus Theorem 2.4 shows that \mathcal{T}_B is an evolution semigroup induced by an exponentially bounded evolution family \mathcal{U}_B .

Fix $f \in \mathcal{D}$ and $t \geq 0$. Equation (3.1) implies

$$\mathcal{R}(-t)T_B(t)f = \mathcal{R}(-t)T(t)f + \int_0^t u(\tau)d\tau,$$

where $u(\tau) := \mathcal{R}(-t)T_B(t-\tau)\mathcal{B}T(\tau)f$, $0 \leq \tau \leq t$. The function $u : [0, t] \rightarrow E$ is continuous. For each $\tau \in [0, t]$ we have

$$(u(\tau))(s) = \chi_I(s+t) U_B(s+t, s+\tau) B(s+\tau) U(s+\tau, s) f(s) =: \tilde{u}(\tau, s)$$

for a.e. $s \in I$. Since the mapping $\tilde{u} : [0, t] \times I \rightarrow X$ is measurable, Lemma 3.2 implies

$$U_B(s+t, s)f(s) = U(s+t, s)f(s) + \int_s^{s+t} U_B(s+t, \tau)B(\tau)U(\tau, s)f(s) d\tau \quad (3.10)$$

for $f \in \mathcal{D}$, $t \geq 0$, and a.e. $s \in I$ with $s+t \in I$. Next we show that in fact (3.10) holds for all $s \in I$ such that $s+t \in I$.

For this it suffices to prove that the integral in (3.10) is continuous from the left with respect to s . First notice that $f(s) \in Y_s$ for $s \in I$ and $f \in \mathcal{D}$ so that due to (3.6) the integral in (3.10) is defined for all $(s+t, s) \in D$. Clearly, we only have to consider functions of the form $f(\cdot) = \varphi(\cdot)U(\cdot, r_0)y$ for $r_0 \in I$, $y \in Y_{r_0}$, and $\varphi \in C_c^1(I)$ with $\varphi(r) = 0$ for $r \leq r_0$. Further we may assume $t > 0$ and $s > r_0$. Let $s' \in [r_0, s]$ with $s'+t \geq s$. Then we obtain

$$\begin{aligned} & \int_s^{s+t} U_B(s+t, \tau)B(\tau)U(\tau, s)f(s) d\tau - \int_{s'}^{s'+t} U_B(s'+t, \tau)B(\tau)U(\tau, s')f(s') d\tau \\ &= \int_s^{s+t} \chi_{[s'+t, s+t]}(\tau) U_B(s+t, \tau)B(\tau)U(\tau, s)f(s) d\tau \\ & \quad + \int_s^{s+t} \chi_{[s, s'+t]}(\tau) (U_B(s+t, \tau) - U_B(s'+t, \tau)) B(\tau)U(\tau, s)f(s) d\tau \\ & \quad + \int_s^{s'+t} U_B(s'+t, \tau)B(\tau)U(\tau, s) (f(s) - U(s, s')f(s')) d\tau \\ & \quad - \varphi(s') \int_{r_0}^s \chi_{[s', s]}(\tau) U_B(s'+t, \tau)B(\tau)U(\tau, r_0)y d\tau. \end{aligned}$$

The integrands of the first, second, and fourth term tend to 0 for a.e. τ as $s' \nearrow s$, and they are bounded by $C_1 \|B(\tau)U(\tau, s)f(s)\|$ and $C_2 \|B(\tau)U(\tau, r_0)y\|$, respectively, where C_k are constants independent of s' and τ . So the first, second, and fourth integral converge to 0 as $s' \nearrow s$ due to (3.6) and the dominated convergence theorem. Since $f(s) - U(s, s')f(s') \in Y_s$, by (3.6) the third summand can be estimated from above by $C_3 \|f(s) - U(s, s')f(s')\|$, where C_3 does not depend on s' . Since f is continuous and U strongly continuous this term also tends to 0 as $s' \nearrow s$.

In order to show uniqueness of \mathcal{U}_B , let \mathcal{V} be an exponentially bounded evolution family satisfying (3.7) for $f \in \mathcal{D}$. Let $(S(t))_{t \geq 0}$ be the associated evolution semigroup on E . As above, Lemma 3.2 implies that $(S(t))_{t \geq 0}$ satisfies (3.1) for $f \in \mathcal{D}$ and $t \geq 0$. Consequently, by Theorem 3.1, $S(t) = T_B(t)$, and thus $\mathcal{V} = \mathcal{U}_B$. This establishes (a).

(b) Since $(\mathcal{B}, D(\mathcal{B}))$ is G -bounded, we have $\hat{\mathcal{B}}f = \mathcal{B}f$ for $f \in D(G)$ by Theorem 3.1(a). Also, \mathcal{B} is G_B -bounded by Theorem 3.1(b). The other assertions in (b) follow from Theorem 3.1(a).

(c) Observe that $(\mathcal{B}, D(\mathcal{B}))$ is closed due to our additional assumption. Hence, the first assertion in (c) holds by Theorem 3.1(d),(a). The remaining statements are shown for \mathcal{U} and \mathcal{U}_B , separately.

(i) Let $s \in I$, $d \geq 0$, and c as in (3.6). Since Y_s is dense and $B(t)$ is closed for a.e. $t \in I$ one concludes that $B(\cdot)U(\cdot, s)y \in L^p([s, s+d], X)$ and (3.6) holds for all $y \in X$, cf. the proof of Theorem 3.1(d). This shows (M) with $Y_s = X$. Let $(t, s) \in D$. Since $\{f(s) : f \in \mathcal{D}\}$ is dense in X and all terms in (3.7) are continuous with respect to $x = f(s)$, this equation carries over to all $x \in X$.

(ii) Next, we treat the assertions concerning \mathcal{U}_B . Fix $f \in D(G)$ and $t \geq 0$. We claim that

$$\tilde{h} : [0, t] \times I; (\tau, s) \mapsto \chi_I(s - \tau) B(s)U_B(s, s - \tau)f(s - \tau) \quad \text{is measurable.} \quad (3.11)$$

We define $g(\tau) := T_B(\tau)f$ for $\tau \in [0, t]$ and $\tilde{g}(\tau, s) := \chi_I(s - \tau)U_B(s, s - \tau)f(s - \tau)$ for $(\tau, s) \in [0, t] \times I$. Theorem 3.1(b) implies that $\tau \mapsto g(\tau)$ is continuous with respect to the graph norm of G . Fix $n \in \mathbb{N}$. There exist disjoint intervals $J_{n,k} = [t_{n,k}, t_{n,k+1})$ and functions $h_{n,k} \in \mathcal{D}$, $k = 1, \dots, m(n)$, such that $[0, t] = \bigcup_k \overline{J_{n,k}}$ and

$$\|h_{n,k} - g(\tau)\|_p + \|G(h_{n,k} - g(\tau))\|_p \leq \frac{1}{n} \quad \text{for } \tau \in J_{n,k}. \quad (3.12)$$

Let $h_n(\tau) := h_{n,k}$ and $\tilde{h}_n(\tau, s) := h_{n,k}(s)$ for $\tau \in J_{n,k}$, $s \in I$, and $n \in \mathbb{N}$. From (3.12) we infer that \tilde{h}_n converges to \tilde{g} in $L^p([0, t] \times I, X)$. Further, (M) implies that $[0, t] \times I \ni (\tau, s) \mapsto B(s)\tilde{h}_n(\tau, s)$ is measurable. Since \mathcal{B} is G -bounded, inequality (3.12) yields

$$\|\mathcal{B}h_n(\tau) - \mathcal{B}g(\tau)\|_p \leq c \|h_{n,k} - g(\tau)\|_p + c \|G(h_{n,k} - g(\tau))\|_p \leq \frac{c}{n}$$

for each $\tau \in J_{n,k}$ and a constant c . Hence, the function $(\tau, s) \mapsto B(s)\tilde{h}_n(\tau, s)$ converges in $L^p([0, t] \times I, X)$ to a function Φ . By passing to subsequences, we can assume that $\tilde{h}_n(\tau, s) \rightarrow \tilde{g}(\tau, s)$ and $B(s)\tilde{h}_n(\tau, s) \rightarrow \Phi(\tau, s)$ for a.e. (τ, s) as $n \rightarrow \infty$. Since $B(s)$ is closed for a.e. $s \in I$, we obtain that $\tilde{g}(\tau, s) \in D(B(s))$ and $B(s)\tilde{h}_n(\tau, s) \rightarrow B(s)\tilde{g}(\tau, s)$ for a.e. (τ, s) . This proves the claim.

If $f \in D(G)$ and $t \geq 0$, then Theorem 3.1(d) yields

$$\begin{aligned} \mathcal{R}(-t)T_B(t)f &= \mathcal{R}(-t)T(t)f + \int_0^t v(\tau)d\tau, \quad \text{where} \\ v(\tau) &:= \mathcal{R}(-t)T(t - \tau)\mathcal{B}T_B(\tau)f \end{aligned}$$

for $\tau \in [0, t]$. Moreover, $v : [0, t] \rightarrow E$ is integrable and for each $\tau \in [0, t]$ we have

$$v(\tau)(s) = \chi_I(s + t)U(s + t, s + \tau)B(s + \tau)U_B(s + \tau, s)f(s) =: \tilde{v}(\tau, s)$$

for a.e. $s \in I$. By (3.11) the function $\tilde{v} : [0, t] \times I \rightarrow X$ is measurable. Thus,

$$U_B(s+t, s)f(s) = U(s+t, s)f(s) + \int_s^{s+t} U(s+t, \tau)B(\tau)U_B(\tau, s)f(s) d\tau \quad (3.13)$$

for a.e. $s \in I$ with $s+t \in I$ due to Lemma 3.2.

As mentioned above, (M) also holds for $p = 1$ and with β replaced by $\gamma \in [0, 1)$. Therefore, for the remainder of the proof, we may and shall assume $p = 1$. For $f \in D(G)$, the function \tilde{h} defined in (3.11) is measurable, and from (3.3) we derive

$$\int_0^t \int_I \|\tilde{h}(\tau, s)\| ds d\tau = \int_0^t \|\mathcal{B}T_B(\tau)f\|_{L^1(I, X)} d\tau \leq \tilde{c} \|f\|_{L^1(I, X)}. \quad (3.14)$$

Now, fix $f \in L^1(I, X)$. Choose $f_n \in D(G)$ such that $f_n \rightarrow f$ in $L^1(I, X)$, and denote by \tilde{h}_n the measurable function corresponding to f_n . Then (3.14), applied to the difference $f_n - f_m$, shows that the functions \tilde{h}_n converge in $L^1([0, t] \times I, X)$; in particular, a subsequence converges for a.e. (τ, s) . For a subsubsequence, $U_B(s, s-\tau)f_n(s-\tau)$ tends to $U_B(s, s-\tau)f(s-\tau)$ for a.e. (τ, s) . Since $B(s)$ is closed for a.e. $s \in I$ we conclude that $\chi_I(s-\tau)U_B(s, s-\tau)f(s-\tau) \in D(B(s))$ for a.e. (τ, s) and that (3.11) and (3.14) hold for all $f \in L^1(I, X)$. Note that, using Fubini's theorem, equation (3.14) can be rewritten as

$$\int_I \int_0^t \chi_I(s+\tau) \|B(s+\tau)U_B(s+\tau, s)f(s)\| d\tau ds \leq \tilde{c} \|f\|_{L^1(I, X)}. \quad (3.15)$$

Fix $x \in X$ and $s_0 \in I$. For $0 < r < \frac{t}{2}$ with $s_0 + r \in I$ we define $f(s) = \chi_{[s_0, s_0+r]}(s)U_B(s, s_0)x$. With this f , we have $B(s+\tau)U_B(s+\tau, s)f(s) = B(s+\tau)U_B(s+\tau, s_0)x$ for $s \in [s_0, s_0+r]$ and $s+\tau \in I$. Hence, $B(\cdot)U_B(\cdot, s_0)x$ is measurable and estimate (3.15) implies

$$\begin{aligned} \tilde{c} \int_{s_0}^{s_0+r} \|U_B(s, s_0)x\| ds &\geq \int_{s_0}^{s_0+r} \int_0^t \chi_I(s+\tau) \|B(s+\tau)U_B(s+\tau, s_0)x\| d\tau ds \\ &\geq \int_{s_0}^{s_0+r} \int_r^{t-r} \chi_I(s_0+\tau) \|B(s_0+\tau)U_B(s_0+\tau, s_0)x\| d\tau ds \\ &= r \int_r^{t-r} \chi_I(s_0+\tau) \|B(s_0+\tau)U_B(s_0+\tau, s_0)x\| d\tau. \end{aligned}$$

Dividing by r and letting $r \rightarrow 0$ we obtain

$$\int_0^t \chi_I(s_0+\tau) \|B(s_0+\tau)U_B(s_0+\tau, s_0)x\| d\tau \leq \tilde{c} \|x\|. \quad (3.16)$$

Applying (3.13) and (3.16) we can now conclude the proof in the same way as we showed that (3.8) holds for all $t \geq s$ and $x \in X$: First, instead of $f \in \mathcal{D}$ we

take functions of the form $g = \varphi(\cdot)U_B(\cdot, s)x$, where $x \in X$, $s \in I$, and $\varphi \in C_c^1(I)$ with $\varphi(r) = 0$ for $r \leq s$. By Lemma 3.3 we have $g \in D(G_B) = D(G)$. Further, we only used (3.10), strong continuity of \mathcal{U} , and estimates as in (M) in the proof of (3.8). Since the corresponding hypotheses are satisfied this proof implies (3.9). \square

The above result can be extended to closed intervals $I \subseteq \mathbb{R}$ with $a = \min I \in \mathbb{R}$. Assume that (M) holds for an exponentially bounded evolution family $\mathcal{U} = (U(t, s))_{(t, s) \in D_I}$ on X and linear operators $(B(t), D(B(t)))$, $t \in I$. Set $J := (a - 1, \infty) \cap I$. Define $\tilde{B}(t) := B(t)$ and $\tilde{Y}_t := Y_t$ for $t \in I$ and $\tilde{B}(t) := 0$ and $\tilde{Y}_t := Y_a$ for $t \in (a - 1, a)$. Define an evolution family $\tilde{\mathcal{U}}$ with index set D_J by setting

$$\tilde{U}(t, s) := \begin{cases} Id, & a \geq t \geq s > a - 1, \\ U(t, \max\{s, a\}), & \text{otherwise.} \end{cases}$$

(See Section 4 for a different extension of \mathcal{U} .) Then (M) also holds for $\tilde{\mathcal{U}}$, $\tilde{B}(\cdot)$, and \tilde{Y}_t . Let $(\tilde{G}, D(\tilde{G}))$ be the generator of the evolution semigroup on $L^p(J, X)$ associated with $\tilde{\mathcal{U}}$.

Corollary 3.5. *Let \mathcal{U} , $\tilde{\mathcal{U}}$, $B(\cdot)$, and $\tilde{B}(\cdot)$ be as considered above. Then there is an evolution family \mathcal{U}_B with index set D_J satisfying the conclusions of Theorem 3.4 for $\tilde{\mathcal{U}}$, $\tilde{B}(\cdot)$, and \tilde{G} on D_J . In particular, if $B(t)$ is closed for a.e. $t \in I$, then \mathcal{U}_B satisfies (3.8) and (3.9) for $x \in X$ and $(t, s) \in D_I$.*

Remark 3.6. *Let $I_n := I \cap (-n, n]$, $n \in \mathbb{N}$. We weaken the hypotheses of Theorem 3.4 by only assuming that \mathcal{U} is bounded on D_{I_n} and (M) holds on I_n for all $n \in \mathbb{N}$. In addition, let $B(t)$ be closed for a.e. $t \in I$. Then there exists an evolution family $\mathcal{U}_{n,B}$ with index set D_{I_n} fulfilling Theorem 3.4(a)–(c) for I_n . By uniqueness, $U_B(t, s) := U_{n,B}(t, s)$ for $(t, s) \in D_{I_n}$ is a (well-defined) evolution family with index set D_I which satisfies (3.8) and (3.9) for $x \in X$ and $(t, s) \in D_I$.*

4. Differentiability in the parabolic case

In this section we suppose that the evolution family \mathcal{U} solves the Cauchy problem related to operators $(A(t), D(A(t)))$, $t \geq 0$, which satisfy the following assumption. We set $\Sigma = \Sigma_\phi := \{0\} \cup \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| < \phi\}$ for $0 < \phi < \pi$.

(P1) $D(A(t))$ is dense, $\rho(A(t)) \supseteq \Sigma_\phi$ for some $\frac{\pi}{2} < \phi < \pi$, $\|R(\lambda, A(t))\| \leq \frac{M}{1+|\lambda|}$, and

$$\|A(t)R(\lambda, A(t))(A(t)^{-1} - A(s)^{-1})\| \leq L \frac{|t - s|^\mu}{1 + |\lambda|^\nu}$$

for $\lambda \in \Sigma$, $t, s \in I$, and constants $M, L \geq 0$ and $\mu, \nu \in (0, 1]$ with $\mu + \nu > 1$.

In view of applications, cf. Section 5, this situation is referred to as the *parabolic case*. Observe that $A(\cdot)^{-1}$ is Hölder continuous and $A(t)$ generates a bounded analytic semigroup. Condition (P1) was introduced (in a somewhat weaker form) by P. Acquistapace and B. Terreni, [2]. Assuming (P1), there is a unique evolution family \mathcal{U} solving (CP) on the spaces $D(A(t))$, [1, Thm. 2.3], see also [3], [31], and the references therein. (To be precise, strong continuity of \mathcal{U} at (s, s) follows from the proof of [1, Thm. 2.3(ii)].) Moreover, \mathcal{U} is exponentially bounded, see (4.1) below.

We extend $A(\cdot)$ to the interval $I := (-1, \infty)$ by setting $A(t) := A(0)$ for $t \in (-1, 0)$. Clearly, the extension still satisfies (P1) with the same constants. Further, we define

$$U(t, s) := \begin{cases} e^{(t-s)A(0)}, & 0 \geq t \geq s > -1, \\ U(t, 0)e^{-sA(0)}, & t > 0 > s > -1. \end{cases}$$

Then $\mathcal{U} = (U(t, s))_{t \geq s > -1}$ is an exponentially bounded evolution family solving (CP) on I . Let $(G, D(G))$ be the generator of the evolution semigroup \mathcal{T} on $E = L^p(I, X)$ associated with \mathcal{U} (where $p \in [1, \infty)$ is specified below in condition (B)).

For $\theta \in [0, 1]$ and $t > -1$, let Y_t be the domain of the fractional power $(-A(t))^\theta$ endowed with the norm $\|x\|_{Y_t} := \|(-A(t))^\theta x\|$. (See, e.g., [22, §2.6] for the basic theory of fractional powers.) By (P1) we obtain $\sup_{t \in I} \|(-A(t))^{-\theta}\| < \infty$ and $\sup_{t \in I} \|(-A(t))^\theta A(t)^{-1}\| < \infty$ (see [22], proof of Lemma 2.6.3). It is shown in [6, Thm. 2.3] that $U(t, s)X \subseteq Y_t$ for $t > s > -1$ and

$$\|(-A(t))^\theta U(t, s)\| \leq C e^{w(t-s)} (t-s)^{-\theta} \quad (4.1)$$

for $(t, s) \in D' = D'_I := \{(t, s) \in D_I : t > s\}$ and constants $C \geq 1$ and $w \in \mathbb{R}$.

We denote by $L^q_{loc,u}(I)$ the space of locally uniformly q -integrable functions on I endowed with the norm

$$\|\varphi\|_{L^q_{loc,u}} := \sup_{s \in I} \left(\int_s^{s+1} |\varphi(\tau)|^q d\tau \right)^{\frac{1}{q}}.$$

On the perturbations $B(t)$, $t \geq 0$, we impose the following condition.

- (B) Let $p \geq 1$ and $0 < \theta < 1$ with $p\theta < 1$. Let $q = \left(\frac{1}{p} - \frac{\theta}{\rho}\right)^{-1}$ for some $\rho \in (p\theta, 1)$. Let $B(t) \in \mathcal{L}(Y_t, X)$ and $\|B(t)\|_{\mathcal{L}(Y_t, X)} \leq \psi(t)$ for a.e. $t \geq 0$, where $\psi \in L^q_{loc,u}(\mathbb{R}_+)$. Finally, let $B(\cdot)U(\cdot, s)y$ be measurable for $s \geq 0$ and $y \in Y_s$.

Notice that $q > p(1 - p\theta)^{-1}$ and $B(t)A(t)^{-1} \in \mathcal{L}(X)$. We remark that in our application $B(t)$ is a differential operator and the assumptions on q, θ, p, ρ lead to

an integrability condition for the coefficients of this operator. Let $B(t) := 0$ and $\psi(t) := 0$ for $t \in (-1, 0)$. On $E = L^p(I, X)$ we define a linear operator by setting $\mathcal{B}f := B(\cdot)f(\cdot)$ for $f \in D(\mathcal{B}) := \{f \in E : f(t) \in Y_t \text{ for a.e. } t \in I, B(\cdot)f(\cdot) \in E\}$. In the next lemma we check the assumptions of Theorem 3.4(a),(b) for \mathcal{U} and $B(\cdot)$.

Lemma 4.1. *Assume (P1) and (B). Then (M) holds for \mathcal{U} , $B(\cdot)$, and Y_t on I . Assume, in addition, that $B(\cdot)A(\cdot)^{-1}x$ is measurable for all $x \in X$. Then $(\mathcal{B}, D(\mathcal{B}))$ is G -bounded.*

Proof. (i) For the first assertion we only have to verify (3.5). Consider $x \in X$, $s > -1$, $t, r > 0$, and $\mu \in \mathbb{R}$. Notice that by (B) and (4.1) we obtain

$$\|B(s+t)U(s+t, s)x\| \leq Ct^{-\theta}e^{wt}\psi(s+t)\|x\| \quad (4.2)$$

for a.e. $t > 0$ and all $x \in X$. Set $\varepsilon := (\mu - w)p$. Then (4.2) and Hölder's inequality imply

$$\begin{aligned} & \int_0^r \|e^{-\mu t}B(s+t)U(s+t, s)x\|^p dt \\ & \leq C^p \|x\|^p \int_0^r e^{(w-\mu)pt} t^{-p\theta} \psi(s+t)^p dt \\ & \leq C^p \|x\|^p \left(\int_0^r \left(e^{-\frac{\varepsilon t}{2}} t^{-p\theta} \right)^{\frac{p}{p\theta}} dt \right)^{\frac{p\theta}{p}} \left(\int_0^r \left(e^{-\frac{\varepsilon t}{2}} \psi(s+t)^p \right)^{(1-\frac{p\theta}{p})^{-1}} dt \right)^{1-\frac{p\theta}{p}} \\ & = C^p \|x\|^p \left(\int_0^r e^{-\varepsilon_1 t} t^{-\rho} dt \right)^{\frac{p\theta}{p}} \left(\int_0^r e^{-\varepsilon_2 t} \psi(s+t)^q dt \right)^{\frac{p}{q}} \end{aligned} \quad (4.3)$$

for some constants $\varepsilon_k \in \mathbb{R}$. For $r \leq r_0$ and $\mu = 0$, estimate (4.3) yields

$$\begin{aligned} \int_0^r \|B(s+t)U(s+t, s)x\|^p dt & \leq C_1 \|x\|^p \left(\int_0^r t^{-\rho} dt \right)^{\frac{p\theta}{p}} \left(\int_s^{s+r} \psi(t)^q dt \right)^{\frac{p}{q}} \\ & \leq C_2 \|\psi\|_{L_{loc,u}^q}^p r^\kappa \|x\|^p \end{aligned}$$

for some constants $C_k, \kappa > 0$. Letting $r \rightarrow 0$ establishes (3.5). Further, let $\mu > w$. Hence, $\varepsilon_k > 0$. We derive from (4.3) that

$$\begin{aligned} & \int_0^\infty \|e^{-\mu t}B(s+t)U(s+t, s)x\|^p dt \\ & \leq C^p \|x\|^p \left(\int_0^\infty \frac{e^{-\varepsilon_1 t}}{t^\rho} dt \right)^{\frac{p\theta}{p}} \left(\sum_{k=0}^\infty e^{-\varepsilon_2 k} \int_{s+k}^{s+k+1} \psi(t)^q dt \right)^{\frac{p}{q}} \leq C_3 \|\psi\|_{L_{loc,u}^q}^p \end{aligned} \quad (4.4)$$

for a constant $C_3 > 0$.

(ii) Next assume that $B(\cdot)A(\cdot)^{-1}x$, $x \in X$, is measurable. Consider the space

$$F := \{f \in E : f(t) \in Y_t \text{ for a.e. } t \in I, (-A(\cdot))^\theta f(\cdot) \in E\}$$

endowed with the norm $\|f\|_F := \|(-A(\cdot))^\theta f(\cdot)\|_E$. Observe that F is a Banach space which is continuously embedded in E . Further we consider \mathcal{B} as an operator on $D(\mathcal{B}_F) := \{f \in F : \mathcal{B}f \in E\}$. It can easily be verified that $\mathcal{B} : D(\mathcal{B}_F) \subseteq F \rightarrow E$ is a closed operator. Let $f \in E$ and $t > 0$. By (4.1) we obtain

$$\|(-A(s))^\theta (T(t)f)(s)\| \leq \|(-A(s))^\theta U(s, s-t)\| \|\mathcal{R}(t)f(s)\| \leq C t^{-\theta} e^{wt} \|\mathcal{R}(t)f(s)\|.$$

Thus, $T(t) \in \mathcal{L}(E, F)$ for $t > 0$. For $t, t' \geq t_0 > 0$ and $f \in E$ we have $T(t)f - T(t')f = T(t_0)(T(t-t_0)f - T(t'-t_0)f)$, and hence the function $u : (0, \infty) \rightarrow F : t \mapsto e^{-\lambda t}T(t)f$ is continuous, where $\lambda > w$ and $f \in E$. Moreover,

$$\|e^{-\lambda t}T(t)f\|_F \leq C e^{(w-\lambda)t} t^{-\theta} \left(\int_I \|\mathcal{R}(t)f(s)\|^p ds \right)^{\frac{1}{p}} \leq C e^{(w-\lambda)t} t^{-\theta} \|f\|_E.$$

As a consequence, $u : \mathbb{R}_+ \rightarrow F$ is integrable. Since λ dominates the growth bound of \mathcal{T} , we infer that $R(\lambda, G) : E \rightarrow F$ is bounded for $\lambda > w$.

It follows from [1, Thm. 2.3] that $D' \ni (t, s) \mapsto A(t)U(t, s)x$ is continuous for $x \in X$. Hence, by the measurability of $B(\cdot)A(\cdot)^{-1}x$, $x \in X$, the mapping

$$D' \ni (t, s) \mapsto B(t)U(t, s)f(s) = B(t)A(t)^{-1}A(t)U(t, s)f(s)$$

is measurable for $f \in E$. Now let $f \in C_c(I, X)$ with support J . Then

$$\|B(s)(T(t)f)(s)\|^p \leq C^p e^{wpt} t^{-p\theta} \|f\|_\infty^p \psi(s)^p \chi_J(s-t),$$

for $t > 0$. Because of $q > p$ we have $\psi \in L_{loc}^p(I)$, and thus $T(t)f \in D(\mathcal{B}_F) \subseteq D(\mathcal{B})$, $t > 0$. Moreover,

$$\begin{aligned} \|\mathcal{B}T(t)f - \mathcal{B}T(t')f\|_E^p &\leq \int_I \psi(s)^p \|(-A(s))^\theta U(s, s-t_0)\|^p \\ &\quad \|U(s-t_0, s-t)(\mathcal{R}(t)f)(s) - U(s-t_0, s-t')(\mathcal{R}(t')f)(s)\|^p ds \end{aligned}$$

for $t', t \geq t_0 > 0$. Clearly, the integrand converges to 0 for a.e. $s \in I$ as $t' \rightarrow t$. Also, the integrand is bounded by $C_4 \psi(\cdot)^p \chi_K(\cdot)$ for a constant C_4 and a compact interval $K \subseteq I$. Therefore the mapping $(0, \infty) \ni t \mapsto \mathcal{B}T(t)f \in E$ is continuous for $f \in C_c(I, X)$ due to the dominated convergence theorem. Further, we derive for $\lambda > \mu > w$ and $u(t) = e^{-\lambda t}T(t)f$, $f \in C_c(I, X)$, that

$$\int_0^\infty \|\mathcal{B}u(t)\|_E dt = \int_0^\infty e^{(\frac{\mu}{p}-\lambda)t} \left(\int_I e^{-\mu t} \|B(s)U(s, s-t)\chi_I(s-t)f(s-t)\|^p ds \right)^{\frac{1}{p}} dt$$

$$\begin{aligned}
&\leq C_5 \left(\int_0^\infty \int_I e^{-\mu t} \|B(s)U(s, s-t)\chi_I(s-t)f(s-t)\|^p ds dt \right)^{\frac{1}{p}} \\
&= C_5 \left(\int_I \int_0^\infty e^{-\mu t} \|B(s+t)U(s+t, s)f(s)\|^p dt ds \right)^{\frac{1}{p}} \\
&\leq C_6 \left(\int_I \|f(s)\|^p ds \right)^{\frac{1}{p}} = C_6 \|f\|_E \tag{4.5}
\end{aligned}$$

for constants $C_k > 0$. (Here we have used Hölder's inequality, Fubini's theorem and estimate (4.4).) Thus, $\mathcal{B}u : \mathbb{R}_+ \rightarrow E$ is integrable. Now Hille's theorem, [8, Thm. 3.7.12], yields $R(\lambda, G)f = \int_0^\infty u(t) dt \in D(\mathcal{B}_F) \subseteq D(\mathcal{B})$ and $\mathcal{B}R(\lambda, G)f = \int_0^\infty \mathcal{B}u(t) dt$. In particular, by (4.5) we have $\|\mathcal{B}R(\lambda, G)f\|_E \leq C_6 \|f\|_E$ for $f \in C_c(I, X)$.

Finally, we show that $(\mathcal{B}, D(\mathcal{B}))$ is G -bounded. Let $g \in D(G)$ and $f = (\lambda - G)g \in E$. Choose a sequence $(f_n) \subseteq C_c(I, X)$, converging to f . Then $g_n := R(\lambda, G)f_n \in D(\mathcal{B}_F)$ tends to g in the norm of F as $n \rightarrow \infty$. Moreover, there exists $E - \lim_{n \rightarrow \infty} \mathcal{B}R(\lambda, G)f_n =: h$. Since \mathcal{B} is closed in $F \times E$, we infer that $g \in D(\mathcal{B}_F) \subseteq D(\mathcal{B})$ and $\mathcal{B}g = h$. Hence, $\|\mathcal{B}g\|_E \leq C_6 \|f\|_E = C_6 \|(\lambda - G)g\|_E$ for $g \in D(G)$. \square

So we can apply the results of the preceding section. In fact, we can even improve them considerably and show that the function $t \mapsto U_B(t, s)x$, $s \in I$, $x \in X$, solves (pCP) for a.e. $t \in (s, \infty)$. For this we have to introduce some new concepts.

Let $1 < p, d < \infty$ and let $X = L^d(\Omega)$ for a σ -finite measure space (Ω, μ) . Then $E = L^p(I, X)$ is a UMD-space, see e.g. [3, III.4.5.2]. We assume (P1) and that the operators $(A(t), D(A(t)))$, $t \geq 0$, satisfy the following condition, where ϕ is determined by (P1).

(P2) The operators $(A(t), D(A(t)))$ admit *bounded imaginary powers* $(-A(t))^{ir}$ and $\|(-A(t))^{ir}\| \leq c e^{(\pi-\phi)|r|}$ for $t \geq 0$, $r \in \mathbb{R}$, and a constant c ,

compare [3, 14] and the references cited therein. Clearly, the extension of $A(\cdot)$ to $I = (-1, \infty)$ given by $A(t) = A(0)$, $-1 < t < 0$, also satisfies (P2). In Section 5 we will see that (P1) and (P2) can be verified for a large class of elliptic operators $A(t)$. We define a linear operator \mathcal{A} on E by setting $\mathcal{A}f := A(\cdot)f(\cdot)$ for $f \in D(\mathcal{A}) := \{f \in E : f(s) \in D(A(s)) \text{ for a.e. } s > -1, A(\cdot)f(\cdot) \in E\}$. Notice that $(\mathcal{A}, D(\mathcal{A}))$ is closed. Consider the first derivative $G_0f = -f'$ on E with domain $D(G_0) = \{f \in W^{1,p}(I, X) : f(-1) = 0\}$. Then it was shown in [14], Cor. 2, Thm. 2, that the operator $(G_0 + \mathcal{A}, D(G_0) \cap D(\mathcal{A}))$ is closed and $(\omega, \infty) \subseteq \rho(G_0 + \mathcal{A})$ for some $\omega \in \mathbb{R}$. Further, we set

$$\mathcal{D}_1 := \text{lin} \{f \in E : f(\cdot) = \varphi(\cdot)U(\cdot, r)y; r \in I, y \in D(A(r)), \varphi \in C_c^1(I) \text{ with } \varphi(s) = 0, s \leq r\}.$$

Since the Cauchy problem (CP) related to $A(\cdot)$ is well-posed, it follows from [28, Prop. 1.13], cf. [11, Prop. 2.9], that $\mathcal{D}_1 \subseteq D(\mathcal{A}) \cap D(G_0)$ and the generator $(G, D(G))$ of the evolution semigroup is the closure of $(G_0 + \mathcal{A}, \mathcal{D}_1)$. Hence, $G_0 + \mathcal{A}$ extends G . Since there exists $\lambda \in \rho(G) \cap \rho(G_0 + \mathcal{A})$, this implies

$$(G, D(G)) = (G_0 + \mathcal{A}, D(G_0) \cap D(\mathcal{A})). \quad (4.6)$$

This fact plays a central role in the proof of the next theorem.

Theorem 4.2. *Assume that the operators $(A(t), D(A(t)))$ and $(B(t), Y_t)$, $t \geq 0$, on $X = L^d(\Omega)$ satisfy (P1), (P2), and (B) for $1 < p, d < \infty$. Moreover, assume that $B(\cdot)A(\cdot)^{-1}x$ is measurable for $x \in X$. Then there is a perturbed evolution family $\mathcal{U}_B = (U_B(t, s))_{t \geq s > -1}$ satisfying the conclusions of Theorem 3.4(a) and (b). Set $u(t) = U_B(t, s)x$ for $t \geq s \geq 0$ and $x \in X$. Then $u \in C([s, \infty), X)$ and $u(s) = x$. Further, we have $u(t) \in D(A(t))$ for a.e. $t \in (s, \infty)$, $u \in W_{loc}^{1,1}((s, \infty), X)$, $(A(\cdot) + B(\cdot))u(\cdot) \in L_{loc}^1((s, \infty), X)$, and*

$$\frac{d}{dt} u(t) = (A(t) + B(t))u(t) \quad \text{for a.e. } t \in (s, \infty).$$

Proof. (1) The existence of $\mathcal{U}_B = (U_B(t, s))_{t \geq s > -1}$ follows from Lemma 4.1 and Theorem 3.4. Let \mathcal{T}_B be the associated evolution semigroup on $E = L^p(I, X)$ with generator $G_B = G + \mathcal{B}$, where $D(G_B) = D(G) \subseteq D(\mathcal{B})$ and $\mathcal{B} = B(\cdot)$. Observe that the generator $(\tilde{G}_0, D(\tilde{G}_0))$ of the left translation semigroup $(\mathcal{R}(-t))_{t \geq 0}$ is given by $\tilde{G}_0 g = g'$ for $g \in D(\tilde{G}_0) = W^{1,p}(I, X)$. Let $f \in D(G)$. We have $T_B(t)f \in D(G) = D(\mathcal{A}) \cap D(G_0) \subseteq D(\tilde{G}_0)$ and $G_B = G_0 + \mathcal{A} + \mathcal{B}$ due to (4.6). Hence,

$$\begin{aligned} \frac{d}{d\tau} \mathcal{R}(-\tau)T_B(\tau)f &= \mathcal{R}(-\tau)G_B T_B(\tau)f + \mathcal{R}(-\tau)\tilde{G}_0 T_B(\tau)f \\ &= \mathcal{R}(-\tau)(\mathcal{A} + \mathcal{B})T_B(\tau)f =: v(\tau), \end{aligned}$$

and $v : [0, t] \rightarrow E$ is continuous. Integration from 0 to t yields

$$\mathcal{R}(-t)T_B(t)f = f + \int_0^t \mathcal{R}(-\tau)(\mathcal{A} + \mathcal{B})T_B(\tau)f d\tau. \quad (4.7)$$

Moreover, for $\tau \in [0, t]$ we have

$$v(\tau)(s) = (A(s + \tau) + B(s + \tau))U_B(s + \tau, s)f(s) =: \tilde{v}(\tau, s)$$

for a.e. $s > -1$. Since $(\mathcal{A}, D(\mathcal{A}))$ is closed and $\mathcal{A}T_B(\tau)f = \mathcal{A}\mathcal{R}(\lambda, G_B)T_B(\tau)(\lambda - G_B)f$, we derive from (4.6) the continuity of $\tau \mapsto \mathcal{A}T_B(\tau)f$. By Lemma 3.2 there

is a measurable function $\Phi : [0, t] \times I \rightarrow X$ such that for a.e. $\tau \in [0, t]$ we have $(\mathcal{R}(-\tau)\mathcal{A}T_B(\tau)f)(s) = \Phi(\tau, s)$ for a.e. $s > -1$. That is, for a.e. $\tau \in [0, t]$

$$A(s + \tau)U_B(s + \tau, s)f(s) = \Phi(\tau, s), \quad \text{and hence} \quad (4.8)$$

$$U_B(s + \tau, s)f(s) = A(s + \tau)^{-1}\Phi(\tau, s) \quad (4.9)$$

for a.e. $s > -1$. Observe that both sides of (4.9) are measurable with respect to (τ, s) . Thus (4.9) and, hence, (4.8) hold for a.e. (τ, s) , and so the left-hand side of (4.8) is measurable on $[0, t] \times I$. Then the function

$$[0, t] \times I \ni (\tau, s) \mapsto B(s + \tau)U_B(s + \tau, s)f(s) = B(s + \tau)A(s + \tau)^{-1}A(s + \tau)U_B(s + \tau, s)f(s)$$

is measurable by the assumption. As a consequence, $\tilde{v} : [0, t] \times I \rightarrow X$ is measurable. From Lemma 3.2 we deduce that $\tilde{v}(\cdot, s)$ is integrable for a.e. $s > -1$ and (4.7) implies after a change of variables

$$U_B(t, s)f(s) = f(s) + \int_s^t (A(\tau) + B(\tau))U_B(\tau, s)f(s)d\tau \quad (4.10)$$

for $s \in (-1, \infty) \setminus N(f)$, $t \in [s, \infty)$, and a null set $N(f)$. (Notice that both sides of (4.10) are continuous with respect to t .)

(2) Now fix $x \in X$ and $s \geq 0$. Choose $r_n \searrow s$ and $\varphi_n \in C_c^1(I)$ with $\varphi_n \equiv 0$ on $(-1, s]$ and $\varphi_n \equiv 1$ on $[r_n, r_n + 1]$. Set $f_n(\cdot) := \varphi_n(\cdot)U_B(\cdot, s)x$. By Lemma 3.3 we have $f_n \in D(G_B) = D(G)$. There exist $s_n \in [r_n, r_n + 1]$ such that $s_n \notin N(f_n)$ and $s_n \searrow s$. From the first part of the proof follows that $U_B(t, s_n)f_n(s_n) \in D(A(t))$ for a.e. $t \in [s_n, \infty)$, $(A(\cdot) + B(\cdot))U_B(\cdot, s_n)f_n(s_n) \in L_{loc}^1([s_n, \infty), X)$ and (4.10) holds for $t \in [s_n, \infty)$. As a consequence, $U_B(\cdot, s_n)f_n(s_n) \in W_{loc}^{1,1}([s_n, \infty), X)$ and

$$\frac{d}{dt}U_B(t, s_n)f_n(s_n) = (A(t) + B(t))U_B(t, s_n)f_n(s_n)$$

for a.e. $t \in [s_n, \infty)$. Set $u(t) := U_B(t, s)x$ for $t \geq s$. Since $u(t) = U_B(t, s_n)f_n(s_n)$ for $t \geq s_n$ the theorem is established. \square

Other perturbation results in the parabolic case can be found, for instance, in [10, Thm. 6.1] or [7, Thm. 7.1.3]. There $B(\cdot)$ satisfies certain Hölder conditions which lead to stronger regularity properties of \mathcal{U}_B .

5. An application

In this section we investigate the following initial value problem with mixed boundary conditions:

$$\begin{aligned}
D_t u(t, \xi) &= \sum_{k,l=1}^n D_k (a_{kl}(t, \xi) D_l u(t, \xi)) + \sum_{k=1}^n b_k(t, \xi) D_k u(t, \xi) + c(t, \xi) u(t, \xi), \quad t > s, \quad \xi \in \Omega, \\
u(t, \xi) &= 0, \quad \xi \in \Gamma_0, \quad t > s, \\
\sum_{k,l=1}^n n_k(\xi) a_{kl}(t, \xi) D_l u(t, \xi) &= 0, \quad \xi \in \Gamma_1, \quad t > s, \\
u(s, \xi) &= x(\xi), \quad \xi \in \Omega.
\end{aligned} \tag{IVP}$$

Here $D_t = \frac{\partial}{\partial t}$ and $D_k = \frac{\partial}{\partial \xi_k}$ denote derivatives in the sense of distributions and the boundary conditions are understood in the sense of traces. Moreover, we consider the following hypotheses:

(A1) $\Omega \subseteq \mathbb{R}^n$ is a bounded domain with compact C^2 -boundary $\partial\Omega$, Γ_i are open and closed in $\partial\Omega$ such that $\partial\Omega = \Gamma_0 \cup \Gamma_1$, $\Gamma_0 \cap \Gamma_1 = \emptyset$, $\Gamma_0 \neq \emptyset$, and $n(\xi)$ is the outer normal of $\partial\Omega$ at $\xi \in \partial\Omega$.

(A2) $a_{kl} = a_{lk} : \mathbb{R}_+ \times \bar{\Omega} \rightarrow \mathbb{R}$ is continuous with bounded partial derivatives with respect to $\xi \in \Omega$, $a_{kl}, D_j a_{kl} \in C^\mu(\mathbb{R}_+, C(\bar{\Omega}))$, $1 \leq k, l, j \leq n$, with $\mu > \frac{1}{2}$, and

$$\frac{1}{\delta} |\eta|^2 \geq \sum_{k,l=1}^n a_{kl}(t, \xi) \eta_k \eta_l \geq \delta |\eta|^2$$

for a constant $\delta > 0$, $\eta \in \mathbb{R}^n$, $t \geq 0$, and $\xi \in \Omega$.

(A3) $b_k, c \in L_{loc,u}^q(\mathbb{R}_+, L^r(\Omega))$, where $r > n$ and $q > \left(\frac{1}{2} - \frac{n}{2r}\right)^{-1}$.

(A3') $b_k \equiv 0$ and $c \in L_{loc,u}^q(\mathbb{R}_+, L^r(\Omega))$, where $2r > n$, $r > 1$, and $q > \left(1 - \frac{n}{2r}\right)^{-1}$.

Assumptions (A1) and (A2) were already considered in [14, §6].

On the space $X = L^d(\Omega)$, $1 < d < \infty$, $d \leq r$, we define the second order operator

$$A(t)x = \sum_{k,l=1}^n D_k (a_{kl}(t, \cdot) D_l x(\cdot)), \quad t \geq 0,$$

$$D(A(t)) = \{x \in W^{2,d}(\Omega) : x(\xi) = 0, \xi \in \Gamma_0, \sum_{k,l=1}^n n_k(\xi) a_{kl}(t, \xi) D_l x(\xi) = 0, \xi \in \Gamma_1\}.$$

Then it is shown in [14, §6] that the operators $(A(t), D(A(t)))$, $t \geq 0$, satisfy (P1) (with constants μ and $\nu = \frac{1}{2}$) and (P2). (See also [1] and [31] for the verification

of (P1) and the references in [14] concerning (P2).) Denote by \mathcal{U} the evolution family solving the Cauchy problem (CP) corresponding to $A(\cdot)$.

Next, we introduce the (formal) first order operator

$$B(t)x = \sum_{k=1}^n b_k(t, \cdot) D_k x(\cdot) + c(t, \cdot)x(\cdot), \quad t \geq 0.$$

Assuming (A3), resp. (A3'), one can see that there are numbers $\theta > \frac{1}{2} + \frac{n}{2r}$, resp. $\theta > \frac{n}{2r}$, $p > 1$, and $\rho < 1$ such that $1 > \rho > p\theta$ and $q = \left(\frac{1}{p} - \frac{\theta}{\rho}\right)^{-1}$. As in the preceding section Y_t is defined as the domain of $(-A(t))^\theta$ endowed with the norm $\|\cdot\|_{Y_t}$. Let $\frac{1}{\vartheta} = \frac{1}{d} - \frac{1}{r} \geq 0$. Then $\vartheta \geq d$ and $\frac{j}{2} + \frac{n}{2r} = \frac{j}{2} + \frac{n}{2d} - \frac{n}{2\vartheta}$ for $j \in \{0, 1\}$. Since $\theta > \frac{j}{2} + \frac{n}{2r}$, we obtain by [7, Thm. 1.6.1] that $Y_t \subseteq W^{j, \vartheta}(\Omega)$ and $\|x\|_{W^{j, \vartheta}} \leq K\|x\|_{Y_t}$. The proof of [7, Thm. 1.6.1] and the estimate

$$\|A(t)^{-1}x\|_{W^{2, d}} \leq K'\|x\|_d, \quad x \in X, \quad (5.1)$$

(see for instance [14, (6.6)] or [3, (IV.2.6.19)]) imply that the constant K does not depend on t . Moreover, we have $\frac{1}{r} + \frac{1}{d'} + \frac{1}{\vartheta} = 1$, where $\frac{1}{d} + \frac{1}{d'} = 1$. Hence, by Hölder's inequality we derive for $x \in Y_t$ and $y \in L^{d'}(\Omega)$

$$\begin{aligned} |\langle B(t)x, y \rangle| &\leq \int_{\Omega} \left(|c(t, \xi)x(\xi)y(\xi)| + \sum_{k=1}^n (|b_k(t, \xi)D_k x(\xi)y(\xi)|) \right) d\xi \\ &\leq \sum_{k=1}^n (\|D_k x\|_{\vartheta} \|b_k(t)\|_r \|y\|_{d'}) + \|x\|_{\vartheta} \|c(t)\|_r \|y\|_{d'} \\ &\leq \tilde{\psi}(t) \|x\|_{W^{1, \vartheta}} \|y\|_{d'} \\ &\leq K\tilde{\psi}(t) \|x\|_{Y_t} \|y\|_{d'}, \end{aligned}$$

where $\tilde{\psi}(t) := \max\{\|c(t)\|_r, \|b_k(t)\|_r, k = 1, \dots, n\}$. Thus $B(t) \in \mathcal{L}(Y_t, X)$, $\|B(t)\|_{\mathcal{L}(Y_t, X)} \leq \psi(t) := K\tilde{\psi}(t)$, and $\psi \in L_{loc, u}^q(\mathbb{R}_+)$.

Now we show that $B(\cdot)A(\cdot)^{-1}x$, $x \in X$, is measurable. The Gagliardo–Nirenberg inequality (see e.g. [22, Lemma 8.4.1]), (5.1), and (P1) imply

$$\begin{aligned} \|D_k(A(t)^{-1} - A(s)^{-1})x\|_d &\leq C_1 \|(A(t)^{-1} - A(s)^{-1})x\|_d^{\frac{1}{2}} \|(A(t)^{-1} - A(s)^{-1})x\|_{W^{2, d}}^{\frac{1}{2}} \\ &\leq C_2 \|x\|_d |t - s|^{\frac{t}{2}} \end{aligned}$$

for constants C_l . Hence, $B(\cdot)A(\cdot)^{-1}x$ is measurable if $b_k, c \in L^\infty(\mathbb{R}_+ \times \Omega)$, cf. [26, Lemma 2.4]. The general case follows by truncation of the coefficients b_k and c and an application of the dominated convergence theorem. Finally, from $B(t)U(t, s)x = B(t)A(t)^{-1}A(t)U(t, s)x$ we derive the measurability of $B(\cdot)U(\cdot, s)x$ for $x \in X$.

As a consequence, Theorem 4.2 yields the following result which extends [24, Ex. 4.6], where the case $b_k \equiv 0$ and $A(t) \equiv A$ was considered.

Proposition 5.1. *Assume (A1), (A2), and (A3), resp. (A3'). Let $X = L^d(\Omega)$, $1 < d < \infty$, $d \leq r$, $x \in X$, and $s \geq 0$. Then there is a function $u \in W_{loc}^{1,1}((s, \infty), X) \cap C([s, \infty), X)$ with $u(s) = x$ such that $u(t) \in D(A(t))$ for a.e. $t \in (s, \infty)$, $(A(\cdot) + B(\cdot))u(\cdot) \in L_{loc}^1((s, \infty), X)$, and u satisfies (IVP) for a.e. $t \in (s, \infty)$ and $\xi \in \Omega$.*

In [4] D.G. Aronson has studied a more general version of (IVP) (however with Dirichlet boundary conditions) and obtained weak solutions of (IVP) by PDE methods. We point out that in [4] the conditions $r > n$ and $q > \left(\frac{1}{2} - \frac{n}{2r}\right)^{-1}$, cf. (A3), also play a crucial role.

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