Strang splitting for a semilinear Schrödinger equation with damping and forcing ☆

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Abstract

We propose and analyze a Strang splitting method for a cubic semilinear Schrödinger equation with forcing and damping terms and subject to periodic boundary conditions. The nonlinear part is solved analytically, whereas the linear part – space derivatives, damping and forcing – is approximated by the exponential trapezoidal rule. The necessary operator exponentials and ϕ-functions can be computed efficiently by fast Fourier transforms if space is discretized by spectral collocation. Under natural regularity assumptions, we first show global existence of the problem in $H^4(\mathbb{T})$ and establish global bounds depending on properties of the forcing. The main result of our error analysis is first-order convergence in $H^1(\mathbb{T})$ and second-order convergence in $L^2(\mathbb{T})$ on bounded time-intervals.

Keywords: Nonlinear Schrödinger equation, Lugiato-Lefever equation, Strang splitting, error analysis, stability, well-posedness, regularity.

1. Introduction

Nonlinear Schrödinger equations (NLS) occur in many different forms and describe a multitude of different phenomena, such as Bose-Einstein condensates, small-amplitude surface water waves, Langmuir waves in hot plasmas, or signal processing through optical fibers, to name but a few. The intriguing properties – for example conservation of norm, energy, and momentum, near-conservation of actions over long times, existence of solitary waves, or possible blow-up – have inspired and challenged mathematicians for a long time. Surveys about these topics can be found, e.g., in the monographs [3] and [22].

In most applications, the solution of the NLS has to be approximated by a numerical scheme. For problems on the d-dimensional torus $\mathbb{T}^d$, splitting methods with spectral collocation in space are particularly popular. These integrators are based on the observation that the linear and the nonlinear part of the NLS can be solved at low computational costs in the absence of the other part. The splitting approach can also be applied to the Gross-Pitaevskii equation, a NLS on $\mathbb{R}^d$ with constraining polynomial potential, by using the basis of Hermite functions for the space

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discretization. The accuracy of such integrators has been analyzed, e.g., in [1, 6, 7, 8, 10, 15, 17, 18, 20, 23]. The long-time behavior of numerical solutions, in particular the (near-)conservation of invariants over long times and the stability of plane waves, has been investigated in [8, 9, 11], and for exponential integrators in [5].

In this article we consider the *Lugiato-Lefever equation*, a cubic, focusing NLS which, in contrast to the "classical" NLS, contains a damping and a forcing term; cf. [19]. This equation has been proposed as a model for the formation of Kerr-frequency combs in microresonators coupled to optical waveguides and driven by an external pump tuned to a resonance wavelength; see [4] and [13]. The frequency combs generated by such a device can be used as optical sources for high-speed data transmission. In the mathematical model, the forcing term represents the external pump, whereas the radiation into the waveguide is modeled by the damping term. In practice it is typically not clear which parametrization generates a suitable frequency comb. As a consequence, the Lugiato-Lefever equation has to be solved many times with different parametrizations, which requires a reliable and efficient simulation method. So far, however, both the properties of the exact solution and the performance of numerical integrators for its approximation are only poorly understood.

In this article, we develop the Strang splitting approach for the Lugiato-Lefever equation and provide an error analysis for this method. The linear inhomogeneous part (including the space derivatives and the forcing/damping terms) is propagated by an exponential integrator whereas the nonlinear part is solved exactly as for the standard NLS. For initial data in $H^4(\mathbb{T})$ and a forcing function in $C^{2-j}([0,T]; H^j(\mathbb{T}))$ for $j = 0, 1, 2$, we prove that the method converges on bounded time-intervals with the classical order 2 in $L^2(\mathbb{T})$, and with order 1 in $H^1(\mathbb{T})$; see Theorem 2 below. The proof consists of several steps which are formulated as self-contained results. As in [18], the classical argument "consistency plus stability yields convergence" must be suitably adapted, because the stability result (Theorem 5 below) assumes the numerical solution to be bounded in $H^1(\mathbb{T})$. This a-priori bound is needed to control the nonlinearity. In the proof of Theorem 2 this bound is verified by means of an error bound of second order for the local error in $H^1(\mathbb{T})$, in addition to the local (third order) error bound in $L^2(\mathbb{T})$ required for consistency (see Theorems 3 and 4, respectively). The error analysis requires a global solution of the Lugiato-Lefever equation in $H^4(\mathbb{T})$. This fact is shown under the above regularity assumptions by means of a modified energy functional taking into account the forcing. Our approach also provides global bounds for the solution on the time interval $\mathbb{R}_+$ which are of independent interest.

Following [18], many authors have used the calculus of Lie derivatives and commutator bounds in their error analysis, e.g. [6, 8, 10, 15, 17, 20, 23]. In contrast to these works, we avoid the notationally rather involved Lie derivatives, because in case of the Lugiato-Lefever equation the iterated commutators between the linear and nonlinear part are not the only source of error: An additional difficulty arising in our situation is the fact that the forcing term is coupled to the space derivatives and to the nonlinear part in a complicated way. Instead of Lie derivatives, our error formulas are based on iterated variation-of-constants formulas, the calculus of ϕ-functions (see Section 4) and the exponential trapezoidal rule (see formula (25)). The resulting error formulas are quite involved, and it requires a considerable effort to keep track of the necessary regularity of the exact and the numerical solutions.

In the next section, we introduce the Lugiato-Lefever equation and present the analytical framework. Existence and uniqueness of a global solution to this equation is shown in Theorem 1 in Section 3, which is the foundation of the numerical analysis in later sections. The splitting method for the Lugiato-Lefever equation is introduced in Section 4, and we formulate the error bounds for the global error (Theorem 2) along with the results required for its proof (bounds of the local error in $L^2(\mathbb{T})$ and $H^1(\mathbb{T})$ and stability of the scheme). All following sections are devoted to the proofs of these assertions. In Section 5, we prove stability of the numerical scheme, and we compile a number of auxiliary results. The bounds of the local errors are shown in Sections 6 and 7, respectively, and the proof of the global error bound follows in Section 8.

2. The Lugiato-Lefever equation

The cubic semilinear Schrödinger equation

$$\partial_t u(t,x) = -u(t,x) + \mathrm{i}\partial_x^2 u(t,x) + \mathrm{i}|u(t,x)|^2 u(t,x) + g(t,x), \qquad t > 0 \tag{1a}$$

$$u(0,x) = u_0(x) \tag{1b}$$

on the one-dimensional torus $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ is known as the Lugiato-Lefever equation in physics and electronic engineering. The terms $-u(t,x)$ and $g(t,x)$ model damping and external forcing, respectively, and do not appear in the "classical" NLS. Clearly, these terms destroy the Hamiltonian structure, and in general the energy, momentum and norm of the solution do not remain constant in time. Solely the one-dimensional torus is considered, because this is the relevant setting for modeling frequency comb generation; cf. [4] and [13]. In the literature, the Lugiato-Lefever equation is sometimes stated in the form

$$\partial_t w(t,x) = -(1+\mathrm{i}\zeta)w(t,x) + \mathrm{i}d\partial_x^2 w(t,x) + \mathrm{i}|w(t,x)|^2 w(t,x) + f(t,x) \tag{2}$$

with additional parameters ζ and d. Since d does not have any significant impact on the results of this paper, we set $d = 1$. Then, equation (2) is equivalent to (1a) via $u(t) = e^{\mathrm{i}\zeta t} w(t,x)$ and $g(t,x) = e^{\mathrm{i}\zeta t} f(t,x)$.

The evolution equation (1) is considered on $L^2(\mathbb{T})$, i.e. on the Hilbert space of square integrable functions with the inner product

$$\langle v, w\rangle = \int_{\mathbb{T}} v(x)\overline{w(x)}\,\mathrm{d}x, \qquad v, w \in L^2(\mathbb{T})$$

and induced norm $\|v\|_{L^2} = \sqrt{\langle v, v\rangle}$. The Sobolev space of all functions $v : \mathbb{T} \to \mathbb{C}$ with partial derivatives up to order $k \in \mathbb{N}_0$ in $L^2(\mathbb{T})$ is denoted by $H^k(\mathbb{T})$. For every k, $H^k(\mathbb{T})$ is a Hilbert space with norm

$$\|v\|_{H^k}^2 = \sum_{j=0}^{k} \|\partial_x^j v\|_{L^2}^2 .$$

In particular, we identify $H^0(\mathbb{T}) = L^2(\mathbb{T})$. We assume the regularity

$$u_0 \in H^4(\mathbb{T}) \quad \text{and} \quad g \in C^{2-j}([0,T]; H^j(\mathbb{T})), \quad j = 0,1,2, \tag{3}$$

for the initial data u_0 and the forcing g, respectively, where $T > 0$ is fixed. It will be shown in Theorem 1 that these assumptions guarantee the global existence and uniqueness of a sufficiently smooth solution.

Henceforth, we will usually omit the space variable and write $u(t)$ instead of $u(t,x)$, and so on. Throughout the paper, $C > 0$ and $C(\cdot) > 0$ denote universal constants, possibly taking different values at various appearances. The notation $C(\cdot)$ means that the constant depends only on the values specified in the brackets.

3. Existence, uniqueness and regularity of solutions

The error analysis of the numerical method presented in Section 4 requires a unique solution of (1) in $H^4(\mathbb{T})$ on $[0, T]$. In the next theorem, we show that the desired global solution exists under assumption (3), and we also establish additional results on the long-time behavior on the time interval $\mathbb{R}_+$.

Theorem 1. *Let $u_0 \in H^4(\mathbb{T})$ and $g \in C^2(\mathbb{R}_+; L^2(\mathbb{T})) \cap C(\mathbb{R}_+; H^2(\mathbb{T}))$. Then,* (1) *has a unique solution*

$$u \in C(\mathbb{R}_+; H^4(\mathbb{T})) \cap C^1(\mathbb{R}_+; H^2(\mathbb{T})) \cap C^2(\mathbb{R}_+; L^2(\mathbb{T})). \tag{4}$$

Under additional assumptions on g, the following bounds hold.

(a) If $g \in L^\infty(\mathbb{R}_+; L^2(\mathbb{T}))$, then $\|u(t)\|_{L^2} \le C$ for $t \ge 0$.
(b) If $g, \partial_t g \in L^\infty(\mathbb{R}_+; L^2(\mathbb{T}))$, then $\|u(t)\|_{H^1} \le C\sqrt{1+t}$ for $t \ge 0$.
(c) If $g \in L^2(\mathbb{R}_+; L^2(\mathbb{T}))$, then $u \in L^2(\mathbb{R}_+; L^2(\mathbb{T}))$.
(d) If $g \in H^1(\mathbb{R}_+; L^2(\mathbb{T}))$, then $\|u(t)\|_{H^1} \le C$ for $t \ge 0$.

Here C only depends on $\|u_0\|_{L^2}$ in (a) and on $\|u_0\|_{H^1}$ in (b) and (d), as well as on the respective norms of g.

Proof of Theorem 1.
For every $k \in \mathbb{Z}$, the operator

$$A := \mathrm{i}\partial_x^2 - I \qquad \text{with domain} \qquad H^{k+2}(\mathbb{T}) \tag{5}$$

generates a strongly continuous group $(e^{tA})_{t\in\mathbb{R}}$ in $H^k(\mathbb{T})$. The level k of regularity is not expressed in our notation since the respective operators are restrictions of each other. We prove Theorem 1 in three steps. First, we construct a unique solution of (1) with the desired regularity, but, on bounded time intervals. The proof of the global existence in the second step is then based on an energy estimate which allows us to bound the $H^1(\mathbb{T})$-norm of the solution, compare e.g. Section 6.1 of [3] for the case $g = 0$. Lastly, we derive the results on the long-time behavior of the solution on the time interval $\mathbb{R}_+$.

Step 1. For the case $g = 0$ and for the base space $\mathbb{R}$ instead of $\mathbb{T}$, Theorem 4.10.1 in [3] yields a unique maximal solution $u \in C([0, a); H^4(\mathbb{R}))$ of (1) for some $a \in (0, \infty]$. Moreover, by this theorem the sup-norm $\|u(t)\|_\infty$ blows up as $t \to a$ if $a < \infty$. As a result, if $\|u(t)\|_{H^1}$ is bounded for t in each bounded interval in $[0, a)$, then $a = \infty$. The proof in [3] works in $H^4(\mathbb{T})$ in the same way. We can also replace here the group $e^{\mathrm{i}t\partial_x^2}$ by e^{tA}. To treat nonzero g, we add the term

$$w(t) = \int_0^t e^{(t-s)A} g(s)\,\mathrm{d}s = \int_0^t e^{sA} g(t-s)\,\mathrm{d}s, \qquad t \in \mathbb{R}_+,$$

to the fixed point map $\mathcal{H}$ on p.137 in [3]. By standard semigroup theory, see e.g. Corollary 4.2.5 in [21], the function w belongs $C(\mathbb{R}_+; H^2(\mathbb{T})) \cap C^1(\mathbb{R}_+; L^2(\mathbb{T}))$ and satisfies $\partial_t w(t) = Aw(t) + g(t)$ for all $t \ge 0$ because $g \in C^1(\mathbb{R}_+; L^2(\mathbb{T}))$. Next, the derivative

$$\partial_t w(t) = e^{tA} g(0) + \int_0^t e^{sA} \partial_t g(t-s)\,\mathrm{d}s, \qquad t \in \mathbb{R}_+,$$

is also an element of $C(\mathbb{R}_+; H^2(\mathbb{T})) \cap C^1(\mathbb{R}_+; L^2(\mathbb{T}))$ since $g \in C^2(\mathbb{R}_+; L^2(\mathbb{T}))$ and $g(0) \in H^2(\mathbb{T})$. As a result, Aw is contained in $C(\mathbb{R}_+; H^2(\mathbb{T}))$, and hence $w = A^{-1}Aw$ in $C(\mathbb{R}_+; H^4(\mathbb{T}))$. One can now proceed as in the proof of Theorem 4.10.1 in [3] and obtain a unique maximal solution $u \in C([0, a); H^4(\mathbb{T}))$ of (1) for u_0 and g from (3), with the above blow-up criterion in $H^1(T)$. Because of $g \in C(\mathbb{R}_+; H^2(\mathbb{T}))$, the equation (1) yields that u belongs to $C^1(\mathbb{R}_+; H^2(\mathbb{T}))$. Differentiating the right-hand side of (1) in time, we deduce that u is also contained in $C^2(\mathbb{R}_+; L^2(\mathbb{T}))$.

Step 2. We next show global existence by controlling the norm of $u(t)$ in $H^1(\mathbb{T})$. Using (1) and integrating by parts, we first compute

$$\begin{aligned} \partial_t \|u(t)\|_{L^2}^2 &= 2 \operatorname{Re} \int_{\mathbb{T}} \overline{u}(t) \partial_t u(t) \, \mathrm{d}x \\ &= 2 \operatorname{Re} \int_{\mathbb{T}} \overline{u}(t) \Big(\mathrm{i}\partial_x^2 u(t) - u(t) + \mathrm{i}|u(t)|^2 u(t) + g(t) \Big) \, \mathrm{d}x \\ &= -2 \|u(t)\|_{L^2}^2 + 2 \operatorname{Re} \int_{\mathbb{T}} \overline{u}(t) g(t) \, \mathrm{d}x \\ &\leq -2 \|u(t)\|_{L^2}^2 + 2 \|u(t)\|_{L^2} \|g(t)\|_{L^2} \\ &\leq -\|u(t)\|_{L^2}^2 + \|g(t)\|_{L^2}^2 \end{aligned} \tag{6}$$

for $t \in [0, a)$. Hence, $\partial_t (e^t \|u(t)\|_{L^2}^2) \leq e^t \|g(t)\|_{L^2}^2$ and integration yields

$$\begin{aligned} \|u(t)\|_{L^2}^2 &\leq e^{-t} \|u_0\|_{L^2}^2 + \int_0^t e^{s-t} \|g(s)\|_{L^2}^2 \, \mathrm{d}s \\ &\leq \|u_0\|_{L^2}^2 + \sup_{0 \leq s \leq b} \|g(s)\|_{L^2}^2 =: C_0(b) \end{aligned} \tag{7}$$

for $0 \leq t \leq b < a$. We further need the modified energy of (1) given by

$$\mathcal{E}(t, v) = \tfrac{1}{2} \|\partial_x v\|_{L^2}^2 - \tfrac{1}{4} \|v\|_{L^4}^4 + \operatorname{Re} \int_{\mathbb{T}} \mathrm{i} g(t) \overline{v} \, \mathrm{d}x \tag{8}$$

for $v \in H^1(\mathbb{T})$ and $t \geq 0$. Proceeding as above, we obtain

$$\begin{aligned} \partial_t \mathcal{E}(t, u(t)) &= \operatorname{Re} \int_{\mathbb{T}} \Big[\partial_x u(t) \partial_{tx} \overline{u}(t) - (|u(t)|^2 u(t) - \mathrm{i} g(t)) \partial_t \overline{u}(t) + \mathrm{i} \overline{u}(t) \partial_t g(t) \Big] \, \mathrm{d}x \\ &= \operatorname{Re} \int_{\mathbb{T}} \Big[\big(-\partial_x^2 u(t) - |u(t)|^2 u(t) + \mathrm{i} g(t)) \partial_t \overline{u}(t) + \mathrm{i} \overline{u}(t) \partial_t g(t) \Big] \, \mathrm{d}x \\ &= \operatorname{Re} \int_{\mathbb{T}} \mathrm{i} \Big[(\partial_t u(t) + u(t)) \partial_t \overline{u}(t) + \overline{u}(t) \partial_t g(t) \Big] \, \mathrm{d}x \\ &= \operatorname{Re} \int_{\mathbb{T}} \mathrm{i} \Big[u(t) (\overline{g}(t) - \mathrm{i}\partial_x^2 \overline{u}(t) - \overline{u}(t) - \mathrm{i}|u(t)|^2 \overline{u}(t)) + \overline{u}(t) \partial_t g(t) \Big] \, \mathrm{d}x \\ &= \operatorname{Re} \int_{\mathbb{T}} \Big[\mathrm{i} \, (u(t) \overline{g}(t) + \overline{u}(t) \partial_t g(t)) + |u(t)|^4 - |\partial_x u(t)|^2 \Big] \, \mathrm{d}x \end{aligned} \tag{9}$$

for $0 \leq t \leq b < a$. On the other hand, Sobolev's embedding theorem and complex interpolation (see Sect. 7.4.2 and 7.4.5 in [24]) yield

$$\|v\|_{L^4} \leq C \|v\|_{H^{1/4}} \leq C \|v\|_{L^2}^{\frac{3}{4}} \|v\|_{H^1}^{\frac{1}{4}},$$

and with Young's inequality, we obtain

$$\|v\|_{L^4}^4 \le \|\partial_x v\|_{L^2}^2 + \|v\|_{L^2}^2 + C\,\|v\|_{L^2}^6. \tag{10}$$

We thus deduce

$$\begin{aligned}\partial_t \mathcal{E}(t,u(t)) &\le \operatorname{Re}\int_{\mathbb{T}} \mathrm{i}(u(t)\overline{g}(t) + \overline{u}(t)\partial_t g(t))\,\mathrm{d}x + \|u(t)\|_{L^2}^2 + C\,\|u(t)\|_{L^2}^6 \\ &\le 2\,\|u(t)\|_{L^2}^2 + \tfrac{1}{2}\,\|g(t)\|_{L^2}^2 + \tfrac{1}{2}\,\|\partial_t g(t)\|_{L^2}^2 + C\,\|u(t)\|_{L^2}^6 \\ &\le 2C_0(b) + CC_0(b)^3 + C_1(b) \end{aligned} \tag{11}$$

with $C_1(b) := \frac{1}{2}\sup_{0\le t\le b}(\|g(t)\|_{L^2}^2 + \|\partial_t g(t)\|_{L^2}^2)$, where we also used (7). The estimate (10) further leads to the lower bound

$$\mathcal{E}(t,u(t)) \ge \tfrac{1}{4}\,\|\partial_x u(t)\|_{L^2}^2 - \tfrac{3}{4}C_0(b) - CC_0(b)^3 - \tfrac{1}{2}\,\|g(t)\|_{L^2}^2. \tag{12}$$

Combining (7), (11) and (12), we arrive at

$$\|u(t)\|_{H^1}^2 \le 4\mathcal{E}(0,u_0) + C(1+b)(C_0(b) + C_0(b)^3 + C_1(b)) \tag{13}$$

for $0 \le t \le b < a$. If a was finite, we could take here $b = a$ and obtain a contradiction to the blow-up condition stated in Step 1. Hence, $a = \infty$.

Step 3. If g is bounded in $L^2(\mathbb{T})$, then (7) shows that $u(t)$ is bounded in $L^2(\mathbb{T})$ for $t \ge 0$. In particular, we can replace $C_0(b)$ by C in this case.

If also $\partial_t g$ is bounded in $L^2(\mathbb{T})$, then (13) implies that $u(t)$ grows at most as $\sqrt{1+t}$ in $H^1(\mathbb{T})$.

Next, if g is contained in $L^2(\mathbb{R}_+; L^2(\mathbb{T}))$, then we infer from the line before (7) and Young's convolution inequality that $u \in L^2(\mathbb{R}_+; L^2(\mathbb{T}))$.

If even $g \in H^1(\mathbb{R}_+; L^2(\mathbb{T}))$, then g is also bounded in $L^2(\mathbb{T})$. Interpolating the previous steps we see that u belongs $L^6(\mathbb{R}_+; L^2(\mathbb{T}))$. Integrating in t, we now deduce from the line before (11) that $\mathcal{E}(t,u(t))$ is uniformly bounded. Hence, the boundedness of $u(t)$ in $H^1(\mathbb{T})$ follows from (12). □

4. Strang splitting for the Lugiato-Lefever equation

In order to formulate a numerical method for (1), it is convenient to define the nonlinear mapping

$$B\colon L^2(\mathbb{T}) \longrightarrow L^1(\mathbb{T}), \qquad B(w) = \mathrm{i}|w|^2.$$

If $w \in H^1(\mathbb{T})$, then $B(w) \in L^\infty(\mathbb{T})$ due to the Sobolev embedding $H^1(\mathbb{T}) \hookrightarrow L^\infty(\mathbb{T})$. For a *fixed* $w \in H^1(\mathbb{T})$, the function $x \mapsto B(w)(x) = \mathrm{i}|w(x)|^2$ will be identified with the multiplication operator

$$B(w)\colon L^2(\mathbb{T}) \to L^2(\mathbb{T}), \quad B(w)v = \mathrm{i}|w|^2 v$$

which generates a unitary group $(e^{tB(w)})_{t\in\mathbb{R}}$ on $L^2(\mathbb{T})$. The same Sobolev embedding implies the inequality

$$\|wv\|_{L^2} \le \|w\|_{L^\infty}\|v\|_{L^2} \le C\|w\|_{H^1}\|v\|_{L^2}, \qquad w \in H^1(\mathbb{T}),\ v \in L^2(\mathbb{T}). \tag{14}$$

If we define

$$k^* = \max\{1, k\} \quad \text{for} \quad k \in \mathbb{N}_0,$$

then the bound

$$\|wv\|_{H^k} \le C\|w\|_{H^{k^*}}\|v\|_{H^k}, \qquad w \in H^{k^*}(\mathbb{T}),\ v \in H^k(\mathbb{T}) \tag{15}$$

follows from (14). In particular, we obtain

$$\|B(w)v\|_{H^k} \le C\|w\|^2_{H^{k^*}}\|v\|_{H^k}\,, \quad k \ge 0\,. \tag{16}$$

As in Section 2 we let $A := \mathrm{i}\partial_x^2 - I$. Then, the Lugiato-Lefever equation (1) reads

$$\partial_t u = Au + B(u)u + g, \tag{17a}$$

$$u(0) = u_0. \tag{17b}$$

The solution is supposed to be approximated on the time-interval $[0, T]$ for $T > 0$.

4.1. ϕ-functions

In the construction and analysis of the splitting method for (1) we use the operator-valued functions $\phi_j(tA)$ defined by

$$\phi_j(tA)v = \int_0^1 \frac{\theta^{j-1}}{(j-1)!} e^{(1-\theta)tA} v \,\mathrm{d}\theta, \quad j \in \mathbb{N}, \qquad \phi_0(tA)v = e^{tA}v, \tag{18}$$

cf. [14]. For every $j, k \in \mathbb{N}_0$ and $t \ge 0$, the operator $\phi_j(tA)\colon H^k(\mathbb{T}) \to H^k(\mathbb{T})$ is bounded. In particular, we have

$$\|\phi_j(tA)v\|_{H^k(\mathbb{T})} \le \frac{1}{j!}\|v\|_{H^k(\mathbb{T})} \qquad \text{for all } v \in H^k(\mathbb{T})$$

due to

$$\|e^{tA}v\|_{H^k(\mathbb{T})} \le e^{-t}\|v\|_{H^k(\mathbb{T})} \quad \text{for all } v \in H^k(\mathbb{T}),\ k \in \mathbb{N}_0, \quad t \ge 0\,. \tag{19}$$

For every $v \in L^2(\mathbb{T})$ and $t > 0$, the recurrence relation

$$\phi_{j+1}(tA)v = (tA)^{-1}\left(\phi_j(tA)v - \frac{1}{j!}v\right), \qquad j \in \mathbb{N}_0 \tag{20}$$

follows from (18) via integration by parts. This recursion yields the Taylor expansions

$$e^{tA}v = \phi_0(tA)v = \sum_{k=0}^{m-1} \frac{t^k}{k!}A^k v + (tA)^m \phi_m(tA)v \tag{21}$$

for $m \in \mathbb{N}$ and $v \in D(A^m)$. Similar to (18) we define for $j \in \mathbb{N}$, $w \in H^1(\mathbb{T})$ and $v \in L^2(\mathbb{T})$

$$\phi_j(tB(w))v = \int_0^1 \frac{\theta^{j-1}}{(j-1)!} e^{(1-\theta)tB(w)} v \,\mathrm{d}\theta, \qquad \phi_0(tB(w))v = e^{tB(w)}v. \tag{22}$$

Equation (21) still holds if A is replaced by $B(w)$.

4.2. Time-integration scheme

Strang splitting methods for (17) are based on the observation that solving each of the two sub-problems

$$\partial_t v(t) = Av(t) + g(t), \tag{23}$$

$$\partial_t w(t) = B(w(t))w(t), \tag{24}$$

is much easier than solving (17a). Let $t_n = n\tau$ with step-size $\tau > 0$. Applying the variation-of-constants formula to (23) yields

$$v(t_{n+1}) = e^{\tau A} v(t_n) + \int_0^\tau e^{(\tau-s)A} g(t_n + s)\,\mathrm{d}s.$$

After $s \mapsto g(t_n + s)$ has been approximated by the linear interpolation

$$s \mapsto g(t_n) + s\frac{g(t_{n+1}) - g(t_n)}{\tau},$$

the integral can be computed analytically via integration by parts, and we obtain the exponential trapezoidal rule

$$v_{n+1} = e^{\tau A} v_n + \tau\Big(\phi_1(\tau A)g(t_n) + \phi_2(\tau A)\big(g(t_{n+1}) - g(t_n)\big)\Big) \tag{25}$$

which yields approximations $v_n \approx v(t_n)$ to the solution of (23); cf. [14].

The sub-problem (24) can even be solved exactly: Since

$$\partial_t\left(|w(t)|^2\right) = 2\mathrm{Re}\,(\overline{w}(t)\partial_t w(t)) = 2\mathrm{Re}\left(\mathrm{i}|w(t)|^4\right) = 0\,,$$

it follows that $|w(t)| = |w(0)|$ is time invariant, and hence the solution of (24) is given explicitly by

$$w(t) = e^{tB(w(0))}w(0)\,.$$

This is a well-known fact; see [8]. Approximations $u_n \approx u(t_n)$ to the solution of the *full* problem (17) can now be computed recursively with the Strang splitting

$$u_n^+ = e^{\tau B(u_n)/2}u_n, \tag{26a}$$

$$u_n^* = e^{\tau A}u_n^+ + \tau\Big(\phi_1(\tau A)g(t_n) + \phi_2(\tau A)\big(g(t_{n+1}) - g(t_n)\big)\Big), \tag{26b}$$

$$u_{n+1} = e^{\tau B(u_n^*)/2}u_n^*\,. \tag{26c}$$

Every time-step $u_n \mapsto u_{n+1}$ of the Strang splitting consists of three sub-steps. First, (24) is solved over the interval $[t_n, t_n + \frac{\tau}{2}]$ with initial data $w(t_n) = u_n$, which yields an update $u_n^+ = w(t_n + \frac{\tau}{2})$. Then, one step of the exponential trapezoidal rule (25) with step-size τ and $v_n = u_n^+$ is carried out, which turns u_n^+ into u_n^*. Finally, (24) is propagated over the interval $[t_n + \frac{\tau}{2}, t_{n+1}]$, which gives the new approximation $u_{n+1} \approx u(t_{n+1})$. Note that for $A = \mathrm{i}\Delta$ and $g(t) \equiv 0$, (26) reduces to the method considered in [18] for solving the NLS in absence of damping and forcing.

For every $\theta \geq 0$, the result after $n \in \mathbb{N}_0$ steps of the Strang splitting (26) with step-size $\tau > 0$ starting at time θ with initial data z will be denoted by

$$\Phi^n_{\tau,\theta}(z).$$

If $n = 1$, then we simply write $\Phi_{\tau,\theta}(z)$ instead of $\Phi^1_{\tau,\theta}(z)$. For any $\tau > 0$ and $n \in \mathbb{N}$, the relations

$$\Phi^0_{\tau,\theta}(z) = z, \qquad \Phi^n_{\tau,0}(z) = \Phi_{\tau,t_{n-1}}\left(\Phi^{n-1}_{\tau,0}(z)\right) = \Phi^{n-1}_{\tau,t_1}\left(\Phi_{\tau,0}(z)\right)$$

follow directly from the definition. In addition to the numerical flow $\Phi^n_{\tau,\theta}(z)$, we also consider the exact flow given by the exact solution of (17a)

$$t \mapsto \Psi_{t,\theta}(z)$$

with initial data $u(\theta) = z$ at time θ.

For the discretization of space the spectral collocation method can be used, i.e. the solution $u(t) = u(t,x)$ is approximated by a trigonometric polynomial which satisfies (1a) in $m \in \mathbb{N}$ equidistant collocation points $x_k = 2\pi k/m$; see [8] for details. If p is such a trigonometric polynomial, then $e^{\tau A}p$ can be easily computed by means of the fast Fourier transform. Terms like $e^{\tau B(p)/2}p$ are approximated with a trigonometric polynomial which interpolates the values $e^{\mathrm{i}\tau|p(x_k)|^2/2}p(x_k)$ in the collocation points. Hence, all terms in (26) can be evaluated quickly at low computational costs. In this paper, however, only the semidiscretization in time with the Strang splitting (26) and without any approximation in space will be analyzed.

4.3. Error analysis: main results

Our goal is to prove that the Strang splitting converges with order 1 in $H^1(\mathbb{T})$ and with order 2 in $L^2(\mathbb{T})$ on bounded time-intervals. In order to state our results, we define the abbreviations

$$m^k_u := \sup_{t\in[0,T]} \|u(t)\|_{H^k}, \qquad m^k_g := \sup_{t\in[0,T]} \|g(t)\|_{H^k},$$
$$m^k_{g'} := \sup_{t\in[0,T]} \|\partial_t g(t)\|_{H^k}, \qquad m^k_{g''} := \sup_{t\in[0,T]} \|\partial^2_t g(t)\|_{H^k}.$$

Observe that for $k \le 4$ the number m^k_u is finite by Theorem 1 and assumption (3). An inspection of the proof of Theorem 1 shows that m^4_u only depends on the norms of u_0 and g in the spaces involved in (3). For a solution $u(t) \in H^4(\mathbb{T})$ of (17) we immediately obtain the estimates

$$\sup_{t\in[0,T]} \|\partial_t u(t)\|_{H^k} \le C\left(m^{k+2}_u, m^k_g\right), \qquad \text{for } 0 \le k \le 2, \tag{27}$$

$$\sup_{t\in[0,T]} \|\partial^2_t u(t)\|_{L^2} \le C\left(m^4_u, m^2_g, m^0_{g'}\right). \tag{28}$$

The following theorem is the main result of the error analysis.

Theorem 2. *Let $u(t) = \Psi_{t,0}(u_0)$ be the exact solution of* (17) *and assume that the initial data u_0 and the forcing g have the regularity* (3)*. Then, the global error of the splitting method* (26) *is bounded by*

$$\|\Phi^n_{\tau,0}(u_0) - u(t_n)\|_{H^1} \le \tau C\left(T, m^3_u, m^1_g, m^1_{g'}\right), \tag{29}$$

$$\|\Phi^n_{\tau,0}(u_0) - u(t_n)\|_{L^2} \le \tau^2 C\left(T, m^4_u, m^2_g, m^1_{g'}, m^0_{g''}\right) \tag{30}$$

for all $n \in \mathbb{N}$ with $t_n = n\tau \le T$ and sufficiently small $\tau > 0$.

Theorem 2 is shown in Section 8. In (61) and (63) we give upper bounds for the step-size τ, but we remark that this step-size restriction is typically too pessimistic in practice. The outline of the proof is taken from [18]. The first two ingredients are bounds for the local error of (26) in $H^1(\mathbb{T})$ and in $L^2(\mathbb{T})$, respectively. They are established in Section 6 and 7.

Theorem 3 (Local error in $H^1(\mathbb{T})$). *Let $n \in \mathbb{N}$ with $t_{n+1} = t_n + \tau \leq T$. If $u(t_n) \in H^3(\mathbb{T})$ and if $g \in C^1([0,T]; H^1(\mathbb{T}))$, then the error after one step of the splitting method* (26) *is bounded by*

$$\left\| \Phi_{\tau,t_n}(u(t_n)) - \Psi_{\tau,t_n}(u(t_n)) \right\|_{H^1} \leq \tau^2 C(m_u^3, m_g^1, m_{g'}^1).$$

Theorem 4 (Local error in $L^2(\mathbb{T})$). *Let $n \in \mathbb{N}$ with $t_{n+1} = t_n + \tau \leq T$. Under assumption* (3) *the error after one step of the splitting method* (26) *is bounded by*

$$\left\| \Phi_{\tau,t_n}(u(t_n)) - \Psi_{\tau,t_n}(u(t_n)) \right\|_{L^2} \leq \tau^3 C(m_u^4, m_g^2, m_{g'}^1, m_{g''}^0).$$

The error bound for the global error of (26) is obtained by combining the bounds for the local error with the following stability result, proved in Section 5.

Theorem 5 (Stability). *Let $n \in \mathbb{N}$ with $t_{n+1} = t_n + \tau \leq T$. For $v, w \in H^1(\mathbb{T})$ with $\|v\|_{H^1} \leq M$ and $\|w\|_{H^1} \leq M$, the splitting method* (26) *satisfies*

$$\|\Phi_{\tau,t_n}(v) - \Phi_{\tau,t_n}(w)\|_{H^k} \leq e^{C\cdot(M_*^2 + M^2 - 1)} \|v - w\|_{H^k}, \qquad k = 0, 1 \tag{31}$$

with constant

$$M_* = e^{(CM^2-1)\tau} M + \tau C m_g^1. \tag{32}$$

5. Stability and auxiliary results

Now we state three lemmas which will be used frequently throughout the paper. The first lemma asserts a stability result for the mapping $v \mapsto e^{tB(v)}v$. As before, we let $k^* = \max\{1, k\}$.

Lemma 1. *If $v, w \in H^{k^*}(\mathbb{T})$ with $\|v\|_{H^{k^*}} \leq M$ and $\|w\|_{H^{k^*}} \leq M$ for some $k \in \mathbb{N}_0$, then*

$$\|e^{tB(v)}v - e^{tB(w)}w\|_{H^k} \leq e^{CM^2 t} \|v - w\|_{H^k}, \qquad t \geq 0, \tag{33a}$$

$$\|e^{tB(v)}v\|_{H^k} \leq M e^{CM^2 t}, \qquad t \geq 0. \tag{33b}$$

Note that for the stability in $L^2(\mathbb{T})$ (i.e. $k = 0$ and $k^* = 1$) the functions v and w have to belong to $H^1(\mathbb{T})$.

Proof. The proof uses ideas of [18]. Let $k \in \mathbb{N}_0$ and let $v, w \in H^{k^*}(\mathbb{T})$ with $\|v\|_{H^{k^*}} \leq M$ and $\|w\|_{H^{k^*}} \leq M$. Then, the functions $x(t) = e^{tB(v)}v$ and $y(t) = e^{tB(w)}w$ are the solutions of the initial value problems

$$\begin{aligned} x'(t) &= B(v)x(t), \quad x(0) = v, \quad t \geq 0, \\ y'(t) &= B(w)y(t), \quad y(0) = w, \quad t \geq 0, \end{aligned} \tag{34}$$

respectively, cf. Section 4.2. The inequality (15) implies that $B(v) \in H^k(\mathbb{T})$ and hence $e^{tB(v)} \in H^k(\mathbb{T})$, and applying (15) once again shows that $x(t) = e^{tB(v)}v \in H^k(\mathbb{T})$ for every $t \in [0, T]$. The same arguments yield that $y(t) \in H^k(\mathbb{T})$ for every $t \in [0, T]$.

First, we examine $\|x(t)\|_{H^k}$. From (16) we derive the estimate

$$\|B(v)x(t)\|_{H^k} \leq CM^2\|x(t)\|_{H^k}\,, \quad t \geq 0, \tag{35}$$

and hence

$$\|x(t)\|_{H^k} \leq \|x(0)\|_{H^k} + \int_0^t \|B(v)x(s)\|_{H^k}\,\mathrm{d}s \leq M + CM^2\int_0^t \|x(s)\|_{H^k}\,\mathrm{d}s\,.$$

Gronwall's lemma now yields

$$\|x(t)\|_{H^k} \leq Me^{CM^2t}, \tag{36}$$

which proves (33b). In order to show (33a), we consider the difference

$$\begin{aligned} B(v)x(t) - B(w)y(t) &= \mathrm{i}|v|^2x(t) - \mathrm{i}|w|^2y(t) \\ &= \mathrm{i}(v-w)\bar{v}x(t) + \mathrm{i}w(\bar{v}-\bar{w})x(t) + \mathrm{i}w\bar{w}\big(x(t)-y(t)\big)\,. \end{aligned}$$

Using also (15) and (36), we derive

$$\begin{aligned} \|B(v)x(t)-B(w)y(t)\|_{H^k} &\leq C[2M\|x(t)\|_{H^{k^*}}\|v-w\|_{H^k} + M^2\|x(t)-y(t)\|_{H^k}] \\ &\leq C[2M^2e^{CM^2t}\|v-w\|_{H^k} + M^2\|x(t)-y(t)\|_{H^k}]. \end{aligned}$$

The equations (34) thus imply

$$\begin{aligned} \|x(t)-y(t)\|_{H^k} &\leq \|v-w\|_{H^k} + \int_0^t \|B(v)x(s) - B(w)y(s)\|_{H^k}\,\mathrm{d}s \\ &\leq \left(1 + 2CM^2\int_0^t e^{CM^2s}\,\mathrm{d}s\right)\|v-w\|_{H^k} \\ &\quad + CM^2\int_0^t \|x(s)-y(s)\|_{H^k}\,\mathrm{d}s \end{aligned}$$

for $t \geq 0$. Since $0 \leq (e^{CM^2t}-1)^2$ yields $2e^{CM^2t} - 1 \leq e^{2CM^2t}$, it follows that

$$1 + 2CM^2\int_0^t e^{CM^2s}\,\mathrm{d}s = 1 + 2(e^{CM^2t}-1) \leq e^{2CM^2t}.$$

Applying Gronwall's lemma once again, we arrive at

$$\|e^{tB(v)}v - e^{tB(w)}w\|_{H^k} = \|x(t)-y(t)\|_{H^k} \leq e^{\widehat{C}M^2t}\|v-w\|_{H^k}$$

with $\widehat{C} = 3C$. □

The next lemma concerns technical estimates regarding the quantity u^* in the splitting method (26).

Lemma 2. *For $n \in \mathbb{N}_0$, a given v and $\tau \in [0,T]$ we define*

$$v^*(\tau) = e^{\tau A}e^{\tau B(v)/2}v + \tau\Big(\phi_1(\tau A)g(t_n) + \phi_2(\tau A)\big(g(t_n+\tau) - g(t_n)\big)\Big). \tag{37}$$

(i) If $v, u_n \in H^1(\mathbb{T})$ with $\|v\|_{H^1}, \|u_n\|_{H^1} \le M$, then

$$\|v^*(\tau)\|_{H^k} \le e^{(CM^2-1)\tau} M + \tau C m_g^k \qquad \textit{for } k = 0, 1\,.$$

(ii) If $k \in \{0, 1\}$ and $v, u_n \in H^{k+2}(\mathbb{T})$ with $\|v\|_{H^{k+2}}, \|u_n\|_{H^{k+2}} \le M$, then

$$\|\partial_\tau v^*(\tau)\|_{H^k} \le C(T, M, m_g^k, m_{g'}^k)\,.$$

(iii) If $v, u_n \in H^4(\mathbb{T})$ with $\|v\|_{H^4}, \|u_n\|_{H^4} \le M$, then

$$\|\partial_\tau^2 v^*(\tau)\|_{L^2} \le C(T, M, m_g^2, m_{g'}^0, m_{g''}^0).$$

Remark. If $v = u_n$, then $v^*(\tau)$ coincides with u_n^* defined in (26b). In the error analysis below, however, Lemma 2 will sometimes also be applied with $v = u(t_n)$.

Proof. The first assertion follows from Lemma 1 and the boundedness of the operators $\phi_j(\tau A)$. For the proof of (ii) and (iii), it is useful to represent the derivative $\partial_\tau \phi_j(\tau A)$ in terms of $\phi_{j-1}(\tau A)$ and $\phi_j(\tau A)$: If $v \in D(A)$, then (18) yields

$$\begin{aligned}
\partial_\tau \phi_j(\tau A) v &= \int_0^1 \frac{\theta^{j-1}}{(j-1)!}(1-\theta) A e^{(1-\theta)\tau A} v \,\mathrm{d}\theta \\
&= \int_0^1 \frac{\theta^{j-1}}{(j-1)!} A e^{(1-\theta)\tau A} v \,\mathrm{d}\theta - j \int_0^1 \frac{\theta^j}{j!} A e^{(1-\theta)\tau A} v \,\mathrm{d}\theta \\
&= (\phi_j(\tau A) - j\phi_{j+1}(\tau A)) A v,
\end{aligned} \tag{38}$$

and with (20) we obtain

$$\begin{aligned}
\partial_\tau \phi_0(\tau A) v &= e^{\tau A} A v\,, && v \in D(A), \quad \tau \ge 0, \\
\partial_\tau \phi_j(\tau A) v &= \frac{1}{\tau}(\phi_{j-1}(\tau A) - j\phi_j(\tau A)) v\,, && v \in L^2(\mathbb{T}), \quad j > 0, \quad \tau > 0.
\end{aligned}$$

Now (ii) and (iii) can be shown with straightforward calculations using (16), Lemma 1, the boundedness of $\phi_j(\tau A)$, and the fact that $\tau \le T$. □

After these preparations we are ready to prove stability of the Strang splitting scheme (26). In order to simplify notation (in particular in Sections 6 and 7), we define

$$B_{1/2}(u) = \frac{\mathrm{i}}{2}|u|^2 = \frac{1}{2} B(u). \tag{39}$$

Proof of Theorem 5. Let $v, w \in H^1(\mathbb{T})$ with $\|v\|_{H^1} \le M$ and $\|w\|_{H^1} \le M$. As in (26), we define

$$\begin{aligned}
v^* &= e^{\tau A} e^{\tau B_{1/2}(v)} v + \tau\Big(\phi_1(\tau A) g(t_n) + \phi_2(\tau A)\big(g(t_n+\tau) - g(t_n)\big)\Big), \\
w^* &= e^{\tau A} e^{\tau B_{1/2}(w)} w + \tau\Big(\phi_1(\tau A) g(t_n) + \phi_2(\tau A)\big(g(t_n+\tau) - g(t_n)\big)\Big)\,.
\end{aligned}$$

According to Lemma 2, we have $\|v^*\|_{H^1} \leq M_*$ and $\|w^*\|_{H^1} \leq M_*$ with M_* defined in (32). Applying Lemma 1 and using (19) results in the estimates

$$\begin{aligned}
\|\Phi_{\tau,t_n}(v) - \Phi_{\tau,t_n}(w)\|_{H^k} &= \|e^{\tau B_{1/2}(v^*)}v^* - e^{\tau B_{1/2}(w^*)}w^*\|_{H^k} \\
&\leq e^{CM_*^2\tau}\|v^* - w^*\|_{H^k} \\
&= e^{CM_*^2\tau}\|e^{\tau A}e^{\tau B_{1/2}(v)}v - e^{\tau A}e^{\tau B_{1/2}(w)}w\|_{H^k} \\
&\leq e^{(CM_*^2-1)\tau}\|e^{\tau B_{1/2}(v)}v - e^{\tau B_{1/2}(w)}w\|_{H^k} \\
&\leq e^{C\cdot(M_*^2+M^2-1)\tau}\|v - w\|_{H^k}
\end{aligned}$$

for $k \in \{0, 1\}$. □

The last lemma in this subsection will be useful in the proofs of Theorems 3 and 4.

Lemma 3. *For a given $n \in \mathbb{N}$ and $\tau \in [0, T]$ let*

$$b(\tau) = B\big(u(t_n + \tau)\big) - B\big(v^*(\tau)\big)\,,$$

where $v^*(\tau)$ is defined by (37) *with $v = u(t_n)$. Under the assumption* (3)*, we have*

$$b(\tau) = \int_0^\tau \partial_\tau b(s)\,\mathrm{d}s = \int_0^\tau \int_0^s \partial_\tau^2 b(r)\,\mathrm{d}r\,\mathrm{d}s.$$

Proof. The fundamental theorem of calculus gives

$$b(\tau) = b(0) + \int_0^\tau \partial_\tau b(s)\,\mathrm{d}s = b(0) + \tau\partial_\tau b(0) + \int_0^\tau \int_0^s \partial_\tau^2 b(r)\,\mathrm{d}r\,\mathrm{d}s.$$

As $v^*(0) = v = u(t_n)$ by assumption, it is clear that $b(0) = 0$. Hence, we only have to show that $\partial_\tau b(0) = 0$. By definition, we have

$$\partial_\tau b(\tau) = 2\mathrm{iRe}\,(\overline{u}(t_n + \tau)\partial_\tau u(t_n + \tau)) - 2\mathrm{iRe}\left(\overline{v^*}(\tau)\partial_\tau v^*(\tau)\right).$$

From $u(t_n) = v = v^*(0)$ and

$$\begin{aligned}
\partial_\tau u(t_n + \tau)\Big|_{\tau=0} &= Au(t_n) + B(u(t_n))u(t_n) + g(t_n), \\
\partial_\tau v^*(\tau)\Big|_{\tau=0} &= Av + B_{1/2}(v)v + g(t_n),
\end{aligned}$$

we deduce

$$\begin{aligned}
\partial_\tau b(0) &= 2\mathrm{iRe}\,(\overline{u}(t_n)\partial_\tau u(t_n)) - 2\mathrm{iRe}\,(\overline{u}(t_n)\partial_\tau v^*(0)) \\
&= 2\mathrm{iRe}\,(\overline{u}(t_n)B(u(t_n))u(t_n)) - 2\mathrm{iRe}\,(\overline{u}(t_n)B_{1/2}(u(t_n))u(t_n)) = 0
\end{aligned}$$

because $\overline{u}(t_n)B(u(t_n))u(t_n) = \mathrm{i}|u(t_n)|^4$ and $\overline{u}(t_n)B_{1/2}(u(t_n))u(t_n) = \frac{\mathrm{i}}{2}|u(t_n)|^4$ are both purely imaginary functions; cf. [2]. □

6. Local error in $H^1(\mathbb{T})$: Proof of Theorem 3

Without loss of generality we assume that $n = 0$, i.e. $u(t_n) = u(0) = u_0$ and $\Psi_{\tau,t_n}(u(t_n)) = \Psi_{\tau,0}(u_0) = u(\tau)$.

Step 1. The variation-of-constants formula yields the representation

$$u(\tau) = e^{\tau A}u_0 + \int_0^\tau e^{(\tau-s)A}\Big(B(u(s))u(s) + g(s)\Big)\,\mathrm{d}s$$

of the exact solution of (17). Substituting the formula a second time for $u(s)$, we obtain

$$u(\tau) = e^{\tau A}u_0 + I_1 + I_2 + R_1\,, \tag{40}$$

where we set

$$I_1 = \int_0^\tau e^{(\tau-s)A}B(u(s))e^{sA}u_0\ \mathrm{d}s, \tag{41}$$

$$I_2 = \int_0^\tau e^{(\tau-s)A}g(s)\ \mathrm{d}s, \tag{42}$$

$$R_1 = \int_0^\tau \int_0^s e^{(\tau-s)A}B(u(s))e^{(s-\sigma)A}\left[B(u(\sigma))u(\sigma) + g(\sigma)\right]\ \mathrm{d}\sigma\ \mathrm{d}s.$$

Using (16), it can be shown that

$$\|R_1\|_{H^1} \le \tau^2 C(m_u^1, m_g^1)\,. \tag{43}$$

The approximation $u_1 = \Phi_{\tau,0}(u_0)$ of the numerical method after one step reads

$$\begin{aligned} u_1 &= e^{\tau B_{1/2}(u_0^*)}e^{\tau A}e^{\tau B_{1/2}(u_0)}u_0 \\ &\quad + \tau e^{\tau B_{1/2}(u_0^*)}\Big(\phi_1(\tau A)g(0) + \phi_2(\tau A)\big(g(\tau) - g(0)\big)\Big) \end{aligned} \tag{44}$$

with $B_{1/2}(\cdot)$ defined in (39) and

$$u_0^* = e^{\tau A}e^{\tau B_{1/2}(u_0)}u_0 + \tau\Big(\phi_1(\tau A)g(0) + \phi_2(\tau A)\big(g(\tau) - g(0)\big)\Big).$$

We consider u_0^* as a function of $\tau > 0$. Using further the expansion

$$e^{\tau B_{1/2}(u_0)}v = \sum_{k=0}^{m-1}\frac{\tau^k}{k!}B_{1/2}^k(u_0)v + \tau^m B_{1/2}^m(u_0)\phi_m(\tau B_{1/2}(u_0))v$$

with $m \in \{1,2\}$, see (21), we derive

$$u_1 = e^{\tau A}u_0 + T_1 + T_2 + R_2\,, \tag{45}$$

with

$$T_1 = \tau\Big(B_{1/2}(u_0^*)e^{\tau A} + e^{\tau A}B_{1/2}(u_0)\Big)u_0, \tag{46}$$

$$T_2 = \tau\Big(\phi_1(\tau A)g(0) + \phi_2(\tau A)\big(g(\tau) - g(0)\big)\Big), \tag{47}$$

$$\begin{aligned} R_2 = \tau^2\Big[&e^{\tau A}B_{1/2}^2(u_0)\phi_2(\tau B_{1/2}(u_0))u_0 + B_{1/2}(u_0^*)e^{\tau A}B_{1/2}(u_0)\phi_1(\tau B_{1/2}(u_0))u_0 \\ &+ B_{1/2}^2(u_0^*)\phi_2(\tau B_{1/2}(u_0^*))e^{\tau A}e^{\tau B_{1/2}(u_0)}u_0 \\ &+ B_{1/2}(u_0^*)\phi_1(\tau B_{1/2}(u_0^*))\Big(\phi_1(\tau A)g(0) + \phi_2(\tau A)\big(g(\tau) - g(0)\big)\Big)\Big]\,. \end{aligned}$$

Estimate (16) and Lemmas 1 and 2 imply

$$\|R_2\|_{H^1} \leq \tau^2 C(T, m_u^1, m_g^1) . \tag{48}$$

Step 2. We compare the exact solution (40) with the numerical solution (45). Using (43) and (48), we infer

$$\|u(\tau) - u_1\|_{H^1} \leq \|I_1 - T_1\|_{H^1} + \|I_2 - T_2\|_{H^1} + \tau^2 C(m_u^1, m_g^1).$$

Our goal is now to bound the terms $\|I_1 - T_1\|_{H^1}$ and $\|I_2 - T_2\|_{H^1}$. With the abbreviations

$$h_1(s) = e^{(\tau - s)A} B(u(s)) e^{sA} u_0 , \tag{49}$$

$$b(\tau) = B(u(\tau)) - B(u_0^*(\tau)) , \tag{50}$$

the first term can be represented as $I_1 - T_1 = Q_1 + E_1$, where

$$Q_1 = \int_0^\tau h_1(s)\,\mathrm{d}s - \frac{\tau}{2}(h_1(0) + h_1(\tau)) \tag{51}$$

is the local quadrature error of the trapezoidal rule and

$$E_1 = \frac{\tau}{2} b(\tau) e^{\tau A} u_0 \tag{52}$$

is a remainder term. The order of the trapezoidal rule is two, and hence its local error scales like $O\left(\tau^3\right)$ if the integrand is smooth enough. For the proof of Theorem 3, however, the bound

$$\|Q_1\|_{H^1} \leq \tau^2 C \sup_{s \in [0,\tau]} \|\partial_s h_1(s)\|_{H^1} \tag{53}$$

is sufficient. Applying Lemma 3 with $n = 0$ and $t_n = 0$, the remainder term E_1 can be bounded by

$$\|E_1\|_{H^1} \leq \tau^2 C m_u^1 \sup_{s \in [0,\tau]} \|\partial_s b(s)\|_{H^1} .$$

The difference

$$I_2 - T_2 = \int_0^\tau e^{(\tau - s)A} g(s)\ \mathrm{d}s - \tau\Big(\phi_1(\tau A) g(0) + \phi_2(\tau A)(g(\tau) - g(0))\Big)$$

is the local error of the exponential trapezoidal rule so that

$$\|I_2 - T_2\|_{H^1} \leq \tau^2 C m_{g'}^1 ,$$

see Theorem 2.7 in [14].

Step 3. To complete the proof of Theorem 3, it remains to show that the terms

$$\sup_{s \in [0,\tau]} \|\partial_s h_1(s)\|_{H^1} \quad \text{and} \quad \sup_{s \in [0,\tau]} \|\partial_s b(s)\|_{H^1}$$

are bounded. The equations (16) and (27) yield

$$\sup_{s \in [0,\tau]} \|\partial_s h_1(s)\|_{H^1} \leq C\left(m_u^3, m_g^1\right).$$

Finally, in view of (27) and Lemma 2, the term

$$\partial_s b(s) = 2\mathrm{i}\left(\mathrm{Re}(\overline{u}(s)\partial_s u(s)) - \mathrm{Re}(\overline{u_0^*}(s)\partial_s u_0^*(s))\right)$$

can be estimated

$$\sup_{s\in[0,\tau]} \|\partial_s b(s)\|_{H^1} \le C\left(T\,, m_u^3\,, m_g^1\,, m_{g'}^1\right)\,,$$

which completes the proof of Theorem 3. □

7. Local error in $L^2(\mathbb{T})$: Proof of Theorem 4

To prove the third-order local error bound in $L^2(\mathbb{T})$, we mimic the proof for the second-order bound of the local error in $H^1(\mathbb{T})$. However, we have to expand the analytical solution and the numerical scheme to a higher order. As before, we assume without any loss of generality that $n = 0$ and let $u(\tau) = \Psi_{\tau,0}(u_0)$.

Step 1. We expand the exact solution further by inserting the variation-of-constants formula for $u(\sigma)$ into (40). It follows that

$$u(\tau) = e^{\tau A}u_0 + I_1 + I_2 + I_3 + I_4 + \hat{R}_1,$$

where I_1 and I_2 have been defined in (41) and (42), respectively, and we introduce

$$\begin{aligned}
I_3 &= \int_0^\tau \int_0^s e^{(\tau-s)A}B(u(s))e^{(s-\sigma)A}B(u(\sigma))e^{\sigma A}u_0 \,\mathrm{d}\sigma\,\mathrm{d}s,\\
I_4 &= \int_0^\tau \int_0^s e^{(\tau-s)A}B(u(s))e^{(s-\sigma)A}g(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}s,\\
\hat{R}_1 &= \int_0^\tau \int_0^s \int_0^\sigma e^{(\tau-s)A}B(u(s))e^{(s-\sigma)A}\\
&\qquad\qquad \times\; B(u(\sigma))e^{(\sigma-\xi)A}[B(u(\xi))u(\xi) + g(\xi)]\,\mathrm{d}\xi\,\mathrm{d}\sigma\,\mathrm{d}s.
\end{aligned}$$

The estimate (16) yields

$$\|\hat{R}_1\|_{L^2} \le \tau^3 C(m_u^1\,, m_g^0)\,. \tag{54}$$

Substituting the expansion

$$e^{\tau B_{1/2}(\cdot)} = I + \tau B_{1/2}(\cdot) + \frac{\tau^2}{2}B_{1/2}^2(\cdot) + \tau^3 B_{1/2}^3(\cdot)\phi_3(\tau B_{1/2}(\cdot))$$

into the splitting method (44), we derive

$$u_1 = e^{\tau A}u_0 + T_1 + T_2 + T_3 + T_4 + \hat{R}_2$$

with T_1, T_2 from (46), (47) and

$$\begin{aligned}
T_3 &= \frac{\tau^2}{2}\left(B_{1/2}^2(u_0^*)e^{\tau A} + 2B_{1/2}(u_0^*)e^{\tau A}B_{1/2}(u_0) + e^{\tau A}B_{1/2}^2(u_0)\right)u_0\\
T_4 &= \tau^2 B_{1/2}(u_0^*)\Big(\phi_1(\tau A)g(0) + \phi_2(\tau A)(g(\tau)-g(0))\Big)\\
\hat{R}_2 &= \tau^3\Big[e^{\tau A}B_{1/2}^3(u_0)\phi_3(\tau B_{1/2}(u_0))u_0 + B_{1/2}(u_0^*)e^{\tau A}B_{1/2}^2(u_0)\phi_2(\tau B_{1/2}(u_0))u_0\\
&\qquad + \frac{1}{2}B_{1/2}^2(u_0^*)e^{\tau A}B_{1/2}(u_0)\phi_1(\tau B_{1/2}(u_0))u_0\\
&\qquad + B_{1/2}^3(u_0^*)\phi_3(\tau B_{1/2}(u_0^*))e^{\tau A}e^{\tau B_{1/2}(u_0)}u_0\\
&\qquad + B_{1/2}^2(u_0^*)\phi_2(\tau B_{1/2}(u_0^*))\Big(\phi_1(\tau A)g(0) + \phi_2(\tau A)(g(\tau)-g(0))\Big)\Big].
\end{aligned}$$

Inequality (16) and Lemma 1 imply

$$\|\hat{R}_2\|_{L^2} \le \tau^3 C(m_u^1, m_g^0)\,. \tag{55}$$

Step 2. Comparing the exact solution with the numerical solution and using (54) and (55), we estimate

$$\begin{aligned}
\|u(\tau) - u_1\|_{L^2} \le &\|I_1 - T_1\|_{L^2} + \|I_2 - T_2\|_{L^2} + \|I_3 - T_3\|_{L^2}\\
&+ \|I_4 - T_4\|_{L^2} + \tau^3 C(m_u^1, m_g^0).
\end{aligned}$$

As before, the terms of the numerical solution are splitted into a suitable quadrature formula and a remainder term. In addition to $h_1(s)$ defined in (49) and $b(\tau)$ defined in (50), we employ the abbreviations

$$\begin{aligned}
h_2(s,\sigma) &= e^{(\tau-s)A}B(u(s))e^{(s-\sigma)A}B(u(\sigma))e^{\sigma A}u_0\,,\\
h_3(s) &= e^{(\tau-s)A}B(u(s))\phi_1(sA)g(0).
\end{aligned}$$

We still use the decomposition $I_1 - T_1 = Q_1 + E_1$ with the quadrature error Q_1 from (51) and the remainder E_1 from (52). Since now we aim at a local error in $L^2(\mathbb{T})$ of third order, we replace the error bound (53) by

$$\|Q_1\|_{L^2} \le \tau^3 C \sup_{s\in[0,\tau]} \|\partial_s^2 h_1(s)\|_{L^2}\,,$$

see [16]. Lemma 3 implies

$$\|E_1\|_{L^2} \le \tau^3 C m_u^1 \sup_{s\in[0,\tau]} \|\partial_s^2 b(s)\|_{L^2}\,.$$

The difference

$$I_2 - T_2 = \int_0^\tau e^{(\tau-s)A}g(s)\,\mathrm{d}s - \tau\Big(\phi_1(\tau A)g(0) + \phi_2(\tau A)(g(\tau)-g(0))\Big)$$

is the local error of the exponential trapezoidal rule. We thus acquire

$$\|I_2 - T_2\|_{L^2} \le \tau^3 C m_{g''}^0\,,$$

see [14]. For the third error term we use the partition $I_3 - T_3 = Q_3 + E_3$ with

$$Q_3 = \int_0^\tau \int_0^s h_2(s,\sigma)\,\mathrm{d}\sigma\,\mathrm{d}s - \frac{\tau^2}{8}\big(h_2(0,0) + 2h_2(\tau,0) + h_2(\tau,\tau)\big),$$
$$E_3 = \frac{\tau^2}{8} b(\tau)\Big(2e^{\tau A}B(u_0) + [B(u(\tau)) + B(u_0^*(\tau))]e^{\tau A}\Big)u_0,$$

and $b(\tau)$ defined in (50). We identify Q_3 as the error of a cubature formula which integrates constant functions exactly. It follows

$$\|Q_3\|_{L^2} \le C\tau^3 \left(\sup_{\triangle}\|\partial_s h_2(s,\sigma)\|_{L^2} + \sup_{\triangle}\|\partial_\sigma h_2(s,\sigma)\|_{L^2}\right),$$

where $\triangle$ is the triangle $0 \le s \le \tau\,, 0 \le \sigma \le s$, see p. 362 in [16]. From Lemma 3 we infer

$$\|E_3\|_{L^2} \le \tau^3 C m_u^1 \sup_{s\in[0,\tau]} \|\partial_s b(s)\|_{L^2}\,.$$

The fourth term is decomposed into three parts

$$I_4 - T_4 = E_4^1 + E_4^2 + E_4^3$$

given by

$$E_4^1 = \int_0^\tau e^{(\tau-s)A}B(u(s))F(s)\,\mathrm{d}s,$$
$$F(s) = \int_0^s e^{(s-\sigma)A}g(\sigma)\,\mathrm{d}\sigma - s\phi_1(sA)g(0),$$
$$E_4^2 = \int_0^\tau s h_3(s)\,\mathrm{d}s - \tau^2 B_{1/2}(u_0^*)\phi_1(\tau A)g(0),$$
$$E_4^3 = -\tau^2 B_{1/2}(u_0^*)\phi_2(\tau A)\big(g(\tau) - g(0)\big).$$

Since $F(s)$ is the local error of the exponential Euler rule, we can estimate

$$\sup_{s\in[0,\tau]} \|F(s)\|_{L^2} \le \tau^2 C m_{g'}^0\,,$$

see [14]. This inequality and (16) lead to

$$\|E_4^1\|_{L^2} \le \tau^3 C(m_u^1\,, m_{g'}^0)\,.$$

Integrating by parts, we calculate

$$\int_0^\tau s h_3(s)\,\mathrm{d}s = \frac{\tau^2}{2}h_3(\tau) - \frac{1}{2}\int_0^\tau s^2 \partial_s h_3(s)\,\mathrm{d}s,$$

so that

$$E_4^2 = \frac{\tau^2}{2} b(\tau)\phi_1(\tau A)g(0) - \frac{1}{2}\int_0^\tau s^2\partial_s h_3(s)\,\mathrm{d}s.$$

Lemma 3 then yields

$$\|E_4^2\|_{L^2} \le C\tau^3\Big(m_g^0 \sup_{s\in[0,\tau]} \|\partial_s b(s)\|_{L^2} + \sup_{s\in[0,\tau]} \|\partial_s h_3(s)\|_{L^2}\Big).$$

Exploiting the regularity of g, we obtain

$$\|g(\tau) - g(0)\|_{L^2} \le \tau m_{g'}^0 \, .$$

Estimate (16), Lemma 2 and the boundedness of $\phi_j(\tau A)$ finally imply

$$\|E_4^3\|_{L^2} \le \tau^3 C(m_u^1\,, m_g^0\,, m_{g'}^0) \, .$$

Step 3. To complete the proof of Theorem 4, it remains to show that the terms

$$\sup_{s\in[0,\tau]} \|\partial_s^2 h_1(s)\|_{L^2}\,, \quad \sup_{\triangle}\|\partial_s h_2(s,\sigma)\|_{L^2}\,, \quad \sup_{\triangle}\|\partial_\sigma h_2(s,\sigma)\|_{L^2}\,,$$
$$\sup_{s\in[0,\tau]} \|\partial_s h_3(s)\|_{L^2} \quad \text{and} \quad \sup_{s\in[0,\tau]} \|\partial_s^2 b(s)\|_{L^2}$$

are bounded. Formulas (16), (2) and (38) yield

$$\begin{aligned}
\sup_{s\in[0,\tau]} \|\partial_s^2 h_1(s)\|_{L^2} &\le C(m_u^4\,, m_g^2\,, m_{g'}^0),\\
\sup_{\triangle}\|\partial_s h_2(s,\sigma)\|_{L^2} &\le C(m_u^2\,, m_g^0),\\
\sup_{\triangle}\|\partial_\sigma h_2(s,\sigma)\|_{L^2} &\le C(m_u^2\,, m_g^0),\\
\sup_{s\in[0,\tau]} \|\partial_s h_3(s)\|_{L^2} &\le C(m_u^2\,, m_g^2).
\end{aligned}$$

We then apply (27), (28) and Lemma 2 to

$$\partial_s^2 b(s) = 2i\Big(|\partial_s u(s)|^2 + \mathrm{Re}(\overline{u}(s)\partial_s^2 u(s)) - |\partial_s u_0^*(s)|^2 - \mathrm{Re}(\overline{u_0^*}(s)\partial_s^2 u_0^*(s))\Big),$$

and conclude the last bound

$$\sup_{s\in[0,\tau]} \|\partial_s^2 b(s)\|_{L^2} \le C\left(T\,, m_u^4\,, m_g^2\,, m_{g'}^1\,, m_{g''}^0\right). \qquad \square$$

8. Global error: Proof of Theorem 2

In order to prove the global error estimates in Theorem 2, the local error bounds from Theorems 3 and 4 are combined with the stability result from Theorem 5 in the classical construction known as Lady Windermere's fan, see [12]. However, the stability result (31) can only be applied if the numerical solution $\Phi_{\tau,0}^n(u_0)$ stays bounded in $H^1(\mathbb{T})$ for all $n \in \mathbb{N}$ with $\tau n \le T$. This condition can be shown by the following induction argument. Let $u_0 \in H^3(\mathbb{T})$ and assume that there is a constant $\widehat{M} > m_u^1$ such that

$$\|\Phi_{\tau,t_\ell}^k(u(t_\ell))\|_{H^1} \le \widehat{M} \qquad \text{for all } \ell \in \mathbb{N}_0, \quad k = 0,\ldots,n-1, \quad t_{\ell+k} \le T. \tag{56}$$

We will prove that

$$\|\Phi^n_{\tau,t_\ell}(u(t_\ell))\|_{H^1} \le \widehat{M} \qquad \text{for all } \ell \in \mathbb{N}_0, \quad t_{\ell+n} \le T \tag{57}$$

provided that the step-size τ is sufficiently small. Since the argument is the same for all ℓ, we assume that $\ell = 0$ with no loss of generality. Representing $\Phi^n_{\tau,0}(u_0)$ by the telescoping sum

$$\Phi^n_{\tau,0}(u_0) = u(t_n) + \sum_{j=0}^{n-1} \Phi^{n-j}_{\tau,t_j}(u(t_j)) - \Phi^{n-j-1}_{\tau,t_{j+1}}(u(t_{j+1})) \tag{58}$$

with $u(t_n) = \Psi_{t_n,0}(u_0)$ and $u(t_0) = u_0$ gives

$$\|\Phi^n_{\tau,0}(u_0)\|_{H^1} \le \|u(t_n)\|_{H^1} + \sum_{j=0}^{n-1} \|\Phi^{n-j}_{\tau,t_j}(u(t_j)) - \Phi^{n-j-1}_{\tau,t_{j+1}}(u(t_{j+1}))\|_{H^1}. \tag{59}$$

According to (56), Theorem 5 can be applied and yields for $n - j - 1 \ge 1$ that

$$\begin{aligned} &\left\|\Phi^{n-j}_{\tau,t_j}(u(t_j)) - \Phi^{n-j-1}_{\tau,t_{j+1}}(u(t_{j+1}))\right\|_{H^1} \\ &= \left\|\Phi_{\tau,t_{n-1}}\left(\Phi^{n-j-1}_{\tau,t_j}(u(t_j))\right) - \Phi_{\tau,t_{n-1}}\left(\Phi^{n-j-2}_{\tau,t_{j+1}}(u(t_{j+1}))\right)\right\|_{H^1} \\ &\le e^{C\cdot(\widehat{M}_*^2+\widehat{M}^2-1)}\|\Phi^{n-j-1}_{\tau,t_j}(u(t_j)) - \Phi^{n-j-2}_{\tau,t_{j+1}}(u(t_{j+1}))\|_{H^1} \end{aligned} \tag{60}$$

with constant

$$\widehat{M}_* = e^{(C\widehat{M}^2-1)\tau}\widehat{M} + \tau C m_g^1,$$

cf. (32). If τ is sufficiently small, then $\widehat{M}_* \le C\widehat{M}$ so that $e^{C\cdot(\widehat{M}_*^2+\widehat{M}^2-1)} \le e^{C\widehat{M}^2}$. Applying (60) recursively, we then obtain

$$\begin{aligned} \|\Phi^{n-j}_{\tau,t_j}(u(t_j)) - \Phi^{n-j-1}_{\tau,t_{j+1}}(u(t_{j+1}))\|_{H^1} &\le e^{C\widehat{M}^2(n-j-1)\tau}\|\Phi_{\tau,t_j}(u(t_j)) - u(t_{j+1})\|_{H^1} \\ &\le e^{CT\widehat{M}^2} C_{loc}\tau^2 \end{aligned}$$

due to Theorem 3, with the constant C_{loc} from the local error bound. So (59) yields

$$\|\Phi^n_{\tau,0}(u_0)\|_{H^1} \le \|u(t_n)\|_{H^1} + n e^{CT\widehat{M}^2} C_{loc}\tau^2 \le m_u^1 + e^{CT\widehat{M}^2} C_{loc} T\tau.$$

If τ is so small that

$$\tau \le \frac{\widehat{M} - m_u^1}{C_{loc}T} e^{-CT\widehat{M}^2}, \tag{61}$$

then $\|\Phi^n_{\tau,0}(u_0)\|_{H^1} \le \widehat{M}$, as required.

It is now easy to show the bound for the global error in $L^2(\mathbb{T})$. The telescoping sum (58) yields

$$\|\Phi^n_{\tau,0}(u_0) - u(t_n)\|_{L^2} \le \sum_{j=0}^{n-1} \|\Phi^{n-j}_{\tau,t_j}(u(t_j)) - \Phi^{n-j-1}_{\tau,t_j}(u(t_{j+1}))\|_{L^2},$$

and with Theorem 5 and Theorem 4 we obtain similar as before

$$\|\Phi^n_{\tau,0}(u_0) - u(t_n)\|_{L^2} \le n e^{CT\widehat{M}^2} \tilde{C}_{loc}\tau^3 \le e^{CT\widehat{M}^2} \tilde{C}_{loc} T\tau^2 \tag{62}$$

with $\tilde{C}_{loc}$ denoting the constant from the local error bound in Theorem 4. The bound for the global error in $H^1(\mathbb{T})$ is obtained upon replacing $\|\cdot\|_{L^2}$ by $\|\cdot\|_{H^1}$, τ^p by τ^{p-1}, and $\tilde{C}_{loc}$ by C_{loc}. □

Remark. According to (62) the global error is small if

$$\tau^2 \ll \frac{1}{\tilde{C}_{loc} T} e^{-CT\widehat{M}^2}. \tag{63}$$

Hence, even if one could avoid the step-size restriction (61) imposed by *stability*, there is still a similar step-size restriction imposed by *accuracy*. Of course, both (61) and (63) are usually too pessimistic in practice. These step-size restrictions are *not* a characteristic property of the equation (1) nor of the splitting method (26). For example, the error bound for the global error of Runge-Kutta methods for solving *ordinary* differential equations is similar to (62); cf. Theorem 3.6 in chapter II in [12].

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