Local Operator Methods and Time Dependent Parabolic Equations on Non-Cylindrical Domains

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Abstract:
We investigate time dependent parabolic problems of diffusion type on open subsets of $\mathbb{R}^{N+1}$ and on networks, where the domains are possibly unbounded or non-cylindrical. The coefficients are assumed to be continuous and may be singular or degenerate at the boundary. We are looking for solutions which belong locally to suitable Sobolev spaces and vanish at the boundary. The well-posedness of the homogeneous linear problem is characterized by a barrier condition which is verified for a large class of highly singular domains. Using this result, we solve the inhomogeneous linear equation and obtain global solutions for Lipschitz nonlinearities of, e.g., logistic type. These applications are based on an abstract approach in the framework of local operators. In this context we derive maximum principles and characterize the well-posedness of the Cauchy problem by excessive barriers and Cauchy barriers. In the parabolic case, we construct a 'variable space propagator' using an associated 'space-time semigroup'. The propagator then allows to solve the above mentioned problems.

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1 Introduction and general information

1.1 Introduction

In this work we investigate time dependent, inhomogeneous, linear and semilinear parabolic problems of diffusion type with Dirichlet boundary conditions. In our principal applications the problems are given on open subsets of $\mathbb{R}^N \times [S,T]$ and on networks. Here we allow for unbounded and non-cylindrical domains with possibly irregular boundaries and consider elliptic operators with continuous coefficients which may be singular or degenerate on the boundary.

Our approach is based on the following idea which we describe in the case of a linear, homogeneous equation on a cylindrical domain $V \times [0,T]$ for an open, bounded subset V of $\mathbb{R}^N$. We are looking for solutions of the problem

$$\begin{aligned} \frac{\partial}{\partial t} u(x,t) &= A(x,t,D)u(x,t), \quad (x,t) \in V \times (0,T], \\ &= \sum_{k,l=1}^{N} a_{kl}(x,t) \frac{\partial^2}{\partial x_k \partial x_l} u(x,t) + \sum_{k=1}^{N} b_k(x,t) \frac{\partial}{\partial x_k} u(x,t) + c(x,t)u(x,t), \\ u(x,t) &= 0, \quad (x,t) \in \partial V \times [0,T], \\ u(\cdot,0) &= f \in C_0(V). \end{aligned} \tag{1.1}$$

We want to transform (1.1) into an abstract autonomous Cauchy problem. To that purpose, we set

$$L(x,t,D) = A(x,t,D) - \tfrac{\partial}{\partial t} \quad \text{and} \quad F(t) = f,\ 0 \le t \le T,$$

and let L be a suitable realization of $L(x,t,D)$ in $C_0(V \times [0,T])$. We then study the Cauchy problem

$$\begin{aligned} \frac{d}{d\sigma} v(\sigma) &= L\, v(\sigma), \quad \sigma > 0, \\ v(0) &= F \end{aligned} \tag{1.2}$$

in $C_0(V \times [0,T])$. This problem is well-posed if and only if L generates a semigroup[1] on $C_0(V \times [0,T])$. Given a solution v of (1.2), we define $u(x,t) = (v(t))(x,t) = v(t,x,t)$ for $(x,t) \in V \times [0,T]$. The function u satisfies the initial and boundary conditions of (1.1) and we have, formally,

$$\begin{aligned} \frac{\partial}{\partial t} u(x,t) &= \frac{\partial}{\partial \sigma} v(\sigma,x,t)\Big|_{\sigma=t} + \frac{\partial}{\partial \tau} v(t,x,\tau)\Big|_{\tau=t} \\ &= A(x,t,D)v(t,x,t) - \frac{\partial}{\partial \tau} v(t,x,\tau)\Big|_{\tau=t} + \frac{\partial}{\partial \tau} v(t,x,\tau)\Big|_{\tau=t} \\ &= A(x,t,D)u(x,t). \end{aligned}$$

[1] "Semigroup" always means "strongly continuous one-parameter semigroup of bounded linear operators".

(See also [27] and [54, p.290].) Of course, this heuristic computation can only be justified under strong regularity assumptions. Therefore we proceed in a somewhat different and more abstract way. First, we have to define L in a precise way. Second, we give conditions for L to be a generator. Third, we look for a representation of the semigroup generated by L. Using this representation, we then obtain solutions of (1.1) (and the more general equations indicated above). All this will be done in the framework of the theory of local operators as introduced in [30, 31, 32]. We point out that this approach allows to treat diffusion problems on networks at the same time.

We briefly outline the contents of the present paper, see also the sketch given in [40]. In Section 2.1 and 2.2, we recall the definition and basic properties of (parabolic) local operators. A local operator A is, roughly speaking, a collection of linear operators A^V acting in $C(V)$ for open subsets V of a metric space Ω such that the operators A^V respect restriction to subsets. Parabolic local operators are defined on 'space-time' domains $\underline{V} \subset \Omega \times J$ for an interval $J \subset \mathbb{R}$. We construct several extensions of a given local operator A, in particular the 'local closure' $\bar{A}$ (cf. Theorem 2.12), where we assume a condition of 'local dissipativity'. Then we establish maximum and comparison principles for locally dissipative, parabolic, local operators in Section 2.3 generalizing results from [41].

In Chapter 3, we show that the Cauchy problem for the part $\bar{A}_V$ of $\bar{A}$ in $C_0(V)$ is well-posed if and only if V possesses a 'Cauchy barrier' in the sense of Definition 3.1. This notion of a barrier is tailored to the Cauchy problems considered in this work. To obtain this characterization, we assume that the local operator A is locally dissipative and 'real', that $\bar{A}_\Omega$ is a generator, and that the domains of A^V are 'large enough'. Our arguments rely on methods from potential and semigroup theory and make heavy use of the concepts of 'excessive barriers', cf. [16], and 'local semigroups', cf. [55, 56]. Our main result Theorem 3.25 is a counterpart of the results in [30, 31, 32] where somewhat different hypotheses were supposed, compare Theorem 3.14 and 3.27. The present formulation is suited to parabolic problems on non-cylindrical domains. However, in the applications to degenerate problems we will use both types of results.

In a third step we investigate the semigroup generated by $\bar{L}_{\underline{V}}$ for a parabolic local operator L. It turns out, Section 4.2, that this so-called 'space-time semigroup' is induced by a variable-space propagator which satisfies the homogeneous equation. Using this representation, we then solve the corresponding inhomogeneous problem in Section 4.3 extending [42]. In the cylindrical case, similar results were obtained by L. Paquet [47, 48, 50]. Space-time semigroups were introduced by J. Howland, [27], in 1974 for an L^2–setting, see also [17, 44]. More recently, they have been used to study well-posedness and perturbation theory of non-autonomous Cauchy problems, see e.g. [38, 39, 51, 52], and asymptotic properties of solutions, see e.g. [9, 28, 53, 57].

Combining these facts we can establish existence and uniqueness of solutions of

$$\begin{cases} \bar{L}u = F & \text{on } \underline{V}, \\ u(x,t) = 0 & \text{on the 'lateral boundary' of } \underline{V}, \\ u(\cdot,S) = f & \text{on the 'bottom' of } \underline{V}, \end{cases}$$

if $\underline{V} \subset \Omega \times [S,T]$ possesses a Cauchy barrier with respect to $\bar{L}$, see Theorem 4.15, 4.16, and 4.24. Based on this result, we then show existence and uniqueness of global solutions of

$$\begin{cases} \bar{L}u + \Phi \circ u &= 0 \quad \text{on } \underline{V}, \\ u(x,t) &= 0 \quad \text{on the 'lateral boundary' of } \underline{V}, \\ u(\cdot, S) &= f \quad \text{on the 'bottom' of } \underline{V}, \end{cases}$$

where $0 \leq f \leq 1$ and Φ is locally Lipschitz and satisfies a sign condition, see Section 5.

This theory is used in Chapter 6 to solve parabolic problems on open subsets of $\mathbb{R}^N \times [S,T]$ and networks. Here we assume that the coefficients are real, continuous, and elliptic, but not necessarily bounded or uniformly elliptic. We allow for unbounded and non-cylindrical domains with possibly irregular boundaries. The class of admissible non-linearities contains, for instance, the logistic equation. By determining $\bar{L}$ explicitely, we can show that the solutions belong locally to suitable Sobolev spaces. In Section 6.1 we construct Cauchy barriers for several problems in $\mathbb{R}^N \times [S,T]$. This is done in the uniformly elliptic case for finite intersections of C^2–domains. For degenerate coefficients we consider regular boundaries, but also degeneracies taking place on irregular parts like isolated points or 'entering faces'.

For the convenience of the reader we add lists of hypotheses and notation.

1.2 List of hypotheses and definitions

1.3 List of symbols and notation

2 Local operators in general and parabolic local operators

2.1 Basic notations, closure and completion of local operators

Here (and in the following section) we introduce basic concepts and notation related to local operators which are used throughout this work.

By Ω we denote a locally compact Hausdorff space having a countable base. Let $\mathcal{O}(\Omega)$ be the collection of non-empty, open subsets V of Ω and let $\mathcal{O}_c(\Omega)$ consist of those $V \in \mathcal{O}(\Omega)$ being relatively compact. For $V \in \mathcal{O}(\Omega)$, we consider the spaces $C(V)$, $C_b(V)$, $C_0(V)$, $C_c(V)$ of continuous functions $f : V \to \mathbb{C}$ which are bounded, vanish at infinity, or have compact support, respectively. The space $C_b(V)$ is endowed with the sup-norm $\|f\| = \sup_V |f|$. We designate by $C^{\mathbb{R}}(V)$ the space of real valued $f \in C(V)$ and analogously for the other function spaces. Also, f^+ (f^-) is the positive (negative) part of $f \in C^{\mathbb{R}}(V)$. By $f|M$ we denote the restriction of a function f to a set M. If $V_n \subset V_{n+1}$ for $n \in \mathbb{N}$ and $\bigcup_{n\in\mathbb{N}} V_n = V$, then we write $V_n \uparrow V$.

Notice that Ω is metrizable and that there are sets $\Omega_n \in \mathcal{O}_c(\Omega)$ such that $\Omega_n \uparrow \Omega$ and $\overline{\Omega_n} \subset \Omega_{n+1}$. Further, if $V_n, V \in \mathcal{O}(\Omega)$ and $V_n \uparrow V$, then for each compact $K \subset V$ we have $K \subset V_n$ for $n \geq n_0(K)$. Throughout, $\mathcal{A}$ denotes a base of $\mathcal{O}(\Omega)$. We use the the following assumption.

(A) If $V, W \in \mathcal{A}$, then $V \cap W \in \mathcal{A}$.
For each compact $K \subset V \in \mathcal{A}$, there is $W \in \mathcal{A} \cap \mathcal{O}_c(\Omega)$ with $K \subset W \subset \overline{W} \subset V$.

Lemma 2.1. *Let* (A) *hold and* $V, V_n \in \mathcal{A}$ *satisfy* $V_n \uparrow V$. *Then there are* $W_k \in \mathcal{A} \cap \mathcal{O}_c(\Omega)$ *such that* $W_k \uparrow V$, $\overline{W_k} \subset W_{k+1}$, *and* $\overline{W_k} \subset V_{n_k}$ *for a subsequence* (n_k).

Proof. There exist sets $G_n \in \mathcal{O}_c(\Omega)$ such that $G_n \uparrow V$ and $\overline{G_n} \subset G_{n+1}$. Take n_1 with $\overline{G_1} \subset V_{n_1}$. Using (A), we find $W_1 \in \mathcal{A} \cap \mathcal{O}_c(\Omega)$ with $\overline{G_1} \subset W_1 \subset \overline{W_1} \subset V_{n_1}$. Next choose k_2 and n_2 such that $\overline{W_1} \subset G_{k_2} \subset \overline{G_{k_2}} \subset V_{n_2}$ and proceed inductively. □

Our approach relies on the concept of a local operator defined as follows, see e.g. [30], [31], [32], [36].

Definition 2.2. A *local operator* A defined on a base $\mathcal{A}$ of $\mathcal{O}(\Omega)$ is a collection of linear operators $A^V : D(A, V) \subset C(V) \to C(V)$ for $V \in \mathcal{A}$ such that $0 \in D(A, V)$ and for $V, W \in \mathcal{A}$ with $W \subset V$ and $f \in D(A, V)$ we have $f|W \in D(A, W)$ and $A^W(f|W) = (A^V f)|W$.

If it causes no confusion, we omit the superscript "V" and write f instead of $f|V$. By $(A_V, D(A_V))$ we denote the *part* of A^V in $C_0(V)$, that is,

$$A_V f = A^V f \quad \text{for } f \in D(A_V) = \{f \in C_0(V) \cap D(A, V) : A^V f \in C_0(V)\}.$$

We give some examples of local operators in order to illustrate the above definition. These operators will be used below to discuss some of our basic notions.

Example 2.3. *For* $\Omega = \mathbb{R}^n$ *and* $V \in \mathcal{O}(\mathbb{R}^n)$ *set* $A_k = \Delta$ *for* $k = 1, 2, 3$, *and* $D(A_1, V) = C^2(V)$, $D(A_2, V) = \{f \in W^{2,p}(V) : \Delta f \in C(V)\}$, $D(A_3, V) = \{f \in W^{2,p}_{loc}(V) : \Delta f \in C(V)\}$, *where* $W^{2,p}_{loc}(V)$ *is the usual Sobolev space and* $p > n$ *is fixed.*

We note that $D(A_3, V) = \{f \in C(V) : \Delta f \in C(V)\}$ for the Laplacian in the sense of distributions, cf. [11, II.3, Prop. 6, 8].

Example 2.4. *For* $\Omega = \mathbb{R}$, $V \in \mathcal{O}(\mathbb{R})$, *and an unbounded function* $m \in C(\mathbb{R})$ *define* $A_4 f = mf$ *and* $D(A_4, V) = \{f \in C(V) : mf \in C_b(V)\}$.

Example 2.5. *Let* $\Omega = \{z \in \mathbb{C} : 1 < |z| < 2\}$. *Let* $V \subset \Omega$ *be a generic open rectangle with sides parallel to the real and imaginary axis. Let* $\mathcal{A}$ *consist of* Ω *and of finite unions of such* V. *Also, for a compact* $K \subset \mathbb{C}$ *we denote by* $P(K)$ *the uniform closure of the polynomials* $p(z)$ *defined on* K. *We set* $A_5 f = f'$ *and* $D(A, W) = \{f \in C(W) : f|K \in P(K)$ *for all compact* $K \subset W\}$ *for* $W \in \mathcal{A}$. *Notice that (A) holds.*

In many applications, local operators are given by a partial differential operator on an open subset Ω of $\mathbb{R}^n$. We refer to [33], [46], and the papers cited above concerning elliptic operators and to [40], [48], [49], and Section 6.1 concerning parabolic operators. However, local operators can also be used to study evolution equations on networks and ramified spaces, see e.g. [34], [35], [37], and Section 6.2. Moreover, the concept of a local operator turns out to be the appropriate framework for various extensions of a given local operator A which we construct in this and the following section. These extensions will be essential for our work.

Next, we introduce classes of local operators for which, in some sense, A^V is determined by A^W for $W \subset V$.

Definition 2.6. Let A be a local operator on $\mathcal{A}$.

1. A is called *complete* if, for $V \in \mathcal{A}$, a function $f \in C(V)$ belongs to $D(A,V)$ provided that for all $x \in V$ there is $V_x \in \mathcal{A}$ such that $x \in V_x \subset V$ and $f|V_x \in D(A,V_x)$.

2. A is called *semi-complete* if, for $V \in \mathcal{A}$, a function $f \in C(V)$ belongs to $D(A,V)$ provided that there are $\mathcal{A} \ni V_n \uparrow V$ such that $f|V_n \in D(A,V_n)$.

Of course, the operator A_1 in Example 2.3 is complete and, a fortiori, semi-complete. The operator A_5 in Example 2.5 is semi-complete but not complete. In fact, it is easy to see that A_5 is semi-complete. On the other hand, take $g(z) = \frac{1}{z}$ for $z \in \Omega$ and $K = \{z \in \mathbb{C} : |z| = r\} \subset \Omega$. Then $g \notin P(K)$ since $\int_K g(z)\,dz \neq 0$ whereas $\int_K z^n\,dz = 0$ for $n \in \mathbb{N}$. Hence, $g \notin D(A,\Omega)$. However, $g|V$ belongs to $D(A,V)$ for each sufficiently small rectangle $V \subset \Omega$ since $g|V$ is given by a power series.

We say that a local operator A' on $\mathcal{A}'$ *extends* a local operator A defined on $\mathcal{A}$ if $\mathcal{A} \subset \mathcal{A}'$ and, for $f \in D(A,V)$ and $V \in \mathcal{A}$, we have $f \in D(A',V)$ and $Af = A'f$ on V. In this case we write $A \subset A'$. We recall that for each local operator A defined on $\mathcal{A}$ there exists the smallest complete extension of A, the *completion* $\hat{A}$, defined on $\mathcal{O}(\Omega)$ which is given by

$$D(\hat{A},V) = \{f \in C(V) : \forall\, x \in V\ \exists\, V_x \in \mathcal{A} \text{ with } x \in V_x \subset V,\ f|V_x \in D(A,V_x)\},$$
$$(\hat{A}^V f)(x) = (A^{V_x} f|V_x)(x) \quad \text{for} \quad V \in \mathcal{O}(\Omega),$$

cf. [30, p.412]. It is easy to see that $\widehat{A_2} = A_3$ in Example 2.3. In the remainder of this section we focus on the definition of a 'local closure'.

Definition 2.7. Let A be a local operator on $\mathcal{A}$.

1. We say that A is *locally closed* if the part of A^V in $C_b(V)$ is closed for all $V \in \mathcal{A}$; that is, if $f_n \to f$ and $A^V f_n \to g$ in $C_b(V)$ for $f_n \in D(A,V) \cap C_b(V)$, then $f \in D(A,V)$ and $A^V f = g$.

2. A is *locally closed u.c.* if, for $V \in \mathcal{A}$ and $f, g \in C(V)$, the existence of $\mathcal{A} \ni V_n \uparrow V$ and $f_n \in D(A,V_n)$ such that $f_n \to f$ and $Af_n \to g$ uniformly on compact subsets of V (u.c.) implies that $f \in D(A,V)$ and $A^V f = g$.

3. A is called *locally dissipative* if, for all $V \in \mathcal{A}$, compact $K \subset V$, and $f \in D(A,V)$ with $\sup_{V\setminus K}|f| < \sup_V |f|$, there exists $x_0 \in V$ such that $|f(x_0)| = \sup_V |f|$ and $Re\,((Af)(x_0)\,\overline{f(x_0)}) \leq 0$ (where $\sup_\emptyset |f| := -\infty$).

4. A is a *real* local operator if $f \in D(A,V)$ for $V \in \mathcal{A}$ implies that $\overline{f} \in D(A,V)$ and $A\overline{f} = \overline{Af}$ on V.

It is obvious that if A is real, then the part A_V in $C_0(\Omega)$ (or the part in $C_b(\Omega)$) is a real operator, i.e., for $f \in D(A_V)$ we have $\overline{f} \in D(A_V)$ and $A_V\overline{f} = \overline{A_V f}$. A straightforward application of Lemma 2.1 shows

Lemma 2.8. *Let A be a local operator on $\mathcal{A}$. If A is locally closed u.c., then A is locally closed and semi-complete. The converse holds if* (A) *is satisfied.*

It can be shown that the local operator A_3 defined in Example 2.3 is locally closed u.c., see [33], [46], and also Section 6.1. Notice that A_1 is a (semi-)complete local operator which is not locally closed if $n \geq 2$, cf. [11, II.3, Rem. 5]. On the other hand, A_4 from Example 2.4 is locally closed but not semi-complete.

The next lemma shows that local dissipativity is equivalent to a maximum modulus principle provided that some mild seperation assumptions hold for the local operator A on $\mathcal{A}$. In the following conditions, we denote by $\mathcal{A}_0$ a base of $\mathcal{O}(\Omega)$ which is contained in $\mathcal{A}$.

(S) For $W \in \mathcal{A}_0 \cap \mathcal{O}_c(\Omega)$ and $x_0 \in W$, there is $h \in D(A,W) \cap C(\overline{W})$ such that $h(x_0) > \sup_{\partial W}|h|$ and $Ah \in C_b(W)$ if $\partial W \neq \emptyset$ and $h(x_0) > 0$ if $\partial W = \emptyset$.

(S1) For $W \in \mathcal{A}_0$ and $x_0 \in W$, there is $h \in D(A_W) \subset C_0(W)$ such that $h(x_0) > 0$.

(S2) $D(A_W)$ is dense in $C_0(W)$ for $W \in \mathcal{A}_0$.

By $h \in D(A,W) \cap C(\overline{W})$ we mean that the function $h \in D(A,W)$ has a continuous extension to $\overline{W}$ usually denoted by the same symbol. Clearly, (S2) $\Rightarrow$ (S1) $\Rightarrow$ (S) and each extension of A satisfies the same separation assumption.

Lemma 2.9. *Let A be a local operator on $\mathcal{A}$. If, for $V \in \mathcal{A}$, the local maximum modulus principle*

$$f \in D(A,V),\ x_0 \in V,\ |f(x_0)| = \sup_V |f| \quad \Longrightarrow \quad Re\,((Af)(x_0)\,\overline{f(x_0)}) \leq 0 \tag{2.1}$$

holds, then A is locally dissipative. The converse is true provided that (S) *is satisfied.*

Proof. (1) Assume that (2.1) holds. Let $V \in \mathcal{A}$, K be a compact subset of V, and $f \in D(A,V)$ satisfy $\sup_{V\setminus K}|f| < \sup_V |f|$. Then there exists $x_0 \in K$ such that $|f(x_0)| = \sup_V |f|$. Thus, (2.1) yields $\mathrm{Re}\,(Af(x_0)\,\overline{f(x_0)}) \leq 0$, i.e., A is locally dissipative.

(2) Now assume that A is locally dissipative and that (S) holds. Let V, x_0, f be as in (2.1) and suppose that $\operatorname{Re}(Af(x_0)\,\overline{f(x_0)}) > 0$. Then we can find $W \in \mathcal{A} \cap \mathcal{O}_c(\Omega)$ and h_0 satisfying (S) such that $x_0 \in W \subset \overline{W} \subset V$ and

$$\operatorname{Re}(Af(x)\,\overline{f(x)}) \geq \alpha > 0 \tag{2.2}$$

for $x \in W$. If $\partial W = \emptyset$, then $W = \overline{W}$ is compact and, by local dissipativity, there is $x_1 \in W$ such that $\operatorname{Re}(Af(x_1)\,\overline{f(x_1)}) \leq 0$. This contradicts (2.2). So assume $\partial W \neq \emptyset$. We have $f(x_0) = \lambda\,|f(x_0)|$ for some $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. Set $h = \lambda h_0$ and $g_\sigma = f + \sigma h$ for $\sigma > 0$. The functions f, h, g_σ have continuous extensions to $\overline{W}$ which are denoted by the same symbols. From (S) and $f(x_0) = \sup_V |f|$ we derive

$$\begin{aligned} |g_\sigma(x_0)| &= |\,\lambda\,|f(x_0)| + \sigma\lambda h_0(x_0)| = |f(x_0)| + \sigma h_0(x_0) \\ &> \beta > |f(x)| + \sigma|h(x)| \geq |g_\sigma(x)| \end{aligned}$$

for $x \in \partial W$ and a constant β. In particular, the set $K = \{x \in W : |g_\sigma(x)| \geq \beta\} \neq \emptyset$ is compact in W. Since A is locally dissipative, there is $x_2 = x_2(\sigma) \in W$ such that

$$\operatorname{Re}(Ag_\sigma(x_2)\,\overline{g_\sigma(x_2)}) \leq 0. \tag{2.3}$$

On the other hand, (S) and (2.2) imply

$$\begin{aligned} \operatorname{Re}((Ag_\sigma)\,\overline{g_\sigma}) &= \operatorname{Re}((Af)\,\overline{f}) + \sigma \operatorname{Re}((Af)\,\overline{h}) + \sigma \operatorname{Re}((Ah)\,\overline{f}) + \sigma^2 \operatorname{Re}((Ah)\,\overline{h}) \\ &= \operatorname{Re}((Af)\,\overline{f}) + O(\sigma) \geq \tfrac{\alpha}{2} > 0 \end{aligned}$$

on W for $\sigma > 0$ small enough. This contradicts (2.3), and thus (2.1) holds. □

We note some consequences of the above result.

Corollary 2.10. *Let A be a local operator on $\mathcal{A}$ which satisfies* (S).

1. *If A is locally dissipative, then the part A_V of A in $C_0(V)$ is dissipative for $V \in \mathcal{A}$.*

2. *Let A be real. Then, A is locally dissipative if and only if the local positive maximum principle*

$$f \in D(A,V) \cap C^{\mathbb{R}}(V),\ x_0 \in V,\ 0 < f(x_0) = \sup\nolimits_V f \implies (Af)(x_0) \leq 0 \tag{2.4}$$

holds for all $V \in \mathcal{A}$.

Proof. 1. For $f \in D(A_V) \subset C_0(V)$ there is $x_0 \in V$ such that $|f(x_0)| = \|f\|$. So the dissipativity of A_V follows by considering the functional $\varphi_f = \overline{f(x_0)}\,\delta_{x_0}$.

2. Let A be real. First, assume that (2.4) holds. Let $V \in \mathcal{A}$, $f \in D(A,V)$, and $x_0 \in V$ such that $|f(x_0)| = \sup_V |f|$. We may assume that $|f(x_0)| > 0$. There is $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$ and $|f(x_0)| = \lambda f(x_0)$. Let $u = \operatorname{Re}(\lambda f)$ and $v = \operatorname{Im}(\lambda f)$.

Hence, $0 < u(x_0) = \sup_V u$ and $v(x_0) = 0$. Since A is real, we have $u, v \in D(A, V)$ and Au, Av are real. Now (2.4) implies

$$\begin{aligned}\mathrm{Re}\,(Af(x_0)\,\overline{f(x_0)}) &= \mathrm{Re}\,(A(\lambda f)(x_0)\,\overline{\lambda f(x_0)}) = \mathrm{Re}\,(Au(x_0)\,u(x_0) + iAv(x_0)\,u(x_0)) \\ &= Au(x_0)\,u(x_0) \le 0.\end{aligned}$$

Second, assume that A is locally disipative. Let $0 < f(x_0) = \sup_V f$ for $f \in D(A, V) \cap C^{\mathbb{R}}(V)$ and $V \in \mathcal{A}$. Suppose that $Af(x_0) > 0$. Choose $W \in \mathcal{A}$ such that $x_0 \in W \subset V$ and $f \ge 0$ on W. An application of (2.1) shows that $Af(x_0)f(x_0) \le 0$ which is impossible. □

As a consequence, the operator A_1 from Example 2.3 is locally dissipative due to the classical (strong) maximum principle. In fact, more general elliptic or parabolic operators yield locally dissipative local operators, see [33], [46] for the elliptic case and [40], [48], Chapter 6 for the parabolic case.

It is well known that a densely defined, dissipative operator possesses a dissipative closure. In the sequel, we show an analogue of this result for local operators using the mild seperation condition (S). Here we generalize [32, Thm. 3] where (S2) was used.

Lemma 2.11. *Assume that A is a locally dissipative, local operator on $\mathcal{A}$ which satisfies* (S). *Let $V, V_n \in \mathcal{A}$ and $f_n \in D(A, V_n)$ such that $V_n \uparrow V$ and $f_n \to f$, $Af_n \to g$ u.c. in $C(V)$ as $n \to \infty$.*

1. *If there is $x_0 \in V$ with $|f(x_0)| = \sup_V |f|$, then* $\mathrm{Re}\,(g(x_0)\overline{f(x_0)}) \le 0$.
2. *If $f = 0$, then $g = 0$.*

Proof. Let V, V_n, f, f_n, g be as in the statement.

1. Let $|f(x_0)| = \sup_V |f|$ for some $x_0 \in V$. We may assume that $|f(x_0)| > 0$. Suppose that $\mathrm{Re}\,(g(x_0)\overline{f(x_0)}) > 0$. Then there exists $W \in \mathcal{A} \cap \mathcal{O}_c(\Omega)$ such that $\mathrm{Re}\, g\overline{f} \ge \alpha > 0$ on W for some constant α and there is $\tilde{h}$ satisfying (S) for $x_0 \in W \subset \overline{W} \subset V$. Without loss of generality, let $\overline{W} \subset V_1$. Set $\lambda = f(x_0)\,|f(x_0)|^{-1}$ and $h = \lambda\tilde{h}$. Define $h_\sigma = \sigma h$ for $\sigma > 0$ and $\varphi_n = f_n - f$ and $\psi_n = Af_n - g$ on $\overline{W}$. Then $\varphi_n, \psi_n \to 0$ in $C(\overline{W})$. For each $\varepsilon > 0$, we estimate

$$\begin{aligned}\mathrm{Re}(A(f_n + h_\sigma)\,(\overline{f_n + h_\sigma})) &= \mathrm{Re}(g + \psi_n + \sigma Ah)\,(\overline{f} + \overline{\varphi_n} + \sigma\overline{h})) \\ &= \mathrm{Re}(g\overline{f}) + \mathrm{Re}(g\overline{\varphi_n}) + \mathrm{Re}(\psi_n\overline{f}) + \mathrm{Re}(\psi_n\overline{\varphi_n}) + \sigma^2\mathrm{Re}(\overline{h}Ah) \\ &\quad + \sigma[\mathrm{Re}(g\overline{h}) + \mathrm{Re}(\psi_n\overline{h}) + \mathrm{Re}(\overline{f}Ah) + \mathrm{Re}(\overline{\varphi_n}Ah)] \\ &\ge \mathrm{Re}(g\overline{f}) + O(\sigma) - \varepsilon \ge \alpha + O(\sigma) - \varepsilon\end{aligned}$$

on W for n large enough. Therefore we can find $\sigma_0 > 0$ and $n_0 \in \mathbb{N}$ such that

$$\mathrm{Re}\,(A(f_n + h_\sigma)\,(\overline{f_n + h_\sigma})) \ge \tfrac{\alpha}{2} > 0 \tag{2.5}$$

on W for $n \geq n_0$ and $\sigma \in (0, \sigma_0]$. On the other hand, for fixed $\sigma \in (0, \sigma_0]$ condition (S) yields

$$\begin{aligned}|(f_n + h_\sigma)(x_0)| &= |\,\lambda\,|f(x_0)| + \varphi_n(x_0) + \lambda\sigma\tilde{h}(x_0)| \geq |f(x_0)| + \sigma\tilde{h}(x_0) - |\varphi_n(x_0)| \\ &> \beta > \gamma > |f(x)| + \sigma\tilde{h}(x)\end{aligned}$$

for $x \in \partial W$, constants $\beta, \gamma > 0$, and $n \geq n_1 \geq n_0$. Moreover,

$$|(f_n + h_\sigma)(x)| \leq |f(x)| + |\varphi_n(x)| + \sigma\tilde{h}(x) \leq \gamma$$

on ∂W for $n \geq n_2 \geq n_1$. Fix $n \geq n_2$. Notice that the set $K = \{x \in W : |f_n(x) + h_\sigma(x)| \geq \beta\} \neq \emptyset$ is compact in W. Since A is locally dissipative, we find $x_1 \in W$ such that

$$\mathrm{Re}\,(A(f_n + h_\sigma)\,(\overline{f_n + h_\sigma}))(x_1) \leq 0$$

This violates (2.5), and so the first assertion is established.

2. Let $f = 0$. Suppose that $|g(x_0)| > 0$ for some $x_0 \in V$. Then there is $W \in \mathcal{A} \cap \mathcal{O}_c(\Omega)$ such that

$$|g(x)| > 0 \qquad \text{for} \quad x \in W \tag{2.6}$$

and there is h satisfying (S) for $x_0 \in W \subset \overline{W} \subset V$. Choose $x_1 \in W$ with $|h(x_1)| = \sup_W |h| > 0$. Without loss of generality, let $\overline{W} \subset V_1$. We have $f_n + \lambda\sigma h \to \lambda\sigma h$ and $A(f_n + \lambda\sigma h) \to g + \lambda\sigma Ah$ in $C(\overline{W})$ as $n \to \infty$ for all $\sigma > 0$ and $\lambda \in \mathbb{C}$. Also, $\sigma|\lambda h(x_1)| = \sup_W |\sigma\lambda h|$. So the first part of the proof shows

$$0 \geq \mathrm{Re}\,((g + \lambda\sigma Ah)(x_1)\,\sigma\overline{\lambda h(x_1)}) = \sigma\left[\mathrm{Re}\,(\overline{\lambda} g(x_1)\overline{h(x_1)}) + \sigma|\lambda|^2\,\mathrm{Re}\,(\overline{h(x_1)}Ah(x_1))\right].$$

Dividing by σ and letting $\sigma \to 0$, yields $\mathrm{Re}\,(\overline{\lambda} g(x_1)\overline{h(x_1)}) \leq 0$ for all $\lambda \in \mathbb{C}$. Taking $\lambda = g(x_1)\overline{h(x_1)}$, we obtain $g(x_1) = 0$. This contradicts (2.6), and hence $g = 0$. □

Given a local operator A on $\mathcal{A}$, we set

$$\begin{aligned} D(\bar{A}, V) = \{f \in C(V) &: \text{there are } V_n \in \mathcal{A},\ f_n \in D(A, V_n),\ g \in C(V) \text{ such that} \\ &V_n \uparrow V,\ f_n \to f, Af_n \to g \ \text{ u.c. in } C(V)\}, \\ \bar{A}^V f = g \qquad &\text{on} \ \ V \in \mathcal{A}, \end{aligned} \tag{2.7}$$

compare [32]. To show that $\bar{A}^V$ is well defined, we assume that A is locally dissipative and that (A) and (S) hold. Let $f, g, g' \in C(V)$, $V_n, V_n' \in \mathcal{A}$, $f_n \in D(A, V_n)$, and $f_n' \in D(A, V_n')$ such that $V_n \uparrow V$, $V_n' \uparrow V$, and $f_n \to f$, $f_n' \to f$, $Af_n \to g$, $Af_n' \to g'$ u.c. in $C(V)$. Then $V_n \cap V_n' \neq \emptyset$ for n large enough and $W_n = V_n \cap V_n' \in \mathcal{A}$. Since $W_n \uparrow V$ and $f_n - f_n' \to 0$, $A(f_n - f_n') \to g - g'$ u.c. in $C(V)$, Lemma 2.11 yields that $g = g'$. Similarly, one verifies that $\bar{A}^V$ is linear and that $\bar{A}$ is a local operator. Moreover, we have

Theorem 2.12. *Let A be a locally dissipative, local operator defined on $\mathcal{A}$ satisfying* (A) *and* (S)*. Then $\bar{A}$ is a locally dissipative, local operator on $\mathcal{A}$ and the smallest semi-complete and locally closed extension of A. If A is real, then $\bar{A}$ is also real.*

Proof. By Lemma 2.11 and 2.9, $\bar{A}$ is locally dissipative. Let $D(\bar{A}, V_n) \ni f_n \to f$ and $\bar{A}f_n \to g$ u.c. in $C(V)$, where $V, V_n \in \mathcal{A}$ and $V_n \uparrow V$. We may assume that $\overline{V_n}$ is compact and contained in V_{n+1} due to Lemma 2.1. There exist $V_{nk} \in \mathcal{A}$ and $f_{nk} \in D(A, V_{nk})$ such that $V_{nk} \uparrow V_n$ and $f_{nk} \to f_n$, $Af_{nk} \to \bar{A}f_n$ u.c. in $C(V_n)$ as $k \to \infty$. For $n \in \mathbb{N}$, there is k_n such that

$$\overline{V}_{n-1} \subset V_{nk_n}\,, \ \ \sup_{\overline{V}_{n-1}} |f_n - f_{nk_n}| \le \frac{1}{n}\,, \ \ \text{and} \ \ \sup_{\overline{V}_{n-1}} |\bar{A}f_n - Af_{nk_n}| \le \frac{1}{n}\,.$$

Set $W_n := V_{nk_n}$ and $h_n := f_{nk_n} \in D(A, W_n)$. Then $W_n \uparrow V$ and $h_n \to f$, $Ah_n \to g$ u.c. in $C(V)$. That is, $f \in D(\bar{A}, V)$ and $\bar{A}f = g$. As a result, $\bar{A}$ is locally closed u.c. and hence, by Lemma 2.8, semi-complete and locally closed. The other assertions are clear. □

Definition 2.13. Let A be a local operator defined on a base $\mathcal{A}$. If (2.7) defines a local operator on $\mathcal{A}$, then we call it the *local closure* $\bar{A}$ of A.

We can combine the completion and the local closure of A.

Theorem 2.14. *Let A be a locally dissipative, local operator defined on $\mathcal{A}$ satisfying* (A) *and* (S)*. Then $\bar{\hat{A}}$ exists and is a locally dissipative, locally closed, and semi-complete extension of A defined on $\mathcal{O}(\Omega)$. If A is real, then $\bar{\hat{A}}$ is also real.*

Proof. In view of Theorem 2.12 we only have to show that $\hat{A}$ (which always exists) is locally dissipative. To apply Lemma 2.9, let $f \in D(\hat{A}, V)$, $V \in \mathcal{O}(\Omega)$, and $x_0 \in V$ with $|f(x_0)| = \sup_V |f|$. Then there is $W \in \mathcal{A}$ such that $\hat{A}^V f(x_0) = A^W f(x_0)$. We conclude $\operatorname{Re}(\hat{A}f(x_0)\,\overline{f(x_0)}) \le 0$ by the local dissipativity of A. □

2.2 Parabolic extension of local operators in space-time

At first, we recall the concept of a *parabolic* local operator as introduced in [40], [41], [48], [49], see also [38], [39], [42], [50]. For the remainder of this chapter we let $J \in \{\,]S,T[\,,]S,T], [S,T]\}$, where $S = -\infty$ and $T = +\infty$ is possible if $S \notin J$ and $T \notin J$, respectively. The collection of non-empty, open subintervals of J is designated by $\mathcal{J}$. Set $\underline{\Omega} = \Omega \times J$. We denote by $\underline{V}$ a generic non-empty, open subset of $\underline{\Omega}$ and by $\underline{x} = (x,t)$ a generic element of $\underline{\Omega}$. Further,

$$\underline{\mathcal{A}} = \{\underline{V} = V \times I : V \in \mathcal{O}(\Omega),\, I \in \mathcal{J}\}$$

is a base of $\mathcal{O}(\underline{\Omega})$ satisfying (A). For $\underline{M} \subset \underline{\Omega}$ and $t \in J$, define

$$M(t) = \{x \in \Omega : (x,t) \in \underline{M}\}, \ \ \underline{M}(t) = M(t) \times \{t\}, \ \ \underline{M}_t = \{(x,s) \in \underline{M} : s > t\}$$
$$I_{\underline{M}} = \{t \in J : M(t) \ne \emptyset\}, \ \ S_{\underline{M}} = \inf\{t \in \mathbb{R} : t \in I_{\underline{M}}\}, \ \ T_{\underline{M}} = \sup\{t \in \mathbb{R} : t \in I_{\underline{M}}\}.$$

Note that $S_{\underline{V}} < T_{\underline{V}}$ and that $I_{\underline{V}}$ and $V(t)$ are open in J and V, respectively, for $\underline{V} \in \mathcal{O}(\underline{\Omega})$.

The following definition is motivated by the properties of the local operator given by $L = \Delta_x - \frac{d}{dt}$ on $D(L, \underline{V}) = C^{2,1}(\underline{V})$ with $\underline{V} \in \mathcal{O}(\mathbb{R}^n \times \mathbb{R})$.

Definition 2.15. A local operator L defined on $\underline{\mathcal{A}}$ (or $\mathcal{O}(\underline{\Omega})$) is called *parabolic* if, for $F \in D(L, \underline{V})$, $\underline{V} \in \underline{\mathcal{A}}$ (or $\underline{V} \in \mathcal{O}(\underline{\Omega})$), and $\varphi \in C^1(I_{\underline{V}})$ with $\varphi'(S) = 0$ if $S \in I_{\underline{V}}$, we have $\varphi F \in D(L, \underline{V})$ and $L(\varphi F) = \varphi\, LF - \varphi' F$.
If $S \in J$ and, in addition, $LF = 0$ on $\underline{V}(S)$ for all $F \in D(L, \underline{V})$, then we say that L is *standard parabolic*.

Here, $\varphi F \in C(\underline{V})$ is defined by $(\varphi F)(\underline{x}) = \varphi(t)F(x,t)$. Examples of parabolic local operators are discussed in the above references and in Chapter 6. We first show that parabolicity is preserved under completion and local closure.

Proposition 2.16. *Assume that L is a (standard) parabolic local operator defined on $\underline{\mathcal{A}}$ (or $\mathcal{O}(\underline{\Omega})$). Then the local operators $\hat{L}$ and $\bar{L}$ (if the latter exists) are also (standard) parabolic.*

Proof. Assume that $\bar{L}$ exists. Let $F \in D(\bar{L}, \underline{V})$, $\underline{V} \in \underline{\mathcal{A}}$ (or $\underline{V} \in \mathcal{O}(\underline{\Omega})$), and $\varphi \in C^1(I_{\underline{V}})$ with $\varphi'(S) = 0$ if $S \in I_{\underline{V}}$. There are $\underline{V}_n \uparrow \underline{V}$, $\underline{V}_n \in \underline{\mathcal{A}}$ (or $\underline{V}_n \in \mathcal{O}(\underline{\Omega})$), and $F_n \in D(L, \underline{V}_n)$ such that $F_n \to F$ and $LF_n \to \bar{L}F$ u.c. in $C(\underline{V})$. Denote the restriction of φ to $I_{\underline{V}_n}$ by $\varphi_n \in C^1(I_{\underline{V}_n})$. Then $\varphi_n F_n \to \varphi F$ u.c. in $C(\underline{V})$, $\varphi_n F_n \in D(L, \underline{V}_n)$, and

$$L(\varphi_n F_n) = \varphi_n LF_n - \varphi_n' F_n \;\to\; \varphi \bar{L}F - \varphi' F$$

u.c. in $C(\underline{V})$. Hence, $\bar{L}$ is parabolic. The other claims can be verified similarly. □

The following construction will be crucial for our characterization of the solvability of Cauchy problems of diffusion type in Chapter 3. Let $I =]a,b[$ for possibly infinite a, b and $K \subset \mathbb{Z}$. A set $\{s_k, k \in K\} \subset \bar{I}$ is called a *partition* of I if $s_k < s_{k+1}$ for $k \in K$, $\{s_k, k \in K\}$ is locally finite, and $\bigcup_k]s_k, s_{k+1}[= I$. We set $I_k =]s_k, s_{k+1}[$ and $\underline{V}_k = V \times I_k$. Obviously, if I is bounded, then K has to be finite and $s_0 = a$ and $s_n = b$ (where we let $0 = \min K$ and $n = \max K$). If $I = \mathbb{R}$, then $K = \mathbb{Z}$ and $\lim_{k\to\pm\infty} s_k = \pm\infty$; and similarly for semibounded intervals. In the following definition we consider a local operator L defined on $\underline{\mathcal{A}}$. The analogous concept for a local operator on $\mathcal{O}(\underline{\Omega})$ is introduced in Definition 2.24.

Definition 2.17. Let L be a local operator on $\underline{\mathcal{A}}$, where $J =]S,T[$. The *parabolic extension* L_p of L is defined by

$$\begin{aligned} D(L_p, \underline{V}) = \{F \in C(\underline{V}) : {}& \text{there is a partition } \{s_k, k \in K\} \text{ of } I \text{ and } G \in C(\underline{V}) \text{ such} \\ & \text{that } F_k = F|\underline{V}_k \in D(L, \underline{V}_k) \text{ and } LF_k = G|\underline{V}_k \;\text{ for } k \in K\}, \\ L_p F = G \qquad & \text{on } \;\underline{V} = V \times I \in \underline{\mathcal{A}}. \end{aligned}$$

The union of two partitions of I is a refinement of both partitions. In view of this fact, it is easy to see that $L_p^{\underline{V}}$ is a well-defined linear operator in $C(\underline{V})$ for all $\underline{V} \in \underline{\mathcal{A}}$. Similarly, a partition of I induces a partition of any open subinterval $I' \subset I$. This implies that L_p is a local operator on $\underline{\mathcal{A}}$. Thus we have

Proposition 2.18. *Definition* 2.17 *yields a local operator L_p on $\underline{\mathcal{A}}$ extending L. If L is real, then L_p is also real.*

At first, we consider permanence properties of the parabolic extension.

Proposition 2.19. *Let L be a parabolic local operator on $\underline{\mathcal{A}}$ with $J =]S, T[$. Then L_p is parabolic.*

Proof. Let $\underline{V} = V \times I \in \underline{\mathcal{A}}$, $F \in D(L_p, \underline{V})$ with a partition $\{s_k\}$, and $\varphi \in C^1(I)$. Then, $(\varphi F)_k = (\varphi|I_k)\, F_k \in D(L, \underline{V}_k)$ and

$$L(\varphi F)_k = (\varphi|I_k)\, LF_k - (\varphi|I_k)'\, F_k = (\varphi\, L_p F - \varphi' F)|\underline{V}_k$$

since L is parabolic. So $\varphi F \in D(L_p, \underline{V})$ and $L_p(\varphi F) = \varphi\, L_p F - \varphi' F$. □

Proposition 2.20. *Let L be a locally disspative, parabolic, local operator on $\underline{\mathcal{A}}$ with $J =]S, T[$. Then L_p is locally dissipative.*

Proof. Let $\underline{V} = V \times I \in \underline{\mathcal{A}}$, $0 \neq F \in D(L_p, \underline{V})$ with a partition $\{s_k\}$, $\hat{\underline{K}} \subset \underline{V}$ be compact, and

$$|F(\underline{x})| \leq \beta < \sup_{\underline{V}} |F| = \|F\| \qquad \text{for } \underline{x} \in \underline{V} \setminus \hat{\underline{K}}$$

and a constant β. Clearly, $\|F\| = \max_{\hat{\underline{K}}} |F|$ and $\underline{V}$ is not compact since I is open. Define $\underline{V}_k' = \underline{V}_k \cup (V \times \{s_{k+1}\})$ and $\underline{M} = \{\underline{x} \in \underline{V} : |F(\underline{x})| = \|F\|\}$. Then $\underline{M}$ is compact, not empty and contained in $\hat{\underline{K}}$. Since the partition is locally finite, there is a lowest index m with $\underline{V}_m' \cap \underline{M} \neq \emptyset$. Notice that $|F(\underline{x}_n)| \to \|F\|$ for $\underline{x}_n \in \underline{V}$ implies that a subsequence $\underline{x}_k$ converges to some $\underline{x}_0 \in \underline{M}$ since $\underline{x}_n \in \hat{\underline{K}}$ for large n. Therefore, there is a constant $\beta < \alpha < \|F\|$ and a compact set $\underline{K} \subset \hat{\underline{K}}$ such that $\underline{M} \subset \underline{K}$,

$$|F(\underline{x})| \leq \alpha < \sup_{\underline{V}} |F| = \|F\| \qquad \text{for } \underline{x} \in \underline{V} \setminus \underline{K}, \tag{2.8}$$

and $\underline{K} \cap \underline{V}_k' = \emptyset$ for $k < m$. Set $\underline{K}_m = \underline{K} \cap \underline{V}_m'$. For every sufficiently small $\varepsilon > 0$, there exists $\bar{s}_1 \in]s_m, s_{m+1}[$ such that

$$\sup_{\underline{x} \in \underline{V}_m \setminus (V \times]\bar{s}_1, s_{m+1}[)} |F(\underline{x})| \geq \|F\| - \varepsilon > \alpha. \tag{2.9}$$

Choose $\bar{s}_2 \in]\bar{s}_1, s_{m+1}[$ and $\varphi \in C^1[s_m, s_{m+1}]$ with support in $[s_m, \bar{s}_2[$ satisfying $0 \leq \varphi \leq 1$, $\varphi' \leq 0$, and $\varphi = 1$ on $[s_m, \bar{s}_1]$. Set $\tilde{\underline{K}}_m = \underline{K} \cap (V \times]s_m, \bar{s}_2]) \subset \underline{K}_m$. (We note that $\bar{s}_1, \bar{s}_2, \varphi, \tilde{\underline{K}}_m$ may depend on ε.) From $\underline{K} \cap (V \times \{s_m\}) = \emptyset$ follows that $\tilde{\underline{K}}_m$ is compact in $\underline{V}_m$. On $\underline{V}_m \setminus \tilde{\underline{K}}_m = (\underline{V}_m \setminus \underline{K}) \cup (V\times]\bar{s}_2, s_{m+1}[)$ we have either $|F| \le \alpha$, by (2.8), or $\varphi = 0$; thus $|\varphi F| \le \alpha$ on $\underline{V}_m \setminus \tilde{\underline{K}}_m$. So (2.9) yields

$$\sup_{\tilde{\underline{K}}_m} |\varphi F| > \alpha \ge |(\varphi F)(\underline{x})| \qquad \text{for } \underline{x} \in \underline{V}_m \setminus \tilde{\underline{K}}_m. \tag{2.10}$$

By (2.9), (2.10), and the local dissipativity of L, there exists $\underline{x}_\varepsilon = (x_\varepsilon, t_\varepsilon) \in \tilde{\underline{K}}_m$ satisfying

$$\|F\| \ge |(\varphi F_m)(\underline{x}_\varepsilon)| = \sup_{\underline{V}_m} |\varphi F_m| \ge \|F\| - \varepsilon \qquad \text{and} \tag{2.11}$$

$$\operatorname{Re}\left(L(\varphi F_m)(\underline{x}_\varepsilon)\, \overline{(\varphi F_m)(\underline{x}_\varepsilon)}\right) \le 0, \tag{2.12}$$

where $F_m = F|\underline{V}_m$. We may assume that $\underline{x}_\varepsilon \to \underline{x}_0 \in \underline{K}_m$ as $\varepsilon \to 0$. Now (2.11) gives

$$1 - \frac{\varepsilon}{\|F\|} \le \varphi(t_\varepsilon)\, \frac{|F_m(\underline{x}_\varepsilon)|}{\|F\|} \le \varphi(t_\varepsilon) \le 1,$$

and therefore $\varphi(t_\varepsilon) \to 1$ and $|F_m(\underline{x}_\varepsilon)| \to |F(\underline{x}_0)| = \|F\|$ as $\varepsilon \to 0$. From (2.12) and the parabolicity of L we infer

$$0 \ge \operatorname{Re}\Big[(\varphi^2 \bar{F}_m\, LF_m)(\underline{x}_\varepsilon) - (\varphi\varphi' |F_m|^2)(\underline{x}_\varepsilon)\Big] \ge \varphi(t_\varepsilon)^2 \operatorname{Re}\Big[(LF_m)(\underline{x}_\varepsilon)\, \overline{F_m(\underline{x}_\varepsilon)}\Big].$$

Finally, $LF_m(\underline{x}_\varepsilon) = L_pF(\underline{x}_\varepsilon) \to L_pF(\underline{x}_0)$, and the assertion follows. □

An inspection of the above proof shows the following result which is needed in Section 6.1.

Corollary 2.21. *Let L be a parabolic local operator on $\mathcal{O}(\Omega\times]S,T])$. Assume that the restriction L_0 of L to $\mathcal{O}(\Omega\times]S,T[)$ is locally dissipative. Then L is locally dissipative.*

Combining Theorem 2.12, Proposition 2.16, 2.18, 2.19, 2.20, and Theorem 2.14, we obtain

Theorem 2.22. *Let L defined on $\underline{\mathcal{A}}$ (with $J =]S,T[$) be a parabolic, locally dissipative, local operator satisfying* (S). *Then there exists the local operator $((\bar{L})_p)^\frown)^-$ which is a locally dissipative, parabolic, locally closed, semi-complete extension of L on $\mathcal{O}(\underline{\Omega})$. If L is real, then $((\bar{L})_p)^\frown)^-$ is real.*

Definition 2.23. Let L be a local operator defined on $\underline{\mathcal{A}}$ (with $J =]S,T[$). If the local operator $L_{\bar{\hat{p}}} := ((\bar{L})_p)^\frown)^-$ exists, then it is called the *closed completed parabolic extension* of L.

Next, we consider the parabolic extension of an operator defined on $\mathcal{O}(\underline{\Omega})$. For $I_k \in \mathcal{J}$ and $\underline{V} \in \mathcal{O}(\underline{\Omega})$, we set $\underline{\Omega}_k = \Omega \times I_k$ and $\underline{V}_k = \underline{V} \cap \underline{\Omega}_k$.

Definition 2.24. Let L be a local operator on $\mathcal{O}(\underline{\Omega})$, where $J =]S, T[$. The *parabolic extension* L_p of L is defined by

$$\begin{aligned} D(L_p, \underline{V}) = \{F \in C(\underline{V}) :& \text{ there is a partition } \{s_k, k \in K\} \text{ of } I_{\underline{V}} \text{ and } G \in C(\underline{V}) \text{ such} \\ & \text{that } F_k = F|\underline{V}_k \in D(L, \underline{V}_k) \text{ and } LF_k = G|\underline{V}_k \text{ for } k \in K\}, \\ L_p F = G \qquad & \text{on } \underline{V} \in \mathcal{O}(\underline{\Omega}). \end{aligned}$$

As in the case of Definition 2.17, one checks that L_p is a well-defined local operator extending L. Also, one sees that L_p is parabolic if L is parabolic as in Proposition 2.19. The next result relates both definitions of the parabolic extension.

Proposition 2.25. *Let L be a parabolic, locally dissipative, local operator on $\mathcal{O}(\underline{\Omega})$ (with $J =]S, T[$) which satisfies* (S) *on a base $\underline{\mathcal{A}}_0$ whose intersection with $\underline{\mathcal{A}}$ is still a base of $\mathcal{O}(\underline{\Omega})$. Then the local operator $(\bar{L}_p)^-$ on $\mathcal{O}(\underline{\Omega})$ exists and is parabolic, locally dissipative, locally closed, and semi-complete. Further, $(L_0)_{\bar{p}}$ extends $(\bar{L}_p)^-$, where L_0 is the restriction of L to $\underline{\mathcal{A}}$.*

Proof. First, notice that L_0 satisfies the assumptions of Theorem 2.22. Let $\underline{V} \in \underline{\mathcal{A}}$ and $F \in D(\bar{L}, \underline{V})$. We find $\underline{V}_n \in \underline{\mathcal{A}}$ and $F_n \in D(L, \underline{V}_n)$ such that $\underline{V}_n \uparrow \underline{V}$ and $F_n \to F$, $LF_n \to \bar{L}F$ u.c. in $C(\underline{V})$. Thus, $F \in D(\overline{L_0}, \underline{V})$ and $\overline{L_0}F = \bar{L}F$ on $\underline{V}$. Using the definition of the parabolic extension, we then derive

$$\bar{L}_p \subset \widehat{(\overline{L_0})_p} \subset (L_0)_{\bar{p}}.$$

In particular, $\bar{L}_p$ is locally dissipative. So the local closure of $\bar{L}_p$ exists and has the asserted properties. □

We will need the concept of a 'translation invariant' local operator on space-time in the next chapter. To that purpose, we define the set $\underline{V}_{-\tau} = \{(x, t) \in \underline{\Omega} : (x, t+\tau) \in \underline{V}\}$ for $\tau \in \mathbb{R}$ and $\underline{V} \in \mathcal{O}(\underline{\Omega})$ and the continuous function $(T_\tau F)(x, t) = F_\tau(x, t) = F(x, t + \tau)$ for $(x, t) \in \underline{V}_{-\tau}$ and $F \in C(\underline{V})$, where $J = \mathbb{R}$.

Definition 2.26. Let L be a local operator defined on $\underline{\mathcal{A}}$ (or $\mathcal{O}(\underline{\Omega})$), where $J = \mathbb{R}$. We say that L is *translation invariant* if, for $\underline{V} \in \underline{\mathcal{A}}$ (or $\underline{V} \in \mathcal{O}(\underline{\Omega})$), $\tau \in \mathbb{R}$, and $F \in D(L, \underline{V})$, one has $F_{-\tau} \in D(L, \underline{V}_\tau)$ and $LF_{-\tau} = (LF)_{-\tau}$ on $\underline{V}_\tau$.

A standard example for this notion is furnished by the local operator $L = \Delta_x - \frac{d}{dt}$ on $C^{2,1}(V \times I)$, $V \in \mathcal{O}(\mathbb{R}^n)$. We note the following permanence properties.

Proposition 2.27. *Let L be a translation invariant, local operator defined on $\underline{\mathcal{A}}$ or $\mathcal{O}(\underline{\Omega})$ (where $J = \mathbb{R}$). Then the local operators $\bar{L}$, $\hat{L}$, L_p and $L_{\bar{p}}$ are also translation invariant (if they exist).*

Proof. Let $\tau \in \mathbb{R}$, $F \in D(L_p, \underline{V})$ and $F_k = F|\underline{V}_k$, where $\underline{V}_k$ are the sets given by the partition $\{s_k\}$ for F. We denote by the subscript k also the kth subdivision of the translated partition $\{s_k + \tau\}$ of $\underline{V}_\tau$. Then, $(F_{-\tau})_k = (F_k)_{-\tau} \in D(L, (\underline{V}_k)_\tau)$ and $L(F_{-\tau})_k = (LF_k)_{-\tau} = (L_pF)_{-\tau}$ on $(\underline{V}_k)_\tau = (\underline{V}_\tau)_k$ by the translation invariance of L. As a result, $F_{-\tau} \in D(L_p, \underline{V}_\tau)$ with partition $\{s_k + \tau\}$ and $L_p(F_{-\tau}) = (L_pF)_{-\tau}$; that is, L_p is translation invariant. The other claims can be verified in the same way. □

2.3 Maximum principles for parabolic local operators

In this section we prove one of the most important features of parabolic, locally dissipative, local operators: they satisfy a parabolic maximum principle. Among other things, we will use this fact in Chapter 4 to show uniqueness of solutions to Cauchy problems of diffusion type on non-cylindrical domains $\underline{V}$.

Let L be a local operator defined on $\mathcal{O}(\underline{\Omega})$, $V \in \mathcal{O}(\Omega)$, and $\underline{V} \in \mathcal{O}(\underline{\Omega})$. Since $\underline{V}$ is not required to be relatively compact, we have to consider the one point compactification $\Omega^* = \Omega \cup \{\infty\}$ of Ω and the set $\underline{\Omega}^* = \Omega^* \times J$. We identify $\underline{V}$ (or V) with a subset $\underline{V}^*$ (or V^*) of $\underline{\Omega}^*$ (or Ω^*). Further, $\overline{\underline{V}^*}$ (or $\overline{V^*}$) is the closure of $\underline{V}$ (or V) in $\underline{\Omega}^*$ (or Ω^*). The boundaries are given by $\partial\underline{V}^* = \overline{\underline{V}^*} \setminus \underline{V}^*$ and $\partial V^* = \overline{V^*} \setminus V^*$, respectively. Also, for $F \in C(\overline{\underline{V}^*})$, we denote the restriction of F to $\underline{V}$ by the same symbol and write $F \in C(\overline{\underline{V}^*}) \cap D(L, \underline{V})$ if the restriction belongs $D(L, \underline{V})$. Recall that $J \in \{\,]S,T[\,,]S,T], [S,T]\,\}$ with possibly infinite $S < T$ if $S \notin J$ or $T \notin J$. Therefore, $\overline{\underline{V}^*}$ is compact in $\underline{\Omega}^*$ if $S_{\underline{V}}$ and $T_{\underline{V}}$ are finite and belong to J.

In order to define a 'parabolic boundary', we always assume that $S_{\underline{V}}$ and $T_{\underline{V}}$ are finite and $S_{\underline{V}} \in J$ for a given $\underline{V} \in \mathcal{O}(\underline{\Omega})$. Set $\underline{\Omega}^*(t) = \Omega^* \times \{t\}$ for $t \in \mathbb{R}$, and let $Z_{\underline{V}}$ be the largest subset of $\underline{\Omega}^*(T_{\underline{V}}) \cap \partial\underline{V}^*$ which is open in $\partial\underline{V}^*$. Then we define by

$$\delta_p\underline{V} = (\partial\underline{V}^* \setminus Z_{\underline{V}}) \cup \underline{V}(S) \quad \text{and} \quad \partial_p\underline{V} = (\partial\underline{V}^* \setminus \underline{\Omega}^*(T_{\underline{V}})) \cup \underline{V}(S) \tag{2.13}$$

the (maximal) *closed parabolic boundary* $\delta_p\underline{V}$ and the (maximal) *parabolic boundary* $\partial_p\underline{V}$, respectively, where $\underline{V}(S) = \emptyset$ if $S = -\infty$. Of course, $Z_{\underline{V}}$ or $\underline{V}(S)$ may be empty and the unions in (2.13) need not to be disjoint. However, $\partial_p\underline{V} \neq \emptyset$ since either $\underline{V}(S) = \underline{V}(S_{\underline{V}}) \neq \emptyset$ or there is $(x, S_{\underline{V}}) \in \partial\underline{V}^*$ due to $S_{\underline{V}} \in J$. To illustrate the above notions, let $\underline{V} = V \times\,]S,T[$ and $\underline{V}' = V \times [S,T]$ belong to $\mathcal{O}(\Omega \times [S,T])$. Then $\delta_p\underline{V} = \delta_p\underline{V}' = (V \times \{S\}) \cup (\partial V^* \times [S,T])$ and $\partial_p\underline{V} = \partial_p\underline{V}' = (V \times \{S\}) \cup (\partial V^* \times [S,T[\,)$. Since $\delta_p\underline{V}$ is the closure of $\partial_p\underline{V}$ in $\partial\underline{V}^*$, we have

$$\sup_{\delta_p\underline{V}} |F| = \sup_{\partial_p\underline{V}} |F| \tag{2.14}$$

for $F \in C_b(\delta_p\underline{V})$. Observe that $\delta_p\underline{V}$ is compact in $\underline{\Omega}^*$ if $T_{\underline{V}} \in J$.

For relatively compact $\underline{V}$, the following results are essentially contained in [41]. There, however, a somewhat smaller 'parabolic boundary' is used, namely the set of those $\underline{x} \in \partial_p\underline{V}$ "which can be reached from below", cf. [20, §2.1]. The next lemma provides the essential step in the proofs of our main theorems below.

Lemma 2.28. *Let* $\underline{V} \in \mathcal{O}(\underline{\Omega})$ *with finite* $S_{\underline{V}}, T_{\underline{V}} \in J$. *Let* L *be a locally dissipative, parabolic, local operator defined on* $\mathcal{O}(\underline{\Omega})$. *Assume that* $F \in C(\overline{\underline{V}^*}) \cap D(L, \underline{V})$. *If* $LF = 0$ *on* $\underline{V}$, *then*

$$\sup_{\delta_p \underline{V}} |F| = \sup_{\underline{V}} |F| . \tag{2.15}$$

Moreover, if $F > 0$ *and* $LF \geq 0$ *on* $\underline{V}$, *then*

$$\sup_{\delta_p \underline{V}} F = \sup_{\underline{V}} F . \tag{2.16}$$

Proof. (1) Let $F \in C(\overline{\underline{V}^*}) \cap D(L, \underline{V})$ and $LF = 0$ on $\underline{V}$. We may assume that $\|F\| = \sup_{\underline{V}} |F| > 0$. Suppose that (2.15) does not hold. Choose $\varphi \in C^1[S_{\underline{V}}, T_{\underline{V}}]$ such that $0 < \varphi \leq 1$, $\varphi'(S_{\underline{V}}) = 0$, $\varphi'(s) < 0$ for $S_{\underline{V}} < s \leq T_{\underline{V}}$, and $\varphi(T_{\underline{V}}) \|F\| > \varphi(S_{\underline{V}}) \sup_{\delta_p \underline{V}} |F|$. In particular, $F_1 = \varphi F \in C(\overline{\underline{V}^*}) \cap D(L, \underline{V})$ also violates (2.15). Set $\underline{K} = \{\underline{x} \in \overline{\underline{V}^*} : |F_1(\underline{x})| = \|F_1\|\} \neq \emptyset$. Then $\underline{K} \cap \delta_p \underline{V} = \emptyset$, and we have either

$$\text{(a)}\ \ \underline{K} \cap (\partial \underline{V}^* \cup \underline{V}(S)) = \emptyset \qquad \text{or} \qquad \text{(b)}\ \ \emptyset \neq \underline{K} \cap (\partial \underline{V}^* \cup \underline{V}(S)) \subset Z_{\underline{V}}.$$

First, suppose that (a) holds. For $\|F_1\| > \alpha > \sup_{\partial \underline{V}^* \cup \underline{V}(S)} |F_1|$, we define

$$\underline{K}_\alpha = \{\underline{x} \in \overline{\underline{V}^*} : |F_1(\underline{x})| \geq \alpha\} \neq \emptyset.$$

Observe that $\underline{K}_\alpha \subset \underline{V}$ and $\underline{K}_\alpha$ is closed in $\overline{\underline{V}^*}$. Therefore, $\underline{K}_\alpha$ is compact in $\underline{V}$. Using local dissipativity of L, we find $\underline{x}_1 = (x_1, t_1) \in \underline{K}_\alpha$ satisfying $|F_1(\underline{x}_1)| = \|F_1\|$ and $\operatorname{Re}((LF_1)(\underline{x}_1)\overline{F_1(\underline{x}_1)}) \leq 0$. Since L is parabolic and $LF = 0$, this implies

$$0 \geq \operatorname{Re}(-\varphi' F \overline{F_1})(\underline{x}_1) = -\varphi'(t_1)\varphi(t_1)|F(\underline{x}_1)|^2.$$

Due to $\underline{K}_\alpha \cap (\partial \underline{V}^* \cup \underline{V}(S)) = \emptyset$, we have $t_1 > S_{\underline{V}}$ and, hence, $-\varphi'(t_1)\varphi(t_1) > 0$. Consequently, $|F(\underline{x}_1)| = \|F_1\| = 0$, which is impossible.

Second, suppose that (b) holds. There exists $\beta > 0$ with

$$\|F_1\| > \beta > \sup_{\delta_p \underline{V}} |F_1|. \tag{2.17}$$

Define

$$\underline{K}_\beta = \{\underline{x} \in \overline{\underline{V}^*} : |F_1(\underline{x})| \geq \beta\} \neq \emptyset.$$

Let $0 < \varepsilon < \|F_1\| - \beta$. Since $F_1 \in C(\overline{\underline{V}^*})$ and $\underline{V}$ is open, there is $\underline{x}_2 = (x_2, t_2) \in \underline{V}$ (depending on ε) such that $|F_1(\underline{x}_2)| \geq \|F_1\| - \varepsilon$ and $S_{\underline{V}} < t_2 < T_{\underline{V}}$. Take $t_3 \in]t_2, T_{\underline{V}}[$ and $\psi_\varepsilon \in C^1[S_{\underline{V}}, T_{\underline{V}}]$ satisfying $0 \leq \psi_\varepsilon \leq 1$, $\psi_\varepsilon' \leq 0$, $\psi_\varepsilon = 1$ on $[S_{\underline{V}}, t_2]$, and $\psi_\varepsilon = 0$ on $[t_3, T_{\underline{V}}]$. The set

$$\underline{K}_\beta' = \{(x, t) \in \underline{K}_\beta : t \leq t_3\} \neq \emptyset$$

is closed in $\overline{\underline{V}^*}$ and $\underline{K}_\beta' \cap Z_{\underline{V}} = \emptyset$. Therefore, (2.17) implies that $\underline{K}_\beta'$ is a compact subset of $\underline{V}$. Also,

$$\sup_{\underline{K}_\beta'} |\psi_\varepsilon F_1| \geq |F_1(\underline{x}_2)| \geq \|F_1\| - \varepsilon > \beta > 0. \tag{2.18}$$

On the other hand, $\underline{V} \setminus \underline{K}'_\beta = (\underline{V} \setminus \underline{K}_\beta) \cup ((\underline{K}_\beta \cap \underline{V}) \setminus \underline{K}'_\beta)$. On $\underline{V} \setminus \underline{K}_\beta$ we have $|\psi_\varepsilon F_1| \le |F_1| < \beta$, while on $\underline{K}_\beta \setminus \underline{K}'_\beta$ the function ψ_ε vanishes. As a result,

$$\sup_{\underline{V} \setminus \underline{K}'_\beta} |\psi_\varepsilon F_1| \le \beta < \sup_{\underline{K}'_\beta} |\psi_\varepsilon F_1| . \tag{2.19}$$

By local dissipativity of L, there exists $\underline{x}_\varepsilon = (x_\varepsilon \, , t_\varepsilon) \in \underline{K}'_\beta$ such that $|\psi_\varepsilon F_1(\underline{x}_\varepsilon)| = \|\psi_\varepsilon F_1\|$ and

$$\begin{aligned} 0 &\ge \operatorname{Re}\Big[L(\psi_\varepsilon F_1)(\underline{x}_\varepsilon)\,\overline{(\psi_\varepsilon F_1)(\underline{x}_\varepsilon)}\Big] \\ &= \operatorname{Re}\Big[(\psi_\varepsilon^2\,\overline{F_1} L F_1)(\underline{x}_\varepsilon) - (\psi'_\varepsilon \psi_\varepsilon\,|F_1|^2)(\underline{x}_\varepsilon)\Big] \\ &= -\psi_\varepsilon^2(t_\varepsilon)\varphi'(t_\varepsilon)\varphi(t_\varepsilon)\,|F(\underline{x}_\varepsilon)|^2 - \psi'_\varepsilon(t_\varepsilon)\psi_\varepsilon(t_\varepsilon)\,|F_1(\underline{x}_\varepsilon)|^2 \\ &\ge -\psi_\varepsilon^2(t_\varepsilon)\,\frac{\varphi'(t_\varepsilon)}{\varphi(t_\varepsilon)}\,|F_1(\underline{x}_\varepsilon)|^2 \,, \end{aligned} \tag{2.20}$$

where we have used that L is parabolic and $LF = 0$. Further, by (2.17), there exists a constant $\gamma > 0$ such that $t_\varepsilon \ge S_{\underline{V}} + \gamma$ for all $\varepsilon > 0$. Hence, $-\frac{\varphi'(t_\varepsilon)}{\varphi(t_\varepsilon)} \ge \delta > 0$ for some constant δ. Moreover, we infer from (2.18) that

$$\begin{aligned} \|F_1\| - \varepsilon \le |(\psi_\varepsilon F_1)(\underline{x}_\varepsilon)| &\le \|F_1\|, \qquad \text{and so} \\ 1 - \frac{\varepsilon}{\|F_1\|} \le \psi_\varepsilon(t_\varepsilon)\,\frac{|F_1(\underline{x}_\varepsilon)|}{\|F_1\|} &\le \psi_\varepsilon(t_\varepsilon) \le 1. \end{aligned}$$

Thus, $\psi_\varepsilon(t_\varepsilon) \to 1$ as $\varepsilon \to 0$, and (2.20) implies $F_1(\underline{x}_\varepsilon) = 0$ for small $\varepsilon > 0$. This contradicts $\underline{x}_\varepsilon \in \underline{K}_\beta$ and so the first assertion is established.

(2) Now assume that $F > 0$ and $LF \ge 0$ on $\underline{V}$ and that (2.16) does not hold. Then,

$$\sup_{\delta_p \underline{V}} F = \sup_{\delta_p \underline{V}} |F| < \sup_{\underline{V}} |F| = \sup_{\underline{V}} F .$$

We proceed as in the first part of the proof and consider $F_1 = \varphi F$ and the alternatives (a) and (b). Observe that

$$F_1\, LF_1 = -\varphi' \varphi\, F^2 + \varphi^2 F\, LF \ge -\varphi' \varphi\, F^2 > 0 \tag{2.21}$$

on $\underline{V} \setminus \underline{V}(S_{\underline{V}})$ since L is parabolic and $F > 0$, $LF \ge 0$.

In case (a), we define $\underline{K}_\alpha$ as before and obtain by local dissipativity a point $\underline{x}_1 \in \underline{V} \setminus \underline{V}(S_{\underline{V}})$ such that $F_1(\underline{x}_1)\, LF(\underline{x}_1) \le 0$ which violates (2.21).

In case (b), we consider again ψ_ε, $\underline{K}'_\beta$, and $\underline{x}_\varepsilon \in \underline{V} \setminus \underline{V}(S_{\underline{V}})$ for $0 < \varepsilon < \|F_1\| - \beta$. The same arguments as above yield

$$\begin{aligned} 0 &\ge (L(\psi_\varepsilon F_1))(\underline{x}_\varepsilon)\,(\psi_\varepsilon F_1)(\underline{x}_\varepsilon) \\ &= \psi_\varepsilon^2(t_\varepsilon)(F_1\, LF_1)(\underline{x}_\varepsilon) - \psi'_\varepsilon(t_\varepsilon)\psi_\varepsilon(t_\varepsilon) F_1(\underline{x}_\varepsilon)^2 \\ &\ge \psi_\varepsilon^2(t_\varepsilon)(F_1\, LF_1)(\underline{x}_\varepsilon). \end{aligned}$$

As in the first part of the proof, we obtain $\psi_\varepsilon(t_\varepsilon) > 0$ for sufficiently small $\varepsilon > 0$. This leads to a contradiction with (2.21). □

Theorem 2.29. *Let $\underline{V} \in \mathcal{O}(\underline{\Omega})$ with finite $S_{\underline{V}} \in J$ and $T_{\underline{V}}$. Let L be a locally dissipative, parabolic, local operator defined on $\mathcal{O}(\underline{\Omega})$. Assume that $F \in C_b(\underline{V}^* \cup \partial_p \underline{V}) \cap D(L, \underline{V})$ and $LF = 0$ on $\underline{V}$. Then,*

$$\sup_{\underline{V}} |F| = \sup_{\partial_p \underline{V}} |F| \,.$$

Proof. Since $\underline{V}$ is open, for all $\varepsilon \in]0, T_{\underline{V}} - S_{\underline{V}}[$ there is $t_\varepsilon \in]T_{\underline{V}} - \varepsilon, T_{\underline{V}}[$ such that $\underline{V}(t_\varepsilon) \neq \emptyset$. Set

$$\underline{V}_\varepsilon = \{(x,t) \in \underline{V} : t < t_\varepsilon\} \quad \text{and} \quad Y_\varepsilon = \partial \underline{V}_\varepsilon^* \cap \underline{\Omega}^*(t_\varepsilon).$$

Then, $\partial \underline{V}_\varepsilon^* \subset \partial_p \underline{V}_\varepsilon \cup Y_\varepsilon$ and $\partial_p \underline{V}_\varepsilon \subset \partial_p \underline{V}$. Further, $Y_\varepsilon \cap \underline{\Omega}^*(T_{\underline{V}}) = \emptyset$, $Y_\varepsilon \cap \underline{V}(S) = \emptyset$, and

$$\overline{\underline{V}_\varepsilon^*} \subset \underline{V}_\varepsilon^* \cup \partial_p \underline{V}_\varepsilon \cup Y_\varepsilon \subset \underline{V}^* \cup \partial_p \underline{V}.$$

Hence, $F \in C(\overline{\underline{V}_\varepsilon^*})$. As a consequence of Lemma 2.28 and (2.14), we obtain

$$\sup_{\underline{V}_\varepsilon} |F| = \sup_{\partial_p \underline{V}_\varepsilon} |F| \le \sup_{\partial_p \underline{V}} |F| \le \sup_{\underline{V}} |F|.$$

The theorem now follows from $\sup_{\varepsilon>0} \sup_{\underline{V}_\varepsilon} |F| = \sup_{\underline{V}} |F|$. □

Theorem 2.30. *Let $\underline{V} \in \mathcal{O}(\underline{\Omega})$ with finite $S_{\underline{V}} \in J$ and $T_{\underline{V}}$. Let L be a locally dissipative, parabolic, local operator defined on $\mathcal{O}(\underline{\Omega})$. Assume that $F \in C_b^{\mathbb{R}}(\underline{V}^* \cup \partial_p \underline{V}) \cap D(L, \underline{V})$ and $LF \ge 0$ on $\underline{V}$. Then,*

$$\sup_{\underline{V}} F \le \sup_{\partial_p \underline{V}} F^+ \,.$$

Proof. Consider t_ε and $\underline{V}_\varepsilon$ as in the proof of the preceding theorem. Again we have $F \in C(\overline{\underline{V}_\varepsilon^*})$. Suppose that

$$\sup_{\underline{V}_\varepsilon} F > \alpha > \sup_{\partial_p \underline{V}_\varepsilon} F^+ \ge 0 \tag{2.22}$$

for some constant α. Then the set $\underline{W}_\varepsilon = \{\underline{x} \in \underline{V}_\varepsilon : F(\underline{x}) > \alpha\}$ is open and not empty. Moreover, $F \in C(\overline{\underline{W}_\varepsilon^*})$ and $F > 0$ on $\underline{W}_\varepsilon$. Since $\partial \underline{W}_\varepsilon^* \cap \partial_p \underline{V}_\varepsilon = \emptyset$, we have $F = \alpha$ on $\partial_p \underline{W}_\varepsilon$. So Lemma 2.28 and (2.14) imply $F \le \alpha$ on $\underline{W}_\varepsilon$ which contradicts the definition of $\underline{W}_\varepsilon$. Therefore, (2.22) is false and

$$\sup_{\underline{V}} F = \sup_{\varepsilon>0} \sup_{\underline{V}_\varepsilon} F \le \sup_{\varepsilon>0} \sup_{\partial_p \underline{V}_\varepsilon} F^+ = \sup_{\partial_p \underline{V}} F^+ \,. \quad □$$

For parabolic operators L defined on $\underline{\mathcal{A}}$ we immediately obtain the following result by considering the completion $\hat{L}$ defined on $\mathcal{O}(\underline{\Omega})$.

Corollary 2.31. *Let L be a locally dissipative, parabolic, local operator defined on $\underline{\mathcal{A}}$ with $J = [S,T]$. Let $\underline{V} = V \times]s,t[$ for $V \in \mathcal{O}(\Omega)$ and $S < s < t < T$. Assume that $F \in C(\overline{V^*} \times [s,t]) \cap D(L, \underline{V})$. If $LF = 0$ on $\underline{V}$, then*

$$\sup_{\underline{V}} |F| = \sup_{\partial_p \underline{V}} |F| \,.$$

Moreover, if F is real and $LF \ge 0$ on $\underline{V}$, then

$$\sup_{\underline{V}} F \le \sup_{\partial_p \underline{V}} F^+ \,.$$

We conclude this chapter with a comparison principle for semilinear equations which is needed in Chapter 5.

Theorem 2.32. *Let $\underline{V} \in \mathcal{O}(\underline{\Omega})$ with finite $S_{\underline{V}} \in J$ and $T_{\underline{V}}$. Let L be a locally dissipative, parabolic, local operator defined on $\mathcal{O}(\underline{\Omega})$ and $\Phi : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz. Assume that $F, G \in C_b^{\mathbb{R}}(\underline{V}^* \cup \partial_p \underline{V}) \cap D(L, \underline{V})$ satisfy*

$$LF + \Phi \circ F \le LG + \Phi \circ G \quad \text{on } \underline{V} \quad \text{and} \quad F \ge G \quad \text{on } \partial_p \underline{V}.$$

Then $F \ge G$ on $\underline{V}$.

Proof. Define $u = G - F$. Suppose that the open set

$$\underline{V}^0 = \{\underline{x} \in \underline{V} : u(\underline{x}) > 0\}$$

is not empty. Clearly, $\underline{V}^0(S) = \emptyset$ and $u \le 0$ on $\partial_p \underline{V}^0$. Set $e_\lambda(t) = e^{-\lambda t}$ for $\lambda > 0$ and $t \in I_{\underline{V}^0}$. The parabolicity of L yields $e_\lambda u \in D(L, \underline{V}^0)$ and

$$L(e_\lambda u) = e_\lambda \, Lu + \lambda e_\lambda u \ge e_\lambda \, (\lambda u - (\Phi \circ G - \Phi \circ F)).$$

There is a constant $k \ge 0$ (depending on $\max\{\|F\|, \|G\|\}$) such that

$$|\Phi(G(\underline{x})) - \Phi(F(\underline{x}))| \le k \, |G(\underline{x}) - F(\underline{x})| = k \, u(\underline{x})$$

for $\underline{x} \in \underline{V}^0$. Combining these inequalities, we obtain for fixed $\lambda \ge k$

$$L(e_\lambda u) \ge (\lambda - k) u \ge 0 \qquad \text{on } \underline{V}_0.$$

Therefore, Theorem 2.30 implies

$$\sup\nolimits_{\underline{V}^0} e_\lambda u \le \sup\nolimits_{\partial_p \underline{V}^0} (e_\lambda u)^+ = 0.$$

This violates the definition of $\underline{V}^0$. As a result, $F \ge G$ on $\underline{V}$. □

3 Localization and well-posedness of Cauchy problems

3.1 Introduction

Let A be a real, locally dissipative, local operator defined on $\mathcal{O}(\Omega)$ which satisfies (S). Then the local closure $\bar{A}$ exists by Theorem 2.12. Given a set $V \in \mathcal{O}(\Omega)$, we study the Cauchy problem

$$(CP_V) \quad \begin{cases} \frac{d}{dt} u(t) &= \bar{A}_V \, u(t), \qquad t \ge 0, \\ u(0) &= f \in D(\bar{A}_V) \end{cases}$$

in $C_0(V)$, where $(\bar{A}_V, D(\bar{A}_V))$ is the part of $\bar{A}$ in $C_0(V)$. The Cauchy problem (CP_V) is called *well-posed* if for all $f \in D(\bar{A}_V)$ there is a unique C^1–function $u = u(\cdot, f)$ such that $u(t) \in D(\bar{A}_V)$ and u satisfies (CP_V) for all $t \geq 0$ and if $u(t, f_n) \to 0$ uniformly for $0 \leq t \leq T$ as $D(\bar{A}_V) \ni f_n \to 0$ in $C_0(V)$. By the closedness of $\bar{A}_V$, the Cauchy problem is well-posed if and only if $\bar{A}_V$ generates a semigroup $(e^{t\bar{A}_V})_{t\geq 0}$ on $C_0(V)$; and then $u(t, f) = e^{t\bar{A}_V} f$. However, much more can be said in our situation since $\bar{A}_V$ is dissipative and satisfies a positive maximum principle by virtue of Corollary 2.10: $\bar{A}_V$ generates a *Feller semigroup*, i.e., $e^{t\bar{A}_V}$ is a positive contraction on $C_0(V)$ for $t \geq 0$, if and only if $D(\bar{A}_V)$ and $(\lambda - \bar{A}_V)D(\bar{A}_V)$ are both dense in $C_0(V)$ for some $\lambda \geq 0$, see e.g. [8, Thm. 2.2]. The latter condition can be checked for a large class of elliptic operators A and sufficiently regular V. In fact, in this case one has:

(∗) There is an exhaustive base $\mathcal{B} \subset \mathcal{O}_c(\Omega)$ of $\mathcal{O}(\Omega)$ such that $D(\bar{A}_W)$ and $(\lambda - \bar{A}_W)D(\bar{A}_W)$ are dense in $C_0(W)$ for some $\lambda \geq 0$ and all $W \in \mathcal{B}$,

cf. [33] and [46]. Here “exhaustive” means that for each compact subset K of $V \in \mathcal{O}(\Omega)$ there exists $G \in \mathcal{B}$ such that $K \subset G \subset \bar{G} \subset V$. Further, if (∗) holds, then the well-posedness of (CP_V) is equivalent to the density of $D(\bar{A}_V)$ and the existence of a ‘Cauchy barrier’ for V, [32, Thm. 6] (see also [30], [31], [36], [48], [55], Theorem 3.14 and 3.27 for similar results). Here we use the following definition.

Definition 3.1. Let A be a local operator defined on $\mathcal{O}(\Omega)$. A set $V \in \mathcal{O}(\Omega)$ possesses a *Cauchy barrier* h (with respect to A) if there exists a compact subset K of V and a function $h \in D(A, V \setminus K)$ such that $h > 0$ and $(A - \lambda)h \leq 0$ on $V \setminus K$ for some $\lambda \geq 0$ and for all $\varepsilon > 0$ there is a compact set K_ε with $K \subset K_\varepsilon \subset V$ and $0 \leq h \leq \varepsilon$ on $V \setminus K_\varepsilon$.

We refer to [30, §6], [43, §6,7], [49], and Section 6.1 for examples and applications of this concept, in particular to degenerate problems. The main purpose of this chapter is to characterize well-posedness of (CP_V) by the existence of a Cauchy barrier without assuming (∗). Instead, in Theorem 3.25, we will suppose that $\bar{A}_\Omega$ is a generator on $C_0(\Omega)$ and that $\bar{A}$ satisfies some mild additional conditions.

Before proceeding, let us briefly discuss the concept of a Cauchy barrier. First, let $h \in C^{\mathbb{R}}(\bar{V}) \cap D(A, V)$ satisfy $(A - \lambda)h \leq 0$ for $V \in \mathcal{O}_c(\Omega)$ and $\lambda \geq 0$. Then h is a ‘barrier’ in the sense that, for $u \in C^{\mathbb{R}}(\bar{V}) \cap D(A, V)$ with $Au = \lambda u$ and $u \leq h$ on ∂V, we have $u \leq h$ on V, see [30, Thm. 5.3] or [31, Thm. 2.9]. For later use we state the easy part of the above mentioned characterization of well-posedness.

Proposition 3.2. *Let B be the generator of a bounded, positive semigroup $(P(t))_{t\geq 0}$ on $C_0(\Omega)$. Then, for each $\lambda > 0$, there exists $h \in D(B)$ such that $h(x) > 0$ and $(B - \lambda)h(x) < 0$ for $x \in \Omega$.*

Proof. An application of Urysohn's lemma (in the version of [15, Cor. VII.4.2]) to the G_δ–set $\{\infty\}$ yields a strictly positive function $f \in C_0(\Omega)$. Let $h = (\lambda - B)^{-1} f$ for a given $\lambda > 0$. Since $h(x) = \int_0^\infty e^{-\lambda t}(P(t)f)(x)\, dt$ and $P(t)f \geq 0$, we have $h(x) > 0$ for $x \in \Omega$. Moreover, $(B - \lambda)h(x) = -f(x) < 0$ for $x \in \Omega$. □

We further want to relate Cauchy barriers with the classical notion of 'local barriers' used in partial differential equations and potential theory. To that purpose, we introduce the following concepts.

Definition 3.3. Let A be a local operator defined on $\mathcal{O}(\Omega)$ and $V \in \mathcal{O}_c(\Omega)$.

1. V is called *weakly* (A-)*Dirichlet regular* if for all $f \in C(\partial V)$ there is unique $u \in C(\bar{V}) \cap D(A, V)$ such that $Au = 0$ on V and $u|\partial V = f$ (i.e., u solves the Dirichlet problem).

2. A function h_x is a *local barrier* for A at $x \in \partial V$ if there is a neighbourhood $U \in \mathcal{O}(\Omega)$ of x such that $h_x \in D(A, V \cap U) \cap C(\overline{V \cap U})$, $h_x > 0$ on $\overline{V \cap U} \setminus \{x\}$, $h_x(x) = 0$, and $Ah_x \leq \lambda h_x$ for some $\lambda \geq 0$.

For sufficiently regular elliptic problems, $V \in \mathcal{O}_c(\mathbb{R}^n)$ is weakly Dirichlet regular if and only if there exists a local barrier h_x for all $x \in \partial V$, see e.g. [6, Chap. VIII], [11, §II.4], [21], [22]. In [6, Chap. VIII] or [20, §3.4] one can find analogous results for parabolic equations. Moreover, for $A = \Delta$ it is not necessary to require strict positivity of h_x on $\partial(V \cap U) \setminus \{x\}$ due to [22, Thm. 8.18, 8.22], cf. [6, Chap. VII]. Note that a Cauchy barrier is a local barrier in this weakened sense.

In case of unbounded domains this equivalence needs some modifications, compare [22, p.193]. In particular, there can exist a Cauchy barrier (and local barriers) for A and V but the corresponding Dirichlet problem is not solvable for all $f \in C(\partial V^*)$. This can be seen by the following example taken from [30, p.423]. Let $\Omega =]-1, \infty[$, $V =]0, \infty[$, and A be the second derivative. Then $h(x) = x$ for $0 < x < 1$ and $h(x) = e^{-x}$ for $x > 1$ is a Cauchy barrier for V. But there is no function $u \in C^2]0, \infty[\cap C[0, \infty]$ such that $u'' = 0$ and $u(0) \neq u(\infty)$.

On the other hand, for uniformly elliptic operators in divergence form the well–posedness of (CP_V) is equivalent to the existence of local barriers for $V \in \mathcal{O}(\Omega)$ by [1, Thm. 4.1]. Together with Theorem 3.25 or 3.27 this shows, roughly speaking, that the existence of local barriers and Cauchy barriers are equivalent in this case. However, our main interest is directed to parabolic problems in Chapter 6. Moreover, we point out that Cauchy barriers allow to handle degenerate problems, see [30], [43], [49], and Section 6.1.

Finally, we observe that under mild conditions weak Dirichlet regularity implies well-posedness of (CP_V). We denote by $f^{\#} \in C_0(\Omega)$ the extension by 0 of a function $f \in C_0(V)$.

Proposition 3.4. *Let A be a locally dissipative, local operator defined on $\mathcal{O}(\Omega)$ satisfying* (S). *Assume that $\bar{A}_\Omega$ is a generator on $C_0(\Omega)$. If $V \in \mathcal{O}_c(\Omega)$ is weakly $(\bar{A} - 1)$–Dirichlet regular and $D(\bar{A}_V)$ is dense, then $\bar{A}_V$ is a generator on $C_0(V)$ and, by Proposition* 3.2*, V possesses a Cauchy barrier.*

Proof. It suffices to show that $1 - \bar{A}_V$ is surjective. For $f \in C_0(V)$, set $g_1 = (\bar{A}_\Omega - 1)^{-1} f^{\#}$ and $g = g_1|\bar{V}$. Since V is weakly Dirichlet-regular, there exists $h \in D(A, V) \cap$

$C(\bar{V})$ such that $(\bar{A}-1)h = 0$ on V and $h = g$ on ∂V. Then, $u = h - g \in D(A, V) \cap C(\bar{V})$, $u \in C_0(V)$, and $(1 - \bar{A})u = f$ on V. □

3.2 Excessive barriers and approximate solutions

Throughout the remainder of this chapter we make use of the following hypothesis.

(H) A is a real, locally dissipative, local operator defined on $\mathcal{O}(\Omega)$ such that (S) holds and $\bar{A}_\Omega$ generates a (Feller) semigroup $(P(t))_{t\geq 0}$ on $C_0(\Omega)$.

In order to characterize well-posedness of (CP_V), we will work to a large extend with a 'barrier' for $P(t)$ rather than for A itself; see [16] for a similar concept in the context of Markov processes.

Definition 3.5. Assume that (H) holds.

1. Let $K \subset \Omega$ be compact and $\eta > 0$. A function $0 \leq h \in C_0(\Omega)$ is called *locally excessive* (K, η) (with respect to $P(t)$ or A) if $P(t)h(x) \leq h(x)$ for $x \in K$ and $0 \leq t \leq \eta$.

2. Let $V \in \mathcal{O}(\Omega)$. A function $0 \leq h \in C_0(\Omega)$ is said to be *locally excessive in* V (w.r.t. $P(t)$ or A) if for all compact $K \subset V$ there is $\eta_K > 0$ such that h is locally excessive (K, η_K).

3. Let $V \in \mathcal{O}(\Omega)$. A function $h \in C_0(V)$ is an *excessive barrier for* V (w.r.t. $P(t)$ or A) if it is strictly positive on V and $h^{\#} \in C_0(\Omega)$ is locally excessive in V for $e^{-\lambda t}P(t) = e^{t(\bar{A}_\Omega - \lambda)}$ and some $\lambda \geq 0$. If we can take $\lambda = 0$, then h is a *regular excessive barrier for* V (w.r.t. $P(t)$ or A).

Let $A_\lambda = A - \lambda$ for $\lambda \in \mathbb{C}$. Notice that $(\overline{A_\lambda})_V = \bar{A}_V - \lambda$ for $V \in \mathcal{O}(\Omega)$. Thus, V has an excessive barrier w.r.t. A if and only if V admits a regular excessive barrier w.r.t. A_λ for some $\lambda \geq 0$. We will see in Remark 3.20 and Lemma 3.24 that a Cauchy barrier is also an excessive barrier under weak additional conditions.

Given a compact subset K of Ω and a function h being locally excessive (K, η), we define the spaces

$$\begin{aligned}
\mathcal{C} &= \{f \in C_0^{\mathbb{R}}(\Omega) : -h \leq f \leq h\}, \\
\mathcal{C}^K &= \{f \in C_0^{\mathbb{R}}(\Omega) : -h(x) \leq f(x) \leq h(x) \text{ for } x \in K\}, \\
\mathcal{N}^K &= \{f \in C_0^{\mathbb{R}}(\Omega) : f(x) = 0 \text{ for } x \in K\},
\end{aligned}$$

and the (nonlinear) mappings $S(t), \Delta(t) : C_0^{\mathbb{R}}(\Omega) \to C_0^{\mathbb{R}}(\Omega)$ given by

$$S(t)f = \sup\{-h, \inf\{h, P(t)f\}\} \in \mathcal{C} \qquad \text{and} \qquad \Delta(t) = S(t) - P(t)$$

for $t \geq 0$ and $f \in C_0^{\mathbb{R}}(\Omega)$. The positivity of $P(t)$ implies

$$P(t) : \mathcal{C} \to \mathcal{C}^K \quad \text{and} \quad \Delta(t) : \mathcal{C} \to \mathcal{N}^K \quad \text{for} \ \ 0 \leq t \leq \eta_K \, , \quad \text{and so} \tag{3.1}$$

$$\Delta(t - [\tfrac{t}{\sigma}])\, S^{[\frac{t}{\sigma}]}(\sigma) f \in \mathcal{N}^K \quad \text{for} \ \ 0 < \sigma \leq \eta_K \, , \ f \in C_0^{\mathbb{R}}(\Omega), \ t \geq \sigma. \tag{3.2}$$

Here, $[\frac{t}{\sigma}]$ is the integer part of $\frac{t}{\sigma}$. Fix $\sigma > 0$ and write $t = k\sigma + \tau$ for $t \geq 0$, where $k = [\frac{t}{\sigma}]$ and $0 \leq \tau < \sigma$. We define a mapping

$$U_\sigma(t) : C_0^{\mathbb{R}}(\Omega) \to \mathcal{C} \quad \text{by} \quad U_\sigma(t) = S(\tau)\, S(\sigma)^k \, . \tag{3.3}$$

Observe that $U_\sigma(\cdot)$ is strongly continuous on $[0, \infty[$. This map can be expanded in the following way. We set $P_+(t) = P(t)$ for $t \geq 0$ and $P_+(t) = 0$ for $t < 0$.

Lemma 3.6. *For $U_\sigma(t)$ defined in* (3.3)*, we have*

$$U_\sigma(t) = P(t) + \Delta(t - [\tfrac{t}{\sigma}]\sigma)\, S^{[\frac{t}{\sigma}]}(\sigma) + \sum_{n=1}^{\infty} P_+(t - n\sigma)\, \Delta(\sigma)\, S^{n-1}(\sigma) \tag{3.4}$$

for $t \geq 0$ and $\sigma > 0$, where we have used the notation introduced above.

Proof. For $0 \leq t < \sigma$, we have $P_+(t - n\sigma) = 0$ for $n = 1, 2, \cdots$, and thus (3.4) follows from $S(t) = P(t) + \Delta(t)$. Further, from $S(\sigma) = \Delta(\sigma) + P(\sigma)$ one easily derives by induction that

$$S(\sigma)^k = P(\sigma)^k + \sum_{l=1}^{k} P(\sigma)^{l-1}\, \Delta(\sigma)\, S(\sigma)^{k-l}$$

for $k = 1, 2, \cdots$. This yields, for $k\sigma \leq t < (k+1)\sigma$ and $\tau = t - k\sigma$,

$$\begin{aligned} U_\sigma(t) &= (\Delta(\tau) + P(\tau))\, S(\sigma)^k \\ &= P(\tau + k\sigma) + \Delta(\tau) S(\sigma)^k + \sum_{l=1}^{k} P(\tau + (l-1)\sigma)\, \Delta(\sigma)\, S(\sigma)^{k-l} \, , \end{aligned}$$

which is (3.4). □

We will use the operators $U_\sigma(t)$ to construct 'approximate solutions' of an 'extension' of (CP_V). To that purpose, let $J = \mathbb{R}$ and define a local operator L on $\underline{\mathcal{A}}$ by setting

$$\begin{aligned} D(L, \underline{V}) = \{ &F \in C(\underline{V}) : F(\cdot, t) \in D(\bar{A}, V) \text{ for } t \in I, \ F(x, \cdot) \in C^1(I) \text{ for } x \in V, \\ &\underline{V} \ni (x,t) \mapsto G(x,t) := (\bar{A}F(\cdot,t))(x) - (\tfrac{d}{dt} F(x, \cdot))(t) \text{ is continuous} \}, \\ LF = G \quad &\text{on } \underline{V} = V \times I \in \underline{\mathcal{A}} \, . \end{aligned} \tag{3.5}$$

Lemma 3.7. *If* (H) *holds, then* (3.5) *defines a real, parabolic, translation invariant, locally dissipative, local operator L on $\underline{\mathcal{A}}$ satisfying* (S).

Proof. Clearly, L is a real, parabolic, translation invariant, local operator. To check (S), let $\underline{x}_0 = (x_0, t_0) \in \underline{W} = W \times I \in \underline{\mathcal{A}}$, where I is bounded and (S) holds for A, h, and $x_0 \in W \in \mathcal{O}_c(\Omega)$. Choose $\varphi \in C_c^1(I)$ with $0 \le \varphi \le 1$ and $\varphi(t_0) = 1$. It is easy to see that $F = \varphi h$ satisfies (S) for $\underline{x}_0 \in \underline{W}$. So we can use Corollary 2.10 to show local dissipativity of L. Let $\underline{V} = V \times I \in \underline{\mathcal{A}}$ and $F \in D(L, \underline{V})$ be real with $0 < F(\underline{x}_0) = \sup_{\underline{V}} F$ for some $\underline{x}_0 = (x_0, t_0) \in \underline{V}$. Since $t \mapsto F(x_0, t)$ has a maximum at $t = t_0$, we have $(\frac{d}{dt} F(x_0, \cdot))(t_0) = 0$. Simlarly, $\bar{A}F(x_0, t_0) \le 0$ because $\bar{A}$ is locally dissipative and real. As a result, $LF(\underline{x}_0) \le 0$. □

By virtue of Theorem 2.22, there exists the closed completed parabolic extension M of L, i.e.,

$$M = L_{\bar{\hat{p}}} = ((\bar{L})_p)^\frown)^- \qquad \text{defined on } \mathcal{O}(\underline{\Omega}), \tag{3.6}$$

and M is real. Since L is translation invariant, the local operator M is transalation invariant by Proposition 2.27. In view of the expansion (3.4), the next result turns out to be useful.

Lemma 3.8. *Assume that* (H) *holds. Define the local operator M as above. Let $I =]a,b[$, $V \in \mathcal{O}(\Omega)$, and $f \in C_0(\Omega)$ with $f|V = 0$. Set $F(x,t) = (P_+(t-s)f)(x)$ for $(x,t) \in V \times I = \underline{V}$ and some $s \in \mathbb{R}$. Then, $F \in D(M, \underline{V})$ and $MF = 0$ on $\underline{V}$. If $s \notin I$, the lemma holds for all $f \in C_0(\Omega)$.*

Proof. First, let $s \le a$ and $f \in D(\bar{A}_\Omega)$. Then, $F(\cdot, t) \in D(\bar{A}, V)$ and

$$(\bar{A}F(\cdot,t))(x) = (\bar{A}_\Omega P(t-s)f)(x) = [\tfrac{d}{dt}(P(\cdot - s)f(x))](t) = [\tfrac{d}{dt} F(x,\cdot)](t)$$

for all $x \in V$ and $t \in I$. Hence, $F \in D(L, \underline{V})$ and $LF = 0$ in this case. For $f \in C_0(\Omega)$ and $s \le a$, approximate f in $C_0(\Omega)$ by $f_n \in D(\bar{A}_\Omega)$. Setting $F_n(x,t) = (P(t-s)f_n)(x)$ on $\underline{V}$, we obtain $F_n \to F$ in $C_b(\underline{V})$ and $LF_n = 0$ on $\underline{V}$. Therefore, $F \in D(\bar{L}, \underline{V})$ and $\bar{L}F = 0$ on $\underline{V}$. Further, $F = 0$ if $s \ge b$. So we have shown the lemma for the case $s \notin I$.

Now, let $s \in I$ and $f|V = 0$. For the partition $\{a, s, b\}$ of I, set $I_1 =]a,s[$, $I_2 =]s,b[$, and $\underline{V}_k = V \times I_k$ for $k = 1,2$. By the first part of the proof, $F_2 = F|\underline{V}_2 \in D(\bar{L}, \underline{V}_2)$ and $\bar{L}F_2 = 0$. Also, $F_1 = F|\underline{V}_1 = 0$. Since $f|V = 0$, we have $\lim_{t \searrow s} F_2(t,\cdot) = f = 0$ in $C_b(V)$, and so $F \in C(\underline{V})$. Consequently, $F \in D(\bar{L}_p, \underline{V})$ and $\bar{L}_p F = 0$. □

Corollary 3.9. *Set $\underline{V} = V \times I$ for an open interval $I \subset]0,\infty[$ with $s = \inf I$ and $V \in \mathcal{O}(\Omega)$. Let h be locally excessive in V with respect to $P(t)$. Take $f \in C_0^{\mathbb{R}}(\Omega)$ with $-h \le f \le h$, $G \in \mathcal{O}_c(\Omega)$ with $G \subset \bar{G} \subset V$, and $0 < \sigma \le \eta_{\bar{G}}$. Define $u_\sigma(x,t) = (U_\sigma(t-s)f)(x)$ for $(x,t) \in V \times \bar{I}$, cf.* (3.3)*. Then, $u_\sigma \in D(M, \underline{G}) \cap C(V \times \bar{I})$ satisfies*

$$\begin{cases} Mu_\sigma &= 0 & \text{on } \underline{G} := G \times I, \\ |u_\sigma(x,t)| &\le h(x) & \text{for } (x,t) \in V \times I, \\ u_\sigma(x,s) &= f(x) & \text{for } x \in V. \end{cases} \tag{3.7}$$

Proof. Obviously, $-h \le u_\sigma \le h$ and $u_\sigma(\cdot,s) = S(0)f = f$. By (3.1) and (3.2), the second term in the expansion (3.4) vanishes on G. Similarly, $\Delta(\sigma)S^{n-1}(\sigma)f = 0$ on G for $n = 1, 2\cdots$. So Lemma 3.8 shows that the summands in (3.4) belong to the kernel of $M^{\underline{G}}$. If I is bounded, the sum in (3.4) is finite and so $u_\sigma \in D(M, \underline{G})$ and $Mu_\sigma = 0$ on $\underline{G}$. For unbounded I, take bounded intervals I_n such that $I_n \uparrow I$ and $s = \inf I_n$. Then $u_\sigma \in \ker M^{\underline{G}}$ by the semi-completeness of M. □

We remark that the above computations motivated the construction of the closed completed parabolic extension in Section 2.2. We use the function u_σ to approximate the solutions of the problem

$$\begin{cases} Mu = 0 & \text{on } V\times\,]0,\infty[\,, \\ u(x,t) = 0 & \text{for } x \in \partial V^* \text{ and } t \ge 0, \\ u(x,0) = f(x) & \text{for } x \in V, \end{cases} \tag{3.8}$$

where $f \in C_0(V)$ and $V \in \mathcal{O}(\Omega)$. Here we say that a function u *solves* (3.8) if u belongs to $C(\overline{V^*} \times [0,\infty[) \cap D(M, V\times\,]0,\infty[)$ and satisfies (3.8).

Proposition 3.10. *Let* (H) *hold and let h be a regular excessive barrier for $V \in \mathcal{O}(\Omega)$ with respect to $P(t)$. Then, for all $f \in C_0^{\mathbb{R}}(V)$, there is unique solution $u \in C(\overline{V^*} \times [0,\infty[) \cap D(M, V\times\,]0,\infty[)$ of* (3.8) *and $|u| \le \|f\|$ on $V \times [0,\infty[$.*

Proof. (1) First, let $f \in C_c^{\mathbb{R}}(V)$. After replacing h by αh for a suitable constant $\alpha > 0$, we can assume that $-h \le f \le h$. Choose $G_n \in \mathcal{O}_c(\Omega)$ with $\overline{G_n} \subset G_{n+1}$ and $G_n \uparrow V$. There are $\eta_n := \eta_{\overline{G_n}}$ such that h is locally excessive $(\overline{G_n}, \eta_n)$. Set $U_n(t) = U_{\eta_n}(t)$, and $u_n(x,t) = (U_n(t)f^{\#})(x)$ for $(x,t) \in \underline{V} = V\times\,]0,\infty[$ and $n \in \mathbb{N}$. Corollary 3.9 shows that $u_n \in C(V \times [0,\infty[) \cap D(M, G_n\times\,]0,\infty[)$ satisfies (3.7) for $\underline{G}_n = G_n\times\,]0,\infty[$. Let $K \subset V$ be compact and $T > 0$. Then, $K \subset G_N$ for some N. For $m, n \ge \tilde{n} \ge N$, we have

$$u_n - u_m = 0 \;\text{ on } G_{\tilde{n}} \times \{0\}, \quad |u_n - u_m| \le 2\max_{\partial G_{\tilde{n}}} h \;\text{ on } \partial G_{\tilde{n}} \times [0,T],$$
$$M(u_n - u_m) = 0 \;\text{ on } \underline{G}_{\tilde{n}}.$$

Therefore the parabolic maximum principle, Theorem 2.29, implies

$$\sup_{G_{\tilde{n}}\times[0,T]} |u_n - u_m| \le 2\max_{\partial G_{\tilde{n}}} h.$$

Due to $h \in C_0(V)$, for a given $\varepsilon > 0$ we can choose a sufficiently large $\tilde{n}$ such that $\sup_{K\times[0,T]} |u_n - u_m| \le \varepsilon$ for $n, m \ge \tilde{n}$. That is, u_n converges u.c. in $C(V \times [0,\infty[)$ to a function $u \in C(V \times [0,\infty[)$. Since $Mu_n = 0$ on $\underline{G}_n$ and M is locally closed u.c., we derive $u \in D(M, V\times\,]0,\infty[)$ and $Mu = 0$ on $V\times\,]0,\infty[$. Further, $u(\cdot,0) = f$ and $-h \le u(\cdot,t) \le h$ for $t \ge 0$. Hence we can extend u by 0 to a function $u \in C(\overline{V^*} \times [0,\infty[)$. Another application of Theorem 2.29 gives $|u| \le \|f\|$ on $V \times [0,\infty[$.

(2) For arbitray $f \in C_0^{\mathbb{R}}(V)$, take $f_k \in C_c^{\mathbb{R}}(V)$ which converge to f in $C_0(V)$. In part (1) we have constructed a solution u^k of (3.8) with intial value f_k. By Theorem 2.29, $|u^k - u^l| \le \|f_k - f_l\|$ on $V\times[0,\infty[$ and so (u^k) converges in $C_b(V\times[0,\infty[)$ to

a function u. This shows that $u \in C(\overline{V^*} \times [0,\infty[) \cap D(M, V \times]0,\infty[)$ solves (3.8) and $|u| \le \|f\|$ on $V \times [0,\infty[$. Uniqueness is an immediate consequence of Theorem 2.29. $\square$

We want to prove that the solutions of (3.8) are given by a Feller semigroup on $C_0(V)$. To construct this semigroup, we suppose that (H) holds and that (3.8) has a solution $(x,t) \mapsto u(x,t;\, f)$ for all $f \in C_0(V)$. The latter condition is satisfied if V possesses a regular excessive barrier due to Proposition 3.10; but it will be important later that we do not make this assumption.

First, observe that the maximum principle, Theorem 2.29, yields uniqueness of solutions of (3.8). We define a linear operator on $C_0(V)$ by $Q(t)f = u(\cdot,t;\, f)$ for $t \ge 0$. Again by Theorem 2.29, $Q(t)$ is a contraction. An application of the positive maximum principle Theorem 2.30 to $-u$ yields positivity of $Q(t)$. Since $u \in C(\overline{V^*} \times [0,\infty[)$, $Q(\cdot)$ is strongly continuous. Also, $Q(0) = Id$. For $\tau \ge 0$ and $f \in C_0(V)$, define

$$v(x,t) = u(x, t+\tau;\, f) = (Q(t+\tau)f)(x) \quad \text{on} \quad V \times [0,\infty[\,.$$

Observe that v is the restriction of u_τ to $V \times [0,\infty[$. Thus, the translation invariance of M implies that v is a solution of (3.8) with initial value $Q(\tau)f$, i.e., $v(\cdot,t) = Q(t)Q(\tau)f$. So we have shown

Proposition 3.11. *Let* (H) *hold and $V \in \mathcal{O}(\Omega)$. Assume that there is a solution $u(\cdot,\cdot;\, f)$ of* (3.8) *for all $f \in C_0(V)$.* (*The latter holds if V has a regular excessive barrier by Proposition* 3.10.) *Then there is a Feller semigroup $Q(t) = e^{tE_V}$ on $C_0(V)$ such that $u(\cdot,t;\, f) = Q(t)f$ for $t \ge 0$.*

3.3 Spatial parabolic extension

Of course, we now have to determine the generator E_V of the semigroup obtained in Proposition 3.11. In what follows, we identify E_V with the 'spatial parabolic extension' A_{pV} of $\bar{A}$. Then, in Theorem 3.14 and the next section, we give conditions implying that $\bar{A}_V = A_{pV} = E_V$ which yields the well-posedness of (CP_V).

To construct the local operator A_p on $\mathcal{O}(\Omega)$, we assume that (H) holds. Let $M = L_{\bar{p}}$ be given by (3.5) and (3.6). Set $(f \otimes 1)(x,t) = f(x)$ for $(x,t) \in \underline{V} = V \times \mathbb{R}$, $V \in \mathcal{O}(\Omega)$, and $f \in C_0(V)$. If $f \otimes 1 \in D(M, \underline{V})$, then we have $(M(f\otimes 1))_t = M(f \otimes 1)$ for $t \in \mathbb{R}$ since M is translation invariant and $f \otimes 1 = (f \otimes 1)_t$. In particular,

$$(M(f \otimes 1))(\cdot,t) = (M(f \otimes 1))(\cdot,0) =: g \in C(V)$$

for $t \in \mathbb{R}$. So we can define a local operator on $\mathcal{O}(\Omega)$ by setting

$$D(A_p, V) = \{f \in C(V) : f \otimes 1 \in D(M, V \times \mathbb{R})\},$$
$$A_p f = g, \qquad \text{where} \;\; M(f \otimes 1) = g \otimes 1.$$

We call A_p the *spatial parabolic extension* of $\bar{A}$ and write A_{pV} for $(A_p)_V$.

Lemma 3.12. *Assume that* (H) *holds. Then* A_p *is a real, locally dissipative, locally closed u.c. extension of* $\bar{A}$.

Proof. It is clear that A_p is a real local operator. For $f \in D(\bar{A}, V)$ and $V \in \mathcal{O}(\Omega)$, we have $f \otimes 1 \in D(L, V \times \mathbb{R}) \subset D(M, V \times \mathbb{R})$ and $M(f \otimes 1) = \bar{A}f \otimes 1$ because of (3.5) and $L \subset M$. Hence, A_p extends $\bar{A}$ and satisfies (S). Local dissipativity of A_p is an immediate consequence of Lemma 2.9 and the local dissipativity of M. Further, consider $\mathcal{O}(\Omega) \ni V_n \uparrow V$ and $f_n \in D(A_p, V_n)$ such that $f_n \to f$ and $A_p f_n \to g$ u.c. in $C(V)$. Then, $f_n \otimes 1 \to f \otimes 1$ and $M(f_n \otimes 1) \to g \otimes 1$ u.c. in $C(V \times \mathbb{R})$. Since M is locally closed u.c., we infer $f \otimes 1 \in D(M, V \times \mathbb{R})$ and $M(f \otimes 1) = g \otimes 1$; that is, A_p is locally closed u.c.. □

Now assume that (3.8) has a solution $(x,t) \mapsto u(x,t;f) = (e^{tE_V} f)(x)$ for all $f \in C_0(V)$. To relate A_{pV} and E_V, we define on $\mathcal{X} = C_0(\mathbb{R}, C_0(V))$ the operator

$$D(\mathcal{E}_V) = \{F \in C^1(\mathbb{R}, C_0(V)) \cap \mathcal{X} : F(t) \in D(E_V) \text{ for } t \in \mathbb{R};\ F', E_V F(\cdot) \in \mathcal{X}\},$$
$$\mathcal{E}_V F = E_V F(\cdot) - F',$$

and the 'space-time semigroup' given by

$$(\mathcal{Q}(t)F)(s) = e^{tE_V} F(s-t) \qquad \text{for } t \geq 0,\ F \in \mathcal{X},\ s \in \mathbb{R},$$

compare Section 4.2 and the references given in the introduction. Clearly, $(\mathcal{Q}(t))_{t\geq 0}$ is a Feller semigroup on $\mathcal{X}$ generated by the closure of $\mathcal{E}_V$ (notice that $\mathcal{Q}(\cdot)$ is the product of two commuting Feller semigroups on $\mathcal{X}$). Further, set $\check{U}(t,s) = e^{(t-s)E_V}$ for $t \geq s$ and $\check{U}(t,s) = Id$ for $t < s$. We use the space

$$\mathcal{Z} = \operatorname{lin}\{F \in \mathcal{X} : F(t) = \varphi(t)\check{U}(t,s)f \text{ for } t,s \in \mathbb{R},\ \varphi \in C_c^1(\mathbb{R}),\ \operatorname{supp}\varphi \subset]s,\infty[\,,\ f \in D(E_V)\}. \tag{3.9}$$

Then, $\mathcal{Z}$ is dense in $\mathcal{X}$, $\mathcal{Q}(t)\mathcal{Z} \subset \mathcal{Z} \subset D(\mathcal{E}_V)$ and

$$(\mathcal{E}_V F)(t) = -\sum_{k=1}^{n} \varphi_k'(t)\check{U}(t,s_k)f_k$$

for $F \in \mathcal{Z}$, see [50, Thm. 2.3] or [28, Prop. 2.9]. In particular, the generator $\overline{\mathcal{E}_V}$ is the closure of $(\mathcal{E}_V, \mathcal{Z})$.

On the other hand, let s, φ, f be given as in (3.9). Define $u(x,t) = (\check{U}(t,s)f)(x)$ for $(x,t) \in \underline{V} = V \times \mathbb{R}$. By Proposition 3.11 and the translation invariance of M, we have $u \in D(M, V \times]s,\infty[)$ and $Mu = 0$ on $V \times]s,\infty[$. So we can find $V_n \times]s_n, t_n[\in \underline{\mathcal{A}}$ and $u_n \in D((\bar{L}_p)\hat{}, V_n \times]s_n, t_n[)$ satisfying $V_n \uparrow V$, $s_n \downarrow s$, $t_n \uparrow \infty$, and $u_n \to u$, $(\bar{L}_p)\hat{} u_n \to 0$ u.c. in $C(V \times]s,\infty[)$. Fix $n_0 \in \mathbb{N}$ such that $\operatorname{supp}\varphi \subset]s_n, t_n[$ for $n \geq n_0$. Set $\underline{V}_n = V_n \times \mathbb{R}$ and

$$F_n(x,t) = \begin{cases} \varphi(t)u_n(x,t), & (x,t) \in V_n \times]s_n, t_n[\,, \\ 0, & (x,t) \in V_n \times (]-\infty, s_n] \cup [t_n, \infty[). \end{cases}$$

Clearly, $F_n \in C(\underline{V}_n)$ for $n \geq n_0$. Using parabolicity, we obtain

$$\varphi u_n \in D(\widehat{\bar{L}_p}, V_n \times]s_n, t_n[) \quad \text{and} \quad \widehat{\bar{L}_p}(\varphi u_n) = \varphi \widehat{\bar{L}_p} u_n - \varphi' u_n \text{ on } V_n \times]s_n, t_n[\,.$$

Also, $F_n(x,t) = 0$ for $x \in V_n$ and $t \notin \operatorname{supp} \varphi$. Hence, from the completeness of $(\bar{L}_p)^\frown$ follows that $F_n \in D((\bar{L}_p)^\frown, \underline{V}_n)$ and

$$(\widehat{\bar{L}_p} F_n)(x,t) = \begin{cases} \varphi(t)\,(\widehat{\bar{L}_p} u_n)(x,t) - \varphi'(t) u_n(x,t), & (x,t) \in V_n \times]s_n, t_n[\,, \\ 0, & x \in V_n,\ t \notin \operatorname{supp} \varphi. \end{cases}$$

Thus, $F_n \to \varphi u$ and $MF_n \to -\varphi' u$ u.c. in $C(\underline{V})$. Since M is locally closed u.c., we have $\varphi u \in D(M, \underline{V})$ and $M(\varphi u) = -\varphi' u$ on $\underline{V}$. After identifying $\mathcal{X}$ with $C_0(\underline{V}) \subset C(\underline{V})$, we see that $M_{\underline{V}}$ extends $(\mathcal{E}_V, \mathcal{Z})$. This implies that

$$\overline{\mathcal{E}_V} \subset M_{\underline{V}} \tag{3.10}$$

since $M_{\underline{V}}$ is a closed operator in $C_0(\underline{V})$.

Next, let $f \in D(E_V)$ and $\psi_n \in C_c^1(\mathbb{R})$ with $0 \leq \psi_n \leq 1$ and $\psi_n = 1$ on $[-n, n]$. Set $(f \otimes \psi_n)(x,t) = f(x)\psi_n(t)$ for $(x,t) \in \underline{V}$. Then, $f \otimes \psi_n \in D(\mathcal{E}_V) \subset D(M, \underline{V})$ and

$$M(f \otimes \psi_n) = \mathcal{E}_V(f \otimes \psi_n) = (E_V f) \otimes \psi_n - f \otimes \psi_n' \qquad \text{on } \underline{V}$$

due to (3.10). Using that M is locally closed u.c., we obtain $f \otimes 1 \in D(M, \underline{V})$ and $M(f \otimes 1) = (E_V f) \otimes 1$; that is, $E_V \subset (A_p)_V = A_{pV}$. By Lemma 3.12 and Corollary 2.10, A_{pV} is dissipative in $C_0(V)$, and hence $E_V = A_{pV}$. Summarizing we have

Proposition 3.13. *Let* (H) *hold and* A_p *be the spatial parabolic extension of* $\bar{A}$*. If, for some* $V \in \mathcal{O}(\Omega)$*, the problem* (3.8) *has solutions for all* $f \in C_0(V)$*, then* $A_{pV} = (A_p)_V$ *coincides with the generator* E_V *obtained in Proposition* 3.11*. In particular, the latter assumption is satisfied if* V *admits an excessive barrier w.r.t.* $P(t)$*.*

Proof. By the above considerations, it remains to show the last assertion. The excessive barrier w.r.t. $\bar{A}$ is a regular excessive barrier w.r.t. to $\overline{A_\lambda} = \bar{A}_\lambda$ for some $\lambda \geq 0$. Let M_λ be the closed completed parabolic extension of $L_\lambda = \bar{A} - \lambda - \frac{d}{dt}$, cf. (3.5), and let $e^{tE_{\lambda V}}$ solve the problem (3.8) corresponding to M_λ. We have $(A_\lambda)_{pV} = E_{\lambda V}$ by Proposition 3.11 and the first assertion. Moreover, it is easy to see that $M_\lambda = M - \lambda$, and thus $(A_\lambda)_p = A_p - \lambda$. Since M is parabolic, $e^{\lambda t} e^{tE_{\lambda V}}$ solves (3.8) corresponding to M. As a consequence, $E_V - \lambda = E_{\lambda V}$ and $E_V = A_{pV}$. □

From Lemma 3.12 we know that $\bar{A} \subset A_p$. We can show equality if there is a base of weakly Dirichlet regular sets, cf. Definition 3.3.

Theorem 3.14. *Assume that* (H) *holds and that* $\bar{A}$ *is complete. If there is a base* $\mathcal{B} \subset \mathcal{O}_c(\Omega)$ *of weakly* $\bar{A}_\lambda$*–Dirichlet regular sets for some* $\lambda > 0$*, then* $A_p = \bar{A}$*. If, in addition,* $V \in \mathcal{O}(\Omega)$ *has an excessive barrier, then* $\bar{A}_V = A_{pV} = E_V$ *is a generator on* $C_0(V)$ *and* (CP_V) *is well-posed.*

Proof. By Lemma 3.12 and Proposition 3.13, it suffices to show that $D(A_p, V) \subset D(\bar{A}, V)$ for $V \in \mathcal{O}(\Omega)$. So let $f \in D(A_p, V)$ for some $V \in \mathcal{O}(\Omega)$. Take $G \in \mathcal{B}$ with $\bar{G} \subset V$. Choose $\psi \in C_c(V)$ such that $0 \le \psi \le 1$ and $\psi = 1$ on $\bar{G}$. Set $g_1 = [\psi(A_p - \lambda)f]^{\#}$. Due to (H) and $\bar{A} \subset A_p$, we find $f_1 \in D(\bar{A}_\Omega)$ such that $g_1 = (\bar{A} - \lambda)f_1 = (A_p - \lambda)f_1$. Thus, $(A_p - \lambda)(f - f_1) = 0$ on G. By assumption, there exists $u \in D(\bar{A}, G) \cap C(\bar{G})$ with $(\bar{A} - \lambda)u = 0$ on G and $u|\partial G = (f - f_1)|\partial G$. Hence, $(A_p - \lambda)(f - f_1 - u) = 0$ on G and $f - f_1 - u = 0$ on ∂G. Since A_{pV} is dissipative in $C_0(V)$, this implies $f = f_1 + u \in D(\bar{A}, G)$. Using completeness of $\bar{A}$, we can now derive $f \in D(\bar{A}, V)$. □

3.4 Local Feller semigroups

We now give another condition which allows to characterize well-posedness of (CP_V) by the existence of an excessive barrier, see Theorem 3.19. This will lead to our final result, Theorem 3.25, in the next section. The following concepts were introduced by J.P. Roth, [55]. We remark that Definition 3.15(1) gives a condition of Lindeberg type which is closely related to the continuity of trajectories of associated Markov processes, cf. [18, p.333]. We set

$$\|f\|_K = \|f|K\|_{C(K)} = \sup_{x \in K} |f(x)|$$

for $f \in C(\Omega)$ and a compact subset K of Ω.

Definition 3.15. Let $(Q(t))_{t \ge 0}$ be a semigroup on $C_0(\Omega)$ and $V \in \mathcal{O}(\Omega)$.

1. $(Q(t))_{t \ge 0}$ is called *local* if for each compact $K \subset \Omega$ and $f \in C_0(\Omega)$ vanishing on a neighbourhood of K we have $\|Q(t)f\|_K = o(t)$ as $t \searrow 0$.

2. A semigroup $(Q_1(t))_{t \ge 0}$ on $C_0(V)$ is *tangent* to $(Q(t))_{t \ge 0}$ if for all $f \in C_0(V)$ and compact $K \subset V$ we have

$$\|Q_1(t)f - Q(t)f^{\#}\|_K = o(t) \qquad \text{as } t \searrow 0. \tag{3.11}$$

Observe that if a semigroup $Q_1(\cdot)$ on $C_0(V)$ is tangent to a local semigroup $Q(\cdot)$ on $C_0(\Omega)$, then $Q_1(\cdot)$ is local in $C_0(V)$ since

$$\|Q_1(t)f\|_K \le \|Q_1(t)f - Q(t)f^{\#}\|_K + \|Q(t)f^{\#}\|_K = o(t)$$

for $f \in C_0(V)$ vanishing near $K \subset V$. In Theorem 3.21 and Lemma 3.22 we give conditions for the locality of a Feller semigroup which are rather easy to verify in applications. First, we apply the new concepts to the situation of Proposition 3.11.

Lemma 3.16. *Let* (H) *hold and let* (3.8) *have solutions for some* $V \in \mathcal{O}(\Omega)$ *and each* $f \in C_0(V)$. *Suppose that* $P(\cdot) = (e^{t\bar{A}_\Omega})_{t \ge 0}$ *is local. Then the semigroup* $(e^{tE_V})_{t \ge 0}$ *is tangent to* $P(\cdot)$ *and, hence, local in* $C_0(V)$.

Proof. It suffices to verify (3.11) for $(e^{tE_V})_{t\geq 0}$, $P(\cdot)$, $0 \leq f \in C_0(V)$, and a compact subset K of V. Choose $G \in \mathcal{O}_c(\Omega)$ with $K \subset G \subset \bar{G} \subset V$. Take $\varphi \in C_c(\Omega)$ satisfying $0 \leq \varphi \leq 1$, $\varphi = 1$ on ∂G, and $\varphi = 0$ on a neighbourhood of K. Define $u(x,t) = (P(t)f^{\#})(x)$ and $u_1(x,t) = (e^{tE_V} f)(x)$ for $(x,t) \in V \times]0,T[= \underline{V}$ and some $T > 0$. By Lemma 3.8 and Proposition 3.11 both functions belong to the kernel of $M^{\underline{V}}$. Then, $v = u - u_1 \in D(M, \underline{V}) \cap C(\overline{V^*} \times [0,T])$ satisfies $Mv = 0$ on $\underline{V}$, $v = 0$ on $V \times \{0\}$, and $v = u \geq 0$ on $\partial V^* \times [0,T]$. As a consequence of the positive maximum principle, Theorem 2.30, we obtain $v \geq 0$; that is,

$$0 \leq (P(t)f^{\#})(x) - (e^{tE_V} f)(x) \qquad \text{for } \ (x,t) \in V \times [0,T].$$

By Lemma 3.8, the function $w = (P(\cdot)\varphi)|\underline{V} - v$ belongs to $D(M, \underline{V})$ and $Mw = 0$. We have $w = \varphi \geq 0$ on $G \times \{0\}$. Moreover, $v(\cdot,t)$ converges to 0 in $C_b(V)$ and $(P(t)\varphi)(x)$ tends to 1 uniformly for $x \in \partial G$ as $t \searrow 0$. Therefore, $w \geq 0$ on $\partial G \times [0,T]$ if we choose $T > 0$ small enough. Another application of Theorem 2.30 gives $w \geq 0$ on $G \times [0,T]$. As a result,

$$0 \leq (P(t)f^{\#})(x) - (e^{tE_V} f)(x) \leq (P(t)\varphi)(x) \qquad \text{for } \ (x,t) \in G \times [0,T].$$

This establishes (3.11) since $P(\cdot)$ is local. □

Proposition 3.17. *Let* (H) *hold and let* (3.8) *have solutions for some* $V \in \mathcal{O}(\Omega)$ *and each* $f \in C_0(V)$. *Suppose that* $P(\cdot) = (e^{t\bar{A}_\Omega})_{t\geq 0}$ *is local. Then,* $E_V = A_{pV} = \bar{A}_V$. *The Feller semigroup on* $C_0(V)$ *generated by this operator is tangent to* $P(\cdot)$ *and, hence, local.*

Proof. In view of Proposition 3.11, Lemma 3.12, Proposition 3.13, and Lemma 3.16, it remains to show that $A_{pV} \subset \bar{A}_V$. So let $f \in D(A_{pV})$. By Proposition 3.13 we have $A_{pV} = E_V$, and so Lemma 3.16 yields

$$\tfrac{1}{t} \, \|(P(t)f^{\#} - f^{\#}) - (e^{tA_{pV}} f - f)\|_K \to 0$$

as $t \searrow 0$ for compact subsets K of V. As a consequence,

$$\lim\nolimits_{t\searrow 0} \|\tfrac{1}{t}(P(t)f^{\#} - f^{\#}) - A_{pV} f\|_K = 0.$$

Set $f_t^{\#} = \frac{1}{t}\int_0^t P(s)f^{\#}\,ds \in D(\bar{A}_\Omega)$ for $t > 0$. Notice that $f_t^{\#}$ tends to $f^{\#}$ in $C_0(\Omega)$ as $t \searrow 0$. Moreover,

$$\bar{A}_\Omega \, f_t^{\#} = \tfrac{1}{t}(P(t)f^{\#} - f^{\#})$$

and $(\bar{A} f_t^{\#})|K$ converges to $(A_p f)|K$ in $C(K)$ as $t \searrow 0$. Take $\mathcal{O}_c(\Omega) \ni G_n \uparrow V$ with $\overline{G_n} \subset V$. Since $\bar{A}$ is locally closed, the above argument shows that $f \in D(\bar{A}, G_n)$ and $\bar{A}f = A_p f$ on G_n. Thus, $f \in D(\bar{A}, V)$ and $\bar{A}f = A_p f$ on V because $\bar{A}$ is semi-complete. □

In the next corollary we collect some consequences of the above result.

Corollary 3.18. *Assume that* (H) *holds and that* $P(\cdot) = (e^{t\bar{A}_\Omega})_{t\geq 0}$ *is local. Each of the following two conditions implies that* $E_V = A_{pV} = \bar{A}_V$ *generates a local Feller semigroup on* $C_0(V)$ *being tangent to* $P(\cdot)$.

1. V *admits an excessive barrier with respect to* $P(t)$.
2. $\bar{A}_V$ *is a generator on* $C_0(V)$.

Proof. The first case was proved in Proposition 3.13 and 3.17. If $\bar{A}_V$ is a generator, then one shows as in Lemma 3.8 that the function $(x,t) \mapsto (e^{t\bar{A}_V} f)(x)$ solves (3.8) on $V \times [0,\infty)$ for $f \in C_0(V)$. Again, Proposition 3.17 implies the assertion. □

The main result of this section now follows easily.

Theorem 3.19. *Assume that* (H) *holds and that* $P(\cdot) = (e^{t\bar{A}_\Omega})_{t\geq 0}$ *is local. For* $V \in \mathcal{O}(\Omega)$, *the Cauchy problem* (CP_V) *is well-posed if and only if* V *has an excessive barrier. In this case the solutions of* (CP_V) *are given by the Feller semigroup generated by* $\bar{A}_V$ *on* $C_0(V)$.

Proof. The last assertion is clear. Sufficieny follows from Corollary 3.18. Conversely, if $\bar{A}_V$ is a generator, then there exists a Cauchy barrier $h \in D(\bar{A}_V)$ for V satisfying $h(x) > 0$ and $(\bar{A}-1)h(x) < 0$ for $x \in V$, see Proposition 3.2. Set $P_{1V}(t) = e^{t(\bar{A}_V - 1)}$. Let $K \subset V$ be compact. Then, $(\bar{A}_V - 1)h \leq \alpha < 0$ on K for a constant α. Hence, there are constants $-\beta, \eta > 0$ such that $P_{1V}(s)(\bar{A}_V - 1)h \leq \beta < 0$ on K for $0 \leq s \leq \eta$. This implies

$$P_{1V}(t)h(x) - h(x) = \int_0^t P_{1V}(s)(\bar{A}_V - 1)h(x)\, ds \leq t\beta \quad \text{for} \;\; x \in K \text{ and } 0 \leq t \leq \eta.$$

Moreover, by Corollary 3.18 the semigroups $(e^{-t}P(t))_{t\geq 0}$ and $P_{1V}(\cdot)$ are tangent. Therefore,

$$\left(e^{-t}P(t)h^{\#} - h^{\#}\right)(x) \leq \|e^{-t}P(t)h^{\#} - P_{1V}(t)h\|_K + \beta t = (\varepsilon_K(t) + \beta)\, t$$

for $x \in K$ and $0 \leq t \leq \eta$, where $\varepsilon_K(t) \to 0$ as $t \searrow 0$. Since $\beta < 0$, we can find $\eta_1 = \eta_1(K) > 0$ such that $e^{-t}P(t)h^{\#} \leq h^{\#}$ on K for $0 \leq t \leq \eta_1$. □

Remark 3.20. *In the above proof we have shown that a Cauchy barrier with* $K = \emptyset$ *is an excessive barrier provided that* (H) *holds and* $P(\cdot)$ *is local.*

Concluding this section, we give a sufficient condition for locality of $P(\cdot)$ in terms of $D(\bar{A}_\Omega)$, see [56].

For all compact $K_1, K_2 \subset \Omega$ with $K_1 \cap K_2 = \emptyset$ there is $0 \leq \varphi \in D(\bar{A}_\Omega)$ such that $\varphi|K_1 > 0$ and $\varphi|K_2 = 0$. (3.12)

Theorem 3.21. *If* (H) *and* (3.12) *hold, then* $P(\cdot)$ *is local.*

Proof. Let K be a compact subset of Ω and let $0 \le f \in C_0(\Omega)$ vanish on a set $G \in \mathcal{O}_c(\Omega)$ containing K. By (3.12), there is $0 \le \varphi \in D(\bar{A}_\Omega)$ such that $\varphi > 0$ on ∂G and $\varphi = 0$ on a neighbourhood of K. Define $u(x,t) = (P(t)\varphi)(x) - (P(t)f)(x)$ for $(x,t) \in \overline{G} \times [0,T]$. Let M be given by (3.6). Then, by Lemma 3.8, we have $u \in D(M, G\times]0,T[)$ and $Mu = 0$ on $G\times]0,T[$ for $T > 0$. Further, $u(\cdot,0) \ge 0$ on G and $u \ge 0$ on $\partial G \times [0,\eta]$ for $\eta > 0$ small enough, see the proof of Lemma 3.16. As a result, $u \ge 0$ on $G \times [0,\eta]$ by Theorem 2.30. This implies

$$0 \le \tfrac{1}{t}(P(t)f)(x) \le \tfrac{1}{t}(P(t)\varphi)(x) \;\to\; (\bar{A}_\Omega\varphi)(x) = 0$$

uniformly for $x \in K$. □

3.5 Cauchy barriers and well-posedness of Cauchy problems

In order to verify (3.12), we strengthen our standing hypothesis.

(OE) For $V \in \mathcal{O}(\Omega)$ and $f \in C_c(V) \cap D(A,V)$, we have $f^\# \in D(A,\Omega)$ and $A^\Omega f^\# = (A^V f)^\#$.

(LS) For $V \in \mathcal{O}(\Omega)$, $x \in V$, and $0 \le f \in D(A,V)$, there exists $0 \le f_1 \in D(A,V) \cap C_c(V)$ such that $f = f_1$ on a neighbourhood of x in V.

(H1) In addition to (H), assume that $\bar{A}$ satisfies (OE) and (LS).

These assumptions of '0-extendability' and 'localization of support' are often quite easy to check in applications, see Chapter 6.

Lemma 3.22. *Assumption* (H1) *implies* (3.12) *with* $\varphi \in C_c(\Omega)$.

Proof. Let $K_1, K_2 \subset \Omega$ be compact and disjoint. For $x \in K_1$, take $V_x \in \mathcal{O}_c(\Omega)$ with $x \in V_x$ and $V_x \cap K_2 = \emptyset$. By Proposition 3.2, there exists $0 < \psi \in D(\bar{A}_\Omega)$. Using (LS), we can find a function $0 \le \varphi_x \in D(\bar{A}, V_x) \cap C_c(V_x)$ such that $\varphi_x = \psi > 0$ on a neighbourhood $W_x \subset V_x$ of x. Moreover, the extension $\varphi_x^\#$ belongs to $D(\bar{A}_\Omega)$ by (OE). We can cover K by $W_{x_1}, \cdots, W_{x_n}$. Then $\varphi = \sum_{k=1}^n \varphi_{x_k}^\#$ satisfies (3.12). □

Lemma 3.23. *Let* (H1) *hold. Then, for* $V \in \mathcal{O}(\Omega)$, $x \in V$, *and* $f \in C_c^{\mathbb{R}}(\Omega) \cap D(\bar{A}, V)$, *there exists* $g \in C_c^{\mathbb{R}}(\Omega) \cap D(\bar{A}_\Omega)$ *such that* $g \ge f$ *on* Ω *and* $g = f$ *on a neighbourhood* $W \in \mathcal{O}_c(V)$ *of* x.

Proof. Let V, x, f be as in the statement of the lemma. We have $0 < \psi \in D(\bar{A}_\Omega)$ by Proposition 3.2. There exists a constant α such that $f + \alpha\psi > 0$ on V. Using (LS) and (OE), we construct positive functions $\psi_1, f_1 \in D(\bar{A}_\Omega)$ with compact support in V satisfying $\psi_1 = \psi$ and $f_1 = f + \alpha\psi$ on a neighbourhood $G \in \mathcal{O}_c(V)$ of x. The function $f_2 = f_1 - \alpha\psi_1 \in D(\bar{A}_\Omega)$ has compact support in V and $f_2 = f$ on G. Let K be a compact set containing $\bar{G}$ and the supports of f and f_2. Set $K_1 = K \setminus G$ and $K_2 = \bar{W}$ for an open neighbourhood W of x with $\bar{W} \subset G$. Due to Lemma 3.22, there is a function $0 \le \varphi \in D(\bar{A}_\Omega) \cap C_c(\Omega)$ such that $\varphi|K_1 \ge 1$ and $\varphi|K_2 = 0$. Finally, let $c = \|f_2\| + \|f\|$ and define $g = f_2 + c\varphi \in D(\bar{A}_\Omega) \cap C_c(\Omega)$. Then, $g = f$ on W and $g = f + c\varphi \ge f$ on G. On $K \setminus G$, we have

$$g \ge c\varphi - \|f_2\| \ge c - \|f_2\| = \|f\| \ge f.$$

Also, $g = c\varphi \ge 0 = f$ off K. □

The following lemma improves Remark 3.20 and provides the essential step of the proof of Theorem 3.25.

Lemma 3.24. *Assume that* (H1) *holds. Let* $W \in \mathcal{O}_c(\Omega)$, K_0 *be a compact subset of* W, $G \in \mathcal{O}_c(\Omega)$ *with* $K_0 \subset G \subset \bar{G} \subset W$, *and let* $h \in D(\bar{A}, W \setminus K_0) \cap C(\bar{W} \setminus K_0)$ *satisfy* $h > 0$ *and* $\bar{A}_\lambda h < 0$ *on* $W \setminus K_0$ *for some* $\lambda > 0$. *Then there exists a positive function* $h_W \in C_c(\Omega)$ *such that* $h_W > 0$ *on* W *and* $h_W \le h$ *on* $W \setminus \bar{G}$ *which is locally excessive in* W *w.r.t.* $\bar{A}_\lambda$. *Moreover, if* h *vanishes on* ∂W, *then* $h_W|W$ *is an excessive barrier for* $\bar{A}$ *and* W.

Proof. Let W, K_0, h, and $G =: G_1$ be as in the statement.

(1) Choose $G_0, G_2 \in \mathcal{O}_c(\Omega)$ with

$$K_0 \subset G_0 \subset \overline{G_0} \subset G_1 \subset \overline{G_1} \subset G_2 \subset \overline{G_2} \subset W.$$

Take a function $0 \le h_1 \in C_c(\Omega)$ satisfying $h_1 = h$ on $W \setminus G_0$, where $h_1 = 0$ on $\Omega \setminus W$ if $h = 0$ on ∂W. There exists $0 < \psi \in D(\bar{A}_\Omega)$ with $(\bar{A}_\Omega - \lambda)\psi < 0$ by Proposition 3.2. Since $h > 0$ on the compact set $\overline{G_2} \setminus G_1$, there is a constant $\beta > 0$ such that $h \ge \beta\psi =: \psi_1$ on $\overline{G_2} \setminus G_1$. We now define

$$h_W(x) = \begin{cases} \inf\{h_1(x), \psi_1(x)\}, & x \in \Omega \setminus \overline{G_1}\,, \\ \psi_1(x), & x \in G_2\,. \end{cases}$$

On the open set $G_2 \setminus \overline{G_1}$ we have $h_1 = h \ge \psi_1$ and so h_W is a well-defined positive element of $C_c(\Omega)$.

(2) Let K be a compact subset of W. For $x \in K$, we find a neighbourhood $W_x \in \mathcal{O}_c(\Omega)$ of x such that either

$$(a)\ \ h_W = \inf\{\psi_1\,, h_1\} \ \text{ on } W_x \ \text{ and } \ \overline{W_x} \subset W \setminus \overline{G_1} \qquad \text{or}$$
$$(b)\ \ h_W = \psi_1 \ \text{ on } W_x \ \text{ and } \ \overline{W_x} \subset G_2\,.$$

(2a) In case (a), we have $h = h_1 \ge h_W$ on $\overline{W_x}$. The positive function $(h - h_W)|W_x$ can be extended to a function $0 \le k_x \in C_c(\Omega)$. Hence, $h_x = k_x + h_W \in C_c(\Omega)$ coincides with h on $\overline{W_x}$ and dominates h_W on Ω. In particular, $h_x \in D(\bar{A}, W_x)$. Due to Lemma 3.23, there exist $W_x' \in \mathcal{O}_c(\Omega)$ and $g_x \in D(\bar{A}_\Omega)$ satisfying

$$x \in W_x' \subset \overline{W_x'} \subset W_x \subset \overline{W_x} \subset W \setminus \overline{G_1}\,, \quad g_x \ge h_x \ge h_W \text{ on } \Omega, \quad g_x = h_x = h \text{ on } \overline{W_x'}\,.$$

In the same way, we construct a function $\hat{g}_x \in D(\bar{A}_\Omega)$ such that

$$\hat{g}_x \geq h_W \text{ on } \Omega \quad \text{and} \quad \hat{g}_x = \psi_1 \text{ on } \overline{W'_x}$$

(where we have replaced W'_x by an open subset containing x if necessary). Set $P_\lambda(t) = e^{-\lambda t} P(t)$. Notice that

$$P_\lambda(t) g_x - g_x = \int_0^t P_\lambda(s) \bar{A}_\lambda \, g_x \, ds.$$

Since $g_x = h$ on $\overline{W'_x}$ and $\bar{A}_\lambda h < 0$, there is a constant $\eta_x > 0$ such that $P_\lambda(s) \bar{A}_\lambda \, g_x \leq 0$ on $\overline{W'_x}$ for $0 \leq s \leq \eta_x$. Hence,

$$P_\lambda(t) h_W \leq P_\lambda(t) g_x \leq g_x = h \quad \text{on } \overline{W'_x} \ \text{ for } 0 \leq t \leq \eta_x \, .$$

The same argument yields (after replacing η_x if necessary)

$$P_\lambda(t) h_W \leq P_\lambda(t) \hat{g}_x \leq \hat{g}_x = \psi_1 \quad \text{on } \overline{W'_x} \ \text{ for } 0 \leq t \leq \eta_x \, .$$

As a result, for case (a) we have shown

$$P_\lambda(t) h_W \leq \inf\{h, \psi_1\} = h_W \quad \text{on } \overline{W'_x} \ \text{ for } 0 \leq t \leq \eta_x \, . \tag{3.13}$$

(2b) In the case (b), we have $h_W = \psi_1$ on $\overline{W_x}$. By Lemma 3.23, there exist a neighbourhood $W'_x \in \mathcal{O}_c(\Omega)$ of x with $\overline{W'_x} \subset W_x$ and a function $g_x \in D(\bar{A}_\Omega)$ satisfying $g_x \geq h_W$ on Ω and $g_x = \psi_1$ on $\overline{W'_x}$. (Note that we have used the same symbols as in step (2a) although the designated objects may differ.) As in (2a), we derive

$$P_\lambda(t) h_W \leq P_\lambda(t) g_x \leq g_x = \psi_1 = h_W \quad \text{on } \overline{W'_x} \ \text{ for } 0 \leq t \leq \eta_x \tag{3.14}$$

for a constant $\eta_x > 0$.

(3) By (3.13) and (3.14), for all $x \in K$ we have found $\eta_x > 0$ and a relatively compact neighbourhood W'_x such that $P_\lambda(t) h_W \leq h_W$ on W'_x for $0 \leq t \leq \eta_x$. Now cover K by finitely many W'_{x_k} and set $\eta_K = \min_k \eta_{x_k} > 0$. Thus, $P_\lambda(t) h_W \leq h_W$ on K for $0 \leq t \leq \eta_K$; that is, h_W is locally excessive in W. Moreover, h_W is strictly positive on W since $\psi_1 > 0$ on Ω and $h_1 = h > 0$ on $W \setminus G_0$. Further, on $\overline{G_2} \setminus G_1$ we have $h_W \leq \psi_1 \leq h$, while $h_W \leq h_1 = h$ on $W \setminus G_2$. That is, $h_W \leq h$ on $W \setminus G$. Finally, $h_W = 0$ on $\Omega \setminus W$ if h is a Cauchy barrier, and h_W is an excessive barrier in this case. □

We now come to main theorem of this chapter. Similar results can be found in [30], [31], [32], [48], and [55]. In these papers, it was always assumed that there exists a base of 'regular' sets in $\mathcal{O}_c(\Omega)$. Below we replace this type of condition by supposing that $\bar{A}_\Omega$ is a generator in $C_0(\Omega)$ (besides the mild hypotheses (OE) and (LS)).

Theorem 3.25. *Assume that* (H1) *holds. The following assertions are equivalent for* $V \in \mathcal{O}(\Omega)$.

1. *The Cauchy problem* (CP_V) *is well-posed.*
2. *The set* V *possesses a Cauchy barrier with respect to* $\bar{A}$.
3. *The set* V *possesses an excessive barrier with respect to* $e^{t\bar{A}_\Omega}$.

If one of these conditions hold, the solutions of (CP_V) *are given by the Feller semigroup generated by* $\bar{A}_V$.

Proof. The last assertion follows from the discussion in Section 3.1. The equivalence "1. ⇔ 3." was proved in Theorem 3.19, Theorem 3.21, and Lemma 3.22. Proposition 3.2 yields "1. ⇒ 2.". To show "2. ⇒ 1.", let h be a Cauchy barrier. This means that $h \in D(\bar{A}, V \setminus K_0)$ satisfies $h > 0$ and $\bar{A}_\lambda h < 0$ on $V \setminus K_0$ for a compact subset K_0 of V and, for each $\varepsilon > 0$, there is a compact set K_ε such that $K_0 \subset K_\varepsilon \subset V$ and $0 \leq h \leq \varepsilon$ on $V \setminus K_\varepsilon$. For relatively compact V, assertion 3., and hence 1., is a consequence of Lemma 3.24. In the general case, take $G_0, G_1, G_2, W_n \in \mathcal{O}_c(\Omega)$ such that $W_n \uparrow V$ and

$$K_0 \subset G_0 \subset \overline{G_0} \subset G_1 \subset \overline{G_1} \subset G_2 \subset \overline{G_2} \subset W_n \subset \overline{W_n} \subset W_{n+1}$$

for $n \in \mathbb{N}$. Let $h_{W_n} =: h_n$ be a function satisfying the assertions of Lemma 3.24 for $G := G_1$ and $W := W_n$. Notice that for the construction of h_{W_n} in the proof of Lemma 3.24 we can use the same sets K_0, G_0, G_1, G_2 and the same function ψ_1 for all $n \in \mathbb{N}$. Further, the functions $h_1 = h_1(n)$ defined in the proof of Lemma 3.24 are equal to h on $W_n \setminus G_0$. Therefore,

$$h_n(x) = \begin{cases} \inf\{h(x), \psi_1(x)\}, & x \in W_n \setminus \overline{G_1}\,, \\ \psi_1(x), & x \in G_2\,. \end{cases}$$

We now solve (3.8) for $M_\lambda = M - \lambda$, compare the proof of Proposition 3.13. First, let $f \in C_c^{\mathbb{R}}(V)$ and $\operatorname{supp} f \subset W_n$ for $n \geq n_0$. Then there is a constant $\alpha > 0$ (depending on f) such that $-\alpha h_n \leq f \leq \alpha h_n$ on V for $n \geq n_0$. So the assumptions of Corollary 3.9 hold for $W_n \subset \overline{W_n} \subset W_{n+1}$ and the function αh_{n+1} being locally excessive in W_{n+1} for $\bar{A}_\lambda$. As a result, there exists an approximate solution $u_n := u_{\sigma_n}$ of (3.7); that is, $u_n \in D(M, W_n \times]0,\infty[) \cap C(W_{n+1} \times [0,\infty[)$ satisfying

$$\begin{cases} M_\lambda u_n &= 0 & \text{on } W_n \times]0,\infty[\,, \\ |u_n(x,t)| &\leq \alpha h_{n+1}(x) & \text{for } (x,t) \in W_{n+1} \times]0,\infty[\,, \\ u_n(x,0) &= f(x) & \text{for } x \in W_{n+1}. \end{cases}$$

Therefore the parabolic maximum principle, Theorem 2.29, implies

$$\|u_k - u_l\|_{C(\overline{W_m} \times [0,T])} \leq \alpha \max\nolimits_{\partial W_m}(h_{k+1} + h_{l+1}) \leq 2\alpha \max\nolimits_{\partial W_m} h \tag{3.15}$$

for $k, l \geq m$ and $T \geq 0$. Hence, u_n converges u.c. to a function u in $C(V \times [0,\infty[)$. Consequently, $u \in D(M, V \times]0,\infty[)$ and $M_\lambda u = 0$ since M is locally closed u.c.. Also,

$u(\cdot,0)=f$ and $|u(x,t)|\le \alpha h(x)$ for $x\in V\setminus \bar G$ and $t\ge 0$. Thus, $u(\cdot,t)\in C_0(V)$ and $u\in C(\overline{V^*}\times[0,\infty[)$. So we have solved problem (3.8) for $f\in C_c(V)$ and M_λ.

If $f\in C_0(V)$, take $f_k\in C_c(V)$ converging to f in $C_0(V)$. For the solutions u^k of (3.8) with initial value f_k we obtain the estimate

$$\sup_{V\times[0,T]}|u^k-u^l|\le \|f_k-f_l\|$$

by Theorem 2.29. It is now easy to see that $u=\lim_k u^k$ solves (3.8) for f and M_λ.

Finally, Proposition 3.17, Lemma 3.22, and Theorem 3.21 imply that $(\overline{A_\lambda})_V=\bar A_V-\lambda$ is a generator in $C_0(V)$ and, hence, (CP_V) is well-posed. □

Corollary 3.26. *Assume that* (H1) *holds. Let $V,V_k\in\mathcal O(\Omega)$ such that $V=\bigcap_{k=1}^n V_k$ and V_k admits a Cauchy barrier w.r.t. $\bar A$. Then the Cauchy problem (CP_V) is well-posed.*

Proof. By Theorem 3.25 each V_k has an excessive barrier h_k. It is easy to see that $h=\inf_k h_k|V$ is an excessive barrier for V. □

A result similar to Corollary 3.26 was shown in [31, Thm. 3.3] using a weaker notion of a Cauchy barrier. There it was also noted that a Cauchy problem need not to be well-posed on $V\cup W$ if it is well-posed on V and W.

We conclude with a characterization of well-posedness in the case that V is the union of regular sets V_n without assuming that $\bar A_\Omega$ is a generator. In particular, we do not suppose (H1). This result is useful for problems with degenerate coefficients, see [30, §6] and Section 6.1. Our argument is an adoption of the proof of [30, Thm. 5.4].

Theorem 3.27. *Let $V\in\mathcal O(\Omega)$ and A be a real, locally dissipative, local operator on $\mathcal O(\Omega)$ satisfying* (S). *Assume there exist $V_n\in\mathcal O_c(\Omega)$ such that $V_n\uparrow V$ and $\bar A_{V_n}$ is a generator on $C_0(V_n)$. Then (CP_V) is well-posed if and only if $D(\bar A_V)$ is dense and V has a Cauchy barrier w.r.t. $\bar A$.*

Proof. Necessity is clear due to Proposition 3.2. For sufficiency, we have to find for a given $0\le g\in C_c(V)$ a function $f\in D(\bar A_V)$ with $(1-\bar A)f=g$. Let $\tilde K=\operatorname{supp} g$ and let the Cauchy barrier h be defined outside the compact set K. We may assume that

$$(K\cup\tilde K)\subset V_1\subset\overline{V_1}\subset V_2\subset\cdots\subset V.$$

By assumption there exist $0\le f_n\in D(\bar A_{V_n})$ with

$$(1-\bar A)f_n=g\qquad\text{on }\ V_n. \tag{3.16}$$

Define $h_1=\|g\|\,(\inf_{\partial V_1}h)^{-1}\,h$ on $V\setminus K$. Then we obtain $0\le f_n\le h_1$ on $\partial(V_n\setminus\overline{V_1})=\partial V_n\cup\partial V_1$. Using $(1-\bar A)f_n=g=0$ on $V_n\setminus\overline{V_1}$, we now derive from [30, Thm. 5.3] that $0\le f_n\le h_1$ on $V_n\setminus\overline{V_1}$ and thus

$$0\le f_n^{\#}\le h_1\qquad\text{on }\ V\setminus\overline{V_1}\,. \tag{3.17}$$

We further have $(1-\bar{A})(f_n - f_m) = 0$ on V_k for $n, m \geq k$. So the maximum principle [30, Thm. 5.3'] shows that

$$\sup_{V_k} |f_n - f_m| \leq \max_{\partial V_k} |f_n - f_m| \leq 2 \max_{\partial V_k} h_1 \, .$$

Hence, f_n converge u.c. in $C(V)$ to a function $f \in C(V)$ since $h_1 \in C_0(V \setminus V_1)$. Moreover, $f \in C_0(V)$ due to (3.17). The assertion now follows from (3.16) and the local closedness u.c. of $\bar{A}$. □

4 Time dependent parabolic problems on non-cylindrical domains

4.1 Introduction

The results of the previous chapter are now applied to parabolic problems of the type

$$\left\{ \begin{array}{rcll} \bar{L}u & = & F & \text{on } \underline{V}_s, \\ u(x,t) & = & 0 & \text{on the `lateral boundary' of } \underline{V}_s, \\ u(x,s) & = & f(x) & \text{for } x \in V(s), \end{array} \right. \tag{4.1}$$

where $\underline{V} \in \mathcal{O}(\underline{\Omega})$ may be non-cylindrical (i.e., not of the form $V \times I$), F is continuous, and L is a parabolic, real, locally dissipative, local operator. In our applications to partial differential equations in Chapter 6, $\bar{L}$ can be calculated explicitly and so we obtain that the solution u belongs to appropriate Sobolev spaces.

At first, we specify the setting of the problem (4.1) more precisely. In this chapter, we consider $\underline{V} \in \mathcal{O}(\underline{\Omega})$ with finite $S \leq S_{\underline{V}}$ and $T_{\underline{V}} = T \in J$. So we have the possibilities

$$\begin{array}{lll} (I) & J_0 =]S,T] \text{ and } \underline{\Omega}_0 = \Omega \times J_0 & \text{or} \\ (II) & J = [S,T] \text{ and } \underline{\Omega} = \Omega \times J \text{ with } V(S) \neq \emptyset. & \end{array}$$

To avoid trivial situations, we always assume that $V(s) \neq \emptyset$ for $S_{\underline{V}} < s \leq T_{\underline{V}}$. (In later sections we will further assume that $S = S_{\underline{V}}$.) We set $\underline{V}_0 = \underline{V} \cap \underline{\Omega}_0$ and use the function spaces $\underline{X} = C_0(\underline{V})$ and $\underline{X}_0 = C_0(\underline{V}_0)$. The space $\underline{X}_0$ is identified with the subspace $\check{\underline{X}}_0$ of $\underline{X}$ which consists of functions vanishing on $\underline{V}(S)$. A linear operator E on $\underline{X}_0$ induces a linear operator $\check{E}$ on $\check{\underline{X}}_0$ in an obvious way. Let $F(t) = F(t,\cdot)$ for $F \in \underline{X}$ or $\underline{X}_0$ and $t \in J$ or J_0. Notice that $F(t) \in C_0(V(t)) =: X(t)$. Given $F \in \underline{X}$, we define by

$$\tilde{F}(t) = \left\{ \begin{array}{ll} F(t), & S \leq t \leq T, \\ F(S), & t < S, \end{array} \right.$$

a function on $\tilde{J} =]-\infty, T]$, and analogously for $\underline{X}_0 \cong \check{\underline{X}}_0$. Also,

$$\tilde{X}(t) = \left\{ \begin{array}{ll} X(t), & S \leq t \leq T, \\ X(S), & t < S, \end{array} \right.$$

where $X(S) := \{0\}$ in the case J_0.

For a function f in $C_0(V)$, $\underline{X}$, or $\underline{X}_0$ we write f^* for the canonical extension to a function in $C_0(V^*)$, $C_0(\underline{V}^*) =: \underline{X}^*$, or $C_0(\underline{V}_0^*) =: \underline{X}_0^*$, respectively. Also, for these functions the superscript $^\#$ designates the extension by 0 to functions defined on Ω, Ω^*, $\underline{\Omega}$, $\underline{\Omega}^* := \Omega^* \times [S,T]$, $\underline{\Omega}_0$, $\underline{\Omega}_0^* := \Omega^* \times]S,T]$. In this way, $C_0(V)$, $\underline{X}$, $\underline{X}_0$ are isometrically identified with subspaces $X^\#$, $X^{*\#}$, $\underline{X}^\#$, $\underline{X}^{*\#}$, $\underline{X}_0^\#$, $\underline{X}_0^{*\#}$ of $C_0(\Omega)$, $C(\Omega^*)$, $C_0(\underline{\Omega})$, $C(\underline{\Omega}^*)$, $C_0(\underline{\Omega}_0)$, $C_0(\underline{\Omega}_0^*)$, respectively. The canonical extension of an operator Q defined on one of these function spaces is denoted accordingly, e.g., $Q^\# F^\# := (QF)^\#$. In particular, a semigroup $Q(\cdot)$ on $\underline{X}$ induces semigroups $Q^\#(\cdot)$ and $Q^{*\#}(\cdot)$ on $\underline{X}^\#$ and $\underline{X}^{*\#}$, respectively.

Next, we examine various parts of the boundary of $\underline{V}$ which are relevant to problem (4.1). For $S \le S_{\underline{V}} \le s < T_{\underline{V}} = T$ and $\underline{V} \in \mathcal{O}(\underline{\Omega})$ or $\mathcal{O}(\underline{\Omega}_0)$, we set

$$\Gamma = \partial \underline{V}^*, \quad \Gamma^s = \partial \underline{V}_s^* \setminus \underline{V}(s)\,, \quad \Gamma_s' = \{(x,t) \in \Gamma : t \ge s\},$$
$$\Gamma_s = \{(x,t) \in \Gamma : t > s\}, \quad \text{and} \quad \Gamma(s+0) = \Gamma(s) \cap \overline{\underline{V}_s^*}\,,$$

where $\underline{V}_s$ was defind on p.69 and the closures and boundaries are understood in the topology of $\underline{\Omega}^*$ and $\underline{\Omega}_0^*$, respectively. Observe that for relatively compact $\underline{V}$ the above sets coincide with the analogous sets defined in $\underline{\Omega}$ and $\underline{\Omega}_0$, respectively. Since $T_{\underline{V}} = T$, the set Γ^s can be interpreted as the lateral boundary of $\underline{V}_s$.

Proposition 4.1. *Let $\underline{V} \in \mathcal{O}(\underline{\Omega})$ or $\mathcal{O}(\underline{\Omega}_0)$ and $S \le S_{\underline{V}} \le s < T_{\underline{V}}$. Then the following assertions hold (with disjoint unions).*

1. $\partial \underline{V}_s^* = \underline{V}(s) \cup (\Gamma_s' \cap \overline{\underline{V}_s^*}) = \underline{V}(s) \cup \Gamma_s \cup \Gamma(s+0)$.
2. $\Gamma^s = \Gamma_s' \cap \overline{\underline{V}_s^*} = \Gamma_s \cup \Gamma(s+0)$.
3. $\partial \underline{V}_S^* = \underline{V}(S) \cup \Gamma$ *and* $\Gamma = \Gamma^S = \Gamma_S'$ *if* $s = S = S_{\underline{V}} \in J$ *and*
   $\partial \underline{V}_{S_{\underline{V}}}^* = \Gamma = \Gamma^{S_{\underline{V}}} = \Gamma_{S_{\underline{V}}}'$ *if* $s = S_{\underline{V}} > S$.
4. $\partial_p \underline{V}_s \subset \Gamma^s \cup \underline{V}(s)$, *where* $\partial_p \underline{V}_S = \partial_p \underline{V}$ *if* $s = S = S_{\underline{V}} \in J$.

Proof. (a) Let $S_{\underline{V}} < s < T_{\underline{V}}$. For $(x,s) \in \underline{V}(s)$ there is an open neigbourhood in $\underline{V}$ of the form $W \times]s-\delta, s+\delta[$. Thus $\underline{V}(s) \subset \partial \underline{V}_s^*$. Further,

$$\Gamma_s' \cap \overline{\underline{V}_s^*} \;\subset\; \Gamma \cap \overline{\underline{V}_s^*} \;\subset\; \partial \underline{V}_s^*$$

since $\Gamma \cap \underline{V}_s \subset \Gamma \cap \underline{V} = \emptyset$. Hence,

$$\underline{V}(s) \cup (\Gamma_s' \cap \overline{\underline{V}_s^*}) \subset \partial \underline{V}_s^*\,.$$

Conversely, if $(x,t) \in \partial \underline{V}_s^*$ with $t > s$, then $(x,t) \in \partial \underline{V}^* = \Gamma$ and so $(x,t) \in \Gamma_s' \cap \overline{\underline{V}_s^*}$. So let $(x,s) \in \partial \underline{V}_s^*$. In each neighbourhood of (x,s) there is a point $(y,r) \notin \underline{V}_s$. If we can always find $(y,r) \notin \underline{V}$, then $(x,s) \in \Gamma$ and, hence, $(x,s) \in \Gamma_s' \cap \overline{\underline{V}_s^*}$. Otherwise,

(x,s) belongs to $\underline{V}(s)$. This yields the first equality in the first assertion for $s > S_{\underline{V}}$. Since $\Gamma'_s \cap \underline{V}(s) = \emptyset$, we have

$$\Gamma^s = \Gamma'_s \cap \overline{\underline{V}^*_s}\,. \tag{4.2}$$

To show 1. and 2. in the case $s > S_{\underline{V}}$, it remains to verify

$$\Gamma^s = \Gamma_s \cup \Gamma(s+0)\,. \tag{4.3}$$

If $(x,t) \in \Gamma^s$ with $t > s$, then $(x,t) \in \Gamma$, and so $(x,t) \in \Gamma_s$. Clearly, $\Gamma_s \subset \Gamma^s \cap \underline{\Omega}^*_s$. Hence, $\Gamma_s = \Gamma^s \cap \underline{\Omega}^*_s$. Moreover,

$$\Gamma(s+0) = \Gamma'_s \cap \underline{\Omega}^*(s) \cap \overline{\underline{V}^*_s} = \Gamma^s \cap \underline{\Omega}^*(s)$$

by (4.2). As a consequence,

$$\Gamma^s = (\Gamma^s \cap \underline{\Omega}^*_s) \cup (\Gamma^s \cap \underline{\Omega}^*(s)) = \Gamma_s \cup \Gamma(s+0)\,.$$

(b) We now let $s = S_{\underline{V}} > S$. Then, $\underline{V}_s = \underline{V}$ and $\underline{V}(s) = \emptyset$. Hence, $\Gamma = \partial \underline{V}^*_s = \Gamma^s = \Gamma'_s$ and the equations (4.2) and (4.3) hold in this case, too. This yields Assertions 1. and 2. for $s = S_{\underline{V}} > S$ and the second part of 3.

Finally, let $s = S_{\underline{V}} = S \in J$. Obviously, $\Gamma = \Gamma'_S$. For points (x,t) with $t > S$ it is clear that $(x,t) \in \Gamma$ if and only if $(x,t) \in \partial \underline{V}^*_S$. So consider $(x,S) \in \Gamma$. There are points $(y,t) \in \underline{V}$ with $t > S$ in each neighbourhood of (x,S) in $\underline{\Omega}^*$; that is, $(x,S) \in \partial \underline{V}^*_S$. Conversely, if $(x,S) \in \partial \underline{V}^*_S \setminus \underline{V}(S)$, then $(x,S) \notin \underline{V}$ and so $(x,S) \in \Gamma$. This establishes $\Gamma^S = \Gamma$ and, since $V(S) \subset \partial \underline{V}^*_S$, the third assertion. Further, (4.2) and (4.3), and thus 1., also hold in this case.

(c) The last assertion follows from the previous ones and the definition of $\partial_p \underline{V}$. □

4.2 Parabolic operators and space-time semigroups

Theorem 3.25 provides us with a characterization of well-posedness of (4.1) for $F = 0$ in the space $C_0(\underline{V})$. In particular, then the solutions are given by the semigroup $Q(\cdot)$ generated by $\bar{L}_{\underline{V}}$ in $C_0(\underline{V})$. We will now represent $Q(\sigma)$ in terms of a 'variable space propagator' $U(t,s) \in \mathcal{L}(C_0(V(s)), C_0(V(t)))$. By means of this representation, we can solve the inhomogeneous problem (4.1) in the next section and a class of related semilinear equations in Chapter 5.

As in the cases (I) and (II) introduced on p. 96, we throughout consider $\underline{V} \in \mathcal{O}(\underline{\Omega})$, resp. $\underline{V}_0 \in \mathcal{O}(\underline{\Omega}_0)$, such that $I_{\underline{V}} = [S,T] = J$, resp. $I_{\underline{V}_0} =]S,T] = J_0$, for finite S, T. First, we recall the following definition from [38], [42], [47], or [50].

Definition 4.2. A linear operator E on $\underline{X} = C_0(\underline{V})$ is called *parabolic* if, for $F \in D(E)$ and $\varphi \in C^1(J)$ with $\varphi'(S) = 0$, we have $\varphi F \in D(E)$ and $E(\varphi F) = \varphi\, EF - \varphi' F$. If, in addition, $EF = 0$ on $\underline{V}(S)$ for all $F \in D(E)$, then we say that L is *standard parabolic.*

A linear operator E on $\underline{X}_0 = C_0(\underline{V}_0)$ is *parabolic* if the induced operator $\breve{E}$ on $\breve{\underline{X}}_0$ is

parabolic.
If a parabolic operator E generates a semigroup $Q(\cdot)$, then $Q(\cdot)$ is called *space-time semigroup.*

Observe that E on $\underline{X}_0$ is parabolic if and only if, for $F \in D(E)$ and $\varphi \in C_0^1(J) = \{\varphi \in C^1(J) : \varphi(S) = \varphi'(S) = 0\}$, we have $\varphi F \in D(E)$ and $E(\varphi F) = \varphi\, EF - \varphi' F$. Some preliminary facts are stated in the following lemmas.

Lemma 4.3. *The following assertions hold for $F \in \underline{X}$.*

1. *The mapping $J \to C_0(\Omega)$; $t \mapsto F^\#(t) = F(t)^\#$ is continuous.*
2. *If $F(t) = 0$ for $t \in J$, then for all $\varepsilon > 0$ there exists $\varphi \in C^1(J)$ such that $\varphi'(S) = 0$, $\varphi = 0$ on a neighbourhood of t in J, and $\|F - \varphi F\| = \|F^\# - \varphi F^\#\| \le \varepsilon$.*

Proof. 1. If the first assertion were false, there would exist $\delta > 0$ and $J \ni t_n \to t$ with

$$|F^\#(x_n, t_n) - F^\#(x_n, t)| \ge \delta \tag{4.4}$$

for some $x_n \in \Omega$. There is a compact subset $\underline{K}$ of $\underline{V}$ such that $|F(\underline{x})| \le \delta/4$ for $\underline{x} \notin \underline{K}$. Consequently, for each n, at least one of the points (x_n, t_n) and (x_n, t) is contained in $\underline{K}$. So we obtain a subsequence (x_k, t_k) or (x_k, t) converging in $\underline{K}$. Hence, $x_k \to x$ for some $(x, t) \in \underline{K}$. On the other hand, (4.4) gives

$$|F^\#(x_k, t_k) - F^\#(x_k, t)| \ge \delta,$$

contradicting the continuity of $F^\#$ at (x, t).

2. Let $F(t) = 0$ and $\varepsilon > 0$. By the first part, there exists $\delta > 0$ such that $\|F^\#(s)\| \le \varepsilon$ for $s \in J$ with $|t - s| \le \delta$ where we can choose $\delta < t - S$ if $t \ne S$. Take $\varphi \in C^1(J)$ satisfying $0 \le \varphi \le 1$, $\varphi = 1$ on $J \cap (\mathbb{R} \setminus]t - \delta, t + \delta[)$, and $\varphi = 0$ on $J \cap]t - \delta_0, t + \delta_0[$ for some $0 < \delta_0 < \delta$. It is easy to see that φ has the properties asserted in 2. □

Lemma 4.4. *Let $t \in J$ and $f \in C_c(V(t))$ for $\underline{V} \in \mathcal{O}(\underline{\Omega})$. Then there exists $F \in \underline{X}$ such that $F(t) = f$ and $\|F\| = \|f\|$. If $t > S$, then we can take $F \in \check{\underline{X}}_0$. Moreover, F can be chosen positive if f is positive.*

Proof. Let $K = \operatorname{supp} f \subset V(t)$. Since $K \times \{t\}$ is compact in $\underline{V}$, we find sets $W_k \in \mathcal{O}_c(\Omega)$ and $I_k = J \cap]t - \delta_k, t + \delta_k[$ for $\delta_k > 0$ and $k = 1, \cdots, n$ such that

$$K \times \{t\} \subset \bigcup_{k=1}^{n} (W_k \times I_k) \subset \underline{V}$$

Take $0 < \delta \le \min_k \delta_k$ with $\delta < t - S$ if $t \ne S$. Set $W = \bigcup_k W_k$ and $I = J \cap]t - \delta, t + \delta[$. Choose $\varphi \in C^1(J)$ such that $0 \le \varphi \le 1$, $\varphi(t) = 1$, and $\operatorname{supp} \varphi \subset I$. Define $F(x, s) = \varphi(s) f(x)$ for $s \in I$ and $x \in W$ and $F(x, s) = 0$ on $\underline{V} \setminus (W \times I)$. Clearly, $F \in \underline{X}$ meets the requirements. □

In the proof of the next result we use an idea from [50, Thm. 1.11]. We set $\tilde{\varphi}_\sigma = \tilde{\varphi}(\cdot + \sigma)|J$ and $\tilde{\varphi}_\sigma = \tilde{\varphi}(\cdot + \sigma)|J_0$, respectively, for $\sigma \in \mathbb{R}$.

Theorem 4.5. *Let $Q(\cdot)$ be a semigroup on $\underline{X}$ generated by a parabolic operator E. The following assertions hold for $F \in \underline{X}$, $t \in J$, and $\sigma \geq 0$.*

1. *For $\varphi \in C(J)$ we have*
$$Q(\sigma)\,(\varphi F) = \tilde{\varphi}_{-\sigma}\, Q(\sigma) F \tag{4.5}$$
2. *If $\tilde{F}(t-\sigma) = 0$, then $(Q(\sigma)F)(t) = 0$.*
3. *If, in addition, E is standard parabolic, then*
$$(Q(\sigma)F)(t) = (Q(t-S)F)(t) \qquad \text{for } \; t - \sigma \leq S. \tag{4.6}$$

Moreover, Assertions 1. and 2. still hold if we replace $\underline{X}$ by $\underline{X}_0$ and consider φ with $\varphi(S) = 0$ in 1. In particular, $(Q(\sigma)F)(t) = 0$ for $t - \sigma \leq S$ and $F \in \underline{X}_0$.

Proof. We only consider the interval J since the proof for J_0 follows the same lines.

1. Let $\varphi \in C^1(J)$ with $\varphi'(S) = 0$ and $F \in D(E)$. Define
$$u(\sigma) = Q(\sigma)(\varphi F) - \tilde{\varphi}_{-\sigma}\, Q(\sigma) F \in \underline{X}$$
for $\sigma \geq 0$. Then $u(0) = 0$ and, by parabolicity, $u(\sigma) \in D(E)$ and
$$Eu(\sigma) = EQ(\sigma)(\varphi F) - \tilde{\varphi}_{-\sigma}\, EQ(\sigma)F + \tilde{\varphi}'_{-\sigma}\, Q(\sigma)F = \tfrac{d}{d\sigma}\, u(\sigma).$$
Therefore, $u(\sigma) = 0$ for $\sigma \geq 0$ since E is a generator. The first assertion now follows by an obvious approximation argument.

2. If $\tilde{F}(t-\sigma) = 0$, then either $t - \sigma > S$ and $F(t-\sigma) = \tilde{F}(t-\sigma) = 0$ or $t - \sigma \leq S$ and $F(S) = \tilde{F}(t-\sigma) = 0$. In both cases, by Lemma 4.3 there is $\varphi \in C(J)$ such that $F_1 = \varphi F \in \underline{X}$, $\tilde{\varphi}(t-\sigma) = 0$, and $\|F - F_1\| \leq \varepsilon$ for each given $\varepsilon > 0$. Moreover, the first assertion implies $(Q(\sigma)F_1)\,(t) = 0$, and so
$$\|(Q(\sigma)F)\,(t)\| = \|(Q(\sigma)F)\,(t) - (Q(\sigma)F_1)\,(t)\| \leq \|Q(\sigma)\|\,\|F - F_1\| \leq \varepsilon\,\|Q(\sigma)\|.$$
This establishes 2.

3. Let E be standard parabolic. For $F \in D(E)$ and $t \in J$, we have
$$(Q(\sigma)F)(t) = F(t) + \int_0^\sigma (Q(\tau)EF)(t)\, d\tau.$$
Since $(EF)(S) = 0$, we obtain $(Q(\tau)EF)(t) = 0$ for $t - \tau \leq S$ by the second assertion. So we conclude that
$$(Q(\sigma)F)(t) = F(t) + \int_0^{t-S} (Q(\tau)EF)(t)\, d\tau = (Q(t-S)F)(t)$$
for $0 \leq t - S \leq \sigma$, and obtain the third assertion by the density of $D(E)$. □

We now express $Q(\sigma)$ by a propagator $U(t,s)$ using of the above theorem. Set $\Delta_I = \{(t,s) \in I^2 : t \geq s\}$ for an interval $I \subset \mathbb{R}$. Define $D(s) = \{f \in X(s) : \exists F \in \underline{X}$ with $F(s) = f\}$ and

$$U(t,s)f = (Q(t-s)F)(\cdot,t) \in X(t) \qquad \text{for } f \in D(s),\ f = F(s),\ (t,s) \in \Delta_J.$$

The linear operator $U(t,s) : D(s) \to X(t)$ is well-defined by virtue of Theorem 4.5. Due to Lemma 4.4, $C_c(V(s))$ is contained in $D(s)$ and for $f \in C_c(V(s))$ there is $F \in \underline{X}$ with $F(s) = f$ and

$$\|U(t,s)f\| \leq \|Q(t-s)\|\,\|F\| = \|Q(t-s)\|\,\|f\| \leq M\,\|f\|$$

for $(t,s) \in \Delta_J$ and $M := \sup\{\|Q(\sigma)\| : 0 \leq \sigma \leq T-S\}$. Therefore $U(t,s)$ can be extended to a bounded operator $U(t,s) : X(s) \to X(t)$. For $f \in D(s)$ and $F \in \underline{X}$ with $F(s) = f$, we have $(Q(r-s)F)(\cdot,r) = U(r,s)f$, i.e., $U(r,s)f \in D(r)$, and

$$U(t,r)U(r,s)f = \left(Q(t-r)Q(r-s)F\right)(\cdot,t) = U(t,s)f.$$

This identity also holds on $X(s)$ since $D(s)$ is dense. As a result,

$$U(t,s) \in \mathcal{L}(X(s),X(t)), \qquad \|U(t,s)\| \leq M, \tag{4.7}$$

$$U(t,s) = U(t,r)U(r,s), \qquad U(s,s) = Id_{X(s)} \tag{4.8}$$

for $S \leq s \leq r \leq t \leq T$. If we consider the induced semigroup $Q^{\#}(\sigma)$ on $\underline{X}^{\#} \subset C_0(\underline{\Omega})$ and construct operators $U^{\#}(t,s)$ by the same procedure, then it is immediate that

$$U^{\#}(t,s)f^{\#} = (U(t,s)f)^{\#}$$

and that (4.7) and (4.8) hold on $X^{\#}(s) := X(s)^{\#}$. Moreover, for a parabolic operator E on $\underline{X}_0$ we define $U(t,s)$ and $U^{\#}(t,s)$ in the same way and derive (4.7) and (4.8) on Δ_{J_0}. Finally, we obtain the analogous results for the operators $U^{*\#}(t,s)$ on $X^{*\#}(s) := X(s)^{*\#}$ by considering the induced semigroup $Q^{*\#}(\cdot)$ on $\underline{X}^{*\#}$ and $\underline{X}_0^{*\#}$. Next, we study the continuity of the operators $U(t,s)$ with respect to t and s.

Lemma 4.6. *Let $U(t,s)$, $(t,s) \in \Delta_J$, be as constructed above and $f \in X(s)$ for $s \geq S$. Then the functions*

$$u : (x,t) \mapsto \left(U(t,s)f\right)(x) \quad \text{and} \quad u^{*\#} : (x,t) \mapsto \left(U^{*\#}(t,s)f^{*\#}\right)(x)$$

belong to $C_0(\underline{V}_s \cup \underline{V}(s))$ and $C(\Omega^* \times [s,T])$, respectively. If $s > S$, the same is true with J replaced by J_0.

Proof. Let $s \geq S$, $f \in D(s)$, and $F \in \underline{X}$ with $F(s) = f$. Then

$$u^{*\#}(x,t) = (Q^{*\#}(t-s)F^{*\#})(x,t) \qquad \text{on } \Omega^* \times [s,T].$$

For $(x,t),(x',t') \in \underline{\Omega}^*$ with $t,t' \geq s$, we estimate

$$|u^{*\#}(x',t') - u^{*\#}(x,t)| \leq \|Q^{*\#}(t'-s)F^{*\#} - Q^{*\#}(t-s)F^{*\#}\| \\ + |Q^{*\#}(t-s)F^{*\#}(x',t') - Q^{*\#}(t-s)F^{*\#}(x,t)|.$$

Since $Q^{*\#}(\cdot)$ is strongly continuous on $\underline{X}^{*\#}$ and $Q^{*\#}(t-s)F^{*\#} \in \underline{X}^{*\#}$, both terms on the right hand side tend to 0 as $(x',t') \to (x,t)$. Further, $u^{*\#}$ vanishes off $\underline{V}_s \cup \underline{V}(s)$. Thus the assertions concerning J are verified for $f \in D(s)$. Due to the uniform boundedness of $U(t,s)$ and $U^{*\#}(t,s)$, the assertions follow by approximation for all $f \in X(s)$. The last claim can be shown in the same way. □

Lemma 4.7. *Let $U(t,s)$, $(t,s) \in \Delta_J$, be as constructed above and $F \in \underline{X}$. Then the functions*

$$\Delta_J \ni (t,s) \mapsto U^{\#}(t,s)F^{\#}(s) \in C_0(\Omega) \quad \text{and} \quad \Delta_J \ni (t,s) \mapsto U^{*\#}(t,s)F^{*\#}(s) \in C(\Omega^*)$$

are continuous. The same is true with J and $\underline{X}$ replaced by J_0 and $\underline{X}_0$.

Proof. Let $(t,s),(t',s') \in \Delta_J$ and $F \in \underline{X}$. Then the estimate

$$\begin{aligned} &\|U^{\#}(t,s)F^{\#}(s) - U^{\#}(t',s')F^{\#}(s')\| \\ &= \|(Q^{\#}(t-s)F^{\#})(t) - (Q^{\#}(t'-s')F^{\#})(t')\| \\ &\leq \|(Q^{\#}(t-s)F^{\#})(t) - (Q^{\#}(t-s)F^{\#})(t')\| + \|Q^{\#}(t-s)F^{\#} - Q^{\#}(t'-s')F^{\#}\| \end{aligned}$$

yields the first claim. The other ones can be proved analogously. □

We collect the above properties in the following definition.

Definition 4.8. Let I be an interval in $\mathbb{R}$ and $Y(t)$, $t \in I$, be Banach spaces which are isomorphic to subspaces $Y^{\#}(t)$ of a fixed Banach space $Y^{\#}$. A family of operators $U(t,s) \in \mathcal{L}(Y(s),Y(t))$, $(t,s) \in \Delta_I$, is called *variable space propagator* if

$$\begin{aligned} &U(s,s) = Id_{Y(s)}, \qquad U(t,r)U(r,s) = U(t,s), \quad \text{and} \\ &\Delta_I \ni (t,s) \mapsto (U(t,s)F(s))^{\#} \quad \text{is continuous in } Y^{\#} \end{aligned}$$

for $t \geq r \geq s$ in I and all functions $t \mapsto F(t) \in Y(t)$ such that $I \ni t \mapsto F^{\#}(t) \in Y^{\#}$ is continuous. If $\|U(t,s)\| \leq M$, then the propagator is called *bounded*.

We further extend a given variable space propagator U with parameter set Δ_J or Δ_{J_0} to the index set $\Delta_{\tilde{J}}$ by setting

$$\tilde{U}(t,s) = \begin{cases} U(t,s), & S \leq s \leq t \leq T, \\ U(t,S), & s < S \leq t \leq T, \\ Id_{X(S)}, & s \leq t < S, \end{cases} \quad \text{and} \quad \tilde{U}(t,s) = \begin{cases} U(t,s), & S < s \leq t \leq T, \\ 0, & s \leq S, \end{cases}$$

on $\tilde{X}(s)$ for J and J_0, respectively. It is easy to see that $\tilde{U}(t,s)$ is a variable space propagator on $\tilde{X}(s)$. One defines the propagators $\tilde{U}^{\#}(t,s)$ and $\tilde{U}^{*\#}(t,s)$ in the same way. The main theorem of this section is now an immediate consequence of the previous results.

Theorem 4.9. *Let $Q(\cdot)$ be a semigroup on $\underline{X}$ (on $\underline{X}_0$) generated by a standard parabolic (parabolic) operator E. Then there is a bounded variable space propagator $U(t,s)$ such that*

$$
\begin{aligned}
(Q(\sigma)F)(t) &= \tilde{U}(t, t-\sigma)\tilde{F}(t-\sigma) \qquad \text{and} \\
(Q^{*\#}(\sigma)F^{*\#})(t) &= \tilde{U}^{*\#}(t, t-\sigma)\tilde{F}^{*\#}(t-\sigma)
\end{aligned}
\tag{4.9}
$$

for $t \in J$ and $F \in \underline{X}$ ($t \in J_0$ and $F \in \underline{X}_0$) and $\sigma \geq 0$.

Remark 4.10.

1. *In this section we have extended results from* [42] *to sets $\underline{V}$ which are not relatively compact. For cylindrical domains $\underline{V} = V \times I$ and the intervals $J =]S,T]$, $]S,\infty[$, $\mathbb{R}$, analogous results can be found in* [17], [44], [47], *and* [50]. *In* [17], [27], [44], [51], *and* [52] *the same problem was investigated in an L^p–context. In these papers, as well as in* [38] *and* [39], *applications to perturbation theory and well-posedness of non-autonomous Cauchy problems were given, see also the next section.*

2. *Clearly,* (4.9) *implies* (4.5) (*and* (4.6) *in the standard parabolic case*). *Further it is easy to see that* (4.5) (*and* (4.6)) *imply* (*standard*) *parabolicity of the generator E. See also the references cited in the first item for similar characterizations.*

4.3 Well-posedness of parabolic problems on non-cylindrical domains

At first, we apply the results of the preceding section to solve Cauchy problems of the type (4.1) supposing that $\bar{L}_{\underline{V}}$ is a generator. The latter assumption can then be characterized by means of the results presented in Chapter 3. For relatively compact $\underline{V}$, Theorems 4.11, 4.12, and 4.13 below are contained in [42], see also [47], [48], [50] for the cylindrical case.

We start with the standard parabolic case $J = [S,T]$, i.e., case (II) from p.96. For $\underline{V} \in \mathcal{O}(\underline{\Omega})$ and a parabolic, locally dissipative, local operator L on $\mathcal{O}(\underline{\Omega})$, we define a standard parabolic operator E on $\underline{X} = C_0(\underline{V})$ by setting

$$EF = LF \quad \text{for} \quad F \in D(E) = \{F \in D(L_{\underline{V}}) : (LF)(S) = 0\}. \tag{4.10}$$

Therefore, if E is generator, then $Q(\sigma) = e^{\sigma E}$ is given by a variable space propagator $U(t,s)$ by virtue of Theorem 4.9. In the following two results we only consider the initial time $s = S$ as the case $s > S$ can be treated using Theorem 4.13 and 4.14.

Theorem 4.11. *Let $\underline{V} \in \mathcal{O}(\underline{\Omega})$ with $V(s) \neq \emptyset$ for $s \in J = [S,T]$, $f \in C_0(V(S))$, and L be a parabolic, locally dissipative, local operator on $\mathcal{O}(\underline{\Omega})$. Assume that the operator E defined in* (4.10) *is a generator on $\underline{X} = C_0(\underline{V})$. Then there is a unique solution $u \in D(E) \cap C(\overline{\underline{V}^*})$ of the problem*

$$
\left\{
\begin{array}{rcll}
Eu &=& 0 & \text{on } \underline{V}, \\
u(\cdot, S) &=& f & \text{on } V(S), \\
u|\Gamma &=& 0, &
\end{array}
\right.
\tag{4.11}
$$

where $\Gamma = \Gamma^S$. *Further,* u *is given by* $u(x,t) = (U(t,S)f)(x)$ *on* $\underline{V}$ *for the variable space propagator obtained in Theorem* 4.9, *and* $|u| \le \|f\|$.

Proof. The function $u : (x,t) \mapsto (U(t,S)f)(x)$ belongs to $\underline{X}$ by Lemma 4.6. Clearly, $u(\cdot,S) = f$. Moreover, $\partial \underline{V}^* = \Gamma = \Gamma^S$ by Proposition 4.1. So u extends continuously to $\overline{\underline{V}^*}$ with $u|\Gamma = 0$. Using the representation of the semigroup $Q(\sigma) = e^{\sigma E}$ given in Theorem 4.9, we derive

$$\begin{aligned}(Q(\sigma)u)(t) &= \tilde{U}(t,t-\sigma)\tilde{u}(t-\sigma) = \begin{cases} U(t,t-\sigma)U(t-\sigma,S)f, & t-\sigma \ge S, \\ U(t,S)f, & t-\sigma < S, \end{cases} \\ &= U(t,S)f = u(t)\end{aligned}$$

for $t \in J$ and $\sigma \ge 0$. This implies $u \in D(E)$ and $Eu = 0$. Finally, uniqueness and the asserted estimate are immediate consequences of the parabolic maximum principle Theorem 2.29 since $\partial_p \underline{V} \subset \Gamma \cup \underline{V}(S)$ by Proposition 4.1. □

Notice that the condition $F(S) = 0$ is necessary for the solvability of (4.12) in the next result.

Theorem 4.12. *Let* $\underline{V}$, f, L, E *be as in Theorem* 4.11 *and* $F \in C_0(\underline{V}_0)$. *Then there is a unique solution* $u \in D(E) \cap C(\overline{\underline{V}^*})$ *of the problem*

$$\begin{cases} Eu = F & \text{on } \underline{V}, \\ u(\cdot,S) = f & \text{on } V(S), \\ u|\Gamma = 0, \end{cases} \tag{4.12}$$

where $\Gamma = \Gamma^S$. *Further, the solution* u *and its extension* $u^{*\#} \in C(\underline{\Omega}^*)$ *are given by*

$$\begin{aligned} u(x,t) &= (U(t,S)f)(x) - \int_S^t (Q(t-\tau)F)(x,t)\,d\tau \quad \text{on } \underline{V} \text{ and} \\ u^{*\#}(t) &= U^{*\#}(t,S)f^{*\#} - \int_S^t U^{*\#}(t,\tau)F^{*\#}(\tau)\,d\tau \quad \text{for } t \in [S,T] \end{aligned}$$

and the variable space propagator obtained in Theorem 4.9.

Proof. By Lemma 4.7 the function

$$\Delta_J \ni (t,s) \mapsto U^{*\#}(t,s)F^{*\#}(s) = (U(t,s)F(s))^{*\#} \in C(\Omega^*)$$

is continuous. So we can define

$$G^{*\#}(t) = \int_S^t U^{*\#}(t,\tau)F^{*\#}(\tau)\,d\tau \quad \text{and} \quad G^{*\#}(x,t) = (G^{*\#}(t))(x)$$

for $(x,t) \in \underline{\Omega}^*$. Clearly, $G^{*\#} \in \underline{X}^{*\#}$ and $G := G^{*\#}|\underline{V} \in \underline{X}$. We obtain

$$\begin{aligned}(Q^{*\#}(\sigma)G^{*\#})(t) &= \tilde{U}^{*\#}(t,t-\sigma)\tilde{G}^{*\#}(t-\sigma)\\ &= \begin{cases} \int_S^{t-\sigma} U^{*\#}(t,\tau)F^{*\#}(\tau)\,d\tau, & t-\sigma \geq S,\\ 0, & t-\sigma < S,\end{cases}\end{aligned}$$

because $\tilde{G}^{*\#}(t) = 0$ for $t \leq S$. Since also $\tilde{F}^{*\#}(t) = 0$ for $t \leq S$, this yields

$$\begin{aligned}\tfrac{1}{\sigma}\,(Q^{*\#}(\sigma)G^{*\#} - G^{*\#})(t) &= \begin{cases} -\frac{1}{\sigma}\int_{t-\sigma}^{t} U^{*\#}(t,\tau)F^{*\#}(\tau)\,d\tau, & t-\sigma \geq S,\\ -\frac{1}{\sigma}\int_S^t U^{*\#}(t,\tau)F^{*\#}(\tau)\,d\tau, & t-\sigma < S,\end{cases}\\ &= -\tfrac{1}{\sigma}\int_{t-\sigma}^{t} \tilde{U}^{*\#}(t,\tau)\tilde{F}^{*\#}(\tau)\,d\tau\end{aligned}$$

for $t \in J$. So Lemma 4.7 implies

$$\tfrac{1}{\sigma}\,(Q^{*\#}(\sigma)G^{*\#} - G^{*\#})(t) \;\to\; -\tilde{F}^{*\#}(t) = -F^{*\#}(t) \qquad \text{as } \sigma \to 0$$

in $X(t)^{*\#}$ uniformly for $t \in J$. As a consequence, $G \in D(E)$ and $EG = -F$. Thus, by Theorem 4.11, the function

$$u : \underline{V} \ni (x,t) \mapsto (U(t,S)f)(x) - G(x,t)$$

belongs to $D(E)$ and $Eu = F$. Clearly, $u(S) = f$ and u extends continuously to $\overline{\underline{V}^*} = \underline{V} \cup \Gamma$ with $u|\Gamma = 0$. Uniqueness follows again from Theorem 2.29 and Proposition 4.1. Finally, by Theorem 4.9 we have

$$U^{*\#}(t,\tau)F^{*\#}(\tau) = (Q(t-\tau)F)^{*\#}(t) \in X(t)^{*\#}$$

for $S \leq \tau \leq t \leq T$. This easily implies the asserted representations of u. □

We now study the case $J_0 =]S,T]$, cf. (I) from p.96. Notice that the local operator L on $\mathcal{O}(\underline{\Omega}_0)$ in the two following theorems could be the restriction of a local operator L' defined on $\mathcal{O}(\underline{\Omega})$.

Theorem 4.13. *Let $\underline{V}_0 \in \mathcal{O}(\underline{\Omega}_0)$ with $V(t) \neq \emptyset$ for $t \in J_0$, $S < s < T$, and $f \in C_0(V(s))$. Let L be a semi-complete, locally dissipative, parabolic, local operator on $\mathcal{O}(\underline{\Omega}_0)$. Assume that the part $L_{\underline{V}_0}$ of L in $\underline{X}_0 = C_0(\underline{V}_0)$ is a generator. Then there is a unique solution $u \in D(L,\underline{V}_s) \cap C(\overline{\underline{V}_s^*})$ of the problem*

$$\begin{cases} Lu = 0 & \text{on } \underline{V}_s,\\ u(\cdot,s) = f & \text{on } V(s),\\ u|\Gamma^s = 0. \end{cases} \tag{4.13}$$

Further, u is given by $u(x,t) = (U(t,s)f)(x)$ on $\underline{V}_s \cup \underline{V}(s)$ for the variable space propagator obtained in Theorem 4.9, and $|u| \leq \|f\|$.

Proof. For the induced propagator $U^{*\#}(t,s) : X^{*\#}(s) \to X^{*\#}(t)$, we define

$$\check{U}^{*\#}(t,s) = \begin{cases} U^{*\#}(t,s), & S < s \le t \le T, \\ U^{*\#}(s,s), & -\infty < t \le s \le T. \end{cases}$$

Take $\varphi \in C^1]-\infty, T]$ with support in $]s,T]$. Then the function

$$u_\varphi^\# :]-\infty, T] \ni t \mapsto \varphi(t)\check{U}^{*\#}(t,s)f^{*\#} \in C(\Omega^*)$$

is continuous by Lemma 4.7. The restriction to J_0 is denoted by the same symbol. We set $u_\varphi(x,t) = (u_\varphi^\#(t))(x)$ for $(x,t) \in \underline{V}_0$. Notice that $u_\varphi \in \underline{X}_0$. Theorem 4.9 yields

$$\begin{aligned}(Q^{*\#}(\sigma)u_\varphi^{*\#})(t) &= \tilde{U}^{*\#}(t,t-\sigma)u_\varphi^\#(t-\sigma) \\ &= \begin{cases} \varphi(t-\sigma)\, U^{*\#}(t,t-\sigma)U^{*\#}(t-\sigma,s)f^{*\#}, & t-\sigma \ge s, \\ 0, & t-\sigma < s, \end{cases} \\ &= \varphi(t-\sigma)\check{U}^{*\#}(t,s)f^{*\#}\end{aligned}$$

for $-\infty < t \le T$ and $\sigma \ge 0$. Since $U^{*\#}(t,s)$ is uniformly bounded, we obtain that

$$\tfrac{1}{\sigma}\left(Q^{*\#}(\sigma)u_\varphi^{*\#} - u_\varphi^{*\#}\right)(t) \;\to\; -\varphi'(t)\,\check{U}^{*\#}(t,s)f^{*\#} \qquad \text{as } \sigma \to 0$$

in $C(\Omega^*)$ uniformly for $t \in J_0$. As a consequence,

$$u_\varphi \in D(L_{\underline{V}_0}) \quad \text{and} \quad Lu_\varphi\,(x,t) = -\varphi'(t)(U(t,s)f)\,(x) \ \text{ on } \ \underline{V}_s\,.$$

On the other hand, we define the function

$$\overline{\underline{\Omega}_s^*} \ni (x,t) \mapsto u^{*\#}(x,t) = (U^{*\#}(t,s)f^{*\#})(x) = (U(t,s)f)^{*\#}(x).$$

By Lemma 4.6, $u^{*\#}$ is continuous and vanishes off $\underline{V}(s) \cup \underline{V}_s$. In particular, the restriction u to $\overline{\underline{V}_s^*}$ is continuous and $u|\Gamma^s = 0$. Clearly, $u(s) = f$. For small $\varepsilon > 0$, choose $\varphi \in C^1\,]-\infty, T]$ with support in $]s,T]$ and $\varphi = 1$ on $[s+\varepsilon, T]$. Then, $u|\underline{V}_{s+\varepsilon} = u_\varphi|\underline{V}_{s+\varepsilon} \in D(L, \underline{V}_{s+\varepsilon})$ and $Lu = 0$ on $\underline{V}_{s+\varepsilon}$. Letting $\varepsilon \to 0$, we obtain $u \in D(L, \underline{V}_s)$ and $Lu = 0$ on $\underline{V}_s$ by the semi-completeness of L. Finally, uniqueness and the asserted estimate are immediate consequences of Theorem 2.29 and Proposition 4.1. □

Theorem 4.14. *Let $\underline{V}_0$, s, f, and L be as in Theorem 4.13 and $F \in C(\overline{\underline{V}_s^*})$ with $F|\Gamma^s = 0$. In addition, assume that L is locally closed u.c.. Then there is a unique solution $u \in D(L, \underline{V}_s) \cap C(\overline{\underline{V}_s^*})$ of the problem*

$$\begin{cases} Lu &= F \quad \text{on } \underline{V}_s, \\ u(\cdot, s) &= f \quad \text{on } V(s), \\ u|\Gamma^s &= 0. \end{cases} \tag{4.14}$$

Further, u is the restriction of the continuous function

$$u^{*\#}(\cdot,t) = U^{*\#}(t,s)f^{*\#} - \int_s^t U^{*\#}(t,\tau)F^{*\#}(\tau)\,d\tau \quad \text{on } \overline{\underline{\Omega}_s^*}\,. \tag{4.15}$$

If $F_0 \in C_0(\underline{V}_0)$ is an extension of F to $\underline{V}_0$, then

$$u(x,t) = (U(t,s)f)(x) - \int_s^t (Q(t-\tau)F_0)(x,t)\,d\tau \quad \text{on } \underline{V}_0\,, \tag{4.16}$$

where $Q(\sigma)$ and $U(t,s)$ are given by Theorem 4.9.

Proof. The set

$$\underline{W}_0^* = (\underline{\Omega}_0^* \setminus \underline{V}_0^*) \cup \overline{\underline{V}_s^*} = (\underline{\Omega}_0^* \setminus \overline{\underline{V}_0^*}) \cup \partial\underline{V}_0^* \cup \overline{\underline{V}_s^*}$$

is closed in $\underline{\Omega}_0^*$. Extend F to a function F_1 on $\underline{W}_0^*$ by setting $F_1(\underline{x}) = 0$ for $\underline{x} \in \partial\underline{V}_0^* \setminus \overline{\underline{V}_s^*}$ and $\underline{x} \in \underline{\Omega}_0^* \setminus \overline{\underline{V}_0^*}$. Using Proposition 4.1, it is easy to see that F_1 is continuous on $\underline{W}_0^*$. So we can extend F_1 to a continuous function $F_0^{*\#}$ on $\underline{\Omega}_0^*$ with support in $\underline{\Omega}_r^*$ for some $S < r < s$. Clearly, the restriction F_0 to $\underline{V}_0$ belongs to $C_0(\underline{V}_0)$. The function

$$(t,\tau) \mapsto (Q^{*\#}(t-\tau)F_0^{*\#})\,(t) = U^{*\#}(t,\tau)F^{*\#}(\tau) \in C(\Omega^*)$$

is continuous for $s \le \tau \le t \le T$ by Lemma 4.7. So we can define

$$G^{*\#}(t) = \int_s^t U^{*\#}(t,\tau)F^{*\#}(\tau)\,d\tau = \int_s^t (Q^{*\#}(t-\tau)F_0^{*\#})(t)\,d\tau \in X(t)^{*\#} \tag{4.17}$$

for $s \le t \le T$ and $G^{*\#}(x,t) = (G^{*\#}(t))(x)$ for $(x,t) \in \Omega^* \times [s,T]$. Let G be the restriction to $\overline{\underline{V}_s^*}$. Notice that $G^{*\#}$ and G are continuous, $G(s) = 0$, and $G|\Gamma^s = 0$.

For small $\varepsilon > 0$, choose $\varphi \in C^1]-\infty,T]$ with support in $]s,T]$ and $\varphi = 1$ on $[s+\varepsilon,T]$, and set

$$G_\varphi^{*\#}(t) = \int_s^t U^{*\#}(t,\tau)\varphi(\tau)F^{*\#}(\tau)\,d\tau \quad \text{for } \ s < t \le T$$

and $G_\varphi^{*\#}(t) = 0$ for $S < t \le s$. Define $G_\varphi^{*\#}(x,t) = (G_\varphi^{*\#}(t))(x)$ on $\underline{\Omega}_0^*$. Then $G_\varphi^{*\#} \in \underline{X}_0^{*\#}$ and

$$\begin{aligned}(Q^{*\#}(\sigma)G_\varphi^{*\#})(t) &= \tilde{U}^{*\#}(t,t-\sigma)\tilde{G}_\varphi^{*\#}(t-\sigma) \\ &= \begin{cases} \int_s^{t-\sigma} U^{*\#}(t,\tau)\varphi(\tau)F^{*\#}(\tau)\,d\tau, & t-\sigma \ge s, \\ 0, & t-\sigma < s, \end{cases}\end{aligned}$$

for $t \in J_0$ and $\sigma \geq 0$. Subsequently,

$$\frac{1}{\sigma}\left(Q^{*\#}(\sigma)G_{\varphi}^{*\#} - G_{\varphi}^{*\#}\right)(t) = -\tfrac{1}{\sigma}\int_{t-\sigma}^{t} \tilde{U}^{*\#}(t,\tau)\,\varphi(\tau)\tilde{F}_0^{*\#}(\tau)\,d\tau$$
$$\to -\varphi(t)F_0^{*\#}(t) \qquad \text{as } \sigma \to 0$$

uniformly in $t \in J_0$ by Lemma 4.7. Hence, the restriction G_φ belongs to $D(L, \underline{V}_s)$ and $LG_\varphi = -\varphi F$ on $\underline{V}_s$. So we obtain $G \in D(L, \underline{V}_s)$ and $LG = -F$ on $\underline{V}_s$ since L is locally closed u.c.. Finally, set

$$u(x,t) = (U(t,s)f)(x) - G(x,t) \quad \text{on } \underline{V}_s \cup \underline{V}(s) \qquad \text{and}$$
$$u^{*\#}(x,t) = (U^{*\#}(t,s)f^{*\#})(x) - G^{*\#}(x,t) \quad \text{on } \overline{\underline{\Omega}_s^*}.$$

By the previous results and Theorem 4.13, we have $u^{*\#}|\underline{V}_s = u$, $u^{*\#}$ is continuous, and $u \in D(L, \underline{V}_s) \cap C(\overline{\underline{V}_s^*})$ satisfies (4.14). Equalities (4.15) and (4.16) follow from (4.17). Theorem 2.29 and Proposition 4.1 yield uniqueness. □

We now combine the above results with Corollary 3.26 using the following assumption.

(H2) Let L be a real, locally dissipative, (standard) parabolic, local operator defined on $\mathcal{O}(\underline{\Omega}_0)$ (on $\mathcal{O}(\underline{\Omega})$) such that (S) holds and $\bar{L}$ satisfies (OE) and (LS). Assume that $\bar{L}_{\underline{\Omega}_0}$ ($\bar{L}_{\underline{\Omega}}$) is a generator on $\underline{X}_0$ (on $\underline{X}$).

Recall that $\bar{L}$ exists and is real, (standard) parabolic, locally disspative, and locally closed u.c.. In the standard parabolic case, $\bar{L}_{\underline{V}}$ coincides with the operator E given by (4.10). It is easy to see that (H2) for $\underline{\Omega}$ implies (H2) for $\underline{\Omega}_0$. In particular, $\bar{L}_{\underline{V}_0}$ is a generator on $\underline{X}_0$ if $\bar{L}_{\underline{V}}$ is a generator on $\underline{X}$ for some $\underline{V} \in \mathcal{O}(\underline{\Omega})$. In fact, in this case there exists $G \in D(\bar{L}_{\underline{V}})$ with $(1 - \bar{L})G = F$ for a given $F \in \underline{X}_0 \cong \check{\underline{X}}_0$. Hence, $G(S) = F(S) + (\bar{L}G)(S) = 0$ and $G \in D(\bar{L}_{\underline{V}_0})$. Moreover, for $F \in \underline{X}_0$ there are $F_n \in D(\bar{L}_{\underline{V}})$ converging to F. Choosing suitable functions $\varphi_n \in C_c^1]S,T]$ with $0 \leq \varphi_n \leq 1$ and $\varphi_n = 1$ for $t \geq S + \delta_n$, one sees that $\varphi_n F_n \in D(\bar{L}_{\underline{V}_0})$ and $\varphi_n F_n \to F$ in $\underline{X}_0$.

Theorem 4.15. *Assume that* (H2) *holds for* $J_0 =]S,T]$. *Let* $s \in]S,T[$, $\underline{V}_0 \in \mathcal{O}(\underline{\Omega}_0)$ *with* $V(t) \neq \emptyset$ *for* $t \in J_0$, $f \in C_0(V(s))$, *and* $F \in C(\overline{\underline{V}_s^*})$ *with* $F|\Gamma^s = 0$. *Suppose that* $\underline{V}_0 = \bigcap_{k=1}^n \underline{W}_k$ *for sets* $\underline{W}_k \in \mathcal{O}(\underline{\Omega}_0)$ *admitting a Cauchy barrier with respect to* $\bar{L}$. *Then there exists a unique function* $u \in C(\overline{\underline{V}_s^*})$ *satisfying the conclusions of Theorem* 4.14 *with* L *replaced by* $\bar{L}$. *If* f *and* $-F$ *are positive, then* u *is positive.*

Proof. By virtue of Corollary 3.26 and Theorem 4.14, we only have to show the last assertion which is verified by an application of Theorem 2.30 to the function $-u$. □

A similar result was shown in [48] for cylindrical $\underline{V}$. Further Corollary 3.26, Theorem 4.12, Theorem 4.14, and Theorem 2.30 yield

Theorem 4.16. *Assume that* (H2) *holds for* $J = [S, T]$. *Let* $s \in [S, T[$, $\underline{V} \in \mathcal{O}(\underline{\Omega})$ *with* $V(t) \neq \emptyset$ *for* $t \in J$, *and* $f \in C_0(V(s))$. *Suppose that* $\underline{V} = \bigcap_{k=1}^{n} \underline{W}_k$ *for sets* $\underline{W}_k \in \mathcal{O}(\underline{\Omega})$ *admitting a Cauchy barrier with respect to* $\bar{L}$. *If* $S < s < T$, *let* $F \in C(\overline{\underline{V}_s^*})$ *with* $F|\Gamma^s = 0$; *if* $s = S$, *let* $F \in C_0(\underline{V}_0)$. *Then there exists a unique function* $u \in C(\overline{\underline{V}_s^*})$ *belonging to* $D(\bar{L}, \underline{V}_s)$ *and, if* $s = S$, *to* $D(\bar{L}, \underline{V})$ *such that*

$$\begin{cases} \bar{L}u = F & \text{on } \underline{V}_s \quad (\text{on } \underline{V} \text{ if } s = S), \\ u(\cdot, s) = f & \text{on } V(s), \\ u|\Gamma^s = 0 & (\Gamma = \Gamma^S \text{ if } s = S). \end{cases} \tag{4.18}$$

Further, u *is given as in Theorem* 4.12 *if* $s = S$ *and as in Theorem* 4.14 *if* $s > S$. *If* f *and* $-F$ *are positive, then* u *is positive.*

We conclude this section with a characterization of 'well-posedness' of the homogeneous Cauchy problem

$$\begin{cases} \bar{L}u = 0 & \text{on } \underline{V}_s \quad (\text{on } \underline{V} \text{ if } s = S), \\ u(\cdot, s) = f & \text{on } V(s), \\ u|\Gamma^s = 0 & (\Gamma = \Gamma^S \text{ if } s = S), \end{cases} \tag{4.19}$$

where we require that the solution $u = u(\cdot;\, s, f)$ is contained in $C(\overline{\underline{V}_s^*}) \cap D(\bar{L}, \underline{V}_s)$ if $S < s < T$, resp. in $C(\overline{\underline{V}^*}) \cap D(\bar{L}, \underline{V})$ if $s = S$. Throughout, we assume that L satisfies (H2) and consider $f \in C_0(V(s))$ and $\underline{V} \in \mathcal{O}(\underline{\Omega})$ or $\underline{V}_0 \in \mathcal{O}(\underline{\Omega}_0)$ with $V(t) \neq \emptyset$ for $t \in J$ or $t \in J_0$, respectively.

Definition 4.17. The Cauchy problem (4.19) is called *uniformly well-posed* on $\underline{V}_0$ (or $\underline{V}$) if, for $I = J_0$ or J,

1. for all $s \in I \setminus \{T\}$ and $f \in C_0(V(s))$ there exists a unique solution $u : (x, t) \mapsto u(x, t;\, s, f)$ of (4.19);
2. $\|u(\cdot, t;\, s, f)\| \leq M\, \|f\|$ for $(t, s) \in \Delta_I$ and $f \in C_0(V(s))$;
3. $\Delta_I \ni (t, s) \mapsto u^{\#}(\cdot, t;\, s, F(s)) \in C_0(\Omega)$ is continuous for $F \in \underline{X}_0$ (or $F \in \underline{X}$).

If $\bar{L}_{\underline{V}_0}$ (or $\bar{L}_{\underline{V}}$) is a generator, then Theorem 4.13 (or 4.11) in combination with Lemma 4.7 yield uniform well-posedness of (4.19), where $M = 1$ and $u(\cdot, t;\, s, f) = U(t, s)f$. Conversely, if (4.19) is uniformly well-posed, set $U(t, s)f = u(\cdot, t;\, s, f)$ for $f \in X(s)$ and $(t, s) \in \Delta_{J_0}$ (or $(t, s) \in \Delta_J$), where $U(T, T) := Id_{X(T)}$. Then we have

Proposition 4.18. *Assume that* (4.19) *is well-posed on* $\underline{V}_0$ (*or* $\underline{V}$). *Let* (H2) *hold and* $\bar{L}$ *be complete. Then the operators* $U(t, s)$ *defined above yield a variable space propagator on* $X(s)$ *and the space-time semigroup given by* (4.9) *on* $\underline{X}_0$ (*or* $\underline{X}$) *is generated by* $\bar{L}_{\underline{V}_0}$ (*or* $\bar{L}_{\underline{V}}$).

Proof. We only consider $\underline{V}$ since $\underline{V}_0$ can be treated in the same way. It is straightforward to see that uniform well-posedness implies that $U(t,s) \in \mathcal{L}(X(s), X(t))$ is a variable space propagator. Define

$$(Q(\sigma)F)\,(t,x) = (\tilde{U}(t,t-\sigma)\tilde{F}(t-\sigma))\,(x) \qquad \text{on } \underline{V}$$

for $F \in \underline{X}$ and $\sigma \geq 0$. Notice that $t \mapsto (Q(\sigma)F)^{\#}(t)$ is continuous in $C_0(\Omega)$ by Definition 4.17(3). Moreover, $(Q(\sigma)F)(t) \in C_0(V(t))$ and

$$\begin{aligned}
&|(Q(\sigma)F)^{\#}(x',t') - (Q(\sigma)F)^{\#}(x,t)| \\
&\quad \leq \; |(Q(\sigma)F)^{\#}(x',t') - (Q(\sigma)F)^{\#}(x',t)| + |(Q(\sigma)F)^{\#}(x',t) - (Q(\sigma)F)^{\#}(x,t)| \\
&\quad \leq \; \|(Q(\sigma)F)^{\#}(t') - (Q(\sigma)F)^{\#}(t)\| \\
&\qquad\qquad + |\tilde{u}^{\#}(x',t;\, t-\sigma, \tilde{F}(t-\sigma)) - \tilde{u}^{\#}(x,t;\, t-\sigma, \tilde{F}(t-\sigma))| \\
&\quad \to \; 0 \qquad \text{as } (x',t') \to (x,t) \text{ in } \underline{\Omega},
\end{aligned}$$

where $\tilde{u}(\cdot,t;\, s,f) = u(\cdot,t;\, s,f)$ for $t \geq s \geq S$ and $\tilde{u}(\cdot,t;\, s,f) = u(\cdot,t;\, S,f)$ for $t \geq S > s$. Hence, $Q(\sigma)F \in C_0(\underline{V})$. Also, $Q(0) = Id$, $Q(\tau+\sigma) = Q(\tau)Q(\sigma)$, $\|Q(\sigma)\| \leq M$, and

$$|(Q(\sigma)F)\,(t,x) - F(t,x)| \leq \|\tilde{u}^{\#}(\cdot,t;\, t-\sigma, \tilde{F}(t-\sigma)) - \tilde{F}^{\#}(t)\| \to 0$$

as $\sigma \to 0$ uniformly for $t \in J$. That is, $Q(\sigma)$ is a semigroup.

It remains to show that the generator E of $Q(\cdot)$ is equal to $\bar{L}_{\underline{V}}$. To this end, we consider the subspace $\underline{Z}$ of $\underline{X}$, cf. p.86, spanned by the functions

$$\underline{V} \ni (x,t) \mapsto \varphi(t)\,(\check{U}(t,s)f)(x) =: \varphi(t)\,u(x,t),$$

where $s \in J$, $f \in X(s)$, and $\varphi \in C^1(J)$ with support in $]s,T]$ if $s > S$ and $\varphi'(S) = 0$ if $s = S$. Notice that $u \in D(\bar{L}, \underline{V}_s)$, resp. $u \in D(\bar{L}, \underline{V})$ if $s = S$, and $\bar{L}u = 0$. Since $\bar{L}$ is parabolic, we obtain that $\bar{L}(\varphi u) = -\varphi' u$ on $\underline{V}_s$ or $\underline{V}$. For $s = S$, this yields $\varphi u \in D(\bar{L}_{\underline{V}})$. For $s > S$, we have $\varphi u(\underline{x}) = 0$ for $(x,t) \in \underline{V}$ with $t \leq s + \varepsilon$ for small $\varepsilon > 0$. Therefore, $\varphi u \in D(\bar{L}_{\underline{V}})$ due to the completeness of $\bar{L}$. In particular, $\bar{L}_{\underline{V}}(\varphi u) = -\varphi' u$. and $\underline{Z} \subset D(\bar{L}_{\underline{V}})$.

On the other hand, we have $Q(\sigma)(\varphi u) = \tilde{\varphi}_{-\sigma} u$. This implies $Q(\sigma)\underline{Z} \subset \underline{Z} \subset D(E)$ and $E(\varphi u) = -\varphi' u$. As in [50, Thm. 2.3] or [28, Prop. 2.9] one sees that $\underline{Z}$ is dense in $\underline{X}$. As a result, $\underline{Z}$ is a core of E and $\bar{L}_{\underline{V}} = E$ on $\underline{Z}$. The closedness of $\bar{L}_{\underline{V}}$ yields $E \subset \bar{L}_{\underline{V}}$. Since $\bar{L}_{\underline{V}}$ is dissipative and $\lambda - E$ is surjective for large $\lambda > 0$, we derive $E = \bar{L}_{\underline{V}}$. □

The above results and Theorem 3.25 imply

Theorem 4.19. *Assume that* (H2) *holds and that* $\bar{L}$ *is complete. For* $\underline{V}_0 \in \mathcal{O}(\underline{\Omega}_0)$ (*or* $\underline{V} \in \mathcal{O}(\underline{\Omega})$) *with* $V(t) \neq \emptyset$ *for* $t \in J_0$ ($t \in J$), *the following assertions are equivalent.*

1. *The Cauchy problem* (4.19) *is uniformly well-posed.*
2. *The operator* $\bar{L}_{\underline{V}_0}$ (*or* $\bar{L}_{\underline{V}}$) *is a generator on* $\underline{X}_0$ (*or* $\underline{X}$).
3. *The set* $\underline{V}_0$ (*or* $\underline{V}$) *admits a Cauchy barrier with respect to* $\bar{L}$.

4.4 Parabolic and standard parabolic problems

In applications it is much easier to check condition (LS) and to construct Cauchy barriers in the (general) parabolic situation, i.e., $J_0 =]S,T]$, than for standard parabolic problems, i.e., $J = [S,T]$, see Chapter 6. So the question arises whether the semigroup generated by $\bar{L}_{\underline{V}_0}$ on $C_0(\underline{V}_0) = \underline{X}_0$ can be extended to a semigroup on $C_0(\underline{V}) = \underline{X}$. To study this problem, let $\underline{V} \in \mathcal{O}(\underline{\Omega})$ with $V(t) \neq \emptyset$ for $t \in J$ and let L be a local operator on $\underline{\Omega}_0$ such that (H2) holds and $\bar{L}_{\underline{V}_0}$ is a generator on $\underline{X}_0$. Recall that $\underline{X}_0$ is identified with the subspace $\check{\underline{X}}_0$ of $\underline{X}$. We work with the operator on $\underline{X}$ given by

$$D(\tilde{\bar{L}}_{\underline{V}_0}) = \{F \in \underline{X} \cap D(\bar{L}, \underline{V}_0) : \bar{L}F \in \underline{X}_0\},$$

$$\tilde{\bar{L}}_{\underline{V}_0} F = \begin{cases} \bar{L}F & \text{on } \underline{V}_0\,, \\ 0 & \text{on } \underline{V}(S). \end{cases}$$

It is easy to see that $\tilde{\bar{L}}_{\underline{V}_0}$ is a standard parabolic, dissipative, closed operator on $\underline{X}$. Moreover, we have the following fact.

Lemma 4.20. *Under the above assumptions, $\tilde{\bar{L}}_{\underline{V}_0}$ generates a semigroup on $\underline{X}$ if and only if $D(\tilde{\bar{L}}_{\underline{V}_0})$ is dense in $\underline{X}$.*

Proof. It suffices to show that the range of $1 - \tilde{\bar{L}}_{\underline{V}_0}$ is dense in $\underline{X}$ provided that $D(\tilde{\bar{L}}_{\underline{V}_0})$ is dense in $\underline{X}$. So consider a finite regular Borel measure μ on $\underline{V}$ such that

$$\int_{\underline{V}} (1 - \tilde{\bar{L}}_{\underline{V}_0}) F \, d\mu = 0 \qquad \text{for all } \; F \in D(\tilde{\bar{L}}_{\underline{V}_0}). \tag{4.20}$$

Since $(1 - \bar{L}_{\underline{V}_0}) D(\bar{L}_{\underline{V}_0}) = \underline{X}_0$, this implies that μ vanishes on $\underline{V}_0$. Due to $\tilde{\bar{L}}_{\underline{V}_0} F = 0$ on $\underline{V}(S)$, equation (4.20) thus reduces to

$$\int_{\underline{V}(S)} F \, d\mu = 0 \qquad \text{for all } \; F \in D(\tilde{\bar{L}}_{\underline{V}_0}).$$

The density of $D(\tilde{\bar{L}}_{\underline{V}_0})$ now yields $\mu = 0$. □

Thus we shall examine conditions ensuring that $D(\tilde{\bar{L}}_{\underline{V}_0})$ is dense in $\underline{X}$. To that purpose, we introduce the set

$$\begin{aligned} \tilde{D}_L(S) = \{ & f \in C_c(V(S)) : \exists\, F \in C_c(\underline{V}) \text{ such that } F(S) = f,\ \operatorname{supp} F \subset \underline{V} \setminus \underline{V}(T), \\ & F|\underline{V}_0 \in D(\bar{L}, \underline{V}_0),\ \bar{L}F \in C_b(\underline{V}_0)\}. \end{aligned} \tag{4.21}$$

We will also make use of the fact that the operator $\overline{p\bar{L}_{\underline{V}_0}}$ is again a dissipative generator on $\underline{X}_0$, see [14] and [29, Thm. 2], where $p(t) = t - S$. We write $(t-S)\bar{L}_{\underline{V}_0}$ for $p\bar{L}_{\underline{V}_0}$. Further, define the local operator $(t-S)\bar{L}$ given by

$$D((t-S)\bar{L}, \underline{W}) = D(\bar{L}, \underline{W}) \quad \text{and} \quad ((t-S)\bar{L})F = p\bar{L}F \quad \text{on } \underline{W}$$

for $\underline{W} \in \mathcal{O}(\underline{\Omega_0})$. Note that the local operator $(t-S)\bar{L}$ is locally disspative, and so $((t-S)\bar{L})_{\underline{V}_0}$ is dissipative in $\underline{X}_0$. Actually, the following holds.

Lemma 4.21. *Under the above assumptions, we have* $\overline{(t-S)\bar{L}_{\underline{V}_0}} = ((t-S)\bar{L})_{\underline{V}_0}$ *so that the latter is a generator in* $\underline{X}_0$.

Proof. It suffices to show $\overline{(t-S)\bar{L}_{\underline{V}_0}} \subset ((t-S)\bar{L})_{\underline{V}_0}$. So take $F_n \in D(\bar{L}_{\underline{V}_0})$ such that

$$F_n \;\to\; F \in D(\overline{(t-S)\bar{L}_{\underline{V}_0}}) \quad \text{and} \quad ((t-S)\bar{L}_{\underline{V}_0})F_n \;\to\; G = \overline{(t-S)\bar{L}_{\underline{V}_0}}F$$

in $\underline{X}_0$. Thus, $\bar{L}F_n \to \frac{1}{t-S}\,G$ on $\underline{V}_{S+\varepsilon}$ for $\varepsilon > 0$. This gives $F \in D(\bar{L}, \underline{V}_{S+\varepsilon})$ and $(t-S)\bar{L}F = G$ on $\underline{V}_{S+\varepsilon}$. Since $\bar{L}$ is semi-complete, we infer that $F \in D(\bar{L}, \underline{V}_0)$ and $(t-S)\bar{L}F = G$ on $\underline{V}_0$. As a result, $F \in D(((t-S)\bar{L})_{\underline{V}_0})$ and $((t-S)\bar{L})_{\underline{V}_0}F = \overline{(t-S)\bar{L}_{\underline{V}_0}}F$. □

Next, let $f \in \tilde{D}_L(S)$ and F as in (4.21). Take $0 < \alpha < 1$ and define, for $F_0 = F|\underline{V}_0$, the function $G_0 \in \underline{X}_0$ by

$$G_0(x,t) = -(t-S)^{\alpha}\,(\bar{L}F_0)(x,t) \qquad \text{on } \underline{V}_0\,.$$

By Lemma 4.21, there exists

$$G_1 = \Big[((t-S)\bar{L})_{\underline{V}_0} - (1-\alpha)\Big]^{-1} G_0 \;\in\; D(\bar{L}_{\underline{V}_0}).$$

Define a function $G \in \underline{X}$ by setting

$$G = \begin{cases} (t-S)^{1-\alpha}\,G_1 + F_0 & \text{on } \underline{V}_0, \\ f & \text{on } \underline{V}(S). \end{cases}$$

Due to the parabolicity of $\bar{L}$, we obtain $G \in D(\bar{L}, \underline{V}_{S+\varepsilon})$ and

$$\begin{aligned} \bar{L}G &= (t-S)^{1-\alpha}\,\bar{L}G_1 - (1-\alpha)(t-S)^{-\alpha}\,G_1 + \bar{L}F_0 \\ &= (t-S)^{-\alpha}\,[(t-S)\bar{L} - (1-\alpha)]G_1 + \bar{L}F_0 = 0 \end{aligned}$$

on $\underline{V}_{S+\varepsilon}$ for $\varepsilon > 0$. Therefore, $G \in D(\tilde{\bar{L}}_{\underline{V}_0})$ and

$$\tilde{D}_L(S) \subset \{G|V(S) : G \in D(\tilde{\bar{L}}_{\underline{V}_0})\} =: D(S). \tag{4.22}$$

The importance of this fact relies on

Lemma 4.22. *Under the above assumptions,* $D(\tilde{\bar{L}}_{\underline{V}_0})$ *is dense in* $\underline{X}$ *if and only if* $D(S)$ *in* (4.22) *is dense in* $C_0(V(S))$.

Proof. Necessity is clear. So assume that $D(S)$ is dense in $C_0(V(S))$. Let μ be a finite Borel measure being orthogonal to $D(\tilde{\bar{L}}_{\underline{V}_0})$. Since $\bar{L}_{\underline{V}_0}$ is a generator, μ has to vanish on $\underline{V}_0$. But then $\mu = 0$ since $D(S)$ is dense. □

The next result is an immediate consequence of (4.22), Lemma 4.20, and Lemma 4.22.

Proposition 4.23. *Assume that L satisfies* (H2) *on $\mathcal{O}(\underline{\Omega}_0)$ and that $\bar{L}_{\underline{V}_0}$ is a generator on $\underline{X}_0$. If $\tilde{D}_L(S)$ is dense in $C_0(V(S))$, then $\tilde{\bar{L}}_{\underline{V}_0}$ is a generator on $\underline{X}$.*

Since $\tilde{\bar{L}}_{\underline{V}_0}$ is standard parabolic, we can derive the main result of this section in the same way as Theorem 4.16. Observe that $\tilde{\bar{L}}_{\underline{V}_0}$ coincides with $\bar{L}$ on $\underline{V}_0$.

Theorem 4.24. *Assume that L satisfies* (H2) *on $\mathcal{O}(\underline{\Omega}_0)$. Let $\underline{V} \in \mathcal{O}(\underline{\Omega})$ with $V(t) \neq \emptyset$ for $t \in J$, $f \in C_0(V(S))$, and $F \in C_0(\underline{V}_0)$. Suppose that $\tilde{D}_L(S)$ is dense in $C_0(V(S))$ and that $\underline{V}_0 = \bigcap_{k=1}^{n} \underline{W}_k$ for sets $\underline{W}_k \in \mathcal{O}(\underline{\Omega}_0)$ admitting a Cauchy barrier with respect to $\bar{L}$. Then the conclusions of Theorem* 4.16 *hold for $s = S$ and $\tilde{\bar{L}}_{\underline{V}_0}$ instead of $\bar{L}$.*

5 Semilinear parabolic problems on non-cylindrical domains

We now use the results on linear problem (4.14) to solve a certain class of semilinear equations containing, e.g., the logistic equation with diffusion, see Section 6.1. To that purpose, let L satisfy (H2) for $\underline{\Omega}_0 = \Omega \times]S, T]$ and finite $S < T$ and let $\underline{V}_0 \in \mathcal{O}(\underline{\Omega}_0)$ satisfy $V(t) \neq \emptyset$ for $t \in J_0 =]S, T]$. We also assume that $\bar{L}_{\underline{V}_0}$ is a generator on $\underline{X}_0 = C_0(\underline{V}_0)$, that is, $\underline{V}_0 = \bigcap_{k=1}^{n} \underline{W}_k$ for sets $\underline{W}_k \in \mathcal{O}(\underline{\Omega}_0)$ admitting a Cauchy barrier relative to $\bar{L}$, cf. Theorem 3.25 and Corollary 3.26. Let $U(t, s)$, $S < s \leq t \leq T$, be the variable space propagator corresponding to $e^{t\bar{L}_{\underline{V}_0}}$, see Theorem 4.9. For the nonlinearity we require

$$\Phi : \mathbb{R} \to \mathbb{R} \text{ is locally Lipschitz, } \Phi(0) = 0, \text{ and } \Phi(\xi) < 0 \text{ for } \xi > 1. \tag{5.1}$$

In particular, these conditions are satisfied by $\Phi(\xi) = c\xi(1 - \xi)$ for $c > 0$. For $v \in C(\underline{V}_0)$ we define $Nv \in C(\underline{V}_0)$ by setting

$$(Nv)(\underline{x}) = \Phi(\operatorname{Re} v(\underline{x})) + i\Phi(\operatorname{Im} v(\underline{x}))$$

for $\underline{x} \in \underline{V}_0$. We are looking for solutions $u \in D(\bar{L}, \underline{V}_s) \cap C(\overline{\underline{V}_s^*})$ of the semilinear problem

$$\begin{cases} (\bar{L} + N)u &= 0 \quad \text{on } \underline{V}_s, \\ u(\cdot, s) &= f \quad \text{on } V(s), \\ u|\Gamma^s &= 0, \end{cases} \tag{5.2}$$

where $f \in C_0(V(s))$ and $S < s < T$. It will be necessary to consider (5.2) also on time intervals $J_0' =]S, T']$ for $S < T' < T$. To this end, let $\underline{\Omega}_0' = \Omega \times J_0'$ and $\underline{W}_{00}' = \underline{W}_0' \setminus \underline{W}_0'(T')$ for $\underline{W}_0' \in \mathcal{O}(\underline{\Omega}_0')$. Notice that $\partial(\underline{W}_0')^* \subset \partial\underline{W}_0^*$ if $\underline{W}_0' = \underline{W}_0 \cap \underline{\Omega}_0'$ for some $\underline{W}_0 \in \mathcal{O}(\underline{\Omega}_0)$. We define

$$D(\bar{L}', \underline{W}_0') = \{F \in C(\underline{W}_0') : F|\underline{W}_{00}' \in D(\bar{L}, \underline{W}_{00}'),\ \bar{L}F = G \text{ on } \underline{W}_{00}' \text{ for } G \in C(\underline{W}_0')\},$$
$$\bar{L}'F = G \qquad \text{on } \underline{W}_0' \in \mathcal{O}(\underline{\Omega}_0').$$

(We let $\bar{L}' = \bar{L}$ if $T' = T$.) Clearly, $\bar{L}'$ is a real, parabolic, local operator on $\mathcal{O}(\underline{\Omega}_0')$. Corollary 2.21 implies the local dissipativity of $\bar{L}'$. It is straightforward to see that $\bar{L}'$ is locally closed u.c.. Further, take $G \in C_0(\underline{V}_0')$ and extend it to a function $G_1 \in C_0(\underline{V}_0)$. Then there exists $F_1 = (1-\bar{L}_{\underline{V}_0})^{-1}G_1$. Thus, $F = F_1|\underline{V}_0' \in D(\bar{L}', \underline{V}_0') \cap C_0(\underline{V}_0')$ and $(1-\bar{L}')F = G$. Given a function $F \in C_0(\underline{V}_0')$, we can approximate an extension F_1 on $\underline{V}_0$ in $C_0(\underline{V}_0)$ by $G_n \in D(\bar{L}_{\underline{V}_0})$ so that $D(\bar{L}'_{\underline{V}_0'})$ is dense. As a result, $\bar{L}'_{\underline{V}_0'}$ generates a semigroup $Q'(\cdot)$ on $C_0(\underline{V}_0')$. Due to the uniqueness of solutions of (4.13), $Q'(\cdot)$ is given by the restriction of $U(t,s)$ to the index set $\Delta_{]S,T']}$.

We further need the Banach space (with sup-norm)

$$\underline{X}_s = \underline{X}_s(T') = \{v \in C(\overline{(\underline{V}_s')^*}) : v|\Gamma^s = 0\}, \quad S < s < T' \le T,$$

which is canonically identified with the subspace $\underline{X}_s(T')^{*\#}$ of $C(\Omega^* \times [s,T'])$ consisting of continuous functions vanishing off $\underline{V}_s' \cup \underline{V}(s)$. An application of Theorem 4.14 with $F = -Nu \in \underline{X}_s(T')$ shows that u solves (5.2) if and only if u satisfies the integral equation

$$u^{*\#}(t) = U^{*\#}(t,s)f^{*\#} + \int_s^t U^{*\#}(t,\tau)(Nu)^{*\#}(\tau)\,d\tau \tag{5.3}$$

in $\underline{X}_s(T')^{*\#}$. Motivated by this fact, we define a mapping $S^{*\#}$ on $\underline{X}_s(T')^{*\#}$ by setting

$$(S^{*\#}v^{*\#})(t) = \int_s^t U^{*\#}(t,\tau)(Nv)^{*\#}(\tau)\,d\tau$$

for $v^{*\#} \in \underline{X}_s(T')^{*\#}$. (Use Lemma 4.7 to see that indeed $S^{*\#}v^{*\#} \in \underline{X}_s(T')^{*\#}$.) We remark that if $u \in \underline{X}_s(r)$ and $v \in \underline{X}_r(T')$ with $s < r < T'$ satisfy (5.3) on $[s,r]$ with initial value f and on $[r,T']$ with initial value $u(r)$, respectively, then the function

$$w(t) = \begin{cases} u(t), & s \le t \le r, \\ v(t), & r \le t \le T', \end{cases}$$

solves (5.3) in the space $\underline{X}_s(T')^{*\#}$ with initial value f. Our main result reads as follows.

Theorem 5.1. *Assume that L satisfies* (H2) *on $\mathcal{O}(\underline{\Omega}_0)$. Let $\underline{V}_0 \in \mathcal{O}(\underline{\Omega}_0)$ be given such that $V(t) \neq \emptyset$ for $t \in J_0 =]S,T]$ and $\underline{V}_0 = \bigcap_{k=1}^n \underline{W}_k$ for sets $\underline{W}_k \in \mathcal{O}(\underline{\Omega}_0)$ admitting a Cauchy barrier relative to $\bar{L}$. Suppose that Φ satisfies* (5.1)*. Let $f \in C_0(V(s))$ for $S < s < T$ and $0 \le f \le 1$. Then there is a unique* (*global*) *solution $u \in D(\bar{L}, \underline{V}_s) \cap C(\overline{\underline{V}_s^*})$ of* (5.2) *with $0 \le u \le 1$.*

Proof. Since $\bar{L}_{\underline{V}_0}$ generates a Feller semigroup $Q(\cdot)$, we have $0 \le U(t,s)f \le 1$ for the propagator $U(t,s)$ given by Theorem 4.9 (use Lemma 4.4). Set

$$w^{*\#}(x,t) = (U^{*\#}(t,s)f^{*\#})(x,t) \qquad \text{for } (x,t) \in \overline{\underline{\Omega}_s^*}.$$

Then, $0 \le w^{*\#} \le 1$. By (5.3), the solution u is given by $(1 - S^{*\#})u^{*\#} = w^{*\#}$. We first solve this equation for $T' \in]s, T]$ sufficiently close to s.

(1) Fix $0 < \delta < 1$. For $v \in \underline{X}_s(T)$ with $-\delta \le v \le 1 + \delta$ we have

$$\|Nv\| \le \tilde{M}_\delta := \sup_{-\delta \le \xi \le 1+\delta} |\Phi(\xi)|.$$

Consequently, $\|S^{*\#}v^{*\#}\| \le (T' - s)\tilde{M}_\delta$ for $s < T'$ and $v \in \underline{Y}_s(T') := \{v \in \underline{X}_s(T') : -\delta \le v \le 1 + \delta\}$. In particular,

$$\|S^{*\#}v^{*\#}\| \le \delta \qquad \text{for} \quad v \in \underline{Y}_s(T_1) \quad \text{and} \quad s < T_1 \le s + \frac{\delta}{\tilde{M}_\delta}\,. \tag{5.4}$$

By (5.1), there is a constant M_δ such that

$$|\Phi(\xi) - \Phi(\eta)| \le M_\delta\,|\xi - \eta| \qquad \text{for} \;\; \xi, \eta \in [-\delta, 1 + \delta].$$

For $v, w \in \underline{Y}_s(T_2)$ and $s < T_2 \le T_1$, we estimate

$$\begin{aligned} \|S^{*\#}v^{*\#} - S^{*\#}w^{*\#}\| &\le (T_2 - s) \sup_{s \le \tau \le T_2} \|(Nv)^{*\#}(\tau) - (Nw)^{*\#}(\tau)\| \\ &\le (T_2 - s)\, M_\delta\, \|v^{*\#} - w^{*\#}\| \le \tfrac{1}{2}\, \|v^{*\#} - w^{*\#}\| \end{aligned} \tag{5.5}$$

if we choose $d := T_2 - s \le (2M_\delta)^{-1}$. Estimates (5.4) and (5.5) imply that $S^{*\#}$ is a strict contraction on $\underline{Y}_s(T_2)^{*\#}$. Therefore there exists a unique function $u \in \underline{Y}_s(T_2)$ such that $u^{*\#} = w^{*\#} + S^{*\#}u^{*\#}$.

(2) Let $u, v \in \underline{X}_s(T')$ solve (5.2) on $[s, T']$ for some $S < s < T' \le T$. Then an application of the comparison principle Theorem 2.32 to $\bar{L}'$ shows that $u = v$. Thus local and gobal solutions of (5.2) and (5.3) are unique.

(3) We will now show that the local solution u of (5.3) constructed in part (1) satisfies $0 \le u(x, t) \le 1$ for $s \le t \le T_2$. Having established this fact, we can apply part (1) to overlapping time intervals $I_k = J_0 \cap [s + kd, s + (k+1)d]$ with fixed length $d > 0$. As observed above, this gives a solution v of (5.3) on J_0, and hence a solution of (5.2).

First, notice that $w = 0$ is a solution of (5.2) for $f = 0$ since $\Phi(0) = 0$. So Theorem 2.32 implies that u is positive. Second, suppose that $u(\underline{x}) > 1$ for some $\underline{x} = (x, t) \in \underline{V}_s$ with $t \le T_2$. Since u is positive on $\underline{V}'_s$ and smaller than 1 on $\partial\underline{V}'_s$, $|u|$ attains its maximum on $\underline{V}'_s$ which strictly larger than 1. Hence, the local dissipativity of $\bar{L}'$ yields a point $\underline{x}_0 \in \underline{V}'_s$ such that $u(\underline{x}_0) = \max_{\underline{V}'_s} |u|$ and

$$0 \ge (\bar{L}'u)(\underline{x}_0)\, u(\underline{x}_0) = -\Phi(u(\underline{x}_0))\, u(\underline{x}_0) > 0$$

because $\Phi(u(\underline{x}_0)) < 0$. This contradiction proves $u \le 1$. □

Remark 5.2. *The above proof shows that Theorem* 5.1 *remains valid for* $T = +\infty$ *provided that* L *and* $U(t, s)$ *are defined for* $J =]S, \infty[$.

Remark 5.3. *Using the same arguments one can treat nonlinearities given by continuous functions* $\Phi : \underline{\Omega}^* \times \mathbb{R} \to \mathbb{R}$ *satisfying* $\Phi(\underline{x}, 0) = 0$ *and* $\Phi(\underline{x}, \xi) < 0$ *for* $x \in \underline{\Omega}$, $1 < \xi < 1 + \delta'$, *a constant* $\delta' > 0$, *and* $|\Phi(\underline{x}, \xi) - \Phi(\underline{x}, \eta)| \le k\,|\xi - \eta|$ *for* $\xi, \eta \in [-a, a]$, $\underline{x} \in \underline{\Omega}$ *and a constant* k *depending on* a. *This holds, e.g., if* $\Phi(\underline{x}, \xi) = m(\underline{x})\xi(1 - \xi)$ *for* $m \in C(\underline{\Omega}^*, \mathbb{R})$ *and* $m > 0$ *on* $\underline{\Omega}$.

6 Applications

6.1 Parabolic partial differential equations of second order

We consider the partial differential operator

$$L(\underline{x}, D) = \sum_{k,l=1}^{N} a_{kl}(x,t)\,\frac{\partial^2}{\partial x_k \partial x_l} + \sum_{k=1}^{N} b_k(x,t)\,\frac{\partial}{\partial x_k} + c(x,t) - \frac{\partial}{\partial t} \tag{6.1}$$

on $\underline{\Omega} = \Omega \times [S, T]$ for an open subset Ω of $\mathbb{R}^N$. We assume that the coefficients a_{kl}, b_k, and $c \le 0$ are real-valued, continuous, and elliptic; that is,

$$\sum_{k,l=1}^{N} a_{kl}(x,t)\, y_k\, y_l > 0 \tag{6.2}$$

for all $y \in \mathbb{R}^N$ and $(x,t) \in \underline{\Omega}$. Notice that on every compact subset $\underline{K}$ of $\underline{\Omega}$ the coeffients are bounded, uniformly continuous, and satisfy

$$\sum_{k,l=1}^{N} a_{kl}(x,t)\, y_k\, y_l \ge \mu_{\underline{K}}\, |y|^2 \tag{6.3}$$

for a constant $\mu_{\underline{K}} > 0$ and $(x,t) \in \underline{K}$. However, the coefficients may be singular or degenerate at the boundary. Set $\underline{\Omega}_{00} = \Omega \times]S, T[$ and $\underline{V}_{00} = \underline{V} \cap \underline{\Omega}_{00}$ for $\underline{V} \in \mathcal{O}(\underline{\Omega})$. As in [40], [48], and [49], we define

$$\begin{aligned}
D(L, \underline{V}_0) &= \{F \in C(\underline{V}_0) \cap W^{2,1}_{p,loc}(\underline{V}_0) : L(\underline{x}, D)F = G \text{ a.e. on } \underline{V}_{00} \text{ for } G \in C(\underline{V}_0)\},\\
D(L, \underline{V}) &= \{F \in C(\underline{V}) : F|\underline{V}_0 \in D(L, \underline{V}_0),\ G = L(\underline{x}, D)F \text{ extends continuously}\\
&\qquad \text{to } \underline{V}(S) \text{ by } 0\},\\
LF = G &\qquad \text{on } \underline{V}_0 \in \mathcal{O}(\underline{\Omega}_0),\ \text{resp. } \underline{V} \in \mathcal{O}(\underline{\Omega}).
\end{aligned} \tag{6.4}$$

Here, $p > N + 2$ is fixed and $W^{2,1}_{p,loc}(\underline{V}_0)$ denotes the space of functions F such that $F|\underline{W}_{00}$ belongs to the Sobolev space

$$W^{2,1}_p(\underline{W}_{00}) = \{F \in L^p(\underline{W}_{00}) : \frac{\partial F}{\partial x_k},\ \frac{\partial^2 F}{\partial x_k \partial x_l},\ \frac{\partial F}{\partial t} \in L^p(\underline{W}_{00}),\ k, l = 1, \ldots, N\}$$

for all $\underline{W} \in \mathcal{O}_c(\underline{V}_0)$.

It is immediate that L is a local operator on $\mathcal{O}(\underline{\Omega}_0)$, resp. $\mathcal{O}(\underline{\Omega})$. Moreover, L does not depend on $p > N+2$. In fact, it suffices to show that if $F \in D(L, \underline{V}_0)$, then $F|\underline{W} \in W_q^{2,1}(\underline{W})$ for all $q > N+2$ and for all sets $\underline{W} \in \mathcal{O}(\underline{\Omega}_{00})$ of the form

$$\underline{W} = W \times I \text{ with } W \in \mathcal{O}_c(\Omega),\ \bar{W} \times \bar{I} \subset \underline{V}_0 \text{ and } \partial W \text{ is of class } C^3. \tag{6.5}$$

(Here the regularity of ∂W is understood in the sense of [24, §IV.4].) For such F and $\underline{W}$, let $G = LF$ and let $W' \supset \bar{W}$ and $I' \supset I$ satisfy (6.5) and $a := \inf I' < \inf I$. Take $\varphi \in C^1(\overline{I'})$ and $\psi \in C_c^2(W')$ with $\psi|W = 1$, $\varphi|I = 1$, and $\varphi(a) = \varphi'(a) = 0$. Set $F_1 = \varphi\psi F$ on $\underline{W}' = W' \times I$. Then,

$$\begin{aligned} G_1 = LF_1 =& \varphi\psi G - \varphi'\psi F + \varphi F \sum_{k=1}^{N} b_k \frac{\partial \psi}{\partial x_k} \\ &+ \varphi \sum_{k,l=1}^{N} a_{kl} \left(\frac{\partial F}{\partial x_k}\frac{\partial \psi}{\partial x_l} + \frac{\partial F}{\partial x_l}\frac{\partial \psi}{\partial x_k} + F \frac{\partial^2 \psi}{\partial x_k \partial x_l} \right). \end{aligned} \tag{6.6}$$

By [24, Lemma II.3.3], $\frac{\partial F}{\partial x_k}$ is Hölder continuous on $\overline{\underline{W}'}$ if $p > N+2$. Hence, $G_1 \in C(\overline{\underline{W}'}) \subset L^q(\underline{W}')$ for each $q > N+2$. Moreover, $F_1 = 0$ on $\partial_p \underline{W}'$. Now, Theorem IV.9.1 of the monograph [24] by O.A. Ladyzenskaja, V.A. Solonnikov, and N.N. Ural'ceva yields $F_1 \in W_q^{2,1}(\underline{W}')$, and so $F \in W_q^{2,1}(\underline{W})$.

Further, it is easy to see that L is real, complete, and (standard) parabolic on $\mathcal{O}(\underline{\Omega}_0)$ ($\mathcal{O}(\underline{\Omega})$). To show that L is locally closed, let $F_n \in D(L, \underline{V}_0)$, $G_n = LF_n$, and $F_n \to F$, $G_n \to G$ in $C_b(\underline{V}_0)$. Consider $\underline{W}$ as in (6.5) and $\underline{W}' = W' \times I'$ as above. By virtue of the local a priori estimate [24, IV(10.12)], we have

$$\begin{aligned} \|F_n - F_m\|_{W_p^{2,1}(\underline{W})} &\leq C_1 \left(\|F_n - F_m\|_{L^p(\underline{W}')} + \|G_n - G_m\|_{L^p(\underline{W}')}\right) \\ &\leq C_2 \left(\|F_n - F_m\|_\infty + \|G_n - G_m\|_\infty\right). \end{aligned}$$

Hence, $F_n|\underline{W}$ converges in $W_p^{2,1}(\underline{W})$ to F. Further,

$$\|LF_n - LF\|_{L^p(\underline{W})} \leq C_3 \|F_n - F\|_{W_p^{2,1}(\underline{W})}$$

which shows that $LF = G$ a.e. on $\underline{W}$. As a consequence, $F \in D(L, \underline{V}_0)$ and $LF = G$. This also implies local closedness of L on $\mathcal{O}(\underline{\Omega})$. In particular, $L = \bar{L}$ in both cases.

Clearly, L satisfies (OE). We check (LS) and (S2) only for $\underline{V}_0 \in \mathcal{O}(\underline{\Omega}_0)$. Extend a given function $F \in C_c(\underline{V}_0)$ to a function $F' \in C_c(\underline{V}_0')$ for an open subset $\underline{V}_0'$ of $\Omega \times]S, T'[$ with $T' > T$ and $\underline{V}_0' \cap \underline{\Omega}_0 = \underline{V}_0$. Then there are functions $F_n' \in C_c^\infty(\underline{V}_0')$ converging to F' in $C_0(\underline{V}_0')$. As a result, $D(L_{\underline{V}_0})$ is dense in $C_0(\underline{V}_0)$. Using [24, Lemma II.3.3], one shows that $F_1 = \psi F$ satisfies (LS) for $\psi \in C_c^{2,1}(\underline{V}_{00})$ with $\psi = 1$ in a neighbourhood of $\underline{x} \in \underline{V}_{00}$, cf. (6.6). A similar argument works if $\underline{x} = (x, T)$.

On $\mathcal{O}(\underline{\Omega}_{00})$, local dissipativity is an immediate consequence of the parabolic maximum principle in $W_p^{2,1}$ proved in [13, Thm. VII.28]. So Corollary 2.21 yields local dissipativity on $\mathcal{O}(\underline{\Omega}_0)$. On $\mathcal{O}(\underline{\Omega})$ it is then obvious.

It remains to verify that the part $L_{\underline{\Omega}_0}$ of L in $C_0(\underline{\Omega}_0)$ is a generator. Since $L_{\underline{\Omega}_0}$ is closed, dissipative, and densely defined. we only have to show that the range of $L_{\underline{\Omega}_0}$ is dense. In fact, an application of Theorem IV.9.1 of [24] yields that $L^p(\underline{\Omega}_0) \cap C_0(\underline{\Omega}_0)$ is contained in the range provided that the coefficients of L are bounded and satisfy (6.3) on $\underline{\Omega}$ and $\partial\Omega$ belongs to C^2, cf. [24, §IV.4], or $\Omega = \mathbb{R}^N$. More generally, we have checked the assumptions of Theorem 3.27 for suitable $(V_n \times J_0) \uparrow \underline{\Omega}_0$. As a consequence, $L_{\underline{\Omega}_0}$ is a generator if and only if there is a Cauchy barrier for L on $\underline{\Omega}_0$. (See also [30, 31, 32, 48, 55].) In any case, Theorem 4.15 and 4.24 imply

Theorem 6.1. *Let L be given as in* (6.1) *and* (6.4) *for continuous, real-valued coefficients satisfying $c \leq 0$ and* (6.2). *Assume that $L_{\underline{\Omega}_0}$ is a generator on $C_0(\underline{\Omega}_0)$. Let $\underline{V} \in \mathcal{O}(\underline{\Omega})$ with $V(t) \neq \emptyset$ for $S \leq t \leq T$. Suppose that $\underline{V}_0 = \bigcap_{k=1}^n \underline{W}_k$ for sets $\underline{W}_k \in \mathcal{O}(\underline{\Omega}_0)$ admitting a Cauchy barrier with respect to L. Let $S \leq s < T$ and $f \in C_0(V(s))$. If $S < s < T$, let $F \in C(\overline{\underline{V}_s^*})$ with $F|\Gamma^s = 0$; if $s = S$, let $F \in C_0(\underline{V}_0)$. Finally, assume that $\tilde{D}_L(S)$ is dense if $s = S$. Then there is a unique function $u \in C(\overline{\underline{V}_s^*}) \cap W_{p,loc}^{2,1}(\underline{V}_s)$ such that*

$$\begin{cases} L(\underline{x}, D)u = F & \text{on } \underline{V}_s, \\ u(\cdot, s) = f & \text{on } V(s), \\ u|\Gamma^s = 0 & (\Gamma = \Gamma^S \text{ if } s = S). \end{cases} \tag{6.7}$$

Further, u is given as in Theorem 4.12 *for $s = S$ and as in Theorem* 4.14 *for $s > S$. If $F = 0$, then $|u| \leq \|f\|$. If f and $-F$ are positive, then u is positive.*

We point out that, in the interior, we have the same regularity as one obtains for regular, cylindrical $\underline{V}$ and bounded, uniformly elliptic coefficients, cf. [24, §IV.9]. Notice that $p \in]N+2, \infty[$ can be chosen arbitrarily large. If the coefficients are Hölder continuous, bounded, and uniformly elliptic, problem (6.7) was solved on non-cylindrical domains in [20, Chap. 3] or [58, Chap. VI], see also e.g. [4, 7, 25, 26].

Further, Theorem 5.1 provides a result on the existence and uniqueness of global solutions for a large class of semilinear equations on non-cylindrical domains, containing the logistic equation with diffusion, i.e., $\Phi(t) = ct(1-t)$ for $c > 0$. Let us mention [5], [19], and [23] among the vast literature on this topic .

Theorem 6.2. *Let L be given as in* (6.1) *and* (6.4) *for continuous, real-valued coefficients satisfying $c \leq 0$ and* (6.2). *Assume that $L_{\underline{\Omega}_0}$ is a generator on $C_0(\underline{\Omega}_0)$. Suppose that $\underline{V}_0 \in \mathcal{O}(\underline{\Omega})$ satisfies $V(t) \neq \emptyset$ for $S < t \leq T$ and $\underline{V}_0 = \bigcap_{k=1}^n \underline{W}_k$ for sets $\underline{W}_k \in \mathcal{O}(\underline{\Omega}_0)$ admitting a Cauchy barrier with respect to L. Let $S < s < T$ and $f \in C_0(V(s))$ with $0 \leq f \leq 1$. Finally, assume that $\Phi : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz, $\Phi(0) = 0$, and $\Phi(\xi) < 0$ for $\xi > 1$ (or, more generally, let Φ satisfy the conditions of*

Remark 5.3). *Then there is a unique function* $u \in C(\overline{\underline{V}_s^*}) \cap W^{2,1}_{p,loc}(\underline{V}_s)$ *such that*

$$\begin{cases} L(\underline{x}, D)u + \Phi \circ u &= 0 \qquad \text{on } \underline{V}_s\,, \\ u(\cdot, s) &= f \qquad \text{on } V(s), \\ u|\Gamma^s &= 0. \end{cases} \tag{6.8}$$

Further, $0 \le u \le 1$ *and* u *is given by* (5.3).

The next result is a refinement of Theorem 3.27 which we need below.

Theorem 6.3. *Let* $\underline{V} \in \mathcal{O}(\underline{\Omega}_0)$ *and* L *be given as in* (6.1) *and* (6.4) *for continuous, real-valued coefficients satisfying* $c \le 0$ *and* (6.2). *Assume there is a compact subset* $\underline{K}$ *of* $\underline{V}$ *and a function* $H \in W^{2,1}_{p,loc}(\underline{V} \setminus \underline{K})$ *for some* $p > N+1$ *such that* $H > 0$ *and* $L(\underline{x}, D)H(\underline{x}) \le H(\underline{x})$ (*a.e.*) *on* $\underline{V} \setminus \underline{K}$ *and* H *extends continuously by 0 on* $\partial \underline{V}^*$. *If there are sets* $\underline{V}_n \in \mathcal{O}_c(\underline{V}_0)$ *such that* $\underline{V}_n \uparrow \underline{V}$ *and* $L_{\underline{V}_n}$ *is a generator on* $C_0(\underline{V}_n)$, *then* $L_{\underline{V}}$ *is a generator on* $C_0(\underline{V})$.

Proof. We procced as in the proof of Theorem 3.27. Let $0 \le G \in C_c(\underline{V})$ and

$$(\operatorname{supp} G \cup \underline{K}) \subset \underline{V}_1 \subset \overline{\underline{V}_1} \subset \underline{V}_2 \subset \cdots \subset \underline{V}.$$

By the assumption, there are $0 \le F_n \in D(L_{\underline{V}_n})$ such that $(1-L)F_n = G$ on $\underline{V}_n$. Set

$$H_1 = \|G\| \,(\inf\nolimits_{\partial \underline{V}_1} H)^{-1}\, H \qquad \text{on } \ \underline{V} \setminus \underline{K}.$$

Thus, $0 \le F_n \le H_1$ on $\partial(\underline{V}_n \setminus \overline{\underline{V}_1}) = \partial \underline{V}_n \cup \partial \underline{V}_1$. Suppose that the set

$$\underline{W}_n = \{\underline{x} \in \underline{V}_n \setminus \overline{\underline{V}_1} : H_1(\underline{x}) - F_n(\underline{x}) < 0\}$$

is not empty. Then. $H_1 - F_n \in C_0(\underline{W}_n)$ and $H_1 - F_n$ attains a negative minimum $\underline{x}_0$ on $\underline{W}_n$. Hence, on a cylindrical neighbourhood $\underline{U} \in \mathcal{O}_c(\underline{W}_n)$ of $\underline{x}_0$ we have

$$\operatorname{ess}\liminf_{\underline{x} \to \underline{x}_0} L(\underline{x}, D)(H_1(\underline{x}) - F_n(\underline{x})) \ge 0$$

due to [13, Thm. VII.28]. On the other hand,

$$0 \ge (L(\underline{x}, D) - 1)(H_1(\underline{x}) - F_n(\underline{x})) \ge L(\underline{x}, D)(H_1(\underline{x}) - F_n(\underline{x})) + \delta$$

a.e. on $\underline{U}$ for a constant $\delta > 0$. This contradiction yields $H_1 \ge F_n$ on $\underline{V}_n$. The assertion now follows as in the proof of Theorem 3.27. □

In the remainder of the section, we construct Cauchy barriers for the local operator L given as in (6.1) and (6.4) (and for variants of L). For simplicity, we assume (unless explicitly stated) that

$$\begin{aligned} &\underline{V} \in \mathcal{O}_c(\mathbb{R}^N \times [0,T]) \text{ with } V(t) \ne \emptyset \text{ for } 0 \le t \le T,\ a_{kl}, b_k, c \in C_b(\mathbb{R}^N \times [0,T], \mathbb{R}), \\ &c \le 0, \text{ and } (6.3) \text{ holds on } \mathbb{R}^N \times [0,T]. \end{aligned} \tag{6.9}$$

We start with the regular case, where we suppose that

(R) $\partial \underline{V}$ is given by $x_i = \phi_k(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, t)$ for some $1 \le i \le n$ and finitely many $C^{2,1}$ mappings ϕ_k defined on open subsets of $\mathbb{R}^{N-1} \times [0,T]$, and $\underline{V}$ is locally on one side of $\partial \underline{V}$.

Notice that in this case no tangent plane of $\partial \underline{V}$ is given by $t = const$. Moreover, the set $V(t)$ has the C^2–boundary $\partial V(t) = (\partial \underline{V})(t)$. Using (the proof of) Lemma 1 in the Appendix of [21], we see that the distance function $d(\underline{x}) = d(\underline{x}, \partial V(t))$ belongs to $C_b^{2,1}(\underline{W})$ for a set $\underline{W} = \{\underline{x} \in \underline{V} : d(\underline{x}) < d_0\}$ and a suitable constant $d_0 > 0$. Moreover, $|\nabla_x(d(\underline{x}))| = 1$.

Proposition 6.4. *Assume that* (6.9) *holds and that* $\partial \underline{V}$ *satisfies* (*R*). *Then* $\underline{V}_0$ *possesses a Cauchy barrier w.r.t. L. In particular, the conclusions of Theorem* 6.1 *and* 6.2 *hold for finite intersections of sets* $\underline{V}_k$ *satisfying* (*R*).

Proof. We let $0 < \varepsilon < 2\mu \, \|(L-c)d\|^{-1}_{C_b(\underline{W})}$, $\phi(\xi) = \varepsilon\xi - \xi^2$, and $\underline{W}_1 = \{\underline{x} \in \underline{W} : d(\underline{x}) < \varepsilon/2\}$, where $\mu > 0$ is the ellipticity constant of a_{kl} on $\mathbb{R}^N \times [0,T]$. Take $\varphi \in C^1[0,T]$ with $\varphi(0) = 0$ and $\varphi' > 0$. We define $F(\underline{x}) = \varphi(t)\, \phi(d(\underline{x}))$ for $\underline{x} \in \underline{W}_1$. Clearly, $0 < F \in D(L, (\underline{W}_1)_0)$ and $F(\underline{x}) \to 0$ as $\underline{x} \to \partial \underline{V}_0$. We further compute (with obvious notation)

$$\begin{aligned} LF(\underline{x}) = {} & -\varphi'(t)\, \phi(d(\underline{x})) - \varphi(t)\, (\varepsilon - 2d(\underline{x}))\, \partial_t d(\underline{x}) + \varphi(t) c(\underline{x}) d(\underline{x}) + \varphi(t)\, (\varepsilon - 2d(\underline{x}))\cdot \\ & \Big(\sum\nolimits_k b_k(\underline{x})\, \partial_k d(\underline{x}) + \sum\nolimits_{kl} a_{kl}(\underline{x})\, \partial_{kl} d(\underline{x})\Big) - 2\varphi(t) \sum\nolimits_{kl} a_{kl}(\underline{x})\, \partial_k d(\underline{x})\, \partial_l d(\underline{x}) \\ \le {} & \varphi(t) \Big(\varepsilon \, \|(L-c)d\|_{C_b(\underline{W})} - 2\mu\, |\nabla_x d(\underline{x})|^2\Big) \le 0. \end{aligned}$$

The second assertion is then clear. □

We point out that this result applies to a variety of Lipschitz domains. Clearly, the above proof extends to suitable unbounded $\underline{V}$.

The following propositions are concerned with degenerate coefficients, compare e.g. [12], [13], and the references therein. The first one is a partial extension of [14] and [29] to the non–autonomous situation. For $A(\underline{x}, D) = L(\underline{x}, D) + \frac{d}{dt}$ and $0 \le p \in C_b(\mathbb{R}^N \times [0,T])$, we define the local operator L^p induced by $p(\underline{x}) A(\underline{x}, D) - \frac{d}{dt}$ as in (6.1) and (6.4).

Proposition 6.5. *Assume that* (6.9) *holds and that* $\partial \underline{V}$ *satisfies* (*R*). *Let* $\frac{\partial}{\partial t}\, d(\underline{x}) \ge 0$ *and* $0 \le p \in C_b(\mathbb{R}^N \times [0,T])$ *with* $p > 0$ *on* $\underline{V}$. *Then* $\underline{V}_0$ *possesses a Cauchy barrier w.r.t.* L^p *and the conclusions of Theorem* 6.1 *and* 6.2 *hold for* L^p *and* $\underline{V}$.

Proof. We proceed as in the proof of Proposition 6.4 and define (for a possibly smaller $\varepsilon > 0$) the function $F(\underline{x}) = \varphi(t)\, (\varepsilon d(\underline{x}) - d(\underline{x})^2)$ for $x \in \underline{W}_1 \subset \underline{V}$, where $\overline{\underline{V} \setminus \underline{W}_1} \subset \underline{V}$, $0 < F \in C_b^{2,1}(\underline{W}_1)$, $F(\underline{x}) \to 0$ as $\underline{x} \to \partial \underline{V}_0$, and $A(\underline{x}, D) F(\underline{x}) \le 0$. Therefore we obtain

$$L^p F(\underline{x}) = -\varphi'(t)\, (\varepsilon d(\underline{x}) - d(\underline{x})^2) - \varphi(t)\, (\varepsilon - 2d(\underline{x}))\, \tfrac{\partial}{\partial t} d(\underline{x}) + p(\underline{x}) A(\underline{x}, D) F(\underline{x}) \le 0.$$

Further, there exist sets $\underline{V}_n \in \mathcal{O}_c(\underline{V}_0)$ satisfying (R) and $\underline{V}_n \uparrow \underline{V}_0$. Extending $p|\underline{V}_n$ to a strictly positive, continuous, and bounded function p_n on $\mathbb{R}^N \times [0,T] = \underline{\Omega}$, we see that $L^{p_n}_{\underline{\Omega}_0}$ is a generator, compare the discussion before Theorem 6.1. Thus, $L^p_{\underline{V}_n}$ is a generator on $C_0(\underline{V}_n)$ by Proposition 6.4 and Theorem 3.27 yields that $L^p_{\underline{V}_0}$ is a generator on $C_0(\underline{V}_0)$. So the second assertion can be shown as Theorem 4.15, 4.24, and 5.1. (Notice that in these results the fact that $L_{\underline{\Omega}_0}$ is a generator is only used to derive that $L_{\underline{V}_0}$ is generator if there is a Cauchy barrier on $\underline{V}_0$.) □

We remark that $\frac{\partial}{\partial t} d(\underline{x}) \geq 0$ holds in particular for cylindrical domains $\underline{V}$. Next, we present two results where the coefficients vanish on irregular parts of the boundary like isolated points or entering faces, see Example 6.8. Here we (partially) generalize [30, §6] and [43, §7] to the nonautonomous case. We point that our growth conditions on the coefficients cannot be weakened in general, see [30, §6] and [43, §7]. First, the irregularity is of a similar form as in the previous proposition.

Proposition 6.6. *Let* (6.9) *hold and let* $\underline{V}$ *be the finite intersection of sets* $\underline{V}_k \in \mathcal{O}_c(\mathbb{R}^N \times [0,T])$ *satisfying* (*R*). *Assume that there are sets* $\underline{M} = M \times [0,T] \subset \underline{W} = W \times [0,T] \subset \overline{\underline{W}} \subset \underline{V}$ *with* $W \in \mathcal{O}_c(\mathbb{R}^N)$ *such that the coefficients of* $A(\underline{x}, D)$ *do not depend on* t *for* $\underline{x} \in \underline{W}$ *and* $M = \{x \in W : p_0(x) = 0\}$ *for a function* $0 \leq p_0 \in C_b(W)$. *Suppose that* $p_0(x) \leq C_K\, d(x,K)^2$ *for* $x \in W \setminus M$ *and all compact subsets* K *of* M. *Moreover, let* $0 \leq p \in C_b(\mathbb{R}^N \times [0,T])$ *with* $p = 0$ *on* $\underline{M}$, $p \geq \delta > 0$ *off* $\underline{W}$, *and* $0 < p(\underline{x}) \leq C\, p_0(x)$ *for* $\underline{x} \in \underline{W} \setminus \underline{M}$ *and constants* $C, \delta > 0$. *Then* $\underline{V} \setminus \underline{M}$ *possesses a Cauchy barrier for* L^p *and the conclusions of Theorem* 6.1 *and* 6.2 *hold for* L^p *and* $\underline{V} \setminus \underline{M}$.

Proof. We may assume that ∂W is of class C^3 (in the sense of [59]). Applying Proposition 6.4, Corollary 3.26, and Proposition 3.2 on $\underline{V} \setminus \underline{W}$, we see that it suffices to construct a Cauchy barrier for L^p near $\underline{M}$ in order to find a Cauchy barrier on $\underline{V} \setminus \underline{M}$. Let $A(x,D) = A(\underline{x},D)$ on $\underline{W}$ and define the local operator A by

$$
\begin{aligned}
D(A,U) &= \{f \in W^{2,q}_{loc}(U) : A(x,D)f = g \ \text{ a.e. for } g \in C(U)\}, \\
Af &= g \qquad \text{on} \ \ U \in \mathcal{O}(W),
\end{aligned}
$$

for some $q > N$, see [46]. Then A_W generates a contraction semigroup on $C_0(W)$ by [46] or [59, Thm. 5]. So Theorem 6.4 and 6.3 of [30] show the existence of a Cauchy barrier h for $p_0 A$ on $W \setminus M$ which is defined on some open set $U \subset W \setminus M$ and satisfies $p_0 A h \leq h$. Take $\varphi \in C^1[0,T]$ with $\varphi(0) = 0$ and $\varphi' > 0$. Set $F(x,t) = \varphi(t)\, h(x)$ for $(x,t) \in U \times [0,T] = \underline{U}$. Clearly, $0 < F \in D(L^p, \underline{U}_0)$, $F(x,t) \to 0$ as $t \to 0$ or $x \to M$, and

$$
L^p F(x,t) = -\varphi'(t)\, h(x) + \varphi(t)\, \frac{p(x,t)}{p_0(x)}\, p_0(x) A h(x) \leq C F(x,t)
$$

on $\underline{U}_0$. The second assertion can be established as in Proposition 6.5. □

Similarly one can treat the situation that $\underline{\overline{M}}$ intersects with a cylindrical, closed and open subset of $\partial\underline{V}$. Next, we study a more general degenerate operator under some additional assumptions on the irregular part of the boundary. To that purpose we weaken the assumptions on the coefficients made before Proposition 6.4. The function h used below will be a suitable modification of the distance function d in the applications, see Example 6.8.

Proposition 6.7. *Let $\underline{V} \in \mathcal{O}_c(\mathbb{R}^N \times [0,T])$ with $V(t) \neq \emptyset$ for $0 \leq t \leq T$ and let $\partial\underline{V}$ be the disjoint union of closed sets $\Gamma_1 \neq \emptyset$ and Γ_2, where Γ_2 satisfies (R). Assume that there exists a function $0 < h \in W^{2,1}_{p,loc}(\underline{V}_1) \cap C_b(\underline{V}_1)$ for some $p > N+2$, an open subset $\underline{V}_1$ of $\underline{V}$ with $\Gamma_1 \subset \partial\underline{V}_1$ and $\Gamma_2 \cap \partial\underline{V}_1 = \emptyset$ satisfying*

$$\lim_{\underline{x}\to\Gamma_1} h(\underline{x}) = 0, \; \frac{\partial h}{\partial t}(\underline{x}) \geq 0, \; \left|\frac{\partial h}{\partial x_k}(\underline{x})\right| \leq C_1, \; \left|\frac{\partial^2 h}{\partial x_k \partial x_l}(\underline{x})\right| \leq \frac{C_2}{h(\underline{x})}$$

for (a.e.) $x \in \underline{V}_1$, $k,l = 1,\ldots,N$, and constants $C_k > 0$. Moreover, we suppose that $a_{kl}, b_k, c \in C_b(\mathbb{R}^N \times [0,T])$, (6.3) holds on $(\mathbb{R}^N \times [0,T]) \setminus \underline{V}_1$, and

$$|b_k(\underline{x})| \leq C_3\, h(\underline{x}), \; |a_{kl}(\underline{x})| \leq C_4\, h(\underline{x})^2, \; \sum\nolimits_{kl} a_{kl}(\underline{x})\, \frac{\partial h}{\partial x_k}(\underline{x})\, \frac{\partial h}{\partial x_l}(\underline{x}) \geq \alpha\, h(\underline{x})^2$$

for $\underline{x} \in \underline{V}_1$, $k,l = 1,\ldots,N$, and constants $\alpha, C_k > 0$. Then the conclusions of Theorem 6.1 and 6.2 hold for finite intersections of sets $\underline{V}_k$ of the above type if Γ_1 is a closed and open subset of all $\partial\underline{V}_k$ and there are relatively compact sets $\underline{V}_k^n \uparrow \underline{V}_k$ satisfying (R).

Proof. It suffices to contruct a Cauchy barrier near Γ_1 by Proposition 6.4. Set $\beta = NC_1C_3 + N^2C_2C_4 - \alpha$. Fix $\lambda \in]0, \alpha + \beta[$. Then there is a root $\kappa \in]0,1[$ of the polynomial $\alpha t^2 + \beta t - \lambda$. Set $F(\underline{x}) = \varphi(t)h(\underline{x})^\kappa$ on $\underline{V}_1$, where $\varphi \in C^1[0,T]$ with $\varphi(0) = 0$ and $\varphi' > 0$. Then $0 < F \in W^{2,1}_{p,loc}(\underline{V}_1)$, $F(\underline{x}) \to 0$ as $\underline{x} \to \Gamma_1$ or $t \to 0$, and

$$\begin{aligned} LF(\underline{x}) &= -\varphi'(t)h(\underline{x})^\kappa - \kappa\varphi(t)h(\underline{x})^{\kappa-1}\,\partial_t h(\underline{x}) + \varphi(t)c(\underline{x})h(\underline{x})^\kappa \\ &\quad + \kappa\varphi(t)h(\underline{x})^{\kappa-1}\sum\nolimits_k b_k(\underline{x})\,\partial_k h(\underline{x}) + \kappa\varphi(t)h(\underline{x})^{\kappa-1}\sum\nolimits_{kl} a_{kl}(\underline{x})\,\partial_{kl}h(\underline{x}) \\ &\quad + \kappa(\kappa-1)\varphi(t)h(\underline{x})^{\kappa-2}\sum\nolimits_{kl} a_{kl}(\underline{x})\,\partial_k h(\underline{x})\,\partial_l h(\underline{x}) \\ &\leq \varphi(t)h(\underline{x})^\kappa\left(\kappa NC_1C_3 + \kappa N^2C_2C_4 + \kappa(\kappa-1)\alpha\right) = \lambda F(\underline{x}) \end{aligned}$$

a.e. on $(\underline{V}_1)_0$. Now the assertion follows as in Proposition 6.5 using Theorem 6.3. □

Let us present two examples for the function h used in Proposition 6.7. In these examples the last assumption in Proposition 6.7 can easily be verified.

Example 6.8.

1. $\underline{V}_1 = \{(x,t) \in \mathbb{R}^N \times [0,T] : 0 < |x| < \delta\}$, $\Gamma_1 = \{0\} \times [0,T]$, *and* $h(x,t) = |x|$.

2. $\underline{V} = \{(x,t) \in \mathbb{R}^N \times [0,T] : 0 < |x| < R(t), x \notin H_j\}$ *for* $H_j = \{x \in \mathbb{R}^N : x_1 = \cdots = x_j = 0,\ x_N \geq 0\}$, $j \in \{1, \cdots, N-1\}$, *and* $0 < R \in C^1[0,T]$ *with* $R' \geq 0$; $\Gamma_1 = \partial \underline{V}$, *and*

$$h(x,t) = \begin{cases} (x_1^2 + \cdots + x_j^2)^{1/2}\,(R(t)^2 - |x|^2), & \underline{x} \in \underline{V},\ x_N > 0, \\ (x_1^2 + \cdots + x_j^2 + x_N^2)^{1/2}\,(R(t)^2 - |x|^2), & \underline{x} \in \underline{V},\ x_N \leq 0. \end{cases}$$

6.2 Diffusion problems on networks

The theory of local operators also provides a convenient framework to study diffusion problems on ramified spaces or networks. Using these methods the autonomous case was investigated in [34, 35, 37], see also [3], [45], and the references therein. Here we indicate how to extend these results to time dependent coefficients and non-cylindrical domains in space-time. We refer to [2] for a different approach. For simplicity, we concentrate on one-dimensional networks Ω in $\mathbb{R}^N$. These are given by at most countably many simple C^2-curves connected by a set of 'ramification nodes'.

More precisely, let $\mathcal{I}$ be a finite or countable index set. Consider simple C^2 curves

$$\phi_i : [0, l_i] \to \mathbb{R}^N, \qquad 0 < l_i < \infty, \ \ i \in \mathcal{I},$$

with $N_i^1 := \phi_i(0) \neq \phi_0(l_i) =: N_i^2$. We assume that the curves ϕ_i are parametrized by arc length (starting at N_i^1). Set $\Omega_i = \phi_i(0, l_i)$ and $\Omega_i^* = \phi_i[0, l_i]$ for $i \in \mathcal{I}$. Further, $E_i = \{N_i^1, N_i^2\}$ and $N(\Omega) = \{N_i^j : i \in \mathcal{I}, j = 1, 2\}$ denote the sets of the end points or *nodes*. The network is then determined by the *branches* Ω_i and a subset $N_e(\Omega)$ of $N(\Omega)$ containing the *exterior nodes*. The set of *ramification nodes* $N_r(\Omega)$ is the complement of $N_e(\Omega)$ in $N(\Omega)$. Finally, define the sets

$$\Omega' = \bigcup_{i \in \mathcal{I}} \Omega_i^* \quad \text{and} \quad \Omega = \Omega' \setminus N_e(\Omega)$$

endowed with the relative topology induced by $\mathbb{R}^N$. We call Ω a *network* if the following conditions hold:

1. Ω is a connected subset of $\mathbb{R}^N$;

2. $\Omega_i^* \cap \Omega_j^* \subset E_i \cap E_j$ for $i \neq j \in \mathcal{I}$ and for each $i \in \mathcal{I}$ we have $\Omega_i^* \cap \Omega_j^* \neq \emptyset$ for at most finitely many $j \in \mathcal{I}$;

3. if $N \in E_i$ for exactly one $i \in \mathcal{I}$, then $N \in N_e(\Omega)$;

4. for each compact $K \subset \mathbb{R}^N$ we have $K \cap \Omega_i^* \neq \emptyset$ for at most finitely many $i \in \mathcal{I}$.

In other words, a branch can only be joined at the end points with at most finitely many other branches and the network is connected and locally finite. Clearly, Ω is locally compact with respect to the relative topology induced by $\mathbb{R}^N$.

Further, for $J = [S,T]$ we set $\underline{\Omega} = \Omega \times J$ and $\underline{\Omega}_i^{(*)} = \Omega_i^{(*)} \times J$, and analogously for $J_0 =]S,T]$. Define $V_i = \phi_i^{-1}(V \cap \Omega_i) \subset (0,l_i)$ and $\underline{V}_i = \{(s,t) \in (0,l_i) \times J : (\phi_i(s),t) \in \underline{V} \cap \underline{\Omega}_i\}$ for $V \subset \Omega$, $\underline{V} \subset \underline{\Omega}$, and $i \in \mathcal{I}$. For a function $F : \underline{V} \to \mathbb{C}$ and $\underline{V} \in \mathcal{O}(\underline{\Omega})$ with $\underline{V}_i \neq \emptyset$, let $F_i(s,t) = F(\phi_i(s),t)$ for $(s,t) \in \underline{V}_i$. Notice that F is continuous if and only if all the functions F_i are continuous up to the ramification nodes contained in $\underline{V}$ and coincide on those nodes.

The network is endowed with the analytical structure induced by the arc length s_i on Ω_i. Morover, if f is differentiable near a ramification node N, then the derivative $d/d\nu_{Ni}$ at N 'along' Ω_i is defined by the derivation with respect to the arc length s_{Ni} on Ω_i starting at N and evaluated at $s_{Ni} = 0$. We say that $F \in W^{2,1}_{p,loc}(\underline{V})$ for $\underline{V} \in \mathcal{O}(\underline{\Omega})$ if $F_i \in W^{2,1}_p((\underline{W}_i)_{00})$ for all subsets $\underline{W} \in \mathcal{O}_c(\underline{\Omega})$ of $\underline{V}$ and $i \in \mathcal{I}$ with $\underline{W}_i \neq \emptyset$, where $(\underline{W}_i)_{00} = \underline{W}_i \setminus (\underline{W}_i(S) \cup \underline{W}_i(T))$. For $p > 3$ and $F \in W^{2,1}_{p,loc}(\underline{V})$ the function $F(\cdot,t)$ is continuously differentiable on Ω_i^* near a ramification node $(N,t) \in \underline{V}$ with $t \in J$ due to [24, Lemma II.3.3].

The diffusion on Ω is described by local operators L_i with coefficients $a_i, b_i, c_i \in C^{\mathbb{R}}([0,l_i] \times J)$ for $i \in \mathcal{I}$ satisfying $c_i \leq 0$ and $a_i \geq \delta_i > 0$ for constants δ_i. Moreover we assume that the functions c_i coincide on ramification nodes. Then L_i, $i \in \mathcal{I}$, is defined by

$$(L_iF_i)(s,t) = a_i(s,t)\,\frac{\partial^2}{\partial s^2}F_i(s,t) + b_i(s,t)\,\frac{\partial}{\partial s}F_i(s,t) + c_i(s,t)\,F_i(s,t) - \frac{\partial}{\partial t}F_i(s,t)$$

on $(\underline{V}_i)_{00}$ for $F \in W^{2,1}_{p,loc}(\underline{V})$, a fixed $p > 3$, and $\underline{V} \in \mathcal{O}(\underline{\Omega})$. These local operators are connected at the ramification nodes N via the operators

$$(B_NF)(t) = \sum_{i:N\in E_i,(N,t)\in\underline{V}} d_{Ni}(t)\,\frac{d}{d\nu_{Ni}}F(\cdot,t), \quad t \in J_0, \tag{6.10}$$

where $d_{Ni} : J \to \mathbb{R}$ is continuously differentiable for $N \in E_i \cap N_r(\Omega)$. Finally, for a fixed $p > 3$ we define

$$\begin{aligned} D(L,\underline{V}) = \{F \in W^{2,1}_{p,loc}(\underline{V}) \cap C(\underline{V}) : \exists G \in C(\underline{V}) \text{ with } G(S) = 0 \text{ if } S \in I_{\underline{V}}, \\ L_iF_i = G_i \text{ a.e. on } (\underline{V}_i)_{00},\ B_NF(t) = 0 \text{ for } (N,t) \in \underline{V}_0, N \in N_r(\Omega)\}, \\ LF = G \qquad \text{on } \underline{V} \in \mathcal{O}(\underline{\Omega})\ (\mathcal{O}(\underline{\Omega}_0)). \end{aligned} \tag{6.11}$$

We remark that the condition $B_NF = 0$ corresponds to Kirchhoff's law. It is easy to see that L is a real, standard parabolic (parabolic), complete, local operator satisfying (OE). Let $\underline{V} \in \mathcal{O}(\underline{\Omega}_0)$ and $(x,t) \in \underline{V}$. Clearly, (S1) holds if x is not a node. If $x \in N_r(\Omega)$, then we can find neighbourhoods $U \subset \bar{U} \subset W$ of x in $V(t)$ containing no other nodes and a smooth function f compactly supported in W such that $0 \leq f \leq 1$ and $f = 1$ on U. Choose $\varphi \in C^1_c(I)$ with $0 \leq \varphi \leq 1$ and $\varphi(t) = 1$. Then $F = \varphi f \in D(L,\underline{V})$ since the functions c_i coincide at (N,s), and so F satisfies (S1). The condition (LS) can be verified similarly on $\underline{\Omega}_0$. We further have

Lemma 6.9. *In addition to the above assumptions, assume that*

$$\text{either } d_{Ni}(t) > 0 \text{ or } d_{Ni}(t) < 0 \text{ for all } i \in \mathcal{I}. \tag{6.12}$$

Then the local operator L given by (6.11) *is locally closed, locally dissipative, and does not depend on $p > 3$.*

Proof. Clearly, it suffices to consider $\underline{V} \in \mathcal{O}(\underline{\Omega}_0)$. First, let $F^n \to F$ and $LF^n \to G$ in $C_b(\underline{V})$ for $F^n \in D(L, \underline{V})$. The results of the preceding section yield $F_i \in W^{2,1}_{p,loc}(\underline{V}_i)$ and $L_i F_i = G_i$ on $\underline{V}_i$. Further, if $(N, t) \in \underline{V}$ for a ramification node N, then there are $W, W' \in \mathcal{O}_c(V(t))$ containing no other node and open subintervals I, I' of J_0 such that

$$(N, t) \in \underline{W} \subset \overline{\underline{W}} \subset \underline{W}' \subset \overline{\underline{W}'} \subset \underline{V},$$

where $\underline{W} = W \times I$ and $\underline{W}' = W' \times I'$. As in [2, §4,7] one can transform the equation $LF^n = G^n$ on $\underline{W}'$ into a boundary value problem for a parabolic system which is well stated in the sense of [24, Chap. VII] or [58]. Therefore the interior a priori estimate proved in [58, Thm. 5.7] yields

$$\sum\nolimits_i \|F_i^n - F_i^m\|_{W_p^{2,1}(\underline{W}_i)} \le c \sum\nolimits_i \|G_i^n - G_i^m\|_{L^p(\underline{W}'_i)} + \|F_i^n - F_i^m\|_{L^p(\underline{W}'_i)}$$

since $B_N F^n = 0$. Thus, $F^n \to F$ in $W_p^{2,1}(\underline{W})$ which easily implies $F \in D(L, \underline{V})$ and $LF = G$ (use [24, Lemma II.3.3] to check $B_N F(t) = 0$).

We apply Corollary 2.10 to verify local dissipativity. So let $\underline{x} \in \underline{V}$ such that $0 < F(\underline{x}) = \sup_{\underline{V}} F$ for a real function $F \in D(L, \underline{V})$. If $\underline{x} \in \underline{\Omega}_i$, then $L_i F_i(\underline{x}) \le 0$ since L_i is locally dissipative by the results of the preceding section. If $x \in N_r(\Omega)$ and $LF(\underline{x}) > 0$, then

$$\frac{d}{d\nu_{Ni}} F(\cdot, t) < 0$$

by [10, Prop. 13.2]. But in combination with (6.12) this violates $B_N F(t) = 0$ so that $LF(\underline{x}) \le 0$ in this case, too. Hence, L is locally dissipative. The p-independence of L follows from the results of Section 6.1. □

Now, the main results of this section are immediate consequences of Theorem 4.15, 4.24, and 5.1. Recall that $p \in]3, \infty[$ can be chosen arbitrarily large.

Theorem 6.10. *Let Ω be a C^2-network with branches Ω_i^*, $i \in \mathcal{I}$, ramification nodes $N \in N_r(\Omega)$ and connecting operators B_N given by* (6.10) *with continuously differentiable functions $d_{Ni} : J \to \mathbb{R}$ satisfying* (6.12). *Let L be defined by* (6.11) *for coefficients $a_i, b_i, c_i \in C^{\mathbb{R}}([0, l_i] \times J)$ satisfying $c_i \le 0$ and $a_i \ge \delta_i > 0$, where the functions c_i coincide on ramification nodes. Assume that $L_{\underline{\Omega}_0}$ is a generator on $C_0(\underline{\Omega}_0)$. Let $\underline{V} \in \mathcal{O}(\underline{\Omega})$ with $V(t) \ne \emptyset$ for $S \le t \le T$. Suppose that $\underline{V}_0 = \bigcap_{k=1}^n \underline{W}_k$ for sets $\underline{W}_k \in \mathcal{O}(\underline{\Omega}_0)$ admitting a Cauchy barrier with respect to L. Let $S \le s < T$ and $f \in C_0(V(s))$. If $S < s < T$, let $F \in C(\overline{\underline{V}_s^*})$ with $F|\Gamma^s = 0$; if $s = S$, let*

$F \in C_0(\underline{V}_0)$. Finally, assume that $\tilde{D}_L(S)$ is dense if $s = S$. Then there is a unique function $u \in C(\overline{\underline{V}_s^*}) \cap W^{2,1}_{p,loc}(\underline{V}_s)$ such that

$$\begin{cases} L_i u_i = F_i & \text{on } (\underline{V}_s)_i\,,\ i \in \mathcal{I}, \\ B_N u(t) = 0 & \text{for } (N,t) \in \underline{V}_s, \\ u(\cdot,s) = f & \text{on } V(s), \\ u|\Gamma^s = 0 & (\Gamma = \Gamma^S \text{ if } s = S). \end{cases} \tag{6.13}$$

Further, u is given as in Theorem 4.12 for $s = S$ and as in Theorem 4.14 for $s > S$. If $F = 0$, then $|u| \le \|f\|$. If f and $-F$ are positive, then u is positive.

Theorem 6.11. *Let Ω and L be given as in Theorem 6.10 Assume that $L_{\underline{\Omega}_0}$ is a generator on $C_0(\underline{\Omega}_0)$. Suppose that $\underline{V}_0 \in \mathcal{O}(\underline{\Omega})$ satisfies $V(t) \ne \emptyset$ for $S < t \le T$ and $\underline{V}_0 = \bigcap_{k=1}^n \underline{W}_k$ for sets $\underline{W}_k \in \mathcal{O}(\underline{\Omega}_0)$ admitting a Cauchy barrier with respect to L. Let $S \le s < T$ and $f \in C_0(V(s))$ with $0 \le f \le 1$. Finally, assume that $\Phi : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz, $\Phi(0) = 0$, and $\Phi(\xi) < 0$ for $\xi > 1$. Then there is a unique function $u \in C(\overline{\underline{V}_s^*}) \cap W^{2,1}_{p,loc}(\underline{V}_s)$ such that*

$$\begin{cases} L_i u_i + \Phi \circ u_i = 0 & \text{on } (\underline{V}_s)_i\,,\ i \in \mathcal{I}, \\ B_N u(t) = 0 & \text{for } (N,t) \in \underline{V}_s \\ u(\cdot,s) = f & \text{on } V(s), \\ u|\Gamma^s = 0. \end{cases} \tag{6.14}$$

Further, $0 \le u \le 1$ and u is given by (5.3).

We remark that for the analogous elliptic problem Cauchy barriers were constructed for finite unions of connected sets $V \in \mathcal{O}_c(\Omega)$ in [34, Thm. 4.1] and, under a certain growth condition on the coefficients, for arbitrary open subsets $V \ne \emptyset$ of Ω in [37, Thm. 2.1]. The Cauchy problem (6.13) was solved by a reduction to a parabolic system in [2] for Hölder continuous time dependent coefficients on a cylindrical domain.

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