L^p-spectrum of degenerate hypoelliptic Ornstein-Uhlenbeck operators

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Abstract

We describe the spectrum of degenerate hypoelliptic Ornstein-Uhlenbeck operators $\mathcal{A} = \sum_{i,j=1}^n q_{ij} D_{ij} + \sum_{i,j=1}^n b_{ij} x_j D_i$ in $L^p(\mathbb{R}^n)$, $1 \le p < +\infty$, and in $C_0(\mathbb{R}^n)$. We show that the spectrum of $\mathcal{A}$ is the sum of $(-\infty, 0]$ and the spectrum of the drift term. Our result gives a complete picture of the spectral properties of Ornstein-Uhlenbeck operators in L^p spaces.

Mathematics subject classification (2000): 35P05, 35J70, 35K65, 47D06.

1 Introduction

The aim of the present paper is the full description of the spectrum of possibly degenerate Ornstein-Uhlenbeck operators

$$\mathcal{A} = \sum_{i,j=1}^{n} q_{ij} D_{ij} + \sum_{i,j=1}^{n} b_{ij} x_j D_i = \mathrm{Tr}(QD^2) + \langle Bx, D\rangle, \qquad x \in \mathbb{R}^n, \tag{1.1}$$

in $L^p(\mathbb{R}^n)$, $1 \le p < +\infty$, and in $C_0(\mathbb{R}^n)$. Here $Q = (q_{ij})$ is a real, constant, symmetric and positive semidefinite matrix and $B = (b_{ij})$ is a nonzero real matrix. The semidefinitess of the matrix Q is responsible for the possible degeneracy of $\mathcal{A}$. Throughout we assume that $\mathcal{A}$ is hypoelliptic, which can be stated as follows: the symmetric matrices

$$Q_t = \int_0^t e^{sB} Q e^{sB^T}\, ds$$

have nonzero determinant for some (equivalently, for all) $t > 0$. In the literature one can find several equivalent conditions for hypoellipticity. In particular, on p. 148 of [13] it is pointed out that the hypoellipticity of $\mathcal{A}$ is equivalent to the property

$$\ker(Q) \textit{ does not contain nontrivial subspaces which are invariant for } B^T, \tag{1.2}$$

see also [16, Appendix]. Here nontrivial means different from $\{0\}$.

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The hypoellipticity assumption implies that the associated Markov semigroup $(T(t))_{t\geq 0}$ has the following explicit representation formula (first found by Kolmogorov [15] in a particular case and then proved by Hörmander [13] in the general case)

$$(T(t)f)(x) = \frac{1}{(4\pi)^{n/2}(\det Q_t)^{1/2}} \int_{\mathbb{R}^n} e^{-\langle Q_t^{-1}y,y\rangle/4} f(e^{tB}x - y)\, dy, \quad x \in \mathbb{R}^n,\ t > 0, \quad (1.3)$$

see also [9, Proposition 4.2]. The parabolic equation $u_t = \mathcal{A}u$, known as Kolmogorov equation, is solved by the function $u(t,x) = T(t)f(x)$ for a large class of initial data f. In recent years, both the semigroup $(T(t))_{t\geq 0}$ and its generator $\mathcal{A}$ have extensively been studied. Several applications in physics and finance for the operator $\mathcal{A}$ and its evolutionary counterpart $\mathcal{A} - D_t$ can be found in the survey [24]. They were also used in the context of rotating fluids, see e.g. [12]. These operators were also the leading example for an intensive research on elliptic and parabolic problems with unbounded coefficients, see e.g. [17].

In the analytical study of $\mathcal{A}$, even in the nondegenerate case the classical L^p and Schauder estimates do not apply because of the unboundedness of the first order coefficients. Regularity properties in spaces of continuous functions were proved in [4] in the nondegenerate case and in [18] in the degenerate case. Schauder estimates can then be deduced by means of interpolation techniques. Moreover, L^p estimates were established in [23] and in [22] in the nondegenerate case, by a semigroup approach, and in [1] in the degenerate case. Fractional powers and the extension problem are studied in [10], whereas Riesz potentials and Sobolev embeddings are treated in [11] under the assumption $\mathrm{tr}(B) \geq 0$.

The underlying stochastic process admits an invariant measure μ if and only if all eigenvalues of the drift matrix B have negative real parts. This means that μ is a probability measure satisfying

$$\int_{\mathbb{R}^n} \big(T(t)f\big)(x)\, d\mu(x) = \int_{\mathbb{R}^n} f(x)\, d\mu(x)$$

for every $t \geq 0$ and continuous and bounded function f on $\mathbb{R}^n$. The invariant measure is unique and given by the Gaussian measure $d\mu = \rho\, dx$ with density

$$\rho(x) = \frac{1}{(4\pi)^{n/2}(\det Q_\infty)^{1/2}} e^{-\langle Q_\infty^{-1}x,x\rangle/4} \quad \text{where} \quad Q_\infty = \int_0^\infty e^{sB}Qe^{sB^T}\, ds,$$

see [5, Chapter II. 6].

The semigroup $(T(t))_{t\geq 0}$ and its generator $\mathcal{A}$ have widely been investigated in the weighted spaces $L^p(\mathbb{R}^n, d\mu)$, if $\sigma(B) \subset \mathbb{C}_-$. Here the unboundedness of the coefficients of $\mathcal{A}$ is balanced by the exponential decay of the density ρ which leads to a much better behavior in several respects. For instance, the generator has compact resolvent in $L^p(\mathbb{R}^n, d\mu)$ if $p \in (1, \infty)$, which is not true in the unweighted spaces L^p. The domain of the generator in $L^p(\mathbb{R}^n, d\mu)$ was computed in [19] for $p = 2$ and in [23] for $p \in (1, \infty)$ in the nondegenerate case. See also [2, 3] for the analogous problem on an infinite-dimensional Hilbert space E instead of $\mathbb{R}^n$. In the degenerate case a sharp inclusion for the domain was shown in [8] for $p = 2$, whereas the case $p \neq 2$ is still an open problem, indicating that the general picture of Ornstein-Uhlenbeck operators is still not complete.

In [21] the spectrum of $\mathcal{A}$ in $L^p(\mathbb{R}^n, d\mu)$ was completely described also in the degenerate case, provided that $\sigma(B) \subset \mathbb{C}_-$. The situation is much different in the spaces $L^p = L^p(\mathbb{R}^n)$ with respect to the Lebesgue measure, e.g., since $\mathcal{A}$ does not have a compact resolvent here. For some choices of B the spectrum of $\mathcal{A}$ was computed in L^p in [20]. This paper is the starting point of our investigation.

The operator $\mathcal{A}$ can be seen as the sum of the diffusion term $\sum_{i,j=1}^n q_{ij}D_{ij}$ and of the drift term $\mathcal{L} = \sum_{i,j=1}^n b_{ij}x_jD_i$. The spectral properties of the drift term are fully understood by [20]. There it was proved that the spectrum of the realization $\mathcal{L}_p$ of $\mathcal{L}$ in L^p is the line $-\mathrm{tr}(B)/p + i\mathbb{R}$ unless B is similar to a diagonal matrix with purely imaginary eigenvalues. In this last case the spectrum of $\mathcal{L}_p$ can be either $i\mathbb{R}$ or a discrete, explicitly given subgroup G of $i\mathbb{R}$, see Theorem 2.2 and Proposition 2.3.

In [20] it is further shown that the boundary spectrum of the realization $\mathcal{A}_p$ of $\mathcal{A}$ in L^p contains the spectrum of $\mathcal{L}_p$ without further assumptions on the matrices Q and $B \neq 0$. Here $\mathcal{A}_p$ is defined as the generator of $(T(t))_{t\geq 0}$ in L^p, see Proposition 2.4. The spectrum of $\mathcal{A}_p$ has been computed in [20] if $\sigma(B)$ is contained in the left or in the right open half-plane. In this case, $\sigma(\mathcal{A}_p)$ is equal to $\{\mu \in \mathbb{C} : \mathrm{Re}\,\mu \leq -\,\mathrm{tr}(B)/p\}$. So it depends on p and is far from being discrete. In addition, and this is the main step in [20], if all the eigenvalues of B have positive real parts, then the open half-plane $\{\mu \in \mathbb{C} \,|\, \mathrm{Re}\,\mu < -\,\mathrm{tr}(B)/p\}$ consists of eigenvalues.

In this paper we complete the picture computing the spectrum of $\mathcal{A}_p$ without any further restriction on $Q = Q^T \geq 0$ and $B \neq 0$, apart from hypoellipticity. We prove that $\sigma(\mathcal{A}_p)$ is given as the sum of the spectra of its diffusion part (i.e., $(-\infty, 0]$) and of the drift term $\mathcal{L}_p$.

Theorem 1.1. *Let* (1.2) *be true and* $p \in [1, \infty]$. *Then the spectrum of* $\mathcal{A}_p$ *is given by*

$$\sigma(\mathcal{A}_p) = (-\infty, 0] + \sigma(\mathcal{L}_p).$$

In particular, we have either $\sigma(\mathcal{A}_p) = (-\infty, 0] + G$ *or* $\sigma(\mathcal{A}_p) = \{\mu \in \mathbb{C} \,|\, \mathrm{Re}\,\mu \leq -\mathrm{tr}(B)/p\}$, *according to* $\sigma(\mathcal{L}_p)$ *being a discrete subgroup* $G = \frac{2\pi i}{\tau}\mathbb{Z}$ *of* $i\mathbb{R}$ *or the whole line* $-\mathrm{tr}(B)/p + i\mathbb{R}$.

Actually, we also show a (weak) spectral mapping theorem for $(T(t))_{t\geq 0}$, see Theorem 4.1. We note that for $p = 2$ the spectral mapping theorem was proved for perturbed Ornstein-Uhlenbeck operators with $Q = I$ and $B = 2I$ by completely different methods in [6].

If $B = B^T$ and $QB = BQ$, by separation of variables one can transform the Ornstein-Uhlenbeck operator into the form $\mathcal{A} = \Delta + \sum_{i=1}^n b_ix_iD_i$. Here the problem can be reduced to one dimensional problems, see [20, Theorem 5.1]. However, this is far from being the general case. We also stress that $\mathcal{A}_2$ does not possess eigenvalues if B has an eigenvalue with negative real part or if B is skew-symmetric and $Q = I$, as we will see in Section 3. So we have to proceed in a different way than in [20] or [21], where eigenfunctions played a crucial role.

Instead, we start by reducing $\mathcal{A}$ to a canonical form with an upper quasi triangular drift matrix whose diagonal is formed by 1×1 and 2×2 blocks containing the real and complex conjugate eigenvalues of B, respectively. The transformation is made through a linear change of variables that leaves the spectrum unchanged.

The second step consists in a scaling procedure leading to a new operator $\mathcal{C}$ in the limit which is the sum of an Ornstein-Uhlenbeck operator in one or two variables and a drift operator acting in the remaining ones. The scaling and the limit allow us to get rid of the upper off-diagonal blocks of the drift matrix of $\mathcal{A}$ and to separate the variables. We can recover the spectrum of $\mathcal{A}_p$ from that of the limit operator $\mathcal{C}$.

The main part of the proof is thus devoted to the investigation of the spectrum of $\mathcal{C}$. Here we can assume that B has an eigenvalue with nonnegative real part, since the other case is already covered by the main result in [20]. The above splitting then reduces the problem to Ornstein-Uhlenbeck operators in $\mathbb{R}$ or in $\mathbb{R}^2$ where B has one nonnegative eigenvalue or two complex conjugate eigenvalues with nonnegative real parts. We further have to treat eigenvalues in $i\mathbb{R}$ and with positive real part separately. The detailed study of these four cases is mainly based on the construction of approximate eigenfuctions.

The paper is structured as follows. In Section 2 we recall the known generator properties of the drift operator $\mathcal{L} = \sum_{i,j=1}^n b_{ij} x_j D_i$ and its spectrum, as computed in [20]. We provide further details in the case where the generated group is periodic. We also collect the known properties on $\mathcal{A}$. Most of the results are contained in [20], where it is assumed that Q is positive definite. However, we explain why they continue to hold with minor modifications in the degenerate hypoelliptic setting. Corollary 2.7 and Proposition 2.8 establish the inclusion $\sigma(\mathcal{A}_p) \subseteq (-\infty, 0] + \sigma(\mathcal{L}_p)$ by means of general spectral theory of semigroups. In Section 3 we show that there are no eigenvalues of $\mathcal{A}_2$ in many cases. Finally, Section 4 is devoted to the proof of Theorem 4.1 that is the expanded version of Theorem 1.1. We mainly have to establish the other inclusion $(-\infty, 0] + \sigma(\mathcal{L}_p) \subseteq \sigma(\mathcal{A}_p)$ which requires a sophisticated analysis of the four cases indicated above. The (weak) spectral mapping theorem then follows mostly from general theory.

Notation. L^p stands for $L^p(\mathbb{R}^n)$ if $p \in [1, \infty)$ and C_c^∞ for $C_c^\infty(\mathbb{R}^n)$. Throughout the whole paper, we write L^∞ for $C_0(\mathbb{R}^n)$, which is the space of continuous functions on $\mathbb{R}^n$ vanishing at infinity, endowed with the supremum norm.

The spectrum and the resolvent set of a linear operator $\mathcal{B}$ are denoted by $\sigma(\mathcal{B})$ and $\rho(\mathcal{B})$, respectively. The *spectral bound* of $\mathcal{B}$ is defined by $s(\mathcal{B}) = \sup\{\operatorname{Re}\mu \,|\, \mu \in \sigma(\mathcal{B})\}$ and the *boundary spectrum* is $\sigma(\mathcal{B}) \cap \{\mu \in \mathbb{C} \,|\, \operatorname{Re}\mu = s(\mathcal{B})\}$. The *approximate point spectrum* $\sigma_{ap}(\mathcal{B})$ of $\mathcal{B}$ is the subset of $\sigma(\mathcal{B})$ of all complex numbers μ for which there is a sequence (v_n) in its domain $D(\mathcal{B})$ such that $\|v_n\| = 1$ and $\|\mathcal{B}v_n - \mu v_n\| \to 0$ as $n \to \infty$. The sequence (v_n) is called an *approximate eigenvector* relative to the *approximate eigenvalue* μ. The topological boundary of the spectrum of $\mathcal{B}$ is always contained in $\sigma_{ap}(\mathcal{B})$ (see [7, Proposition IV.1.10]).

We write $\mathcal{B}_p$ to indicate a realization of a (differential) operator $\mathcal{B}$ in L^p, that is when $\mathcal{B}$ is provided with a specific domain in L^p. However, we sometimes omit the suffix p in the proofs, to shorten the notation.

If B is a matrix, B^T denotes its transpose. We set $\mathbb{C}_+ = \{\mu \in \mathbb{C} \,|\, \operatorname{Re}\mu > 0\}$ and $\mathbb{C}_- = \{\mu \in \mathbb{C} \,|\, \operatorname{Re}\mu < 0\}$. When $p = \infty$, by $1/p$ we mean 0.

Acknowledgement. The authors wish to thank Enrico Priola for helpful discussions during the preparation of the paper. D.P. is member of GNAMPA of the Italian Istituto Nazionale di Alta Matematica (INdAM) and has been partially supported by the PRIN project "Deterministic and stochastic evolution equations" of the Italian Ministry of Education MIUR and by Italian INFN.

2 Basic and known results

We collect background material from [20] and prove auxiliary results concerning the drift term and the Ornstein-Uhlenbeck operator.

2.1 Properties of $\mathcal{L}$

Let $B = (b_{ij}) \neq 0$ be a real $n \times n$ matrix and consider the drift operator

$$\mathcal{L} = \sum_{i,j=1}^n b_{ij} x_j D_i$$

defined on its maximal domain

$$D(\mathcal{L}_p) = \{u \in L^p \,|\, \mathcal{L}u \in L^p\}$$

in L^p for $1 \le p \le \infty$, where $\mathcal{L}u$ is understood in the sense of distributions. We write $\mathcal{L}_p$ for $(\mathcal{L}, D(\mathcal{L}_p))$ and recall the following results, whose proofs can be found in [20, Section 2].

Proposition 2.1. *Let $1 \le p \le \infty$. The operator $\mathcal{L}_p$ generates the C_0-group $(S(t))_{t\in\mathbb{R}}$ in L^p defined by*

$$(S(t)f)(x) = f(e^{tB}x), \tag{2.1}$$

the space C_c^∞ is a core of $\mathcal{L}_p$, and we have

$$\|S(t)f\|_p = e^{-\frac{t}{p}\operatorname{tr}(B)}\|f\|_p \tag{2.2}$$

for $f \in L^p$ and $t \in \mathbb{R}$.

We next describe the spectrum of $\mathcal{L}_p$ distinguishing several cases.

Theorem 2.2. *Let $1 \le p \le \infty$.*

(a) Let $\operatorname{tr}(B) \neq 0$. Then $\sigma(\mathcal{L}_p) = -\operatorname{tr}(B)/p + i\mathbb{R}$.

(b) Let $\operatorname{tr}(B) = 0$ and B be not similar to a diagonal matrix with purely imaginary eigenvalues. Then $\sigma(\mathcal{L}_p) = i\mathbb{R}$.

(c) Let B be similar to a diagonal matrix with nonzero eigenvalues $\pm i\sigma_1, \pm i\sigma_2, \dots, \pm i\sigma_k$ in $i\mathbb{R}$ and possibly 0, where $\sigma_r\sigma_s^{-1} \notin \mathbb{Q}$ for some $r, s \in \{1, \dots, k\}$. Then $\sigma(\mathcal{L}_p) = i\,\mathbb{R}$.

(d) Let B be similar to a diagonal matrix with nonzero eigenvalues $\pm i\sigma_1, \pm i\sigma_2, \dots, \pm i\sigma_k$ in $i\mathbb{R}$ and possibly 0, where $\sigma_r\sigma_s^{-1} \in \mathbb{Q}$ for all $r, s \in \{1, \dots, k\}$. Then $(S(t))_{t\in\mathbb{R}}$ is periodic and $\sigma(\mathcal{L}_p)$ is the discrete subgroup $G = \{i(n_1\sigma_1 + \cdots + n_k\sigma_k) \,|\, (n_1, \dots, n_k) \in \mathbb{Z}^k\}$.

In the sequel we need more information about case (d) above in which $(S(t))_{t\in\mathbb{R}}$ is periodic.

Proposition 2.3. *Let B be similar to a diagonal matrix with nonzero eigenvalues $\pm i\sigma_1, \pm i\sigma_2, \dots, \pm i\sigma_k$ in $i\mathbb{R}$ and possibly 0, with $2k \le n$. Assume that for every $j \in \{2, \dots, k\}$ we have $\sigma_j = \frac{p_j}{q_j}\sigma_1$ for some coprime integers p_j and q_j. Then $(S(t))$ is periodic with period $\tau = 2\pi N\sigma_1^{-1}$, where N is the least common multiple of $q_2, \dots, q_k$. Moreover, the set G from Theorem 2.2 is given by $G = \frac{\sigma_1}{N}i\mathbb{Z} = \frac{2\pi}{\tau}i\mathbb{Z}$.*

Proof. We denote a point in $\mathbb{R}^n$ by $x = (x_1, y_1, \dots, x_k, y_k, w_{2k+1}, \dots, w_n)$ and set $z_j = (x_j, y_j)$. Possibly after a change of variables we obtain

$$S(t)f(x) = f(e^{it\sigma_1}z_1, \dots, e^{it\sigma_k}z_k, w_{2k+1}, \dots, w_n), \tag{2.3}$$

see Theorem 2.6 of [20]. If $0 \notin \sigma(B)$, the components w_j are not present. Formula (2.3) yields $S(\tau)f = f$.

We prove that the set G defined in Theorem 2.2 (d) coincides with $\frac{\sigma_1}{N}i\mathbb{Z}$. The inclusion $\subseteq$ easily follows from the form of the numbers σ_j and the definition of N. To show the other inclusion, we first observe that the greatest common divisor of $N, Np_2/q_2, \dots, Np_k/q_k$ is equal to 1. Indeed, otherwise there would exist a prime number p dividing $N, \dots, Np_k/q_k$. Let $\alpha \in \mathbb{N}$ be the greatest exponent for which p^α divides N. Then p^α occurs in the prime factorization of some q_j. Since p_j and q_j are coprime, p cannot divide Np_j/q_j, and this is a contradiction. As a result, each integer m can be written as

$$m = m_1 N + m_2 \frac{Np_2}{q_2} + \cdots + m_k \frac{Np_k}{q_k},$$

for suitable $m_j \in \mathbb{Z}$. This is equivalent to saying that the element $\frac{\sigma_1}{N} m$ can be written as $m_1\sigma_1 + \cdots + m_k\sigma_k$ and concludes the proof. $\square$

2.2 Properties of $\mathcal{A}$

We turn our attention to the Ornstein-Uhlenbeck operator defined in (1.1) and to the associated semigroup $(T(t))_{t\geq 0}$ given by (1.3). We always assume the hypoellipticity condition (1.2) and $1 \leq p \leq \infty$. We do not need the full description of the domain of the generator, but only the fact that smooth functions with compact support are a core. We point out, however, that the domain has been described in [23, Section 4] and in [22] in the nondegenerate case and in [1] in the degenerate one.

Proposition 2.4. *The semigroup $(T(t))_{t\geq 0}$ is strongly continuous on L^p, $1 \leq p \leq \infty$, and satisfies the estimate*

$$\|T(t)\| \leq e^{-\frac{t}{p}\operatorname{tr}(B)} \tag{2.4}$$

for $t \geq 0$. Moreover, C_c^∞ is a core for the generator $\mathcal{A}_p$.

Proof. If the diffusion matrix Q is positive definite, the stated properties and a partial description of the domain of the generator have been proved in Section 3 of [20]. However, the same proofs hold in the degenerate hypoelliptic case. We only sketch them and refer to [20] for more details. To show (2.4), we write $T(t)f = S(t)(g_t * f)$ where

$$g_t(y) = \frac{1}{(4\pi)^{n/2}(\det Q_t)^{1/2}} e^{-\langle Q_t^{-1}y, y\rangle/4}$$

and $S(t)$ is defined in (2.1). The estimate (2.4) then follows from (2.2), Young's inequality for convolutions, and $\|g_t\|_1 = 1$. Since $T(t)f \to f$ in L^p as $t \to 0^+$ for $f \in C_c^\infty$, by density (2.4) implies the strong continuity of $(T(t))_{t\geq 0}$ in L^p for every $1 \leq p \leq \infty$.

Let $\mathcal{A}_p$ and $\mathcal{S}(\mathbb{R}^n)$ denote the generator of $(T(t))_{t\geq 0}$ in L^p and the Schwartz class, respectively. One easily checks that $\mathcal{S}(\mathbb{R}^n) \subseteq D(\mathcal{A}_p)$ and $\mathcal{A}_p f = \mathcal{A} f$ for every $f \in \mathcal{S}(\mathbb{R}^n)$. Moreover, $\mathcal{S}(\mathbb{R}^n)$ is dense in L^p and invariant for $(T(t))_{t\geq 0}$ by (1.3). Therefore it is a core of $\mathcal{A}_p$. By a truncation argument we conclude that C_c^∞ is a core for $\mathcal{A}_p$. $\square$

We recall Theorem 3.3 and Corollary 3.5 of [20].

Proposition 2.5. *The boundary spectrum of $\mathcal{A}_p$ contains the spectrum of the drift $\mathcal{L}_p$.*

Corollary 2.6. *The growth bound of $(T(t))_{t\geq 0}$ in L^p is $\omega_p = -\operatorname{tr}(B)/p$.*

Standard semigroup theory then yields first inclusions of the spectra.

Corollary 2.7. *The spectrum of $\mathcal{A}_p$ belongs to the half-plane $\{\mu \in \mathbb{C} \mid \operatorname{Re}\mu \leq -\operatorname{tr}(B)/p\}$, and that of $T(t)$ to the closed ball $\overline{B}(0, e^{-\frac{t}{p}\operatorname{tr}(B)})$.*

If $\sigma(\mathcal{L}_p)$ is the whole line $-\mathrm{tr}(B)/p + i\mathbb{R}$, the half-plane $\{\mu \in \mathbb{C} \,|\, \mathrm{Re}\,\mu \le -\mathrm{tr}(B)/p\}$ coincides with the sum $(-\infty, 0] + \sigma(\mathcal{L}_p)$. This is not the case if $\sigma(\mathcal{L}_p)$ is a discrete subgroup of $i\mathbb{R}$, which occurs when the group generated by the drift part is periodic (see Theorem 2.2 (d)). However, also in this case we have the inclusion $\sigma(\mathcal{A}_p) \subseteq (-\infty, 0] + \sigma(\mathcal{L}_p)$ as proved in the next proposition.

Proposition 2.8. *Let B be similar to a diagonal matrix with nonzero eigenvalues $\pm i\sigma_1, \cdots, \pm i\sigma_k$ in $i\mathbb{R}$ and possibly 0. Assume that the quotient $\sigma_r\sigma_s^{-1}$ is rational for all r and s. Then $\sigma(\mathcal{A}_p) \subseteq (-\infty, 0] + \sigma(\mathcal{L}_p)$.*

Proof. Let $\tau > 0$ be the period of $(S(t))_{t\in\mathbb{R}}$, see Proposition 2.3. Also e^{tB^T} is τ-periodic and in particular $e^{\tau B^T} = I$. By the representation formula (1.3) we have

$$(T(\tau)f)(x) = \frac{1}{(4\pi)^{n/2}(\det Q_\tau)^{1/2}} \int_{\mathbb{R}^n} e^{-\langle Q_\tau^{-1}y, y\rangle/4} f(x-y)\, dy,$$

showing that $T(\tau) = T_\tau(1)$ where $(T_\tau(t))_{t\ge 0}$ is the semigroup generated by the diffusion operator $A_\tau = \mathrm{Tr}(Q_\tau D^2)$, whose spectrum is $(-\infty, 0]$. Take $\mu = a + ib \in \sigma(\mathcal{A}_p)$. The spectral inclusion Theorem IV.3.6 of [7] yields that $e^{\tau(a+ib)}$ belongs to $\sigma(T(\tau)) = \sigma(T_\tau(1))$. Since $(T_\tau(t))_{t\ge 0}$ is analytic, from Corollary IV.3.12 of [7] we infer the identity $\sigma(T_\tau(1)) \setminus \{0\} = e^{\sigma(A_\tau)} = (0, 1]$. It follows that $a \le 0$ and $\tau b = 2m\pi$ for some $m \in \mathbb{Z}$ and hence $ib \in \frac{2\pi}{\tau} i\mathbb{Z} = G$, using also Proposition 2.3. Therefore $\sigma(\mathcal{A}_p)$ is contained in $(-\infty, 0] + G$. $\square$

The spectrum of the Ornstein-Uhlenbeck operators has been computed in [20, Section 4] if either $\sigma(B) \subset \mathbb{C}_-$ or $\sigma(B) \subset \mathbb{C}_+$. The proofs in that paper are written only in the uniformly elliptic case where Q is positive definite, but in the introduction of [20] it is pointed out that they also work only assuming the hypoellipticity condition (1.2).

To explain why this condition suffices, we recall that the spectrum of $\mathcal{A}_p$ is determined in [20] at first under the assumption $\sigma(B) \subset \mathbb{C}_+$ by exhibiting explicit eigenfunctions for the eigenvalues $\mu < -\mathrm{tr}(B)/p$. These are computed using the matrix

$$\widetilde{Q}_\infty = \int_0^\infty e^{-sB} Q e^{-sB^T}\, ds.$$

The above integral converges since the matrix semigroup $(e^{-sB})_{s\ge 0}$ is exponentially stable. Moreover $\widetilde{Q}_\infty$ is nondegenerate under condition (1.2). Since $\widetilde{Q}_\infty$, and not Q, enters all calculations, all results still hold in the hypoelliptic setting provided that $\sigma(B) \subset \mathbb{C}_+$, including the extreme cases $p = 1, \infty$.

The case $\sigma(B) \subset \mathbb{C}_-$ follows from the preceding one by a simple duality argument, which we describe now. The formal adjoint of $\mathcal{A}$ is given by

$$\mathcal{A}^* = \sum_{i,j=1}^n q_{ij} D_{ij} - \sum_{i,j=1}^n b_{ij} x_j D_i - \mathrm{tr}(B).$$

Let $\mathcal{A}^*_{p'}$ be the realization of $\mathcal{A}^*$ in $L^{p'}$, $1/p + 1/p' = 1$, as the generator of the semigroup (1.3) with $-B$ replacing B (also in the definition of Q_t) multiplied by the exponential factor $e^{-t\,\mathrm{tr}(B)}$. Notice that the spectrum of the drift matrix is now contained in $\mathbb{C}_+$. Therefore, for every $\mu \in \mathbb{C}$ with $\mathrm{Re}\,\mu < \mathrm{tr}(B)/p' - \mathrm{tr}(B) = -\mathrm{tr}(B)/p$ the operator $\mu - \mathcal{A}^*_{p'}$ is not injective. Let $({\mathcal{A}_p}', D({\mathcal{A}_p}'))$ denote the adjoint of $\mathcal{A}_p$ in $L^{p'}$. Recalling that C_c^∞ is a core for

$\mathcal{A}_{p'}^*$ and $\mathcal{A}_p$, it is easily seen that $D(\mathcal{A}_{p'}^*) \subseteq D(\mathcal{A}_p{}')$ and $\mathcal{A}_p{}' f = \mathcal{A}_{p'}^* f$ for every $f \in D(\mathcal{A}_{p'}^*)$. Since $\mu - \mathcal{A}_{p'}^*$ is not injective, it follows that $\mu - \mathcal{A}_p$ is not surjective and hence $\mu \in \sigma(\mathcal{A}_p)$. The other inclusion is provided by Corollary 2.7. Note that this works in the extreme cases $p = 1, \infty$ as well.

We state the results discussed above.

Theorem 2.9. *Let $1 \le p \le \infty$ and (1.2) be true. If either $\sigma(B) \subset \mathbb{C}_-$ or $\sigma(B) \subset \mathbb{C}_+$, then $\sigma(\mathcal{A}_p) = \{\mu \in \mathbb{C} \,|\, \operatorname{Re}\mu \le -\operatorname{tr}(B)/p\}$. In the latter case, every μ with $\operatorname{Re}\mu < -\operatorname{tr}(B)/p$ is an eigenvalue.*

3 Preliminary considerations

3.1 Non-existence of eigenvalues

In contrast to [20] we cannot use eigenvalues in the proof of our main result. To show this we rule out eigenvalues of $\mathcal{A}$ if the spectrum of B intersects $\mathbb{C}_-$ or if B is skew-symmetric and $Q = I$, where we assume that $p = 2$.

First case. We assume that some eigenvalue of B has a negative real part. Suppose that $\mu \in \mathbb{C}$ with $\operatorname{Re}\mu < -\frac{1}{2}\operatorname{tr}(B)$ was an eigenvalue of $\mathcal{A}_2$ with eigenfunction $f \in L^2 \setminus \{0\}$. The spectral mapping theorem for the point spectrum shows that $T(t)f = e^{\mu t} f$ for every $t \ge 0$, see Theorem IV.3.7 and Corollary IV.3.8 of [7]. Denoting by

$$\hat{f}(\xi) = \int_{R^n} f(x) e^{ix\cdot\xi}\, dx$$

the Fourier transform of f, the representation formula (1.3) implies that the equation $T(t)f = e^{\mu t} f$ is equivalent to

$$\hat{f}(e^{-tB^T}\xi) = e^{(\mu+\operatorname{tr}(B))t} e^{|Q_t^{1/2} e^{-tB^T}\xi|^2} \hat{f}(\xi), \tag{3.1}$$

where $t \ge 0$, see Section 4 of [20]. We compute

$$\begin{aligned} |Q_t^{1/2} e^{-tB^T}\xi|^2 = \langle Q_t e^{-tB^T}\xi, e^{-tB^T}\xi\rangle &= \int_0^t \langle e^{sB} Q e^{sB^T} e^{-tB^T}\xi, e^{-tB^T}\xi\rangle\, ds \\ &= \int_0^t |Q^{1/2} e^{(s-t)B^T}\xi|^2\, ds = \int_0^t |Q^{1/2} e^{-sB^T}\xi|^2 ds \end{aligned} \tag{3.2}$$

for $\xi \in \mathbb{R}^n$. Take $\lambda \in \sigma(B) = \sigma(B^T)$ with $\operatorname{Re}\lambda < 0$. Let P be the spectral projection of B^T corresponding to λ. Fix $\varepsilon > 0$ with $\operatorname{Re}\lambda + \varepsilon < 0$. Then there exists a constant $M > 0$ such that $\|e^{sB^T} P\| \le M e^{(\operatorname{Re}\lambda+\varepsilon)s}$ for every $s \ge 0$. Observe that also $-B$ satisfies (1.2), so that there is a constant $\nu > 0$ with

$$\int_0^1 |Q^{1/2} e^{-sB^T}\xi|^2 ds = \Big\langle \int_0^1 e^{-sB} Q e^{-sB^T}\xi\, ds, \xi \Big\rangle \ge \nu|\xi|^2.$$

Let $t \in [m, m+1)$ for some $m \in \mathbb{N}_0$. Inserting P in (3.2), it follows

$$\begin{aligned}
|Q_t^{1/2} e^{-tB^T}\xi|^2 &\geq \int_0^m |Q^{1/2} e^{-sB^T}\xi|^2 ds = \sum_{k=0}^{m-1} \int_0^1 |Q^{1/2} e^{-rB^T} e^{-kB^T}\xi|^2 dr \\
&\geq \sum_{k=0}^{m-1} \nu \|P\|^{-2} \, |Pe^{-kB^T}\xi|^2 \geq \sum_{k=0}^{m-1} \nu (M\|P\|)^{-2} e^{-2(\operatorname{Re}\lambda+\varepsilon)k} \, |P\xi|^2 \\
&\geq c e^{-2(\operatorname{Re}\lambda+\varepsilon)t} |P\xi|^2
\end{aligned}$$

for some constant $c > 0$.

Integrating (3.1) on $\mathbb{R}^n$, we derive

$$\begin{aligned}
\int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 \, d\xi &= e^{2t\,(\operatorname{Re}\mu + \frac{1}{2}\operatorname{tr}(B))} \int_{\mathbb{R}^n} e^{2|Q_t^{1/2} e^{-tB^T}\xi|^2} |\hat{f}(\xi)|^2 \, d\xi \\
&\geq \exp\left(2c\alpha^2 e^{-2(\operatorname{Re}\lambda+\varepsilon)t}\right) e^{2t\,(\operatorname{Re}\mu + \frac{1}{2}\operatorname{tr}(B))} \int_{\{|P\xi| \geq \alpha\}} |\hat{f}(\xi)|^2 \, d\xi,
\end{aligned}$$

for every $t \geq 0$ and $\alpha > 0$. Letting $t \to +\infty$, the right hand side blows up unless $\int_{\{|P\xi|\geq\alpha\}} |\hat{f}(\xi)|^2 \, d\xi = 0$. Since $\alpha > 0$ is arbitrary and the set $\{P\xi = 0\}$ has measure 0, this would imply $\hat{f} = 0$ and thus $f = 0$ in L^2, which is a contradiction.

Second case. We assume that $B = -B^T$ and $Q = I$. Noting $\operatorname{tr}(B) = 0$, we now suppose there was an eigenvalue μ of $\mathcal{A}_2$ with $\operatorname{Re}\mu < 0$. Arguing as before, we rewrite (3.1) as

$$e^{-|Q_t^{1/2} e^{-tB^T}\xi|^2} \hat{f}(e^{-tB^T}\xi) = e^{\mu t} \hat{f}(\xi), \quad t \geq 0,$$

and then integrate over $\mathbb{R}^n$ to obtain

$$\int_{\mathbb{R}^n} e^{-2|Q_t^{1/2}\xi|^2} |\hat{f}(\xi)|^2 \, d\xi = e^{2\operatorname{Re}\mu\, t} \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 d\xi.$$

Observing

$$|Q_t^{1/2}\xi|^2 = \langle Q_t\xi, \xi\rangle = \int_0^t \langle e^{sB} e^{sB^T}\xi, \xi\rangle \, ds = \int_0^t \langle e^{s(B+B^T)}\xi, \xi\rangle \, ds = t\,|\xi|^2,$$

we derive

$$\begin{aligned}
\int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 d\xi &= \int_{\mathbb{R}^n} e^{-2t(|\xi|^2 + \operatorname{Re}\mu)} |\hat{f}(\xi)|^2 \, d\xi \\
&= \int_{\{|\xi|^2 > -\operatorname{Re}\mu\}} e^{-2t(|\xi|^2 + \operatorname{Re}\mu)} |\hat{f}(\xi)|^2 \, d\xi + \int_{\{|\xi|^2 < -\operatorname{Re}\mu\}} e^{-2t(|\xi|^2 + \operatorname{Re}\mu)} |\hat{f}(\xi)|^2 \, d\xi.
\end{aligned}$$

The first integral in the last line tends to 0 as $t \to +\infty$ by dominated convergence. The second integral tends to $+\infty$ by monotone convergence, if $\int_{\{|\xi|^2 < -\operatorname{Re}\mu\}} |\hat{f}(\xi)|^2 \, d\xi > 0$. Therefore we have either $\|\hat{f}\|_2 = +\infty$ or $\|\hat{f}\|_2 = 0$, and we get a contradiction in any case.

By duality one deduces from the above examples that, if $\sigma(B)$ intersects both $\mathbb{C}_-$ and $\mathbb{C}_+$, a point λ can be in the spectrum of $\mathcal{A}_2$ even though $\lambda - \mathcal{A}_2$ is injective and has dense range. Approximate eigenvalues will thus play a central role.

3.2 Basic transformations

In order to describe the spectrum of $\mathcal{A}$, we will reduce the drift matrix B to a quasi triangular upper matrix. This is done as follows. If M is an invertible real $n \times n$ matrix, we define the change of variables

$$\Phi_M : L^p \to L^p, \qquad (\Phi_M u)(y) = u(M^{-1}y). \tag{3.3}$$

Setting $v = \Phi_M u$, one easily calculates that $\mathcal{A}u(x) = \mathcal{A}_0 v(Mx)$ for $x \in \mathbb{R}^n$, where

$$\mathcal{A}_0 v = \mathrm{Tr}(Q_0 D^2 v) + \langle B_0 y, Dv \rangle$$

with $y = Mx$, $Q_0 = MQM^T$, and $B_0 = MBM^{-1}$. We conclude

$$\mathcal{A} = \Phi_M^{-1} \mathcal{A}_0 \Phi_M \qquad \text{with} \;\; D(\mathcal{A}_p) = \Phi_M^{-1} D(\mathcal{A}_{0,p}).$$

We observe that the new operator $\mathcal{A}_0$ is still hypoelliptic, see (1.2), and that the spectrum is invariant under this transformation.

Applying Schur's theorem for real matrices (see e.g. Theorem 2.3.4 in [14]), we can now choose a real orthogonal matrix M such that $MBM^{-1} = T$ with

$$T = \begin{pmatrix} B_1 & * & * & \cdots & * \\ 0 & B_2 & * & \cdots & * \\ 0 & 0 & B_3 & * & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & B_l \end{pmatrix} \tag{3.4}$$

where each B_j is a real 1×1 matrix with a real eigenvalue of B, or a real 2×2 matrix with a pair of nonreal complex conjugate eigenvalues $\alpha_j \pm i\beta_j$. The diagonal blocks B_j may be arranged in any prescribed order. By * we denote an arbitrary block.

4 The spectrum of $\mathcal{A}_p$

The spectrum of the Ornstein-Uhlenbeck operators $\mathcal{A}_p$ depends on the spectrum of the drift operator $\mathcal{L}_p$ which in turn is determined by B. If B is similar to a diagonal matrix with nonzero eigenvalues $\pm i\sigma_1, \pm i\sigma_2, \dots, \pm i\sigma_k$ in $i\mathbb{R}$ and possibly 0 and if all ratios $\sigma_r \sigma_s^{-1}$ belong to $\mathbb{Q}$, then Theroem 2.2 shows that $\sigma(\mathcal{L}_p)$ is a discrete subgroup $G = \frac{2\pi i}{\tau}\mathbb{Z}$ of $i\mathbb{R}$, where $S(\tau) = I$. In this case we prove that $\sigma(\mathcal{A}_p) = (-\infty, 0] + G$. In all the remaining cases, the spectrum of $\mathcal{L}_p$ is the vertical line $-\mathrm{tr}(B)/p + i\mathbb{R}$ and we show that $\sigma(\mathcal{A}_p)$ is the half-plane $\{\mu \in \mathbb{C} \,|\, \mathrm{Re}\,\mu \le -\mathrm{tr}(B)/p\}$. These results are the main achievement of the paper and have been stated in Theorem 1.1, which we reformulate here with an additional result.

Theorem 4.1. *Let* (1.2) *be true and* $p \in [1, \infty]$. *Then the spectrum of* $\mathcal{A}_p$ *is given by*

$$\sigma(\mathcal{A}_p) = (-\infty, 0] + \sigma(\mathcal{L}_p). \tag{4.1}$$

In particular, we have either $\sigma(\mathcal{A}_p) = (-\infty, 0] + G$ *or* $\sigma(\mathcal{A}_p) = \{\mu \in \mathbb{C} \,|\, \mathrm{Re}\,\mu \le -\mathrm{tr}(B)/p\}$, *according to* $\sigma(\mathcal{L}_p)$ *being the discrete subgroup* $G = \frac{2\pi i}{\tau}\mathbb{Z}$ *of* $i\mathbb{R}$ *or the whole line* $-\mathrm{tr}(B)/p + i\mathbb{R}$. *Moreover, the semigroup* $(T(t))_{t \ge 0}$ *satisfies the weak spectral mapping theorem*

$$\sigma(T(t)) = \overline{\exp(t\sigma(\mathcal{A}_p))}, \qquad t \ge 0. \tag{4.2}$$

We even have $\sigma(T(t)) \setminus \{0\} = e^{t\sigma(\mathcal{A}_p)}$ *except for the case that* $\sigma(\mathcal{L}_p) = G = \frac{2\pi i}{\tau}\mathbb{Z}$ *and* t/τ *is irrational.*

Proof. Theorem 2.9 shows the equality (4.1) if $\sigma(B) \subset \mathbb{C}_-$ or $\sigma(B) \subset \mathbb{C}_+$. Moreover, by Corollary 2.7 and Proposition 2.8 the inclusion $\sigma(\mathcal{A}_p) \subseteq (-\infty, 0] + \sigma(\mathcal{L}_p)$ always holds. Therefore we only have to prove the other inclusion $(-\infty, 0] + \sigma(\mathcal{L}_p) \subseteq \sigma(\mathcal{A}_p)$ in two remaining cases: one eigenvalue of B has a positive real part or one eigenvalue of B lies on the imaginary axis. Note that these cases may overlap and that the first one includes situations covered by Theorem 2.9. The inclusion $(-\infty, 0] + \sigma(\mathcal{L}_p) \subseteq \sigma(\mathcal{A}_p)$ is established in these two cases in the following two subsections.

To prove the (weak) spectral mapping theorem, we take (4.1) for granted. Let $t > 0$. The spectral inclusion Theorem IV.3.6 of [7] and Corollary 2.7 show that

$$e^{t\sigma(\mathcal{A}_p)} \subseteq \sigma(T(t)) \setminus \{0\} \subseteq \{\mu \in \mathbb{C} \,|\, 0 < |\mu| \leq e^{-t\,\mathrm{tr}(B)/p}\} =: B_t.$$

We thus even obtain $\sigma(T(t)) \setminus \{0\} = e^{t\sigma(\mathcal{A}_p)}$ if $\sigma(\mathcal{A}_p) = \{\mu \in \mathbb{C} \,|\, \mathrm{Re}\,\mu \leq -\mathrm{tr}(B)/p\}$. In the other case Proposition 2.3 yields $\sigma(\mathcal{L}_p) = G = \frac{2\pi i}{\tau}\mathbb{Z}$. Now $\mathrm{tr}(B) = 0$. By (4.1) we can now write

$$e^{t\sigma(\mathcal{A}_p)} = \{e^{ta}e^{2\pi imt/\tau} \,|\, a \leq 0,\ m \in \mathbb{Z}\} = (0, 1] \cdot S_t,$$

with $S_t = \{e^{2\pi imt/\tau} \,|\, m \in \mathbb{Z}\}$. There are two subcases.

First, let t/τ be irrational. Then the set S_t is dense in the unit circle and it follows that $\overline{\exp(t\sigma(\mathcal{A}_p))}$ is equal to $\overline{B_0}$; i.e., (4.2) is true.

Second, let $t/\tau = j/k$ for coprime $j, k \in \mathbb{N}$. Then S_t coincides with the set Γ_k of kth unit roots so that $e^{t\sigma(\mathcal{A}_p)} = (0, 1] \cdot \Gamma_k$ On the other hand, we have $S(t)^k = S(j\tau) = I$. As in the proof of Proposition 2.8, we deduce that $T(t)^k = T(kt) = T_{kt}(1)$ for the analytic semigroup $(T_{kt}(s))_{s\geq 0}$ generated by $\mathrm{Tr}(Q_{kt}D^2)$. The spectrum of $T_{kt}(1)$ is thus equal to $[0, 1]$ and hence $\sigma(T(t)) \setminus \{0\} = (0, 1] \cdot \Gamma_k$ as required. □

4.1 The case $\sigma(B) \cap \mathbb{C}_+ \neq \emptyset$

We show the remaining inclusion in the proof of Theorem 4.1 in the first case.

Proposition 4.2. *Let $\sigma(B) \cap \mathbb{C}_+ \neq \emptyset$. Then $(-\infty, 0] + \sigma(\mathcal{L}_p) \subseteq \sigma(\mathcal{A}_p)$.*

In the proof we use degenerate Ornstein-Uhlenbeck operators depending on different sets of variables, as we explain now. We let $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^m$ and write a point $z \in \mathbb{R}^n$ accordingly as $z = (x, y)$. Let B_1 and B_2 be real $k \times k$ and $m \times m$ matrices, respectively, and Q_2 a real, symmetric and positive semidefinite $m \times m$ matrix. We consider the operators

$$\mathcal{L}^{(1)} = \langle B_1 x, D_x\rangle \quad \text{and} \quad \mathcal{A}^{(2)} = \mathrm{Tr}(Q_2 D^2) + \langle B_2 y, D_y\rangle. \tag{4.3}$$

Here $\mathcal{L}^{(1)}$ is a drift operator on $L^p(\mathbb{R}^k)$ and $\mathcal{A}^{(2)}$ is an Ornstein-Uhlenbeck operator on $L^p(\mathbb{R}^m)$, which is assumed to be hypoelliptic (recall that L^∞ means C_0). Let $(S_1(t))_{t\geq 0}$ and $(T_2(t))_{t\geq 0}$ be the semigroups generated by $\mathcal{L}^{(1)}$ and $\mathcal{A}^{(2)}$ respectively. Then $(S_1(t) \otimes T_2(t))_{t\geq 0}$ acting on $L^p(\mathbb{R}^k) \otimes L^p(\mathbb{R}^m)$ can be extended to a C_0-semigroup on $L^p(\mathbb{R}^n)$, whose generator is the L^p-closure $\mathcal{C}_p$ of $\mathcal{C} = \mathcal{L}^{(1)} + \mathcal{A}^{(2)}$ initially defined on $D(\mathcal{L}_p^{(1)}) \otimes D(\mathcal{A}_p^{(2)})$. Since $C_c^\infty(\mathbb{R}^k)$ and $C_c^\infty(\mathbb{R}^m)$ are cores for $\mathcal{L}^{(1)}$ and $\mathcal{A}^{(2)}$, respectively, it follows that $C_c^\infty(\mathbb{R}^n)$ is a core for $\mathcal{C}_p$.

Proof of Proposition 4.2. Let $\lambda = \alpha + i\beta \in \sigma(B)$ with $\alpha > 0$. As explained at the end of Section 3, using a change of variables we can assume that our operator is given by

$$\mathcal{A} = \mathrm{Tr}(Q_0 D^2) + \langle Tx, D\rangle,$$

where T is in the quasi triangular form (3.4), its last block B_l corresponds to λ, and Q_0 is the transformed diffusion matrix. We distinguish between the cases $\beta = 0$ and $\beta \neq 0$. (Below we tacitly assume that $T \neq B_l$ since the easier case $T = B_l$ can be treated analogously.)

Case 1. $\boldsymbol{\beta = 0}$. Denote the nonreal eigenvalues of B by $\{\alpha_1 \pm i\beta_1, \ldots, \alpha_k \pm i\beta_k\}$ with $0 \leq 2k < n$ and the real ones by $\{\lambda_{2k+1}, \ldots, \lambda_n\}$ with $\lambda_n = \alpha > 0$. We write a point in $\mathbb{R}^n$ as $x = (x_1, y_1, \ldots, x_k, y_k, w_{2k+1}, \ldots, w_n)$. We use a scaling argument in which the variables $z_j = (x_j, y_j)$ relative to conjugated eigenvalues are coupled and which leaves the last variable unscaled. Let $D_{z_j} = (D_{x_j}, D_{y_j})$ for $j = 1, \ldots, k$. The scaling operator is defined by

$$I_r u(x) = u\left(\frac{z_1}{r^{\gamma_1}}, \frac{z_2}{r^{\gamma_2}}, \ldots, \frac{z_k}{r^{\gamma_k}}, \frac{w_{2k+1}}{r^{\gamma_{2k+1}}}, \ldots, \frac{w_{n-1}}{r^{\gamma_{n-1}}}, w_n\right)$$

for $r > 0$ and with $\gamma_1 = 1$ and $\gamma_i > \gamma_j > 0$ for $i < j$. Observe that $\|I_r^{-1}\| = \|I_r\|^{-1}$ on L^p. Let $u \in C_c^\infty$. Computing $I_r^{-1}\mathcal{A}I_r u$, one finds that

$$\lim_{r\to+\infty} I_r^{-1}\mathcal{A}I_r u = \mathcal{C}u \qquad \text{in} \quad L^p,$$

for the limit operator

$$\mathcal{C}u = \nu D_{w_n}^2 u + \lambda_n w_n D_{w_n} u + \sum_{j=1}^{k}\langle B_j z_j, D_{z_j} u\rangle + \sum_{j=2k+1}^{n-1} \lambda_j w_j D_{w_j} u.$$

The constant ν is the component $\langle Q_0 e_n, e_n\rangle$ of Q_0 where $e_n = (0, \ldots, 0, 1)$. It is positive, which can be explained as follows. The last row vector in the matrix T is $\lambda_n e_n$. This means that the transpose of T maps e_n to $\lambda_n e_n$. Let X be the one-dimensional subspace spanned by e_n. It is invariant for the transpose of T. Since $\mathcal{A}$ is hypoelliptic, X is not contained in the kernel of Q_0. It follows $Q_0 e_n \neq 0$ and hence $\nu = |Q_0^{1/2} e_n|^2 > 0$.

Note that we can write $\mathcal{C} = \mathcal{L}^{\lim} + \mathcal{A}^{\lim}$ with

$$\mathcal{A}^{\lim} = \nu D_n^2 + \lambda_n w_n D_n, \qquad \mathcal{L}^{\lim} = \sum_{j=1}^{k}\langle B_j z_j, D_{z_j}\rangle + \sum_{j=2k+1}^{n-1} \lambda_j w_j D_{w_j}.$$

We endow $\mathcal{C}$ with the domain described before the proof and call it $\mathcal{C}_p$. We first establish a crucial spectral property of $\mathcal{C}_p$.

Claim. *Every μ with $\operatorname{Re}\mu < -\operatorname{tr}(B)/p$ is an approximate eigenvalue for $\mathcal{C}_p$.*

Since $\lambda_n > 0$, every λ with $\operatorname{Re}\lambda < -\lambda_n/p$ is an eigenvalue of the one dimensional operator $\mathcal{A}_p^{\lim}$ by Theorem 2.9. Theorem 2.2 shows that $\mathcal{L}_p^{\lim}$ possesses the approximate eigenvalue $-c/p$, where

$$c = 2\sum_{j=1}^{k}\alpha_j + \sum_{i=2k+1}^{n-1}\lambda_i = \operatorname{tr}(B) - \lambda_n.$$

Now, fix $\mu \in \mathbb{C}$ with $\operatorname{Re}\mu < -\operatorname{tr}(B)/p$ and set $\lambda = \mu + c/p$. Note that $\operatorname{Re}\lambda < -\lambda_n/p$. Choose an eigenfunction $u_1 = u_1(w_n)$ of $\mathcal{A}_p^{\lim}$ for λ with $\|u_1\|_{L^p(\mathbb{R})} = 1$. Given $\varepsilon > 0$, there is a function $u_2 = u_2(z_1, \ldots, z_k, w_{2k+1}, \ldots, w_{n-1})$ in $D(\mathcal{L}_p^{\lim})$ with $\|u_2\|_{L^p(\mathbb{R}^{n-1})} = 1$ and $\|\mathcal{L}^{\lim}u_2 + \frac{c}{p}u_2\|_{L^p(\mathbb{R}^{n-1})} \leq \varepsilon$. The function $u = u_1 u_2$ thus belongs to $D(\mathcal{C}_p)$, has norm one in L^p and satisfies

$$\mathcal{C}u - \mu u = \big(\mathcal{A}^{\lim}u_1 - \lambda u_1\big)u_2 + \big(\mathcal{L}^{\lim}u_2 + \tfrac{c}{p}u_2\big)u_1 = \big(\mathcal{L}^{\lim}u_2 + \tfrac{c}{p}u_2\big)u_1\,, \qquad (4.4)$$

which yields $\|\mathcal{C}u - \mu u\|_p \le \varepsilon$. So the claim is proved.

Take $\lambda_0 \in \rho(\mathcal{A}_p)$. By similarity, λ_0 belongs to $\rho(I_r^{-1}\mathcal{A}I_r)$ with resolvent $R(\lambda_0, I_r^{-1}\mathcal{A}_pI_r) = I_r^{-1}R(\lambda_0,\mathcal{A}_p)I_r$ for all $r>0$. It follows $\|R(\lambda_0, I_r^{-1}\mathcal{A}_pI_r)\| \le \|R(\lambda_0,\mathcal{A}_p)\| =: C$ and

$$\|u\|_p = \|R(\lambda_0, I_r^{-1}\mathcal{A}_pI_r)(\lambda_0 - I_r^{-1}\mathcal{A}I_r)u\|_p \le C\,\|(\lambda_0 - I_r^{-1}\mathcal{A}I_r)u\|_p \tag{4.5}$$

for $u \in C_c^\infty$. Letting $r \to +\infty$, we infer $\|u\|_p \le C\,\|(\lambda_0 - \mathcal{C})u\|_p$. Since C_c^∞ is a core for $\mathcal{C}_p$, this shows that λ_0 cannot be an approximate eigenvalue of $\mathcal{C}_p$, and hence $\operatorname{Re}\lambda_0 \ge -\operatorname{tr}(B)/p$ by the claim. This means that $(-\infty,0] + \sigma(\mathcal{L}_p)$ is contained in $\sigma(\mathcal{A}_p)$.

Case 2. $\boldsymbol{\beta \neq 0}$. We rearrange the blocks in (3.4) such that the first blocks contain the real eigenvalues $\lambda_1,\dots,\lambda_k$ with $0 \le k < n$ and the other blocks contain the complex ones $\alpha_{k+1} \pm i\beta_{k+1},\dots,\alpha_{k+m} \pm i\beta_{k+m}$ for $n = k+2m$, where $\alpha_{k+m} = \alpha > 0$ and $\beta_{k+m} = \beta \neq 0$. As a consequence, a point of $\mathbb{R}^n$ is denoted by $x = (w_1, w_2, \dots, w_k, z_{k+1}, \dots, z_{k+m})$ with $z_j = (x_j, y_j)$. The scaling operator is now defined by

$$J_r u(x) = u\left(\frac{w_1}{r^{\gamma_1}}, \frac{w_2}{r^{\gamma_2}}, \dots, \frac{w_k}{r^{\gamma_k}}, \frac{z_{k+1}}{r^{\gamma_{k+1}}}, \dots, \frac{z_{k+m-1}}{r^{\gamma_{k+m-1}}}, z_{k+m}\right) \tag{4.6}$$

with $\gamma_1 = 1$ and $\gamma_i > \gamma_j > 0$ for $i < j$. For every $u \in C_c^\infty$ we have

$$\lim_{r\to+\infty} J_r^{-1}\mathcal{A}J_r u = \mathcal{C}u \qquad \text{in } L^p,$$

where the limit operator is given by

$$\begin{aligned}\mathcal{C}u = {}& \operatorname{Tr}(Q_0^\dagger D_{k+m}^2 u) + \langle B_{k+m} z_{k+m}, D_{z_{k+m}} u\rangle \\ & + \sum_{j=1}^{k} \lambda_j w_j D_{w_j} u + \sum_{j=1}^{m-1} \langle B_{k+j} z_{k+j}, D_{z_{k+j}} u\rangle.\end{aligned} \tag{4.7}$$

Here $Q_0^\dagger$ is the lower right 2×2 submatrix of Q_0.

As before we introduce $\mathcal{C}_p$ and claim that the open half-plane $\{\mu \in \mathbb{C} \,|\, \operatorname{Re}\mu < -\operatorname{tr}(B)/p\}$ is contained in its approximate spectrum.

To prove the claim, we split $\mathcal{C}$ as the sum $\mathcal{A}^{\lim} + \mathcal{L}^{\lim}$ for the Ornstein-Uhlenbeck operator

$$\mathcal{A}^{\lim} = \operatorname{Tr}(Q_0^\dagger D_{k+m}^2) + \langle B_{k+m} z_{k+m}, D_{z_{k+m}}\rangle$$

in the last two variables and the drift operator

$$\mathcal{L}^{\lim} = \sum_{j=1}^{k} \lambda_j w_j D_{w_j} + \sum_{j=1}^{m-1} \langle B_{k+j} z_{k+j}, D_{z_{k+j}}\rangle$$

acting in the remaining variables.

We show that $\mathcal{A}^{\lim}$ is hypoelliptic by verifying (1.2). Let Y be a real subspace of $\mathbb{R}^2$ which is invariant for B_{k+m}^T. Suppose that $\dim Y = 1$. Then there would exist a real eigenvalue for B_{k+m}^T, but this is not the case as $\sigma(B_{k+m}^T) = \{\alpha \pm i\beta\}$. We thus have either $Y = \{0\}$ or $Y = \mathbb{R}^2$. Suppose that $\mathbb{R}^2 \subseteq \ker(Q_0^\dagger)$. In this case the real subspace of $\mathbb{R}^n$ spanned by $e_{n-1} = (0,\dots,0,1,0)$ and $e_n = (0,\dots,0,1)$ would be invariant for the transpose of the drift matrix T and it would be contained in $\ker(Q_0)$. This contradicts the hypoellipticity of $\mathcal{A}$.

Since $\alpha_{k+m} > 0$, by Theorem 2.9 every λ with $\operatorname{Re}\lambda < -2\alpha_{k+m}/p$ is an eigenvalue of $\mathcal{A}_p^{\lim}$. Moreover, Theorem 2.2 yields that $\sigma_{ap}(\mathcal{L}_p^{\lim})$ contains the number $-c/p$ with

$$c = \sum_{j=1}^{k} \lambda_j + \sum_{j=1}^{m-1} 2\alpha_{j+k}.$$

Take $\mu \in \mathbb{C}$ with $\operatorname{Re}\mu < -\operatorname{tr}(B)/p$ and set $\lambda = \mu + c/p$. Take an eigenfunction $u_1 = u_1(x_{k+m}, y_{k+m})$ of $\mathcal{A}_p^{\lim}$ for λ with $\|u_1\|_{L^p(\mathbb{R}^2)} = 1$. Given $\varepsilon > 0$, we have a function $u_2 = u_2(w_1, \ldots, w_k, z_{k+1}, \ldots, z_{k+m-1})$ in $D(\mathcal{L}_p^{\lim})$ satisfying $\|u_2\|_{L^p(\mathbb{R}^{n-2})} = 1$ and $\|\mathcal{L}^{\lim}u_2 + \frac{c}{p}u_2\|_{L^p(\mathbb{R}^{n-2})} \leq \varepsilon$. As in (4.4), $u = u_1 u_2$ is an approximate eigenfunction for $\mathcal{C}_p$ with approximate eigenvalue μ. We can then proceed as at the end of Case 1, see (4.5). □

4.2 The case $\sigma(B) \cap i\mathbb{R} \neq \emptyset$

To deal with imaginary eigenvalues of B, we need a second type of transformation. We introduce an isometry $S : L^p \to L^p$ by

$$Su(x) = e^{is\phi(x)}u(x), \tag{4.8}$$

where $s \in \mathbb{R}$ and the function $\phi : \mathbb{R}^n \to \mathbb{R}$ is chosen below. For $u \in C_c^\infty$, say, the operator $\mathcal{A}$ given by (1.1) is transformed into

$$S^{-1}\mathcal{A}Su(x) = \mathcal{A}u - s^2\langle QD\phi, D\phi\rangle u + 2is\langle QD\phi, Du\rangle + is\operatorname{Tr}(QD^2\phi)u + is\langle Bx, D\phi\rangle u. \tag{4.9}$$

Let $\lambda \in \sigma(B) \cap i\mathbb{R}$. Then we have either $\lambda = 0$ or $\lambda = i\beta \neq 0$. The next two propositions show the spectral inclusion needed for Theorem 4.1 separately for these two cases.

Proposition 4.3. *Let $0 \in \sigma(B)$. Then $(-\infty, 0] + \sigma(\mathcal{L}_p) \subseteq \sigma(\mathcal{A}_p)$.*

Proof. In the proof we write $\mathcal{A}$ for $\mathcal{A}_p$ and similarly for the other operators involved. Observe that the kernel of B^T is a nontrivial subspace which is invariant for B^T. Condition (1.2) thus yields a vector $\xi \in \ker(B^T)$ with $Q\xi \neq 0$. Then $\langle Q\xi, \xi\rangle = |Q^{1/2}\xi|^2 \neq 0$. We set $\phi(x) = \xi \cdot x$ for $x \in \mathbb{R}^n$. Identity (4.9) then becomes

$$S^{-1}\mathcal{A}Su(x) = \mathcal{A}u - s^2\langle Q\xi, \xi\rangle u + 2is\langle Q\xi, Du\rangle + is\langle x, B^T\xi\rangle u = \tilde{\mathcal{A}}u - s^2\langle Q\xi, \xi\rangle u \tag{4.10}$$

where we have defined $\tilde{\mathcal{A}}u = \mathcal{A}u + 2is\langle Q\xi, Du\rangle$ and used $B^T\xi = 0$. Let $k \in \mathbb{N}$ and the isometry $V_k : L^p \to L^p$ be given by

$$V_k u(x) = k^{-n/p}u(k^{-1}x). \tag{4.11}$$

For $u \in C_c^\infty$, we compute

$$V_k^{-1}\tilde{\mathcal{A}}V_k u = k^{-2}\operatorname{Tr}(QD^2u) + \langle Bx, Du\rangle + k^{-1}2is\langle Q\xi, Du\rangle \;\longrightarrow\; \mathcal{L}u = \langle Bx, Du\rangle,$$

as $k \to +\infty$. Set $\tilde{\mathcal{A}}_k = V_k^{-1}\tilde{\mathcal{A}}V_k$. Then $\rho(\tilde{\mathcal{A}}_k) = \rho(\tilde{\mathcal{A}})$ by similarity, where we omit the subscript p. We want to show the inclusion $\sigma(\mathcal{L}) \subseteq \sigma(\tilde{\mathcal{A}})$, for which we need the next fact.

Claim. Let $\lambda \in \rho(\tilde{\mathcal{A}}) \cap \rho(\mathcal{L})$ and $f \in L^p$. We then obtain

$$R(\lambda, \tilde{\mathcal{A}}_k)f \to R(\lambda, \mathcal{L})f \quad \text{in } L^p \qquad \text{as } k \to \infty. \tag{4.12}$$

Since C_c^∞ is a core of $(\mathcal{L}, D(\mathcal{L}_p))$ by Proposition 2.1, it suffices to prove the convergence on the dense subspace $(\lambda - \mathcal{L})C_c^\infty$. Let $f = \lambda u - \mathcal{L}u$ for some $u \in C_c^\infty$. Using the identity

$$R(\lambda, \tilde{\mathcal{A}}_k)f - R(\lambda, \mathcal{L})f = R(\lambda, \tilde{\mathcal{A}}_k)(\mathcal{L} - \tilde{\mathcal{A}}_k)R(\lambda, \mathcal{L})f$$

we deduce

$$\|R(\lambda, \tilde{\mathcal{A}}_k)f - R(\lambda, \mathcal{L})f\|_p \le \|R(\lambda, \tilde{\mathcal{A}}_k)\| \, \|\mathcal{L}u - \tilde{\mathcal{A}}_k u\|_p \le \|R(\lambda, \tilde{\mathcal{A}})\| \, \|\mathcal{L}u - \tilde{\mathcal{A}}_k u\|_p$$

and the claim follows.

Now, let $\lambda_0 \in \sigma(\mathcal{L})$. Suppose that $\lambda_0 \in \rho(\tilde{\mathcal{A}})$. Then there exists a radius $r > 0$ such that $\lambda \in \rho(\tilde{\mathcal{A}}) = \rho(\tilde{\mathcal{A}}_k)$ whenever $|\lambda - \lambda_0| < r$. Take λ with $|\lambda - \lambda_0| < r$ and $\operatorname{Re}\lambda > \operatorname{Re}\lambda_0$. Then λ also belongs to $\rho(\mathcal{L})$ by Theorem 2.2. The formula (4.12) thus yields

$$\|R(\lambda, \mathcal{L})f\|_p \le \liminf_{k\to\infty} \|R(\lambda, \tilde{\mathcal{A}}_k)f\|_p \le \|R(\lambda, \tilde{\mathcal{A}})\| \, \|f\|_p.$$

for every $f \in L^p$. In the limit $\lambda \to \lambda_0$ the left-hand side blows up, whereas the right-hand side remains bounded. By this contradiction, λ_0 belongs to $\sigma(\tilde{\mathcal{A}})$ and consequently $\lambda_0 - s^2\langle Q\xi, \xi\rangle$ to $\sigma(\mathcal{A})$, see (4.10). As $s \in \mathbb{R}$ is arbitrary, we conclude that $\sigma(\mathcal{L}) + (-\infty, 0] \subseteq \sigma(\mathcal{A})$. $\square$

We next treat the remaining case.

Proposition 4.4. *Let $\beta \neq 0$ and $i\beta \in \sigma(B)$, then $(-\infty, 0] + \sigma(\mathcal{L}_p) \subseteq \sigma(\mathcal{A}_p)$.*

In order to show this proposition, we proceed as in Case 2 of the proof of Proposition 4.2 obtaining the same limit operator $\mathcal{C} = \mathcal{A}^{\lim} + \mathcal{L}^{\lim}$, see (4.7). But now we cannot use Theorem 2.9 to determine the spectrum of $\mathcal{A}^{\lim}$, since the 2×2 drift matrix of $\mathcal{A}^{\lim}$ has the purely imaginary eigenvalues $\pm i\beta$. Instead we directly compute the spectrum of $\mathcal{A}^{\lim}$. We start with a first-order operator that will appear in a scaling limit.

Lemma 4.5. *Let $b, s, \mu_1, \mu_2 \in \mathbb{R}$ and set $\mathcal{T}_\infty u = bx_2D_1u - bx_1D_2u - s^2\big(\mu_1\frac{x_1^2}{|x|^2} + \mu_2\frac{x_2^2}{|x|^2}\big)u$. Let $\mathcal{T}_{\infty,p}$ be the realization of $\mathcal{T}_\infty$ in $L^p(\mathbb{R}^2)$ endowed with domain $D(\mathcal{T}_\infty) = \{u \in L^p(\mathbb{R}^2) \,|\, \mathcal{T}_\infty u \in L^p(\mathbb{R}^2)\}$, where $\mathcal{T}_\infty u$ is understood in the sense of distributions. Then, for every $m \in \mathbb{Z}$, the number $imb - s^2(\mu_1+\mu_2)/2$ is an eigenvalue of $\mathcal{T}_{\infty,p}$ possessing an eigenfunction u in $C_c^\infty(\mathbb{R}^2 \setminus \{0\})$.*

Proof. In polar coordinates (ρ, θ), our operator is expressed by

$$\mathcal{T}_\infty u = -b\partial_\theta u - s^2(\mu_1 \cos^2\theta + \mu_2 \sin^2\theta)u.$$

Let $\varphi \in C_c^\infty(0,\infty)$ and $m \in \mathbb{Z}$. Set $u(x) = \varphi(|x|)e^{im\theta}e^{-s^2(\mu_1-\mu_2)\sin(2\theta)/(4b)}$ for $x \in \mathbb{R}^2$. Then u belongs to $C_c^\infty(\mathbb{R}^2 \setminus \{0\})$ and straightforward computations show that $\mathcal{T}_\infty u = \lambda u$ with $\lambda = imb - s^2(\mu_1+\mu_2)/2$. $\square$

Lemma 4.6. *Let $\mathcal{A}^\diamond$ be a hypoelliptic Ornstein-Uhlenbeck operator on $\mathbb{R}^2$ whose drift matrix $B^\diamond$ has the eigenvalues $\pm i\beta$ for $\beta \in \mathbb{R}\setminus\{0\}$. Then $(-\infty, 0] + i\beta\mathbb{Z} = \sigma_{ap}(\mathcal{A}_p^\diamond) = \sigma(\mathcal{A}_p^\diamond)$.*

Proof. We divide the proof in four steps.
1) Put $\mathcal{A}^\diamond$ in a canonical form. Let μ_1 and μ_2 be the two nonnegative eigenvalues of the diffusion matrix $Q^\diamond$ of $A^\diamond$. There is an invertible matrix $M_1 \in \mathbb{R}^{2\times 2}$ such that

$$M_1 B^\diamond M_1^{-1} = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}.$$

Then take an orthogonal $U_2 \in \mathbb{R}^{2\times 2}$ such that $U_2(M_1 Q^\diamondsuit M_1^T)U_2^T = D$ for the diagonal matrix D with diagonal elements μ_1 and μ_2. Since U_2 is 2×2 orthogonal, we obtain

$$U_2 \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix} U_2^T = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} =: B^\circ,$$

where $b = \pm\beta$. The change of variables (3.3) with $M = U_2 M_1$ thus yields

$$\mathcal{A}^\circ = \Phi_M \mathcal{A}^\diamondsuit \Phi_M^{-1} = \mu_1 D_{11} u + \mu_2 D_{22} u + b x_2 D_1 u - b x_1 D_2 u.$$

with $D(\mathcal{A}_p^\circ) = \Phi_M D(\mathcal{A}_p^\diamondsuit)$.

We observe that there are two possible cases: either μ_1 and μ_2 are both positive, or one of them is positive and the other one zero. In the first case $\mathcal{A}^\circ$ is a nondegenerate Ornstein-Uhlenbeck operator, in the second one it is a degenerate hypoelliptic operator.

2) Scale $\mathcal{A}^\circ$ by the isometry (4.8). We now set $\phi(x) = |x|$ for all $x = (x_1, x_2) \in \mathbb{R}^2$ in (4.8). Observe that $D\phi(x) = \frac{1}{|x|}x$ and $\langle B^\circ x, D\phi\rangle = 0$. For $u \in C_c^\infty(\mathbb{R}^2 \setminus \{0\})$ and $s \in \mathbb{R}$ the formula (4.9) thus yields

$$\begin{aligned} S^{-1}\mathcal{A}^\circ S u(x) = \mathcal{A}^\circ u - s^2 \left(\mu_1 \frac{x_1^2}{|x|^2} + \mu_2 \frac{x_2^2}{|x|^2}\right) u + 2is\left(\mu_1 \frac{x_1}{|x|} D_1 u + \mu_2 \frac{x_2}{|x|} D_2 u\right) \\ + is\left(\mu_1 \frac{x_2^2}{|x|^3} + \mu_2 \frac{x_1^2}{|x|^3}\right) u =: \mathcal{T} u. \end{aligned}$$

By similarity, it is enough to treat $\mathcal{T}$.

3) Show that $i\beta\mathbb{Z} + (-\infty, 0] \subseteq \sigma(\mathcal{T}_p) = \sigma(\mathcal{A}_p^\diamondsuit)$. We scale the operator $\mathcal{T}$ through the isometries V_k in (4.11), obtaining

$$\begin{aligned} V_k^{-1}\mathcal{T} V_k u = \frac{1}{k^2}\mu_1 D_{11} u + \frac{1}{k^2}\mu_2 D_{22} u + b x_2 D_1 u - b x_1 D_2 u - s^2\left(\mu_1 \frac{x_1^2}{|x|^2} + \mu_2 \frac{x_2^2}{|x|^2}\right) u \\ + \frac{1}{k} 2is \left(\mu_1 \frac{x_1}{|x|} D_1 u + \mu_2 \frac{x_2}{|x|} D_2 u\right) + \frac{1}{k} is \left(\mu_1 \frac{x_2^2}{|x|^3} + \mu_2 \frac{x_1^2}{|x|^3}\right) u. \end{aligned}$$

Set $\mathcal{T}_k = V_k^{-1}\mathcal{T} V_k$. With the limit operator $\mathcal{T}_\infty$ from Lemma 4.5, it follows

$$\mathcal{T}_k u \;\longrightarrow\; \mathcal{T}_\infty u = b x_2 D_1 u - b x_1 D_2 u - s^2 \left(\mu_1 \frac{x_1^2}{|x|^2} + \mu_2 \frac{x_2^2}{|x|^2}\right) u$$

in L^p as $k \to \infty$, for every $u \in C_c^\infty(\mathbb{R}^2 \setminus \{0\})$.

We now argue as in the proof of Proposition 4.2. Take $\lambda_0 \in \rho(\mathcal{T}_p)$. By similarity, we have $\lambda_0 \in \rho(\mathcal{T}_k)$ and $\|R(\lambda_0, \mathcal{T}_k)\| \le C$ for every $k \in \mathbb{N}$. For $u \in C_c^\infty(\mathbb{R}^2 \setminus \{0\})$ we derive $\|u\|_p \le C\,\|(\lambda_0 - \mathcal{T}_k)u\|_p$, and thus $\|u\|_p \le C\,\|(\lambda_0 - \mathcal{T}_\infty)u\|_p$ letting $k \to +\infty$. Recalling that $b = \pm\beta$, Lemma 4.5 implies that $\lambda_0 \notin i\beta\mathbb{Z} - s^2(\mu_1+\mu_2)/2$ for all $s \in \mathbb{R}$. We have this shown the inclusion $i\beta\mathbb{Z} + (-\infty, 0] \subseteq \sigma(\mathcal{T}_p) = \sigma(\mathcal{A}_p^\diamondsuit)$.

4) Compute the spectrum of $\mathcal{A}_p^\diamondsuit$. Theorem 2.2 (d) shows that $i\beta\mathbb{Z}$ is the spectrum of the drift operator $\mathcal{L}^\diamondsuit = \langle B^\diamondsuit x, D\rangle$. From Proposition 2.8 we deduce $\sigma(\mathcal{A}_p^\diamondsuit) \subseteq (-\infty, 0] + i\beta\mathbb{Z}$ and hence $\sigma(\mathcal{A}_p^\diamondsuit) = i\beta\mathbb{Z} + (-\infty, 0]$ by step 3). In particular, $\sigma(\mathcal{A}_p^\diamondsuit)$ coincides with its topological boundary so that $\sigma(\mathcal{A}_p^\diamondsuit) = \sigma_{ap}(\mathcal{A}_p^\diamondsuit)$. □

Proof of Proposition 4.4. As already pointed out, we proceed as in Case 2 of the proof of Proposition 4.2. The operator $\mathcal{A}$ thus has the form $\mathcal{A} = \mathrm{Tr}(Q_0 D^2) + \langle Tx, D\rangle$ with T as in (3.4). As after (4.6), the functions $J_r^{-1}\mathcal{A}J_r u$ tend to $\mathcal{C}u$ in L^p as $r \to +\infty$ for $u \in C_c^\infty$, where $\mathcal{C} = \mathcal{A}^{\lim} + \mathcal{L}^{\lim}$ is defined in (4.7) and it is split into the same operators $\mathcal{A}^{\lim}$ and $\mathcal{L}^{\lim}$. In particular, $\mathcal{A}^{\lim}$ is hypoelliptic. Lemma 4.6 yields the spectral identity $\sigma(\mathcal{A}_p^{\lim}) = \sigma_{ap}(\mathcal{A}_p^{\lim}) = (-\infty, 0] + i\beta\mathbb{Z}$.

We next want to show the equality $\sigma(\mathcal{L}_p^{\lim}) + i\beta\mathbb{Z} = \sigma(\mathcal{L}_p)$. Let $\bar{B}$ denote the coefficient matrix of $\mathcal{L}^{\lim}$. Observe that it is diagonalizable since it has $n-2$ eigenvalues (counted with multiplicities) and that $\sigma(B) = \sigma(\bar{B}) \cup \{\pm i\beta\}$. Hence, case (b) of Theorem 2.2 does not occur for $\mathcal{L}^{\lim}$. Let $\mathcal{L}^{\lim}$ fall under cases (a) or (c) of Theorem 2.2 (so that $\mathcal{L}$ cannot fall under case (d)). Theorem 2.2 then leads to

$$\sigma(\mathcal{L}_p^{\lim}) = -\mathrm{tr}(\bar{B})/p + i\mathbb{R} = -\mathrm{tr}(B)/p + i\mathbb{R}.$$

The asserted equality thus follows from Theorem 2.2. In case (d), Theorem 2.2 and Proposition 2.3 yield

$$\sigma(\mathcal{L}_p^{\lim}) = \{i(n_1\sigma_1 + \cdots + n_{m-1}\sigma_{m-1}) \,|\, (n_1, \cdots, n_{m-1}) \in \mathbb{Z}^{m-1}\}.$$

If β/σ_1 is rational, we infer from these results

$$\begin{aligned}\sigma(\mathcal{L}_p) &= \{i(n_1\sigma_1 + \cdots + n_m\beta) \,|\, (n_1, \cdots, n_m) \in \mathbb{Z}^m\} \\ &= \{i(n_1\sigma_1 + \cdots + n_{m-1}\sigma_{m-1}) \,|\, (n_1, \cdots, n_{m-1}) \in \mathbb{Z}^{m-1}\} + i\beta\mathbb{Z} \\ &= \sigma(\mathcal{L}^{\lim}) + i\beta\mathbb{Z}.\end{aligned}$$

Otherwise, it follows $\sigma(\mathcal{L}_p) = i\mathbb{R} = \sigma(\mathcal{L}^{\lim}) + i\beta\mathbb{R}$ as well.

Observe that also $\sigma(\mathcal{L}_p^{\lim}) = \sigma_{ap}(\mathcal{L}_p^{\lim})$. Take $\lambda_1 \in \sigma(\mathcal{A}_p^{\lim})$ and $\lambda_2 \in \sigma(\mathcal{L}_p^{\lim})$. As in (4.4), we check that $\mu = \lambda_1 + \lambda_2$ is an approximate eigenvalue for $\mathcal{C}_p$; i.e., $(-\infty, 0] + i\beta\mathbb{Z} + \sigma(\mathcal{L}^{\lim}) = (-\infty, 0] + \sigma(\mathcal{L}_p)$ is contained in $\sigma_{ap}(\mathcal{C}_p)$. Arguing as in (4.5), we finally see that $\sigma_{ap}(\mathcal{C}_p)$ is a subset of $\sigma(\mathcal{A}_p)$. $\square$

Propositions 4.2, 4.3 and 4.4 conclude the proof of Theorem 4.1.

References

[1] M. Bramanti, G. Cupini, E. Lanconelli, E. Priola, Global L^p estimates for degenerate Ornstein-Uhlenbeck operators, *Math. Z.* **266** (2010), 789–816.

[2] A. Chojnowska-Michalik, B. Goldys, Symmetric Ornstein-Uhlenbeck semigroups and their generators, *Probab. Theory Related Fields* **124** (2002), 459–486.

[3] A. Chojnowska-Michalik, B. Goldys, Generalized Ornstein-Uhlenbeck semigroups: Littlewood-Paley-Stein inequalities and the P. A. Meyer equivalence of norms, *J. Funct. Anal.* **182** (2001), 243–279.

[4] G. Da Prato, A. Lunardi, On the Ornstein-Uhlenbeck operator in spaces of continuous functions, *J. Funct. Anal.* **131** (1995), 94–114.

[5] G. Da Prato, J. Zabczyk, *Ergodicity for Infinite Dimensional Systems,* Cambridge University Press, 1996.

[6] R. Donninger, B. Schörkhuber, A spectral mapping theorem for perturbed Ornstein-Uhlenbeck operators on $L^2(\mathbb{R}^d)$, *J. Funct. Anal* **268** (2015), 247–2524.

[7] K. Engel, R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations,* Graduate Texts in Mathematics 194, Springer, 2000.

[8] B. Farkas, A. Lunardi, Maximal regularity for Kolmogorov operators in L^2 spaces with respect to invariant measures, *J. Math. Pures Appl.* **86** (2006) 310–321.

[9] N. Garofalo, G. Tralli, Mehler met Ornstein and Uhlenbeck: the geometry of Carnot groups of step two and their heat kernels, *preprint, arXiv: 2007.10862.*

[10] N. Garofalo, G. Tralli, A class of nonlocal hypoelliptic operators and their extensions, *Indiana Univ. Math. J., to appear.*

[11] N. Garofalo, G. Tralli, Hardy-Littlewood-Sobolev inequalities for a class of non-symmetric and non-doubling hypoelliptic semigroups, *preprint, arXiv: 1904.12982.*

[12] M. Geissert, H. Heck, M. Hieber, L^p-theory of the Navier-Stokes flow in the exterior of a moving or rotating obstacle, *J. Reine Angew. Math.* **596** (2006), 45–62.

[13] L. Hörmander, Hypoelliptic second order differential equations, *Acta Math.* **119** (1967), 147–171.

[14] R.A. Horn, C.R. Johnson, *Matrix Analysis,* Cambridge University Press, 1985.

[15] A. Kolmogorov, Zufällige Bewegungen (zur Theorie der Brownschen Bewegung), *Ann. of Math.* **35** (1934), no. 1, 116–117.

[16] E. Lanconelli, S. Polidoro On a class of hypoelliptic evolution operators, *Rend. Sem. Mat. Univ. Pol. Torino* **52** (1994), 29–63.

[17] L. Lorenzi, *Analytical Methods for Markov Semigroups,* 2^{nd} *ed.,* Chapman & Hall/CRC, 2017.

[18] A. Lunardi, Schauder estimates for a class of degenerate elliptic and parabolic operators with unbounded coefficients in $\mathbb{R}^n$, *Ann. Sc. Norm. Sup. Pisa Cl. Sci. (4)* **24** (1997), 133–164.

[19] A. Lunardi, On the Ornstein-Uhlenbeck operator in L^2 spaces with respect to invariant measures, *Trans. Amer. Math. Soc.* **349** (1997), 155–169.

[20] G. Metafune, L^p-spectrum of Ornstein-Uhlenbeck operators, *Ann. Sc. Norm. Sup. Pisa Cl. Sci. (4)* **30** (2001), 97–124.

[21] G. Metafune, D. Pallara, E. Priola, Spectrum of Ornstein-Uhlenbeck operators in L^p spaces with respect to invariant measures, *J. Funct. Anal.* **196** (2002), 40–60.

[22] G. Metafune, D. Pallara, V. Vespri, L^p-estimates for a class of elliptic operators with unbounded coefficients $\mathbb{R}^N$, *Houston J. Math.* **31** (2005), 605–620.

[23] G. Metafune, J. Prüss, A. Rhandi, R. Schaubelt, The domain of the Ornstein-Uhlenbeck operator on an L^p-space with invariant measure, *Ann. Sc. Norm. Sup. Pisa Cl. Sci. (5)* **1** (2002), 471–485.

[24] A. Pascucci, *Kolmogorov equations in physics and in finance. Elliptic and parabolic problems,* Progr. Nonlinear Differential Equations Appl. 63, Birkhäuser, 2005, pp. 353–364.