STABILITY OF PERIODIC SOLUTIONS TO PARABOLIC PROBLEMS WITH NONLINEAR BOUNDARY CONDITIONS

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Abstract. We investigate non–autonomous quasilinear systems of parabolic partial differential equations with fully nonlinear boundary conditions. We establish local wellposedness and study the time and space regularity of the solutions. Our main results give principles of linearized (orbital) stability and instability for solutions in the vicinity of a periodic solution. Our approach relies on a detailed study of regularity properties of the linearized nonautonomous problem and its evolution family.

1. Introduction

In the qualitative theory of evolution equations, a first basic task is the investigation of the behavior of solutions in a neighborhood of an equilibrium. One typically looks for stability, convergence or locally invariant manifolds such as the stable, unstable and center manifolds. These local properties can be quite often tackled using spectral information about the linearization at the equilibrium. If the given problem is autonomous, then the linearized one is also autonomous so that one can use the well developed semigroup theory.

One of the next steps is to study the vicinity of a given non–constant τ–periodic orbit u_* in an analogous way. However, here the linearization is non–autonomous even if the given nonlinear problem is autonomous. So one cannot use semigroup theory anymore. It has to be replaced by the theory of evolution families $U(t,s)$, $t \geq s$, which are the solution operators of non–autonomous linear evolution equations. In this paper we will focus on τ–periodic or even time independent coefficients so that the linearized problem is τ–periodic. In this case the spectrum of the *monodromy operator* $U(\tau,0)$ determines much of the asymptotic behavior of $U(\cdot,\cdot)$ which in turn should govern the qualitative properties of the nonlinear equation near u_*. But the presence of the periodic orbit causes further difficulties. If the given coefficients are autonomous, then $U(\tau,0)$ always has the eigenvalue 1 with the eigenvector $u_*'(0)$, see Section 5. Correspondingly, we can obtain at most orbital stability of u_* with asymptotic phase; i.e., for each initial value u_0 near u_* with solution u there is a $\theta \in [0,\tau]$ such that $u(t) - u_*(t+\theta)$ decays as $t \to \infty$.

In this paper we study quasilinear parabolic systems on a domain $\Omega \subset \mathbb{R}^n$ with fully nonlinear boundary conditions

$$\begin{aligned} \partial_t u(t) + A(t,u(t))u(t) &= F(t,u(t)), \quad \text{on } \Omega, \ \ t>0, \\ B_j(t,u(t)) &= 0, \quad \text{on } \partial\Omega, \ \ t \geq 0, \ \ j \in \{1,\dots,m\}, \\ u(0) &= u_0, \quad \text{on } \Omega. \end{aligned} \tag{1.1}$$

Here, the solution $u(t,x)$ belongs to $\mathbb{C}^N$, the map $A(t,u_0)$ is an elliptic differential operator of even order $2m$ in non divergence form whose coefficients depend on the solution u and its derivatives up to order $2m-1$, F is an substitution operator also acting on u and its derivatives up to order $2m-1$, and B_j are substitution

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operators at the boundary depending on u and its derivatives up to order $m_j \leq 2m-1$. We only assume local smoothness of the coefficients and impose no growth restrictions. Moreover, we require that the linearized boundary value problems $(A(t,v), B_1'(t,v), \cdots, B_m'(t,v))$ are parameter elliptic and satisfy the Lopatinski–Shapiro conditions, see [3], [7], [8], and Section 2.

The equations (1.1) are a model case for fully nonlinear boundary conditions arising naturally in the treatment of free boundary problems, see e.g. [5], [13] or [22]. Such problems often contain nonlinear dynamical boundary conditions, cf. e.g. [6] and the references therein.

We establish local wellposedness and smoothing properties of (1.1) in Section 3. To some extend we can follow here the approach of [15] and [16], where the autonomous version of (1.1) and its behavior near an equilibrium was studied in detail. For the reader's convenience we briefly recall the necessary information about the setting and several auxiliary results in Section 2. At this point we just note that we work in an L^p–setting and that we use maximal regularity estimates on bounded time intervals for the linear problem (1.2) below, proved in [8]. In the proofs of Sections 2 and 3, we focus on the parts which differ significantly from those in [15] because of the time dependence of the coefficients in the present paper.

In Section 4 we study in detail the linear inhomogeneous boundary value problem

$$\begin{aligned} \partial_t v(t) + A_*(t)v(t) &= g(t) \qquad \text{on } \Omega, \quad t \in J, \\ B_{j*}(t)v(t) &= h_j(t) \qquad \text{on } \partial\Omega, \quad t \in J, \quad j \in \{1, \dots, m\}, \\ v(t_0) &= v_0, \qquad \text{on } \Omega, \end{aligned} \tag{1.2}$$

see (2.16) and (2.18), which is the linearization of (1.1) at the τ–periodic orbit u_*. We assume that the coefficients in (1.1) are τ–periodic. We first treat the homogeneous case where we have $g = h_j = 0$ in (1.2). This problem is governed by the linear operators $A_0(t) = A_*(t)|\ker B_*(t)$, see (4.16). In Lemma 4.4 and Proposition 4.6 we prove that the operators $A_0(t)$, $t \in \mathbb{R}$, satisfy the Aquistapace–Terreni conditions, i.e., $A_0(t)$ are sectorial of uniform type and their resolvents satisfy a certain Hölder estimate (see (4.17) and (4.18), as well as [1], [2] and Section 4). These conditions imply that $A_0(\cdot)$ generates a parabolic evolution family $U(\cdot,\cdot)$ solving the homogeneous linearized problem (1.2) with $g = h_j = 0$. Moreover, for the study of the asymptotic properties of (1.1) we need a variation of constants formula for the linearized problem stated in Proposition 4.9. This formula relies on the extension of $U(\cdot,\cdot)$ to the extrapolation spaces corresponding to $A_0(t)$ (or equivalently, on regularity properties of the adjoints $U(t,s)^*$), see Section 4. In our previous papers [18] and [19] we have developed an extrapolation theory in the framework of the Aquistapace–Terreni conditions. However, this theory is not fully applicable here due to the limited regularity of u_*. Fortunately, using the structure of problem (1.2) we can establish a somewhat improved version of the Hölder estimate of Acquistapace and Terreni, stated in (4.31), which allows to derive the extrapolation theory needed for the investigation of (1.1), see Propositions 4.7 and 4.10. Based on these results we can then prove the fundamental linear maximal regularity estimates for the time intervals $\mathbb{R}_+$ and $\mathbb{R}_-$ in Propositions 4.11 and 4.12.

Now we have all tools at hand to prove in Propositions 5.1 and 5.2 the principles of linearized (in-)stability for the τ–periodic solution u_* of (1.1) in the case of τ–periodic coefficients. Finally, if the coefficients do not depend on time and if the spectrum of $U(\tau, 0)$ consists of a part being strictly in the open unit disk and of the simple eigenvalue 1, then Theorem 5.3 shows that u_* is orbitally stable for (1.1) with asymptotic phase. An analogous result was shown for nonlinear problems with linear boundary conditions in Theorem 9.3.7 of [17] in a C^α setting. To our knowledge there are no related theorems for nonlinear boundary conditions. Only for the special case of quasilinear boundary conditions, there are different stability results for periodic orbits in the context of Hopf bifurcation, see e.g. [24]. On the other hand, the paper [5] treats related stability properties of travelling wave solutions of a certain free boundary value problem (which leads to a problem with nonlinear boundary conditions). Finally, we want to point out that the theory developed in this paper should be the basis for investigations of the qualitative behavior near u_* beyond the stable case. In particular, we want to study stable, unstable and center manifolds near an periodic orbit in future work.

Notation. Let $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\} = \mathbb{C}_+$ and $J \subset \mathbb{R}$ be a closed interval. We set $D_k = -i\partial_k = -i\partial/\partial x_k$ and use the multi index notation. The k–tensor of the partial derivatives of order k is denoted by ∇^k, and we let $\underline{\nabla}^k u = (u, \nabla u, \dots, \nabla^k u)$. For an operator A on a Banach space we write $\operatorname{dom}(A)$, $\ker(A)$, $\operatorname{ran}(A)$, $\sigma(A)$ and $\rho(A)$ for its domain, kernel, range, spectrum, and resolvent set, respectively. $\mathcal{B}(X,Y)$ (resp., $\mathcal{B}_2(X,Y)$) is the space of bounded linear (resp., bilinear) operators between two Banach spaces X and Y, and we put $\mathcal{B}(X) = \mathcal{B}(X,X)$. For an open set U with boundary ∂U, we denote by $C^k(U)$ ($BC^k(U)$, or $BUC^k(U)$, or $C_0^k(U)$, respectively) the space of k–times continuously differentiable functions u on U (such that u and its derivatives up to order k are bounded, or bounded and uniformly continuous, or vanish at ∂U (and at infinity if U is unbounded), respectively), where $BC^k(U)$ is endowed with its canonical norm. For $C^k(\overline{U})$ and $BC^k(\overline{U})$ we require in addition that u and its derivatives up to order k have a continuous extension to ∂U. For unbounded U, we write $C_0^k(\overline{U})$ for the space of $u \in C^k(\overline{U})$ such that u and its derivatives up to order k vanish at infinity. By $H_p^k(U)$ we designate the Sobolev spaces. A generic constant depending on K will be denoted by $c = c(K)$. Similarly, $\varepsilon_K = \varepsilon : \mathbb{R}_+ \to \mathbb{R}_+$ is a generic nondecreasing function with $\varepsilon(r) \to 0$ as $r \to 0$.

2. Setting and preliminaries

Let $\Omega \subset \mathbb{R}^n$ be an open connected set with a compact boundary $\partial\Omega$ of class C^{2m} and outer unit normal $\nu(x)$, where $m \in \mathbb{N}$. Throughout this paper, we fix an exponent p with

$$n + 2m < p < \infty. \tag{2.1}$$

Let $E = \mathbb{C}^N$ with $\mathcal{B}(E) = \mathbb{C}^{N\times N}$ for some fixed $N \in \mathbb{N}$, and let $t_0 \in \mathbb{R}$. For a $\mathbb{C}^N$–valued function $u(t) = u(t,x)$, $t \ge t_0$, $x \in \overline{\Omega}$, we investigate the non-autonomous quasilinear initial boundary value problem with fully nonlinear boundary conditions

$$\begin{aligned} \partial_t u(t) + A(t,u(t))u(t) = F(t,u(t)), &\quad \text{on } \Omega, \ \ t > t_0, \\ B_j(t,u(t)) = 0, &\quad \text{on } \partial\Omega, \ \ t \ge t_0, \ \ j \in \{1,\dots,m\}, \\ u(t_0) = u_0, &\quad \text{on } \Omega. \end{aligned} \tag{2.2}$$

Of particular interest are maps A, B_j and F which do not depend explicitly on t. The operators in (2.2) are given by

$$\begin{aligned} [A(t,u)v](x) &= \sum_{|\alpha|=2m} a_\alpha(t,x,u(x),\nabla u(x),\dots,\nabla^{2m-1}u(x))\, D^\alpha v(x), \quad x \in \Omega, \\ [F(t,u)](x) &= f(t,x,u(x),\nabla u(x),\dots,\nabla^{2m-1}u(x))), \quad x \in \Omega, \\ [B_j(t,u)](x) &= b_j(t,x,(\gamma u)(x),(\gamma\nabla u)(x),\dots,(\gamma\nabla^{m_j}u)(x)), \quad x \in \partial\Omega, \end{aligned} \tag{2.3}$$

for some $m_j \in \{0,1,\dots,2m-1\}$ and all $j \in \{1,\dots,m\}$, $v \in H_p^{2m}(\Omega;\mathbb{C}^N)$ and $u \in BC^{2m-1}(\overline{\Omega};\mathbb{C}^N)$, resp. $u \in C^{m_j}(\overline{\Omega};\mathbb{C}^N)$ in the last line of (2.3). In B_j we have used the spatial trace operator γ which we usually omit from the notation. We set $\mathcal{E}_l = E \times E^n \times \cdots \times E^{(n^l)}$ for $l \in \mathbb{N}_0$. For each $k \in \mathbb{N}_0$, we fix an order of the multi indeces $\beta \in \mathbb{N}_0^n$ with $|\beta| = k$. We order the n^k components of a k–tensor in the same way, thus using β as the label for the component corresponding to $\beta \in \mathbb{N}_0^n$ with $|\beta| = k$. For a function w depending on $z \in E^{(n^k)}$, we denote by $\partial_\beta w$ its partial derivative with respect to β-th argument. Throughout, the coefficients of (2.2) are assumed to satisfy the following regularity conditions.

(R) $a_\alpha \in C^1(\mathbb{R}\times\mathcal{E}_{2m-1}; BC(\overline{\Omega};\mathcal{B}(E)))$, $\partial_\beta a_\alpha \in C^1(\mathbb{R}\times\mathcal{E}_{2m-1}; BC(\overline{\Omega};\mathcal{B}_2(E^2,E)))$ for all $k=\{0,1,\cdots,2m-1\}$ and $\alpha,\beta\in\mathbb{N}_0^n$ with $|\alpha|=2m$, $|\beta|=k$, and $a_\alpha(t,x,0)\to a_\alpha(t,\infty)$ in $\mathcal{B}(E)$ for each $t\in\mathbb{R}$ as $|x|\to\infty$, if Ω is unbounded; $f\in C^1(\mathbb{R}\times\mathcal{E}_{2m-1}; BC(\overline{\Omega};E))$ and $\partial_\beta f\in C^1(\mathbb{R}\times\mathcal{E}_{2m-1}; BC(\overline{\Omega};\mathcal{B}(E)))$ for all $k=\{0,1,\cdots,2m-1\}$ and $\beta\in\mathbb{N}_0^n$ with $|\beta|=k$, and $f(t,\cdot,0)$ belongs to $L^p(\Omega;E)$ for each $t\in\mathbb{R}$ if Ω is unbounded;
$b_j\in C^{2m-m_j+2}(\partial\Omega\times\mathcal{E}_{m_j}; BC(\mathbb{R};E))$ and it exist $\partial_t b_j\in C^{2m-m_j+1}(\partial\Omega\times\mathcal{E}_{m_j}; BC(\mathbb{R},E)))$ and $\partial_{tt}b_j\in C(\mathbb{R}\times\partial\Omega\times\mathcal{E}_{m_j};E)$ for each $j\in\{1,\dots,m\}$.

Rooughly speaking, we require that the coefficients in the interior are C^2 and that those on the boundary belong to C^{2m-m_j+2}, as in the hypothesis (RR) of [16]. In addition, we mostly require that

(P) all functions in (R) are periodic in t with a common period $\tau>0$.

Given $u_0\in C^{m_j}(\overline{\Omega};\mathbb{C}^N)$, we further define

$$\begin{aligned}[B_j'(t,u_0)v](x) &= (\partial_z b_j)(t,x,u_0(x),\nabla u_0(x),\dots,\nabla^{m_j}u_0(x))\cdot\gamma\underline{\nabla}^{m_j}v(x) \qquad (2.4)\\ &=\sum_{k=0}^{m_j}\sum_{|\beta|=k} i^k\,(\partial_\beta b_j)(t,x,u_0(x),\nabla u_0(x),\dots,\nabla^{m_j}u_0(x))\,\gamma D^\beta v(x)\end{aligned}$$

for $x\in\partial\Omega$, $v\in C^{m_j}(\overline{\Omega};\mathbb{C}^N)$, and $j\in\{1,\dots,m\}$. We set $B=(B_1,\dots,B_m)$ and $B'=(B_1',\dots,B_m')$. The symbols of the principal parts of the linear differential operators are the matrix–valued functions given by

$$\mathcal{A}_\#(t,x,z,\xi)=\sum_{|\alpha|=2m}a_\alpha(t,x,z)\,\xi^\alpha,\quad \mathcal{B}_{j\#}(t,x,z,\xi)=\sum_{|\beta|=m_j}i^{m_j}(\partial_\beta b_j)(t,x,z)\,\xi^\beta$$

for $t\in\mathbb{R}$, $x\in\overline{\Omega}$, $z\in\mathcal{E}_{2m-1}$ and $\xi\in\mathbb{R}^n$, resp. $x\in\partial\Omega$, $z\in\mathcal{E}_{m_j}$ and $\xi\in\mathbb{R}^n$. We further set $\mathcal{A}_\#(t,\infty,\xi)=\sum_{|\alpha|=2m}a_\alpha(t,\infty)\,\xi^\alpha$ if Ω is unbounded. One defines the *normal ellipticity* and the *Lopatinskii–Shapiro condition* for $A(t,u_0)$ and $B'(t,u_0)$ at a function $u_0\in C_0^{2m-1}(\overline{\Omega};\mathbb{C}^N)$ as follows:

(E) $\sigma(\mathcal{A}_\#(t,x,\underline{\nabla}^{2m-1}u_0(x),\xi))\subset\mathbb{C}_+$ and (if Ω is unbounded) $\sigma(\mathcal{A}_\#(t,\infty,\xi))\subset\mathbb{C}_+$ for all $t\in\mathbb{R}$, $x\in\overline{\Omega}$ and $\xi\in\mathbb{R}^n$ with $|\xi|=1$.

(LS) Let $t\in\mathbb{R}$, $x\in\partial\Omega$, $\xi\in\mathbb{R}^n$, and $\lambda\in\overline{\mathbb{C}_+}$ with $\xi\perp\nu(x)$ and $(\lambda,\xi)\neq(0,0)$. The function $\varphi=0$ is the only solution in $C_0(\mathbb{R}_+;\mathbb{C}^N)$ of the ode system

$$\begin{aligned}\lambda\varphi(y)+\mathcal{A}_\#(t,x,\underline{\nabla}^{2m-1}u_0(x),\xi+i\nu(x)\partial_y)\varphi(y)&=0,\quad y>0,\\ \mathcal{B}_{j\#}(t,x,\underline{\nabla}^{m_j}u_0(x),\xi+i\nu(x)\partial_y)\varphi(0)&=0,\quad j\in\{1,\dots,m\}.\end{aligned}$$

These conditions are crucial for the linear regularity results from [8], stated in Theorem 2.1, which are the basis for our approach. We refer to [3], [7], [8], and the references therein for more information concerning (E) and (LS).

We discuss the function spaces and trace theorems needed below. First, we put

$$X_0=L_p(\Omega;\mathbb{C}^N),\quad X_1=H_p^{2m}(\Omega;\mathbb{C}^N),\quad X_{1-1/p}=W_p^{2m(1-1/p)}(\Omega;\mathbb{C}^N),$$

and denote the norms of these spaces by $|\cdot|_0$, $|\cdot|_1$, and $|\cdot|_{1-1/p}$, respectively. On the Slobodetskii spaces W_p^s we use the 'intrinsic' norm given by

$$|v|^p_{W_p^s(\Omega)^N}=|v|^p_{L_p(\Omega)^N}+\sum_{|\alpha|=k}[\partial^\alpha v]^p_{W_p^\sigma(\Omega)^N},\quad [w]^p_{W_p^\sigma(\Omega)^N}=\iint_{\Omega^2}\frac{|w(y)-w(x)|^p}{|y-x|^{n+\sigma p}}\,dx\,dy,$$

for $s=k+\sigma$ with $k\in\mathbb{N}_0$ and $\sigma\in(0,1)$. Occasionally we use without further notice that W_p^s coincides with the real interpolation space $(L_p,W_p^l)_{s/l,p}$ if $l\in\mathbb{N}$ and $s\in(0,l)$ is not an integer. (See [25, §4.4].) We note that $X_1\hookrightarrow X_{1-1/p}\hookrightarrow X_0$,

$$X_{1-1/p}\hookrightarrow C_0^{2m-1}(\overline{\Omega};\mathbb{C}^N),\qquad\text{and}\qquad X_{1-1/p}\hookrightarrow H_p^{2m-1}(\overline{\Omega};\mathbb{C}^N)\qquad (2.5)$$

by (2.1) and standard properties of Sobolev spaces, cf. [25, §4.6.1]. From the above expression of $[w]_{1-1/p} := [w]^p_{W_p^{1-1/p}(\Omega)^N}$ we deduce that

$$[uv]_{1-1/p} \le c\,(|u|_{L^\infty}\,[v]_{1-1/p} + [u]_{1-1/p}\,|v|_{L^\infty}) \le c\,[u]_{W_p^{1-1/p}}\,[v]_{W_p^{1-1/p}}\,, \tag{2.6}$$

where we also used Sobolev's embedding theorem and (2.1).

Let $I \subset \mathbb{R}$ be an interval (maybe, not closed) containing more than a point. Then we introduce the function spaces

$$\mathbb{E}_0(I) = L_p(I; L_p(\Omega;\mathbb{C}^N)) = L_p(I; X_0),$$
$$\mathbb{E}_1(I) = H^1_p(I; L_p(\Omega;\mathbb{C}^N)) \cap L_p(I; H^{2m}_p(\Omega;\mathbb{C}^N)) = H^1_p(I; X_0) \cap L_p(I; X_1),$$

equipped with the natural norms. Mostly, we deal with closed intervals which are denoted by J instead of I. Since we want to insert functions $u \in \mathbb{E}_1(I)$ into the nonlinearities, we need the embedding

$$\mathbb{E}_1(I) \hookrightarrow BUC(I; X_{1-1/p}) \hookrightarrow BUC(I; C_0^{2m-1}(\overline{\Omega};\mathbb{C}^N)), \tag{2.7}$$

see [4, Thm.III.4.10.2] for the first and (2.5) for the second embedding. There is a constant $c_0(T_0)$ which is larger than the norms of the first embedding in (2.7) and of $\mathbb{E}_1(I) \hookrightarrow BUC(I; C_0^{2m-1}(\overline{\Omega};\mathbb{C}^N))$, for all intervals I of length greater than a fixed $T_0 > 0$, see [4, Lem.III.4.10.1]. Moreover, one can choose an I–independent constant c_0 for functions vanishing at the left end point of I, see e.g. [15, §2].

Due to (2.7), the temporal trace operator γ_0 at time $t = 0$ belongs to $\mathcal{B}(\mathbb{E}_1([0,1]), X_{1-1/p})$. Recall that the spatial trace operator γ at $\partial\Omega$ induces continuous maps

$$\gamma : W^s_p(\Omega;\mathbb{C}^N) \to W^{s-1/p}_p(\partial\Omega;\mathbb{C}^N) \tag{2.8}$$

for $1/p < s \le 2m$ if $s - 1/p$ is not an integer, cf. [25, §4.7.1]. Here we let $W^k_p = H^k_p$ for $k \in \mathbb{N}_0$, and the Slobodetskii spaces $W^s_p(\partial\Omega)$ are defined via local charts and have the analogous properties as $W^s_p(\Omega)$, cf. [25, §3.6.1]. We further set

$$Y_0 = L_p(\partial\Omega;\mathbb{C}^N), \quad Y_{j,1} = W^{2m\kappa_j}_p(\partial\Omega;\mathbb{C}^N), \quad Y_{j,1-1/p} = W^{2m\kappa_j - 2m/p}_p(\partial\Omega;\mathbb{C}^N)$$

for $j \in \{1,\dots,m\}$, introducing the number

$$\kappa_j := 1 - \frac{m_j}{2m} - \frac{1}{2mp}\,.$$

Note that $\kappa_j > \frac{1}{p}$ due to (2.1). We let $Y_r = Y_{1,r} \times \cdots \times Y_{m,r}$ for $r = 1, 1-1/p$. The boundary data of our linearized equations will be contained in the spaces

$$\begin{aligned}\mathbb{F}_j(J) &= W^{\kappa_j}_p(J; L_p(\partial\Omega;\mathbb{C}^N)) \cap L_p(J; W^{2m\kappa_j}_p(\partial\Omega;\mathbb{C}^N)) \\ &= W^{\kappa_j}_p(J; Y_0) \cap L_p(J; Y_{j,1}), \qquad j \in \{1,\dots,m\},\end{aligned} \tag{2.9}$$

endowed with the natural norms, where $\mathbb{F}(J) := \mathbb{F}_1(J) \times \cdots \times \mathbb{F}_m(J)$. It holds

$$\mathbb{F}_j(J) \hookrightarrow BUC(J; Y_{j,1-1/p}) \hookrightarrow BUC(J \times \partial\Omega), \tag{2.10}$$

so that $\gamma_0 \in \mathcal{B}(\mathbb{F}_j([0,1]), Y_{j,1-1/p})$. Here the second embedding follows from Sobolev's embedding theorem using (2.1), and the first one is a consequence of Proposition 3 in [20], see [15, §2]. The norms of the embeddings in (2.10) depend on J as described after (2.7). Finally, Lemmas 3.5 and 3.8 of [8] yield

$$\gamma\partial^\beta : \mathbb{E}_1(J) \to \mathbb{F}_j(J) \tag{2.11}$$

for $|\beta| \le m_j \le 2m$.

Let $u_0, v \in BC^{2m-1}(\overline{\Omega};\mathbb{C}^N)$ with $|u_0|_{BC^{2m-1}} \le R$ for any $R > 0$, $w \in X_1$, and $t \in \mathbb{R}$ with $|t| \le R$. We introduce the linear operators $F'(t,u_0)$ and $A'(t,u_0)w$ by setting

$$[F'(t,u_0)v](x) = \sum_{k=0}^{2m-1} \sum_{|\beta|=k} (\partial_\beta f)(t,x,u_0(x),\nabla u_0(x),\dots,\nabla^{2m-1}u_0(x))\,\partial^\beta v(x),$$

$$[A'(t,u_0)w]v(x) = A'(t,u_0)[v,w](x)$$
$$= \sum_{|\alpha|=2m} \sum_{k=0}^{2m-1} \sum_{|\beta|=k} (\partial_\beta a_\alpha)(t,x,u_0(x),\dots,\nabla^{2m-1}u_0(x))\,[\partial^\beta v(x), D^\alpha w(x)]$$

for $x \in \Omega$, with a similar notation as in (2.4). Note that $\partial_\beta a_\alpha(t,x,z) : E^2 \to E$ is bilinear and that the coefficients of $F'(t,u_0)$ and $A'(t,u_0)$ are uniformly bounded by a constant $c(R)$. Taking also into account (2.5) and (R), we thus obtain

$$|F'(t,u_0)|_{\mathcal{B}(X_{1-1/p},X_0)} \le c(R). \tag{2.12}$$

Similarly, $[v,w] \mapsto A'(t,u_0)[v,w]$ is a bilinear map from $X_{1-1/p} \times X_1$ to X_0 with

$$|A'(t,u_0)[v,w]|_0 \le c(R)\,|v|_{BC^{2m-1}}\,|w|_1 \le c(R)\,|v|_{1-1/p}\,|w|_1\,. \tag{2.13}$$

Moreover, the maps $(t,u_0) \mapsto A'(t,u_0)$ and $(t,u_0) \mapsto F'(t,u_0)$ are uniformly continuous for t in bounded intervals and u_0 in balls of $X_{1-1/p}$. Next, take $v \in X_{1-1/p}$ with $|v|_{1-1/p} \le R$, and let $|t| \le R$. Using (R) and (2.5), we deduce

$$\begin{aligned} |F(t,u_0+v) - F(t,u_0) - F'(t,u_0)v|_0 &\le \varepsilon_R(|v|_{1-1/p})\,|v|_{1-1/p}, \\ |A(t,u_0+v)w - A(t,u_0)w - [A'(t,u_0)w]v|_0 &\le \varepsilon_R(|v|_{1-1/p})\,|v|_{1-1/p}\,|w|_1\,. \end{aligned} \tag{2.14}$$

As a result, $A'(t,\cdot)$ and $F'(t,\cdot)$ are in fact the Fréchet derivatives of the functions $A(t,\cdot) \in C^1(X_{1-1/p};\mathcal{B}(X_1,X_0))$ and $F(t,\cdot) \in C^1(X_{1-1/p};X_0)$, respectively. Finally, we have

$$|[A(t,u_0+v) - A(t,u_0)]w|_0 \le c(R)\,|v|_{1-1/p}\,|w|_1\,. \tag{2.15}$$

We linearize (2.2) at a function $u_* \in \mathbb{E}_1(J)$ obtaining the linear operators

$$\begin{aligned} A_*(t) &:= A(t,u_*(t)) + A'(t,u_*(t))u_*(t) - F'(t,u_*(t)) \in \mathcal{B}(X_1,X_0), \\ B_{j*}(t) &:= B_j'(t,u_*(t)) \in \mathcal{B}(X_{1-1/p},Y_{j,1-1/p}) \cap \mathcal{B}(X_1,Y_{j,1}), \end{aligned} \tag{2.16}$$

for (almost) all $t \in J$ and $j \in \{1,\dots,m\}$. We set $B_*(t) = (B_{1*}(t),\dots,B_{m*}(t))$. (For the mapping properties of $B_j'(t,u_*(t))$) see [15, §2] and also Corollary 2.5 below.)

We are now in a position to state the crucial regularity theorem for the linear initial boundary value problem associated with (2.2). Fix $T>0$ and $t_0 \in \mathbb{R}$, set $J = [t_0,t_0+T]$, and take a function $u_* \in \mathbb{E}_1(J)$. Assume that (R) is true and that (E) and (LS) hold at t and $u_*(t)$ for each $t \in J$. Set $a^*_\alpha(t,x) = a_\alpha(t,x,\underline{\nabla}^{2m-1}u_*(t,x))$ for $|\alpha| = 2m$ and

$$b^*_{j\beta}(t,x) = i^k(\partial_\beta b_j)(t,x,\underline{\nabla}^{m_j}u_*(t,x)) \tag{2.17}$$

for $k = |\beta| \le m_j$ and $j \in \{1,\dots,m\}$. We then have $a^*_\alpha \in BC(J \times \overline{\Omega};\mathcal{B}(E))$ and $a^*_\alpha(t,x) \to a_\alpha(t,\infty)$ as $|x| \to \infty$ for each $t \in J$ since $u_* \in C(J;C_0^{2m-1}(\overline{\Omega};\mathbb{C}^N))$ due to (2.7). As in the proof of Proposition 2.4 one verifies that $b^*_{j\beta} \in \mathbb{F}_j(J)$. Moreover, the lower order terms $A'(t,u_*(t))u_*(t) - F'(t,u_*(t))$ do not enter into (E) and (LS) of [8] and their coefficients are bounded or belong to $L_p(J;L_p(\Omega;\mathcal{B}(E)))$. Thus the differential operators $A_*(t)$ and $B_{j*}(t)$ satisfy assumptions (E), (LS), (SD), (SB) from [8]. So Theorem 2.1 of [8] yields the following result.

Theorem 2.1. *Let $t_0 \in \mathbb{R}$ and $u_* \in \mathbb{E}_1(J)$ for $J = [t_0,t_0+T]$. Assume that (R) is true and that (E) and (LS) hold at t and $u_*(t)$, for each $t \in J$. Define $A(t)$ and $B_{j*}(t)$ by (2.16) for $t \in J$ and $j \in \{1,\dots,m\}$. Then there is a unique function $v =: S(t_0,v_0,g,h) \in \mathbb{E}_1(J)$ satisfying*

$$\begin{aligned} \partial_t v(t) + A_*(t)v(t) &= g(t) \qquad \text{on } \Omega, \quad t \in J, \\ B_{j*}(t)v(t) &= h_j(t) \qquad \text{on } \partial\Omega, \quad t \in J, \quad j \in \{1,\dots,m\}, \\ v(t_0) &= v_0, \qquad \text{on } \Omega, \end{aligned} \tag{2.18}$$

if and only if

$$(v_0,g,h) \in X_{1-1/p} \times \mathbb{E}_0(J) \times \mathbb{F}(J) \quad \text{and} \quad B_*(t_0)v_0 = h(t_0),$$

where $h := (h_1, \dots, h_m)$. *In this case, there is a constant* $c_1 = c_1(J)$ *such that*

$$\|v\|_{\mathbb{E}_1(J)} \le c_1\,(|v_0|_{1-1/p} + \|g\|_{\mathbb{E}_0(J)} + \|h\|_{\mathbb{F}(J)}). \tag{2.19}$$

As in [15, §2] one can check that $c_1 = c_1(T_0, T_1)$ if $T \in [T_0, T_1]$ and $0 < T_0 < T_1 < \infty$, and that $c_1 = c_1(T_1)$ if $h(t_0) = 0$.

For a given function $u \in \mathbb{E}_1([t_0, t_0+T])$ and a given solution $u_* \in \mathbb{E}_1([t_0, t_0+T])$ of (2.2), we set $v(t) = u(t) - u_*(t)$ and $v_0 = u_0 - u_*(t_0)$. Then the problem (2.2) for u is equivalent to the initial–boundary value problem for v given by

$$\begin{aligned} \partial_t v(t) + A_*(t)v(t) = G(t, v(t)) \qquad &\text{on } \Omega, \quad t > t_0,\\ B_{j*}(t)v(t) = H_j(t, v(t)) \qquad &\text{on } \partial\Omega, \quad t \ge t_0, \quad j \in \{1, \dots, m\},\\ v(t_0) = v_0, \qquad &\text{on } \Omega. \end{aligned} \tag{2.20}$$

Here we have used the nonlinear maps G and H defined by

$$\begin{aligned} G(t, v) &:= \big(A(t, u_*(t))v - A(t, u_*(t)+v)v\big) - \big(A(t, u_*(t) + v)u_*(t) - A(t, u_*(t))u_*(t)\\ &\qquad - [A'(t, u_*(t))u_*(t)]v\big) + \big(F(t, u_*(t) + v) - F(t, u_*(t)) - F'(t, u_*(t))v\big),\\ H_j(t, w) &:= B_j'(t, u_*(t))w - B_j(t, u_*(t) + w), \quad j \in \{1, \dots, m\}, \end{aligned} \tag{2.21}$$

for all $v \in X_1$ and $w \in C^{m_j}(\overline{\Omega}; \mathbb{C}^N)$, where $u_* \in \mathbb{E}_1(J)$ is given. As before, we set $H(t, v) = (H_1(t, v), \dots, H_m(t, v))$.

Definition 2.2. We say that a function u *solves* problem (2.2), (2.18) or (2.20) on a (possibly noncompact) interval I containing 0 if u belongs to $\mathbb{E}_1(J)$ for each compact interval $J \subset I$ and satisfies the respective problem for (a.e.) $t \in I$.

For functions u_* and v which belong to $\mathbb{E}_1([a, b])$ for all compact subintervals of an interval I we define the substitution operators $\mathbb{G}(v)(t) = G(t, v(t))$ and $\mathbb{H}_j(v)(t) = H_j(t, v(t))$ for a.e. $t \in J$, setting $\mathbb{H} = (\mathbb{H}_1, \dots, \mathbb{H}_m)$. Their mapping properties will be crucial for our main results. We work on weighted functions spaces when treating the asymptotic behavior. Let $t_0 \in \mathbb{R}$ and $J = [t_0, \infty)$ or $J = (-\infty, t_0]$. We set $e_\delta(t) = e^{\delta(t-t_0)}$ for $t \in \mathbb{R}$ and $\delta \in \mathbb{R}$, and introduce the spaces

$$\mathbb{E}_k(J, \delta) = \{v : e_\delta v \in \mathbb{E}_k(J)\} \quad (k = 0, 1), \qquad \mathbb{F}(J, \delta) = \{v : e_\delta v \in \mathbb{F}(J)\} \tag{2.22}$$

endowed with the norms

$$\|v\|_{\mathbb{E}_k(J,\delta)} = \|e_\delta v\|_{\mathbb{E}_k(J)} \quad (k = 0, 1), \qquad \|v\|_{\mathbb{F}(J,\delta)} = \|e_\delta v\|_{\mathbb{F}(J)}.$$

We recall Lemma 11 from [15] which is used in the next proof.

Lemma 2.3. *Let* Z *be a Banach space,* $\alpha \in (0, 1)$*, and* $\delta \in \mathbb{R}$*. Set* $I(t) = [t-1, t+1] \cap \mathbb{R}_+$ *for* $t \in \mathbb{R}_+$*. Then we have*

$$\begin{aligned} [e_\delta f]_{W_p^\alpha(\mathbb{R}_+;Z)} &\le c\,\|e_\delta f\|_{L_p(\mathbb{R}_+;Z)} + c\Big[\int_{\mathbb{R}_+}\int_{I(t)} e^{\delta tp}\,\frac{|f(t) - f(s)|_Z^p}{|t-s|^{1+\alpha p}}\,ds\,dt\Big]^{\frac{1}{p}}\\ &\le c\,\|e_\delta f\|_{W_p^\alpha(\mathbb{R}_+;Z)}, \end{aligned}$$

where the constants c *can be chosen uniformly for* δ *belonging to compact intervals.*

Proposition 2.4. *Assume that (R) holds. In the case of a compact interval* J*, we take* $u_* \in \mathbb{E}_1(J)$*. Further, let* $t_0 \in \mathbb{R}$ *and set* $J_+ = [t_0, \infty)$ *and* $J_- = (-\infty, t_0]$*. In the case of the intervals* $J_\pm$*, we also assume that (P) holds and take a* τ*- periodic* $u_* \in \mathbb{E}_1([0, \tau])$ *satisfying* $B_*(t, u_*(t)) = 0$ *for all* t*.*

(I) Let $\delta \ge 0$*. Then the following assertions are valid.*

(a) We have $\mathbb{G} \in C^1(\mathbb{E}_1(J_+, \delta); \mathbb{E}_0(J_+, \delta))$*,* $\mathbb{G} \in C^1(\mathbb{E}_1(J_-), -\delta); \mathbb{E}_0(J_-, -\delta))$*, and* $\mathbb{G} \in C^1(\mathbb{E}_1(J); \mathbb{E}_0(J))$*, respectively. Moreover,* $\mathbb{G}(0) = 0$*,* $\mathbb{G}'(0) = 0$*, and*

$$\begin{aligned} [\mathbb{G}'(v)w](t) = {}& [F'(t, u_*(t) + v(t)) - F'(t, u_*(t))]w(t)\\ &+ [A(t, u_*(t)) - A(t, u_*(t) + v(t))]w(t)\\ &+ [A'(t, u_*(t))u_*(t) - A'(t, u_*(t) + v(t))(u_*(t) + v(t))]w(t) \end{aligned} \tag{2.23}$$

for all $v, w \in \mathbb{E}_1(J_\pm, \delta)$ *and* $t \in J_\pm$*, respectively, for all* $v, w \in \mathbb{E}_1(J)$ *and* $t \in J$.

(b) We have $\mathbb{H} \in C^1(\mathbb{E}_1(J_+, \delta); \mathbb{F}(J_+, \delta))$*,* $\mathbb{H} \in C^1(\mathbb{E}_1(J_-, -\delta); \mathbb{F}(J_-, -\delta))$*, and* $\mathbb{H} \in C^1(\mathbb{E}_1(J); \mathbb{F}(J))$*, respectively. Moreover* $\mathbb{H}'(0) = 0$ *and*

$$[\mathbb{H}'(v)w](t) = [B'(t, u_*(t)) - B'(t, u_*(t) + v(t))]w(t) \tag{2.24}$$

for all $v, w \in \mathbb{E}_1(J_\pm, \delta)$ *and* $t \in J_\pm$*, respectively, for all* $v, w \in \mathbb{E}_1(J)$ *and* $t \in J$*. Finally,* $\mathbb{H}(0) = 0$ *holds if and only if* $B(t, u_*(t)) = 0$ *for all* t.

(II) Let $\delta \in \mathbb{R}$*. Take* $v \in \mathbb{E}_1(J_\pm, \delta)$ *with* $|v(t)|_{1-1/p} \le r$ *for all* $t \in J_\pm$ *and some* $r \ge 0$*. Then there is a nondecreasing function* $\varepsilon : \mathbb{R}_+ \to \mathbb{R}_+$ *such that* $\varepsilon(r) \to 0$ *as* $r \to 0$ *and*

$$\begin{aligned} \|\mathbb{G}(v)\|_{\mathbb{E}_0(J_\pm,\delta)} &\le \varepsilon(r)\, \|e_\delta v\|_{L_p(J_\pm;X_1)}\,, \\ \|\mathbb{H}(v)\|_{\mathbb{F}(J_\pm,\delta)} &\le \varepsilon(r)\, \|v\|_{\mathbb{E}_1(J_\pm,\delta)}\,, \\ \|e_\delta \mathbb{H}(v)\|_{L_p(J_\pm;Y_1)} &\le \varepsilon(r)\, \|e_\delta v\|_{L_p(J_\pm;X_1)}\,, \end{aligned} \tag{2.25}$$

where ε *can be chosen uniformly for* $t_0 \in \mathbb{R}$ *and for* δ *in compact intervals.*

Proof. (1) In the proof we restrict ourselves to the case J_+. (The other intervals can be treated in the same way.) Hence, all coefficients and the function u_* are τ–periodic. The periodicity will imply that several estimates are uniform in $t \in J_+$. For simplicity we let $t_0 = 0$, $J_+ = \mathbb{R}_+$ and $\tau = 1$, and sometimes we write $\mathbb{E}_k(\delta)$ instead of $\mathbb{E}_k(\mathbb{R}_+, \delta)$, $k = 0, 1$, etc.. We point out that for $\delta \ge 0$ and $t \ge 0$ we have

$$|w(t)|_{BC^{2m-1}} \le c\,|w(t)|_{1-1/p} \le c\,|e^{\delta t} w(t)|_{1-1/p} \le c\,\|w\|_{\mathbb{E}_1(\mathbb{R}_+,\delta)} \tag{2.26}$$

due to (2.5) and (2.7). The constants do not depend on $w \in \mathbb{E}_1(\delta)$. Moreover, here and below the constants are uniform for $t \in \mathbb{R}_+$ and for δ in compact intervals. We further deduce from (2.7) and the periodicity of u_* that

$$\begin{aligned} \|e_\delta\, |w|_{BC^{2m-1}}\, u_*\|^p_{L_p(\mathbb{R}_+;X_1)} &\le c \sum_{n=1}^\infty \|e_\delta\, |w|_{BC^{2m-1}}\|^p_{BC([n-1,n])}\, \|u_*\|^p_{L_p([0,1];X_1)} \\ &\le c \sum_{n=1}^\infty \|e_\delta w\|^p_{\mathbb{E}_1([n-1,n])} = c\,\|w\|^p_{\mathbb{E}_1(\delta)} \end{aligned} \tag{2.27}$$

for all $w \in \mathbb{E}_1(\delta)$. In the following we use (2.26) and (2.27) without further notice. From now on we take $\delta \ge 0$ unless we are dealing with part (II).

We define $\mathbb{G}'(v)$ by (2.23) for $v \in \mathbb{E}_1(\delta)$. The properties (2.12), (2.13), (2.14), and (2.15) imply that $\mathbb{G}(v) \in \mathbb{E}_0(\delta)$, $\mathbb{G}'(v) \in \mathcal{B}(\mathbb{E}_1(\delta), \mathbb{E}_0(\delta))$ and that the first line of (2.25) holds. Further, $\mathbb{G}'(v)$ is the Fréchet derivative of $\mathbb{G}$ at v due to (2.14), (2.15), $\delta \ge 0$, and the formula

$$\begin{aligned} &G(t, v(t) + w(t)) - G(t, v(t)) - [\mathbb{G}'(v)w](t) \\ &\quad = F(t, u_*(t) + v(t) + w(t)) - F(t, u_*(t) + v(t)) - F'(t, u_*(t) + v(t))w(t) \\ &\qquad - \Big(A(t, u_*(t) + v(t) + w(t)) - A(t, u_*(t) + v(t))\Big) w(t) \\ &\qquad - \Big(A(t, u_*(t) + v(t) + w(t))(u_*(t) + v(t)) - A(t, u_*(t) + v(t))(u_*(t) + v(t)) \\ &\qquad\quad - [A'(t, u_*(t) + v(t))(u_*(t) + v(t))]w(t)\Big). \end{aligned}$$

The continuity of $v \mapsto \mathbb{G}'(v)$ is shown in a similar way.

(2) We give the proof of the assertions concerning $\mathbb{H}_j$ for a fixed $j \in \{1, \dots, m\}$ which will mostly be suppressed from the notation. We fix $v \in \mathbb{E}_1(\delta)$ and take $w \in \mathbb{E}_1(\delta)$ with $\|w\|_{\mathbb{E}_1(\delta)} \le r_0$ for a fixed, but arbitrary $r_0 > 0$. In the following, the constants may depend on v and r_0, but not on w. Define $\mathbb{H}'$ by (2.24). One can verify that $\mathbb{H}(v) \in \mathbb{F}(\delta)$ and $\mathbb{H}'(v) \in \mathcal{B}(\mathbb{E}_1(\delta), \mathbb{F}(\delta))$ by similar, but simpler arguments as used below. In view of (2.4) and (2.21), we can write

$$-[H(t, v(t) + w(t)) - H(t, v(t)) - [\mathbb{H}'(v)w](t)](x)$$

$$
\begin{aligned}
&= [B(t,u_*(t)+v(t)+w(t)) - B(t,u_*(t)+v(t)) - B'(t,u_*(t)+v(t))w(t)](x)\\
&= b(t,x,\underline{\nabla}[u_*(t,x)+v(t,x)+w(t,x)]) - b(t,x,\underline{\nabla}[u_*(t,x)+v(t,x)])\\
&\qquad - (\partial_z b)(t,x,\underline{\nabla}[u_*(t,x)+v(t,x)])\cdot \underline{\nabla} w(t,x)\\
&=: h(t,x,\underline{\nabla}[u_*(t,x)+v(t,x)],\underline{\nabla} w(t,x)) \qquad (2.28)
\end{aligned}
$$

where we set $\underline{\nabla} := \underline{\nabla}^{m_j} = (\nabla^0, \nabla^1, \dots, \nabla^{m_j})$ and ∂_z is the partial derivative of b with respect to the corresponding arguments in $E \times E^n \times \cdots \times E^{(n^{m_j})}$. (Recall that we have suppressed the trace operator in front of all $\underline{\nabla}$ terms.) We set $\xi = \underline{\nabla}[u_*(t,x)+v(t,x)]$ and $\eta = \underline{\nabla} w(t,x)$ for fixed $x \in \partial\Omega$ and $t \ge 0$. Then we obtain

$$
\begin{aligned}
h(t,x,\xi,\eta) &= b(t,x,\xi+\eta) - b(t,x,\xi) - (\partial_z b)(t,x,\xi)\cdot\eta,\\
\partial_\xi h(t,x,\xi,\eta) &= (\partial_z b)(t,x,\xi+\eta) - (\partial_z b)(t,x,\xi) - (\partial_{zz} b)(t,x,\xi)\cdot\eta,\\
\partial_\eta h(t,x,\xi,\eta) &= (\partial_z b)(t,x,\xi+\eta) - (\partial_z b)(t,x,\xi),\\
\partial_t h(t,x,\xi,\eta) &= (\partial_t b)(t,x,\xi+\eta) - (\partial_t b)(t,x,\xi) - (\partial_{tz} b)(t,x,\xi)\cdot\eta,\\
\partial_x h(t,x,\xi,\eta) &= (\partial_x b)(t,x,\xi+\eta) - (\partial_x b)(t,x,\xi) - (\partial_{xz} b)(t,x,\xi)\cdot\eta.
\end{aligned}
$$

Assertion (R) yields

$$
|h(t,x,\xi,\eta)|, |\partial_\xi h(t,x,\xi,\eta)| \le \varepsilon(|\eta|)\,|\eta|, \qquad |\partial_\eta h(t,x,\xi,\eta)| \le c\,|\eta|, \tag{2.29}
$$
$$
|\partial_t h(t,x,\xi,\eta)| \le \varepsilon(|\eta|)\,|\eta|, \tag{2.30}
$$
$$
|\partial_x h(t,x,\xi,\eta)| \le \varepsilon(|\eta|)\,|\eta|, \tag{2.31}
$$

where c and ε do not depend on x and t and are uniform for ξ, η in bounded sets (using also the periodicity). Thanks to (2.26) and $\delta \ge 0$, we derive from (2.29) that

$$
\begin{aligned}
&e^{\delta t}\,|H(t,v(t)+w(t)) - H(t,v(t)) - [\mathbb{H}'(v)w](t)|_{Y_0} \le \varepsilon(|w(t)|_Z)\,|e^{\delta t} w(t)|_Z\,,\\
&\|e_\delta\,[\mathbb{H}(v+w) - \mathbb{H}(v) - \mathbb{H}'(v)w]\|_{L_p(\mathbb{R}_+;Y_0)} \le \varepsilon(\|w\|_{\mathbb{E}_1(\delta)})\,\|e_\delta w\|_{L_p(\mathbb{R}_+;X_1)} \qquad (2.32)
\end{aligned}
$$

where we have set $Z = BC^{2m-1}(\Omega)^N$ for a moment. The corresponding inequality for part (II) is shown similarly.

(3) We now consider the estimate involving $W_p^{\kappa_j}(\mathbb{R}_+;Y_0)$, cf. (2.9). We fix $x \in \partial\Omega$ and omit it in the notation. Then we can compute

$$
\begin{aligned}
&h(t,\underline{\nabla}(u_*(t)+v(t)),\underline{\nabla} w(t)) - h(s,\underline{\nabla}(u_*(s)+v(s)),\underline{\nabla} w(s)) \qquad (2.33)\\
&= \int_0^1 (\partial_t h)\big(s+\theta(t-s),\underline{\nabla}(u_*(s)+v(s)),\underline{\nabla} w(s)\big)\,(t-s)\,d\theta\\
&\quad + \int_0^1 (\partial_\xi h)\big(t,\underline{\nabla}(u_*(s)+v(s)) + \theta[\underline{\nabla}(u_*(t)+v(t)) - \underline{\nabla}(u_*(s)+v(s))],\underline{\nabla} w(s)\big)d\theta\\
&\qquad\quad \cdot \underline{\nabla}[u_*(t)+v(t)-u_*(s)-v(s))]\\
&\quad + \int_0^1 (\partial_\eta h)\big(t,\underline{\nabla}(u_*(t)+v(t)),\underline{\nabla} w(s)+\theta\underline{\nabla}(w(t)-w(s))\big)d\theta \cdot \underline{\nabla}(w(t)-w(s))
\end{aligned}
$$

for $t,s \ge 0$. Set $\varphi(t) = h(t,\underline{\nabla}(u_*(t)+v(t)),\underline{\nabla} w(t))$ and $\psi(t) = \underline{\nabla}[u_*(t)+v(t)]$. Then (2.33), (2.30) and (2.29) yield

$$
\begin{aligned}
|\varphi(t)-\varphi(s)|_{Y_0} &\le \varepsilon(|w(s)|_{BC^{2m-1}})\,|\underline{\nabla} w(s)|_{Y_0}\,|t-s| \qquad (2.34)\\
&\quad + \varepsilon(|w(s)|_{BC^{2m-1}})\,|\underline{\nabla} w(s)|_\infty\,|\psi(t)-\psi(s)|_{Y_0}\\
&\quad + c\,(|w(t)|_{BC^{2m-1}} + |w(s)|_{BC^{2m-1}})\,|\underline{\nabla}(w(t)-w(s))|_{Y_0}
\end{aligned}
$$

for all $t,s \ge 0$. We put $I(s) = [s-1,s+1]\cap\mathbb{R}_+$. Combining (2.34) with Lemma 2.3, (2.32), (2.7), (2.11), $\delta \ge 0$ and the periodicity of u_*, we derive

$$
\begin{aligned}
&[e_\delta\,(\mathbb{H}(v+w) - \mathbb{H}(v) - \mathbb{H}'(v)w)]^p_{W_p^\kappa(\mathbb{R}_+;Y_0)} = [e_\delta\varphi]^p_{W_p^\kappa(\mathbb{R}_+;Y_0)}\\
&\quad \le c\,\|e_\delta\varphi\|^p_{L_p(\mathbb{R}_+;Y_0)} + c\int_0^\infty \int_{I(s)} e^{\delta p s}\,\frac{|\varphi(t)-\varphi(s)|^p_{Y_0}}{|t-s|^{1+\kappa p}}\,dt\,ds
\end{aligned}
$$

$$
\begin{aligned}
&\le c\,\varepsilon(\|w\|_{\mathbb{E}_1(\delta)})^p \Big[\|w\|^p_{\mathbb{E}_1(\delta)} + \int_0^\infty \int_{I(s)} e^{\delta ps}\,|\underline{\nabla} w(s)|^p_{Y_0}\,\frac{|t-s|^p}{|t-s|^{1+\kappa p}}\,dt\,ds\\
&\quad + \int_0^\infty \int_{I(s)} e^{\delta ps}\,\frac{|\underline{\nabla}(w(t)-w(s))|^p_{Y_0}}{|t-s|^{1+\kappa p}}\,dt\,ds\\
&\quad + \int_0^\infty \int_{I(s)} |\underline{\nabla} w(s)|^p_\infty\, e^{\delta ps}\,\frac{|\psi(t)-\psi(s)|^p_{Y_0}}{|t-s|^{1+\kappa p}}\,dt\,ds\Big]\\
&\le c\,\varepsilon(\|w\|_{\mathbb{E}_1(\delta)})^p \Big[\|w\|^p_{\mathbb{E}_1(\delta)} + \int_0^\infty e^{\delta ps}\,|\underline{\nabla} w(s)|^p_{Y_0}\,ds + \|e_\delta \underline{\nabla} w\|_{W^\kappa_p(\mathbb{R}_+;Y_0)}\\
&\quad + \sum_{n=1}^\infty \|e_\delta w\|^p_{\mathbb{E}_1([n-1,n])} \int_{n-1}^n \int_{I(s)} \frac{|\underline{\nabla} u_*(t)-\underline{\nabla} u_*(s)|^p_{Y_0}}{|t-s|^{1+\kappa p}}\,dt\,ds\\
&\quad + \|e_\delta w\|^p_{BC(\mathbb{R}_+;X_{1-1/p})} \int_0^\infty \int_{I(s)} e^{\delta ps}\,\frac{|\underline{\nabla} v(t)-\underline{\nabla} v(s)|^p_{Y_0}}{|t-s|^{1+\kappa p}}\,dt\,ds\Big]\\
&\le c\,\varepsilon(\|w\|_{\mathbb{E}_1(\delta)})^p \Big[\|w\|^p_{\mathbb{E}_1(\delta)} + \|e_\delta \underline{\nabla} w\|^p_{W^{\kappa_j}_p(\mathbb{R}_+;Y_0)} + \|w\|^p_{\mathbb{E}_1(\delta)} \|u_*\|^p_{\mathbb{E}_1([0,\tau])}\Big]\\
&\le c\,\varepsilon(\|w\|_{\mathbb{E}_1(\delta)})^p \|w\|^p_{\mathbb{E}_1(\delta)}.
\end{aligned}
$$

The above estimate and (2.32) show that $v \mapsto \mathbb{H}_j(v) \in W^{\kappa_j}_p(\mathbb{R}_+;Y_0)$ is differentiable. The corresponding inequality in part (II) is shown in the same way.

(4) We further have to prove inequality (2.32) with $L_p(\mathbb{R}_+;Y_0)$ replaced by $L_p(\mathbb{R}_+;Y_1)$. This can be done essentially as in the autonomous case treated in parts (4) and (5) of the proof of [15, Proposition 10], using (2.31). By means of a change of coordinates one can reduce the problem to the unit ball K in $\mathbb{R}^{n-1}$ instead of $\partial\Omega$. We restrict ourselves to the case $m_j = 2m-1$ since the other cases can be treated by an induction argument as in part (5) of the proof of [15, Proposition 10]. In the crucial estimate after (69) in [15] one only has to change the term involving $u_*(t)$, which we now estimate by

$$
\begin{aligned}
&\int_0^\infty \varepsilon(|w(t)|_{BC^{2m-1}})^p\,|e^{\delta t} w(t)|^p_{BC^{2m-1}} \iint_{K^2} \frac{|\underline{\nabla} u_*(t,y)-\underline{\nabla} u_*(t,x)|^p}{|y-x|^{n-2+p}}\,dx\,dy\,dt\\
&\le c\,\varepsilon(\|w\|_{BC(\mathbb{R}_+;BC^{2m-1})})^p \sum_{n=1}^\infty \|e_\delta\, w\|^p_{BC([n-1,n];BC^{2m-1})} \int_{n-1}^n |u_*(t)|^p_1\,dt\\
&\le c\,\varepsilon(\|w\|_{\mathbb{E}_1(\delta)})^p \sum_{n=1}^\infty \|e_\delta\, w\|^p_{\mathbb{E}_1([n-1,n])} \le c\,\varepsilon(\|w\|_{\mathbb{E}_1(\delta)})^p\,\|w\|^p_{\mathbb{E}_1(\delta)}
\end{aligned}
$$

employing (2.8), (2.7), and the periodicity of u_*. The corresponding inequality in part (II) is shown in the same way. Finally, the continuity of $v \mapsto \mathbb{H}'(v)$ can be established by similar methods. ☐

For later use we state Corollary 12 of [15] which could also be deduced from the above proposition.

Corollary 2.5. *Assume that (R) holds. For every $t \in \mathbb{R}$, the maping $u_0 \mapsto B(t,u_0)$ belongs to $C^1(X_{1-1/p};Y_{1-1/p})$ with the derivative $B'(t,u_0)$ given by (2.4).*

3. Local well–posedness and regularity

We start with the basic local existence and uniqueness result for (2.2). Using Proposition 2.4, the proof follows the lines of the proof of Proposition 13 in [15] and it is therefore omitted.

Proposition 3.1. *Let $t_0 \in \mathbb{R}$. Assume that condition (R) holds and that (E) and (LS) hold at t_0 and a function $u_0 \in X_{1-1/p}$ satisfying $B(t_0,u_0) = 0$. Then there* *is a number* $T = T(t_0, u_0) > 0$ *such that the problem* (2.2) *has a unique solution* $u \in \mathbb{E}_1([t_0, t_0 + T])$.

Under the assumptions of Proposition 3.1, let $t^+(t_0, u_0)$ be the supremum of those $T > 0$ such that (2.2) has a solution $u \in \mathbb{E}_1([t_0, t_0 + T])$. Proposition 3.1 implies that $t^+(t_0, u_0) > 0$. This solution is unique provided that (E) and (LS) hold at t and the function $u(t)$ for each $t \in [t_0, t_0 + t^+(t_0, u_0))$. We now establish our main well–posedness result. It says that (2.2) generates a local flow acting on the solution manifolds

$$\mathcal{M}(t) = \{w \in X_{1-1/p} : B(t, w) = 0\}. \tag{3.1}$$

Moreover, the equation possesses a smoothing effect because of the quasilinear structure of the differential equation. To state this property, we write $(t - t_0)u$ for the function $v(t) = (t-t_0)u(t)$ with $t \geq t_0$. For given $u_0 \in X_{1-1/p}$ and $s \in \mathbb{R}$, we define the space

$$X^0_{1-1/p}(s) = \{z_0 \in X_{1-1/p} : B'(s, u_0)z_0 = 0\},$$

which is the tangent space of $\mathcal{M}(s)$ at u_0 if $u_0 \in \mathcal{M}(s)$. Let $s \in \mathbb{R}$ and let $u_0 \in X_{1-1/p}$ satisfy (E) and (LS) at $s \in \mathbb{R}$. Proposition 5 of [15] yields operators

$$\widehat{\mathcal{N}}(s) \in \mathcal{B}(Y_{1-1/p}, X_{1-1/p}) \quad \text{with} \quad B'(s, u_0)\widehat{\mathcal{N}}(s) = I. \tag{3.2}$$

Using also Corollary 2.5, we define the projections $\mathcal{P}(s) \in \mathcal{B}(X_{1-1/p}, X^0_{1-1/p}(s))$ by

$$\mathcal{P}(s) = I - \widehat{\mathcal{N}}(s)B'(s, u_0) \qquad \text{for each} \quad s \in \mathbb{R}.$$

Theorem 3.2. *Let* $t_0 \in \mathbb{R}$. *Assume that condition (R) holds and that (E) and (LS) hold for* t_0 *at a function* $u_0 \in \mathcal{M}(t_0)$. *Let* $u = u(\cdot; t_0, u_0)$ *denote the solution of* (2.2)*, and let (E) and (LS) hold at* t *and the function* $u(t; t_0, u_0)$ *for each* $t \in [t_0, t_0 + t^+(t_0, u_0))$. *Let* $T \in (0, t^+(t_0, u_0))$ *and* $J = [t_0, t_0 + T]$. *Then the following assertions are true.*

(a) There is an open ball $B_\rho(u_0)$ *in* $X_{1-1/p}$ *such that there exists a solution* $w \in \mathbb{E}_1(J)$ *of* (2.2) *for each initial value* $w_0 \in B_\rho(u_0)$ *satisfying* $B(t_0, w_0) = 0$. *Moreover, there is an open ball* W_0 *in* $X^0_{1-1/p}(t_0)$ *centered at 0 and a map* $\Phi(\cdot, t_0) \in C^1(W_0; \mathbb{E}_1(J))$ *with uniformly bounded derivative and* $\Phi(\cdot, t_0)0 = 0$ *such that* $w = u + \Phi(\cdot, t_0)(\mathcal{P}(t_0)(w_0 - u_0))$.

(b) We have $(t - t_0)u \in H^1_p(J; X_1) \cap H^2_p(J; X_0)$*, and thus* $u \in C^1((t_0, t_0 + T]; X_{1-1/p}) \cap C^{2-1/p}((t_0, t_0 + T]; X_0) \cap C^{1-1/p}((t_0, t_0 + T]; X_1)$.

(c) Assume in addition that (E) and (LS) hold for each $u_1 \in \mathcal{M}(t)$ *and all* $t \in \mathbb{R}$. *If the number* $t^+(u_0)$ *is finite, then* $\|u\|_{\mathbb{E}_1([t_0, t_0+t^+(t_0,u_0)))} = \infty$ *and* $u(t)$ *does not converge in* $X_{1-1/p}$ *as* $t \to t^+(t_0, u_0)$.

Proof. Assertion (a) can be shown as part (a) of Theorem 14 in [15], using the new Proposition 2.4. Moreover, (c) is a consequence of a standard argument, see the proof of Theorem 14(c) in [15].

(b) Take numbers $T > 0$ and $\epsilon \in (0, 1)$ such that u is a solution of (2.2) on $[t_0, t_0 + T']$ with $T' = (1 + \epsilon)T$. Let $\lambda \in (1 - \epsilon, 1 + \epsilon)$, and $u_\lambda(t) = u(\lambda t + t_0)$. Then $v = u_\lambda$ is the unique solution of the problem

$$\begin{aligned} \partial_t v(t) + \lambda A(\lambda t + t_0, v(t))v(t) &= \lambda F(\lambda t + t_0, v(t)), \qquad \text{on } \Omega, \;\; t > 0, \\ B(\lambda t + t_0, v(t)) &= 0, \qquad \text{on } \partial\Omega, \;\; t \geq 0, \\ v(0) &= u_0, \qquad \text{on } \Omega, \end{aligned} \tag{3.3}$$

on the time interval $[0, T]$. We define $A_*(t)$ and $B_*(t)$ as in formulas (2.16) but replacing there $u_*(t)$ by $u(t + t_0)$, and we temporarily set $G(\lambda, t, v) = -\lambda A(\lambda t + t_0, v(t))v(t) + A_*(t)v(t) + \lambda F(\lambda t + t_0, v(t))$ and $H(\lambda, t, v) = B_*(t)v(t) - B(\lambda t + t_0, v(t))$ for $t \in [0, T]$ and $v \in \mathbb{E}_1([0, T])$. Then the problem (3.3) is equivalent to

$$\partial_t v(t) + A_*(t)v(t) = G(\lambda, t, v(t)), \qquad \text{on } \Omega, \;\; t > 0,$$

$$B_*(t)v(t) = H(\lambda, t, v(t)), \qquad \text{on } \partial\Omega, \ \ t \geq 0, \tag{3.4}$$
$$v(0) = u_0, \qquad \text{on } \Omega.$$

Observe that the compatibility condition $H(\lambda, 0, u_0) = B_*(0)u_0$ holds. Let $\mathbb{G}(\lambda, \cdot)$ and $\mathbb{H}(\lambda, \cdot)$ be the substitution operators for $G(\lambda, \cdot)$ and $H(\lambda, \cdot)$. We claim that $\mathbb{G} \in C^1((1-\epsilon, 1+\epsilon)\times\mathbb{E}_1([0,T]); \mathbb{E}_0([0,T]))$ with $\partial_2\mathbb{G}(1,u) = 0$ and $\mathbb{H} \in C^1((1-\epsilon, 1+\epsilon)\times\mathbb{E}_1([0,T]); \mathbb{F}([0,T]))$ with $\partial_2\mathbb{H}(1,u) = 0$. Most of this claim can be established as in the proof of Proposition 2.4, using the time regularity of the coefficients assumed in (R). Only the differentiability of $\lambda \mapsto \mathbb{H}_j(\lambda, v)$ requires new arguments. To check this fact, we work in the framework of the proof of Proposition 2.4. We set $\xi = \underline{\nabla} v(t,x)$ for $t \in [0,T]$, $x \in \partial\Omega$ and $v \in \mathbb{E}_1([0,T])$, as well as

$$h(t,x,\xi) = -\big[b_j(\mu t + t_0, x, \xi) - b_j(\lambda t + t_0, x, \xi) - (\mu - \lambda)t(\partial_t b_j)(\lambda t + t_0, x, \xi)\big]$$

for $\lambda, \mu \in (1-\epsilon, 1+\epsilon)$. Since b_j is C^2 in (t,ξ) by (R), we obtain

$$|\partial_t h(t,x,\xi)| \leq \varepsilon(|\lambda - \mu|)\,|\lambda - \mu| \quad \text{and} \quad |\partial_\xi h(t,x,\xi)| \leq \varepsilon(|\lambda - \mu|)\,|\lambda - \mu|.$$

Therefore,

$$|h(t,x,\underline{\nabla} v(t,x)) - h(s,x,\underline{\nabla} v(s,x))| \leq \varepsilon(|\lambda - \mu|)\,|\lambda - \mu|\,\big(|t-s| + |\underline{\nabla} v(t,x) - \underline{\nabla} v(s,x)|\big).$$

Now we can proceed as in part (3) of the proof of Proposition 2.4 to show the required differentiability with respect to $W_p^{\kappa_j}([0,T]; Y_0)$. The differentiability in $L_p(J; Y_{j,1})$ can be deduced as in part (4) of the proof of Proposition 2.4 using the above definition of h instead of the definition given in (2.28).

The function $z_0 = u_0 - \widehat{\mathcal{N}}(t_0)H(\lambda, 0, u_0)$ belongs to $X^0_{1-1/p}(t_0)$. Fixing this z_0, we introduce the map

$$\mathcal{L}_0 : (1-\epsilon, 1+\epsilon) \times \mathbb{E}_1([0,T]) \to \mathbb{E}_1([0,T]);$$
$$\mathcal{L}_0(\lambda, v) = v - S(0, z_0 + \widehat{\mathcal{N}}(t_0)\gamma_0\mathbb{H}(\lambda, v), \mathbb{G}(\lambda, v), \mathbb{H}(\lambda, v)),$$

where S is the solution operator of (2.18) for the present operators $A_*(t)$ and $B_*(t)$. Since u solves (2.2), we have $\mathcal{L}_0(1,u) = 0$. By the above observations, $\mathcal{L}_0$ is a C^1–map with $\partial_2\mathcal{L}_0(1,u) = I$.

The implicit function theorem thus yields an $\epsilon' \in (0,\epsilon)$, a ball $\mathbb{B}_{\rho_0}(u)$ in $\mathbb{E}_1([0,T])$, and a map $\Psi \in C^1((1-\epsilon', 1+\epsilon'); \mathbb{E}_1([0,T]))$ such that $\Psi(1) = u$ and $\Psi(\lambda)$ solves (3.4) with u_0 replaced by $u_0(\lambda) := [\Psi(\lambda)](0)$. We further have

$$u_0(\lambda) = z_0 + \widehat{\mathcal{N}}(t_0)H(\lambda, 0, u_0(\lambda)) = u_0 + \widehat{\mathcal{N}}(t_0)\big(H(\lambda, 0, u_0(\lambda)) - H(\lambda, 0, u_0)\big),$$
$$u_0(\lambda) - u_0 = -\widehat{\mathcal{N}}(t_0)\big(B(t_0, u_0(\lambda)) - B(t_0, u_0) - B'(t_0, u_0)(u_0(\lambda) - u_0)\big).$$

Therefore (3.2), Corollary 2.5 and (2.7) yield

$$\begin{aligned}|u_0(\lambda) - u_0|_{1-1/p} &\leq c\varepsilon(|u_0(\lambda) - u_0|_{1-1/p})\,|u_0(\lambda) - u_0|_{1-1/p}\\ &\leq c\varepsilon(c\,\|\Psi(\lambda) - \Psi(1)\|_{\mathbb{E}_1})\,|u_0(\lambda) - u_0|_{1-1/p}\end{aligned}$$

for constants c and a function ε with $\varepsilon(r) \to 0$ as $r \to 0$ which do not depend on λ. Decreasing $\epsilon' > 0$, we deduce that $u_0(\lambda) = u_0$, and thus $\Psi(\lambda)$ solves (3.3) provided $|\lambda - 1|$ is sufficiently small. So $u_\lambda = \Psi(\lambda)$ by the uniqueness of (3.3).

As a result, $u_\lambda = \Psi(\lambda) \in \mathbb{E}_1([0,T])$ is continuously differentiable in λ with derivative $(\frac{d}{d\lambda}u_\lambda)(t) = t\dot{u}(\lambda t + t_0)$. Taking $\lambda = 1$, we deduce that $(t-t_0)\partial_t u \in \mathbb{E}_1(J)$. Consequently, $\partial_t((t-t_0)u) = (t-t_0)\partial_t u + u \in \mathbb{E}_1(J) \hookrightarrow C(J; X_{1-1/p})$, and hence $(t-t_0)u \in H^2_p(J; X_0) \cap H^1_p(J; X_1) \cap C^1(J; X_{1-1/p})$. Assertion (b) now follows from Sobolev's embedding theorem. □

We will need a quantitative version of Theorem 3.2(b). In order to avoid technical problems, we restrict ourselves to the autonomous setting which is sufficient for our main result Theorem 5.3. So we just recall Theorem A.1 of [16]. We note that the condition (RR) in [16] follows from (R) of our present paper.

Proposition 3.3. *Assume that condition (R) holds for maps $A(t,u) = A(u)$, $F(t,u) = F(u)$ and $B(t,u) = B(u)$ not depending on time t explicitely. Moreover, let (E) and (LS) hold for a function $u_0 \in X_{1-1/p}$ with $B(u_0) = 0$. Let u denote the solution of (2.2) with $t_0 = 0$, and let (E) and (LS) hold at the function $u(t)$ for each $t \in J = [0,T]$, where $T \in (0, t^+(u_0))$ is fixed. Then there exists a $\rho > 0$ such that for each initial value $v_0 \in X_{1-1/p}$ with $B(v_0) = 0$ and $|v_0 - u_0|_{1-1/p} < \rho$ the solution v of (2.2) with initial condition $v(0) = v_0$ satisfies*

$$\|t(v-u)\|_{H^1_p(J;X_1)} + \|t(v-u)\|_{H^2_p(J;X_0)} \le c\,|v_0 - u_0|_{1-1/p}\,,$$

with a uniform constant c for such v_0.

4. Linearization at a periodic solution

In our main results we study the qualitative behavior of (2.2) near a periodic solution u_* based on exponential splittings of the linearization

$$\begin{aligned} \partial_t v(t) + A_*(t)v(t) &= g(t) \qquad \text{on } \Omega, \quad t > t_0, \\ B_{j*}(t)v(t) &= h_j(t) \qquad \text{on } \partial\Omega, \quad t \ge t_0, \quad j \in \{1,\dots,m\}, \\ v(t_0) &= v_0, \qquad \text{on } \Omega. \end{aligned} \tag{4.1}$$

of (2.2) along u_*, cf. (2.16). Here we work under the following hypothesis.

Hypothesis 4.1. Conditions (R) and (P) are true, (2.2) has a τ–periodic solution u_*, and (E) and (LS) hold at t and $u_*(t)$ for each $t \in [0,\tau]$.

As a preparation we have to establish several results on the non-autonomous linear problem (4.1), which are of independent interest. Here and below we assume that Hypothesis 4.1 holds and that the operators $A_*(t)$ and $B_*(t)$ are defined by (2.16) for the given τ–periodic solution u_*. Observe that Theorem 3.2(b) implies the crucial regularity property

$$u_* \in C^{1-1/p}([0,\tau]; X_1) \cap C^1([0,\tau]; X_{1-1/p}) \cap C^{2-1/p}([0,\tau]; X_0). \tag{4.2}$$

As Proposition 4.6 below indicates, we have to modify the boundary operator $B_{j*}(t)$ and the corresponding nonlinearity $H_j(t,v)$ in the case that $m_j = 0$.

Remark 4.2. In the above situation, if $m_j = 0$ for some $j \in \{1,\dots,m\}$, then the boundary condition $b_j(t,x,(u_*(t,x)+v(x))) = 0$ in (2.2) holds on $\partial\Omega$ if and only if

$$\partial_z b_j(t,x,u_*(t,x))v(x) = \partial_z b_j(t,x,u_*(t,x))v(x) - b_j(t,x,u_*(t,x)+v(x)) \tag{4.3}$$

for all $x \in \partial\Omega$, where $t \in \mathbb{R}$ and $v \in X_{1-1/p}$. The corresponding boundary condition in (LS) is given by

$$\partial_z b_j(t,x,u_*(t,x))\varphi(0) = 0. \tag{4.4}$$

Due to Remark 1 in [15] (see also [3] and [7]), the Lopatinskii-Shapiro condition (LS) is equivalent to the surjectivity of a certain linear map $\mathbb{B}(t,x)P(t,x) : \mathbb{C}^{2mN} \to \mathbb{C}^{mN}$. Here, $P(t,x)$ is a projection on $\mathbb{C}^{2mN}$ (called $P_+(b,\sigma)$ at the end of Section 6.1 in [7]) and $\mathbb{B}(t,x) = (\mathbb{B}_1(t,x),\dots,\mathbb{B}_m(t,x))$ is given by $N \times 2mN$ matrices $\mathbb{B}_k(t,x)$ with $\mathbb{B}_j(t,x) = (\partial_z b_j(t,x,u_*(t,x)),0,\dots,0)$. Hence, if $\partial_z b_j(t,x,u_*(t,x))$ were not surjective for some $t \in [0,\tau]$ and $x \in \partial\Omega$, then (LS) would be wrong. The matrices $\partial_z b_j(t,x,u_*(t,x))$ are thus invertible, and the inverses $[\partial_z b_j(t,x,u_*(t,x))]^{-1}$ are uniformly bounded for $t \in \mathbb{R}$ and $x \in \partial\Omega$ by the compactness of $[0,\tau] \times \partial\Omega$. As a result, the boundary condition (4.3) is equivalent to the equation

$$v(x) = v(x) - [\partial_z b_j(t,x,u_*(t,x))]^{-1} b_j(t,x,u_*(t,x)+v(x)), \quad x \in \partial\Omega,$$

and the initial condition (4.4) in (LS) is equivalent to $\varphi(0) = 0$ on $\partial\Omega$. We thus redefine $B_{j*}(t)$ and $H_j(t,v)$ in the case of $m_j = 0$ by setting

$$B_{j*}(t) = \gamma \quad \text{and} \quad H_j(t,v) = \gamma v - [\partial_z b_j(t,\cdot,u_*(t))]^{-1} b_j(t,\cdot,u_*(t)+v). \tag{4.5}$$

We note that the maps in (4.5) satisfy the mapping properties from Proposition 2.4 for $m_j = 0$. This can be seen as in Proposition 2.4 since the function $\tilde{h}(t,x,\xi,\eta) := [\partial_z b_j(t,x,u_*(t,x))]^{-1} h(t,x,\xi,\eta)$ fulfills the same estimates (2.29), (2.30) and (2.31) as the function h defined by (2.28). These modifications are used below without further notice. $\Diamond$

We start with Hölder properties of the operators $A_*(t)$ and $B_*(t)$.

Lemma 4.3. *Let Hypothesis 4.1 hold. Then there is a constant c such that*

$$|(A_*(t) - A_*(s))v|_0 \le c\,|t-s|\,|v|_1 + c\,|t-s|^{1-\frac{1}{p}}\,|v|_{1-1/p}\,, \tag{4.6}$$

$$|(B_*(t) - B_*(s))v|_{Y_1} \le c\,|t-s|\,|v|_1 + c\,|t-s|^{1-\frac{1}{p}}\,|v|_{1-1/p} \tag{4.7}$$

for all $t, s \in \mathbb{R}$ and $v \in X_1$.

Proof. Recall the definition of $A_*(t)$ and $B_*(t)$ in (2.16), (2.4) and Remark 4.2. The first inequality easily follows from (4.2) and the assumptions (R) and (P). (The C^2–condition in (R) for a_α and f is used here for the lower order terms.) Let $t, s \in \mathbb{R}$ and $v \in X_1$. The constants in the following estimates do not depend on t, s or v. Observe that (4.7) trivially holds if $m_j = 0$ since then $B_{j*}(t) = \gamma$ for all t by Remark 4.2. So we can assume that $m_j > 0$. We further have

$$|(B_*(t) - B_*(s))v|_{Y_1} \le c \sum_{j=1, m_j>0}^{m} \sum_{k=0}^{m_j} \sum_{|\beta|=k} |(b^*_{j\beta}(t) - b^*_{j\beta}(s))D^\beta v|_{Y_{j,1}}\,,$$

where $b^*_{j\beta}$ was defined in (2.17). Let $|\beta| \le m_j$. Using (2.6), (2.5) and (2.8), we obtain

$$\begin{aligned}
|(b^*_{j\beta}(t) - b^*_{j\beta}(s))D^\beta v|_{Y_{j,1}} &\le c\,|\underline{\nabla}^{2m-m_j-1}[(b^*_{j\beta}(t) - b^*_{j\beta}(s))D^\beta v]|_{W_p^{1-\frac{1}{p}}(\partial\Omega)^N} \\
&\le c \sum_{l=0}^{2m-m_j-1} \sum_{|\lambda|=l, |\mu|=2m-1-l} |\partial^\lambda (b^*_{j\beta}(t) - b^*_{j\beta}(s))\,\partial^\mu v|_{W_p^{1-\frac{1}{p}}(\partial\Omega)^N} \\
&\le c \sum_{l=0}^{2m-m_j-1} \Big(|b^*_{j\beta}(t) - b^*_{j\beta}(s)|_{W_p^{l+1-\frac{1}{p}}(\partial\Omega)^N} |v|_{BC^{2m-1-l}(\overline{\Omega})^N} \\
&\qquad + |b^*_{j\beta}(t) - b^*_{j\beta}(s)|_{BC^l(\partial\Omega)^N} |v|_{W_p^{2m-l-\frac{1}{p}}(\partial\Omega)^N} \Big) \\
&\le c \sum_{l=0}^{2m-m_j-1} \Big(|b^*_{j\beta}(t) - b^*_{j\beta}(s)|_{W_p^{l+1-\frac{1}{p}}(\partial\Omega)^N} |v|_{1-1/p} + |t-s|\,|v|_1 \Big).
\end{aligned}$$

For the final Lipschitz estimate we also employed (R), (4.2), and formula (4.9) below with $[\,\cdot\,]_{1-1/p}$ replaced by $|\cdot|_\infty$. It remains to show

$$d(t,s) := [b^*_{j\beta}(t) - b^*_{j\beta}(s)]_{W_p^{l+1-\frac{1}{p}}(\partial\Omega)^N} \le c\,|t-s|^{1-\frac{1}{p}} \tag{4.8}$$

for every $|\beta| \le m_j$, $l \le 2m - m_j - 1$ and $j \in \{1, \dots, m\}$. In the following we restrict ourselves to the highest order case $l = 2m - m_j - 1$. By differentiation, one sees that $d(t,s)$ is less than a linear combination of terms of the form

$$\begin{aligned}
D(t,s) := \big[&\partial_x^\lambda\,\partial_z^\mu\,\partial_\beta b_j(t,\cdot,\underline{\nabla}^{m_j} u_*(t)) \cdot \partial_x^\nu \underline{\nabla}^{m_j} u_*(t)) \\
&- \partial_x^\lambda\,\partial_z^\mu\,\partial_\beta b_j(s,\cdot,\underline{\nabla}^{m_j} u_*(s)) \cdot \partial_x^\nu \underline{\nabla}^{m_j} u_*(s))\big]_{1-1/p}
\end{aligned} \tag{4.9}$$

for multi indeces λ, μ and ν with $|\lambda + \mu| \le 2m - m_j - 1$ and $|\nu| \le 2m - m_j - 1$. Here and below we write $[w]_{1-1/p}$ instead of $[w]_{W_p^{1-1/p}(\partial\Omega)^N}$, and we have used the fact that this expression dominates the norm of $W_p^{1-1/p}(\partial\Omega)^N$. Setting

$$b^\bullet(t) = \partial_x^\lambda\,\partial_z^\mu\,\partial_\beta b_j(t,\cdot,\underline{\nabla}^{m_j} u_*(t)) \quad \text{and} \quad u^\bullet_*(t) = \partial_x^\nu \underline{\nabla}^{m_j} u_*(t),$$

we can thus estimate

$$\begin{aligned} D(t,s) &\le [(b^\bullet(t)-b^\bullet(s))\cdot u_*^\bullet(t)]_{1-1/p} + [b^\bullet(s)\cdot(u_*^\bullet(t)-u_*^\bullet(s))]_{1-1/p} \\ &\le c\,[b^\bullet(t)-b^\bullet(s)]_{1-1/p}\,\|u_*\|_{BC([0,\tau];X_1)} + c\sup_{0\le r\le\tau}[b^\bullet(r)]_{1-1/p}\,|u_*^\bullet(t)-u_*^\bullet(s)|_1 \\ &\le c\,[b^\bullet(t)-b^\bullet(s)]_{1-1/p} + c\,|t-s|^{1-1/p}\sup_{0\le r\le\tau}[b^\bullet(r)]_{1-1/p}\,, \end{aligned}$$

thanks to (2.6), (2.8), (4.2) and the periodicity of the coefficients. We next establish the Hölder property

$$[b^\bullet(t)-b^\bullet(s)]_{1-1/p} \le c\,|t-s|^{1-1/p} \tag{4.10}$$

which then implies (4.8) and thus (4.7).

To prove (4.10), we can restrict ourselves to the case that $\partial\Omega$ is the unit ball K in $\mathbb{R}^{n-1}$ by means of a change of coordinates. We set $u^\circ = \underline{\nabla}^{m_j} u_*$ and $b^\circ = \partial_x^\lambda\,\partial_z^\mu\,\partial_\beta b_j$ with $|\lambda+\mu| = 2m-m_j-1$. For $x,y\in K$, we have

$$\begin{aligned} & b^\circ(t,y,u^\circ(t,y)) - b^\circ(s,y,u^\circ(s,y)) - (b^\circ(t,x,u^\circ(t,x)) - b^\circ(s,x,u^\circ(s,x))) \\ &= \int_0^1 \Big(\partial_2 b^\circ(t,x+\theta(y-x),u^\circ(t,y)) - \partial_2 b^\circ(s,x+\theta(y-x),u^\circ(s,y))\Big)(y-x)\,d\theta \\ &\quad + b^\circ(t,x,u^\circ(t,y)) - b^\circ(t,x,u^\circ(t,x)) - \big(b^\circ(s,x,u^\circ(s,y)) - b^\circ(s,x,u^\circ(s,x))\big). \end{aligned}$$

In the above equation we denote the integral term by S_1 and the last line by S_2. Observe that $u^\circ\in C^1([0,\tau];BC(\overline{\Omega})^N)$ by (2.5) and (4.2). This property and assumption (R) yield

$$\iint_{K^2}\frac{|S_1|_{Y_0}^p}{|y-x|^{n-2+p}}\,dy\,dx \le c\,|t-s|^p \iint_{K^2}\frac{|y-x|^p}{|y-x|^{n-2+p}}\,dy\,dx \le c\,|t-s|^p. \tag{4.11}$$

We rewrite the term S_2 as

$$\begin{aligned} S_2 &= \int_0^1 \partial_z b^\circ(t,x,u^\circ(t,x)+\theta(u^\circ(t,y)-u^\circ(t,x)))\cdot[u^\circ(t,y)-u^\circ(t,x)]\,d\theta \\ &\quad - \int_0^1 \partial_z b^\circ(s,x,u^\circ(s,x)+\theta(u^\circ(s,y)-u^\circ(s,x)))\cdot[u^\circ(s,y)-u^\circ(s,x)]\,d\theta \\ &= \int_0^1 \partial_z b^\circ(t,x,u^\circ(t,x)+\theta(u^\circ(t,y)-u^\circ(t,x))) \qquad (4.12)\\ &\qquad\quad \cdot[(u^\circ(t,y)-u^\circ(s,y)) - (u^\circ(t,x)-u^\circ(s,x))]\,d\theta \\ &\quad + \int_0^1 \Big[\partial_z b^\circ(t,x,u^\circ(t,x)+\theta(u^\circ(t,y)-u^\circ(t,x))) \qquad (4.13)\\ &\qquad\quad - \partial_z b^\circ(s,x,u^\circ(s,x)+\theta(u^\circ(s,y)-u^\circ(s,x)))\Big]\cdot[u^\circ(s,y)-u^\circ(s,x)]\,d\theta. \end{aligned}$$

Using again (R), (2.8) and (4.2), we estimate

$$\begin{aligned} \iint_{K^2}\frac{|(4.12)|^p}{|y-x|^{n-2+p}}\,dy\,dx &\le c\,[u^\circ(t)-u^\circ(s)]_{1-1/p} \\ &\le c\,|u_*(t)-u_*(s)|_1 \le c\,|t-s|^{1-\frac{1}{p}}. \end{aligned} \tag{4.14}$$

Using that $\partial_t b_j\in C^{2m-m_j+1}$ due to (R), we similarly obtain

$$\iint_{K^2}\frac{|(4.13)|^p}{|y-x|^{n-2+p}}\,dy\,dx \le c\,|t-s|^{1-\frac{1}{p}}\,[u^\circ(s)]_{1-1/p}^p \le c\,|t-s|^{1-\frac{1}{p}}. \tag{4.15}$$

Combining (4.11), (4.14) and (4.15), we conclude that (4.10) holds. □

Assuming Hypothesis 4.1, we define $A_0(t) = A_*(t)|\ker(B_*(t))$ for each $t\in\mathbb{R}$; i.e.,

$$A_0(t)u = A_*(t)u, \quad u\in D(A_0(t)) = \{u\in X_1 : B_{j*}(t)u = 0,\ j=1,\dots,m\}. \tag{4.16}$$

We show that these operators satisfy the Acquistapace–Terreni conditions from [1] and [2] (which we discuss below): There are constants $\omega \in \mathbb{R}$, $\phi \in (\pi/2, \pi)$, $K > 0$ and $\mu, \nu \in (0,1]$ such that $\mu + \nu > 1$ and

$$\lambda \in \rho(-A_0(t) - \omega), \qquad \|(\lambda + \omega + A_0(t))^{-1}\| \le \frac{K}{1 + |\lambda|}, \tag{4.17}$$

$$\|(A_0(t) + \omega)(\lambda + \omega + A_0(t))^{-1}\,[(\omega + A_0(t))^{-1} - (\omega + A_0(s))^{-1}]\| \le K\,|t - s|^{\mu}\,|\lambda|^{-\nu} \tag{4.18}$$

for all $t, s \in \mathbb{R}$ and $\lambda \in \mathbb{C} \setminus \{0\}$ with $|\arg(\lambda)| \le \phi$.

In Theorem 8.2 of [7] (see also the references therein) it has been proved that the operator $-A_0(t)$ is sectorial on X_0 for each $t \in \mathbb{R}$. We show in the next lemma that the corresponding constants can be chosen uniformly in t, so that (4.17) holds. In this context we also establish the uniformity of various maximal regularity estimates.

Lemma 4.4. *Assume that Hypothesis 4.1 holds. Let $t_0 \in \mathbb{R}$, $T > 0$, $g \in \mathbb{E}_0(\mathbb{R})$ and $h \in \mathbb{F}(\mathbb{R})$. Then the following assertions are true.*

(a) Let $v_0 \in X_{1-1/p}$ with $B_*(t_0)v_0 = h(t_0)$. There is a unique solution $v \in \mathbb{E}_1([0,T])$ of the problem

$$\begin{aligned} \partial_t v(t) + A_*(t + t_0)v(t) &= g(t + t_0), && \text{on } \Omega,\ t \in (0,T], \\ B_*(t + t_0)v(t) &= h(t + t_0), && \text{on } \partial\Omega,\ t \in [0,T], \\ v(0) &= v_0, && \text{on } \Omega. \end{aligned}$$

We have a constant $c > 0$ independent of $t_0 \in \mathbb{R}$ and v_0, g, h such that

$$\|v\|_{\mathbb{E}_1([0,T])} \le c\,(|v_0|_{1-1/p} + \|g\|_{\mathbb{E}_0([t_0,t_0+T])} + \|h\|_{\mathbb{F}([t_0,t_0+T])}). \tag{4.19}$$

(b) In Theorem 2.1 the constant c_1 in the maximal regularity estimate (2.19) *can be chosen to be independent of $t_0 \in \mathbb{R}$.*

(c) Let $s \in \mathbb{R}$ and $v_0 \in X_{1-1/p}$ with $B_*(s)v_0 = h(0)$. There is a unique solution $v \in \mathbb{E}_1([0,T])$ of the autonomous problem

$$\begin{aligned} \partial_t v(t) + A_*(s)v(t) &= g(t), && \text{on } \Omega,\ t \in (0,T], \\ B_*(s)v(t) &= h(t), && \text{on } \partial\Omega,\ t \in [0,T], \\ v(0) &= v_0, && \text{on } \Omega. \end{aligned} \tag{4.20}$$

It satisfies the estimate (4.19) *with $t_0 = 0$ and a constant not depending on $s \in \mathbb{R}$ or on v_0, g, h.*

(d) Condition (4.17) *holds.*

(e) Let $s \in \mathbb{R}$ and $\mu \ge \omega$, where ω is given by (4.17)*, see assertion (d). There is a unique solution $v \in \mathbb{E}_1(\mathbb{R}_-)$ of the autonomous backward problem*

$$\begin{aligned} \partial_t v(t) + (\mu + A_*(s))v(t) &= g(t), && \text{on } \Omega,\ t \le 0, \\ B_*(s)v(t) &= h(t), && \text{on } \partial\Omega,\ t \le 0. \end{aligned}$$

It satisfies the estimate (4.19) *on the interval $\mathbb{R}_-$ with $v_0 = 0$ and a constant not depending on $s \in \mathbb{R}$ or on g, h.*

Proof. 1) We fix $t_0 \in \mathbb{R}$, take $s \in \mathbb{R}$, and assume that $B_*(t_0 + s)v_0 = h(t_0 + s)$. We put $J = [0,T]$. Let $u \in \mathbb{E}_1(J)$ be the solution of the problem

$$\begin{aligned} \partial_t u(t) + A_*(t + t_0 + s)u(t) &= g(t + t_0 + s), && \text{on } \Omega,\ t \in (0,T], \\ B_*(t + t_0 + s)u(t) &= h(t + t_0 + s), && \text{on } \partial\Omega,\ t \in [0,T], \\ u(0) &= v_0, && \text{on } \Omega. \end{aligned}$$

We rewrite this system as

$$\begin{aligned} \partial_t u(t) + A_*(t + t_0)u(t) &= g(t + t_0 + s) + A_*(t + t_0)u(t) - A_*(t + t_0 + s)u(t), \\ B_*(t + t_0)u(t) &= h(t + t_0 + s) + B_*(t + t_0)u(t) - B_*(t + t_0 + s)u(t), \end{aligned}$$

$$u(0) = v_0.$$

Since the compatibility condition $B_*(t_0)v_0 = h(t_0+s)+B_*(t_0)u(0)-B_*(t_0+s)u(0)$ holds, Theorem 2.1 yields

$$\begin{aligned}
\|u\|_{\mathbb{E}_1(J)} \le c(T,t_0)\big(&|v_0|_{1-1/p} + \|g\|_{\mathbb{E}_0([t_0+s,t_0+s+T])} + \|h\|_{\mathbb{F}([t_0+s,t_0+s+T])} \\
&+ \|(A_*(\cdot+t_0) - A_*(\cdot+t_0+s))u(\cdot)\|_{\mathbb{E}_0(J)} \\
&+ \|(B_*(\cdot+t_0+s) - B_*(\cdot+t_0))u(\cdot)\|_{\mathbb{F}(J)}\big).
\end{aligned}$$

Here and below the constants do not depend on s. Lemma 4.3 implies that

$$\|(A_*(\cdot+t_0) - A_*(\cdot+t_0+s))u(\cdot)\|_{\mathbb{E}_0(J)} \le c|s|^{1-\frac{1}{p}}\|u\|_{L_p(J;X_1)} \le c|s|^{1-\frac{1}{p}}\|u\|_{\mathbb{E}_1(J)}$$
$$\|(B_*(\cdot+t_0+s) - B_*(\cdot+t_0))u(\cdot)\|_{L_p(J;Y_1)} \le c|s|^{1-\frac{1}{p}}\|u\|_{L_p(J;X_1)} \le c|s|^{1-\frac{1}{p}}\|u\|_{\mathbb{E}_1(J)}.$$

We observe that the coefficients of B_* belong to $C^1(\mathbb{R}; C(\partial\Omega)^N)$ due to (R) and (4.2). In the following calculations we fix an index $j \in \{1,\dots,m\}$ and omit it from the notation partly. As in (2.6), we can estimate

$$\begin{aligned}
&\|(B_*(\cdot+t_0+s) - B_*(\cdot+t_0))u(\cdot)\|_{W_p^\kappa(J;Y_0)} \\
&\le c\sum_{k=0}^{m_j}\sum_{|\beta|=k} \|(b_\beta^*(\cdot+t_0+s) - b_\beta^*(\cdot+t_0))D^\beta u(\cdot)\|_{W_p^\kappa(J;Y_0)} \\
&\le c\sum_{k=0}^{m_j}\sum_{|\beta|=k} \|b_\beta^*(\cdot+t_0+s) - b_\beta^*(\cdot+t_0)\|_{W_p^\kappa(J;C(\partial\Omega)^N)}\, \|D^\beta u(\cdot)\|_{W_p^\kappa(J;Y_0)} \\
&\le c\varepsilon(s)\, \|u\|_{\mathbb{E}_1(J)}\,,
\end{aligned}$$

where $b_{j\beta}^*$ is given by (2.17) and we also used (2.11) and the fact that translations are strongly continuous on $W_p^\kappa(\mathbb{R}; C(\partial\Omega)^N)$. We further set $J_s = [t_0+s, t_0+s+T]$. Combining the above inequalities, we can fix a $\delta(t_0,T) > 0$ such that

$$\begin{aligned}
\|u\|_{\mathbb{E}_1(J)} &\le c(T,t_0)\left(|v_0|_{1-1/p} + \|g\|_{\mathbb{E}_0(J_s)} + \|h\|_{\mathbb{F}(J_s)}\right) + \frac{1}{2}\|u\|_{\mathbb{E}_1(J)}\,, \\
\|u\|_{\mathbb{E}_1(J)} &\le 2c(T,t_0)\left(|v_0|_{1-1/p} + \|g\|_{\mathbb{E}_0(J_s)} + \|h\|_{\mathbb{F}(J_s)}\right).
\end{aligned}$$

for all $|s| \le \delta(t_0,T)$. Assertion (a) now follows from a compactness argument and the periodicity of the coefficients and of u_*.

2) We derive assertion (b) from (a) by a translation. Assertion (c) is a special case of (a). Following the proof of Proposition 1.2 of [21] and using (c), one can verify assertion (d), see also [11, Theorem 2.2]. The last assertion is a consequence of Proposition 9 in [15] (with $Q = 0$), where the uniformity of the constants can be proved by the arguments given in part 1). □

The following lemma deals with the solution operator of an auxiliary stationary problem which is used below to show condition (4.18), for instance.

Lemma 4.5. *Assume that Hypothesis 4.1 holds. Let $t \in \mathbb{R}$, $\varphi \in X_0$, $\psi = (\psi_1,\dots,\psi_m) \in Y_1$, and $\lambda \ge 0$. Let ω be given by (4.17), see Lemma 4.4(d). Then there is a unique solution $u = \mathcal{R}_{\lambda+\omega}(t)(\varphi,\psi) \in X_1$ of the elliptic boundary value problem*

$$\begin{aligned}
(\lambda + \omega + A_*(t))u &= \varphi \qquad \text{on } \Omega, \\
B_{j*}(t)u &= \psi_j \qquad \text{on } \partial\Omega, \quad j \in \{1,\dots,m\}.
\end{aligned} \tag{4.21}$$

This solution is given by $\mathcal{R}_{\lambda+\omega}(t)(\varphi,\psi) = (\lambda+\omega+A_0(t))^{-1}\varphi + \mathcal{N}_{\lambda+\omega}(t)\psi$, where $\mathcal{N}_{\lambda+\omega}(t)\psi$ solves (4.21) with $\varphi = 0$. For $\lambda \ge 1$, we further have

$$\|\mathcal{R}_\omega(t)\|_{\mathcal{B}(X_0\times Y_1,X_1)} \le c \qquad \text{and} \qquad \|\mathcal{N}_{\lambda+\omega}(t)\|_{\mathcal{B}(Y_1,X_0)} \le c\lambda^{\kappa-1}, \tag{4.22}$$

where $\kappa := \max_{j=1,\dots,m} \kappa_j$ and the constant c does not depend on $\lambda \ge 1$ or $t \in \mathbb{R}$.

Proof. Let $\lambda \geq 0$, $t \in \mathbb{R}$, $\varphi \in X_0$ and $\psi \in Y_1$. The existence of $\mathcal{N}_{\omega+\lambda}(t)$ was shown in Proposition 5 in [15]. Since $\lambda + \omega + A_0(t)$ is invertible by Lemma 4.4(d), there is only one solution of (4.21); and it is easy to check that it is given by $u = (\lambda+\omega+A_0(t))^{-1}\varphi + \mathcal{N}_{\lambda+\omega}(t)\psi$. It remains to show the inequalities (4.22). We first let $\lambda > 0$. Set $f = e_\lambda \varphi \in \mathbb{E}_0(\mathbb{R}_-)$ and $g_j = e_\lambda \psi_j \in \mathbb{F}_j(\mathbb{R}_-)$, where $e_\lambda(t) = e^{\lambda t}$, $t \leq 0$, and $j \in \{1, \dots, m\}$. Lemma 4.4(e) gives a unique solution $v_\lambda \in \mathbb{E}_1(\mathbb{R}_-)$ of the autonomous system

$$\begin{aligned} \partial_t v_\lambda(t) + (A_*(s) + \omega) v_\lambda(t) = f(t) \qquad & \text{on } \Omega, \quad t \leq 0, \\ B_{j*}(s) v_\lambda(t) = g_j(t) \qquad & \text{on } \partial\Omega, \quad t \leq 0, \quad j \in \{1, \dots, m\}, \end{aligned} \tag{4.23}$$

satisfying the estimate

$$\|v_\lambda\|_{\mathbb{E}_1(\mathbb{R}_-)} \leq c \sum_{j=1}^m (\|e_\lambda \varphi\|_{\mathbb{E}_0(\mathbb{R}_-)} + \|e_\lambda \psi_j\|_{\mathbb{F}_j(\mathbb{R}_-)}). \tag{4.24}$$

where c does not depend on $s \in \mathbb{R}$, $\lambda > 0$, $\varphi \in X_0$ and $\psi \in Y_1$. The function $u_\lambda := e_{-\lambda} v_\lambda$ then solves

$$\begin{aligned} \partial_t u_\lambda(t) + (A_*(s) + \lambda + \omega) u_\lambda(t) = \varphi \qquad & \text{on } \Omega, \quad t \leq 0, \\ B_{j*}(s) u_\lambda(t) = \psi_j \qquad & \text{on } \partial\Omega, \quad t \leq 0, \quad j \in \{1, \dots, m\}. \end{aligned} \tag{4.25}$$

Clearly, also $u_\lambda(\cdot + \sigma)$ satisfies (4.25) for each $\sigma \leq 0$, so that $e_\lambda u_\lambda(\cdot + \sigma) \in \mathbb{E}_1(\mathbb{R}_-)$ is a solution of (4.23) for the inhomogeneities f and g. Since the solutions of (4.23) in $\mathbb{E}_1(\mathbb{R}_-)$ are unique, we obtain that $e_\lambda u_\lambda(\cdot + \sigma) = v_\lambda = e_\lambda u_\lambda$ which yields $u_\lambda(\sigma) = u_\lambda(0) =: u_\lambda^0$ for every $\sigma \leq 0$. So (4.25) leads to $u_\lambda^0 = \mathcal{R}_{\lambda+\omega}(s)(\varphi, \psi)$. Inequality (4.24) further implies

$$\begin{aligned} \|e_\lambda u_\lambda\|_{\mathbb{E}_1(\mathbb{R}_-)} &\leq c \sum_{j=1}^m (\|e_\lambda \varphi\|_{\mathbb{E}_0(\mathbb{R}_-)} + \|e_\lambda \psi_j\|_{\mathbb{F}_j(\mathbb{R}_-)}) \\ &\leq c \sum_{j=1}^m (\lambda^{-\frac{1}{p}} |\varphi|_0 + \lambda^{-\frac{1}{p}} |\psi_j|_{Y_{j,1}} + \lambda^{\kappa_j - \frac{1}{p}} |\psi_j|_{Y_0}) \end{aligned}$$

with constants not depending on $s \in \mathbb{R}$, $\lambda > 0$, $\varphi \in X_0$ or $\psi \in Y_1$. (One can estimate the norm of $e_\lambda \psi_j$ in $W_p^{\kappa_j}(\mathbb{R}_-; Y_0)$ by interpolation.) We conclude that

$$\lambda^{-\frac{1}{p}} |u_\lambda^0|_1 + \lambda^{1-\frac{1}{p}} |u_\lambda^0|_0 \leq c \sum_{j=1}^m (\lambda^{-\frac{1}{p}} |\varphi|_0 + \lambda^{-\frac{1}{p}} |\psi_j|_{Y_{j,1}} + \lambda^{\kappa_j - \frac{1}{p}} |\psi_j|_{Y_0}),$$

$$|\mathcal{R}_{\lambda+\omega}(s)(\varphi, \psi)|_1 \leq c \sum_{j=1}^m (|\varphi|_0 + |\psi_j|_{Y_{j,1}} + \lambda^{\kappa_j} |\psi_j|_{Y_0}), \tag{4.26}$$

$$|\mathcal{N}_{\lambda+\omega}(s)\psi|_0 \leq c \sum_{j=1}^m \lambda^{\kappa_j - 1} (|\psi_j|_{Y_{j,1}} + |\psi_j|_{Y_0}) \leq c \lambda^{\kappa - 1} |\psi|_{Y_1}, \tag{4.27}$$

where we assume that $\lambda \geq 1$ in (4.27). So the second part of (4.22) has been shown. Finally, noting that the constants in (4.26) do not depend on λ we can let $\lambda \to 0$ in the first inequality in (4.26) obtaining the first part of (4.22). □

The above lemmas now enable us to establish the second Acquistapace and Terreni condition (4.18).

Proposition 4.6. *Assume that Hypothesis 4.1 holds. The operators $A_0(t)$, $t \in \mathbb{R}$, then satisfy (4.18) with $\mu = 1 - \frac{1}{p}$ and $\nu = 1 - \overline{\kappa}$, where $\overline{\kappa} := \max\{\kappa_j; m_j > 0, j = 1, \dots, m\}$ and $0 := \max \emptyset$. Moreover, the graph norms of $A_0(t)$, $t \in \mathbb{R}$, are uniformly equivalent to $|\cdot|_1$.*

Proof. Let $\omega \geq 0$ be given by (4.17), see Lemma 4.4(d). To verify the last assertion, we observe that $\varphi = (\omega + A_0(t))^{-1}(\omega + A_0(t))\varphi$ for all $\varphi \in D(A_0(t))$ and $t \in \mathbb{R}$. The first part of estimate (4.22) thus gives $|\varphi|_1 \leq c\,|(\omega + A_0(t))\varphi|_0$ for a constant $c > 0$ and all $\varphi \in X_1$ and $t \in \mathbb{R}$. The reverse inequality follows from (R).

Take $\lambda \geq 1$. For $t, s \in \mathbb{R}$ and $\varphi \in X_0$, we set

$$v = -(\omega + A_0(s))^{-1}\varphi \quad \text{and} \quad u = (\lambda + \omega + A_0(t))^{-1}(\lambda + \omega + A_0(s))v.$$

We then obtain

$$u - v = -(A_0(t) + \omega)(\lambda + \omega + A_0(t))^{-1}[(\omega + A_0(t))^{-1} - (\omega + A_0(s))^{-1}]\varphi, \quad (4.28)$$

and this function solves the problem

$$\begin{aligned} (\lambda + \omega)(u - v) + A_*(t)(u - v) &= (A_*(s) - A_*(t))v \qquad \text{on } \Omega \\ B_*(t)(u - v) &= (B_*(s) - B_*(t))v, \qquad \text{on } \partial\Omega. \end{aligned}$$

Using Lemma 4.5, we conclude that

$$u - v = (\lambda + \omega + A_0(t))^{-1}(A_*(s) - A_*(t))v + \mathcal{N}_{\lambda+\omega}(t)(B_*(s) - B_*(t))v.$$

Estimates (4.17) and (4.27) and Remark 4.2 thus yield

$$\begin{aligned} |u - v|_0 &\leq \frac{c}{1 + \lambda}|(A_*(t) - A_*(s))v|_0 + c \sum_{j=1, m_j > 0}^{m} \lambda^{\kappa_j - 1}|(B_{j*}(t) - B_{j*}(s))v|_{Y_{j,1}} \\ &\leq c\lambda^{\overline{\kappa}-1}\big(|(A_*(t) - A_*(s))v|_0 + \sum_{j=1, m_j > 0}^{m} |(B_{j*}(t) - B_{j*}(s))v|_{Y_{j,1}}\big). \end{aligned}$$

Here and below, the constants c do not depend on t, s, φ or λ. Employing Lemma 4.3 and $v = -(\omega + A_0(s))^{-1}\varphi$, one estimates

$$|u - v|_0 \leq c\lambda^{\overline{\kappa}-1}\big(|t - s|^{1-\frac{1}{p}}|(\omega + A_0(s))^{-1}\varphi|_{1-\frac{1}{p}} + |t - s|\,|(\omega + A_0(s))^{-1}\varphi|_1\big). \quad (4.29)$$

The last assertion implies that $(X_0, \operatorname{dom}(A_0(s)))_{1-\frac{1}{p},p}$ is embedded into $X_{1-1/p}$ with uniform embedding constants. So inequality (4.29) leads to

$$|u - v|_0 \leq c\lambda^{\overline{\kappa}-1}|t - s|^{1-\frac{1}{p}}|\varphi|_0$$

for $|t - s| \leq \tau$, and by periodicity for all $t, s \in \mathbb{R}$. In view of (4.28), we have shown (4.18) for $\lambda \geq 1$ with $\mu = 1 - \frac{1}{p}$ and $\nu = 1 - \overline{\kappa}$. By means of (4.17) one can extend this estimate to $\lambda \in \Sigma_\phi$. Observe that $\mu + \nu > 1$ due to (2.1). □

Thanks to (4.17) and (4.18), the operator family $A_0(\cdot)$ generates an evolution family $U(t, s)$, $t \geq s$, $t, s \in \mathbb{R}$, on X_0. More precisely, for all $t > s$, the map $(t, s) \mapsto U(t, s) \in \mathcal{B}(X_0)$ is continuous and continuously differentiable in t, $U(t, s)X_0 \subset \operatorname{dom}(A_0(t))$, and $\partial_t U(t, s) = -A_0(t)U(t, s)$. Moreover, $(t, s) \mapsto U(t, s)$ is strongly continuous for $t \geq s$, $U(t, s)U(s, r) = U(t, r)$ and $U(t, t) = I$ for all $t \geq s \geq r$, and the function $u = U(\cdot, s)x$ is the unique solution in $C([s, \infty), X_0) \cap C^1((s, \infty), X_0)$ with $u(t) \in \operatorname{dom}(A_0(t))$ for all $t > s$ of the problem

$$u'(t) + A_0(t)u(t) = 0, \quad t > s, \qquad u(s) = x, \quad (4.30)$$

for every $s \in \mathbb{R}$ and $x \in X_0$. These facts have been established in [1] and [2], see also [4], [26], [27]. Since $A_0(t + \tau) = A_0(t)$ for all $t \in \mathbb{R}$ and the period $\tau > 0$ from Hypothesis 4.1, the periodicity of the evolution family (i.e., $U(t+\tau, s+\tau) = U(t, s)$ for all $t \geq s$) follows from the uniqueness of (4.30).

We further introduce the inter/extrapolation spaces for $A_0(s)$ for $s \in \mathbb{R}$. By X_{-1}^s we denote the *extrapolation space* for $A_0(s)$; that is, the completion of X_0 with respect to the norm $|u_0|_{-1}^s = |(\omega + A_0(s))^{-1}u_0|_0$. We can extend $-A_0(s)$ to an operator $-A_{-1}(s) : X_0 \to X_{-1}^s$ generating an analytic semigroup $e^{-tA_{-1}(s)}$ on X_{-1}^s

which extends $e^{-tA_0(s)}$. We point out that $A_*(s)u \neq A_{-1}(s)u$ if $u \in X_1 \setminus \mathrm{dom}(A_0(s))$ due to (4.40) below. For $\alpha \in (0,1)$, we use the continuous interpolation spaces

$$X^s_\alpha = (X_0, \mathrm{dom}(A_0(s)))^0_{\alpha,\infty} \qquad \text{and} \qquad X^s_{\alpha-1} = (X^s_{-1}, X_0)^0_{\alpha,\infty}$$

between X_0 and $\mathrm{dom}(A_0(s))$, respectively, between X^s_{-1} and X_0, which are endowed with the norms

$$\begin{aligned} |x|^s_\alpha &= \sup_{r>0} |r^\alpha (\omega + A_0(s))(r + \omega + A_0(s))^{-1} x|_0 \qquad \text{and} \\ |x|^s_{\alpha-1} &= \sup_{r>0} |r^\alpha (r + \omega + A_{-1}(s))^{-1} x|_0 \,. \end{aligned}$$

We also set $X^s_1 = \mathrm{dom}(A_0(s))$ and write $|x|^s_1$ for the graph norm on $\mathrm{dom}(A_0(s))$, and we put $|x|^t_0 := |x|_0$. The restriction $A_{\alpha-1}(s) : X^s_\alpha \to X^s_{\alpha-1}$ of $A_{-1}(s)$ generates an analytic semigroup on $X^s_{\alpha-1}$ which is the restriction of $e^{-tA_{-1}(s)}$ and the extension of $e^{-tA_0(s)}$. We refer to [4], [12], [17] or [18] for the standard properties of these spaces and the corresponding fractional power spaces.

We observe that the inequality (4.29) actually yields the estimate

$$\begin{aligned} &|(\omega + A_0(s))^{-1}\varphi - (\omega + A_0(t))^{-1}\varphi|^t_{1-\overline{\kappa}} \\ &\quad \le c\,|t-s|^{1-\frac{1}{p}} |(\omega + A_0(s))^{-1}\varphi|_{1-1/p} + c\,|t-s|\,|\omega + A_0(s))^{-1}\varphi|_1 \end{aligned} \tag{4.31}$$

for all $t, s \in \mathbb{R}$ and $\varphi \in X_0$, if Hypothesis 4.1 holds. In [18, §2] we have discussed the extension of $U(t,s)$ to $X^s_{\alpha-1}$ for $1-\mu < \alpha \le 1$, based on [27, Theorem 2.1] and just assuming (4.17) and (4.18). Using the additional property (4.31), we can now also treat the case of $\alpha \in (0, 1-\mu]$ in our setting, which is crucial for our approach.

Proposition 4.7. *Assume that Hypothesis 4.1 holds. Let $\alpha \in (0,1]$. Then the following assertions are true.*

(a) The operators $U(t,s)$, $t \ge s$, have locally uniformly bounded extensions $U_{\alpha-1}(t,s) : X^s_{\alpha-1} \longrightarrow X^t_{\alpha-1}$ satisfying also

$$|U_{\alpha-1}(t,s)x|^t_\beta \le c\,(t-s)^{\alpha-1-\beta}\,|x|^s_{\alpha-1} \tag{4.32}$$

for all $x \in X^s_{\alpha-1}$, $\beta \in [0,1]$ and $0 < t-s \le T$, where $c = c(T)$.

(b) Let $\epsilon > 0$, $t \in \mathbb{R}$, and $g : (-\infty, t] \to X_0$ be locally integrable. Then the function $s \mapsto U_{\alpha-1}(t,s)A_{\alpha-1}(s)(\omega + A_0(s))^{-\alpha-\epsilon} g(s)$ is locally integrable from $(-\infty, t]$ to $X^t_{\alpha-1}$.

(c) For $\varphi \in X^s_\alpha$ and $t > s$ there exists

$$\frac{\partial^+}{\partial s} U(t,s)\varphi = U_{\alpha-1}(t,s)A_{\alpha-1}(s)\varphi \qquad \text{in } X_0. \tag{4.33}$$

Proof. (a) By rescaling, we can assume that $\omega = 0$. Recall from Proppposition 4.6 that $\mu = 1 - \frac{1}{p}$, $\nu = 1 - \overline{\kappa}$, and $\overline{\kappa} = \max\{\kappa_j;\, m_j > 0,\, j = 1, \dots, m\}$. Let $\theta \in (\mu, 1)$, $\eta \in (0, \mu - \overline{\kappa})$, $\vartheta := \mu - \eta > \overline{\kappa}$, and $1 - \mu < \overline{\alpha} < 1 - \mu + \eta$. By Lemma A.1 and equation (A.5) in [18] we can extend $U(t,s)$ to a locally uniformly bounded operator $U_{\overline{\alpha}-1}(t,s) : X^s_{\overline{\alpha}-1} \to X^t_{\overline{\alpha}-1}$ for every $t \ge s$, and the maps $\tilde{V}(t,s) := A_{-1}(t)^{-\vartheta} U_{\overline{\alpha}-1}(t,s) A_{-1}(s)^\vartheta$ satisfy

$$\begin{aligned} \tilde{V}(t,s)\varphi = A_0(t)^{-\vartheta} A_0(s)^\vartheta e^{-(t-s)A_0(s)}\varphi + \int_s^t \tilde{V}(t,\sigma) A_0(\sigma)^{1-\vartheta} \\ \cdot [A_0(\sigma)^{-1} - A_0(s)^{-1}]\, A_0(s)^{1+\vartheta} e^{-(\sigma-s)A_0(s)}\varphi\, d\sigma \end{aligned}$$

for all $\varphi \in \mathrm{dom}(A_0(s)^\vartheta)$ and $t \ge s$. It follows that

$$\begin{aligned} U(t,s)\varphi = e^{-(t-s)A_0(s)}\varphi + \int_s^t U_{\overline{\alpha}-1}(t,\sigma) A_{-1}(\sigma) \\ \cdot [A_0(\sigma)^{-1} - A_0(s)^{-1}]\, A_0(s) e^{-(\sigma-s)A_0(s)}\varphi\, d\sigma \end{aligned} \tag{4.34}$$

for all $\varphi \in X_0$. We note that the right hand side of (4.34) is contained in X_0 by Proposition 2.1 of [18] and (4.18). Estimate (4.31) implies that $\psi(\sigma) := A_0(\sigma)^{1-\theta}\,[A_0(\sigma)^{-1} - A_0(s)^{-1}]\,A_0(s)e^{-(\sigma-s)A_0(s)}\varphi$ belongs to $D(A_0(\sigma)^{\theta-\overline{\kappa}-\varepsilon}) \hookrightarrow X_0$ for every $\varepsilon \in (0, \theta-\overline{\kappa})$, $\sigma > s$, and $\varphi \in X_0$. Hence, $A_{-1}(\sigma)^\theta\psi(\sigma) \in X^\sigma_{-\overline{\kappa}-2\varepsilon} \hookrightarrow X^\sigma_{\overline{\alpha}-1}$ for sufficiently small $\varepsilon > 0$, since then $\overline{\alpha} - 1 < -\mu + \eta < -\overline{\kappa} - 2\varepsilon$. We now define

$$V(t,s)\varphi := A_{-1}(t)^{-\theta}U_{\overline{\alpha}-1}(t,s)A_{-1}(s)^\theta\varphi$$

for all $t > s$ and $\varphi \in \mathrm{dom}(A_0(s)^{\theta-\overline{\kappa}-\varepsilon})$. Observe that

$$V(t,s)\varphi = A_0(t)^{-\theta}U(t,s)A_0(s)^\theta\varphi$$

for $t > s$ and $\varphi \in \mathrm{dom}(A_0(s)^\theta)$. So (4.34) yields

$$\begin{aligned} V(t,s)\varphi &= e^{-(t-s)A_0(s)}\varphi + (A_0(t)^{-\theta} - A_0(s)^{-\theta})\,A_0(s)^\theta e^{-(\sigma-s)A_0(s)}\varphi \\ &\qquad + \int_s^t V(t,\sigma)A_0(\sigma)^{1-\theta}\,[A_0(\sigma)^{-1} - A_0(s)^{-1}]\,A_0(s)^{1+\theta}e^{-(\sigma-s)A_0(s)}\varphi\,d\sigma \\ &=: e^{-(t-s)A_0(s)}\varphi + a(t,s)\varphi + \int_s^t V(t,\sigma)k(\sigma,s)\varphi\,d\sigma \end{aligned} \tag{4.35}$$

for all $\varphi \in \mathrm{dom}(A_0(s)^\theta)$, where we also write $(V*k)(t,s)\varphi$ for the integral term and $b(t,s) := e^{-(t-s)A_0(s)} + a(t,s)$. Using $X^\sigma_\nu \hookrightarrow \mathrm{dom}(A_0(\sigma)^{1-\theta})$ and (4.31), we obtain

$$\begin{aligned} |k(t,s)\varphi|_0 &\le c\,|\sigma - s|^{1-\frac{1}{p}}\,|A_0(s)^\theta e^{(s-\sigma)A_0(s)}\varphi|_{1-\frac{1}{p}} + c\,|\sigma - s|\,|\,A_0(s)^\theta e^{(s-\sigma)A_0(s)}\varphi|_1 \\ &\le c\,|\sigma - s|^{-\theta}\,|\varphi|_0. \end{aligned}$$

Here and below we also employ standard properties of analytic semigroups, cf. [17]. For a suitable path Γ in $\mathbb{C}$ (see [4, 12, 17]), we conclude in a similar way that

$$\begin{aligned} |a(t,s)\varphi|_0 &\le \frac{1}{2\pi}\int_\Gamma |\lambda|^{-\theta}|((\lambda + A_0(t))^{-1} - (\lambda + A_0(s))^{-1})A_0(s)^\theta e^{-(t-s)A_0(s)}\varphi|_0\,|d\lambda| \\ &= \frac{1}{2\pi}\int_\Gamma |\lambda|^{-\theta}|A_0(t)(\lambda + A_0(t))^{-1}(A_0(t)^{-1} - A_0(s)^{-1})A_0(s)(\lambda + A_0(s))^{-1} \\ &\qquad\qquad \cdot A_0(s)^\theta e^{-(t-s)A_0(s)}\varphi|_0\,|d\lambda| \\ &\le \frac{c}{2\pi}\int_\Gamma |\lambda|^{-\theta+\overline{\kappa}-1}\Big(|t-s|^{1-\frac{1}{p}}|A_0(s)^{\theta-1}e^{-(t-s)A_0(s)}\varphi|_{1-1/p} \\ &\qquad\qquad + |t-s||A_0(s)^\theta e^{-(t-s)A_0(s)}\varphi|_0\Big)\,|d\lambda| \\ &\le c\,|\varphi|_0, \end{aligned} \tag{4.36}$$

where c depends on T with $|t-s| \le T$. Note that (4.35) yields

$$V(t,s)\varphi = b(t,s)\varphi + \sum_{l=1}^n (b *_l k)(t,s)\varphi + (V *_{n+1} k)(t,s)\varphi \tag{4.37}$$

for all $\varphi \in \mathrm{dom}(A_0(s)^\theta)$ and $l \in \mathbb{N}$, where $*_l$ denotes the l–fold 'convolution' with k. The above estimates and the proof of Lemma II.3.2.1 of [4] imply that the sum in (4.37) is bounded in X_0 by $c\,|\varphi|_0$, uniformly in $n \in \mathbb{N}$ and locally uniformly for $t \ge s$. On the other hand, we deduce

$$|(V*k)(t,\sigma)\varphi|_0 \le c\,|A_0(\sigma)^{\theta-\vartheta}\varphi|_0$$

from $\overline{\alpha} - 1 + \theta > 0$, the local uniform boundedness of $U_{\overline{\alpha}-1}(t,\sigma)A_{-1}(\sigma)^\vartheta : X_0 \to X^t_{\overline{\alpha}-1}$, $X^\sigma_\nu \hookrightarrow \mathrm{dom}(A_0(\sigma)^{1-\vartheta})$, and (4.18). Similarly,

$$|A_0(\sigma)^{\theta-\vartheta}k(\sigma,s)\varphi|_0 \le c\,|\sigma - s|^{\eta-1}|A_0(\sigma)^{\theta-\vartheta}\varphi|_0\,.$$

Using again the proof of Lemma II.3.2.1 of [4], we see that the term $(V *_{n+1} k)(t,s)\varphi$ converges to 0 in X_0 as $n \to \infty$, where $\varphi \in \mathrm{dom}(A_0(s)^\theta)$. Letting $n \to \infty$ in (4.37), we thus conlcude that $V(t,s)$ has a locally bounded extension in $\mathcal{B}(X_0)$.

Taking $\theta = 1 - \alpha \pm \epsilon$ for $\alpha \in (0, 1-\mu)$ and $\epsilon > 0$ with $1 - \alpha \pm \epsilon \in (\mu, 1)$, we deduce the first claim in (a) by reiteration for such α, see e.g. Theorem 1.2.15 and Proposition 2.2.15 in [17]. Since this claim was shown in Lemma A.1 of [18] for $1 - \mu < \alpha \leq 1$, we obtain the first assertion in (a) for all $\alpha \in (0, 1]$ by reiteration.

Starting from (4.34) and using similar arguments as above, we can also show that

$$|U(t,s)A_0(s)^\theta \varphi|_0 \leq c(T)(t-s)^{-\theta}|\varphi|_0 \tag{4.38}$$

for every $\mu < \theta < 1$, $0 < t-s \leq T$ and $\varphi \in D(-A_0(s))^\theta)$. Estimate (4.32) with $\beta = 0$ now follows by reiteration as above, and the general case is an easy consequence of the smoothing properties of $U(\cdot,\cdot)$.

(b) The assertion is clear if we take $g \in \mathrm{dom}(A_0(\cdot))$. The general case can be deduced from (a) by means of the approximations $n(n + A_0(\cdot))^{-1}g$.

(c) Let $\beta \in (0, \alpha)$ be smaller than $1 - \overline{\kappa}$ and $1 - \frac{1}{p}$, and take $s \in \mathbb{R}$, $\tau, h > 0$ and $\varphi \in \mathrm{dom}(A_0(s)^\beta)$. Arguing as in (4.36), we estimate

$$|(A_0(s+h)^{\beta-1} - A_0(s)^{\beta-1})A_0(s)e^{-\tau A_0(s)}\varphi|_0 \leq c\,(h^{1-\frac{1}{p}}\tau^{\beta-1+\frac{1}{p}} + h\tau^{\beta-1})\,|A_0(s)^\beta\varphi|_0.$$

Using this estimate, the last assertion can be proved as Proposition 2.1 in [19] invoking also (4.34), (4.38), and (4.31). □

We next derive the representation formula (4.41) for the solution to (2.18) which is important for the study of the asymptotic behavior. As a preparation, we first collect several relevant facts in a corollary.

Corollary 4.8. *Assume that Hypothesis 4.1 holds. Let* $\kappa = \max\{\kappa_j : j = 1, \cdots, m\}$ *and* $0 < \alpha < 1 - \kappa$. *Then the following assertions are true.*

(a) The operators $\mathcal{N}_\omega(t)$, $t \in \mathbb{R}$, *map* Y_1 *into* X_α^t *with uniformly bounded norms. Moreover,* $X_1 \hookrightarrow X_\alpha^t$.

(b) The map $t \longmapsto \mathcal{N}_\omega(t) \in \mathcal{B}(Y_1, X_1)$ *is globally Hölder continuous on* $\mathbb{R}$.

(c) The operators

$$\Pi(t) := (\omega + A_{\alpha-1}(t))\mathcal{N}_\omega(t) \in \mathcal{B}(Y_1, X_{\alpha-1}^t)$$

are uniformly bounded for $t \in \mathbb{R}$, *and the function*

$$s \longmapsto U_{\alpha-1}(t,s)\Pi(s)h(s) \in X_{\alpha-1}^t$$

is integrable on $[t_0, t]$ *for all* $t_0 < t$ *and* $h \in L_p([t_0,t]; Y_1)$.

Proof. Assertion (a) is a consequence of (4.22) as well as of Proposition 2.2 of [14] and its proof. To check (b), recall that $u(t) = \mathcal{N}_\omega(t)\psi$ solves the elliptic problem (4.21) at time $t \in \mathbb{R}$ for $\lambda = 0$, $\varphi = 0$ and $\psi \in Y_1$. We then have

$$\begin{aligned} (\omega + A_*(t))(u(t) - u(s)) &= (A_*(s) - A_*(t))u(s), \\ B_{j*}(t)(u(t) - u(s)) &= (B_{j*}(s) - B_{j*}(t))u(s), \quad j \in \{1, \dots, m\}. \end{aligned} \tag{4.39}$$

Lemma 4.5 further yields

$$u(t) - u(s) = (\omega + A_0(t))^{-1}(A_*(s) - A_*(t))u(s) + \mathcal{N}_\omega(t)(B_*(s) - B_*(t))u(s),$$

so that (4.22) implies that

$$|\mathcal{N}_\omega(t)\psi - \mathcal{N}_\omega(s)\psi|_1 \leq c\,|(A_*(t) - A_*(s))\mathcal{N}_\omega(s)\psi|_0 + c\,|(B_*(t) - B_*(s))\mathcal{N}_\omega(s)\psi|_{Y_1}.$$

The Hölder property in (b) thus follows from Lemma 4.3 and (4.22). Assertion (a) implies the first part of (c), and the second part is a consequence of part (a) and Proposition 4.7(b). □

Proposition 4.9. *Assume that Hypothesis 4.1 holds. We then have*

$$A_{-1}(t)\varphi = A_*(t)\varphi + (\omega + A_{-1}(t))\mathcal{N}_\omega(t)B_*(t)\varphi \tag{4.40}$$

for all $\varphi \in X_1$ and $t \in \mathbb{R}$. Let $v \in \mathbb{E}_1(J)$, $g \in \mathbb{E}_0(J)$, $h \in L_p(J; Y_1)$, and $v_0 \in X_0$ for $J = [t_0, t_0 + T]$. Consider the equations

$$(a) \begin{cases} \dot v(t) + A_*(t)v(t) = g(t), \\ B_*(t)v(t) = h(t), \\ v(t_0) = v_0, \end{cases} \qquad (b) \begin{cases} \dot v(t) + A_{\alpha-1}(t)v(t) = g(t) + \Pi(t)h(t), \\ v(t_0) = v_0. \end{cases}$$

Then v satisfies (a) for a.e. $t \in J$ if and only if it satisfies (b) for a.e. $t \in J$. If the solution exists, it is given by

$$v(t) = U(t,t_0)v_0 + \int_{t_0}^t U(t,s)g(s)\,ds + \int_{t_0}^t U_{\alpha-1}(t,s)\Pi(s)h(s)\,ds, \quad t \in J. \tag{4.41}$$

Proof. The equation (4.40) and the equivalence of (a) and (b) were shown (in the proof of) Proposition 6 of [15]. Equation (4.41) essentially follows from (b) and (4.33), though the details are a bit technical. If $v \in \mathbb{E}_1(J)$ solves the problem (a) or (b), we must have $v_0 \in X_{1-1/p}$, $h \in \mathbb{F}(J)$ and $B(t_0)v_0 = h(t_0)$ in view of Theorem 2.1. Moreover, (4.33) and equation (b) imply that

$$\tfrac{\partial^+}{\partial s} U(t,s)v(s) = U_{\alpha-1}(t,s)(g(s) + \Pi(s)h(s)) \ \in\ X_0$$

for a.e. $s \in (t_0, t)$ and all $t \in (t_0, t_0 + T]$. If we also had $g \in C((t_0, t_0 + T]; X_0)$ and $h \in C((t_0, t_0 + T]; Y_1)$, then the function $s \mapsto \langle U(t,s)v(s), \phi\rangle$ were continuously differentiable on (t_0, t) for every $\phi \in X_0$. Integrating over $s \in [t_0 + \varepsilon, t - \varepsilon]$ we would obtain

$$U(t, t_0 - \varepsilon)v(t - \varepsilon) - U(t, t_0 + \varepsilon)v(t_0 + \varepsilon) = \int_{t_0+\varepsilon}^{t-\varepsilon} U_{\alpha-1}(t,s)(g(s) + \Pi(s)h(s))\,ds$$

for every $\varepsilon \in (0, (t - t_0)/2)$. Letting $\varepsilon \to 0$, the asserted equation (4.41) then follows in this case.

We will approximate g and h by continuous functions. There are $g_n \in C(J; X_0)$ converging to g in $L^p(J; X_0)$ as $n \to \infty$. When treating h, we must take care of the compatibility condition. Let $L = \Delta^m_{\partial\Omega}$ be the power of the Laplace–Beltrami operator on $\partial\Omega$. We have $h_j(t_0) \in Y_{j,p} = (Y_0, D(L))_{\kappa_j - 1/p, p}$ by (2.10) and real interpolation. Taking into account Theorems 3 and 8 of [9], we obtain that $\psi_j(t) = e^{(t-t_0)L}h_j(t_0)$, $t \in J$, belongs to $\mathbb{F}_j(J) \cap C((t_0, t_0 + T]; Y_{j,1})$ for every $j \in \{1, \cdots, m\}$, and $\psi(t_0) = h(t_0)$. We set $\psi = (\psi_1, \ldots, \psi_m)$ and $h^0 = h - \psi \in \mathbb{F}(J)$. Since $h^0(t_0) = 0$, one finds functions $h^0_n \in C(J; Y_1) \cap \mathbb{F}(J)$ with $h^0_n(t_0) = 0$ for all $n \in \mathbb{N}$ which converge to h^0 in $\mathbb{F}(J)$ by standard cutoff and mollification arguments.

Theorem 2.1 gives a function $w \in \mathbb{E}_1(J)$ satisfying

$$\begin{aligned} \partial_t w(t) + A_*(t)w(t) &= 0 &&\text{on } \Omega, \quad t \in J, \\ B_*(t)w(t) &= \psi(t) &&\text{on } \partial\Omega, \quad t \in J, \\ w(t_0) &= v_0, &&\text{on } \Omega, \end{aligned}$$

and it gives functions $v^0_n \in \mathbb{E}_1(J)$ satisfying

$$\begin{aligned} \partial_t v^0_n(t) + A_*(t)v^0_n(t) &= g_n(t) &&\text{on } \Omega, \quad t \in J, \\ B_*(t)v^0_n(t) &= h^0_n(t) &&\text{on } \partial\Omega, \quad t \in J, \\ v^0_n(t_0) &= 0, &&\text{on } \Omega. \end{aligned}$$

As seen above, we obtain

$$\begin{aligned} w(t) &= U(t,t_0)v_0 + \int_{t_0}^t U_{\alpha-1}(t,s)\Pi(s)\psi(s)\,ds, \\ v^0_n(t) &= \int_{t_0}^t U(t,s)g_n(s)\,ds + \int_{t_0}^t U_{\alpha-1}(t,s)\Pi(s)h^0_n(s)\,ds. \end{aligned}$$

for all $t \in J$ and $n \in \mathbb{N}$, since the inhomogeneities are continuous. From Theorem 2.1 we further deduce that $v_n^0 + w$ converge to the solution v of (a) in $\mathbb{E}_1(J)$ as $n \to \infty$. So (4.41) follows by letting $n \to \infty$ in the above integral formulas. □

We say that the evolution family $U(\cdot,\cdot)$ has an *exponential dichotomy* on $\mathbb{R}$ if there exist ('stable') projections $P(t) \in \mathcal{B}(X_0)$, $t \in \mathbb{R}$, and a dichotomy exponent $\delta_0 > 0$ such that $U(t,s)P(s) = P(t)U(t,s)$, $U(t,s) : \ker(P(s)) \to \ker(P(t))$ has an inverse denoted by $U_Q(s,t)$, and

$$\|U(t,s)P(s)\|\,, \|U_Q(s,t)Q(t)\| \le ce^{-\delta_0(t-s)}$$

for all $t \ge s$, where we set $Q(\cdot) = I - P(\cdot)$. If $P(t) = I$ for all $t \in \mathbb{R}$, then $U(\cdot,\cdot)$ is called *exponentially stable*. Since the evolution family is periodic, its exponential dichotomy is equivalent to the fact that $\rho(U(\tau,0))$ does not intersect the unit circle, see e.g. [17, §6.3] or [23, §3.1]. The projections $Q(t)$ map X_0 to $\mathrm{dom}(A_0(t)) \subseteq X_1$ with uniformly bounded norms for $t \in \mathbb{R}$, because of $A_0(t)Q(t) = A_0(t)U(t,t-1)U_Q(t-1,t)Q(t)$. Therefore the operators $P(t)$, $t \in \mathbb{R}$, are uniformly bounded in $\mathcal{B}(X_1)$ and $\mathcal{B}(X_{1-1/p})$, too. We refer to the survey given in [23] for more information about exponential dichotomy.

In the following result we extend the exponential dichotomy to the extrapolated evolution family, cf. Proposition 2.2 of [18] for the case $\alpha > 1-\mu$.

Proposition 4.10. *Assume that Hypothesis 4.1 holds and that $U(\cdot,\cdot)$ has an exponential dichotomy. Let $\alpha \in (0,1]$. Then the operators $P(t)$ and $Q(t)$ admit uniformly bounded extensions $P_{\alpha-1}(t) : X^t_{\alpha-1} \to X^t_{\alpha-1}$ and $Q_{\alpha-1}(t) : X^t_{\alpha-1} \to X_1$ for $t \in \mathbb{R}$. The following assertions hold for all $t > s$ in $\mathbb{R}$ and a constant $N(\alpha) \ge 0$.*

(1) $Q_{\alpha-1}(t)X^t_{\alpha-1} = Q(t)X_0$;
(2) $U_{\alpha-1}(t,s)P_{\alpha-1}(s) = P_{\alpha-1}(t)U_{\alpha-1}(t,s)$;
(3) $U_{\alpha-1}(t,s) : Q_{\alpha-1}(s)(X^s_{\alpha-1}) \to Q_{\alpha-1}(t)(X^t_{\alpha-1})$ *is invertible with inverse* $U_{Q,\alpha-1}(s,t)$;
(4) $|U_{\alpha-1}(t,s)P_{\alpha-1}(s)x|_0 \le N(\alpha)\max\{(t-s)^{\alpha-1},1\}e^{-\delta_0(t-s)}|x|^s_{\alpha-1}$ *for* $x \in X^s_{\alpha-1}$;
(5) $|U_{\alpha-1}(t,s)P_{\alpha-1}(s)x|^t_{\alpha-1} \le N(\alpha)e^{-\delta_0(t-s)}|x|^s_{\alpha-1}$ *for* $x \in X^s_{\alpha-1}$;
(6) $|U_{Q,\alpha-1}(s,t)Q_{\alpha-1}(t)x|_1 \le N(\alpha)e^{-\delta_0(t-s)}|x|^t_{\alpha-1}$ *for* $x \in X^t_{\alpha-1}$.

Proof. Most of the results can be proved as Proposition 2.2 of [18] now employing Proposition 4.7, except for (5). Clearly, (5) holds for $0 \le t-s \le 2$. For $t > s+2$, we estimate

$$\begin{aligned}|U_{\alpha-1}(t,s)P_{\alpha-1}(s)x|^t_{\alpha-1} &\le c\,|U(t,s+1)P(s+1)U(s+1,s)x|_0\\ &\le ce^{-\delta_0(t-s-1)}\,|U(s+1,s)x|_0 \le ce^{-\delta_0(t-s)}\,|x|^s_{\alpha-1}\end{aligned}$$

using the exponential dichotomy on X_0 and (4.32). □

We will now use the exponential dichotomy of $U(\cdot,\cdot)$ to extend the maximal regularity result Theorem 2.1 to the unbounded time intervals $J_+ = [t_0,\infty)$ and $J_- = (-\infty,t_0]$. Let $\delta \in \mathbb{R}$ and recall the definition (2.22) of the weighted functions spaces. We set $U^\delta(t,s) = e^{\delta(t-s)}U(t,s)$ for $t \ge s$, and assume that $U^\delta(\cdot,\cdot)$ has an exponential dichotomy. Given $(w_0,g,h) \in X_{1-1/p} \times \mathbb{E}_0(J_+,\delta) \times \mathbb{F}(J_+,\delta)$, we introduce

$$\begin{aligned}L^+(t_0,w_0,g,h)(t) = U(t,t_0)w_0 &+ \int_{t_0}^t U(t,s)P(s)g(s)\,ds - \int_t^\infty U_Q(t,s)Q(s)g(s)\,ds\\ &+ \int_{t_0}^t U_{\alpha-1}(t,s)P_{\alpha-1}(s)\Pi(s)h(s)\,ds \qquad (4.42)\\ &- \int_t^\infty U_{Q,\alpha-1}(t,s)Q_{\alpha-1}(s)\Pi(s)h(s)\,ds, \qquad t \ge t_0,\end{aligned}$$

$$\phi_0^+ = \int_{t_0}^{\infty} U_Q(t_0,s)Q(s)g(s)\,ds + \int_{t_0}^{\infty} U_{Q,\alpha-1}(t_0,s)Q_{\alpha-1}(s)\Pi(s)h(s)\,ds. \tag{4.43}$$

Observe that $U_Q(t,s)Q(s) = Q(t)U_Q(t,s)Q(s)$ and that $Q_{\alpha-1}(s)\Pi(s) = Q(s)(\omega + A_0(s))Q(s)\mathcal{N}_\omega(s)$ is a bounded operator from Y_1 into $\mathrm{dom}(A_0(s))$. Taking into account Proposition 4.10, we see that the $Q(\cdot)$–integrals converge even in $\mathrm{dom}(A_0(t))$. We thus omit the index $\alpha - 1$ in the last integrals of (4.42) and (4.43). Similarly one sees that the $P(\cdot)$ integrals converge in $X_{\alpha-1}^t$. Further, let $v_0 \in X_{1-1/p}$ with $B_*(t_0)v_0 = h(t_0)$. Due to Proposition 4.9 and Theorem 2.1 the solution of (2.18) is given by

$$S(t_0, v_0, g, h)(t) := U(t, t_0)v_0 + \int_{t_0}^{t} U(t,s)g(s)\,ds + \int_{t_0}^{t} U_{\alpha-1}(t,s)\Pi(s)h(s)\,ds$$

for $t \ge t_0$. Let L_δ^+ and S_δ be the variants of L^+ and S with U replaced by U^δ.

Proposition 4.11. *Let Hypothesis 4.1 hold, $\delta_2 > \delta_1$, and assume that $U^\delta(\cdot,\cdot)$ has an exponential dichotomy for some $\delta \in [\delta_1, \delta_2]$. Let $t_0 \in \mathbb{R}$, $g \in \mathbb{E}_0([t_0,\infty),\delta)$, $h \in \mathbb{F}([t_0,\infty),\delta)$, and $v_0 \in X_{1-1/p}$ with $B_*(t_0)v_0 = h(t_0)$. Define ϕ_0^+ by (4.43). Then the following assertions are equivalent.*

(1) $S(t_0, v_0, g, h) \in \mathbb{E}_0([t_0,\infty),\delta)$.
(2) $L^+(t_0, v_0 + \phi_0^+, g, h) \in \mathbb{E}_0([t_0,\infty),\delta)$.
(3) $Q(t_0)v_0 = -\phi_0^+$.

In this case, we have $S(t_0, v_0, g, h) = L^+(t_0, P(t_0)v_0, g, h)$ and it holds

$$\|S(t_0, v_0, g, h)\|_{\mathbb{E}_1([t_0,\infty),\delta)} \le c_1' \left(|v_0|_{1-1/p} + \|g\|_{\mathbb{E}_0([t_0,\infty),\delta)} + \|h\|_{\mathbb{F}([t_0,\infty),\delta)}\right). \tag{4.44}$$

The constant c_1' does not depend on t_0, δ, v_0, g and h.

Proof. Observe that $e_\delta\, S(t_0, v_0, g, h) = S_\delta(t_0, v_0, e_\delta g, e_\delta h)$ and $e_\delta L^+(t_0, w_0, g, h) = L_\delta^+(t_0, w_0, e_\delta g, e_\delta h)$. So we can assume that $\delta = 0$ by rescaling. In particular, $U(\cdot,\cdot)$ is assumed to have an exponential dichotomy. (The uniformity of the constant with respect to δ in compact intervals is a consequence of the proof below.) It is straightforward to verify that $L^+(t_0, v_0 + \phi_0^+, g, h) = S(t_0, v_0, g, h)$ which gives the first equivalence. We note that $w_0 := v_0 + \phi_0^+$ belongs to $\mathrm{ran}(P(t_0))$ if and only if $w_0 = P(t_0)v_0$ if and only if $Q(t_0)v_0 = -\phi_0^+$. Moreover, the proof of (4.44) given below yields that the integral terms of $L^+(t_0, v_0 + \phi_0^+, g, h)$ belong to $\mathbb{E}_0([t_0,\infty))$. Thus the second equivalence holds. It remains to check (4.44) for the case $\delta = 0$ if (3) holds. Here we have to modify the proof of Proposition 8 in [15] for the autonomous case since we do not know whether $P(\cdot)$ leaves invariant $\mathbb{E}_1([t_0,\infty))$.

Let $v_0 \in X_{1-1/p}$, $g \in \mathbb{E}_0([t_0,\infty))$, and $h \in \mathbb{F}([t_0,\infty))$ such that $B_*(t_0)v_0 = h(t_0)$ and $Q(t_0)v_0 = -\phi_0^+$. Set $w_0 := P(t_0)v_0 = v_0 + \phi_0^+$ and $v = L^+(t_0, w_0, g, h) = S(t_0, v_0, g, h)$. Theorem 2.1 shows that

$$\|v\|_{\mathbb{E}_1([t_0,t_0+2])} \le c_1 \left(|v_0|_{1-1/p} + \|g\|_{\mathbb{E}_0([t_0,t_0+2])} + \|h\|_{\mathbb{F}([t_0,t_0+2])}\right).$$

In (4.42) we further put

$$I_0(t) = U(t,t_0)P(t_0)v_0, \quad I_1(t) = \int_{t_0}^{t} U(t,s)P(s)g(s)\,ds,$$

$$I_2(t) = \int_{t_0}^{t} U_{\alpha-1}(t,s)P(s)\Pi(s)(s)\,ds, \quad I_3(t) = \int_{t}^{\infty} U_Q(t,s)Q(s)(g(s) + \Pi(s)h(s))\,ds$$

for $t \ge t_0$. Proposition 4.10 and Theorem 2.4 of [10] easily imply that

$$\|I_0 + I_1 - I_3\|_{\mathbb{E}_0([t_0+2,\infty)} \le c \left(|v_0|_0 + \|g\|_{\mathbb{E}_0([t_0+2,\infty))} + \|h\|_{L_p([t_0+2,\infty);Y_1)}\right).$$

(One can use $x = -\int_{t_0}^{\infty} U_Q(t_0,s)Q(s)g(s)ds$ in Theorem 2.4 of [10], and condition (2.9) of [10] follows from Lemma 4.4(c) as explained in [10, §5.1].) Here and below the constants do not depend on t_0, v_0, g and h.

For the term I_2, we take $\chi \in C^\infty([t_0-1, t_0+1])$ with $\chi(t_0-1) = 1$ and $\chi = 0$ on $[t_0-\frac{1}{2}, t_0+1]$. For $n = 2, 3, \ldots$, set $\chi_n(s) = \chi(s-n)$ for $s \in [t_0+n-1, t_0+n+1] =: J_n$ and $h_n = (1-\chi_n)h|J_n$. For $t \in [t_0+n, t_0+n+1]$, we can write

$$\begin{aligned}
I_2(t) &= \int_{t_0+n-1}^{t} U_{\alpha-1}(t,s)\Pi(s)h_n(s)\,ds - \int_{t_0+n-1}^{t} U(t,s)Q(s)\Pi(s)h_n(s)\,ds \\
&\quad + U(t, t_0+n-\tfrac{1}{2}) \int_{t_0+n-1}^{t_0+n-\frac{1}{2}} U_{\alpha-1}(t_0+n-\tfrac{1}{2}, s)\chi_n(s)P(s)\Pi(s)h(s)\,ds \\
&\quad + U(t, t_0+n-1) \int_{t_0}^{t_0+n-1} U_{\alpha-1}(t_0+n-1, s)P(s)\Pi(s)h(s)\,ds \\
&=: I_{21}(t) + I_{22}(t) + I_{23}(t) + I_{24}(t).
\end{aligned}$$

Due to $h_n(t_0+n-1) = 0$, Theorem 2.1 combined with Proposition 4.9 yields

$$\|I_{21}\|_{\mathbb{E}_1(J_n)} \le c\,\|h_n\|_{\mathbb{F}(J_n)} \le c\,\|h\|_{\mathbb{F}(J_n)}\,.$$

We further deduce from the last part of Proposition 4.6, Proposition 4.7, Corollary 4.8 and Proposition 4.10 that

$$\begin{aligned}
&\|I_{22}\|_{\mathbb{E}_1([t_0+n,t_0+n+1])} \le c\,\|h\|_{L^p([t_0+n-1,t_0+n];Y_1)}\,, \\
&\|I_{23}\|_{\mathbb{E}_1([t_0+n,t_0+n+1])} \le c\,\|h\|_{L^p([t_0+n-1,t_0+n];Y_1)}\,, \\
&|I_{24}(t)|_1 + |\partial_t I_{24}(t)|_0 \le c \int_{t_0}^{t_0+n-1} e^{-\delta_0(t-s)}|h(s)|_{Y_1}\,ds \le c \int_{t_0}^{t} e^{-\delta_0(t-s)}|h(s)|_{Y_1}\,ds, \\
&\|I_{24}\|_{\mathbb{E}_1([t_0+n,t_0+n+1])} \le c\,\|h\|_{L^p([t_0+n-1,t_0+n];Y_1)}\,.
\end{aligned}$$

As in [15, Proposition 8] these estimates lead to $\|I_2\|_{\mathbb{E}_1([t_0+2,\infty))} \le c\,\|h\|_{\mathbb{F}([t_0+2,\infty))}$. Combining the above facts, we arrive at

$$\|v\|_{\mathbb{E}_1([t_0,\infty))} \le c_1\,\big(|w_0|_{1-1/p} + \|g\|_{\mathbb{E}_0([t_0,\infty))} + \|h\|_{\mathbb{F}([t_0,\infty))}\big). \qquad \square$$

We further need a variant of Proposition 4.11 for backward solutions of (2.18) on $(-\infty, t_0]$. Given $g \in \mathbb{E}_0((-\infty,t_0],\delta)$, $h \in \mathbb{F}((-\infty,t_0],\delta)$ and $v_0 \in X_0$, we define

$$\begin{aligned}
L^-(t_0, v_0, g, h)(t) := U_Q(t,t_0)Q(t_0)v_0 &+ \int_{-\infty}^{t} U(t,s)P(s)g(s)\,ds \\
&- \int_{t}^{t_0} U_Q(t,s)Q(s)g(s)\,ds + \int_{-\infty}^{t} U_{\alpha-1}(t,s)P_{\alpha-1}(s)\Pi(s)h(s)\,ds \\
&- \int_{t}^{t_0} U_Q(t,s)Q_{\alpha-1}(s)\Pi(s)h(s)\,ds, \qquad t \le t_0,
\end{aligned} \tag{4.45}$$

$$\phi_0^- := \int_{-\infty}^{t_0} U_{\alpha-1}(t_0,s)\,P_{\alpha-1}(s)(g(s) + \Pi(s)h(s))\,ds. \tag{4.46}$$

Proposition 4.12. *Assume that Hypothesis 4.1 holds, $\delta_2 > \delta_1$, and that $U^\delta(\cdot,\cdot)$ has an exponential dichotomy for some $\delta \in [\delta_1, \delta_2]$. Let $t_0 \in \mathbb{R}$, $g \in \mathbb{E}_0((-\infty,t_0],\delta)$, $h \in \mathbb{F}((-\infty,t_0],\delta)$, and $v_0 \in X_0$. Consider problem (2.18) on $(-\infty,t_0]$ with the final value $v(t_0) = v_0$. Then there is a solution v of (2.18) on $(-\infty,t_0]$ belonging to $\mathbb{E}_0((-\infty,t_0],\delta)$ if and only if $P(t_0)v_0 = \phi_0^-$. In this case, $v = L^-(t_0,v_0,g,h)$ is the unique solution of (2.18) in $\mathbb{E}_1((-\infty,t_0],\delta)$ with the final value v_0 and*

$$\|L^-(t_0,v_0,g,h)\|_{\mathbb{E}_1((-\infty,t_0],\delta)} \le c_1'\,(|Q(t_0)v_0|_0 + \|g\|_{\mathbb{E}_0((-\infty,t_0],\delta)} + \|h\|_{\mathbb{F}((-\infty,t_0],\delta)}).$$

The constant c_1' does not depend on t_0, δ, v_0, g or h.

One shows the asserted equivalence as in Proposition 9 of [15]. The final estimate can be proved by straightforward modifications of the proof of Proposition 4.11.

5. The asymptotic stability of periodic solutions

We assume that Hypothesis 4.1 holds for a τ–periodic solution u_* of (2.2). Moreover, let $U(\cdot,\cdot)$ be the τ–periodic evolution family generated by operators $A_0(t)$, $t \in \mathbb{R}$, defined in (4.16). Hence, $U(\cdot,\cdot)$ solves the linearized problem (2.18) with $g = h_j = 0$. We start with the principle of linearized stability.

Proposition 5.1. *Assume that Hypothesis 4.1 holds with $\tau > 0$ and that $r(U(\tau,0)) < e^{-\delta\tau} < 1$ for some $\delta > 0$. Then there exist constants $\rho, c > 0$ such that for all $u_0 \in X_{1-1/p}$ and $t_0 \in \mathbb{R}$ with $|u_0 - u_*(t_0)|_{1-1/p} \le \rho$ and $B(t_0,u_0) = 0$, the solution u of* (2.2) *exists for all $t \ge t_0$ and satisfies $|u(t) - u_*(t)|_{1-1/p} \le ce^{-\delta(t-t_0)}$ for all $t \ge t_0$.*

Based on the theory developed in the previous sections one can establish the above result as Proposition 16 of [15]. So we only sketch the main parts of the proof. By the assumptions, $U(\cdot,\cdot)$ is exponentially stable. Therefore, $\phi_0^+ = 0$ in (4.43) and $S = L^+$ in Proposition 4.11. Using this fact, Propositions 2.4 and 4.11 as well as the contraction mapping principle, we solve (2.20) by a fixed point problem in the set

$$\{v \in \mathbb{E}_1([t_0,\infty),\delta) : v(0) = v_0, \|v\|_{\mathbb{E}_1([t_0,\infty),\delta)} \le c\rho\}$$

where $u_* + v_0 \in \mathcal{M}$ with $|v_0|_{1-1/p} \le \rho$ is given. The solution v of (2.20) yields the required solution $u = u_* + v$ of (2.2).

Working on the time interval $(-\infty, t_0]$ one obtains an analogous instability result if there is a nontrivial spectral gap outside the unit circle.

Proposition 5.2. *Assume that Hypothesis 4.1 holds with $\tau > 0$, that the circle $|\lambda| = e^{\delta\tau}$ belongs to $\rho(U(\tau,0))$ for some $\delta > 0$ and that $\sigma(U(\tau,0)) \cap \{|\lambda| > e^{\delta\tau}\} \ne \emptyset$. Let $t_0 \in \mathbb{R}$. Then there is a solution $u \ne u_*$ of* (2.2) *on the time interval $(-\infty, t_0]$ such that $|u(t) - u_*(t)|_{1-1/p} \le ce^{\delta(t-t_0)}$ for all $t \le t_0$. In particular, u_* is unstable.*

Proof. The rescaled evolution family $U^{-\delta}(t,s) = e^{-\delta(t-s)}U(t,s)$ has an exponential dichotomy with nontrivial stable and unstable projections $P(t)$ and $Q(t)$, respectively. Fix any nonzero $v_0 \in Q(t_0)X_0$ and define

$$\mathcal{L}(v) = L^-(t_0, v_0, \mathbb{G}(v), \mathbb{H}(v)) \qquad \text{for } \ v \in \mathbb{E}_1((-\infty,t_0],-\delta) =: \mathbb{E}_1(-\delta),$$

where $\mathbb{G}(v)$ and $\mathbb{H}(v)$ are given by (2.21) and L^- is given by (4.45). Due to Propositions 2.4 and 4.12, $\mathcal{L}$ is continuously differentiable on $\mathbb{E}_1(-\delta)$ and a fixed point $v = \mathcal{L}(v)$ solves (2.20) on $(-\infty, t_0]$ with final value $v(t_0) = v_0 + \phi_0^-$, where $\phi_0^- \in P(t_0)X_0$ is given by (4.46). Observe that $v(t_0) \ne 0$ and that $u = v + u_*$ then solves (2.2) on $(-\infty, t_0]$. Moreover, (2.7) yields

$$|v(t)|_{1-1/p} \le e^{-\delta(t-t_0)}|v(t)|_{1-1/p} \le c_0\,\|v\|_{\mathbb{E}_1(-\delta)}, \qquad t \le t_0, \tag{5.1}$$

since $-\delta(t-t_0) \ge 0$. So it remains to obtain a fixed point $v = \mathcal{L}(v)$ in $\mathbb{E}_1(-\delta)$.

We consider the closed ball $\overline{B}(\rho)$ in $\mathbb{E}_1(-\delta)$. For $v, w \in \overline{B}(\rho)$, Propositions 2.4 and 4.12 and (5.1) imply that

$$\begin{aligned}
\|\mathcal{L}(v)\|_{\mathbb{E}_1(-\delta)} &\le c\,(|v_0|_0 + \|\mathbb{G}(v)\|_{\mathbb{E}_0(-\delta)} + \|\mathbb{H}(v)\|_{\mathbb{F}(-\delta)}) \\
&\le c\,|v_0|_0 + \varepsilon(\rho)\,\|v\|_{\mathbb{E}_1(-\delta)}\,, \\
\|\mathcal{L}(v) - \mathcal{L}(w)\|_{\mathbb{E}_1(-\delta)} &\le c\,\sup\{\|\mathbb{G}'(z)\|, \|\mathbb{H}'(z)\| : \|z\|_{\mathbb{E}_1(-\delta)} \le \rho\}\,\|v-w\|_{\mathbb{E}_1(-\delta)} \\
&\le \varepsilon(\rho)\,\|v-w\|_{\mathbb{E}_1(-\delta)}\,.
\end{aligned}$$

Now, we first fix a $\rho > 0$ with $\varepsilon(\rho) \le 1/2$ and then choose a $v_0 \in Q(t_0)X_0 \setminus \{0\}$ with $c\,|v_0|_0 \le \rho/2$. As a result $\mathcal{L}$ is a strict contraction on $\overline{B}(\rho)$, and we obtain the desired fixed point. □

If the problem (2.2) is autonomous, i.e., the coefficients do not depend on time, then Proposition 5.1 is never applicable if the periodic orbit u_* is not an equilibrium. In fact, Theorem 3.2 implies that $u_* \in H^1_p((a,b);X_1) \cap H^2_p((a,b);X_0) \cap C^1((a,b);X_{1-1/p})$ for all $a < b$ in $\mathbb{R}$. So we can differentiate (2.2) with respect to t in X_0 and $Y_{1-1/p}$, respectively. As a result, $u'_* \in L_p((a,b);X_1) \cap H^1_p((a,b);X_0)$ satisfies (2.18) with $g = 0$ and $h_j = 0$, so that $u'_*(t) \in \mathrm{dom}(A_0(t))$ for a.e. $t \in \mathbb{R}$ and

$$\partial_t u'_*(t) + A_0(t)u'_*(t) = 0.$$

This means that $U(\tau,0)u'_*(0) = u'_*(\tau) = u'_*(0)$. Since u_* is not an equilibrium, it holds $u'_*(0) \neq 0$ and thus 1 is an eigenvalue of $U(\tau,0)$. However, if this eigenvalue is simple and the rest of the spectrum of $U(\tau,0)$ is strictly contained in the open unit disk, then we can show that the orbit u_* is asymptotically stable with asymptotic phase.

Theorem 5.3. *Let Hypothesis 4.1 hold for a non–constant τ–periodic orbit u_* and for maps $A(t,u) = A(u)$, $F(t,u) = F(u)$ and $B(t,u) = B(u)$ not depending on time t explicitely. Assume that 1 is a simple eigenvalue of $U(\tau,0)$ and that $\max\{|\lambda| : \lambda \in \sigma(U(\tau,0)) \setminus \{1\}\} < e^{-\delta\tau} < 1$ for some $\delta > 0$. Then there exist constants $r, c > 0$ such that for all $u_0 \in X_{1-1/p}$ with $|u_0 - u_*(0)|_{1-1/p} \le r$ and $B(u_0) = 0$, the solution u of (2.2) with $t_0 = 0$ exists for all $t \ge 0$ and there is a $\theta \in \mathbb{R}$ such that $|u(t) - u_*(t+\theta)|_1 \le ce^{-\delta t}$ for all $t \ge 1$.*

Proof. We set $u_\theta(t) = u_*(\theta + t)$ for all $t \in \mathbb{R}$ and any given $\theta \in \mathbb{R}$. Observe that u_θ also solves (2.2) with the initial condition $u_\theta(0) = u_*(\theta)$ since (2.2) is autonomous. Recall from (2.16) and (2.21) the definition of $A_*(t)$, $B_*(t)$, $G(t,v)$ and $H(t,v)$ for the periodic orbit u_*. Let $A_\theta(t)$, $B_\theta(t)$, $G_\theta(t,v)$ and $H_\theta(t,v)$ be given in the same way for u_θ instead of u_*. Let $u_0 \in X_{1-1/p}$ with $B(u_0) = 0$ be given, and let u be the solution of (2.2) with $u(0) = u_0$. Then the function $w = u - u_\theta$ satisfies

$$\begin{aligned}
\partial_t w(t) + A_*(t)w(t) &= (A_*(t) - A_\theta(t))w(t) + G_\theta(t,w(t)) =: \tilde{G}_\theta(t,w(t)) \quad \text{on } \Omega,\ t>0,\\
B_*(t)w(t) &= (B_*(t) - B_\theta(t))w(t) + H_\theta(t,w(t)) =: \tilde{H}_\theta(t,w(t)) \quad \text{on } \partial\Omega,\ t \ge 0,\\
w(0) &= u_0 - u(\theta) =: w_0 \quad \text{on } \Omega.
\end{aligned} \tag{5.2}$$

Let $\tilde{\mathbb{G}}_\theta$ and $\tilde{\mathbb{H}}_\theta$ be the corresponding substitution operators, which are given by

$$\begin{aligned}
\tilde{\mathbb{G}}_\theta(v) &= A_*(\cdot)v + A(u_\theta)u_\theta - A(u_\theta + v)(u_\theta + v) + F(u_\theta + v) - F(u_\theta),\\
\tilde{\mathbb{H}}_\theta(v) &= B'(u_*)v - B(u_\theta + v)
\end{aligned} \tag{5.3}$$

for all $v \in \mathbb{E}_1(\delta)$ and $\theta \in \mathbb{R}$. In the following we write $\mathbb{E}_1(\delta)$ instead of $\mathbb{E}_1(\mathbb{R}_+,\delta)$ etc., where $\delta > 0$ is given by the assumptions. Proposition 2.4 yields that $\mathbb{G}_\theta \in C^1(\mathbb{E}_1(\delta);\mathbb{E}_0(\delta))$ and $\mathbb{H}_\theta \in C^1(\mathbb{E}_1(\delta);\mathbb{F}(\delta))$ with $\mathbb{G}'_\theta(0) = 0$ and $\mathbb{H}'_\theta(0) = 0$. We next check that the multiplication operators $A_*(\cdot) - A_\theta(\cdot)$ and $B_*(\cdot) - B_\theta(\cdot) = B'(u_*) - B'(u_\theta)$ belong to $\mathcal{B}(\mathbb{E}_1(\delta),\mathbb{E}_0(\delta))$ and $\mathcal{B}(\mathbb{E}_1(\delta),\mathbb{F}(\delta))$, respectively. We then obtain that $\tilde{\mathbb{G}}_\theta \in C^1(\mathbb{E}_1(\delta);\mathbb{E}_0(\delta))$, $\tilde{\mathbb{H}}_\theta \in C^1(\mathbb{E}_1(\delta);\mathbb{F}(\delta))$, and

$$\begin{aligned}
\tilde{\mathbb{G}}'_\theta(0) &= A(u_*) - A(u_\theta) + A'(u_*)u_* - A'(u_\theta)u_\theta + F'(u_\theta) - F'(u_*),\\
\tilde{\mathbb{H}}'_\theta(0) &= B'(u_*) - B'(u_\theta) = \mathbb{H}'(u_\theta - u_*) - \mathbb{H}'(0),
\end{aligned} \tag{5.4}$$

using also (2.24). First, $A_*(\cdot) - A_\theta(\cdot) \in \mathcal{B}(\mathbb{E}_1(\delta),\mathbb{E}_0(\delta))$ and the inequality

$$\|\tilde{\mathbb{G}}'_\theta(0)\|_{\mathcal{B}(\mathbb{E}_1(\delta),\mathbb{E}_0(\delta))} \le \varepsilon(|\theta|) \tag{5.5}$$

follow from the properties of A and F stated before Theorem 2.1 and from the estimate $|u_*(t) - u_\theta(t)|_1 \le c\,|\theta|^{1-1/p}$ for all $t,\theta \in \mathbb{R}$, see (4.2). Second, in view of (5.4), Proposition 2.4 and (4.2) yield that

$$\|(B'(u_*) - B'(u_\theta))v\|_{\mathbb{F}(J)} \le \varepsilon(\|u_* - u_\theta\|_{\mathbb{E}_1(J)})\,\|v\|_{\mathbb{E}_1(J)} \le \varepsilon(|\theta|)\,\|v\|_{\mathbb{E}_1(J)}$$

for all $v \in \mathbb{E}_1(J)$ and compact intervals J. Using the periodicity of u_*, one concludes

$$\begin{aligned}\|e_\delta(B'(u_*)-B'(u_\theta))v\|^p_{L_p(\mathbb{R}_+;Y_k)} &= \sum_{n=0}^{\infty}\|(B'(u_*)-B'(u_\theta))e_\delta v\|^p_{L_p([n\tau,(n+1)\tau];Y_k)}\\ &\le \varepsilon(|\theta|)^p\,\|v\|^p_{\mathbb{E}_1(\delta)},\end{aligned}$$

where $k=0,1$, $e_\delta(t)=e^{\delta t}$ and we have fixed one index $j\in\{1,\dots,m\}$ which is omitted from the notation. By means of Lemma 2.3 and writing $f=(B'(u_*)-B'(u_\theta))v$ and $I_n=[n\tau-1,(n+1)\tau+1]\cap\mathbb{R}_+$, we further estimate

$$\begin{aligned}&\|e_\delta(B'(u_*)-B'(u_\theta))v\|^p_{W_p^\kappa(\mathbb{R}_+;Y_0)}\\ &\le \varepsilon(|\theta|)^p\,\|v\|^p_{\mathbb{E}_1(\delta)} + c\sum_{n=0}^{\infty}\int_{n\tau}^{(n+1)\tau}\int_{I(t)} e^{\delta tp}\frac{|f(t)-f(s)|^p_{Y_0}}{|t-s|^{1+\kappa p}}\,ds\,dt\\ &\le \varepsilon(|\theta|)^p\,\|v\|^p_{\mathbb{E}_1(\delta)} + c\sum_{n=0}^{\infty}e^{n\tau\delta p}\,[(B'(u_*)-B'(u_\theta))v]^p_{W_p^\kappa(I_n;Y_0)}\\ &\le \varepsilon(|\theta|)^p\,\|v\|^p_{\mathbb{E}_1(\delta)} + c\sum_{n=0}^{\infty}\varepsilon(|\theta|)^p\,e^{n\tau\delta p}\,\|v\|^p_{\mathbb{E}_1(I_n)}\\ &\le \varepsilon(|\theta|)^p\,\|v\|^p_{\mathbb{E}_1(\delta)},\end{aligned}$$

where $I(t)=[t-1,t+1]\cap\mathbb{R}_+$. Summing up, we have shown that

$$\|B'(u_*)-B'(u_\theta)\|_{\mathcal{B}(\mathbb{E}_1(\delta),\mathbb{F}(\delta))} = \|\tilde{\mathbb{H}}'_\theta(0)\|_{\mathcal{B}(\mathbb{E}_1(\delta),\mathbb{F}(\delta))} \le \varepsilon(|\theta|). \tag{5.6}$$

Let $P(t)$, $t\in\mathbb{R}$, be the stable projections for $U(t,s)$, and $Q(t)=I-P(t)$. Due to $\operatorname{ran}(Q(0))\subset\operatorname{dom}(A_0(0))$, we have $Z_0:=P(0)X^0_{1-1/p}(0)\subset X^0_{1-1/p}(0)$ and thus $P(0)X^0_{1-1/p}(0)=\operatorname{ran}(P(0))\cap X_{1-1/p}\cap\ker(B_*(0))$. Observe that (2.7) yields

$$|v(t)|_{1-1/p}\le e^{\delta t}|v(t)|_{1-1/p}\le c_0\,\|v\|_{\mathbb{E}_1(\delta)} \tag{5.7}$$

for all $v\in\mathbb{E}_1(\delta)$ and $t\ge 0$, since $\delta\ge 0$. Let $\hat{\mathcal{N}}(0)$ be the right inverse of $B_*(0)\in\mathcal{B}(X_{1-1/p},Y_{1-1/p})$ introduced in (3.2). We then have

$$B_*(0)P(0)\hat{\mathcal{N}}(0)=(B_*(0)-B_*(0)Q(0))\hat{\mathcal{N}}(0)=B_*(0)\hat{\mathcal{N}}(0)=I \tag{5.8}$$

on $Y_{1-1/p}$. Using the operator L^+ from (4.42), we define the map

$$\mathcal{L}_\theta: Z_0\times\mathbb{E}_1(\delta)\to\mathbb{E}_1(\delta);\quad \mathcal{L}_\theta(z_0,v)=v-L^+(z_0+P(0)\hat{\mathcal{N}}(0)\gamma_0\tilde{\mathbb{H}}_\theta(v),\tilde{\mathbb{G}}_\theta(v),\tilde{\mathbb{H}}_\theta(v))$$

for any $\theta\in\mathbb{R}$, where we have omitted the argument $t_0=0$ in L^+. Because of (5.8) and $Z_0\subset\ker(B_*(0))$, the compatibility condition in Theorem 2.1 holds. In addition $\tilde{\mathbb{G}}_\theta$ and $\tilde{\mathbb{H}}_\theta$ are C^1, so that we have $\mathcal{L}_\theta\in C^1(Z_0\times\mathbb{E}_1(\delta);\mathbb{E}_1(\delta))$, $\mathcal{L}_\theta(0,0)=0$ and

$$\partial_2\mathcal{L}_\theta(0,0)=I-L^+(P(0)\hat{\mathcal{N}}(0)\gamma_0\tilde{\mathbb{H}}'_\theta(0),\tilde{\mathbb{G}}'_\theta(0),\tilde{\mathbb{H}}'_\theta(0)).$$

The estimates (5.5) and (5.6) combined with Proposition 4.12 now imply that there is an $\eta_0>0$ such that $\partial_2\mathcal{L}_\theta(0,0)\in\mathcal{B}(\mathbb{E}_1(\delta))$ is invertible provided that $|\theta|\le\eta_0$. So the implicit function theorem yields numbers $\rho_0>0$ and a C^1-map Φ_θ from the ball $B(\rho_0):=Z_0\cap B_{1-1/p}(0,\rho_0)$ to $\mathbb{E}_1(\delta)$ such that $\Phi_\theta(0)=0$ and $\mathcal{L}_\theta(z_0,\Phi_\theta(z_0))=0$ for each $z_0\in B(\rho_0)$. Further, possibly after decreasing $\rho_0>0$ and $\eta_0>0$, we obtain that $\Phi'_\theta(z_0)$ is uniformly bounded for $\theta\in[-\eta_0,\eta_0]$ and $z_0\in B(\rho_0)$. This can be seen as in the proof of Theorem 14 in [15] differentiating the fixed point equation $\mathcal{L}_\theta(z_0,\Phi_\theta(z_0))=0$ w.r.t. z_0 and employing (5.5) and (5.6) once more. Due to Proposition 4.11 and (4.43), the function $w=\Phi_\theta(z_0)$ solves problem (5.2) with the initial value

$$\begin{aligned}\varphi(\theta,z_0):=w(0)&=z_0+P(0)\hat{\mathcal{N}}(0)\tilde{H}_\theta(0,w(0))\\ &\quad-Q(0)\int_0^\infty U_Q(0,s)Q(s)\Big(\tilde{G}_\theta(s,w(s))+\Pi(s)\tilde{H}_\theta(s,w(s))\Big)\,ds.\end{aligned} \tag{5.9}$$

where $z_0 \in B(\rho_0) \subset Z_0$ and $|\theta| \le \eta_0$. It holds $w(0) \in X_{1-1/p}$ and $B_*(0)w(0) = \tilde{H}_\theta(0, w(0))$ due to (5.8), $z_0 \in \ker(B_*(0))$ and $Q(0)X_0 \subset \ker(B_*(0))$. We further set

$$\psi_\theta(z_0) = P(0)\hat{\mathcal{N}}(0)\tilde{H}_\theta(0, [\Phi_\theta(z_0)](0)). \tag{5.10}$$

Observe that $u = w + u_\theta = \Phi_\theta(z_0) + u_\theta$ then solves (2.2) with the initial condition $u(0) = w(0) + u_*(\theta)$. From $\Phi_\theta(0) = 0$ and the boundedness of Φ'_θ we also infer that

$$\|\Phi_\theta(z_0)\|_{\mathbb{E}_1(\delta)} \le c\,|z_0|_{1-1/p} \le c\rho_0 \tag{5.11}$$

for all $\theta \in [-\eta_0, \eta_0]$ and $z_0 \in B(\rho_0)$.

Now, let $u_0 \in X_{1-1/p}$ with $B(u_0) = 0$ be given. We look for $\theta \in [-\eta_0, \eta_0]$ and $z_0 \in B(\rho_0)$ such that

$$u_0 = u_\theta(0) + \varphi(\theta, z_0) = u_*(\theta) + \varphi(\theta, z_0). \tag{5.12}$$

If (5.12) holds, the function $u = \Phi_\theta(z_0) + u_\theta$ solves (2.2) with the initial condition $u(0) = u_0$. Moreover, (5.7), (5.11), and Proposition 3.3 imply that $u(t) - u_*(t+\theta) = [\Phi_\theta(z_0)](t)$ decays exponentially in X_1 as asserted. (In order to apply Proposition 3.3 one possibly has to decrease $\rho_0 > 0$.)

So it remains to verify (5.12). As observed before the statement of the theorem, we have $U(\tau, 0)u'_*(0) = u'_*(0) \ne 0$ so that by the spectral assumptions the function $u'_*(0)$ spans $Q(0)X_0$. Hence, we can choose $x^* \in X_0^*$ such that

$$\langle u'_*(0), x^*\rangle = 1 \qquad \text{and} \qquad Q(0)x = \alpha u'_*(0)\,\langle x, x^*\rangle \tag{5.13}$$

for some $\alpha \in \mathbb{C} \setminus \{0\}$ and all $x \in X_0$. Therefore, (5.12) holds if and only if

$$\begin{pmatrix} \theta \\ z_0 \end{pmatrix} = \Psi(\theta, z_0) := \begin{pmatrix} \langle u_0 - u_*(\theta) - \varphi(\theta, z_0) + \theta u'_*(0), x^*\rangle \\ P(0)(u_0 - u_*(\theta)) - \psi_\theta(z_0) \end{pmatrix}$$

for some $\theta \in [-\eta_0, \eta_0]$ and $z_0 \in B(\rho_0)$. We look for $\eta \in (0, \eta_0)$ and $\rho \in (0, \rho_0)$ such that Ψ becomes a strict contraction on $[-\rho, \rho] \times \overline{B}(\rho)$. First, we observe that

$$\Psi(\theta, z_0) - \Psi(\overline{\theta}, \overline{z}_0) = \begin{pmatrix} \langle u_*(\overline{\theta}) - u_*(\theta) + \varphi(\overline{\theta}, \overline{z}_0) - \varphi(\theta, z_0) + (\theta - \overline{\theta})u'_*(0), x^*\rangle \\ P(0)(u_*(\overline{\theta}) - u_*(\theta)) + \psi_{\overline{\theta}}(\overline{z}_0) - \psi_\theta(z_0) \end{pmatrix}$$

for all $\theta, \overline{\theta} \in [-\eta, \eta]$ and $z_0, \overline{z}_0 \in \overline{B}(\rho)$ with $\eta \in (0, \eta_0]$ and $\rho \in (0, \rho_0]$. Since $P(0)u'_*(0) = 0$ and $u'_* \in C(\mathbb{R}; X_{1-1/p})$, we can estimate

$$\begin{aligned} |P(0)(u_*(\overline{\theta}) - u_*(\theta))|_{1-1/p} &\le \int_0^1 |P(0)[u'_*(\theta + s(\overline{\theta} - \theta)) - u'_*(0)]\,(\overline{\theta} - \theta)|_{1-1/p}\,ds \\ &\le \varepsilon(\eta)\,|\theta - \overline{\theta}|. \end{aligned}$$

Similarly, one obtains

$$\begin{aligned} &|\langle u_*(\overline{\theta}) - u_*(\theta) + (\theta - \overline{\theta})u'_*(0), x^*\rangle \\ &\qquad \le c \int_0^1 |u'_*(\theta + s(\overline{\theta} - \theta)) - u'_*(0)|_{1-1/p}\,|\overline{\theta} - \theta|\,ds \le \varepsilon(\eta)\,|\theta - \overline{\theta}|. \end{aligned}$$

To treat the remaining terms, we write $w = \Phi_\theta(z_0)$ and $\overline{w} = \Phi_{\overline{\theta}}(\overline{z}_0)$ and note that

$$w - \overline{w} = L^+(z_0 - \overline{z}_0 + P(0)\hat{\mathcal{N}}(0)\gamma_0(\tilde{\mathbb{H}}_\theta(w) - \tilde{\mathbb{H}}_{\overline{\theta}}(\overline{w})), \tilde{\mathbb{G}}_\theta(w) - \tilde{\mathbb{G}}_{\overline{\theta}}(\overline{w}), \tilde{\mathbb{H}}_\theta(w) - \tilde{\mathbb{H}}_{\overline{\theta}}(\overline{w})).$$

Proposition 4.11, (3.2) and (2.10) then yield

$$\|w - \overline{w}\|_{\mathbb{E}_1(\delta)} \le c\Big(|z_0 - \overline{z}_0|_{1-\frac{1}{p}} + \|\tilde{\mathbb{G}}_\theta(w) - \tilde{\mathbb{G}}_{\overline{\theta}}(\overline{w})\|_{\mathbb{E}_0(\delta)} + \|\tilde{\mathbb{H}}_\theta(w) - \tilde{\mathbb{H}}_{\overline{\theta}}(\overline{w})\|_{\mathbb{F}(\delta)}\Big). \tag{5.14}$$

Taking into account (5.3) and $B(u_*) = 0$, we calculate

$$\begin{aligned} \tilde{\mathbb{H}}_\theta(w) - \tilde{\mathbb{H}}_{\overline{\theta}}(\overline{w}) &= \tilde{\mathbb{H}}_\theta(w) - \tilde{\mathbb{H}}_\theta(\overline{w}) + B(u_{\overline{\theta}} + \overline{w}) - B(u_{\overline{\theta}}) + B(u_\theta) - B(u_\theta + \overline{w}) \\ &= \int_0^t \tilde{\mathbb{H}}'_\theta(w + s(\overline{w} - w))\,(\overline{w} - w)\,ds \end{aligned}$$

$$+\int_0^1 [B'(u_\theta + s\overline{w}) - B'(u_{\overline{\theta}} + s\overline{w})]\overline{w}\,ds.$$

Estimate (5.11) yields $\|w + s(\overline{w} - w)\|_{\mathbb{E}_1(\delta)} \le c\rho$ for all $s \in [0,1]$. Since $\tilde{\mathbb{H}}'_\theta$ is continuous, we can thus deduce from (5.6) that

$$\|\tilde{\mathbb{H}}'_\theta(w + s(\overline{w} - w))\|_{\mathcal{B}(\mathbb{E}_1(\delta),\mathbb{F}(\delta))} \le \varepsilon(\rho) + \varepsilon(|\theta|) \le \varepsilon(\rho) + \varepsilon(\eta).$$

Due to (R) and Lemma A.2 in [16], the map $v \mapsto B'(v) \in \mathcal{B}(\mathbb{E}_1(J), \mathbb{F}(J))$ is locally Lipschitz on $\mathbb{E}_1(J)$ for every compact interval J. As in the calculations leading to (5.6), it then follows that

$$\begin{aligned}\|[B'(u_\theta + s\overline{w}) - B'(u_{\overline{\theta}} + s\overline{w})]\overline{w}\|_{\mathbb{F}(\delta)} &\le c\,\|u_\theta - u_{\overline{\theta}}\|_{\mathbb{E}_1([0,\tau])}\,\|\overline{w}\|_{\mathbb{E}_1(\delta)}\\ &\le c\rho \int_0^1 \|u_*'(\cdot + \theta + s(\overline{\theta} - \theta))\|_{\mathbb{E}_1([0,\tau])}\,|\theta - \overline{\theta}|\,ds\\ &\le c\rho\,|\theta - \overline{\theta}|,\end{aligned}$$

employing again (5.11) and Theorem 3.2. As a result,

$$\|\tilde{\mathbb{H}}_\theta(w) - \tilde{\mathbb{H}}_{\overline{\theta}}(\overline{w})\|_{\mathbb{F}(\delta)} \le (\varepsilon(\eta) + \varepsilon(\rho))\,\|w - \overline{w}\|_{\mathbb{E}_1(\delta)} + c\rho\,|\theta - \overline{\theta}|. \tag{5.15}$$

In a similar way, one derives

$$\|\tilde{\mathbb{G}}_\theta(w) - \tilde{\mathbb{G}}_{\overline{\theta}}(\overline{w})\|_{\mathbb{E}_0(\delta)} \le (\varepsilon(\eta) + \varepsilon(\rho))\,\|w - \overline{w}\|_{\mathbb{E}_1(\delta)} + c\rho\,|\theta - \overline{\theta}|. \tag{5.16}$$

Taking sufficiently small $\eta > 0$ and $\rho > 0$, (5.14) thus leads to

$$\|w - \overline{w}\|_{\mathbb{E}_1(\delta)} \le c\,|z_0 - \overline{z}_0|_{1-1/p} + c\rho\,|\theta - \overline{\theta}|.$$

Inserting this inequality into (5.15) and (5.16), we conclude that

$$\begin{aligned}\|\tilde{\mathbb{G}}_\theta(w) - \tilde{\mathbb{G}}_{\overline{\theta}}(\overline{w})\|_{\mathbb{E}_0(\delta)} &\le (\varepsilon(\eta) + \varepsilon(\rho))\,(|z_0 - \overline{z}_0|_{1-1/p} + |\theta - \overline{\theta}|),\\ \|\tilde{\mathbb{H}}_\theta(w) - \tilde{\mathbb{H}}_{\overline{\theta}}(\overline{w})\|_{\mathbb{F}(\delta)} &\le (\varepsilon(\eta) + \varepsilon(\rho))\,(|z_0 - \overline{z}_0|_{1-1/p} + |\theta - \overline{\theta}|).\end{aligned} \tag{5.17}$$

Using (5.10), (3.2), (2.10) and Lemma 2.3, we arrive at

$$\begin{aligned}|\psi_\theta(z_0) - \psi_{\overline{\theta}}(\overline{z}_0)|_{1-1/p} &\le c\,\|\tilde{\mathbb{H}}_\theta(w) - \tilde{\mathbb{H}}_{\overline{\theta}}(\overline{w})\|_{\mathbb{F}([0,1])} \le c\,\|\tilde{\mathbb{H}}_\theta(w) - \tilde{\mathbb{H}}_{\overline{\theta}}(\overline{w})\|_{\mathbb{F}(\delta)}\\ &\le c\,(\varepsilon(\eta) + \varepsilon(\rho))\,(|z_0 - \overline{z}_0|_{1-1/p} + |\theta - \overline{\theta}|).\end{aligned}$$

Finally, (5.9) and (5.17) yield

$$|Q(0)(\varphi(\overline{\theta}, \overline{z}_0) - \varphi(\theta, z_0)| \le c\,(\varepsilon(\eta) + \varepsilon(\rho))\,(|z_0 - \overline{z}_0|_{1-1/p} + |\theta - \overline{\theta}|).$$

Summing up, we can fix $\eta = \rho \in (0, \eta_0] \cap (0, \rho_0)$ such that Ψ is Lipschitz with constant $1/2$ on $[-\rho, \rho] \times \overline{B}(\rho) =: M$, where we take the norm $\|(\theta, z_0)\| = \max\{|\theta|, |z_0|_{1-1/p}\}$ on $M \subset \mathbb{R} \times X_{1-1/p}$.

To show the invariance of M under Ψ, we first note that $\Psi(0,0) = (\langle u_0 - u_*(0), x^*\rangle, P(0)(u_0 - u_*(0)))$, and hence $\|\Psi(0,0)\| \le cr$, provided that $|u_0 - u_*(0)|_{1-1/p} \le r$. So, for $(\theta, z_0) \in M$ it follows that

$$\|\Psi(\theta, z_0)\| \le cr + \frac{1}{2}\max\{|\theta|, |z_0|_{1-1/p}\} \le cr + \frac{\rho}{2},$$

and $\Psi : M \to M$ if $r > 0$ is chosen small enough. □

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