Perturbation and an abstract characterization of evolution semigroups*

Frank Räbiger, Universität Tübingen, Mathematisches Institut, Auf der Morgenstelle 10, D-72076 Tübingen, Germany.

Abdelaziz Rhandi, Universität Tübingen, Mathematisches Institut, Auf der Morgenstelle 10, D-72076 Tübingen, Germany.†

Roland Schnaubelt, Universität Tübingen, Mathematisches Institut, Auf der Morgenstelle 10, D-72076 Tübingen, Germany.

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Abstract

Evans and Howland used an abstract characterization of evolution semigroups and a perturbation result for $\mathcal{C}_0$–semigroups to treat non–autonomous Schrödinger equations and problems in scattering theory. Here we proceed in a similar way. At first we characterize evolution semigroups on spaces of vector–valued functions $E(X)$ which are induced by a strongly continuous evolution family $(U(t,s))_{t\geq s}$ defined on the Banach space X. From this and a perturbation result of Voigt we obtain a sufficient condition for an operator on $L^p(\mathbb{R},X)$ to be the generator of an evolution semigroup. This will be applied to non–autonomous heat equations with absorption.

1 Evolution Families

The solutions of a non–autonomous Cauchy problem on a Banach space X can be expressed (under appropriate conditions) by a family $(U(t,s))_{t\geq s}$ in the space $\mathcal{L}(X)$ of bounded linear operators on X such that

(E1) the mapping $(t,s)\mapsto U(t,s)$ from $\mathcal{H}:=\{(t,s)\in\mathbb{R}^2 : t\geq s\}$ into $\mathcal{L}(X)$ is strongly continuous,

(E2) $U(s,s)=Id_X$, $U(t,r)U(r,s)=U(t,s)$ for all $t\geq r\geq s$,

(E3) there are constants $M\geq 1$ and $\omega\in\mathbb{R}$ such that $\|U(t,s)\|\leq Me^{\omega(t-s)}$ for all $(t,s)\in\mathcal{H}$

(see e.g. [6], [9], [20], [28]). In the following we call such a family of operators an *evolution family* on X. If $(S(t))_{t\geq 0}$ is a strongly continuous semigroup of operators on X (abbr. $\mathcal{C}_0$–semigroup), then $U(t,s) := S(t-s)$, $(t,s) \in \mathcal{H}$, defines an evolution family.

For the investigation of evolution families $(U(t,s))_{(t,s)\in\mathcal{H}}$ it is useful to associate with $(U(t,s))_{(t,s)\in\mathcal{H}}$ a semigroup $(T(t))_{t\geq 0}$ of operators on the Banach space $F = C_0(\mathrm{I\!R}, X)$ or $L^p(\mathrm{I\!R}, X)$, $1 \leq p < \infty$, by setting

$$T(t)f\,(\cdot) := U(\cdot, \cdot - t)f(\cdot - t)\,, \qquad f \in F,\ t \geq 0$$

(see e.g. [8], [10], [14], [17], [19]). It is easy to verify that $(T(t))_{t\geq 0}$ is a $\mathcal{C}_0$–semigroup on F (cf. [8], [23]). We call $(T(t))_{t\geq 0}$ the *evolution semigroup associated with* $(U(t,s))_{(t,s)\in\mathcal{H}}$. Recently, R. Rau [23], [24], [25], and Y. Latushkin and S. Montgomery–Smith [11], [12] (see also [15], [26] and the references therein) discovered, that there is A close connection between the asymptotic properties of an evolution family and the spectra of the induced evolution semigroup and its generator G. In particular, it is possible to characterize the exponential dichotomy or hyperbolicity of an evolution family by the invertibility of G (see [13, Thm. 3.6], [21, Cor. 1.6]).

In [22] these results were extended to a more general class of X–valued function spaces, containing in particular $L^p(\mathrm{I\!R}, X)$, $1 \leq p < \infty$, and many spaces occuring in interpolation theory (see [22, Cor. 4.5]). In Section 2 we recall the definition of evolution semigroups on such spaces.

In [8], [10], abstract characterizations of evolution semigroups were used in order to apply perturbation results for $\mathcal{C}_0$–semigroups to non–autonomous Schrödinger equations and scattering theory. Here we will proceed in a similar way. First we characterize evolution semigroups on the spaces introduced in Section 2 by means of scalar multiplication operators (Theorem 3.4). This generalizes results obtained in [8], [10], [17], [19]. Together with a perturbation theorem of Voigt [30] this yields sufficient conditions that a linear operator on $L^p(\mathbb{R}, X)$ generates an evolution semigroup. As an application we obtain the existence of an evolution family for a class of non–autonomous heat equations (Section 4).

2 Evolution Semigroups on X–valued Banach Function Spaces

Let (Ω, Σ, μ) be a measure space and let $M(\Omega, \Sigma, \mu)$ be the space of μ–measurable real–valued functions modulo the functions vanishing μ–a.e.. With the canonical order $M(\Omega, \Sigma, \mu)$ is an order complete vector lattice. We denote by χ_A the characteristic function of a set A. A linear subspace E of $M(\Omega, \Sigma, \mu)$ is called an *ideal* if $\varphi \in E$, $\psi \in M(\Omega, \Sigma, \mu)$ and $|\psi| \leq |\varphi|$ implies $\psi \in E$.

Definition 2.1 A Banach lattice E is called a *Banach function space* over (Ω, Σ, μ) if E is an ideal in $M(\Omega, \Sigma, \mu)$, $\chi_A \in E$ for all sets $A \in \Sigma$ of finite measure and each $\varphi \in E$ is locally integrable, i.e., $\int_A |\varphi| \, d\lambda < \infty$ for all sets $A \in \Sigma$ of finite measure.

If E is a Banach function space over (Ω, Σ, μ) and X a Banach space, then the space $E(X)$ of strongly measurable functions $f : \Omega \to X$ such that $\|f(\cdot)\|_X \in E$ endowed with the norm $\|f\|_{E(X)} := \| \, \|f(\cdot)\|_X \, \|_E$ is a Banach space.

Let $\mathcal{B}$ be the Borel algebra of $\mathbb{R}$ and λ the Lebesgue measure. We denote by $C_c(\mathbb{R}, X)$ the space of continuous functions $f : \mathbb{R} \to X$ with compact support. Recall that a Banach function space E has *order continuous norm* if for each decreasing net $(\varphi_\alpha)_{\alpha \in A}$ of positive functions in E such that $\inf_\alpha \varphi_\alpha = 0$ one has $\lim_\alpha \|\varphi_\alpha\| = 0$.

Lemma 2.2 *Let E be a Banach function space over $(\mathbb{R}, \mathcal{B}, \lambda)$ with order continuous norm. Then $C_c(\mathbb{R}, X)$ is dense in $E(X)$.*

Proof. We show that each $f \in E(X)$ can be approximated by functions in $C_c(\mathbb{R}, X)$. Let $A_n := [\|f(\cdot)\|_X \leq n]$. Since E has order continuous norm, the sequence $(\chi_{A_n \cap [-n,n]} f)$ converges in norm to f. Thus we may assume that f has compact support $A := \operatorname{supp} f \subseteq \mathbb{R}$, and is uniformly bounded, say $\|f(\cdot)\|_X \leq M$ a.e.. Let (f_n) be a sequence of simple functions such that $\lim_n f_n(s) = f(s)$ for almost every s. We may assume $\operatorname{supp} f_n \subseteq A$ and $\|f_n(\cdot)\|_X \leq M + 1$ a.e. for all $n \in \mathbb{N}$. Thus $(\|f(\cdot) - f_n(\cdot)\|_X)_{n \in \mathbb{N}}$ is dominated by $(2M+1)\chi_A \in E$ and converges almost everywhere to 0. By Lebesgue's theorem for Banach function spaces with order continuous norm (cf. [3, Prop. 1.3.6]) we obtain $\lim_n \|f - f_n\|_{E(X)} = \lim_n \| \, \|f(\cdot) - f_n(\cdot)\|_X \, \|_E = 0$. Hence f can be approximated in $E(X)$ by simple functions. Thus it remains to show that each function $\chi_C(\cdot)\, x \in E(X)$ with $x \in X$ and $\lambda(C) < \infty$ can be approximated in $E(X)$ by functions belonging to $C_c(\mathbb{R}, X)$. By the regularity of the Lebesgue measure we find compact sets K_n and open sets O_n, $n \in \mathbb{N}$, such that $K_n \subseteq C \subseteq O_n \subseteq O_1$ and $\lambda(O_n \setminus K_n) \leq \frac{1}{n}$ for all $n \in \mathbb{N}$. Choose positive functions $\psi_n \in C_c(\mathbb{R}, \mathbb{R})$ such that $\|\psi_n\|_\infty \leq 1$, $\psi_{n \,|K_n} = 1$ and $\operatorname{supp} \psi_n \subseteq O_n$. Then $(\|\psi_n(\cdot)\, x - \chi_C(\cdot)\, x\|_X)$ converges to 0 almost everywhere and is dominated by $2\,\|x\|\,\chi_C \in E$. Again by Lebesque's theorem we obtain $\lim_n \psi_n(\cdot)\, x = \chi_C(\cdot)\, x$ in $E(X)$. Clearly, $\psi_n(\cdot)\, x \in C_c(\mathbb{R}, X)$ and the proof is finished. □

In order to define evolution semigroups on spaces $E(X)$ we assume that E is a Banach function space over $(\mathbb{R}, \mathcal{B}, \lambda)$ such that

(P1) E has order continuous norm,

(P2) E is translation invariant, i.e., with $\varphi \in E$ we have $R(t)\varphi := \varphi(\cdot - t) \in E$ for all $t \in \mathbb{R}$,

(P3) the group of translations $(R(t))_{t\in\mathbb{R}}$ is strongly continuous on E.

Examples of such spaces are $L^p(\mathbb{R})$, $L^p(\mathbb{R}) \cap L^q(\mathbb{R})$, $1 \le p, q < \infty$, the Lorentz spaces $L_{p,q}(\mathbb{R})$, $1 < p < \infty$, $1 \le q < \infty$, and, more general, rearrangement invariant function spaces over $(\mathbb{R}, \mathcal{B}, \lambda)$ with order continuous norm (see [3, 4.4.6, 2.4]). Let $(U(t,s))_{(t,s)\in\mathcal{H}}$ be an evolution family on the Banach space X. Then for $t \ge 0$

$$T(t)f\,(\cdot) := U(\cdot, \cdot - t)f(\cdot - t)\,, \qquad f \in E(X), \tag{E}$$

is an element of $E(X)$.

Proposition 2.3 *Let E be a Banach function space satisfying (P1)–(P3). Let $T(t)$, $t \ge 0$, be defined as above. Then $T(t) \in \mathcal{L}(E(X))$ and $(T(t))_{t\ge0}$ is a $\mathcal{C}_0$–semigroup.*

Proof. For $f \in E(X)$ we have

$$\begin{aligned} \|T(t)f\|_{E(X)} &= \|\, \|U(\cdot, \cdot - t)f(\cdot - t)\|_X \,\|_E \\ &\le Me^{\omega t}\, \|\, \|f(\cdot - t)\|_X \,\|_E \\ &\le Me^{\omega t}\, \|R(t)\|\, \|f\|_{E(X)} \\ &\le Ne^{\alpha t}\, \|f\|_{E(X)} \end{aligned}$$

for some constants $N \ge 1$ and $\alpha \in \mathbb{R}$. In particular, $(T(t))_{t\ge0}$ is uniformly bounded on compact intervals. For $f \in C_c(\mathbb{R}, X)$ we obtain $T(t)f \in C_c(\mathbb{R}, X)$ and $\lim_{t\to0} \|T(t)f - f\|_\infty = 0$. If $g \in C_c(\mathbb{R}, X)$ has support K, then $\|g\|_{E(X)} \le \|g\|_\infty\, \|\chi_K\|_E$. From this we get $\lim_{t\to0} \|T(t)f - f\|_{E(X)} = 0$. Since $C_c(\mathbb{R}, X)$ is dense in $E(X)$, the strong continuity follows . □

Definition 2.4 Let $(U(t,s))_{(t,s)\in\mathcal{H}}$ be an evolution family on a Banach space X and let E be a Banach function space over $(\mathbb{R},\mathcal{B},\lambda)$ satisfying (P1)–(P3). The $\mathcal{C}_0$–semigroup $(T(t))_{t\geq 0}$ on $E(X)$ defined by (E) is called the *evolution semigroup associated with* $(U(t,s))_{(t,s)\in\mathcal{H}}$. We denote its generator by $(G, D(G))$.

3 Characterization of Evolution Semigroups on X–valued Banach Function Spaces

Let $C_b(\mathbb{R},\mathcal{L}_s(X))$ be the space of functions with values in $\mathcal{L}(X)$ which are bounded and strongly continuous. Endowed with the sup–norm $C_b(\mathbb{R},\mathcal{L}_s(X))$ is a Banach space. Let E be a Banach function space over $(\mathbb{R},\mathcal{B},\lambda)$ with order continuous norm and let $M(\cdot)\in C_b(\mathbb{R},\mathcal{L}_s(X))$. We define a *multiplication operator* $\mathcal{M}$ on $E(X)$ by

$$\mathcal{M}f\,(\cdot) = M(\cdot)f(\cdot), \qquad f\in E(X).$$

Using Lemma 2.2 it is easy to show that $\mathcal{M}$ is a bounded operator on $E(X)$ satisfying $\|\mathcal{M}\| = \|M(\cdot)\|_\infty$. Therefore, $C_b(\mathbb{R},\mathcal{L}_s(X))$ can be isometrically identified with a closed subspace of $\mathcal{L}(E(X))$.

In [8, Prop. 1.3] it is proved that a bounded operator $\mathcal{M}$ on $C_0(\mathbb{R},X)$ is induced by an operator–valued function $M(\cdot)\in C_b(\mathbb{R},\mathcal{L}_s(X))$ if and only if

$$\mathcal{M}(\varphi f) = \varphi\,\mathcal{M}f$$

for each $f\in C_0(\mathbb{R},X)$ and $\varphi\in C_b(\mathbb{R},\mathbb{C})$. An similar result for bounded operators $\mathcal{M}$ on $E(X)$ requires additional assumptions on $\mathcal{M}$ (cf. [8, Thm. 5.7]). To that purpose we introduce the following notion (see also [17, Def. 4.4]).

Definition 3.1 Let E be a Banach function space over $(\mathbb{R}, \mathcal{B}, \lambda)$ and let X be a Banach space. Consider a bounded linear operator $\mathcal{M}$ on $E(X)$. A subspace F of $E(X)$ is called $\mathcal{M}$*–determining* if

(D1) $F \subseteq C_b(\mathbb{R}, X)$ and $\mathcal{M}F \subseteq C_b(\mathbb{R}, X)$,

(D2) F is dense in $E(X)$,

(D3) $F^s := \{f(s) \,:\, f \in F\}$ is dense in X for all $s \in \mathbb{R}$.

Notice that F satisfies (D3) if it is dense in $C_0(\mathbb{R}, X)$.

In the following we make use of subsets Φ of $L^\infty(\mathbb{R})$ with the following property:

(F) for every $s \in \mathbb{R}$ there is a sequence (φ_n) in $\Phi \setminus \{0\}$ such that $I_n :=\operatorname{supp} \varphi_n$, $n \in \mathbb{N}$, is compact and $\lim_{n\to\infty} \sup_{t\in I_n} |s - t| = 0$.

Examples of such sets are $L^\infty(\mathbb{R})$, $C_c(\mathbb{R})$, the space $C_c^1(\mathbb{R})$ of continuously differentiable functions with compact support or the set of characteristic functions with compact support. We state the following lemma. Notice that $\varphi f \in E(X)$ for $\varphi \in L^\infty(\mathbb{R})$ and $f \in E(X)$.

Lemma 3.2 *Let E be a Banach function space over $(\mathbb{R}, \mathcal{B}, \lambda)$ with order continuous norm. Moreover, let $s \in \mathbb{R}$ and (φ_n) a sequence in $L^\infty(\mathbb{R})$ such that condition (F) holds. Then for every continuous function $f : \mathbb{R} \to X$ we have*

$$\|f(s)\| = \lim_{n\to\infty} \frac{1}{\|\varphi_n\|_E} \; \|\varphi_n f\|_{E(X)} \,.$$

Proof. Let $\varepsilon > 0$. Set $I_n := \operatorname{supp} \varphi_n$. Choose $n \in \mathbb{N}$ such that $\|f(s) - f(r)\|_X < \varepsilon$ for all $r \in I_n$. Then

$$\begin{aligned} \|\varphi_n f - \varphi_n(\cdot) f(s)\|_{E(X)} &\leq \sup_{r\in I_n} \|f(s) - f(r)\|_X \;\; \|\varphi_n\|_E \\ &\leq \varepsilon \, \|\varphi_n\|_E \,. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\|f(s)\| - \varepsilon &= \frac{1}{\|\varphi_n\|_E}\ \|\varphi_n(\cdot)f(s)\|_{E(X)} - \varepsilon \\ &\leq \frac{1}{\|\varphi_n\|_E}\ \|\varphi_n f\|_{E(X)} \\ &\leq \frac{1}{\|\varphi_n\|_E}\ \|\varphi_n(\cdot)f(s)\|_{E(X)} + \varepsilon \\ &= \|f(s)\| + \varepsilon\,.\end{aligned}$$

This yields the assertion. □

Now we can show the analogue of [8, Prop. 1.3]. Our proof follows [21, Prop. 2.3].

Proposition 3.3 *Let E be a Banach function space over $(\mathbb{R},\mathcal{B},\lambda)$ with order continuous norm and let X be a Banach space. Consider a subset Φ of $L^\infty(\mathbb{R})$ for which (F) is satisfied. If $\mathcal{M} \in \mathcal{L}(E(X))$, then the following assertions are equivalent.*

(a) There is a function $M(\cdot) \in C_b(\mathbb{R}, \mathcal{L}_s(X))$ such that $\mathcal{M} = M(\cdot)$.

(b) There is a $\mathcal{M}$–determining subspace F of $E(X)$ such that

$$\mathcal{M}(\varphi f) = \varphi\,\mathcal{M} f \tag{3.1}$$

for all $\varphi \in \Phi$ and $f \in F$.

Proof. (a) ⇒ (b) is clear (consider $F = C_c(\mathbb{R}, X)$).
(b) ⇒ (a): Fix $f \in F$ and $s \in \mathbb{R}$. Choose $(\varphi_n) \subseteq \Phi$ such that (F) holds. By (D1) the functions f and $\mathcal{M}f$ are continuous. Therefore, Lemma 3.2 and (3.1) imply

$$\|\mathcal{M}f\,(s)\| = \lim_{n\to\infty} \frac{1}{\|\varphi_n\|_E}\ \|\varphi_n\,\mathcal{M}f\|_{E(X)}$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \frac{1}{\|\varphi_n\|_E} \ \|\mathcal{M}\,(\varphi_n f)\|_{E(X)} \\
&\le \|\mathcal{M}\|_{\mathcal{L}(E(X))} \ \lim_{n\to\infty} \frac{1}{\|\varphi_n\|_E} \ \|\varphi_n f\|_{E(X)} \\
&= \|\mathcal{M}\| \ \|f(s)\|\,. \qquad (3.2)
\end{aligned}
$$

Let $F^s := \{f(s) \ : \ f \in F\}$. Define

$$
M(s) : \ F^s \to X \ : \ x \mapsto (\mathcal{M}f)\,(s)\,,
$$

where $f \in F$ satisfies $f(s) = x$. Then (3.2) implies that $M(s)$ is well–defined. Moreover, $\|M(s)\| \le \|\mathcal{M}\|$. Since F^s is dense in X, $M(s)$ has an unique extension to X with norm less or equal $\|\mathcal{M}\|$. We denote this extension also by $M(s)$.

Now we show $M(\cdot) \in C_b(\mathbb{R}, \mathcal{L}_s(X))$. Fix $s \in \mathbb{R}$, $x \in X$ and $\epsilon > 0$. Choose $y \in F^s$ such that $\|x - y\| < \epsilon$ and $f \in F$ such that $f(s) = y$. Since f and $\mathcal{M}f$ are continuous, there is $\delta > 0$ such that

$$
\|f(s) - f(r)\| < \epsilon \quad \text{and} \quad \|(\mathcal{M}f)(s) - (\mathcal{M}f)(r)\| < \epsilon \qquad \text{for all } |r - s| < \delta.
$$

Then for $|r - s| < \delta$ we have

$$
\begin{aligned}
\|M(s)x - M(r)x\| &\le \|M(s)x - M(s)y\| + \|M(s)y - M(r)f(r)\| \\
&\quad + \|M(r)f(r) - M(r)y\| + \|M(r)y - M(r)x\| \\
&\le \|M(s)\|\,\|x - y\| + \|(\mathcal{M}f)(s) - (\mathcal{M}f)(r)\| \\
&\quad + \|M(r)\|\,\|f(r) - f(s)\| + \|M(r)\|\,\|x - y\| \\
&\le (3\|\mathcal{M}\| + 1)\epsilon.
\end{aligned}
$$

Thus $M(\cdot) \in C_b(\mathbb{R}, \mathcal{L}_s(X))$. Moreover, $\mathcal{M}f\,(\cdot) = M(\cdot)f(\cdot)$ for $f \in F$. Since F is dense in $E(X)$, $M(\cdot)$ represents $\mathcal{M}$. □

We denote by $(R(t))_{t\in\mathbb{R}}$ and $(\mathcal{R}(t))_{t\in\mathbb{R}}$ the (right–)translation groups on E and $E(X)$, respectively. Set $\varphi_t := R(t)\varphi$ for $t \in \mathbb{R}$, $\varphi \in E$. Let $(A, D(A))$ be the generator of $(R(t))_{t\in\mathbb{R}}$, i.e., $A\varphi = -\varphi'$ for $\varphi \in D(A)$. Notice that $C_c^1(\mathbb{R})$ is contained in $D(A)$. Finally, we denote by $\rho(G)$ the *resolvent set* of an operator G and by $R(\lambda, G) := (\lambda - G)^{-1}$, $\lambda \in \rho(G)$, the *resolvent* of G. Now we state the main result of this section. Similar results were obtained in [8, Thm. 1.6] for $C_0(\mathbb{R}, X)$ and in [17, Thm. 4.12] for $L^p([0,T], X)$. In the sequel we apply some ideas used in the proofs of [10, Thm. 1], [19, Thm. I.11].

Theorem 3.4 *Let X be a Banach space and let E be a Banach function space over $(\mathbb{R}, \mathcal{B}, \lambda)$ satisfying (P1)–(P3). Let $(T(t))_{t\geq 0}$ be a $\mathcal{C}_0$–semigroup on $E(X)$ with generator $(G, D(G))$. Assume that there is $\lambda \in \rho(G)$ such that $R(\lambda, G) : E(X) \to C_0(\mathbb{R}, X)$ is continuous and has dense image. Then the following assertions are equivalent.*

(a) $(T(t))_{t\geq 0}$ is an evolution semigroup.

(b) There is a set $\Phi \subseteq L^\infty(\mathbb{R})$ satisfying (F) such that

$$T(t)\,(\varphi f) = \varphi_t\, T(t) f, \qquad t \geq 0,$$

for all $\varphi \in \Phi$ and $f \in E(X)$.

(c) For all $\varphi \in C_c^1(\mathbb{R})$ and $f \in D(G)$ we have $\varphi f \in D(G)$ and

$$G\,(\varphi f) = -\varphi' f + \varphi\, G f.$$

(d) There is $\mu \in \rho(G)$ such that for all $\varphi \in C_c^1(\mathbb{R})$ and $f \in E(X)$ we have

$$R(\mu, G)\,(\varphi'\, R(\mu, G) f) = \varphi\, R(\mu, G) f - R(\mu, G)\,(\varphi f).$$

Proof. (a) $\Rightarrow$ (c): Let $\varphi \in C_c^1(\mathbb{R}) \subseteq D(A)$ and $f \in D(G)$. Then

$$\frac{1}{t}\Big(T(t)\,(\varphi f) - \varphi f\Big) = \frac{1}{t}\,(\varphi_t - \varphi)\, T(t) f + \frac{1}{t}\,\varphi\,(T(t) f - f), \quad t > 0,$$

which implies (c).

(c) $\Rightarrow$ (b): Consider $u(t) := \varphi_t T(t)f$, $t \geq 0$, for $\varphi \in C_c^1(\mathbb{R})$ and $f \in D(G)$. By the assumption $u(t) \in D(G)$ and

$$Gu(t) = -\varphi_t' T(t)f + \varphi_t GT(t)f, \qquad t \geq 0.$$

On the other hand, observe that $u(0) = \varphi f$ and $u(\cdot)$ is continuously differentiable satisfying

$$\frac{d}{dt}u(t) = -\varphi_t' T(t)f + \varphi_t GT(t)f, \qquad t \geq 0.$$

Hence, $u(\cdot)$ solves the well–posed Cauchy problem

$$\frac{d}{dt}v(t) = Gv(t), \quad t \geq 0, \qquad v(0) = \varphi f$$

on $E(X)$. Therefore,

$$\varphi_t T(t)f = u(t) = T(t)(\varphi f), \qquad t \geq 0.$$

Clearly, $\Phi := C_c^1(\mathbb{R})$ satisfies (F), and hence (b) follows.

(b) $\Rightarrow$ (a): Set $\mathcal{M}_t := T(t)\mathcal{R}(-t)$ and $F_t := \mathcal{R}(t)D(G)$, $t \geq 0$. Then (b) yields $\mathcal{M}_t(\varphi f) = \varphi \mathcal{M}_t f$ for all $\varphi \in \Phi$ and $f \in E(X)$. By our assumption there is $\lambda \in \rho(G)$ such that $R(\lambda, G) : E(X) \to C_0(\mathbb{R}, X)$ is continuous and has dense image. Thus, $D(G)$ is a dense subspace of $C_0(\mathbb{R}, X)$. Therefore, $F_t \subseteq C_0(\mathbb{R}, X)$ and F_t fulfills (D2) and (D3), $t \geq 0$. Moreover, $\mathcal{M}_t F_t = T(t)\mathcal{R}(-t)\mathcal{R}(t)D(G) \subseteq D(G) \subseteq C_0(\mathbb{R}, X)$. Hence, F_t is $\mathcal{M}_t$–determining for each $t \geq 0$. By Proposition 3.3 there exist operators $M(t,s) \in \mathcal{L}(X)$, $s \in \mathbb{R}$, $t \geq 0$, such that $\mathcal{M}_t = M(t, \cdot) \in C_b(\mathbb{R}, \mathcal{L}_s(X))$. Moreover,

$$\|M(t,s)\| \leq \sup_{s \in \mathbb{R}} \|M(t,s)\| = \|T(t)\mathcal{R}(-t)\| \leq Ne^{\alpha t}, \quad t \geq 0, \tag{3.3}$$

for some constants $N \geq 1$ and $\alpha \in \mathbb{R}$. Let $f \in D(G) \subseteq C_0(\mathbb{R}, X)$ and $g = (\lambda - G)f$. Then $T(t)f - f = R(\lambda, G)\,(T(t)g - g) \in C_0(\mathbb{R}, X)$ and

$$\|T(t)f - f\|_\infty \leq \|R(\lambda, G)\|_{\mathcal{L}(E(X),C_0)}\,\|T(t)g - g\|_{E(X)}\,, \quad t \geq 0. \tag{3.4}$$

Now we want to show the strong continuity of the function $t \mapsto M(t,s)$ for each $s \in \mathbb{R}$. Fix $t_0 \geq 0$ and $s_0 \in \mathbb{R}$. By (3.3) it suffices to consider the dense subspace $F_{t_0}^{s_0} = \{g(s_0) \in X : \ g \in \mathcal{R}(t_0)D(G)\}$. Let $x \in F_{t_0}^{s_0}$ and $f \in D(G)$ such that $x = f(s_0 - t_0) = (\mathcal{R}(t_0))f(s_0)$. Then by (3.3)

$$\begin{aligned}\|M(t,s_0)x - M(t_0,s_0)x\| &\leq \|M(t,s_0)\,(\mathcal{R}(t_0)f)(s_0) - M(t,s_0)\,(\mathcal{R}(t)f)(s_0)\| \\ &\quad + \|M(t,s_0)\,(\mathcal{R}(t)f)(s_0) - M(t_0,s_0)\,(\mathcal{R}(t_0)f)(s_0)\| \\ &\leq Ne^{\alpha t}\,\|\mathcal{R}(t_0)f - \mathcal{R}(t)f\|_\infty + \|T(t)f - T(t_0)f\|_\infty\,.\end{aligned}$$

Therefore, by (3.4) the mapping $t \mapsto M(t,s_0)x$ is continuous. Thus, $M(\cdot,\cdot)$ is strongly continuous in both variables. We set $U(t,s) := M(t-s,t)$ for $(t,s) \in D$. Hence, $(U(t,s))_{(t,s)\in\mathcal{H}}$ is a family in $\mathcal{L}(X)$ satisfying (E1) and (E3).

Let $t, r \geq 0$. Fix $f \in D(G)$. Then f and $T(t+r)f = T(t)T(r)f$ are continuous. Moreover,

$$\begin{aligned}\Big(T(t+r)f\Big)(s) &= \Big(\mathcal{M}_{t+r}\,\mathcal{R}(t+r)f\Big)(s) \\ &= U(s, s-t-r)f(s-t-r) \\ \Big(T(t)T(r)f\Big)(s) &= \Big(\mathcal{M}_t\mathcal{R}(t)\mathcal{M}_r\mathcal{R}(r)f\Big)(s) \\ &= U(s,s-t)\,U(s-t,s-t-r)f(s-t-r)\end{aligned}$$

for all $s \in \mathbb{R}$. Since $D(G)$ is dense in $C_0(\mathbb{R}, X)$ this implies (E2).

(c) $\Leftrightarrow$ (d) follows by a simple computation. □

Recall that a *core* D of the generator $(A, D(A))$ of a $\mathcal{C}_0$–semigroup is a subspace of $D(A)$ which is dense in the graph norm $\|x\|_A := \|x\| + \|Ax\|$, $x \in D(A)$.

Remark 3.5

(1) In Theorem 3.4 assertion (c) is equivalent to the following condition.

(c') There is a core $\mathcal{D}$ of G such that for all $\varphi \in C_c^1(\mathrm{I\!R})$ and $f \in \mathcal{D}$ we have $\varphi f \in D(G)$ and $G(\varphi f) = -\varphi' f + \varphi G f$.

(2) If E is a Banach function space satisfying (P1)–(P3) on which the group of translations is bounded and G is the generator of an evolution semigroup on $E(X)$, then $R(\lambda, G): E(X) \to C_0(\mathrm{I\!R}, X)$, $\lambda \in \rho(G)$, is always continuous with dense image (see [22, Prop. 3.3]).

(3) In [8, Thm. 6.4] it is shown that for separable Banach spaces X and bounded $\mathcal{C}_0$–groups $(T(t))_{t\in\mathrm{I\!R}}$ on $L^p(\mathrm{I\!R}, X)$, $1 \leq p < \infty$, assertion (b) is equivalent to the existence of a *strongly measurable*, invertible and bounded evolution family which defines $(T(t))_{t\in\mathrm{I\!R}}$.

4 Perturbation of Evolution Semigroups

In this section we apply Theorem 3.4 and a perturbation theorem of Voigt [30, Thm. 1] in order to show that under certain conditions operators of the form $G = -\frac{d}{ds} + A + B(\cdot)$ generate an evolution semigroup on $L^p(\mathrm{I\!R}, X)$, $1 \leq p < \infty$. Here A is the generator of a $\mathcal{C}_0$–semigroup on X and $B(s)$ are (possibly unbounded) linear operators on X. As an example we will consider the heat equation with "absorption"

$$\frac{\partial}{\partial t} u(\xi, t) = \Delta_\xi u(\xi, t) + V(\xi, t) u(\xi, t), \quad \xi \in \mathrm{I\!R}^N,\ t \in \mathrm{I\!R}.$$

First we recall the result of Voigt (cf. [30, Thm. 1, (1.7)]).

Theorem 4.1 *Let $(T_0(t))_{t\geq 0}$ be a $\mathcal{C}_0$–semigroup on a Banach space F with generator*

$(G_0, D(G_0))$. Let $(\mathcal{B}, D(\mathcal{B}))$ be a linear operator on F and assume that there exists a dense subspace $\mathcal{D}$ of F such that the following conditions are satisfied:

(i) $\mathcal{D}$ is invariant under $(T_0(t))_{t\geq 0}$ and contained in $D(G_0) \cap D(\mathcal{B})$ and for every $u \in \mathcal{D}$ the function $[0, \infty[\to F : \; t \mapsto \mathcal{B}T_0(t)u$ is continuous.

(ii) There are constants $\alpha \in]0, \infty]$ and $\gamma \in [0, 1[$ such that

$$\int_0^\alpha \|\mathcal{B}T_0(t)u\| \, dt \leq \gamma \, \|u\|$$

for all $u \in \mathcal{D}$.

Then the closure G of $(G_0 + \mathcal{B})_{|\mathcal{D}}$ is the generator of a $\mathcal{C}_0$–semigroup $(T(t))_{t\geq 0}$ on F and $D(G) = D(G_0)$. Moreover, for sufficiently large $\lambda \in \rho(G_0) \cap \rho(G)$ there exists $C_\lambda \in \mathcal{L}(F)$ such that $R(\lambda, G) = R(\lambda, G_0)C_\lambda$.

Notice that $\mathcal{D}$ is a core of G, since G is the closure of $(G_0 + \mathcal{B})_{|\mathcal{D}}$.

Throughout this section let X be a Banach space and let $(e^{tA})_{t\geq 0}$ be a $\mathcal{C}_0$–semigroup on X with generator $(A, D(A))$. Set $X_A = (D(A), \|\cdot\|_A)$, where $\|\cdot\|_A$ denotes the graph norm of A. On $F = L^p(\mathrm{I\!R}, X)$, $1 \leq p < \infty$, we consider the translation group $(\mathcal{R}(t))_{t\in \mathrm{I\!R}}$ and the multiplication semigroup $\mathcal{M}(t)$ defined by $\mathcal{M}(t)f(\cdot) = e^{tA}f(\cdot)$, $t \geq 0$. Since these semigroups commute,

$$\mathcal{C} := \{u \in L^p(\mathrm{I\!R}, X_A) : \; u \text{ is differentiable a.e. and } u' \in F\}$$

is a core of the generator $(G_0, D(G_0))$ of the evolution semigroup given by

$$T_0(t)f(\cdot) = e^{tA}f(\cdot - t), \qquad t \geq 0\,, \; f \in F.$$

Moreover, $G_0 f(\cdot) = -f'(\cdot) + Af(\cdot)$ for $f \in \mathcal{C}$ (see [15, A-I.3.8]). As shown in [21, Lemma 2.1] the domain of the generator of an evolution semigroup on $L^p(\mathbb{R}, X)$ is a dense subspace of $C_0(\mathbb{R}, X)$. In particular, this holds for $D(G_0)$.

We consider a family $(B(s))_{s\in\mathbb{R}}$ of linear operators on X satisfying the following conditions.

(A) There is a dense subspace D of X such that $D \subseteq D(B(s)) \cap D(A)$ and $e^{tA}D \subseteq D$ for a.e. $s \in \mathbb{R}$ and each $t \geq 0$.

(B) For each $x \in D$ and $t \geq 0$ the function $B(\cdot)e^{tA}x$ is contained in $L^p_{\mathrm{loc}}(\mathbb{R}, X)$ and there exist constants $\alpha > 0$ and $\beta \geq 0$ such that $\alpha^{1/q}\beta < 1$, $\frac{1}{p} + \frac{1}{q} = 1$, and

$$\int_0^\alpha \|B(s+t)e^{tA}x\|^p \, dt \leq \beta^p \, \|x\|^p$$

for all $x \in D$ and a.e. $s \in \mathbb{R}$.

(C) The function $[0, \infty[\to F : \ t \mapsto \chi_I(\cdot)\, B(\cdot)\, e^{tA}x$ is continuous for all $x \in D$ and each compact intervall $I \subseteq \mathbb{R}$.

Later on we will give an example of a family $B(\cdot)$ of unbounded operators satisfying (A)–(C). In the case $A = 0$ condition (C) is trivially fulfilled and we obtain the following simpler conditions.

(A_0) There is a dense subspace D of X such that $D \subseteq D(B(s))$ for a.e. $s \in \mathbb{R}$.

(B_0) For every $x \in D$ the function $B(\cdot)x$ is contained in $L^p_{\mathrm{loc}}(\mathbb{R}, X)$ and there exist constants $\alpha > 0$ and $\beta \geq 0$ such that $\alpha^{1/q}\beta < 1$, $\frac{1}{p} + \frac{1}{q} = 1$, and

$$\int_0^\alpha \|B(s+t)x\|^p \, dt \leq \beta^p \, \|x\|^p$$

for all $x \in D$ and a.e. $s \in \mathbb{R}$.

Now we can state our perturbation result.

Theorem 4.2 *Let A be the generator of the $\mathcal{C}_0$–semigroup $(e^{tA})_{t\geq 0}$ on X and let $(G_0, D(G_0))$ be the generator of the associated evolution semigroup $(T_0(t))_{t\geq 0}$ on $F = L^p(\mathbb{R}, X)$, $1 \leq p < \infty$. Consider a family of linear operators $(B(s))_{s\in\mathbb{R}}$ on X satisfying the conditions (A)–(C). Then the operator $-\frac{d}{ds} + A + B(\cdot)$ has an extension $(G, D(G))$ with $D(G) = D(G_0)$ which generates an evolution semigroup $(T(t))_{t\geq 0}$ on F.*

First we state an immediate consequence of Theorem 4.2. An analogous result in the case $F = L^p([0, T], X)$ was shown by H. Neidhardt [18, Thm. 4.3].

Corollary 4.3 *Let $(B(s))_{s\in\mathbb{R}}$ be a family of linear operators on X satisfying the conditions (A_0) and (B_0). Then the operator $-\frac{d}{ds} + B(\cdot)$ has an extension $(G, D(\frac{d}{ds}))$ which generates an evolution semigroup on $F = L^p(\mathbb{R}, X)$, $1 \leq p < \infty$.*

Proof of Theorem 4.2.
In order to apply Theorem 4.1 we consider the multiplication operator $\mathcal{B}$ on F defined by $\mathcal{B}f := B(\cdot)f(\cdot)$ for $f \in D(\mathcal{B})$ where

$$D(\mathcal{B}) := \{f \in F : \ f(s) \in D(B(s)) \text{ for a.e. } s \in \mathbb{R} \text{ and } B(\cdot)f(\cdot) \in F\}.$$

In addition, we set

$$\mathcal{D} := \{f \in F : \ f = \sum_{i=1}^{n} \varphi_i \otimes x_i \ ; \ x_i \in D, \ \varphi_i \in C_c^1(\mathbb{R}), \ n \in \mathbb{N}\}.$$

We show that G_0, $\mathcal{B}$ and $\mathcal{D}$ satisfy the assumptions (i) and (ii) of Theorem 4.1. Since D is a dense subspace of X, it follows that $\mathcal{D}$ is a dense subspace of F. From the definition of $\mathcal{D}$ and conditions (A) and (B) we obtain $\mathcal{D} \subseteq D(\mathcal{B}) \cap \mathcal{C} \subseteq$

$D(\mathcal{B}) \cap D(G_0)$ and $T_0(t)\mathcal{D} \subseteq \mathcal{D}$ for all $t \geq 0$. In order to check (i) it remains to prove that the function $\mathcal{B}T_0(\cdot)u$ is continuous for all $u = \varphi \otimes x$ where $\varphi \in C_c^1(\mathrm{I\!R})$ and $x \in D$. Fix $t_0 \in \mathrm{I\!R}$. Since φ has compact support, there is a compact interval I_{t_0} such that $\operatorname{supp} \varphi(\cdot - t) \subseteq I_{t_0}$ for all $|t - t_0| < 1$. Then for $|t - t_0| < 1$ we have

$$\begin{aligned}
\|\mathcal{B}T_0(t)u - \mathcal{B}T_0(t_0)u\|_F &= \|\varphi(\cdot - t)B(\cdot)e^{tA}x - \varphi(\cdot - t_0)B(\cdot)e^{t_0A}x\|_F \\
&\leq \|(\varphi(\cdot - t) - \varphi(\cdot - t_0))\, B(\cdot)e^{t_0A}x\|_F \\
&\quad + \|\varphi(\cdot - t)\, B(\cdot)\,(e^{tA}x - e^{t_0A}x)\|_F \\
&\leq \|\varphi(\cdot - t) - \varphi(\cdot - t_0)\|_\infty \, \|\chi_{I_{t_0}}(\cdot)\, B(\cdot)e^{t_0A}x\|_F \\
&\quad + \|\varphi\|_\infty \, \|\chi_{I_{t_0}}(\cdot)\, B(\cdot)(e^{tA}x - e^{t_0A}x)\|_F \,.
\end{aligned}$$

Hence, it follows from (B) and (C) that

$$\lim_{t \to t_0} \|\mathcal{B}T_0(t)u - \mathcal{B}T_0(t_0)u\|_F = 0.$$

Now we show that condition (ii) of Theorem 4.1 is satisfied. Let $u = \sum_{i=1}^n \varphi_i \otimes x_i \in \mathcal{D}$. From Hölder's inequality and Fubini's Theorem we obtain

$$\begin{aligned}
\int_0^\alpha \|\mathcal{B}T_0(t)u\|_F \, dt &= \int_0^\alpha \left(\int_{\mathrm{I\!R}} \left\| \sum_{i=1}^n \varphi_i(s - t)\, B(s)e^{tA}x_i \right\|^p ds \right)^{1/p} dt \\
&= \int_0^\alpha \left(\int_{\mathrm{I\!R}} \left\| \sum_{i=1}^n \varphi_i(s)\, B(s + t)e^{tA}x_i \right\|^p ds \right)^{1/p} dt \\
&\leq \alpha^{1/q} \left(\int_0^\alpha \int_{\mathrm{I\!R}} \left\| \sum_{i=1}^n \varphi_i(s)\, B(s + t)e^{tA}x_i \right\|^p ds\, dt \right)^{1/p} \\
&= \alpha^{1/q} \left[\int_{\mathrm{I\!R}} \left(\int_0^\alpha \left\| B(s + t)e^{tA} \left(\sum_{i=1}^n \varphi_i(s)x_i \right) \right\|^p dt \right) ds \right]^{1/p} .
\end{aligned}$$

Now (B) implies

$$\int_0^\alpha \|\mathcal{B}T_0(t)u\|_F \, dt \leq \alpha^{1/q}\beta \, \|u\|_F \,.$$

Setting $\gamma := \alpha^{1/q}\beta$ this yields (ii).

It follows from Theorem 4.1 that the closure G of $(G_0 + \mathcal{B})_{|\mathcal{D}}$ generates a strongly continuous semigroup $(T(t))_{t\geq 0}$ on F and $D(G) = D(G_0)$. Now we want to apply Theorem 3.4. By Theorem 4.1 there is $\lambda \in \rho(G) \cap \rho(G_0)$ such that $R(\lambda, G) = R(\lambda, G_0)C$ for some $C \in \mathcal{L}(F)$. Since G_0 generates an evolution semigroup on F, the operator $R(\lambda, G_0) : F \to C_0(\mathbb{R}, X)$ is continuous (see [21, Lemma 2.1]). Therefore, $R(\lambda, G) : F \to C_0(\mathbb{R}, X)$ is continuous. Moreover, $\mathcal{D} \subseteq D(G)$ is dense in $C_0(\mathbb{R}, X)$. Thus, $R(\lambda, G)$ has dense image in $C_0(\mathbb{R}, X)$. Let $\varphi \in C_c^1(\mathbb{R})$ and $f \in \mathcal{D}$. Then $\varphi f \in \mathcal{D}$ and

$$\begin{aligned} G(\varphi f) &= -\frac{d}{ds}(\varphi f)(\cdot) + A(\varphi f)(\cdot) + B(\cdot)(\varphi f)(\cdot) \\ &= -\varphi' f - \varphi f' + \varphi A f(\cdot) + \varphi B(\cdot) f(\cdot) \\ &= -\varphi' f + \varphi G f. \end{aligned}$$

Thus from Remark 3.5(1) and Theorem 3.4 the assertion follows. □

Remark 4.4 *Let $(T(t))_{t\geq 0}$ be the perturbed evolution semigroup of Theorem 4.2 and denote by $(U(t,s))_{(t,s)\in\mathcal{H}}$ the corresponding evolution family. Then for every $x \in D$ and almost every $t \in \mathbb{R}$ the function $\{s \in \mathbb{R} : s \leq t\} \to X : s \mapsto U(t,s)x$ is differentiable a.e. and*

$$\frac{\partial}{\partial s} U(t,s)x = -U(t,s)(A + B(s))x \tag{4.1}$$

for almost all $s \leq t$.

Proof. The semigroup $(T(t))_{t\geq 0}$ satisfies

$$T(t)f - f = \int_0^t T(\tau) G f \, d\tau$$

for $f \in D(G)$ and $t \geq 0$ (see [15, A-I.1.6]). Recall that G is the closure of $(G_0 + \mathcal{B})_{|\mathcal{D}}$. Fix $x \in D$ and $\varphi \in C_c^1(\mathbb{R})$. Then $\varphi \otimes x \in \mathcal{D}$ and for a.e. $s \in \mathbb{R}$ we have

$$\begin{aligned}
&T(t)(\varphi \otimes x)\,(s) - (\varphi \otimes x)(s) \\
&\quad = U(s, s-t)\varphi(s-t)x - \varphi(s)x \\
&\quad = \int_0^t U(s, s-\tau)\Big(-\varphi'(s-\tau)x + \varphi(s-\tau)Ax + \varphi(s-\tau)B(s-\tau)x\Big)\, d\tau. \\
&\quad = \int_{s-t}^s U(s, \tau)\Big(-\varphi'(\tau)x + \varphi(\tau)Ax + \varphi(\tau)B(\tau)x\Big)\, d\tau.
\end{aligned}$$

By a change of variables we obtain

$$\varphi(s)U(t,s)x - \varphi(t)x = \int_s^t U(t,\tau)\Big(-\varphi'(\tau)x + \varphi(\tau)Ax + \varphi(\tau)B(\tau)x\Big)\, d\tau.$$

Now the right–hand side of this equation is differentiable for almost every $s \leq t$ and almost every t. Since $\varphi \in C_c^1(\mathbb{R})$, this implies that the function $\{s \in \mathbb{R} : s \leq t\} \to X : \ s \mapsto U(t,s)x$ is differentiable a.e. for almost every t and

$$\varphi(s)\,\frac{\partial}{\partial s}\,U(t,s)x + \varphi'(s)\,U(t,s)x = -U(t,s)\Big(-\varphi'(s)x + \varphi(s)Ax + \varphi(s)B(s)x\Big).$$

We can choose φ such that $\varphi(s) \neq 0$. Thus (4.1) holds for almost all $t \geq s$ and each $x \in D$. □

Example 4.5

We consider the Banach spaces $X = L^r(\mathbb{R}^N)$ and $F = L^p(\mathbb{R}, X)$ where $\frac{N}{2} < \frac{r}{p} < \infty$. Let $A = \Delta$ be the Laplacian on X. Then the Schwartz space $\mathcal{S}(\mathbb{R}^N)$ of rapidly decreasing smooth functions is a core for Δ. We set $D := \mathcal{S}(\mathbb{R}^N)$. It is well–known that $e^{tA}f = K_t * f$, where $K_t(\xi) := (4\pi t)^{-N/2}\exp(-\frac{|\xi|^2}{4t})$ for $\xi \in \mathbb{R}^N$. Let $V(\cdot,\cdot) \in F \cap L^\infty(\mathbb{R}, X)$ and $B(s)g := V(s,\cdot)g(\cdot)$ for a.e. $s \in \mathbb{R}$ and $g \in D(B(s))$, where $D(B(s)) \subseteq X$ is the maximal domain. Note that $B(s)$ is not assumed to be everywhere defined, but we have $L^\infty(\mathbb{R}^N) \subseteq D(B(s))$.

We want to check the assumptions (A)–(C). It easy to verify that $D \subseteq D(B(s)) \cap$

$D(A)$ and $e^{tA}D \subseteq D$ for $t \geq 0$ and a.e. $s \in \mathbb{R}$. Moreover, for $g \in D$

$$\begin{aligned}
\int_{\mathbb{R}} \|B(s)e^{tA}g\|_r^p \, ds &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}^N} |V(s,\xi)\,(e^{tA}g)(\xi)|^r \, d\xi \right)^{\frac{p}{r}} ds \\
&\leq \|e^{tA}g\|_\infty^p \int_{\mathbb{R}} \left(\int_{\mathbb{R}^N} |V(s,\xi)|^r \, d\xi \right)^{\frac{p}{r}} ds \\
&= \|e^{tA}g\|_\infty^p \, \|V\|_F^p
\end{aligned}$$

and hence $B(\cdot)e^{tA}g \in F$. Similarly, one can show (C).

By an easy computation we see that

$$\|K_t\|_{r'} = (4\pi t)^{-\frac{N}{2r}} \, (r')^{-\frac{N}{2r'}} =: C\, t^{-\frac{N}{2r}}$$

where $\frac{1}{r} + \frac{1}{r'} = 1$ and $C > 0$ is a constant depending on N and r. Using this identity and Young's inequality we obtain for $g \in D$

$$\begin{aligned}
\int_0^\alpha \|B(s+t)e^{tA}g\|_r^p \, dt &= \int_0^\alpha \left(\int_{\mathbb{R}^N} |V(s+t,\xi)\,(e^{tA}g)(\xi)|^r \, d\xi \right)^{\frac{p}{r}} dt \\
&\leq \int_0^\alpha \|K_t * g\|_\infty^p \left(\int_{\mathbb{R}^N} |V(s+t,\xi)|^r \, d\xi \right)^{\frac{p}{r}} dt \\
&= \int_0^\alpha \|K_t * g\|_\infty^p \, \|V(s+t,\cdot)\|_r^p \, dt \\
&\leq \|g\|_r^p \int_0^\alpha \|K_t\|_{r'}^p \, \|V(s+t,\cdot)\|_r^p \, dt \\
&= C^p \, \|g\|_r^p \int_0^\alpha t^{-\frac{Np}{2r}} \, \|V(s+t,\cdot)\|_r^p \, dt \\
&\leq C^p \, \|g\|_r^p \, \|V\|_{L^\infty(\mathbb{R},X)}^p \int_0^\alpha t^{-\frac{Np}{2r}} \, dt \\
&= \tilde{C}\, \alpha^{1-\frac{Np}{2r}} \, \|g\|_r^p \, ,
\end{aligned}$$

where $\tilde{C} > 0$ is a constant. For the last equality we used $\frac{N}{2} < \frac{r}{p}$. Now we set $\beta^p := \tilde{C}\, \alpha^{1-\frac{Np}{2r}}$. We can choose $\alpha > 0$ such that $\alpha^{1/q}\,\beta < 1$. Hence, (B) is satisfied.

So by Theorem 4.2 there exists an extension G of $-\frac{d}{ds} + \Delta + V(\cdot,\cdot)$ which generates an evolution semigroup on F. Let $\lambda \in \rho(G)$ and choose $\mathcal{D}$ as in the proof of Theorem 4.2. Then $\tilde{F} := (\lambda - G)\mathcal{D}$ is dense in F. Furthermore, for

$f \in \tilde{F}$ there is a unique $g \in \mathcal{D}$ with $(\lambda - G)g = f$. Therefore, we find for every f contained in the dense subspace $\tilde{F}$ a solution of the inhomogenous heat equation

$$\frac{\partial}{\partial s}\, g(s,\xi) = \Delta_\xi\, g(s,\xi) + V(s,\xi) g(s,\xi) - \lambda g(s,\xi) + f(s,\xi)$$

for a.e. $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$.

Example 4.6

Let $\Omega \subseteq \mathbb{R}^N$ be an open set which is locally regular of class $\mathcal{C}^4$ and uniformly regular of class $\mathcal{C}^2$ in the sense of F.E. Browder [5]. On Ω we consider the differential operator

$$Au := \sum_{i,j=1}^{N} D_i\, (a_{i,j}\, D_j)u$$

where D_i denotes the partial derivative, i.e. $D_i u := \frac{\partial u}{\partial \eta_i}$. We assume that the coefficients $a_{i,j} = a_{j,i}$ of A are bounded and uniformly continuous. For $1 < r < \infty$ we denote by A^D (resp. A^N) the realization of A in $X := L^r(\Omega)$ with Dirichlet boundary conditions (resp. Neumann boundary conditions). This means that we set

$$\begin{aligned} D(A^D) &:= \{u \in W^{2,r} : \ u(x) = 0 \ \text{ for } x \in \partial\Omega\} \qquad \text{and} \\ D(A^N) &:= \{u \in W^{2,r} : \ \frac{\partial u}{\partial n}\,(x) = 0 \ \text{ for } x \in \partial\Omega\}. \end{aligned}$$

Moreover, we assume that A^D (resp. A^N) generates an analytic semigroup $(e^{tA^D})_{t\ge 0}$ (resp. $(e^{tA^N})_{t\ge 0}$) on $L^r(\Omega)$ (sufficient conditions for this assumption can be found in [1]). We denote by $(S_2^D(t))_{t\ge 0}$ (resp. $(S_2^N(t))_{t\ge 0}$) the corresponding semigroup on $L^2(\Omega)$. It follows from [29, Thm. 2] that

$$|S_2^D(t) f| \le c\, e^{t \log c}\, K_{bt} * |f| \quad \text{ and } \quad |S_2^N(t) f| \le c\, e^{t \log c}\, K_{bt} * |f|$$

holds for all $f \in L^2(\Omega)$ and $t \geq 0$ with constants $c \geq 1$ and $b > 0$. Consequently, we obtain

$$|e^{tA^D} f| \leq c\, e^{t \log c}\, K_{bt} * |f| \quad \text{and} \quad |e^{tA^N} f| \leq c\, e^{t \log c}\, K_{bt} * |f|$$

for all $f \in L^r(\Omega)$ and $t \geq 0$ (cf. [2, Sect. 4]). Now we consider $F = L^p(\mathbb{R}, X) = L^p(\mathbb{R}, L^r(\Omega))$ for $\frac{N}{2} < \frac{r}{p} < \infty$. Since $r > \frac{N}{2}$ it follows from Sobolev's imbedding Theorem (cf. [4, Cor. IX.13]) that $W^{2,r}(\Omega) \subseteq L^\infty(\Omega)$. Let $V(\cdot,\cdot)$ and $B(\cdot)$ be defined as in Example 4.5. If we set $D = D(A^D)$ (resp. $D = D(A^N)$), then we can see as in Example 4.5 that $B(\cdot)$ satisfies (A)–(C) for A^D and A^N, respectively.

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