SIAM J. Control Optim.
Vol. 41, No. 4, pp. 1141–1165

FEEDBACKS FOR NONAUTONOMOUS REGULAR LINEAR SYSTEMS*

ROLAND SCHNAUBELT†

Abstract. We introduce nonautonomous well-posed and (absolutely) regular linear systems as quadruples consisting of an evolution family and output, input, and input–output maps subject to natural hypotheses. In the spirit of Weiss' work, these maps are represented in terms of admissible observation and control operators (the latter in an approximate sense) in the time domain. In this setting, the closed-loop system exists for a canonical class of "admissible" feedbacks, and it inherits the absolute regularity and other properties of the given system. In particular, we can iterate feedbacks.

Key words. input–output map, evolution family, Lebesgue extension, representation, closed-loop system, controllable, observable, robustness of exponential dichotomy, input–output stability

AMS subject classifications. Primary, 93C25; Secondary, 47D06

PII. S036301290139169X

1. Introduction.

As a motivation, we first look at the finite dimensional nonautonomous linear system

$$
\begin{aligned}
x'(t) &= A(t)x(t) + B(t)u(t), \quad t \geq s \geq 0,\\
y(t) &= C(t)x(t), \quad t \geq s \geq 0, \quad x(s) = x_0,
\end{aligned}
\tag{1.1}
$$

on the state space X with control operators $B(t) : U \to X$, observation operators $C(t) : X \to Y$, the control space U, and the observation space Y. Let $T(t,s)$, $t \geq s \geq 0$, be the evolution family (propagator) on X generated by $A(\cdot)$. Then the output of (1.1) with $u = 0$, the state of (1.1) with $x_0 = 0$, and the input–output operator of (1.1) are given by

$$
\begin{aligned}
(\Psi_s x_0)(t) &= C(t)T(t,s)x_0, \qquad \Phi_{t,s}u = \int_s^t T(t,\tau)B(\tau)u(\tau)d\tau,\\
(\mathbb{F}_s u)(t) &= C(t)\int_s^t T(t,\tau)B(\tau)u(\tau)d\tau, \qquad t \geq s.
\end{aligned}
\tag{1.2}
$$

If one feeds back the output via $u(t) = \Delta(t)y(t)$, the resulting closed-loop system is described by the perturbed evolution equation

$$
x'(t) = [A(t) + B(t)\Delta(t)C(t)]\,x(t), \quad t \geq s \geq 0, \qquad x(s) = x_0. \tag{1.3}
$$

Of course, $x(t) = T_\Delta(t,s)x_0$ solves (1.3) if T_Δ is generated by $A(t) + B(t)\Delta(t)C(t)$. This evolution family also satisfies the "variation of constants formulas"

$$
T_\Delta(t,s)x = T(t,s)x + \int_s^t T(t,\tau)B(\tau)\Delta(\tau)C(\tau)T_\Delta(\tau,s)x\,d\tau, \tag{1.4}
$$

$$
T_\Delta(t,s)x = T(t,s)x + \int_s^t T_\Delta(t,\tau)B(\tau)\Delta(\tau)C(\tau)T(\tau,s)x\,d\tau \tag{1.5}
$$

*Received by the editors July 2, 2001; accepted for publication (in revised form) April 11, 2002; published electronically October 29, 2002.
http://www.siam.org/journals/sicon/41-4/39169.html
†FB Mathematik and Informatik, Universität Halle, 06099 Halle (Saale), Germany (schnaubelt@mathematik.uni-halle.de).

for $t \geq s$ and $x \in X$. Identity (1.4) is the integrated version of (1.3). To derive (1.5), we perturb T_Δ by $-B(t)\Delta(t)C(t)$. There are formulas analogous to (1.4) and (1.5) relating the maps from (1.2) with the corresponding ones of the closed-loop system. These formulas are needed to show further properties of the closed-loop system. For instance, the closed-loop system is observable (controllable) if and only if the open-loop system is observable (controllable). In this framework, one can also show the equivalence of internal stability with input/output stability, detectability, and stabilizability. We establish infinite dimensional versions of these results in section 5.

If we pass to an infinite dimensional state space X, it is no longer clear that (1.3) possesses differentiable solutions for "many" initial values even if the Cauchy problem for $A(\cdot)$ is well-posed; cf. [7], [9, section VI.9], [10]. Nevertheless, the formulas (1.2) still work, and there is an evolution family T_Δ fulfilling (1.4) and (1.5). Thus $x(t) = T_\Delta(t,s)x_0$ is the "mild" solution of (1.3) [7]. However, point or boundary control and observation lead to input and output operators $B(t) : U \to \overline{X}_t$ and $C(t) : \underline{X}_t \to Y$ for spaces $\underline{X}_t \subsetneqq X \subsetneqq \overline{X}_t$, where $C(t)$ usually is not closable; see, e.g., [3], [16]. In order to solve (1.4) in this more general setting, we may restrict ourselves to "admissible" observation and control operators—roughly speaking, those for which the expressions (1.2) make sense. Then we are also faced with the question of whether the operators $B(t)$ and $C(t)$ are again admissible for the perturbed evolution family T_Δ, which is necessary to verify (1.5) or to iterate feedbacks.

The resulting perturbation problem (1.3) generalizes the settings of both the Desch–Schappacher theorem (where $\Delta(t) = C(t) = I$) and the Miyadera theorem (where $\Delta(t) = B(t) = I$) from semigroup theory [9, section III.3], [19]. In the control literature, there is a rich perturbation theory for the autonomous case (i.e., $A(t) = A$, $B(t) = B$, $C(t) = C$, $\Delta(t) = \Delta$). Linear systems belonging to the *Pritchard–Salamon class* [18] were exhaustively treated in [6]. Salamon and Weiss introduced the larger class of *well-posed linear systems* in [21] and [28], [29], [30], [31]. Here the semigroup T is given, and the operators Φ, Ψ, and $\mathbb{F}$ are defined in an abstract way by certain algebraic relations. One can then construct admissible control and observation operators B and C and obtain formulas such as (1.2) if the system satisfies a quite natural *regularity* hypothesis. Weiss established a powerful feedback theory for regular systems in the Hilbert space situation [32]. We refer to section 4, [3, section 3.3], [17], [33], and, in particular, to Staffans' monograph [25] for further information and literature.

For nonautonomous systems in variational form, there is the well-known approach due to Lions [16]; see also [1] and [3, Chap. 2]. In a general setting, Hinrichsen, Jacob, and Pritchard [10], [12], [14] constructed an evolution family solving (1.4) for initial values x contained in a dense subspace $\underline{X}$ of X under rather weak assumptions covering autonomous regular systems. However, (1.5) and the admissibility of the perturbed system was investigated only in [12] requiring stronger hypotheses of Pritchard–Salamon type.

In the present work, we combine the direct approach of Hinrichsen, Jacob, and Pritchard with some of Weiss' ideas: In Definition 2.6, we introduce "Lebesgue extensions" of given observation operators $C(t)$ (cf. [28]) which allow the study of (1.4) and (1.5) for all $x \in X$ and simplify several technical details of the proofs considerably. For similar reasons, we mostly work with *nonautonomous (absolutely) regular systems*, which are defined in the spirit of Weiss' work (see Definitions 3.6 and 3.10) as opposed to *admissible systems*, which have been used in [10], [12], [14] and are given directly by operators $B(t)$ and $C(t)$ (see Definition 3.8). In Theorem 2.7, Proposition 3.5, and Theorem 3.11, we represent a given regular system similar as in (1.2). It is known

[27, Ex. 6] that (1.3) can only be solved if the feedback is not "too large." We thus introduce *admissible* feedbacks in Definition 4.1; cf. [25, section 7.1], [32, section 3]. In our main theorem, Theorem 4.4, we then establish the existence of an absolutely regular closed-loop system for a given absolutely regular nonautonomous system with admissible time varying feedback.

However, the extension of Weiss' theory to the nonautonomous case is limited by two serious obstacles: One cannot apply transform methods, and, in contrast to semigroups (see, e.g., [2, Chap. V], [9, section II.5]), we do not have a general extrapolation theory for evolution families. The first point excludes the use of transfer functions (being crucial in [32]) but leads us to arguments which work in a Banach space setting (as in [25, Chap. 7]). The second point forces us to employ approximation formulas for the representation of control systems in Proposition 3.5. A similar problem occurs in the computation of the feedback system and in the context of (1.5); cf. Remark 4.7.

In section 5 we derive analogues of (1.4) and (1.5) for the operators given in (1.2). It is also seen that the closed-loop system is controllable (or observable) if and only if the given system is controllable (or observable). Moreover, iterated feedbacks behave as one would expect. We further prove that the feedback system inherits the exponential dichotomy (or stability) of T. Results of this type are important tools in investigating the long-term behavior of evolution equations but have not yet been obtained for perturbations mapping from a subspace of X to a larger space. Finally, the equivalence of internal stability with input–output stability, detectability, and stabilizability is established, extending theorems from [5], [6], [17], [20], [33] to the present setting. As a sample of possible applications, we treat in section 6 a parabolic problem with point observation and control in space dimension $n \le 3$ which can be generalized in various directions.

Notation. We denote the space of bounded linear operators from X to Y by $\mathcal{L}(X,Y)$ and put $\mathcal{L}(X) := \mathcal{L}(X,X)$, where X, Y, U, Z always designate Banach spaces. $C_b(\mathbb{R}_+, \mathcal{L}_s(X,Y))$ and $L^\infty(\mathbb{R}_+, \mathcal{L}_s(X,Y))$ are the spaces of (essentially) bounded strongly continuous and strongly measurable operator-valued functions, respectively. We set $a \vee b = \max\{a,b\}$, $a \wedge b = \min\{a,b\}$, $a^+ = a \vee 0$, and $a^- = (-a)^+$ for $a, b \in \mathbb{R}$ and write $\mathbb{1}_N$ for the characteristic function of $N \subset M$. Unless otherwise stated, p is a number contained in $[1,\infty)$. The spaces $L^p_{loc}([s,\infty), Z)$ and $C([s,\infty), Z)$ are endowed with their standard Fréchet topologies. We mostly use the same symbol for a function on $J \subset \mathbb{R}$ and its restrictions to subintervals.

2. Nonautonomous observation systems.

DEFINITION 2.1. *A set $T = (T(t,s))_{t \ge s \ge 0} \subseteq \mathcal{L}(X)$ is an* evolution family *if*

(E1) $T(t,s) = T(t,r)T(r,s),\;\; T(s,s) = I,$

(E2) $(t,s) \mapsto T(t,s)$ *is strongly continuous, and*

(E3) $\|T(t,s)\| \le Me^{w(t-s)}$

for $t \ge r \ge s \ge 0$ and constants $M \ge 1$ and $w \in \mathbb{R}$. We also define $(\mathbb{K}_s f)(t) = \int_s^t T(t,\tau) f(\tau)\, d\tau$ for $t \ge s \ge 0$ and $f \in L^1_{loc}([s,\infty), X)$ and put $\mathbb{K} = \mathbb{K}_0$.

Evolution families arise as solution operators of nonautonomous evolution equations, although not every evolution family solves such a problem. We refer to [4], [9, section VI.9], and the references therein for further information. Condition (E3) is needed only in the study of asymptotic properties in section 5; see Remark 4.5.

DEFINITION 2.2. *Let T be an evolution family on X and $\Psi_s : X \to L^p_{loc}([s,\infty), Y)$, $s \ge 0$, be linear operators satisfying*

$$\Psi_s x = \Psi_t T(t,s)x \quad \text{on} \quad [t,\infty) \quad \text{and} \quad \int_s^{s+t_0} \|(\Psi_s x)(t)\|_Y^p \, dt \le \gamma^p \, \|x\|_X^p \tag{2.1}$$

for $t \ge s \ge 0$, $x \in X$, *and some* $t_0 > 0$, $\gamma = \gamma(t_0) > 0$. *Then* $(T, \Psi) = (T, \{\Psi_s : s \ge 0\})$ *is a* nonautonomous observation system. *We extend the map* $\Psi_s x$ *by 0 to* $\mathbb{R}$.

LEMMA 2.3. *Let* (T, Ψ) *be a nonautonomous observation system. Then one can replace the constant* t_0 *in* (2.1) *by every* $t_1 > 0$ *and* $\gamma = \gamma(t_0)$ *by* $\gamma(t_1) = c_0 M \gamma(t_0) c(t_1)$, *where* $c(t) = e^{w^+ t}$ *for* $w \neq 0$, $c(t) = (1 + \frac{t}{t_0})^{\frac{1}{p}}$ *for* $w = 0$, *and* c_0 *depends on* t_0, w, p.

Proof. The case $t_1 \le t_0$ is obvious. So let $t_1 = nt_0 + \tau$ for some $n \in \mathbb{N}$ and $\tau \in [0, t_0)$. Setting $I_k = [s + kt_0, s + (k+1)t_0]$, we deduce from Definition 2.2 that

$$\|\Psi_s x\|^p_{L^p([s,s+t_1],Y)} \le \sum_{k=0}^{n} \|\Psi_{s+kt_0} T(s+kt_0, s)x\|^p_{L^p(I_k,Y)} \le M^p \, \gamma(t_0)^p \sum_{k=0}^{n} e^{wpt_0 k}$$

for $x \in X$ and $s \ge 0$. The assertion then follows easily. ∎

DEFINITION 2.4. *Let* T *be an evolution family on* X *and* $C(s) : D(C(s)) \subseteq X \to Y$, $s \ge 0$, *be densely defined linear operators such that* $T(\cdot, s)x \in D(C(\cdot), s) := \{f \in L^p_{loc}([s,\infty), X) : f(t) \in D(C(t))$ *for a.e.* $t \ge s$, $C(\cdot)f(\cdot) \in L^p_{loc}([s,\infty), Y)\}$ *and*

$$\int_s^{s+t_0} \|C(t)T(t,s)x\|^p_Y \, dt \le \gamma^p \, \|x\|^p_X \tag{2.2}$$

for $s \ge 0$, $x \in D(C(s))$, *and some constants* $\gamma, t_0 > 0$. *Then we say that* $C(s)$, $s \ge 0$, *are* (T-)admissible observation operators.

LEMMA 2.5. *Let* $C(s)$, $s \ge 0$, *be* T*-admissible observation operators. Then* (2.2) *holds for all* $t_0 > 0$ *with a possibly different* γ. *Let* $\Psi_s : X \to L^p_{loc}([s,\infty), Y)$, $s \ge 0$, *be the continuous extension of the map* $D(C(s)) \ni x \mapsto C(\cdot)T(\cdot, s)x$. *Then* (T, Ψ) *is a nonautonomous observation system.*

Proof. The first claim can be established as Lemma 2.3; one has only to replace $s + kt_0$ by points $s_k \approx s + kt_0$ such that $T(s_k, s)x \in D(C(s_k))$; see [23, Lem. 4.13]. So we can define Ψ_s as in the claim. Given $t \ge s \ge 0$ and $x \in D(C(s))$, we take $z_n \in D(C(t))$ converging in X to $T(t,s)x$ and $t_n \searrow t$ such that $T(t_n, s)x, T(t_n, t)z_n \in D(C(t_n))$. Since $\Psi_t T(t,s)x = \lim_{n\to\infty} \mathbb{1}_{[t_n, t+t_0]} \, C(\cdot)T(\cdot, t)z_n$ in $L^p([t, t+t_0], Y)$, we obtain

$$\begin{aligned}
&\|\Psi_t T(t,s)x - C(\cdot)T(\cdot,s)x\|^p_{L^p([t,t+t_0],Y)} \\
&= \lim_{n\to\infty} \left[\int_{t_n}^{t+t_0} \|C(\tau)T(\tau, t_n)\,[T(t_n,t)z_n - T(t_n,s)x]\|^p d\tau + \int_t^{t_n} \|C(\tau)T(\tau,s)x\|^p d\tau \right] \\
&\le \gamma^p \lim_{n\to\infty} \|T(t_n,t)z_n - T(t_n,s)x\|^p = 0.
\end{aligned}$$

Therefore, (2.1) holds for $x \in D(C(s))$ and thus for $x \in X$ by approximation. ∎

We note that different admissible observation operators $C_1(s)$ and $C_2(s)$ may yield the same observation system as shown in [28, Ex. 1.2]. However, if the observation operators $C(s)$ are closable, then one easily verifies that $\Psi_s x = \overline{C(\cdot)}T(\cdot,s)x$ for the induced observation system. We now proceed in the converse direction and represent a given observation system by admissible observation operators; cf. [28, Def. 4.1].

DEFINITION 2.6. *For a nonautonomous observation system* (T, Ψ), *we define*

$$C(s)x = \lim_{\tau \searrow 0} \frac{1}{\tau} \int_s^{s+\tau} (\Psi_s x)(\sigma)\, d\sigma \qquad (in\ Y) \tag{2.3}$$

for $x \in \underline{X}_s := \{x \in X :$ *the limit in* (2.3) *exists*$\}$ *and*

$$\|x\|_{\underline{X}_s} = \|x\|_X + \sup_{0<\tau\le 1} \left\| \frac{1}{\tau} \int_s^{s+\tau} (\Psi_s x)(\sigma)\, d\sigma \right\|_Y$$

for $x \in \underline{X}_s$ *and* $s \ge 0$. *The space* $D(C(\cdot), s)$ *is defined as in Definition* 2.4 *by replacing* $D(C(t))$ *with* $\underline{X}_t$.

Clearly, $\|\cdot\|_{\underline{X}_s}$ is a norm on the subspace $\underline{X}_s$ and $C(s) : \underline{X}_s \to Y$ is linear and continuous. As in [28, Prop. 4.3], one verifies that $(\underline{X}_s, \|\cdot\|_{\underline{X}_s})$ is complete.

We say that $t \in \mathbb{R}$ is a *p-Lebesgue point* of $f \in L^p_{loc}(\mathbb{R}, Z)$, $1 \le p < \infty$, if

$$\lim_{|J|\to 0} \frac{1}{|J|} \int_J \|f(s) - f(t)\|^p\, ds = 0,$$

where the limit is taken over compact intervals J containing t (of length $|J|$). If $p = 1$, then t is called the *Lebesgue point.* Recall that a.e. t is a p-Lebesgue point of $f \in L^p_{loc}(\mathbb{R}, Z)$; see, e.g., [31, Lem. 6.1] or [26, section I.1.8]. The next representation theorem extends [28, Thm. 4.5] to nonautonomous observation systems. A different representation of output functions was given in [11] applying Weiss' theory to the "evolution semigroup" on $L^p([0, t_0], X)$ associated with T; cf. [4].

THEOREM 2.7. *Let* (T, Ψ) *be a nonautonomous observation system, and let* $C(s) \in \mathcal{L}(\underline{X}_s, Y)$ *be given as in Definition* 2.6. *Let* $x \in X$ *and* $t \ge s \ge 0$. *Then* $T(t, s)x \in \underline{X}_t$ *if and only if* $1/\tau \int_0^\tau (\Psi_s x)(t + \sigma)\, d\sigma$ *converges as* $\tau \searrow 0$. *If this is the case, then the limit equals* $C(t)T(t, s)x$. *Thus* $(\Psi_s x)(t) = C(t)T(t, s)x$ *for all Lebesgue points* t *of* $\Psi_s x$.

Proof. The theorem follows from the identity

$$\frac{1}{\tau} \int_t^{t+\tau} [\Psi_s x](\sigma)\, d\sigma = \frac{1}{\tau} \int_t^{t+\tau} [\Psi_t T(t, s)x](\sigma)\, d\sigma. \qquad \square$$

This theorem shows that the operators $C(t)$, $t \ge 0$, introduced in Definition 2.6 are admissible observation operators. According to Lemma 2.5, they generate an observation system $(\tilde{\Psi}, T)$. It is easy to see that, in fact, $(\Psi_s x)(t) = (\tilde{\Psi}_s x)(t)$ for each $x \in X$ and a.e. $t \ge s$. We say that the operators $C(t)$ from Definition 2.6 *represent the observation system* (T, Ψ).

In the remainder of this section, we establish several properties of Ψ_s which will be important for our main perturbation result.

LEMMA 2.8. *Let* (T, Ψ) *be a nonautonomous observation system,* $f \in L^p_{loc}(\mathbb{R}_+, X)$, *and* $t_0 > 0$. *Then the map* $[0, t_0] \ni s \mapsto \Psi_s f(s) \in L^p([0, t_0], Y)$ *is measurable, and*

$$\int_0^{t_0} \|\Psi_s f(s)\|^p_{L^p([0,t_0],Y)}\, ds \le \gamma(t_0)^p\, \|f\|^p_{L^p([0,t_0],X)}\,. \tag{2.4}$$

Proof. For $f \in C(\mathbb{R}_+, X)$, the map $s \mapsto \Psi_s f(s)$ is continuous from the right since

$$\|\Psi_s f(s) - \Psi_r f(r)\|^p_{L^p([0,t_0],Y)} = \|\Psi_s(f(s) - T(s, r)f(r))\|^p_{L^p([s,t_0],Y)} + \|\Psi_r f(r)\|^p_{L^p([r,s],Y)}$$

for $0 \le r \le s \le t_0$. Functions $f \in L^p_{loc}(\mathbb{R}_+, X)$ can be treated by approximation. The estimate (2.4) follows from (2.1). $\square$

The nonclosedness of $C(t)$ is a major obstacle for the analysis of observation systems and input–output operators; for instance, it is a priori not clear whether $C(t)$

can be taken out of an integral. As in the autonomous case (see, e.g., [31, section 4]), such problems can be overcome by employing the operators

$$C_\tau(s)x = \frac{1}{\tau}\int_s^{s+\tau} (\Psi_s x)(\sigma)\,d\sigma, \tag{2.5}$$

$x \in X$, $s \geq 0$, and $\tau \in (0,1]$. Due to this definition, $C_\tau(s)$ belongs to $\mathcal{L}(X,Y)$ with norm less than or equal to $\gamma(1)\tau^{-\frac{1}{p}}$, $C_\tau(s)x$ converges as $\tau \to 0$ if and only if $x \in \underline{X}_s$, and then the limit equals $C(s)x$. Let $C_c(\mathbb{R}_+)$ be the space of continuous functions with compact support in $[0,\infty)$. We also define

$$\mathcal{D}_s = \operatorname{span}\{\varphi(\cdot)T(\cdot,r)x : x \in X,\, r \geq s,\, \varphi \in C_c(\mathbb{R}_+),\, \varphi(t) = 0 \text{ for } s \leq t < r\} \tag{2.6}$$

for $s \geq 0$ (setting $T(t,s) := 0$ for $t < s$), and we put $\mathcal{D} = \mathcal{D}_0$. This space is dense in $L^p([s,\infty),X)$ and in $C_0([s,\infty),X)$, the space of continuous functions vanishing at infinity. This fact can be seen by an obvious modification of the proof of [4, Thm. 3.12].

LEMMA 2.9. *Let (T,Ψ) be a nonautonomous observation system represented by $C(s)$, and let $C_\tau(s)$ be given by (2.5). Then $(s,\tau) \mapsto C_\tau(s)x$ is continuous on $\mathbb{R}_+ \times (0,1]$,*

$$\|C_\tau(\cdot)T(\cdot,s)x\|_{L^p([s,s+t_0],Y)} \leq \gamma(t_0+1)\,\|x\|, \quad \text{and} \tag{2.7}$$

$$\Psi_s x = \lim_{\tau\to 0} C_\tau(\cdot)T(\cdot,s)x \quad \text{in } L^p_{loc}([s,\infty),Y) \tag{2.8}$$

for $x \in X$, $s \geq 0$, $\tau \in (0,1]$, and $t_0 > 0$.

Proof. If $f \in \mathcal{D}$, then $(t,\tau) \mapsto C_\tau(t)f(t)$ is continuous since

$$C_\tau(t)f(t) = \sum_{k=1}^n \varphi_k(t)\,\frac{1}{\tau}\int_t^{t+\tau} (\Psi_{r_k}x_k)(\sigma)\,d\sigma$$

for $\tau > 0$, $t \geq 0$, and suitable $n \in \mathbb{N}$, $r_k \geq 0$, $x_k \in X$, $\varphi_k \in C_c(\mathbb{R}_+)$. The first assertion follows by approximation. We further estimate

$$\|C_\tau(\cdot)T(\cdot,s)x\|^p_{L^p([s,s+t_0],Y)}$$

$$\leq \int_s^{s+t_0+\tau} \frac{1}{\tau}\int_{\sigma-\tau}^{\sigma} \|(\Psi_s x)(\sigma)\|^p\,dt\,d\sigma \leq \gamma(t_0+1)^p\,\|x\|^p$$

using Hölder's inequality and Fubini's theorem. Similarly, (2.8) follows from

$$\|C_\tau(\cdot)T(\cdot,s)x - \Psi_s x\|^p_{L^p([s,s+t_0],Y)}$$

$$\leq \frac{1}{\tau}\int_0^\tau \int_s^{s+t_0} \|(\Psi_s x)(t+\sigma) - (\Psi_s x)(t)\|^p\,dt\,d\sigma. \qquad \square$$

We want to show that $C(\cdot)\mathbb{K}_s : L^p([s,s+t_0],X) \to L^p([s,s+t_0],Y)$ is well defined and bounded. This fact is crucial for Theorem 4.4, and its proof is somewhat technical. We set

$$\varphi(t;\tau,\sigma,f) = (C_\tau(t) - C_\sigma(t))\int_0^t T(t,s)f(s)\,ds,$$

$$o_f(t) = \varlimsup_{\tau,\sigma\to 0} \|\varphi(t;\tau,\sigma,f)\| = \varlimsup_{n\to\infty} \sup_{m\geq n} \sup_{\tau,\sigma\in[1/m,1/n]} \|\varphi(t;\tau,\sigma,f)\|$$

for $f \in L^1_{loc}(\mathbb{R}_+, X)$, $t \ge 0$, $\tau, \sigma \in (0,1]$. Observe that o_f is measurable. We further need the *maximal operator* given by

$$M\psi(t) = \sup_{\tau>0} \frac{1}{\tau} \int_t^{t+\tau} |\psi(s)|\, ds \ \in [0,\infty]$$

for all $t \in \mathbb{R}$ and $\psi \in L^1_{loc}(\mathbb{R})$. Recall that

$$\|M\psi\|_{L^p(\mathbb{R})} \le c_p \, \|\psi\|_{L^p(\mathbb{R})} \tag{2.9}$$

for $\psi \in L^p(\mathbb{R})$, $1 < p \le \infty$, and a constant c_p; see [26, Thm. I.1].

LEMMA 2.10. *Let (T, Ψ) be a nonautonomous observation system, $p \in (1,\infty)$, and $f \in L^p_{loc}(\mathbb{R}_+, X)$. Then $C_\tau(t)(\mathbb{K}f)(t) \to C(t)(\mathbb{K}f)(t)$ as $\tau \searrow 0$ for a.e. $t \ge 0$.*

Proof. Take $g \in \mathcal{D}$ and $f \in L^p_{loc}(\mathbb{R}_+, X)$. Observe that $o_g = 0$ a.e. because of

$$C_\tau(t)(\mathbb{K}g)(t) = \sum_{k=1}^n \int_0^t \varphi_k(s)\, ds \; \frac{1}{\tau} \int_t^{t+\tau} (\Psi_{r_k} x_k)(\sigma)\, d\sigma$$

for $\tau > 0$, $t \ge 0$, and suitable $n \in \mathbb{N}$, $r_k \ge 0$, $x_k \in X$, and $\varphi_k \in C_c(\mathbb{R}_+)$. Due to Lemma 2.8, there is a measurable function $\psi_{f-g} : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\text{for a.e. } \ s \ge 0, \qquad \|[\Psi_s(f(s) - g(s))](t)\|_Y = \psi_{f-g}(t,s) \quad \text{for a.e. } \ t \ge s.$$

(Here we set $[\Psi_s(f(s) - g(s))](t) = \psi_{f-g}(t,s) = 0$ for $t < s$, $t > t_0$, or $s > t_0$, where $t_0 > 0$ is fixed but arbitrary.) Employing these facts, we estimate

$$\begin{aligned} o_f(t) \le o_{f-g}(t) &\le \sup_{\tau,\sigma \in (0,1]} \int_0^t \|(C_\tau(t) - C_\sigma(t))\, T(t,s)(f(s) - g(s))\|\, ds \\ &\le 2 \sup_{\tau \in (0,1]} \int_0^t \frac{1}{\tau} \int_t^{t+\tau} \|[\Psi_s(f(s) - g(s))](\rho)\|\, d\rho\, ds \\ &\le 2 \int_0^t [M\psi_{f-g}(\cdot, s)](t)\, ds \end{aligned} \tag{2.10}$$

for t not contained in a set of measure 0 depending on g. Approximating $0 \le \phi \in L^1_{loc}(\mathbb{R}^2)$ by continuous functions, one sees that $(t,s) \mapsto [M\phi(\cdot, s)](t)$ is measurable. We can now use (2.10), Fubini's theorem, and the maximal inequality (2.9) to derive

$$\begin{aligned} |\{t \in [0,t_0] : o_f(t) > \varepsilon\}| &\le \frac{2}{\varepsilon} \int_0^{t_0} \int_0^t [M\psi_{f-g}(\cdot, s)]\,(t)\, ds\, dt \\ &\le \frac{c}{\varepsilon} \int_0^{t_0} \|M\psi_{f-g}(\cdot, s)\|_{L^p(\mathbb{R})}\, ds \\ &\le \frac{c'}{\varepsilon} \int_0^{t_0} \|\psi_{f-g}(\cdot, s)\|_{L^p([s,t_0])}\, ds \\ &= \frac{c'}{\varepsilon} \int_0^{t_0} \|\Psi_s(f(s) - g(s))\|_{L^p([s,t_0],Y)}\, ds \\ &\le \frac{\gamma c'}{\varepsilon}\, \|f - g\|_{L^1([0,t_0],X)} \end{aligned}$$

for each $\varepsilon > 0$ and constants c, c' not depending on f, g, ε. Since g is arbitrary, the set $\{o_f > \varepsilon\}$ has Lebesgue measure 0. This fact implies the assertion. ∎

PROPOSITION 2.11. *Let (T,Ψ) be a nonautonomous observation system represented by $C(t)$, $p\in(1,\infty)$, and let $C_\tau(t)$ be given by* (2.5). *Then $\mathbb{K}_s f\in D(C(\cdot),s)$,*

$$\|C(\cdot)\mathbb{K}_s f\|_{L^p([s,s+t_0],Y)} \le c(t_0)\,\|f\|_{L^p([s,s+t_0],X)}\,,$$

and $C_\tau(\cdot)\mathbb{K}_s f \to C(\cdot)\mathbb{K}_s f$ in $L^p_{loc}([s,\infty),Y)$ as $\tau\to 0$ for $s\ge 0$, $t_0>0$, $f\in L^p_{loc}(\mathbb{R}_+,X)$, and a constant $c(t_0)$ independent of f and s.

Proof. By Lemma 2.10, $C(\cdot)\mathbb{K}_s f$ is a well-defined measurable function. Further,

$$\begin{aligned}\int_s^{s+t_0}\|C_\tau(t)(\mathbb{K}_s f)(t)\|^p\,dt &\le c\int_s^{s+t_0}\int_s^t \|C_\tau(t)T(t,r)f(r)\|^p\,dr\,dt\\ &= c\int_s^{s+t_0}\int_r^{s+t_0}\|C_\tau(t)T(t,r)f(r)\|^p\,dt\,dr\\ &\le c\tilde\gamma^p\,\|f\|^p_{L^p([s,s+t_0],X)} \qquad (2.11)\end{aligned}$$

for a constant c by Hölder's inequality, Fubini's theorem, and Lemma 2.9. Similarly,

$$\|(C_\tau(\cdot)-C_\sigma(\cdot))\mathbb{K}_s f\|_p^p \le c\int_s^{s+t_0}\int_r^{s+t_0}\|(C_\tau(t)-C_\sigma(t))T(t,r)f(r)\|^p\,dt\,dr,$$

and the right side tends to 0 as $\tau,\sigma\to 0$ by Lemma 2.9 and the dominated convergence theorem. Hence $C_\tau(\cdot)\mathbb{K}_s f$ also converges in $L^p([s,s+t_0],Y)$ to $C(\cdot)\mathbb{K}_s f$. The asserted estimate then follows from (2.11). ∎

3. Well-posed and regular nonautonomous systems.

DEFINITION 3.1. *Let T be an evolution family on X, and let $\Phi_{t,s}=\Phi(t,s): L^p_{loc}([s,\infty),U)\to X$, $t\ge s\ge 0$, be linear operators satisfying*

$$\Phi_{t,s}u = \Phi_{t,r}(u|[r,\infty)) + T(t,r)\Phi_{r,s}u,\quad t\ge r\ge s\ge 0,\quad \text{and} \qquad (3.1)$$

$$\|\Phi_{t,s}u\|_X \le \beta\,\|u\|_{L^p([s,t],U)},\quad 0\le t-s\le t_0, \qquad (3.2)$$

for $u\in L^p(\mathbb{R}_+,U)$ and constants $t_0>0$, $\beta=\beta(t_0)>0$. Then $(T,\Phi)=(T,\{\Phi_{t,s}: t\ge s\ge 0\})$ is called a nonautonomous control system.

Observe that the above definition implies that $\Phi_{t,t}=0$ and $\Phi_{t,s}u=\Phi_{t,r}u$ if $u=0$ on $[s,r]\subseteq[s,t]$. Thus the control system is *causal*.

LEMMA 3.2. *Let (T,Φ) be a nonautonomous control system. Then*

$$\|\Phi_{t,s}u\|_X \le c_0' M\beta(t_0)\,c(t-s)\,\|u\|_{L^p([s,t],U)}, \qquad (3.3)$$

$$\|\Phi(\cdot,s)u\|_{L^p([s,t],X)} \le c_0' M\beta(t_0)\,c(t-s)\,\|u\|_{L^p([s,t],U)} \qquad (3.4)$$

for $t\ge s\ge 0$, $u\in L^p([s,t],U)$, and $c_0'=c_0'(t_0,w,p)$ ($c(t)$ was defined in Lemma 2.3*).*

Proof. In Lemma 3.4, we show the measurability of $\Phi(\cdot,s)u$ (of course without referring to (3.4)). The assertion is clear for $s\le t\le s+t_0$. Let $s_k=s+kt_0$ for $k\in\mathbb{N}_0$, let $t\in[s_n,s_{n+1}]$ for some $n\in\mathbb{N}$, and let u_k be the restriction of u to $[s_k,s_{k+1}]\cap[s,t]$ for $k=0,\dots,n$. Then

$$\Phi(t,s)u = \Phi(t,s_n)u_n + \sum_{k=1}^n T(t,s_k)\Phi(s_k,s_{k-1})u_{k-1}\,, \qquad (3.5)$$

$$\|\Phi(t,s)u\| \le \beta\,\|u_n\|_p + M\beta\sum_{k=1}^n e^{w(t-s_k)}\,\|u_{k-1}\|_p \le M\beta e^{w^- t_0}\,(a*b)_n\,,$$

where $a_k = e^{wt_0 k}$ if $k = 0, \dots, n$ and $a_k = 0$ otherwise, $b_k = \|u_k\|_p$, $a = (a_k)_k$, and $b = (b_k)_k$. Young's inequality now implies the lemma. $\square$

DEFINITION 3.3. *Let T be an evolution family on X, and let $\overline{X}_t$, $t \geq 0$, be Banach spaces in which X is densely and continuously embedded. Assume that $T(t,s)$ has a locally uniformly bounded extension $\overline{T}(t,s) : \overline{X}_s \to \overline{X}_t$ (which then satisfies* (E1) *and is strongly continuous with respect to s). We call $B(t) \in \mathcal{L}(U, \overline{X}_t)$, $t \geq 0$,* (T-)admissible control operators *if the function $\overline{T}(t,\cdot)B(\cdot)u(\cdot)$ is integrable in $\overline{X}_t$,*

$$(\overline{\mathbb{K}}_s B(\cdot)u)\,(t) := \int_s^t \overline{T}(t,\tau)B(\tau)u(\tau)\,d\tau \in X,$$

and there are constants $t_0, \beta > 0$ such that

$$\|(\overline{\mathbb{K}}_s B(\cdot)u)(t)\|_X \leq \beta\,\|u\|_{L^p([s,t],U)} \tag{3.6}$$

for all $0 \leq s \leq t \leq s + t_0$ and $u \in L^p([s,t],U)$. (We omit the subscript s if $s = 0$.)

Setting $\Phi_{t,s}u := (\overline{\mathbb{K}}_s B(\cdot)u)(t)$, we obtain, of course, a nonautonomous control system (T,Φ) if $B(t)$, $t \geq 0$, are admissible control operators. Every autonomous control system is given by a T-admissible control operator due to [29, Thm. 3.9], where $\overline{X}_t$, $t \geq 0$, coincide with the extrapolation space X_{-1} of X with respect to the semigroup T (see, e.g., [2, Chap. V], [9, section II.5]). In Proposition 3.5, we extend this result to the time dependent setting but only in an approximate sense because of the lack of an extrapolation theory for evolution families. We first show a preliminary fact.

LEMMA 3.4. *Let (T,Φ) be a nonautonomous control system, and let $u \in L^p_{loc}(\mathbb{R}_+, U)$. Then $t \mapsto \Phi_{t,s}u \in X$ is continuous from the right for $t \geq s$, $s \mapsto \Phi_{t,s}u \in X$ is continuous for $s \in [0,t]$ (locally uniformly in t), and $(t,s) \mapsto \Phi_{t,s}u \in X$ is measurable.*

Proof. Definition 3.1 implies the estimates

$$\begin{aligned}
\|\Phi(t',s)u - \Phi(t,s)u\| &\leq \|\Phi(t',t)u\| + \|(T(t',t) - I)\Phi(t,s)u\| \\
&\leq \beta\,\|u\|_{L^p([t,t'],U)} + \|(T(t',t) - I)\Phi(t,s)u\|, \\
\|\Phi(t,s)u - \Phi(t,s')u\| &= \|T(t,s')\Phi(s',s)u\| \leq M\beta e^{|w|(t-s)}\,\|u\|_{L^p([s,s'],U)}\,,
\end{aligned}$$

where $t' \geq t \geq s' \geq s$. Thus the lemma is established. $\square$

Let $u \in L^p_{loc}(\mathbb{R}, U)$, $t \geq 0 \geq s$, and $n \in \mathbb{N}$. We define $(B_n u)(t) = n\,\Phi(t, t - \frac{1}{n})u$, where $\Phi(t,s)u := \Phi(t,0)u$. Note that $B_n u \in L^\infty_{loc}(\mathbb{R}_+, X)$ because of the above lemma. To approximate Φ, we introduce

$$\Phi^n(t,s)u = \Phi^n_{t,s}u := \int_s^t T(t,\tau)(B_n u)(\tau)\,d\tau = (\mathbb{K}_s B_n u)(t) \tag{3.7}$$

for $t \geq s \geq 0$, $n \in \mathbb{N}$, and $u \in L^p_{loc}(\mathbb{R}, U)$. These operators can be expressed by

$$\begin{aligned}
\Phi^n(t,s)u &= n \int_s^t \left(\Phi\left(t, \tau - \frac{1}{n}\right) u - \Phi(t,\tau)u \right) d\tau \\
&= n \int_{s-\frac{1}{n}}^{s} \Phi(t,\tau)u\,ds - n \int_{t-\frac{1}{n}}^{t} \Phi(t,\tau)u\,d\tau \\
&= \Phi(t,s)u + nT(t,s) \int_{s-\frac{1}{n}}^{s} \Phi(s,\tau)u\,d\tau - n \int_{t-\frac{1}{n}}^{t} \Phi(t,\tau)u\,d\tau
\end{aligned} \tag{3.8}$$

due to (3.1). If we take a function $u \in L^p_{loc}([s,\infty),U)$ and extend it by 0 to $\mathbb{R}$, then

$$\Phi(t,s)u - \Phi^n(t,s)u = n\int_{t-\frac{1}{n}}^{t} \Phi(t,\tau)u\,d\tau =: r_n(t;\,u). \tag{3.9}$$

To represent Φ approximately, we define operators $B_n(t) \in \mathcal{L}(U,X)$ by

$$B_n(t)z := (B_n u_z)(t) = n\Phi\left(t, t-\frac{1}{n}\right)u_z\,, \qquad \text{where} \;\; u_z \equiv z,\; z \in U.$$

PROPOSITION 3.5. *Let (T,Φ) be a nonautonomous control system, $n \in \mathbb{N}$, $0 \le s \le t \le s + t_0$, $t_0 > 0$, $z \in U$, and $u \in L^p_{loc}(\mathbb{R},U)$. Then we have the following:*

1. *$\Phi^n(t,s)u \to \Phi(t,s)u$, and $\|\Phi^n(t,s)u\|_X \le 2\beta(t_0)\|u\|_{L^p([s,t],U)}$.*
2. *$(t,s) \mapsto \Phi(t,s)u$, and $t \mapsto B_n(t)z$ are continuous in X.*
3. *$[\mathbb{K}_s B_n(\cdot)u](t) \to \Phi(t,s)u$, and $\|[\mathbb{K}_s B_n(\cdot)u](t)\|_X \le \beta(t_0+1)\,\|u\|_{L^p([s,t],U)}$.*

Here the limits as $n \to \infty$ are taken in X and are locally uniform in (t,s).

Proof. For $u \in L^p_{loc}(\mathbb{R},U)$, we estimate

$$\left\|n\int_{t-\frac{1}{n}}^{t} \Phi(t,\tau)u\,d\tau\right\|_X \le \sup_{t-\frac{1}{n}\le\tau\le t} \|\Phi(t,\tau)u\|_X \le \beta\,\|u\|_{L^p([t-\frac{1}{n},t],U)}\,,$$

$$\left\|nT(t,s)\int_{s-\frac{1}{n}}^{s} \Phi(s,\tau)u\,d\tau\right\|_X \le M\beta e^{w(t-s)}\,\|u\|_{L^p([s-\frac{1}{n},s],U)}\,,$$

which yields the first part of (1) because of (3.8). This fact implies (2). The second part of (1) follows from (3.9) if we extend $u \in L^p([s,\infty),X)$ by 0 to $\mathbb{R}$. We set $\tilde{u}(\tau,\sigma) = u(\tau)$ and $u^{(n)}(\sigma) = n\int_\sigma^{\sigma+\frac{1}{n}} u(\tau)\,d\tau$ for $\sigma \ge \tau \ge 0$ and $n \in \mathbb{N}$. Taking first $u \in W^{1,p}_{loc}(\mathbb{R},U)$, using Hölder's inequality, and interchanging integrals, we estimate

$$\begin{aligned}
\|B_k u - B_k(\cdot)u\|_{L^1([s,s+t_0],X)} &\le k\int_s^{s+t_0} \left\|\Phi\left(\tau,\tau-\frac{1}{k}\right)[u-\tilde{u}(\tau,\cdot)]\right\|_X d\tau \\
&\le \beta k\int_s^{s+t_0}\left(\int_{\tau-\frac{1}{k}}^{\tau}\left(\int_\sigma^\tau \|u'(\rho)\|_U\,d\rho\right)^p d\sigma\right)^{\frac{1}{p}} d\tau \\
&\le \beta k^{1-\frac{1}{p}}\int_s^{s+t_0}\int_0^{\frac{1}{k}} \|u'(\tau-\rho)\|_U\,d\rho\,d\tau \\
&\le c\left(\int_0^{\frac{1}{k}}\int_s^{s+t_0} \|u'(\tau-\rho)\|_U^p\,d\tau\,d\rho\right)^{\frac{1}{p}}
\end{aligned}$$

so that $[\mathbb{K}_s B_k(\cdot)u](t) \to \Phi(t,s)u$ as $k \to \infty$ locally uniformly in this case. Fix now $t > s \ge 0$, and extend $u \in L^p([s,t],X)$ by 0 to $\mathbb{R}$. Then (3.1) and the above results imply

$$
\begin{aligned}
[\mathbb{K}_s B_n(\cdot)u](t) &= n \int_{s-\frac{1}{n}}^{t} \Phi(t,\tau)\left[\tilde u\left(\tau+\frac{1}{n},\cdot\right)-\tilde u(\tau,\cdot)\right] d\tau \\
&= \lim_{k\to\infty} n \int_{s-\frac{1}{n}}^{t}\int_{\tau}^{t} T(t,\sigma)B_k(\sigma)\left[u\left(\tau+\frac{1}{n}\right)-u(\tau)\right] d\sigma\, d\tau \\
&= \lim_{k\to\infty} n \int_{s-\frac{1}{n}}^{t} T(t,\sigma)B_k(\sigma)\int_{s-\frac{1}{n}}^{\sigma}\left[u\left(\tau+\frac{1}{n}\right)-u(\tau)\right] d\tau\, d\sigma \\
&= \Phi\left(t, s-\frac{1}{n}\right) u^{(n)}.
\end{aligned}
$$

Observing that $\|u^{(n)}\|_{L^p([s,t],U)} \le \|u\|_{L^p([s,t],U)}$, we deduce (3). ◻

We give, as in [13], the nonautonomous analogue of Weiss' definition of a *well-posed system*; see [30, Def. 1.1].

DEFINITION 3.6. *Let* (T,Φ) *and* (T,Ψ) *be nonautonomous control and observation systems. If there are linear operators* $\mathbb{F}_s : L^p_{loc}([s,\infty),U) \to L^p_{loc}([s,\infty),Y)$, $s \ge 0$*, satisfying*

$$\mathbb{F}_s u = \Psi_t \Phi_{t,s} u + \mathbb{F}_t(u|[t,\infty)) \quad \text{on } [t,\infty) \quad \text{and} \tag{3.10}$$

$$\|\mathbb{F}_s u\|_{L^p([s,s+t_0],Y)} \le \kappa\, \|u\|_{L^p([s,s+t_0],U)} \tag{3.11}$$

for $u \in L^p_{loc}([s,\infty),U)$*,* $t \ge s \ge 0$*, and constants* $t_0 > 0$*,* $\kappa = \kappa(t_0) > 0$*, then* $\Sigma = (T,\Phi,\Psi,\mathbb{F}) = (T,\Phi_{t,s},\Psi_s,\mathbb{F}_s)_{t\ge s\ge 0}$ *is called a* well-posed nonautonomous system*, and* $\mathbb{F}_s$*,* $s \ge 0$*, are called* input–output operators*. We put* $\mathbb{F} = \mathbb{F}_0$*.*

The above definition implies that $\mathbb{F}_s u = 0$ on $[s,t]$ and $\mathbb{F}_s u = \mathbb{F}_t(u|[t,\infty))$ on $[t,\infty)$ if u vanishes on $[s,t]$. Hence $\mathbb{F}_s$ is *causal*, and we can define its restriction as

$$\mathbb{F}_{t,s} = \mathbb{F}(t,s) : L^p([s,t],U) \to L^p([s,t],Y), \qquad t \ge s \ge 0.$$

LEMMA 3.7. *A well-posed nonautonomous linear system* Σ *satisfies* (3.11) *with* t_0 *replaced by each* $t_1 > 0$ *and* $\kappa = \kappa(t_0)$ *by* $\kappa(t_1) = c_0''(\kappa(t_0) \vee M\beta(t_0)\gamma(t_0))c(t_1)$*, where* $c_0'' = c_0''(w,t_0)$ *and* $c(t)$ *was defined in Lemma* 2.3*.*

Proof. The assertion is clear for $t_1 \le t_0$. So let $t_1 \in [s_n, s_{n+1})$ for some $n \in \mathbb{N}$, $s_k = s + kt_0$, $I_k = [s_k, s_{k+1}]$, and $u_k = u|I_k$ for $k \in \mathbb{N}_0$, $s \ge 0$, and $u \in L^p_{loc}([s,\infty),U)$. From Definition 3.6 and (3.5), we deduce that

$$\mathbb{F}_s u = \mathbb{F}_{s_k} u_k + \sum_{j=1}^{k} \Psi_{s_k} T(s_k,s_j)\Phi(s_j,s_{j-1})u_{j-1} \quad \text{on} \quad I_k,$$

$$
\begin{aligned}
\|\mathbb{F}_s u\|_{L^p(I_k,Y)} &\le \kappa(t_0)\,\|u_k\|_p + M\beta(t_0)\gamma(t_0)\sum_{j=1}^{k} e^{wt_0(k-j)}\,\|u_{j-1}\|_p \\
&\le (\kappa(t_0) \vee M\beta(t_0)\gamma(t_0))\, e^{w^- t_0}\,(a*b)_k\,,
\end{aligned}
$$

$$\|\mathbb{F}_s u\|_{L^p([s,s+t_1],Y)} \le \left(\sum_{k=0}^{n} \|\mathbb{F}_s u\|^p_{L^p(I_k,Y)}\right)^{\frac{1}{p}} \le (\kappa(t_0) \vee M\beta(t_0)\gamma(t_0))\, e^{w^- t_0}\,\|a*b\|_{\ell^p}\,,$$

where the sequences a and b were defined in the proof of Lemma 3.2. Young's inequality now implies the asserted estimate. ◻

Also, Definition 3.6 is complemented by a concept involving admissible input and output operators; cf. [10], [12, section 1.3], [14].

DEFINITION 3.8. *Let $B(s)$ and $C(s)$, $s \geq 0$, be T-admissible control and observation operators. We call the triple $(T, B(\cdot), C(\cdot))$ an* admissible nonautonomous system *if $\overline{\mathbb{K}}_s B(\cdot)u \in D(C(\cdot), s)$ and $\|C(\cdot)\overline{\mathbb{K}}_s B(\cdot)u\|_{L^p([s,s+t_0],Y)} \leq \kappa \|u\|_{L^p([s,s+t_0],U)}$ for $s \geq 0$, $u \in L^p_{loc}([s,\infty), U)$, and constants $\kappa, t_0 > 0$.*

LEMMA 3.9. *Let $(T, B(\cdot), C(\cdot))$ be an admissible nonautonomous system. Define Ψ_s as in Lemma 2.5, $\Phi_{t,s}u := (\overline{\mathbb{K}}_s B(\cdot)u)(t)$, and $\mathbb{F}_s := C(\cdot)\overline{\mathbb{K}}_s B(\cdot)u$. Then $(T, \Phi, \Psi, \mathbb{F})$ is a well-posed nonautonomous system.*

Proof. In view of Lemma 2.5, we have only to verify (3.10) for $u \in L^p_{loc}([s,\infty), U)$ and $t \geq s \geq 0$. There are $t_n \searrow t$ such that $\Phi_{t_n,t}u, \Phi_{t_n,s}u \in D(C(t_n))$, and hence

$$\mathbb{F}_s u = \mathbb{F}_t u - \Psi_{t_n}\Phi_{t_n,t}u + \Psi_{t_n}\Phi_{t_n,s}u$$

a.e. on $[t_n, \infty)$. The third term on the right-hand side converges in L^p to $\Psi_t\Phi_{t,s}u$ due to Proposition 3.5 and the proof of Lemma 2.8. The assertion then follows from

$$\|\Psi_{t_n}\Phi_{t_n,t}u\|_{L^p([t_n,s+t_0],Y)} \leq \beta\gamma \|u\|_{L^p([t,t_n],U)}. \qquad \square$$

In order to prove a converse to the above lemma, we need the first of the following notions, which extends the corresponding concept due to Weiss [30, Def. 4.1].

DEFINITION 3.10. *A well-posed nonautomonous system $\Sigma = (T, \Phi, \Psi, \mathbb{F})$ is called* regular (with feedthrough $D = 0$) *if*

$$\lim_{\tau \searrow 0} \frac{1}{\tau} \int_t^{t+\tau} (\mathbb{F}_t u_z)(\sigma)\, d\sigma = 0 \tag{3.12}$$

(in Y) and absolutely regular *if*

$$\lim_{\tau \searrow 0} \frac{1}{\tau} \int_t^{t+\tau} \|(\mathbb{F}_t u_z)(\sigma)\|_Y^p\, d\sigma = 0 \tag{3.13}$$

for all $t \geq 0$ and $z \in U$, where $u_z(s) := z$ for $s \geq 0$.

We derive several useful properties of a well-posed system Σ. First, (3.11) yields

$$\left\|\frac{1}{\tau} \int_t^{t+\tau} (\mathbb{F}_t u_z)(\sigma)\, d\sigma\right\|_Y^p \leq \frac{1}{\tau} \int_t^{t+\tau} \|(\mathbb{F}_t u_z)(\sigma)\|_Y^p\, d\sigma \leq \kappa^p \|z\|^p \tag{3.14}$$

for $0 < \tau \leq t_0$, $t \geq 0$, and $z \in U$. Take $u \in L^p_{loc}(\mathbb{R}_+, U)$, and set $\tilde{u}(t,\sigma) = u(t)$ for $\sigma \geq t$ and $t \geq 0$. Then the functions

$$t \mapsto F_\tau(t) = \frac{1}{\tau} \int_t^{t+\tau} (\mathbb{F}_t u)(\sigma)\, d\sigma \quad \text{and} \quad t \mapsto \tilde{F}_\tau(t) = \frac{1}{\tau} \int_t^{t+\tau} (\mathbb{F}_t \tilde{u}(t,\cdot))(\sigma)\, d\sigma$$

are measurable for a fixed $\tau > 0$. Indeed, using (3.10), we can write

$$\begin{aligned} \tau\tilde{F}_\tau(t) - \tau\tilde{F}_\tau(r) = &\int_{r+\tau}^{t+\tau} [\mathbb{F}_t \tilde{u}(t,\cdot)](\sigma)\, d\sigma + \int_t^{r+\tau} [\mathbb{F}_t(\tilde{u}(t,\cdot) - \tilde{u}(r,\cdot))](\sigma)\, d\sigma \\ &- \int_t^{r+\tau} [\Psi_t \Phi_{t,r} \tilde{u}(r,\cdot)](\sigma)\, d\sigma - \int_r^t [\mathbb{F}_r \tilde{u}(r,\cdot)](\sigma)\, d\sigma \end{aligned}$$

for $t \geq r \geq t - \tau$. This identity and the straightforward estimates imply the left continuity of $\tilde{F}_\tau$ if u is continuous. Thus $\tilde{F}_\tau$ is measurable by approximation and

(3.14). The function F_τ can be handled in the same way. If Σ is regular, we deduce from Lebesgue's theorem and (3.14) that

$$\lim_{\tau\searrow 0}\int_0^{t_0} \|\tilde{F}_\tau(t)\|_Y^p\,dt = 0. \tag{3.15}$$

Similarly, $\varphi_u(\cdot,\tau)^{\frac{1}{p}}$ is measurable, and absolute regularity yields

$$\lim_{\tau\searrow 0}\int_0^{t_0} \frac{1}{\tau}\,\varphi_u(t,\tau)\,dt := \lim_{\tau\searrow 0}\int_0^{t_0}\frac{1}{\tau}\int_t^{t+\tau}\|[\mathbb{F}_t\tilde{u}(t,\cdot)](\sigma)\|_Y^p\,d\sigma\,dt = 0. \tag{3.16}$$

We now show a nonautonomous version of Weiss' representation theorem [30, Thm. 4.5].

THEOREM 3.11. *Let* $\Sigma = (T,\Phi,\Psi,\mathbb{F})$ *be a regular nonautonomous system, and let* $C(s)$ *and* $C_\tau(s)$ *be given by Definition* 2.6 *and* (2.5). *Then* $\Phi(\cdot,s)u \in D(C(\cdot),s)$, *and* $\mathbb{F}_s u = C(\cdot)\Phi(\cdot,s)u$ *for* $s\ge 0$ *and* $u\in L^p_{loc}([s,\infty),U)$. *Moreover,* $C_\tau(\cdot)\Phi(\cdot,s)u\to\mathbb{F}_s u$ *in* $L^p_{loc}([s,\infty),Y)$ *as* $\tau\searrow 0$, *and* $\|C_\tau(\cdot)\Phi(\cdot,s)u\|_{L^p([s,s+t_0],Y)}\le c\,\|u\|_{L^p([s,s+t_0],U)}$ *for* $\tau\in(0,1]$ *and a constant* $c=c(t_0)$ *independent of* u *and* s.

Proof. Let $t\in[s,\infty)$ be a p-Lebesgue point of u and $\mathbb{F}_s u$ such that the regularity condition (3.12) holds at this point t. Setting $o_t(\sigma)=u(\sigma)-u(t)$ for $\sigma\ge t$, we have

$$\mathbb{F}_s u = \mathbb{F}_t\tilde{u}(t,\cdot) + \mathbb{F}_t o_t + \Psi_t\Phi_{t,s}u \quad \text{on}\quad [t,\infty)\quad\text{and} \tag{3.17}$$

$$\left\|\frac{1}{\tau}\int_t^{t+\tau}(\mathbb{F}_t o_t)(\sigma)\,d\sigma\right\|^p \le \kappa^p\,\frac{1}{\tau}\int_t^{t+\tau}\|u(\sigma)-u(t)\|^p d\sigma. \tag{3.18}$$

Consequently, $C_\tau(t)\Phi_{t,s}u$ converges in Y to $(\mathbb{F}_s u)(t)$ as $\tau\to 0$ so that the first assertion holds. The estimate (3.18) and Fubini's theorem further yield

(3.19)
$$\int_s^{s+t_0}\left\|\frac{1}{\tau}\int_t^{t+\tau}(\mathbb{F}_t o_t)(\sigma)\,d\sigma\right\|^p dt \le \frac{\kappa^p}{\tau}\int_0^\tau\int_s^{s+t_0}\|u(t+\sigma)-u(t)\|^p dt\,d\sigma,$$
$$\int_s^{s+t_0}\left\|\frac{1}{\tau}\int_t^{t+\tau}[(\mathbb{F}_s u)(\sigma)-(\mathbb{F}_s u)(t)]\,d\sigma\right\|^p dt$$
$$\le \frac{\kappa^p}{\tau}\int_0^\tau\int_s^{s+t_0}\|(\mathbb{F}_s u)(t+\sigma)-(\mathbb{F}_s u)(t)\|^p dt\,d\sigma,$$

where both terms on the right-hand side converge to 0 as $\tau\to 0$. Combining these facts with (3.15) and (3.17), we establish that $\mathbb{F}_s u = \lim_\tau C_\tau(\cdot)\Phi(\cdot,s)u$ in $L^p_{loc}([s,\infty),Y)$. The asserted estimate follows in a similar way. ∎

The next approximation result complements Proposition 3.5 for absolutely regular systems. For technical reasons, we have to use the operators $B_n : L^p_{loc}(\mathbb{R}_+,U)\to L^\infty_{loc}(\mathbb{R}_+,X)$ rather than $B_n(t):U\to X$. Observe that only regularity is used in the proof of estimate (3.20).

PROPOSITION 3.12. *Let* Σ *be an absolutely regular nonautonomous system,* $p\in(1,\infty)$, *and let* $C(s)$ *and* $\Phi^n_{t,s}$ *be given as in Definition* 2.6 *and* (3.7). *Then* $\Phi^n(\cdot,s)u\in D(C(\cdot),s)$, $C(\cdot)\Phi^n(\cdot,s)u\to\mathbb{F}_s u$ *in* $L^p_{loc}([s,\infty),Y)$ *as* $n\to\infty$, *and*

$$\|C(\cdot)\Phi^n(\cdot,s)u\|_{L^p([s,s+t_0],Y)}\le 2\kappa(t_0)\,\|u\|_{L^p([s,s+t_0],U)} \tag{3.20}$$

for $u\in L^p_{loc}([s,\infty),U)$, $s\ge 0$, $n\in\mathbb{N}$, *and* $t_0>0$.

Proof. Due to Proposition 2.11, we have $\Phi^n(\cdot,s)u = \mathbb{K}_s B_n u \in D(C(\cdot),s)$. Formula (3.9), Proposition 2.11, and Theorem 3.11 further yield

$$\mathbb{F}_s u - C(\cdot)\Phi^n(\cdot,s)u = C(\cdot)r_n(\cdot;\,u) = \lim_{\tau\to 0} C_\tau(\cdot)r_n(\cdot;\,u) \quad (\text{in } \ L^p_{loc}([s,\infty),Y)). \tag{3.21}$$

Using Hölder's inequality, Fubini's theorem, and Theorem 3.11, we now derive

$$\begin{aligned} \|C(\cdot)r_n(\cdot;\,u)\|^p_{L^p([s,s+t_0],Y)} &\le \lim_{\tau\to 0} n\int_s^{s+t_0}\int_{t-\frac{1}{n}}^{t} \|C_\tau(t)\Phi_{t,\sigma}u\|_Y^p\,d\sigma\,dt \\ &\le n\int_{s-\frac{1}{n}}^{s+t_0} \|\mathbb{F}_\sigma u\|^p_{L^p([\sigma,\sigma+1/n],Y)}\,d\sigma \\ &\le n\kappa(t_0)\int_0^{\frac{1}{n}}\int_{s-\frac{1}{n}}^{s+t_0}\|u(t+\sigma)\|^p\,d\sigma\,dt \le \kappa(t_0)\,\|u\|_p^p. \end{aligned} \tag{3.22}$$

(Here we have considered a function $u\in L^p([s,s+t_0],U)$ and extended it by 0 to $\mathbb{R}$.) Thus (3.20) holds. The estimate (3.22) also gives

$$\begin{aligned} &\|C(\cdot)r_n(\cdot;\,u)\|_{L^p([s,s+t_0],Y)} \\ &\le \left(\int_{s-\frac{1}{n}}^{s+t_0} n\,\|\mathbb{F}_\sigma o_\sigma\|^p_{L^p([\sigma,\sigma+\frac{1}{n}],Y)}\,d\sigma\right)^{\frac{1}{p}} + \left(\int_{s-\frac{1}{n}}^{s+t_0} n\,\|\mathbb{F}_\sigma \tilde u(\sigma,\cdot)\|^p_{L^p([\sigma,\sigma+\frac{1}{n}],Y)}\,d\sigma\right)^{\frac{1}{p}}. \end{aligned}$$

The right-hand side tends to 0 as in (3.19) and (3.16). ∎

4. The main result and discussion. Let Σ be a regular nonautonomous system, $\Delta(\cdot)\in L^\infty(\mathbb{R}_+,\mathcal{L}_s(Y,U))$, and let $C(s)$ be given by Definition 2.6. For $x\in X$ and $s\ge 0$, we are looking for functions $x(\cdot)\in C([s,\infty),X)\cap D(C(\cdot),s)$ satisfying

$$x(t) = T(t,s)x + \Phi_{t,s}\Delta(\cdot)C(\cdot)x(\cdot), \qquad t\ge s, \tag{4.1}$$

or, if $\Phi(\cdot,s)u = \overline{\mathbb{K}}_s B(\cdot)u(\cdot)$ for admissible control operators $B(s)$,

$$x(t) = T(t,s)x + \int_s^t \to (t,\tau)B(\tau)\Delta(\tau)C(\tau)x(\tau)\,d\tau, \qquad t\ge s. \tag{4.2}$$

As shown by [27, Ex. 6], one cannot allow for every bounded feedback in (4.1) in general. (We note that this example gives rise to an absolutely regular autonomous system with $p=1$ and $\Delta=B=I$.) This fact motivates the next concept.

DEFINITION 4.1. *Let $\Sigma=(T,\Phi,\Psi,\mathbb{F})$ be a well-posed nonautonomous system. We call $\Delta(\cdot)\in L^\infty(\mathbb{R}_+,\mathcal{L}_s(Y,U))$ an* admissible feedback *for Σ if there is $t_0>0$ such that $I-\mathbb{F}(s+t_0,s)\Delta(\cdot)$, $s\ge 0$, have uniformly bounded inverses on $L^p([s,s+t_0],Y)$.*

Of course, $\Delta(\cdot)$ is admissible if

$$\|\Delta(\cdot)\|_\infty < \left[\inf_{t_0>0}\sup_{s\ge 0}\|\mathbb{F}(s+t_0,s)\|\right]^{-1} =: q. \tag{4.3}$$

The right-hand side of this inequality equals ∞ if $B(t)$ and $C(t)$ are of "lower order"; see, e.g., [6] or [23, Ex. 4.11]. We point out that the invertibility of $I-\mathbb{F}(s+t_0,s)\Delta(\cdot)$ is in fact necessary for some properties of the feedback system as shown by Lemma 4.3 and Proposition 5.1. The next lemmas also indicate that our notion of admissibility is quite flexible; see [21, Lem. 4.1], [25, section 7.1], and [32, section 3] for autonomous analogues.

LEMMA 4.2. *Let $\Delta(\cdot) \in L^\infty(\mathbb{R}_+, \mathcal{L}_s(Y,U))$ and Σ be a well-posed nonautonomous system. If $I - \mathbb{F}(s+t_0,s)\Delta(\cdot)$ is invertible on $L^p([s,s+t_0],Y)$ for all $s \geq 0$ and some $t_0 > 0$, then $I - \mathbb{F}(s+t_1,s)\Delta(\cdot)$ is invertible on $L^p([s,s+t_1],U)$ for all $s \geq 0$ and $t_1 > 0$. The notion of an admissible feedback is independent of $t_0 > 0$.*

Proof. We assume that $U = Y$ and $\Delta(s) = I$ for simplicity. First, let $t_1 \leq t_0$. Extend a given $v \in L^p([s,s+t_1],Y)$ by 0 to $\tilde{v} \in L^p([s,s+t_0],Y)$, and set $\tilde{u} = (I - \mathbb{F}(s+t_0,s))^{-1}\tilde{v}$. Then $(I - \mathbb{F}(s+t_1,s))u = v$ for the restriction u of $\tilde{u}$. If $u = \mathbb{F}(s+t_1,s)u$, then there is a function $u_1 \in L^p([s+t_1,s+t_0],U)$ such that $(I - \mathbb{F}(s+t_0,s+t_1))u_1 = \Psi_{s+t_1}\Phi_{s+t_1,s}u$. Set $\tilde{u} = u$ on $[s,s+t_1]$ and $\tilde{u} = u_1$ on $[s+t_1,s+t_0]$. Hence $\tilde{u} = \mathbb{F}(s+t_0,s)\tilde{u}$ so that $u = 0$.

It remains to consider $t_1 = nt_0$ for $n \in \mathbb{N}$. Proceeding by induction, we assume that the assertion is true for $t_1 = nt_0$. It is then clear that $I - \mathbb{F}(s+(n+1)t_0,s)$ is injective. For $v \in L^p([s,s+(n+1)t_0],Y)$, we set $u_1 = (I - \mathbb{F}(s+nt_0,s))^{-1}(v|[s,s+nt_0])$ and $u_2 = (I - \mathbb{F}(s+(n+1)t_0,s+nt_0))^{-1}\{v|[s+nt_0,s+(n+1)t_0] + \Psi_{s+nt_0}\Phi_{s+nt_0,s}u_1\}$. Putting u_1 and u_2 together, one sees that $I - \mathbb{F}(s+(n+1)t_0,s)$ is surjective. □

LEMMA 4.3. *For maps $T : E \to F$ and $V : F \to E$, the following are equivalent:*

1. *$I - VT$ is bijective on E.*
2. *$I - TV$ is bijective on F.*
3. *There is a map $S : E \to F$ such that $S - T = TVS = SVT$.*

Then we have $(I - TV)^{-1} = I + T(I - VT)^{-1}V = I + SV$ and $(I - VT)^{-1} = I + VS$, and S in (3) is uniquely given by $S = (I - TV)^{-1}T = T(I - VT)^{-1}$.

Thus a feedback $\Delta(\cdot)$ is admissible if and only if $I - \Delta(\cdot)\mathbb{F}(s+t_0,s)$, $s \geq 0$, have uniformly bounded inverses for some/all $t_0 > 0$ if and only if for some/all $t_0 > 0$ there are uniformly bounded operators $\mathbb{F}^\Delta(s+t_0,s)$, $s \geq 0$, such that $\mathbb{F}^\Delta(s+t_0,s) - \mathbb{F}(s+t_0,s) = \mathbb{F}^\Delta(s+t_0,s)\Delta(\cdot)\mathbb{F}(s+t_0,s) = \mathbb{F}(s+t_0,s)\Delta(\cdot)\mathbb{F}^\Delta(s+t_0,s)$.

We now solve (4.1) by constructing an evolution family T_Δ on X and show that the feedback system Σ^Δ is again absolutely regular if the unperturbed system is absolutely regular. Proposition 5.1 describes the relations between Σ and Σ^Δ in greater detail.

THEOREM 4.4. *Let $\Sigma = (T, \Phi, \Psi, \mathbb{F})$ be a regular nonautonomous system and $\Delta(\cdot) \in L^\infty(\mathbb{R}_+, \mathcal{L}_s(Y,U))$ be an admissible feedback. Then the following hold:*

(a) *There is an evolution family T_Δ on X such that $T_\Delta(\cdot,s)x \in D(C(\cdot),s)$,*

$$\|C(\cdot)T_\Delta(\cdot,s)x\|_{L^p([s,s+t_0],Y)} \leq \gamma' \|x\|, \tag{4.4}$$

$x(\cdot) = T_\Delta(\cdot,s)x$ is the unique solution of (4.1), and

$$T_\Delta(t,s)x = T(t,s)x + \Phi_{t,s}\Delta(\cdot)C(\cdot)T_\Delta(\cdot,s)x \tag{4.5}$$

for $t \geq s \geq 0$, $x \in X$, and a constant γ'. If, in addition, $\Phi(\cdot,s)u = \overline{\mathbb{K}}_s B(\cdot)u(\cdot)$ for T-admissible control operators $B(t)$, then

$$T_\Delta(t,s)x = T(t,s)x + \int_s^t \to (t,\tau)B(\tau)\Delta(\tau)C(\tau)T_\Delta(\tau,s)x\, d\tau. \tag{4.6}$$

(b) *If the system is absolutely regular and $p \in (1,\infty)$, then*

$$T_\Delta(t,s)x = T(t,s)x + \lim_{n\to\infty}\int_s^t T_\Delta(t,\tau)[B_n(\Delta(\cdot)\Psi_s x)](\tau)\, d\tau \tag{4.7}$$

for $t \ge s \ge 0$ and $x \in X$, where the limit is taken in X and is locally uniform in t. Moreover, $\Sigma^\Delta = (T_\Delta, \Phi^\Delta, \Psi^\Delta, \mathbb{F}^\Delta)$ is an absolutely regular system, where we set

$$\Psi_s^\Delta x = C(\cdot)T_\Delta(\cdot, s)x, \quad \Phi_{t,s}^\Delta u = \lim_{n\to\infty} [\mathbb{K}_s^\Delta B_n u](t),$$

$$\mathbb{F}_s^\Delta u = \lim_{n\to\infty} C(\cdot)\mathbb{K}_s^\Delta B_n u, \quad \mathbb{K}_s^\Delta f\,(t) = \int_s^t T_\Delta(t,\tau) f(\tau)\, d\tau$$

for $t \ge s \ge 0$, $x \in X$, $u \in L^p_{loc}([s,\infty), U)$, and $f \in L^p_{loc}([s,\infty), X)$, where the limits are taken in X and L^p_{loc}, respectively.

Proof. (a) We first prove the uniqueness of solutions to (4.1). If v solves (4.1) with $x = 0$, then $C(\cdot)v = \mathbb{F}_s\Delta(\cdot)C(\cdot)v$ by Theorem 3.11. Since $I - \mathbb{F}(s+t_1, s)\Delta(\cdot)$ is injective, $C(\cdot)v$ has to vanish a.e. on $[s, s+t_1]$, where $t_1 > 0$ can be chosen arbitrarily large by Lemma 4.2. Hence (4.1) implies that $v = 0$. To solve (4.1), we define

$$T_\Delta(t,s)x = T(t,s)x + \Phi_{t,s}\Delta(\cdot)(I - \mathbb{F}(s+t_1, s)\Delta(\cdot))^{-1}\Psi_s x \tag{4.8}$$

for $0 \le t - s \le t_1$ and $x \in X$. Clearly, $T_\Delta(t,s)$ is an exponentially bounded linear operator on X, and $T_\Delta(\cdot, s)x$ is continuous in X by Proposition 3.5. Theorems 2.7 and 3.11 further show that $T_\Delta(\cdot, s)x \in D(C(\cdot), s)$ and

$$C(\cdot)T_\Delta(\cdot, s)x = \Psi_s x + \mathbb{F}_s\Delta(\cdot)(I - \mathbb{F}(s+t_1, s)\Delta(\cdot))^{-1}\Psi_s x \tag{4.9}$$

$$= (I - \mathbb{F}(s+t_1, s)\Delta(\cdot))^{-1}\Psi_s x. \tag{4.10}$$

Hence (4.4) holds. Inserting (4.10) into (4.8), we obtain (4.5) and (4.6) and thus have solved (4.1). One verifies (E1) for T_Δ using the uniqueness of (4.1), formula (3.1), and a standard argument. It remains to establish the strong continuity of T_Δ. We first take $(t_n, s_n) \to (s_0, s_0)$ with $t_n \ge s_n \ge 0$. For $\varepsilon > 0$, $x \in X$, and large n, there is $r \in [0, s_0] \cap [0, s_n]$ such that $\|T(s_0, r)x - x\| \le \varepsilon$. Then (4.8) and (2.1) yield

$$\|T_\Delta(t_n, s_n)x - x\| \le \|T_\Delta(t_n, s_n)(x - T(s_n, r)x)\| + \|T_\Delta(t_n, s_n)T(s_n, r)x - x\|$$
$$\le c\,\|x - T(s_n, r)x\| + \|T(t_n, r)x - x\| + c\left[\int_{s_n}^{t_n} \|[\Psi_r x](\sigma)\|^p d\sigma\right]^{\frac{1}{p}},$$
$$\overline{\lim_{n\to\infty}}\, \|T_\Delta(t_n, s_n)x - x\| \le (c+1)\varepsilon$$

for a constant c. Therefore, T_Δ is strongly continuous at (s_0, s_0). If $(t_n, s_n) \to (t_0, s_0)$ for some $t_0 > s_0$, we may assume that $t_n > s_n$ and $t_n > s_0$. We take $t_n \ge r_n \ge s_n \vee s_0$ with $r_n \to s_0$ and derive (E2) from the above results and the expression

$$T_\Delta(t_n, s_n)x - T_\Delta(t_0, s_0)x = T_\Delta(t_n, r_n)(T_\Delta(r_n, s_n)x - T_\Delta(r_n, s_0)x)$$
$$+ T_\Delta(t_n, s_0)x - T_\Delta(t_0, s_0)x.$$

(b) Define $\mathcal{D}_{\Delta,s}$ as in (2.6) using T_Δ. Then (4.5) and Proposition 3.5 imply that

$$\mathbb{K}_s^\Delta f\,(t) = \mathbb{K}_s f\,(t) + \lim_{n\to\infty} \int_s^t \int_\tau^t T(t,\sigma)B_n(\sigma)\Delta(\sigma)C(\sigma)T_\Delta(\sigma,\tau)f(\tau)\, d\sigma\, d\tau$$

for $f \in \mathcal{D}_{\Delta,s}$ and $s \ge 0$ since the integrand is the sum of functions of the form

$$(\tau, \sigma) \mapsto \gamma(\tau)\, T(t,\sigma)B_n(\sigma)\Delta(\sigma)C(\sigma)T_\Delta(\sigma, r)x. \tag{4.11}$$

For the same reason, $\mathbb{K}_s^{\Delta} f$ belongs to $D(C(\cdot), s)$, and we can apply Fubini's theorem and take $T(t,\sigma)B_n(\sigma)\Delta(\sigma)C(\sigma) \in \mathcal{L}(\underline{X}_\sigma, X)$ out of the inner integral. So we obtain

$$\mathbb{K}_s^{\Delta} f = \mathbb{K}_s f + \lim_{n\to\infty} \mathbb{K}_s B_n(\cdot)\Delta(\cdot)C(\cdot)\mathbb{K}_s^{\Delta} f = \mathbb{K}_s f + \Phi(\cdot,s)\Delta(\cdot)C(\cdot)\mathbb{K}_s^{\Delta} f \tag{4.12}$$

for $f \in \mathcal{D}_{\Delta,s}$ due to Proposition 3.5. Theorem 3.11 now shows that $C(\cdot)\mathbb{K}_s^{\Delta} f = C(\cdot)\mathbb{K}_s f + \mathbb{F}_s\Delta(\cdot)C(\cdot)\mathbb{K}_s^{\Delta} f$. Hence

$$C(\cdot)\mathbb{K}_s^{\Delta} f = (I - \mathbb{F}(s+t_1,s)\Delta(\cdot))^{-1}C(\cdot)\mathbb{K}_s f \tag{4.13}$$

on $[s, s+t_1]$ for each $t_1 > 0$. Inserting (4.13) into (4.12), we conclude that

$$\mathbb{K}_s^{\Delta} f = \mathbb{K}_s f + \Phi(\cdot,s)\Delta(\cdot)(I - \mathbb{F}(s+t_1,s)\Delta(\cdot))^{-1}C(\cdot)\mathbb{K}_s f \tag{4.14}$$

on $[s, s+t_1]$ for $f \in \mathcal{D}_{\Delta,s}$. This identity holds for all $f \in L^p_{loc}([s,\infty), X)$ by Proposition 2.11. So we may take $f = B_n u$ for $u \in L^p_{loc}([s,\infty), U)$ and $n \in \mathbb{N}$, and thus

$$\mathbb{K}_s^{\Delta} B_n u = \mathbb{K}_s B_n u + \Phi(\cdot,s)\Delta(\cdot)(I - \mathbb{F}(s+t_1,s)\Delta(\cdot))^{-1}C(\cdot)\mathbb{K}_s B_n u. \tag{4.15}$$

As a consequence, $\mathbb{K}_s^{\Delta} B_n u \in D(C(\cdot), s)$ and

$$C(\cdot)\mathbb{K}_s^{\Delta} B_n u = (I - \mathbb{F}(s+t_1,s)\Delta(\cdot))^{-1}C(\cdot)\mathbb{K}_s B_n u \tag{4.16}$$

by Proposition 2.11 and Theorem 3.11. In view of Propositions 3.5 and 3.12, we can take the limit as $n \to \infty$ in the formulas (4.15) and (4.16) (in $C([s, s+t_1], X)$ and $L^p([s, s+t_1], Y)$, respectively). It is then easy to see that $\Sigma^{\Delta} = (T_{\Delta}, \Phi^{\Delta}, \Psi^{\Delta}, \mathbb{F}^{\Delta})$ defined in the assertion is a well-posed nonautonomous system. Equation (4.16) and Proposition 3.12 further yield

$$\int_t^{t+\tau} \|(\mathbb{F}_t^{\Delta} u_z)(\sigma)\|^p \, d\sigma \le c \int_t^{t+\tau} \|(\mathbb{F}_t u_z)(\sigma)\|^p \, d\sigma$$

for $t \ge 0$, $\tau > 0$, $u_z \equiv z$, and $z \in U$ so that additionally Σ^{Δ} is absolutely regular.

We now choose $u = \Delta(\cdot)\Psi_s x$ for $x \in X$ and deduce from (4.15), Propositions 3.5 and 3.12, and (4.8) that

$$\begin{aligned}\lim_{n\to\infty} \mathbb{K}_s^{\Delta} B_n \Delta(\cdot)\Psi_s x &= \Phi(\cdot,s)\Delta(\cdot)\Psi_s x + \Phi(\cdot,s)\Delta(\cdot)(I - \mathbb{F}(s+t_1,s)\Delta(\cdot))^{-1}\mathbb{F}_s\Delta(\cdot)\Psi_s x \\ &= \Phi(\cdot,s)\Delta(\cdot)(I - \mathbb{F}(s+t_1,s)\Delta(\cdot))^{-1}\Psi_s x = T_{\Delta}(\cdot,s)x - T(\cdot,s)x,\end{aligned}$$

where the limit is taken in X and is locally uniform in t. Thus (4.7) holds. ∎

We state several variants of Theorem 4.4 and compare them to related results.

REMARK 4.5. *The above theorem remains valid if we do not assume* (E3), *require* β, γ, κ *and* $\|(I - \mathbb{F}(s+t_0,s)\Delta(\cdot))^{-1}\|$ *only to be uniform with respect to* $s \in [0,a]$ *for every* $a > 0$, *and assert for the perturbed problem only the analogous properties. The proof of part* (a) *also works in the case that* T *(and then* T_{Δ}*) is only strongly continuous in* t *and* s *separately. Part* (a) *can be verified for admissible systems, too, if one considers only* $x \in D(C(s))$ *in* (4.2), (4.4), *and* (4.6)*; see* [23, *Thm.* 4.18] *or* (c) *below.*

REMARK 4.6. *Let* Σ *be a nonautonomous regular system with* $p \in (1,\infty)$. *It can be shown that* $\langle C(\cdot)\mathbb{K}_s B_n u, v\rangle \to \langle \mathbb{F}_s u, v\rangle$ *as* $n \to \infty$ *for all* $v \in L^q([s, s+t_0], Y^*)$, $\frac{1}{p} + \frac{1}{q} = 1$. *Thus, if* Y *is reflexive, the conclusions of Theorem* 4.4(b) *hold for merely* *regular systems except that the limits exist only weakly and that it is not clear whether* Σ^Δ *is regular again. In the autonomous case, the regularity of* Σ^Δ *for regular* Σ *was established in* [32, *Thm.* 4.5, 4.7]*, but the proof given there relies on Laplace transforms and a Tauberian theorem* [31, *Thm.* 5.2] *not available here; see also* [25, *section* 7.5].

(a) *Perturbation theory of evolution equations.* Theorem 4.4 is a joint nonautonomous extension of the Desch–Schappacher and Miyadera perturbation theorem from semigroup theory (see, e.g., [9, section III.3]): First, let $B(t)$ be T-admissible control operators, and define $Y = X$, $\Psi_s = T(\cdot, s)$, and $\mathbb{F}_s = \overline{\mathbb{K}}_s B(\cdot)$; i.e., $C(t) \equiv I$. This gives an absolutely regular nonautonomous system with $\kappa(t_0) = \beta t_0^{1/p}$ so that $q = \infty$ in (4.3). Second, let (T, Ψ) be a nonautonomous observation system for $p \in (1, \infty)$ represented by $C(t)$. Setting $U = X$, $\Phi_{t,s}u = (\mathbb{K}_s u)(t)$, and $\mathbb{F}_s = C(\cdot)\mathbb{K}_s$, i.e., $B(t) \equiv I$, we obtain a well-posed nonautonomous system thanks to Proposition 2.11. Approximating u_z by $T(\cdot, t)z$, one verifies that the system is absolutely regular. A nonautonomous Miyadera theorem for closable perturbations $C(t)$ and $p \geq 1$ was proved in [19] by other methods.

(b) *Autonomous controlled systems.* Let $T(t - s) = T(t, s)$ be a C_0-semigroup generated by A and $\Delta(t) \equiv \Delta$. We say that (T, B, C) belongs to the *Pritchard–Salamon class* [18] if (2.2) holds with $\|x\|_X$ replaced by $\|x\|_{\overline{X}}$ and (3.6) holds with $\|\cdot\|_X$ replaced by $\|\cdot\|_{\underline{X}}$. The perturbation theory for this class was developed in detail in [6]. In this case, one can extend $T_\Delta(t)$ to $\overline{X}$, and the number q in (4.3) is equal to ∞.

Weiss introduced autonomous regular systems in [28], [29], [30], [31] similarly as in the above definitions by considering only the initial time $s = 0$. He solved the feedback problem in [32, Thm. 6.1] for a well-posed system with $p = 2$ on Hilbert spaces X, Y, U, allowing for nontrivial feedthrough D and assuming that (roughly speaking) $I - CR(\lambda, A_{-1})B\Delta$ is invertible on a right halfplane; see also [21]. If the system is regular, the feedback system is again regular and can be represented almost in the natural way; see [32, section 7]. The feedback theory for several classes of (non)regular systems is exhaustively studied in Chapter 7 of Staffans' monograph [25] in a Banach space setting and also for $p = 1, \infty$.

The remaining difficulties come from the fact that, in general, $T_\Delta(t)$ cannot be continuously extended to the extrapolation space X_{-1}^A corresponding to T (see [23, Ex. 4.20]); in particular, the extrapolation space X_{-1}^Δ of T_Δ may differ from X_{-1}^A. Weiss constructed subspaces W and W_Δ of X_{-1}^A and X_{-1}^Δ, respectively, such that $Jx := \lim_{\lambda \to \infty} \lambda R(\lambda, A_{-1})x$ (in X_{-1}^Δ) defines an isomorphism $J : W \to W_\Delta$; see [32, Thm. 7.7]. Note that $Jx = x$ for $x \in X$. Then

$$T_\Delta(t)x = T(t)x + \int_0^t T_{\Delta,-1}(t - \tau)JB\Delta CT(\tau)x\, d\tau$$

by (6.11), (6.1), and [32, p. 55]. In other words, Weiss managed to put the limit in (4.7) inside the integral using a different regularization. Identifying B and JB, he represented the feedback system in terms of B and C and computed the generator of T_Δ [32, section 7]; see [25, section 7.4] for a somewhat different approach.

(c) *Nonautonomous controlled systems.* Part (a) of Theorem 4.4 was proved by Hinrichsen, Jacob, and Pritchard for nonautonomous admissible systems in a slightly differing setting; see [10, Thm. 3.2] and [12], [14] also for nonlinear feedback. They work with separately strongly continuous evolution families and have some additional technical assumptions (see, e.g., Hypotheses 4 and 7 of [10]). Moreover, they obtain (4.6) with a pointwise representation of $C(\cdot)T(\cdot, s)x$ only for $x \in D(C(s))$. The issues investigated in Theorem 4.4(b) were not considered in [10] and [14] and were considered in [12, Thm. 3.4.7] only for systems of Pritchard–Salamon type.

REMARK 4.7. *In addition to the assumptions of Theorem* 4.4(b), *we suppose that* $\Phi_{t,s}$ *is given by admissible observation operators* $B(\cdot) \in L^\infty(\mathbb{R}_+, \mathcal{L}_s(U, \overline{X}))$ *for* $\overline{X}_t \equiv \overline{X}$ *and that* $T_\Delta(t,s)$ *has a locally uniformly bounded extension* $\overline{T}_\Delta(t,s) : \overline{X} \to \overline{X}$. *Thus* $\overline{T}_\Delta$ *satisfies* (E1) *and* (E2). *We set* $(\overline{\mathbb{K}}_s^\Delta f)(t) = \int_s^t \overline{T}_\Delta(t,\tau) f(\tau)\, d\tau$ *for* $t \ge s \ge 0$ *and* $f \in L^1_{loc}([s,\infty), \overline{X})$. *Then* $\Phi^\Delta(\cdot, s)u = \overline{\mathbb{K}}_s^\Delta B(\cdot)u$, $\mathbb{F}_s^\Delta u = C(\cdot)\overline{\mathbb{K}}_s^\Delta B(\cdot)u$, *and*

$$T_\Delta(t,s)x = T(t,s)x + \int_s^t \to_\Delta (t,\tau)B(\tau)\Delta(\tau)C(\tau)T(\tau,s)x\, d\tau \tag{4.17}$$

for $u \in L^p_{loc}([s,\infty), U)$, $x \in X$, *and* $t \ge s \ge 0$.

Proof. Observe that $B_n u \to B(\cdot)u$ as $n \to \infty$ in $L^p_{loc}(\mathbb{R}_+, \overline{X})$ for $u \in L^p_{loc}(\mathbb{R}_+, \overline{X})$ because of the inequality

$$\|B_n u - B(\cdot)u\|^p_{L^p([0,t_0],\overline{X})} \le n \int_0^{\frac{1}{n}} \int_0^{t_0} \|\overline{T}(\tau+\sigma,\tau)B(\tau)u(\tau) - B(\tau+\sigma)u(\tau+\sigma)\|^p_{\overline{X}}\, d\tau\, d\sigma,$$

which is a consequence of Hölder's inequality and Fubini's theorem. Thus $\Phi^\Delta(\cdot, s)u = \overline{\mathbb{K}}_s^\Delta B(\cdot)u$, and (4.17) holds. The identities (4.15) and (4.16) then imply that

$$C(\cdot)\overline{\mathbb{K}}_s^\Delta B(\cdot)u = (I - \mathbb{F}(s+t_1, s)\Delta(\cdot))^{-1}\mathbb{F}_s u = \mathbb{F}_s^\Delta u. \qquad \square$$

The above remark and paragraph (b) indicate that an (absolutely) regular nonautonomous system and the corresponding feedback system can be represented by operators $B(t)$ (and not just approximately by $B_n(t)$) whenever we have a decent extrapolation theory for the given problem. It seems to be reasonable to study first the case that T is generated by operators $A(t)$ and consider spaces $\overline{X}_t$ related to $A(t)$. For various results on parabolic evolution equations and extrapolation spaces, we refer to [1], [2, Chap.V], [23, Prop. 2.1].

5. Further properties of the feedback system. In the setting of Theorem 4.4, we study the relationship between the open- and the closed-loop systems in more detail; see [25, Chap. 7] and [32, section 6] for similar results in the autonomous case. To put the formulas in a concise form, we define $\Psi(t,s)x = \mathbb{1}_{[s,t]}\Psi_s x$ and

$$\Sigma(t,s) = \begin{pmatrix} T(t,s) & \Phi(t,s) \\ \Psi(t,s) & \mathbb{F}(t,s) \end{pmatrix} : X \times L^p([s,t],U) \to X \times L^p([s,t],Y), \quad t \ge s \ge 0.$$

PROPOSITION 5.1. *Let* Σ *be an absolutely regular nonautonomous system, let* $p \in (1,\infty)$, *let* $\Delta(\cdot)$ *be an admissible feedback for* Σ, *and let* Σ^Δ *be the feedback system from Theorem* 4.4. *Then*

$$\mathbb{F}_s^\Delta = (I - \mathbb{F}_s\Delta(\cdot))^{-1}\mathbb{F}_s = \mathbb{F}_s(I - \Delta(\cdot)\mathbb{F}_s)^{-1} = C(\cdot)\Phi^\Delta(\cdot, s), \tag{5.1}$$

$$\Sigma^\Delta(t,s) - \Sigma(t,s) = \Sigma(t,s)\begin{pmatrix} 0 & 0 \\ 0 & \Delta(\cdot) \end{pmatrix}\Sigma^\Delta(t,s) = \Sigma^\Delta(t,s)\begin{pmatrix} 0 & 0 \\ 0 & \Delta(\cdot) \end{pmatrix}\Sigma(t,s). \tag{5.2}$$

Proof. The first equality in (5.1) is an immediate consequence of (4.16) and Proposition 3.12. Lemma 4.3 then yields the second equality in (5.1) and the expressions for $\mathbb{F}^\Delta - \mathbb{F}$ in the lower right corner of (5.2). Taking the limit in (4.15) and using the formulas for $\mathbb{F}^\Delta$, we deduce the last equality in (5.1). The identities for $T^\Delta - T$ in the upper left corner of (5.2) were established in Theorem 4.4, and they imply the formulas for $\Psi^\Delta - \Psi$ in the lower left corner in (5.2). The first equality in the upper right corner follows from (4.15). Employing the previous results, we finally obtain

$$\begin{aligned}\Phi^\Delta(\cdot,s)\Delta(\cdot)\mathbb{F}_s &= \Phi(\cdot,s)\Delta(\cdot)\mathbb{F}_s + \Phi(\cdot,s)\Delta(\cdot)\mathbb{F}_s^\Delta\Delta(\cdot)\mathbb{F}_s\\ &= \Phi(\cdot,s)\Delta(\cdot)\mathbb{F}_s^\Delta = \Phi^\Delta(\cdot,s) - \Phi(\cdot,s). \qquad \square\end{aligned}$$

The above result allows us to prove that the following control theoretic properties (cf. [7]) remain unchanged under feedback.

DEFINITION 5.2. (a) *A nonautonomous control system* (T,Φ) *is called* exactly (approximately) controllable on $[s,t]$ *if* $\Phi(t,s)$ *is surjective (has dense range) and it is called* exactly (approximately) null controllable on $[s,t]$ *if* $T(t,s)X$ *is contained in the (closure of)* $\Phi(t,s)L^p([s,t],U)$.

(b) *A nonautonomous observation system* (T,Ψ) *is called* (continuously) initially observable on $[s,t]$ *if* $\Psi(t,s)$ *is injective (bounded from below) and* (continuously) finally observable on $[s,t]$ *if* $\ker\Psi(t,s)\subset\ker T(t,s)$ *(if* $\|T(t,s)x\|\le c\,\|\Psi(t,s)x\|_p$ *for a constant* $c>0$ *and* $x\in X$*).*

PROPOSITION 5.3. *Let* Σ *be an absolutely regular nonautonomous system, let* $p\in(1,\infty)$*, let* $\Delta(\cdot)$ *be an admissible feedback for* Σ*, and let* Σ^Δ *be the corresponding feedback system. Then* Σ *possesses one of the properties in Definition* 5.2 *if and only if* Σ^Δ *has the same property.*

Proof. (1) The assertions concerning exact (approximate) controllability and (continuous) initial observability follow from the formulas

$$\begin{aligned}\Phi^\Delta(t,s) &= \Phi(t,s)(I+\Delta(\cdot)\mathbb{F}_s^\Delta), &\qquad \Phi(t,s) &= \Phi^\Delta(t,s)(I-\Delta(\cdot)\mathbb{F}_s),\\ \Psi^\Delta(t,s) &= (I+\mathbb{F}_s^\Delta\Delta(\cdot))\Psi(t,s), &\qquad \Psi(t,s) &= (I-\mathbb{F}_s\Delta(\cdot))\Psi^\Delta(t,s),\end{aligned}$$

which are immediate consequences of (5.2).

(2) Assume that Σ is null controllable. For $x\in X$, there is $u\in L^p([s,t],U)$ such that $T(t,s)x=\Phi_{t,s}u$. Thus (5.2) yields $T_\Delta(t,s)x=\Phi^\Delta(t,s)[u-\Delta(\cdot)\mathbb{F}_s u + \Delta(\cdot)\Psi_s x]$, and Σ^Δ is null controllable. The converse implication and the equivalence for approximate null controllability are shown in the same way.

(3) Assume that Σ is continuously finally observable. Using (5.2), we estimate

$$\begin{aligned}\|T^\Delta(t,s)x\| &\le \|T(t,s)x\| + \|\Phi(t,s)\Delta(\cdot)\Psi^\Delta(t,s)x\|\\ &\le c\,\|\Psi(t,s)x\|_p + c_1\,\|\Psi^\Delta(t,s)x\|_p\\ &\le (c+c_1)\,\|\Psi^\Delta(t,s)x\|_p + \|\mathbb{F}_s\Delta(\cdot)\Psi^\Delta(t,s)x\|_p \le c_2\,\|\Psi^\Delta(t,s)x\|_p\end{aligned} \tag{5.3}$$

so that Σ^Δ is continuously finally observable. If Σ is finally observable and $\Psi^\Delta(t,s)x=0$, then $\Psi(t,s)x=-\mathbb{F}_s\Delta(\cdot)\Psi^\Delta(t,s)x=0$. Hence $T(t,s)x=0$, and (5.3) yields $T^\Delta(t,s)x=0$. The converse implications are proved similarly. $\square$

Theorem 4.4 also guarantees that repeated feedbacks behave nicely.

PROPOSITION 5.4. *Let* Σ *be an absolutely regular nonautonomous system with* $p\in(1,\infty)$*, let* $\Delta(\cdot)$ *be an admissible feedback for* Σ*, let* Σ^Δ *be the corresponding feedback system, and let* $\tilde\Delta(\cdot)\in L^\infty(\mathbb{R}_+,\mathcal{L}_s(Y,U))$*. Then* $\tilde\Delta(\cdot)$ *is admissible for* Σ^Δ *if and only if* $\Delta(\cdot)+\tilde\Delta(\cdot)$ *is admissible for* Σ*. If this is the case, then* $\Sigma^{\Delta+\tilde\Delta}=(\Sigma^\Delta)^{\tilde\Delta}$*.*

Proof. Proposition 5.1 implies that

$$\begin{aligned}&\mathbb{F}_s\,[I-\tilde\Delta(\cdot)\mathbb{F}_s^\Delta] = [I-\mathbb{F}_s(\Delta(\cdot)+\tilde\Delta(\cdot))]\,\mathbb{F}_s^\Delta \quad\text{and}\\ &[I-\mathbb{F}_s^\Delta\tilde\Delta(\cdot)]\,\mathbb{F}_s = \mathbb{F}_s^\Delta\,[I-(\Delta(\cdot)+\tilde\Delta(\cdot))\mathbb{F}_s].\end{aligned} \tag{5.4}$$

Assume that $\Delta(\cdot) + \tilde{\Delta}(\cdot)$ is admissible for Σ. We then deduce from (5.4) and (5.1) that

$$\mathbb{F}_s^{\Delta+\tilde{\Delta}} - \mathbb{F}_s^{\Delta} = \mathbb{F}_s^{\Delta+\tilde{\Delta}}\tilde{\Delta}(\cdot)\mathbb{F}_s^{\Delta} = \mathbb{F}_s^{\Delta}\tilde{\Delta}(\cdot)\mathbb{F}_s^{\Delta+\tilde{\Delta}}$$

so that $\tilde{\Delta}(\cdot)$ is admissible for Σ^{Δ} by Lemma 4.3. The converse implication is proved in the same way. The second claim follows similarly from (5.2) and Lemma 4.3. ∎

We introduce a basic asymptotic property of evolution equations; see, e.g., [4], [9].

Definition 5.5. *An evolution family T has an* exponential dichotomy *(or is called* hyperbolic*) if there are projections $P(t)$, $t \geq 0$, and constants $N, \delta > 0$ such that $P(\cdot) \in C_b(\mathbb{R}_+, \mathcal{L}_s(X))$ and, for $t \geq s \geq 0$ and $Q(t) = I - P(t)$,*

1. *$T(t,s)P(s) = P(t)T(t,s)$,*
2. *the restriction $T_Q(t,s) : Q(s)X \to Q(t)X$ of $T(t,s)$ has the inverse $T_Q(s,t)$,*
3. *$\|T(t,s)P(s)\| \leq Ne^{-\delta(t-s)}$, and $\|T_Q(s,t)Q(t)\| \leq Ne^{-\delta(t-s)}$.*

If $P(t) \equiv I$, then T is called exponentially stable.

Persistence of dichotomy under perturbations mapping from spaces $\underline{X}_t$ into X has been studied intensively; see [4, section 5.2], [22, section 5], and the references given there. The next result also holds for admissible systems; see [23, Thm. 4.23].

Theorem 5.6. *Assume that $(T, \Phi, \Psi, \mathbb{F})$ is a regular nonautonomous system and that $\Delta(\cdot) \in L^\infty(\mathbb{R}_+, \mathcal{L}_s(Y,U))$ is Σ-admissible with $k(t_0) := \sup_s \|(I - \mathbb{F}(s + t_0, s)\Delta(\cdot))^{-1}\|$. Suppose that T has an exponential dichotomy with constants $N, \delta > 0$ and projections $P(t)$. Then there is a number $\varepsilon_0 = \varepsilon_0(N, \delta, t_0) > 0$ such that*

$$k(t_0)\beta(t_0)\gamma(t_0)\,\|\Delta(\cdot)\|_\infty \leq \varepsilon_0$$

implies that T_Δ is hyperbolic with projections having the same rank as $P(t)$ and $Q(t)$.

Proof. We extend the evolution families T and T_Δ to the time interval $\mathbb{R}$ by setting $T(t,s) = T_\Delta(t,s) = \exp[(t-s)\delta(Q(0) - P(0))]$ for $0 \geq t \geq s$ and $T_{(\Delta)}(t,s) = T_{(\Delta)}(t,0)\exp[-s\delta(Q(0) - P(0))]$. Observe that we can take $k(t_1) = k(t_0)$ for $0 < t_1 \leq t_0$ by the proof of Lemma 4.2 and that $\exp[t\delta(Q(0) - P(0))] = e^{-\delta t}P(0) + e^{\delta t}Q(0)$. Therefore, (4.8) yields

$$\|T_\Delta(s + t_0, s) - T(s + t_0, s)\| \leq (1 + e^{\delta t_0})Nk(t_0)\beta(t_0)\gamma(t_0)\,\|\Delta(\cdot)\|_\infty.$$

The assertion then follows from [24, Prop. 2.3] (see also [4, Thm. 5.23] and the references therein), where $\varepsilon_0 := (1 - e^{\delta t_0})^2\,((1 + e^{\delta t_0})8N^3)^{-1}$. ∎

We finally characterize the exponential stability of T from the perspective of control theory using the following notions; cf. [5], [6], [17], [20], [25, section 8.2].

Definition 5.7. *A nonautonomous control system (T, Φ) is called* stabilizable *if there exists an observation system (T_F, Ψ^F) with an exponentially stable evolution family T_F on X such that $T_F(t,s)x = T(t,s)x + \Phi_{t,s}\Psi_s^F x$ for all $x \in X$ and $t \geq s \geq 0$.*

Definition 5.8. *A nonautonomous observation system (T, Ψ) is called* detectable *if there is a control system (T_K, Φ^K) with an exponentially stable evolution family T_K on X such that $T_K(t,s)x = T(t,s)x + \Phi_{t,s}^K\Psi_s x$ for all $x \in X$ and $t \geq s \geq 0$.*

The following theorem relates the exponential stability of T, i.e., *internal stability*, with the boundedness of $\mathbb{F} : L^p(\mathbb{R}_+, U) \to L^p(\mathbb{R}_+, Y)$, the so-called *input–output stability*. Versions of Theorem 5.9 for the autonomous Hilbert space setting are proved in [6, Thm. 5.8] for the Pritchard–Salamon class, in [20, Cor. 1.8] for regular systems, and in [17, Thm. 5.2] and [33, Thm. 5.3] for well-posed systems. In that case, the input–output stability can be replaced by the equivalent condition that the transfer function $H(\lambda) = CR(\lambda, A_{-1})B$ is bounded for $\operatorname{Re}\lambda > 0$; cf. [31, Thm. 3.1]. In [5, Thm. 4.3], our theorem was shown for bounded control and observation operators using the characterization of exponential stability given in [4, Thm. 3.26]. Here we employ Datko's theorem [8, Thm. 1, Rem. 3] in order to avoid some technical problems. However, we remark that Datko's theorem can be deduced from [4, Thm. 3.26]; see [23, Thm. 1.19]. A variant of the next result holds for admissible systems [23, Thm. 4.29].

THEOREM 5.9. *Let $\Sigma = (T, \Phi, \Psi, \mathbb{F})$ be a regular nonautonomous system. Then the following assertions are equivalent:*

1. *T is exponentially stable.*
2. *(T, Φ) is stabilizable, and $\Phi(\cdot, 0) \in \mathcal{L}(L^p(\mathbb{R}_+, U), L^p(\mathbb{R}_+, X))$.*
3. *(T, Ψ) is detectable, and $\|\Psi_s x\|_{L^p([s,\infty),Y)} \le c\,\|x\|$ for $s \ge 0$ and $x \in X$.*
4. *Σ is detectable and stabilizable, and $\mathbb{F} \in \mathcal{L}(L^p(\mathbb{R}_+, U), L^p(\mathbb{R}_+, Y))$.*

Proof. Let 1 hold. Then Σ is always stabilizable (take $\Psi^F = 0$) and detectable (take $\Phi^K = 0$). The other assertions in 2–4 follow from Lemmas 2.3, 3.2, and 3.7. Extending $u \in L^p([s,\infty), U)$ by 0 to $\mathbb{R}_+$ and using causality, we see that the norms of $\Phi(\cdot, s)$ and $\mathbb{F}_s$ decrease as s increases. The assumptions in 2 and Lemma 2.3 yield $\|T(\cdot, s)x\|_{L^p([s,\infty),X)} \le c\,\|x\|_X$ for $s \ge 0$, $x \in X$, and a constant c. Thus 1 is a consequence of Datko's theorem [8, Thm. 1, Rem. 3]. The implication "3⇒1" can be proved in the same way. If Σ is stabilizable, then Theorems 2.7 and 3.11 show that the operators $C(t)$ representing Ψ are also T_F-admissible, and $\Psi_s x = C(\cdot)T_F(\cdot, s)x - \mathbb{F}_s \Psi_s^F x$ for $s \ge 0$ and $x \in X$. Hence 4 implies 1 by Lemmas 2.3 and 2.5. ∎

6. A parabolic problem with point control and observation. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^2-boundary $\partial\Omega$, and let $a_{kl}, a_k, a_0 : \mathbb{R}_+ \times \overline{\Omega} \to \mathbb{R}$, $k, l = 1, \dots, n$ be bounded and uniformly Hölder continuous such that $\sum_{kl} a_{kl}(t,\xi)v_k v_l \ge \alpha\,|v|^2$ for a constant $\alpha > 0$ and $v \in \mathbb{R}^n$, $t \ge 0$, $\xi \in \overline{\Omega}$. Further, let $b, c : \mathbb{R}_+ \to \Omega$ be uniformly Lipschitz such that $|b(t) - c(t)| \ge \delta > 0$ for $t \ge 0$. Let $\varphi \in C_0(\Omega)$, $s \ge 0$, and $D_k = \frac{\partial}{\partial \xi_k}$. The unique solution $w \in C([s,\infty) \times \overline{\Omega}) \cap C^{1,2}((s,\infty) \times \Omega)$ of the problem

$$\begin{aligned} w_t(t,\xi) &= \sum\nolimits_{kl} a_{kl}(t,\xi) D_k D_l w(t,\xi) + \sum\nolimits_{k} a_k(t,\xi) D_k w(t,\xi) \\ &\qquad + a_0(t,\xi) w(t,\xi), \quad t > s, \\ w(t,\xi) &= 0, \quad \xi \in \partial\Omega,\ t \ge s, \qquad w(s,\xi) = \varphi(\xi), \quad \xi \in \Omega, \end{aligned} \tag{6.1}$$

is given by $w(t,\xi) = \int_\Omega k(t,s,\xi,\eta)\varphi(\eta)\,d\eta$ for a continuous kernel $k(t,s,\xi,\eta)$, $t > s \ge 0$, $\xi \in \overline{\Omega}$, $\eta \in \Omega$, satisfying the Gaussian estimate

$$|k(t,s,\xi,\eta)| \le M(t-s)^{-\frac{n}{2}} \exp\left(-\frac{w|\xi - \eta|^2}{t-s} + \tilde{w}(t-s) \right)$$

for $0 < t - s \le t_0$ and constants $M, w > 0$ and $\tilde{w} \in \mathbb{R}$; see, e.g., [15, section IV.16]. By the uniqueness of solutions, we also have

$$k(t,s,\xi,\eta) = \int_\Omega k(t,r,\xi,\zeta) k(r,s,\zeta,\eta)\,d\zeta, \quad t > r > s \ge 0,\ \xi, \eta \in \Omega. \tag{6.2}$$

We take $p,q\in[1,\infty)$ and set $X=L^q(\Omega)$, $U=Y=\mathbb{C}$, $T(s,s)=I$, $\Phi(s,s)u=0$, and

$$T(t,s)\varphi=\int_\Omega k(t,s,\cdot,\eta)\varphi(\eta)\,d\eta,\qquad \Phi_{t,s}u=\int_s^t k(t,\tau,\cdot,b(\tau))u(\tau)\,d\tau,$$
$$(\Psi_s\varphi)(t)=\int_\Omega k(t,s,c(t),\eta)\varphi(\eta)\,d\eta,\qquad (\mathbb{F}_s u)(t)=\int_s^t k(t,\tau,c(t),b(\tau))u(\tau)\,d\tau$$

for $t>s\geq 0$, $\varphi\in X$, and $u\in L^p_{loc}(\mathbb{R}_+)$. These maps correspond to the PDE (6.1) complemented by the control $B(t)u(t)=\delta_{b(t)}u(t)$ and the output $y(t)=w(t,c(t))$.

The operators $T(t,s)$ yield an evolution family on X due to standard elliptic regularity and, e.g., [2, Thm. II.4.4.1]. Since b is Lipschitz, we have

$$\exp\left(-w\,\frac{|\xi-b(s)|^2}{t-s}\right)\leq c_1\,\exp\left(-w\,\frac{|\xi-b(t)|^2}{t-s}\right) \tag{6.3}$$

for $t>s\geq 0$, $\xi\in\Omega$, and a constant c_1. Hence $|(\mathbb{F}_s u)(t)|\leq \varphi * |u|\,(t)$, $s\leq t\leq s+t_0$, where we have extended $u\in L^p_{loc}([s,\infty))$ by 0 and put $\varphi(t)=c_1\,t^{-\frac{n}{2}}\,\exp(-\frac{w\delta^2}{t}+\tilde{w}t)$ for $t>0$ and $\varphi(t)=0$ otherwise. For $1+\frac{1}{p}=\frac{1}{\lambda}+\frac{1}{\mu}$, Young's inequality yields

$$\|\mathbb{F}_s u\|_{L^p[s,s+t_0]}\leq \|\varphi\|_{L^\lambda[0,t_0]}\,\|u\|_{L^\mu[s,s+t_0]} \tag{6.4}$$

so that (3.11) holds for each $p\in[1,\infty]$. Observe that $\Psi_s\varphi$ is continuous on (s,∞) for each $\varphi\in L^1(\Omega)$ and that $t\mapsto\Phi(t,s)u\in L^1(\Omega)$ is continuous on $[s,\infty)$ for $u\in L^1_{loc}([s,\infty))$. Moreover, (6.2) implies (2.1), (3.1), and (3.10). Using (6.3), Hölder's inequality, and that the norm of the Gaussian kernel in $L^{q'}(\mathbb{R}^n)$ equals $ct^{-n/2q}$, we compute

$$\int_s^t|\Psi_s\varphi(\tau)|^p\,d\tau\leq c\,\|\varphi\|_q^p\int_s^t(\tau-s)^{-\frac{np}{2q}}\,d\tau, \tag{6.5}$$

$$\|\Phi_{t,s}u\|_q\leq c\int_s^t(t-\tau)^{-\frac{n}{2q'}}\,|u(\tau)|\,d\tau \tag{6.6}$$

for $0<t-s\leq t_0$, $1/q+1/q'=1$, and constants c. Thus the operators

$$\Psi_s: L^q(\Omega)\to L^p_{loc}([s,\infty)),\;\; q>\frac{np}{2},\quad \Phi_{t,s}: L^p_{loc}([s,t])\to L^q(\Omega),\;\; q'>\frac{np'}{2}, \tag{6.7}$$

are continuous. As a result, $(T,\Phi,\Psi,\mathbb{F})$ is a well-posed nonautonomous system provided that $n=1$, $q>p/2$, and $q'>p'/2$ (for instance, if $p=q=2$). In view of (6.4), this system is absolutely regular, and every bounded feedback is admissible.

The restriction $n=1$ was needed only to obtain the boundedness of Ψ_s and $\Phi_{t,s}$ for the same values of p and q. We now discuss to what extent the assertions of Theorem 4.4 remain valid for $n=2,3$. All results dealing with Ψ and Φ *separately* are true for the exponents indicated in (6.7). Observe that $\mathbb{F}_s$ satisfies (3.11), (3.13), and the assertions of Lemma 4.2 for all $p\geq 1$ and that every bounded feedback is admissible. Moreover, the proof and assertion of Theorem 3.11 work as before. Proposition 3.12 holds for $n=2$, $u\in L^r_{loc}(\mathbb{R}_+)$ with $r>1$, $X=L^q(\Omega)$ with $q'>r'$, and $p<q$ in the assertions. In fact, we have $B_n u\in L^\infty_{loc}(\mathbb{R}_+,L^q(\Omega))$ for $q'>r'$. So we can apply Proposition 2.11 for $p<q$ to obtain (3.21) and then proceed as before.

We now consider Theorem 4.4, where we restrict ourselves to the case where the state space X equals $L^2(\Omega)$ and the given system Σ is admissible with exponent 2.

We define $T_\Delta(t,s)\varphi$ for $\varphi \in L^q(\Omega)$ with $q > 2n/(4-n)$ as in (4.8). Then $\Psi_s\varphi \in L^p_{loc}([s,\infty))$ for all $p \in (4/(4-n), 2q/n)$, and $T_\Delta(t,s) : L^q(\Omega) \to L^2(\Omega)$ is bounded for $0 \le t-s \le t_0$ due to (6.7). Because of (6.4) with $p = \infty$ and $\mu < 4/n$ and (6.7), we have

$$\|\mathbb{F}_s\Delta(\cdot)(I - \mathbb{F}(s+t_1,s)\Delta(\cdot))^{-1}\Psi_s x\|_{L^\infty[s,s+t_0]} \le c\,\|\Psi_s\varphi\|_{L^\mu[s,s+t_0]} \le c\,\|\varphi\|_2\,.$$

Thus (4.9) and (6.5) yield

$$|C(t)T_\Delta(t,s)\varphi| \le c\,(t-s)^{-\frac{n}{4}}\,\|\varphi\|_2 \tag{6.8}$$

for $0 < t-s \le t_0$. The identity (4.5) (with $\varphi \in L^q(\Omega)$) follows as before. Using (4.5), (6.6), and (6.8), we further estimate $\|T_\Delta(t,s)\varphi\|_2 \le c\,(t-s)^{1-\frac{n}{2}}\,\|\varphi\|_2$. Therefore, we can extend $T_\Delta(t,s)$ and (4.5) to $L^2(\Omega)$. We can now argue as in the proof of Theorem 4.4 and deduce part (a) of the theorem if we replace (4.4) by (6.8), allow for a blow-up of $T_\Delta(t,s)$ as $t \to s$ if $n = 3$, and consider solutions $x(\cdot)$ of (4.1) such that $x(\cdot) \in C([s,\infty), L^1(\Omega))$ and $C(\cdot)x(\cdot) \in L^1_{loc}([s,\infty))$.

Now let $n = 2$. In part (b), we restrict ourselves in (4.11) to cut-off functions γ with compact support in (r,∞). (One checks as in [4, Thm. 3.12] that the set of resulting functions f is still dense in $L^p_{loc}([s,\infty), X)$. Here we need the strong continuity of $T_\Delta(t,s)$ at $t = s$ and must thus exclude $n = 3$.) We proceed as before and deduce (4.15) and (4.16) for $u \in L^p_{loc}([s,\infty))$ with $p > 2$. We can take the limits as $n \to \infty$ in $C([s,s+t_1], L^2(\Omega))$ and $L^2[s,s+t_1]$, respectively, and obtain $\Phi^\Delta u$ and $\mathbb{F}^\Delta u$. Moreover, $\Phi^\Delta u$ satisfies an estimate like (6.6). We can thus apply $\Phi^\Delta_{t,s}$ on $u = \Delta(\cdot)\Psi_s\varphi$ for $\varphi \in L^2(\Omega)$ so that (4.7) holds. The other assertions of Theorem 4.4(b) can be verified as before except that Ψ^Δ and Φ^Δ have the mapping properties from (6.7) with $q = 2$.

Acknowledgments. I started this project together with Yuri Latushkin and Timothy Randolph from the University of Missouri, and I profited greatly from many joint discussions.

REFERENCES

[1] P. Acquistapace and B. Terreni, *Classical solutions of nonautonomous Riccati equations arising in parabolic boundary control problems*, Appl. Math. Optim., 39 (1999), pp. 361–409.

[2] H. Amann, *Linear and Quasilinear Parabolic Problems. Volume* 1: *Abstract Linear Theory*, Birkhäuser Boston, Boston, 1995.

[3] A. Bensoussan, G. Da Prato, M.C. Delfour, and S.K. Mitter, *Representation and Control of Infinite-Dimensional Systems, Volume* I, Birkhäuser Boston, Boston, 1992.

[4] C. Chicone and Y. Latushkin, *Evolution Semigroups in Dynamical Systems and Differential Equations*, AMS, Providence, RI, 1999.

[5] S. Clark, Y. Latushkin, S. Montgomery-Smith, and T. Randolph, *Stability radius and internal versus external stability in Banach spaces: An evolution semigroup approach*, SIAM J. Control Optim., 38 (2000), pp. 1757–1793.

[6] R. Curtain, H. Logemann, S. Townley, and H. Zwart, *Well-posedness, stabilizability, and admissibility for Pritchard–Salamon systems*, J. Math. Systems Estim. Control, 7 (1997), pp. 439–476.

[7] R. F. Curtain and A. J. Pritchard, *Infinite Dimensional Linear System Theory*, Springer-Verlag, New York, 1978.

[8] R. Datko, *Uniform asymptotic stability of evolutionary processes in a Banach space*, SIAM J. Math. Anal., 3 (1972), pp. 428–445.

[9] K. J. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, Springer-Verlag, New York, 2000.

[10] D. Hinrichsen and A. J. Pritchard, *Robust stability of linear evolution operators on Banach spaces*, SIAM J. Control Optim., 32 (1994), pp. 1503–1541.

[11] A. Idrissi and A. Rhandi, *Admissibility of time-varying observations for non-autonomous systems*, J. Comput. Anal. Appl., to appear.

[12] B. Jacob, *Time-Varying Infinite Dimensional State-Space Systems*, Ph.D. thesis, Fachbereich Mathematik/Informatik, Universität Bremen, Bremen, 1995.

[13] B. Jacob, *Optimal control of time-varying well-posed linear systems on a finite time horizon*, in Mathematical Theory of Networks and Systems, Proceedings of the MTNS–98, A. Beghi, I. Finesso, and G. Picci, eds., Il Polografo, Padova, Italy, 1998, pp. 483–486.

[14] B. Jacob, V. Dragan, and A. J. Pritchard, *Robust stability of infinite dimensional time-varying systems with respect to nonlinear perturbations*, Integral Equations Operator Theory, 22 (1995), pp. 440–462.

[15] O. A. Ladyzenskaja, V. A. Solonnikov, and N. N. Ural'ceva, *Linear and Quasilinear Equations of Parabolic Type*, AMS, Providence, RI, 1968.

[16] J.-L. Lions, *Optimal Control of Systems Governed by Partial Differential Equations*, Springer-Verlag, New York, 1971.

[17] K. A. Morris, *Justification of input–output methods for systems with unbounded control and observation*, IEEE Trans. Automat. Control, 44 (1999), pp. 81–85.

[18] A. J. Pritchard and D. Salamon, *The linear quadratic control problem for infinite-dimensional systems with unbounded input and output operators*, SIAM J. Control Optim., 25 (1987), pp. 121–144.

[19] F. Räbiger, A. Rhandi, R. Schnaubelt, and J. Voigt, *Non-autonomous Miyadera perturbations*, Differential Integral Equations, 13 (1999), pp. 341–368.

[20] R. Rebarber, *Conditions for the equivalence of internal and external stability for distributed parameter systems*, IEEE Trans. Automat. Control, 38 (1993), pp. 994–998.

[21] D. Salamon, *Infinite-dimensional linear systems with unbounded control and observation: A functional analytic approach*, Trans. Amer. Math. Soc., 300 (1987), pp. 383–431.

[22] R. Schnaubelt, *Sufficient conditions for exponential stability and dichotomy of evolution equations*, Forum Math., 11 (1999), pp. 543–566.

[23] R. Schnaubelt, *Exponential Dichotomy of Non-Autonomous Evolution Equations*, Habilitation thesis, Mathematische Fakultät, Universität Tübingen, Tübingen, 1999.

[24] R. Schnaubelt, *Asymptotically autonomous parabolic evolution equations*, J. Evol. Equ., 1 (2001), pp. 19–37.

[25] O. J. Staffans, *Well-Posed Linear Systems Part* I: *General Theory*, book manuscript dated July 9, 2001; available online from http://www.abo.fi/˜staffans.

[26] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, NJ, 1970.

[27] J. Voigt, *On the perturbation theory for strongly continuous semigroups*, Math. Ann., 229 (1977), pp. 163–171.

[28] G. Weiss, *Admissible observation operators for linear semigroups*, Israel J. Math., 65 (1989), pp. 17–43.

[29] G. Weiss, *Admissibility of unbounded control operators*, SIAM J. Control Optim., 27 (1989), pp. 527–545.

[30] G. Weiss, *The representation of regular linear systems on Hilbert spaces*, in Control and Estimation of Distributed Parameter Systems (Vorau, 1988), F. Kappel, K. Kunisch, and W. Schappacher, eds., Birkhäuser Verlag, Basel, 1989, pp. 401–416.

[31] G. Weiss, *Transfer functions of regular linear systems* I: *Characterization of regularity*, Trans. Amer. Math. Soc., 342 (1994), pp. 827–854.

[32] G. Weiss, *Regular linear systems with feedback*, Math. Control Signals Systems, 7 (1994), pp. 23–57.

[33] G. Weiss and R. Rebarber, *Optimizability and estimatability for infinite-dimensional linear systems*, SIAM J. Control Optim., 39 (2000), pp. 1204–1232.