

**EXPONENTIAL STABILITY, EXPONENTIAL EXPANSIVENESS,
AND EXPONENTIAL DICHOTOMY
OF EVOLUTION EQUATIONS ON THE HALF-LINE**

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Let $\mathcal{U} = (U(t, s))_{t \geq s \geq 0}$ be an evolution family on the half-line of bounded linear operators on a Banach space X . We introduce operators G_0 , G_X and I_X on certain spaces of X -valued continuous functions connected with the integral equation $u(t) = U(t, s)u(s) + \int_s^t U(t, \xi)f(\xi)d\xi$, and we characterize exponential stability, exponential expansiveness and exponential dichotomy of \mathcal{U} by properties of G_0 , G_X and I_X , respectively. This extends related results known for finite dimensional spaces and for evolution families on the whole line, respectively.

INTRODUCTION

Consider the non-autonomous linear evolution equation

$$\frac{d}{dt}u(t) = A(t)u(t), \quad t \in J = R_+ \text{ or } R, \quad (NCP)$$

on a Banach space X . If the operators $A(t)$, $t \in J$, are bounded (in particular, if X is finite dimensional) there is an extensive literature initiated by the work of Perron which connects asymptotic properties (of the solutions) of (NCP) with specific properties of the operator L defined by

$$Lu(t) = \frac{d}{dt}u(t) - A(t)u(t), \quad t \in J,$$

on a space of X -valued functions. For example, the existence of an exponential dichotomy of (NCP) is related to the invertibility of L if $J = R$ and to Fredholm properties of L if $J = R_+$ (see [BeG], [BGK], [Cop], [DaK], [MaS], [Pal] and the references therein). If the

* This work was done while the first author was visiting the Department of Mathematics of the University of Tübingen. The support of the Alexander von Humboldt Foundation is gratefully acknowledged. The author wishes to thank R. Nagel and the Functional Analysis group in Tübingen for their kind hospitality and constant encouragement.

** Support by Deutsche Forschungsgemeinschaft DFG is gratefully acknowledged.

operators $A(t)$, $t \in J$, are unbounded the situation becomes more delicate and usually one has to impose rather restrictive conditions in order to obtain similar results referring to the operator L (e.g. see [Hen], [Zha]).

A less restrictive assumption is the well-posedness of (NCP) in the sense that the solutions of (NCP) yield an evolution family $\mathcal{U} = (U(t, s))_{t \geq s, t, s \in J}$ of bounded linear operators on X (see [Nic], [Paz], [RRS], [RS2], [Sch] and [Tan] for sufficient conditions and examples). Note that solution does not necessarily mean ‘classical’ solution. Then instead of the operator L one investigates the integral equation

$$u(t) = U(t, s)u(s) + \int_s^t U(t, \xi)f(\xi)d\xi, \quad t \geq s, \quad t, s \in J, \quad (IE)$$

and its connection with asymptotic properties of the evolution family \mathcal{U} (see [Bus], [DaK], [Dat], [LRS], [LeZ], [Zhi]). For instance, if $J = R$ then it is shown in [LRS] that \mathcal{U} has an exponential dichotomy if and only if for every $f \in C^b(R, X)$ there is a unique solution $u \in C^b(R, X)$ of (IE) .

Another approach uses the so-called evolution semigroup $\mathcal{T} = (T(t))_{t \geq 0}$ on a space of X -valued functions induced by the evolution family \mathcal{U} (see [AuM], [LaM], [LaR], [LMR1], [LMR2], [Mi1], [Mi2], [Nic], [RRS], [RS1], [RS2], [Rau], [Sch] and the references therein for this concept). Roughly speaking (if $J = R$) the evolution semigroup \mathcal{T} is given by

$$T(t)f(\xi) = U(\xi, \xi - t)f(\xi - t), \quad t \geq 0,$$

(see (1.1) for the exact definition in the present context) and in certain situations its generator G is an extension of $-L = -\frac{d}{dt} + A(\cdot)$ (see [LMR1, Prop. 2.9], [Sch, Prop. 1.13]). It turns out that in the spirit of Perron’s observations asymptotic properties of \mathcal{U} like exponential stability or exponential dichotomy can be described by spectral properties of the operator G (see [AuM], [LaM], [LaR], [LMR1], [LMR2], [Mi1], [Mi2], [RS1], [Rau], [Sch]). However, we have to point out that most results in this direction are restricted to the line case $J = R$.

In the present paper we characterize exponential stability, exponential expansiveness and exponential dichotomy of an evolution family \mathcal{U} on the half-line $J = R_+$. Our approach is based on the use of (generators of) evolution semigroups and their connection with the integral equation (IE) (see Section 1). Exponential stability and exponential expansiveness of \mathcal{U} is characterized in Theorem 2.2 and Theorem 2.5, respectively. Section 3 deals with exponential stability of individual orbits. Our main interest, however, is directed to the exponential dichotomy of \mathcal{U} which will be characterized in Theorem 4.3 and Theorem 4.5. Concerning the relevance of exponential dichotomy and its far-reaching applications we refer to [SaS] and the references therein.

1. PRELIMINARIES

Recall that a family of operators $\mathcal{U} = (U(t, s))_{t \geq s \geq 0}$ of bounded linear operators on a Banach space X is a (*strongly continuous, exponentially bounded*) *evolution family* on the half-line if

1. $U(t, t) = Id$ and $U(t, r)U(r, s) = U(t, s)$ for $t \geq r \geq s \geq 0$,
2. $(t, s) \mapsto U(t, s)x$ is continuous for every $x \in X$,
3. there are constants $K \geq 0$, $\alpha \in \mathbb{R}$ such that $\|U(t, s)\| \leq Ke^{\alpha(t-s)}$ for $t \geq s \geq 0$.

Then $\omega(\mathcal{U}) := \inf\{\alpha \in \mathbb{R} : \text{there is } K \geq 0 \text{ such that } \|U(t, s)\| \leq Ke^{\alpha(t-s)}, t \geq s \geq 0\}$ is called the *growth bound* of \mathcal{U} .

Throughout the whole paper the following function spaces (endowed with the sup-norm) play an important role.

$$\begin{aligned} C_0 &:= \{v : [0, \infty) \rightarrow X : v \text{ is continuous, } v(0) = 0 = \lim_{t \rightarrow \infty} v(t)\}, \\ C_X &:= \{v : [0, \infty) \rightarrow X : v \text{ is continuous, } \lim_{t \rightarrow \infty} v(t) = 0\}, \\ C_X(t_0) &:= \{v|_{[t_0, \infty)} : v \in C_X\}, \quad t_0 \geq 0, \\ C^b &:= \{v : [0, \infty) \rightarrow X : v \text{ is continuous and bounded}\}. \end{aligned}$$

An evolution family \mathcal{U} on X always defines an *evolution semigroup* $\mathcal{T} = (T(t))_{t \geq 0}$ of bounded linear operators on C_0 and C_X by setting

$$[T(t)v](s) = \begin{cases} U(s, s-t)v(s-t), & s \geq t, \\ U(s, 0)v(0), & 0 \leq s \leq t, \end{cases} \quad (1.1)$$

for v in C_0 and C_X , respectively. It can be easily seen that \mathcal{T} is strongly continuous. We denote the infinitesimal generator of \mathcal{T} on C_0 and C_X by G_0 and G_X , respectively. Note that $D(G_0) = D(G_X) \cap C_0$ and $G_0v = G_Xv$ for $v \in D(G_0)$, i.e. G_0 is the part of G_X in C_0 .

The aim of this paper is to characterize asymptotic properties of a given evolution family \mathcal{U} by (spectral) properties of the generators G_0 and G_X . The following lemma is the key tool in our strategy. It connects the operators G_0 and G_X with the following inhomogeneous integral equation

$$u(t) = U(t, s)u(s) + \int_s^t U(t, \xi)f(\xi)d\xi, \quad t \geq s \geq 0. \quad (1.2)$$

1.1 LEMMA. *a) Let $u, f \in C_0$. Then $u \in D(G_0)$ and $G_0u = -f$ if and only if*

$$u(t) = \int_0^t U(t, \xi)f(\xi)d\xi, \quad t \geq 0. \quad (1.3)$$

b) Let $u \in C_X$ and $f \in C_0$. Then $u \in D(G_X)$ and $G_X u = -f$ if and only if (1.2) holds.

PROOF. We only prove the first assertion. The proof of the second one follows the same lines.

Let $G_0 u = -f$. The general theory of linear semigroups (see e.g. [Paz, p.4-5]) yields

$$T(t)u - u = \int_0^t T(\xi)G_0 u d\xi = - \int_0^t T(\xi)f d\xi \text{ for } t \geq 0.$$

Thus,

$$u = T(t)u + \int_0^t T(\xi)f d\xi.$$

From the definition of T we easily obtain that u is a solution of the integral equation (1.2) and has the form (1.3).

Conversely, if $u, f \in C_0$ and u satisfies (1.3), then by reversing the above argument we obtain

$$T(t)u - u = - \int_0^t T(\xi)f d\xi \text{ for } t \geq 0.$$

In particular, this implies $u \in D(G_0)$ and $G_0 u = -f$.

In the following remark we collect some additional properties of the operators G_0 and G_X . For sake of convenience we set $U(t, s) = 0$ for $0 \leq t < s$.

1.2 REMARKS. a) From Lemma 1.1 we immediately obtain that G_0 is injective and that $\ker G_X = \{u \in C_X : u(t) = U(t, 0)u(0), t \geq 0\}$.

b) The range $R(G_X)$ of G_X is always contained in C_0 (in particular G_X is never invertible). In fact, if $u \in D(G_X)$ then

$$[G_X u](0) = \lim_{t \downarrow 0} \frac{T(t)u(0) - u(0)}{t} = \lim_{t \downarrow 0} \frac{u(0) - u(0)}{t} = 0.$$

c) Let $t_0 \geq 0$, $x \in X$ and $\varphi: [0, \infty) \rightarrow R$ be continuously differentiable such that $\varphi|_{[0, t_0)} = 0$. Set $u(t) = \varphi(t)U(t, t_0)x$, $t \geq 0$, and $f(t) = \varphi'(t)U(t, t_0)x$, $t \geq 0$. If $u \in C_X$ and $f \in C_0$ then an immediate application of Lemma 1.1 yields $u \in D(G_X)$ and $G_X u = -f$.

Next we define an operator I_X on C_X connected with the integral equation (1.2). If $u, f \in C_X$ satisfy (1.2) we set

$$I_X u := f,$$

$$D(I_X) := \{u \in C_X : \text{there is } f \in C_X \text{ such that } u \text{ and } f \text{ satisfy (1.2)}\}.$$

1.3 LEMMA. $(I_X, D(I_X))$ is a well-defined closed linear operator on C_X and an extension of $(-G_X, D(G_X))$.

PROOF. Let $I_X u = f$ and $I_X u = g$ for $u, f, g \in C_X$. Then

$$\int_s^t U(t, \xi)(f(\xi) - g(\xi))d\xi = 0 \text{ for } t \geq s \geq 0.$$

By the continuity of the integrand we obtain $0 = U(t, t)(f(t) - g(t)) = f(t) - g(t)$, $t \geq 0$, i.e. $f = g$.

The closedness of I_X follows immediately from the definition of I_X by the integral equation (1.2). Finally, Lemma 1.1 shows that I_X is an extension of $-G_X$.

1.4 REMARKS. a) In general, $-I_X$ is a proper extension of G_X , in particular, I_X can be surjective (cf. Theorem 4.3). Moreover, the spectra of $-I_X$ and G_X are different. In fact, for $\lambda \in C$ with $\operatorname{Re}\lambda < -\omega(\mathcal{U})$ the function $u_\lambda : t \mapsto e^{\lambda t}U(t, 0)x$, $t \geq 0$, $x \in X$, belongs to C_X and $I_X u_\lambda = \lambda u_\lambda$. Thus $\{\lambda \in C : \operatorname{Re}\lambda > \omega(\mathcal{U})\}$ is contained in the (point) spectrum of $-I_X$ whereas the spectrum of G_X is contained in a left half-plane (see [Paz, Remark 1.5.4]). Note, however, that we always have $\ker I_X = \ker G_X = \{u \in C_X : u(t) = U(t, 0)u(0), t \geq 0\}$.

b) From Lemma 1.1 it easily follows that G_0 is the part of $-I_X$ in C_0 , i.e. $D(G_0) = \{u \in D(I_X) \cap C_0 : I_X u \in C_0\}$ and $G_0 u = -I_X u$ for $u \in D(G_0)$.

2. EXPONENTIAL STABILITY AND EXPONENTIAL EXPANSIVENESS

In this section we characterize exponentially stable and exponentially expansive evolution families $\mathcal{U} = (U(t, s))_{t \geq s \geq 0}$ by means of the operators G_0 and I_X and the integral equation (1.2), respectively. First we give the definition of exponential stability and exponential expansiveness of an evolution family.

2.1 DEFINITION. The evolution family \mathcal{U} on the Banach space X is said to be

- a) *exponentially stable* if there are constants $N, \nu > 0$ such that $\|U(t, s)\| \leq N e^{-\nu(t-s)}$ for all $t \geq s \geq 0$ (or, in other words, if $\omega(\mathcal{U}) < 0$),
- b) *exponentially expansive* if $U(t, s)$ is invertible and there are constants $N, \nu > 0$ such that $\|U(t, s)x\| \geq N e^{\nu(t-s)}\|x\|$ for $x \in X$ and $t \geq s \geq 0$.

We have the following characterization of exponentially stable evolution families.

2.2 THEOREM. Let $\mathcal{U} = (U(t, s))_{t \geq s \geq 0}$ be an evolution family on the Banach space X . Then the following assertions are equivalent:

- (i) \mathcal{U} is exponentially stable.
- (ii) G_0 is invertible.
- (iii) For every $f \in C_0$ the function $t \mapsto u_f(t) = \int_0^t U(t, \xi) f(\xi) d\xi$ belongs to C_0 .
- (iv) For every $f \in C_0$ the function $t \mapsto u_f(t) = \int_0^t U(t, \xi) f(\xi) d\xi$ belongs to C^b .

From the proof of the theorem we separate the following lemma for later use.

2.3 LEMMA. Let $\chi : [t_0, t_1) \rightarrow (0, \infty)$ be a continuous function and let $c > 0$ and $K, \alpha \geq 0$ be constants such that $\chi(t) \leq Ke^{\alpha(t-t_0)}$ and $\int_{t_0}^t \chi(\xi) \chi(\xi)^{-1} d\xi \leq c$ for $t \in [t_0, t_1)$. Then $\chi(t) \leq \max(cK, K)e^{\alpha + \frac{1}{c}(t-t_0)}$.

PROOF of Lemma 2.3. If $t \in [t_0, t_1)$ and $t \leq t_0 + 1$, then $\chi(t) \leq Ke^\alpha \leq Ke^{\alpha + \frac{1}{c}(t-t_0)}$.

Now let $t \in [t_0, t_1)$ and $t > t_0 + 1$. Set $\phi(s) := \int_{t_0}^s \chi(\xi)^{-1} d\xi$ for $s \in [t_0, t_1)$. By our assumption $\phi(s) \leq c\phi'(s)$, and hence $\phi(s_1) \geq \phi(s_0)e^{\frac{1}{c}(s_1-s_0)}$ for $t_1 > s_1 \geq s_0 > t_0$. Thus

$$\chi(t) = \frac{1}{\phi'(t)} \leq \frac{c}{\phi(t_0 + 1)} e^{-\frac{1}{c}(t-t_0-1)} \leq cKe^{\alpha + \frac{1}{c}(t-t_0)}.$$

PROOF of Theorem 2.2. The implications (i) \Rightarrow (iii) \Rightarrow (iv) are obvious. The equivalence of (ii) and (iii) follows from Lemma 1.1.

(iv) \Rightarrow (i): By assumption $B : C_0 \rightarrow C^b : f \mapsto u_f$ is linear and everywhere defined. Moreover, B is closed, and hence B is bounded. Let $c := \|B\|$.

Now fix $t_0 \geq 0$ and $0 \neq y \in X$. Let $t_1 := \sup\{t \geq t_0 : U(t, t_0)y \neq 0\}$. Then $U(t, t_0)y \neq 0$ for $t \in [t_0, t_1)$ and $U(t, t_0)y = 0$ for $t \geq t_1$. Set $\chi(t) := \|U(t, t_0)y\|$, $t \in [t_0, t_1)$. For $n \in \mathbb{N}$ sufficiently large choose a real continuous function φ_n on $[0, \infty)$ such that φ_n has compact support contained in (t_0, t_1) , $0 \leq \varphi_n \leq 1$ and

$$\varphi_n(t) = 1 \quad \text{for } t \in [t_0 + \frac{1}{n}, \min\{n, t_1 - \frac{1}{n}\}].$$

Let

$$f_n(t) = \begin{cases} \varphi_n(t)\chi(t)^{-1}U(t, t_0)y & \text{for } t \in [t_0, t_1), \\ 0 & \text{for } t \in [0, t_0] \cup [t_1, \infty). \end{cases}$$

Then $f_n \in C_0$ and

$$\int_{t_0}^t \frac{\varphi_n(\xi)}{\chi(\xi)} d\xi \chi(t) = \int_{t_0}^t \frac{\varphi_n(\xi)}{\chi(\xi)} d\xi \|U(t, t_0)y\| = \left\| \int_{t_0}^t U(t, \xi) f_n(\xi) d\xi \right\| \leq c\|f_n\| = c$$

for $t \in [t_0, t_1)$. By letting $n \rightarrow \infty$ we obtain $\int_{t_0}^t \chi(t)\chi(\xi)^{-1}d\xi \leq c$ for $t \in [t_0, t_1)$. Moreover, by the exponential boundedness of \mathcal{U} we have $\chi(t) \leq Ke^{\alpha(t-t_0)}\|y\|$ on $[t_0, t_1)$. An application of Lemma 2.3 yields

$$\chi(t) \leq \max\{cK, K\}e^{\alpha+\frac{1}{c}}e^{-\frac{1}{c}(t-t_0)}\|y\|$$

for $t \in [t_0, t_1)$. Thus

$$\|U(t, t_0)y\| \leq \max\{cK, K\}e^{\alpha+\frac{1}{c}}e^{-\frac{1}{c}(t-t_0)}\|y\|$$

for $t \geq t_0$. Since the constants K, α and c are independent of t_0 and y the exponential stability of \mathcal{U} follows.

REMARK. The proof of (iv) \Rightarrow (i) follows ideas used in [DaK, Proof of Theorem IV.3.3]. By different methods the equivalence of (i) and (iv) has been shown by C. Buşe [Bus, Theorem 1] (see also [Dat, Theorem 8] for a related result) and the equivalence of (i), (ii) and (iii) is proved by Y. Latushkin, S. Montgomery-Smith and T. Randolph [LMR2, Theorem 2.2, Corollary 2.3]. Related results in the autonomous case have been shown e.g. by J. van Neerven [Nee] and Vũ Quốc Phóng [Vũ]. Note that Theorem 2.2 also holds if we replace C_0 and C^b by $L^p(R_+, X)$, $1 \leq p < \infty$, and consider the evolution semigroup \mathcal{T} on $L^p(R_+, X)$ induced by \mathcal{U} (see [Dat, Theorem 6] and [LMR2, Theorem 2.2, Corollary 2.3]).

An immediate consequence of Theorem 2.2 is the spectral mapping theorem for the evolution semigroup on C_0 . Recall that $\sigma(B)$ denotes the *spectrum* of a linear operator B and $\rho(B) := C \setminus \sigma(B)$ is the *resolvent set*. Moreover, $s(B) := \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(B)\}$ is the *spectral bound* and $r(B) := \sup\{|\lambda| : \lambda \in \sigma(B)\}$ is the *spectral radius* of B .

2.4 COROLLARY. *Let \mathcal{U} be an evolution family on the Banach space X . Then the evolution semigroup \mathcal{T} on C_0 satisfies the spectral mapping theorem*

$$e^{t\sigma(G_0)} = \sigma(\mathcal{T}(t)) \setminus \{0\}, \quad t \geq 0.$$

Furthermore, $\sigma(G_0) = \{\lambda \in C : \operatorname{Re}\lambda \leq s(G_0)\}$ is a left half-plane and $\sigma(\mathcal{T}(t)) = \{\lambda \in C : |\lambda| \leq r(\mathcal{T}(t))\}$, $t > 0$, is a disc.

PROOF. Let $\lambda \in \rho(G_0)$ and $\mu \in C$ such that $\operatorname{Re}\mu \geq \operatorname{Re}\lambda$. Note that $G_0 - \lambda Id$ is the generator of the evolution semigroup $\mathcal{T}_\lambda = (e^{-\lambda t}T(t))_{t \geq 0}$ on C_0 induced by the evolution family $\mathcal{U}_\lambda = (e^{-\lambda(t-s)}U(t, s))_{t \geq s \geq 0}$. Theorem 2.2 implies that \mathcal{U}_λ and hence \mathcal{U}_μ

is exponentially stable and, as a consequence, that $G_0 - \mu Id$ is invertible, i.e. $\mu \in \rho(G_0)$. Thus $\sigma(G_0) = \{\lambda \in C : \operatorname{Re}\lambda \leq s(G_0)\}$ is a left half-plane.

If $\operatorname{Re}\lambda > s(G_0)$, then again by Theorem 2.2 the evolution family \mathcal{U}_λ is exponentially stable, and hence the induced evolution semigroup \mathcal{T}_λ is exponentially stable as well. In particular $r(e^{-\lambda t}T(t)) < 1$, i.e. $r(T(t)) < e^{\operatorname{Re}\lambda \cdot t}$ for $t > 0$. Thus $r(T(t)) \leq e^{s(G_0) \cdot t}$, $t \geq 0$. Together with the spectral inclusion theorem $e^{t\sigma(G_0)} \subseteq \sigma(T(t))$, $t \geq 0$ (see [Paz, 2.2.3]), it follows that $\sigma(T(t))$, $t > 0$, is a disc and that the spectral mapping theorem holds.

REMARK. The same result (with a different proof) can be found in [LMR2, Theorem 1.2, Theorem 2.2, Corollary 2.3] and [Sch, Theorem 5.3]. The observation that on C_0 the spectrum of $T(t)$, $t \geq 0$, is always a disc goes back to [Rau, Proposition 2].

We now come to the characterization of exponentially expansive evolution families.

2.5 THEOREM. *Let $\mathcal{U} = (U(t, s))_{t \geq s \geq 0}$ be an evolution family on the Banach space X . Then the following assertions are equivalent:*

(i) \mathcal{U} is exponentially expansive.

(ii) For every $t_0 \geq 0$ and every $f \in C_X(t_0)$ there is a unique $u_f \in C_X(t_0)$ such that

$$u_f(t) = U(t, s)u_f(s) + \int_s^t U(t, \xi)f(\xi)d\xi, \quad t \geq s \geq t_0. \quad (2.1)$$

REMARK. Condition (ii) implies that the operator I_X is invertible. However, Example 4.6 shows that the exponential expansiveness of \mathcal{U} (and hence (ii)) is not equivalent to the invertibility of I_X . On the contrary, if $U(t, s)$ is surjective for all $t \geq s \geq 0$, then from the proof of Theorem 2.5 it follows that \mathcal{U} is exponentially expansive if and only if I_X is invertible.

As above we separate from the proof of the theorem the following lemma which is essentially contained in [DaK, Proof of Theorem IV.3.3].

2.6 LEMMA. *Let $\chi : [0, \infty) \rightarrow (0, \infty)$ be a continuous function and let $c > 0$ and $K, \alpha > 0$ be constants such that $\chi(\tau) \leq Ke^{\alpha(\tau-t)}\chi(t)$, $\tau \geq t \geq 0$, and $\int_t^\infty \chi(t)\chi(\tau)^{-1}d\tau \leq c$, $t \geq 0$. Then there exists $N \geq 0$ only dependent on K, α and c such that $\chi(t) \geq Ne^{\frac{1}{c}(t-s)}\chi(s)$ for $t \geq s \geq 0$.*

PROOF. Let $\phi(t) := \int_t^\infty \chi(\tau)^{-1} d\tau$, $t \geq 0$. By our assumption $\phi(t) \leq -c\phi'(t)$. Thus $\phi(t) \leq \phi(s)e^{-\frac{1}{c}(t-s)}$ for $t \geq s \geq 0$. On the other hand the exponential estimate of χ yields

$$\begin{aligned} \chi(t)\phi(t) &= \chi(t) \int_t^\infty \frac{1}{\chi(\tau)} d\tau \geq K^{-1} \int_t^\infty e^{-\alpha(\tau-t)} d\tau \\ &\geq K^{-1} \sum_{k=1}^\infty \int_{t+(k-1)}^{t+k} e^{-\alpha(t+k-t)} d\tau = K^{-1} \sum_{k=1}^\infty e^{-\alpha k} =: \tilde{N}. \end{aligned}$$

Moreover, by our assumption, $\chi(s)\phi(s) \leq c$, $t \geq s \geq 0$. Thus

$$\chi(t) \geq \frac{\tilde{N}}{\phi(t)} \geq \frac{\tilde{N}}{\phi(s)} e^{\frac{1}{c}(t-s)} \geq \frac{\tilde{N}}{c} e^{\frac{1}{c}(t-s)} \chi(s) \quad \text{for } t \geq s \geq 0.$$

Now we are in a position to give the proof of Theorem 2.5.

PROOF of Theorem 2.5. (i) \Rightarrow (ii): Let \mathcal{U} be exponentially expansive and $t_0 \geq 0$. If $f \in C_X(t_0)$ then the function $t \mapsto u_f(t) := -\int_t^\infty U(\xi, t)^{-1} f(\xi) d\xi$ belongs to $C_X(t_0)$. It is a straightforward computation to show that u_f satisfies (2.1). If $u \in C_X(t_0)$ satisfies (2.1) with $f = 0$, then $u(t) = U(t, t_0)u(t_0)$, $t \geq t_0$. Thus $\|u(t)\| \geq Ne^{\nu(t-t_0)}\|u(t_0)\|$ for constants $N, \nu > 0$. This yields $u(t_0) = 0$, and hence $u = 0$.

(ii) \Rightarrow (i): **A)** The assumption implies that I_X is invertible. Since I_X is closed its inverse I_X^{-1} is bounded and $c := \|I_X^{-1}\| > 0$.

B) Fix $0 \neq x \in X$. The unique solvability of (2.1) for $f = 0$ in C_X yields $u(t) := U(t, 0)x \neq 0$ for $t \geq 0$. For each $n \in \mathbb{N}$ choose a real continuous function φ_n on $[0, \infty)$ such that $0 \leq \varphi_n \leq 1$, $\varphi_n = 1$ on $[0, n]$ and $\varphi_n = 0$ on $[n+1, \infty)$. Let $f_n(t) := -\varphi_n(t)\|u(t)\|^{-1}u(t)$ for $t \geq 0$. An easy computation shows that

$$u_n(t) := \int_t^\infty \frac{\varphi_n(\xi)}{\|u(\xi)\|} d\xi u(t), \quad t \geq 0,$$

and f_n solve (1.2), i.e. $I_X u_n = f_n$. Hence, $\|u_n\| \leq c\|f_n\| = c$. Letting $n \rightarrow \infty$ we obtain

$$\int_t^\infty \frac{1}{\|u(\xi)\|} d\xi \leq \frac{c}{\|u(t)\|}, \quad t \geq 0.$$

The exponential boundedness of \mathcal{U} yields $\|u(\tau)\| \leq Ke^{\alpha(\tau-t)}\|u(t)\|$, $\tau \geq t \geq 0$, for constants $K, \alpha > 0$ independent of x . By Lemma 2.6 there is a constant $N \geq 0$ independent of x such that $\|u(t)\| \geq Ne^{\frac{1}{c}(t-s)}\|u(s)\|$ for $t \geq s \geq 0$.

C) It remains to show that $U(t, s)$ is surjective for $t \geq s \geq 0$. Fix $s \geq 0$. Let $y \in X$ and set $v(t) := U(t, s)y$, $t \geq s$. We choose a real continuous function φ on $[0, \infty)$ with compact support such that $|\varphi(t)| \leq 1$, $\text{supp } \varphi \subset (s, \infty)$ and

$$\int_s^\infty \varphi(\xi) d\xi = 1.$$

Now consider the function w defined by

$$w(t) := v(t) \int_t^\infty \varphi(\xi) d\xi, \quad t \geq s.$$

Then $w \in C_X(s)$ and w is a solution of equation (2.1) with $f(t) := -\varphi(t)v(t)$, $t \geq s$. Extend f continuously to $[0, \infty)$ by setting $f(t) = 0$ on $[0, s]$. Then there is a solution $z \in C_X$ of equation (2.1) on $[0, \infty)$. By our assumption $z|_{[s, \infty)} = w$. Thus

$$y = w(s) = z(s) = U(s, 0)z(0) + \int_0^s U(s, \xi)f(\xi)d\xi = U(s, 0)z(0).$$

This proves the surjectivity of $U(s, 0)$ for $s \geq 0$. Together with $U(t, 0) = U(t, s)U(s, 0)$ this yields the surjectivity of $U(t, s)$ for $t \geq s \geq 0$.

3. EXPONENTIAL STABILITY OF BOUNDED ORBITS

In Theorem 2.2 we saw that exponential stability of an evolution family $\mathcal{U} = (U(t, s))_{t \geq s \geq 0}$ is characterized by the invertibility of the generator G_0 of the evolution semigroup \mathcal{T} on C_0 . We now show that a condition on the approximate point spectrum of G_0 implies exponential stability of all bounded orbits $U(t, t_0)x$, $t \geq t_0 \geq 0$. Recall that for an operator B on a Banach space Y the *approximate point spectrum* $A\sigma(B)$ of B is the set of all complex numbers λ such that for every $\epsilon > 0$ there exists $y \in D(B)$ with $\|y\| = 1$ and $\|(\lambda - B)y\| \leq \epsilon$.

As the whole spectrum $\sigma(G_0)$ (see Corollary 2.4) the approximate point spectrum $A\sigma(G_0)$ is invariant under translations along the imaginary axis.

3.1 LEMMA. $A\sigma(G_0) = i\mu + A\sigma(G_0)$ for all $\mu \in R$.

PROOF. Consider the multiplication operator M_μ on C_0 defined by $(M_\mu v)(t) := e^{i\mu t}v(t)$, $t \geq 0$, $v \in C_0$. Then M_μ is an isometry and

$$\begin{aligned} (M_\mu T(t)M_{-\mu}v)(s) &= \begin{cases} e^{i\mu s}U(s, s-t)e^{-i\mu(s-t)}v(s-t), & s \geq t \geq 0, \\ 0, & 0 \leq s \leq t, \end{cases} \\ &= e^{i\mu t}(T(t)v)(s), \end{aligned}$$

$v \in C_0$, $s \geq 0$. Thus $M_\mu T(t)M_{-\mu} = e^{i\mu t}T(t)$, $t \geq 0$, and hence $M_\mu G_0 M_{-\mu} = i\mu + G_0$. From this the assertion immediately follows.

We now come to the main result of this section. A special case of it has been shown in [AuM, Theorems 7 and 7']. Our proof follows the techniques of [AuM].

3.2 THEOREM. Let $\mathcal{U} = (U(t, s))_{t \geq s \geq 0}$ be an evolution family on the Banach space X such that $A\sigma(G_0) \cap iR \neq iR$. Then every bounded orbit of \mathcal{U} is exponentially stable. Precisely, if $\sup_{t \geq t_0} \|U(t, t_0)x\| < \infty$ for fixed $x \in X$ and $t_0 \geq 0$, then there exist constants $N, \nu > 0$ independent of x and t_0 such that

$$\|U(t, t_0)x\| \leq Ne^{-\nu(t-s)}\|U(s, t_0)x\|, \quad t \geq s \geq t_0.$$

PROOF. Let $x \in X$ and $t_0 \geq 0$ such that $\sup_{t \geq t_0} \|U(t, t_0)x\| < \infty$.

A) $\lim_{t \rightarrow \infty} \|U(t, t_0)x\| = 0$:

Choose a real continuous function φ on $[0, \infty)$ with compact support such that $\varphi(t_0 + 1) = 1$ and $\varphi|_{[0, t_0]} = 0$. Define

$$v(t) := \begin{cases} \varphi(t)U(t, t_0)x & \text{for } t > t_0, \\ 0 & \text{for } 0 \leq t \leq t_0. \end{cases}$$

Then $v \in C_0$ and

$$\begin{aligned} & \|T(t+h)v - T(t)v\| \\ &= \max \left\{ \sup_{s \geq t+h} \|U(s, s-t-h)v(s-t-h) - U(s, s-t)v(s-t)\|, \right. \\ & \quad \left. \sup_{t+h \geq s \geq t} \|U(s, s-t)v(s-t)\| \right\} \end{aligned}$$

for $t \geq 0$ and $h > 0$. From

$$U(s, s-t)v(s-t) = \begin{cases} \varphi(s-t)U(s, t_0)x & \text{for } s-t > t_0, \\ 0 & \text{for } 0 \leq s-t \leq t_0, \end{cases}$$

$\varphi(0) = 0$ and the uniform boundedness of $U(t, t_0)x$, $t \geq t_0$, we obtain

$$\limsup_{h \downarrow 0} \sup_{t \geq 0} \sup_{t+h \geq s \geq t} \|U(s, s-t)v(s-t)\| = 0.$$

On the other hand

$$\begin{aligned} & \sup_{s \geq t+h} \|U(s, s-t-h)v(s-t-h) - U(s, s-t)v(s-t)\| \\ &= \sup_{s \geq t+h} \|\varphi(s-t-h)U(s, t_0)x - \varphi(s-t)U(s, t_0)x\| \\ &\leq \sup_{s \geq t+h} |\varphi(s-t-h) - \varphi(s-t)| \sup_{s \geq t_0} \|U(s, t_0)x\|. \end{aligned}$$

By the uniform continuity of φ we have

$$\limsup_{h \downarrow 0} \sup_{t \geq 0} \sup_{s \geq t+h} \|U(s, s-t-h)v(s-t-h) - U(s, s-t)v(s-t)\| = 0.$$

Thus $t \mapsto T(t)v$ is a uniformly continuous C_0 -valued function on $[0, \infty)$. Our assumption on G_0 and Lemma 3.1 imply $A\sigma(G_0) \cap iR = \emptyset$. An application of a result of Batty and Vũ [BaV, Theorem 1] yields $\lim_{t \rightarrow \infty} T(t)v = 0$. In particular,

$$\begin{aligned} 0 &= \lim_{t \rightarrow \infty} \|(T(t)v)(t + t_0 + 1)\| = \lim_{t \rightarrow \infty} \|U(t + t_0 + 1, t_0 + 1)v(t_0 + 1)\| \\ &= \lim_{t \rightarrow \infty} \|U(t + t_0 + 1, t_0)x\|. \end{aligned}$$

B) $\|U(t, t_0)x\| \leq Ne^{-\nu(t-t_0)}\|x\|$, $t \geq t_0$:

Without loss of generality we may assume $\|x\| = 1$. Since $0 \notin A\sigma(G_0)$ there is a constant $\nu > 0$ such that

$$\|G_0v\| \geq \nu\|v\| \text{ for } v \in D(G_0). \quad (3.1)$$

Let $u(t) := U(t, t_0)x$, $t \geq t_0$, and $t_1 := \sup\{t \geq t_0 : U(t, t_0)x \neq 0\}$. The exponential boundedness of \mathcal{U} implies that there are constants $K, \alpha > 0$ (independent of x and t_0) such that $\|u(t)\| \leq Ke^{\alpha(t-t_0)}$, $t \geq t_0$. For $n \in \mathbb{N}$ sufficiently large we choose a real continuous function ψ_n on $[0, \infty)$ such that ψ_n has compact support contained in (t_0, t_1) , $0 \leq \psi_n \leq 1$ and

$$\psi_n(t) = 1 \quad \text{for } t \in [t_0 + \frac{1}{n}, \min\{n, t_1 - \frac{1}{n}\}].$$

Let

$$\varphi_n(t) := \begin{cases} 0 & \text{for } t \in [0, t_0], \\ \int_{t_0}^t \psi_n(\xi) \|u(\xi)\|^{-1} d\xi & \text{for } t \in (t_0, t_1), \\ \int_{t_0}^{t_1} \psi_n(\xi) \|u(\xi)\|^{-1} d\xi & \text{for } t \in [t_1, \infty), \end{cases}$$

$u_n(t) := \varphi_n(t)U(t, t_0)x$, $t \geq 0$, and $f_n(t) := \varphi_n'(t)U(t, t_0)x$, $t \geq 0$, where we set $U(t, t_0) = 0$ for $0 \leq t < t_0$. From A) and the definition of φ_n it follows that u_n and f_n satisfy the assumptions of Remark 1.2 c). Hence $u_n \in D(G_0)$ and $G_0u_n = -f_n$. Thus, by (3.1),

$$1 = \sup_{t \geq t_0} \|f_n(t)\| = \|f_n\| \geq \nu\|u_n\| = \nu \sup_{t \geq t_0} \|u_n(t)\|.$$

In particular,

$$\nu^{-1} \geq \|u_n(t)\| = |\varphi_n(t)| \|u(t)\| = \int_{t_0}^t \psi_n(\xi) \|u(\xi)\|^{-1} d\xi \|u(t)\| \quad \text{for } t \in [t_0, t_1].$$

Letting $n \rightarrow \infty$ we obtain

$$\int_{t_0}^t \|u(t)\| \|u(\xi)\|^{-1} d\xi \leq \nu^{-1}.$$

By Lemma 2.3 there is a constant N only dependent on ν, K and α such that

$$\|U(t, t_0)x\| = \|u(t)\| \leq Ne^{-\nu(t-t_0)}, \quad t \geq t_0.$$

C) Fix $s \geq t_0$ and set $y := U(s, t_0)x$. Then $\sup_{t \geq s} \|U(t, s)y\| < \infty$ and from A) and B) we obtain

$$\|U(t, t_0)x\| = \|U(t, s)y\| \leq Ne^{-\nu(t-s)}\|y\| = Ne^{-\nu(t-s)}\|U(s, t_0)x\|, \quad t \geq s.$$

Denote by $X_0(t_0) := \{x \in X : \lim_{t \rightarrow \infty} U(t, t_0)x = 0\}$ the *stable subspace* corresponding to \mathcal{U} and $t_0 \geq 0$. From Theorem 3.2 we obtain the following properties of $X_0(t_0)$.

3.3 COROLLARY. *Under the conditions of Theorem 3.2 we have*

$$\begin{aligned} X_0(t_0) &= \{x \in X : \sup_{t \geq t_0} \|U(t, t_0)x\| < \infty\} \\ &= \{x \in X : \|U(t, t_0)x\| \leq Ne^{-\nu(t-t_0)}\|x\|\}, \quad t_0 \geq 0, \end{aligned}$$

for fixed constants $N, \nu > 0$, and $X_0(t_0)$ is a closed linear subspace of X .

REMARKS. a) Lemma 3.1 and Lemma 1.1 imply that the following assertions are equivalent:

- (i) $A\sigma(G_0) \cap iR \neq iR$.
- (ii) $0 \notin A\sigma(G_0)$.
- (iii) $A\sigma(G_0) \cap iR = \emptyset$.
- (iv) There is a constant $\nu > 0$ such that for any pair $u, f \in C_0$ with $u(t) = \int_0^t U(t, \xi)f(\xi)d\xi$, $t \geq 0$, one has $\|f\| \geq \nu\|u\|$.

b) An evolution family with an exponential dichotomy (see Definition 4.1) always satisfies $A\sigma(G_0) \cap iR = \emptyset$ (see the Remark after Theorem 4.3).

c) If B is strongly continuous and uniformly bounded function from $[0, \infty)$ into the space of bounded linear operator on X , then there is a unique evolution family $\mathcal{U}_B = (U_B(t, s))_{t \geq s \geq 0}$ satisfying the variation of constants formula

$$U_B(t, s)x = U(t, s)x + \int_s^t U(t, \xi)B(\xi)U_B(\xi, s)xd\xi, \quad t \geq s \geq 0, \quad x \in X$$

(see [RS2], [Sch] and the references therein). The generator of the evolution semigroup \mathcal{T}_B on C_0 induced by \mathcal{U}_B is given by $G_0 + B(\cdot)$. Suppose that G_0 is invertible resp. $0 \notin A\sigma(G_0)$. Then $G_0 + B(\cdot)$ has the same property for every perturbation $B(\cdot)$ such that $\|B(\cdot)\| = \sup_{t \geq 0} \|B(t)\|$ is sufficiently small (cf. [Kat, IV.3.1]). In other words exponential stability of \mathcal{U} resp. of all bounded orbits of \mathcal{U} is robust with respect to small perturbations $B(\cdot)$ of \mathcal{U} (see also [LMR2, Section 3.1], [Sch, Section 5.4]).

4. EXPONENTIAL DICHOTOMY

In this section we characterize the exponential dichotomy of an evolution family \mathcal{U} by properties of the operators G_0, G_X and I_X , and of the operators G_Z and I_Z to be defined below. At first we recall the following definition.

4.1 DEFINITION. An evolution family $\mathcal{U} = (U(t, s))_{t \geq s \geq 0}$ on the Banach space X is said to have an *exponential dichotomy* if there exist bounded linear projections $P(t)$, $t \geq 0$, on X and constants $N, \nu > 0$ such that

- a) $U(t, s)P(s) = P(t)U(t, s)$, $t \geq s \geq 0$,
- b) the restriction $U(t, s)|_{\ker P(s)} : \ker P(s) \rightarrow \ker P(t)$, $t \geq s \geq 0$, is an isomorphism (and we denote its inverse by $U(s, t)|_{\ker P(t)} : \ker P(t) \rightarrow \ker P(s)$),
- c) $\|U(t, s)x\| \leq Ne^{-\nu(t-s)}\|x\|$ for $x \in P(s)X$, $t \geq s \geq 0$,
- d) $\|U(s, t)|_{\ker P(t)}x\| \leq Ne^{-\nu(t-s)}\|x\|$ for $x \in \ker P(t)$, $t \geq s \geq 0$.

In the following lemma we collect some properties of the family $P(t)$, $t \geq 0$. By $\mathcal{L}(Y, X)$ we denote the space of bounded linear operators between the Banach spaces Y and X .

4.2 LEMMA. *Let \mathcal{U} be an evolution family having an exponential dichotomy with corresponding family of projections $P(t)$, $t \geq 0$, and constants $N, \nu > 0$. Then the following holds:*

- a) $M := \sup_{t \geq 0} \|P(t)\| < \infty$,
- b) $[0, t] \ni s \mapsto U(s, t)|_{\ker P(t)} \in \mathcal{L}(\ker P(t), X)$ is strongly continuous for $t > 0$.
- c) $t \mapsto P(t)$ is strongly continuous,
- d) $\|U(t, s)P(s)\| \leq MNe^{-\nu(t-s)}$ for $t \geq s \geq 0$,
- e) $\|U(s, t)|_{\ker P(t)}(I - P(t))\| \leq MNe^{-\nu(t-s)}$ for $t \geq s \geq 0$.

PROOF. a) can be shown as in [DaK, Lemma IV.1.1 and Lemma IV.3.2]. For sake of completeness we present the details. Fix $t_0 \geq 0$. Let $P_0 := P(t_0)$, $P_1 := Id - P(t_0)$ and $X_k := P_k X$, $k = 0, 1$. Set $\gamma_{t_0} := \inf\{\|x_0 + x_1\| : x_k \in X_k, \|x_0\| = \|x_1\| = 1\}$. If $x \in X$ and $P_k x \neq 0$, $k = 0, 1$, then

$$\begin{aligned} \gamma_{t_0} &\leq \left\| \frac{P_0 x}{\|P_0 x\|} + \frac{P_1 x}{\|P_1 x\|} \right\| = \frac{1}{\|P_0 x\|} \|P_0 x\| + \frac{\|P_0 x\|}{\|P_1 x\|} \|P_1 x\| \\ &= \frac{1}{\|P_0 x\|} \|x\| + \frac{\|P_0 x\| - \|P_1 x\|}{\|P_1 x\|} \|P_1 x\| \leq \frac{2\|x\|}{\|P_0 x\|}. \end{aligned}$$

As a consequence $\|P_0\| \leq 2\gamma_{t_0}^{-1}$. It remains to show that there is a constant $c > 0$ (independent of t_0) such that $\gamma_{t_0} \geq c$. For this fix $x_k \in X_k$, $k = 0, 1$, with $\|x_0\| = \|x_1\| = 1$.

By the exponential boundedness of \mathcal{U} we have $\|U(t, t_0)(x_0 + x_1)\| \leq Ke^{\alpha(t-t_0)}\|x_0 + x_1\|$ for $t \geq t_0$ and constants $K, \alpha \geq 0$. Thus

$$\begin{aligned} \|x_0 + x_1\| &\geq K^{-1}e^{-\alpha(t-t_0)}\|U(t, t_0)x_0 + U(t, t_0)x_1\| \\ &\geq K^{-1}e^{-\alpha(t-t_0)}(N^{-1}e^{\nu(t-t_0)} - Ne^{-\nu(t-t_0)}) \\ &=: c_{t-t_0}, \quad t \geq t_0, \end{aligned}$$

and hence $\gamma_{t_0} \geq c_{t-t_0}$. Obviously $c_m > 0$ for m sufficiently large. Thus $0 < c_m \leq \gamma_{t_0}$.

b) Fix $t > 0$, $0 \leq s_0 \leq t$ and $x \in \ker P(t)$, and let (s_n) be a sequence in $[0, t]$ converging to s_0 . There is $y \in \ker P(0)$ such that $U(t, 0)y = x$. By the strong continuity of \mathcal{U} we have

$$\lim_n \|U(s_n, t)|x - U(s_0, t)|x\| = \lim_n \|U(s_n, 0)|y - U(s_0, 0)|y\| = 0.$$

c) Note that

$$\begin{aligned} \|P(t)x - P(s)x\| &\leq \|P(t)x - P(t)U(t, s)x\| + \|U(t, s)P(s)x - P(s)x\| \\ &\leq (\sup_{r \geq 0} \|P(r)\|) \|x - U(t, s)x\| + \|U(t, s)P(s)x - P(s)x\| \end{aligned}$$

for $x \in X$ and $t \geq s \geq 0$. By the strong continuity of \mathcal{U} we obtain that $P(\cdot)$ is strongly continuous from the right. In order to show strong continuity from the left set $Q(\cdot) := Id - P(\cdot)$ and fix $t > 0$ and $x \in X$. For $0 \leq s \leq t$ we have $Q(s)x = U(s, t)|U(t, s)Q(s)x = U(s, t)|Q(t)U(t, s)x$. By b) the family $(U(s, t)|)_{s \in [0, t]} \subseteq \mathcal{L}(\ker P(t), X)$ is strongly continuous and uniformly bounded. The strong continuity of \mathcal{U} yields $\lim_{s \uparrow t} Q(t)U(t, s)x = Q(t)x$. Thus

$$\lim_{s \uparrow t} Q(s)x = \lim_{s \uparrow t} U(s, t)|Q(t)U(t, s)x = Q(t)x,$$

i.e. $Q(\cdot)x$ and hence $P(\cdot)x$ is continuous from the left.

Assertions d) and e) are immediate consequences of a).

We come to our first main result. It characterizes evolution families with an exponential dichotomy by conditions on the operators G_0 , G_X and I_X , respectively.

4.3 THEOREM. *Let $\mathcal{U} = (U(t, s))_{t \geq s \geq 0}$ be an evolution family on the Banach space X . Then the following assertions are equivalent:*

- (i) \mathcal{U} has an exponential dichotomy.
- (ii) The range $R(G_X)$ of G_X coincides with C_0 and $X_0(0)$ is complemented in X .
- (iii) I_X is surjective and $X_0(0)$ is complemented in X .

PROOF. (i) \Rightarrow (ii): Let $P(t)$, $t \geq 0$, be the family of projections given by the exponential dichotomy. Then $X_0(0) = P(0)X$, and hence $X_0(0)$ is complemented. If $f \in C_0$ define $v : [0, \infty) \rightarrow X$ by

$$v(t) = \int_0^t U(t, \xi)P(\xi)f(\xi)d\xi - \int_t^\infty U(t, \xi)(Id - P(\xi))f(\xi)d\xi. \quad (4.1)$$

An easy computation shows that $v \in C_X$ and v is a solution of equation (1.2). By Lemma 1.1 we have $G_X v = -f$. Together with Remark 1.2 b) this implies $R(G_X) = C_0$.

(i) \Rightarrow (iii): We can use the same arguments as in the proof of (i) \Rightarrow (ii). Note that for $f \in C_X$ the function v defined by (4.1) is also in C_X and satisfies (1.2). Hence by the definition of I_X we have $v \in D(I_X)$ and $I_X v = f$, i.e. I_X is surjective.

(ii) \Rightarrow (i): **A)** Let $Z \subseteq X$ be a complement of $X_0(0)$ in X , i.e. $X = X_0(0) \oplus Z$. Set $X_1(t) := U(t, 0)Z$, $t \geq 0$. Clearly,

$$\begin{aligned} U(t, s)X_0(s) &\subseteq X_0(t), \quad t \geq s, \\ U(t, s)X_1(s) &= X_1(t), \quad t \geq s. \end{aligned} \quad (4.2)$$

B) There are constants $N, \nu > 0$ such that

$$\|U(t, 0)x\| \geq N e^{\nu(t-s)} \|U(s, 0)x\| \text{ for } x \in X_1(0) \text{ and } t \geq s \geq 0. \quad (4.3)$$

In fact, let $Y := \{v \in D(G_X) : v(0) \in X_1(0)\}$ endowed with the graph norm $\|v\|_{G_X} := \|v\| + \|G_X v\|$. Then Y is a closed subspace of the Banach space $(D(G_X), \|\cdot\|_{G_X})$, and hence Y is complete. By Remark 1.2 a) we have $\ker G_X = \{v \in C_X : v(t) = U(t, 0)x \text{ for some } x \in X_0(0)\}$. Since $X = X_0(0) \oplus X_1(0)$ and $R(G_X) = C_0$ we obtain that $G_X : Y \rightarrow C_0$ is bijective and hence an isomorphism. Thus there is a constant $\nu > 0$ such that

$$\|G_X v\| \geq \nu \|v\|_{G_X} \geq \nu \|v\| \text{ for } v \in Y. \quad (4.4)$$

Let now $0 \neq x \in X_1(0)$ and set $u(t) := U(t, 0)x$, $t \geq 0$. By Remark 1.2 a) we have $u(t) \neq 0$ for all $t \geq 0$. For each $n \in \mathbb{N}$ choose a real continuous function φ_n on $[0, \infty)$ such that $0 \leq \varphi_n \leq 1$, $\varphi_n = 1$ on $[\frac{1}{n}, n]$ and $\varphi_n = 0$ on $\{0\} \cup [n+1, \infty)$. Set $f_n(t) := -\varphi_n(t)\|u(t)\|^{-1}u(t)$, $t \geq 0$, and

$$u_n(t) := \int_t^\infty \frac{\varphi_n(\xi)}{\|u(\xi)\|} d\xi u(t), \quad t \geq 0.$$

Then $f_n \in C_0$ and $u_n \in C_X$. An easy computation shows that f_n and u_n satisfy (1.2). By Lemma 1.1 we have $u_n \in D(G_X)$ and $G_X u_n = -f_n$. From (4.4) we obtain $\|f_n\| \geq \nu \|u_n\|$. Letting $n \rightarrow \infty$ this yields

$$\int_t^\infty \frac{1}{\|u(\xi)\|} d\xi \leq \nu^{-1} \|u(t)\|^{-1}, \quad t \geq 0.$$

Now the exponential boundedness of \mathcal{U} and Lemma 2.6 imply that there is a constant $N \geq 0$ independent of x such that $\|u(t)\| \geq Ne^{\nu(t-s)}\|u(s)\|$, $t \geq s \geq 0$.

C) $X = X_0(t) \oplus X_1(t)$, $t \geq 0$:

Let $Y \subseteq C_X$ be as in B). Then $D(G_0) = D(G_X) \cap C_0 \subseteq Y$ and (4.4) yields $\|G_0 v\| \geq \nu \|v\|$ for $v \in D(G_0)$. Thus $0 \notin A\sigma(G_0) \cap iR$ and Corollary 3.3 implies that $X_0(t)$ is closed for $t \geq 0$.

From (4.2), (4.3) and the closedness of $X_1(0)$ we derive that $X_1(t)$ is closed and $X_0(t) \cap X_1(t) = \{0\}$ for $t \geq 0$.

Finally, fix $t_0 > 0$ and $x \in X$. Choose a real continuous function φ on $[0, \infty)$ such that φ has compact support contained in $[t_0, \infty)$ and $\int_{t_0}^{\infty} \varphi(\xi) d\xi = 1$. Set $v(t) := \int_t^{\infty} \varphi(\xi) d\xi U(t, t_0)x$ and $f(t) := -\varphi(t)U(t, t_0)x$, $t \geq t_0$. Then v is a solution of equation (2.1), and $v \in C_X(t_0)$. Extend f continuously to $[0, \infty)$ by setting $f|_{[0, t_0)} = 0$. Then $f \in C_0$ and by assumption there exists $w \in C_X$ such that $G_X w = -f$. In view of Lemma 1.1 w is a solution of equation (1.2). In particular, $w|_{[t_0, \infty)}$ satisfies (2.1). Thus

$$v(t) - w(t) = U(t, t_0)(v(t_0) - w(t_0)) = U(t, t_0)(x - w(t_0)), \quad t \geq t_0.$$

Since $v - w|_{[t_0, \infty)} \in C_X(t_0)$ this implies $x - w(t_0) \in X_0(t_0)$. On the other hand $w(0) = w_0 + w_1$ with $w_k \in X_k(0)$, $k = 0, 1$. Then $w(t_0) = U(t_0, 0)w_0 + U(t_0, 0)w_1$ and by (4.2) we have $U(t_0, 0)w_k \in X_k(t_0)$, $k = 0, 1$. Hence $x = x - w(t_0) + w(t_0) \in X_0(t_0) + X_1(t_0)$. This proves C).

D) Let $P(t)$ be the projection from X onto $X_0(t)$ with kernel $X_1(t)$, $t \geq 0$. Then (4.2) implies $P(t)U(t, s) = U(t, s)P(s)$, $t \geq s \geq 0$. From (4.2) and (4.3) we obtain that $U(t, s)|_{\ker P(s)} \rightarrow \ker P(t)$, $t \geq s \geq 0$ is an isomorphism. Finally, by (4.3), Theorem 3.2 and our assumption $A\sigma(G_0) \cap iR \neq iR$ there exist constants $N, \nu > 0$ such that

$$\|U(t, s)x\| \leq Ne^{-\nu(t-s)}\|x\| \text{ for } x \in P(s)X, \quad t \geq s \geq 0,$$

$$\|U(s, t)|x\| \leq Ne^{-\nu(t-s)}\|x\| \text{ for } x \in \ker P(t), \quad t \geq s \geq 0.$$

Thus \mathcal{U} has an exponential dichotomy.

(iii) \Rightarrow (i): We use exactly the same arguments as in (ii) \Rightarrow (i). We only have to replace the operator G_X by I_X .

REMARK. Part C) of the proof shows that for an evolution family \mathcal{U} with an exponential dichotomy the generator G_0 of the evolution semigroup on C_0 always satisfies $0 \notin A\sigma(G_0)$, and hence $A\sigma(G_0) \cap iR = \emptyset$ (see Section 3, Remark a)).

If X is a Hilbert space we only have to assume the closedness of the stable subspace.

4.4 COROLLARY. *Let $\mathcal{U} = (U(t, s))_{t \geq s \geq 0}$ be an evolution family on the Hilbert space H . Then the following assertions are equivalent:*

- (i) \mathcal{U} has an exponential dichotomy.
- (ii) $R(G_H) = C_0$ and $H_0(0)$ is closed.
- (iii) I_H is surjective and $H_0(0)$ is closed.

In the proof of Theorem 4.3, (ii) \Rightarrow (i), we saw that the invertibility of G_X restricted to a certain subspace of C_X plays a crucial role in order to prove the existence of an exponential dichotomy for a given evolution family \mathcal{U} . In our next result we show that the existence of an exponential dichotomy can be even characterized by such an invertibility condition. Let us introduce the following notion. For a closed linear subspace Z of X let

$$C_Z := \{f \in C_X : f(0) \in Z\}.$$

Denote by G_Z the part of G_X in C_Z , i.e. $D(G_Z) = D(G_X) \cap C_Z$ and $G_Z v = G_X v$ for $v \in D(G_Z)$. Then G_0 is the part of G_Z in C_0 . In the same way let I_Z be the restriction of I_X to C_Z , i.e. $D(I_Z) = D(I_X) \cap C_Z$ and $I_Z v = I_X v$ for $v \in D(I_Z)$. Notice that the evolution semigroup \mathcal{T} leaves C_Z invariant and that G_Z is the generator of the restriction of \mathcal{T} to C_Z .

With this notation we obtain the following characterization of evolution families with exponential dichotomy. A similar result is shown in [BGK, Theorem 1.1] for the finite dimensional case (see also [Pal]). We point out that in contrast to Theorem 4.3 we do not have to assume that $X_0(0)$ is complemented.

4.5 THEOREM. *Let $\mathcal{U} = (U(t, s))_{t \geq s \geq 0}$ be an evolution family on the Banach space X and let Z be a closed linear subspace of X . Then the following assertions are equivalent:*

- (i) \mathcal{U} has an exponential dichotomy with $\ker P(0) = Z$.
- (ii) $G_Z : D(G_Z) \subseteq C_Z \rightarrow C_0$ is invertible.
- (iii) $I_Z : D(I_Z) \subseteq C_Z \rightarrow C_X$ is invertible.

PROOF. (i) \Rightarrow (ii) : Let $P(t)$, $t \geq 0$, be a family of projections given by the exponential dichotomy such that $\ker P(0) = Z$. Then $P(0)X = X_0(0)$ and $X = X_0(0) \oplus Z$. Fix $f \in C_0$. By Theorem 4.3 there is $v \in D(G_X)$ such that $G_X v = f$. On the other hand $u : [0, \infty) \rightarrow X : t \mapsto U(t, 0)P(0)v(0)$ belongs to C_X and $G_X u = 0$ (cf. Remark 1.2 a)). Then $v - u \in D(G_Z)$ and $G_Z(v - u) = G_X v = f$. Hence $G_Z : D(G_Z) \rightarrow C_0$ is surjective. If $w \in \ker G_Z$ then $w(t) = U(t, 0)w(0)$, $t \geq 0$ (cf. Remark 1.2 a)). Since $w \in C_X$ we have $w(0) \in Z \cap X_0(0) = \{0\}$ and hence $w = 0$, i.e. G_Z is injective.

(i) \Rightarrow (iii) can be shown by the same arguments. One only has to use Remark 1.4 a) instead of Remark 1.2 a).

(ii) \Rightarrow (i) : Let $G_Z : D(G_Z) \rightarrow C_0$ be invertible. By Remark 1.2 b) we have $C_0 = R(G_Z) = R(G_X)$. The closedness of G_Z implies that G_Z^{-1} is bounded, and hence there exists $\nu > 0$ such that $\|G_Z v\| \geq \nu \|v\|$ for all $v \in D(G_Z)$. Since G_0 is the part of G_Z in C_0 we obtain $0 \notin A\sigma(G_0)$. Finally we show that $X = X_0(0) \oplus Z$. Corollary 3.3 implies that $X_0(0)$ is closed. Now let $x \in X$. If $U(t, 0)x = 0$ for some $t > 0$, then $x \in X_0(0)$. Otherwise $U(t, 0)x \neq 0$ for all $t \geq 0$. Let $\varphi : [0, \infty) \rightarrow \mathbb{R}$ be continuously differentiable such that $\varphi|_{[0,1]} = 1$ and $\varphi|_{[2,\infty)} = 0$. Set $u(t) := \varphi(t)U(t, 0)x$, $t \geq 0$, and $f(t) := \varphi'(t)U(t, 0)x$, $t \geq 0$. By Remark 1.2 c) we have $u \in D(G_X)$ and $G_X u = -f$. On the other hand there exists $v \in D(G_Z)$ such that $G_Z v = -f$. Thus $u - v \in \ker G_X$ and hence

$$(u - v)(t) = U(t, 0)(u(0) - v(0)) = U(t, 0)(x - v(0)), \quad t \geq 0.$$

Since $u - v \in C_X$ this implies $x - v(0) \in X_0(0)$. Thus $x = x - v(0) + v(0) \in X_0(0) + Z$. If $y \in Z \cap X_0(0)$, then w defined by $w(t) := U(t, 0)y$, $t \geq 0$, belongs to $C_Z \cap \ker G_X$ (see Remark 1.2 a)). Hence $G_Z w = 0$ and by the invertibility of G_Z we have $w = 0$. Thus $0 = w(0) = y$, i.e. $X_0(0) \cap Z = \{0\}$. This shows $X = X_0(0) \oplus Z$. The assertion now follows from Theorem 4.3.

(iii) \Rightarrow (i) is shown by the same arguments as (ii) \Rightarrow (i).

REMARK. As a special case of Theorem 4.5 the invertibility of G_X resp. I_X characterizes evolution families with an exponential dichotomy such that $X_0(0) = P(0)X = \{0\}$.

We conclude with an example of an evolution family \mathcal{U} with non-trivial exponential dichotomy such that I_X is invertible. In particular this shows that invertibility of I_X is not equivalent to the exponential expansiveness of \mathcal{U} (cf. Theorem 2.5).

4.6 EXAMPLE. Let $X = L^1[0, \infty)$. For $0 \leq s \leq t \leq 1$ set

$$U_1(t, s)f(\xi) := \begin{cases} 0, & 0 \leq \xi \leq t - s, \\ f(\xi - t + s), & t - s < \xi, \end{cases}$$

and for $1 \leq s \leq t$ set

$$U_2(t, s)f(\xi) := \begin{cases} e^{-(t-s)}f(\xi), & 0 \leq \xi \leq 1, \\ e^{(t-s)}f(\xi), & \xi > 1, \end{cases}$$

$f \in X$. Then $\mathcal{U} = (U(t, s))_{t \geq s \geq 0}$ defined by

$$U(t, s) := \begin{cases} U_1(t, s), & 0 \leq s \leq t \leq 1, \\ U_2(t, 1)U_1(1, s), & s \leq 1 \leq t, \\ U_2(t, s), & 1 \leq s \leq t, \end{cases}$$

is an evolution family on X with an exponential dichotomy. The corresponding family of projections $P(t)$, $t \geq 0$, is given by

$$P(t)f = \chi_{[0, \min\{1, t\}]}f, \quad f \in X,$$

where χ_C denotes the characteristic function of a set C . In particular, $\{0\} = P(0)X = X_0(0)$. Theorem 4.5 implies that I_X is invertible. Since $\|U(t, 1)P(1)\| \leq e^{-(t-1)}$, $t \geq 1$, the evolution family \mathcal{U} cannot be exponentially expansive.

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1991 Mathematics Subject Classification: Primary: 34G10, 47D06, Secondary: 47H20