STABLE AND UNSTABLE MANIFOLDS FOR QUASILINEAR PARABOLIC SYSTEMS WITH FULLY NONLINEAR BOUNDARY CONDITIONS

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Dedicated to Guiseppe Da Prato on the occasion of his 70th birthday

Abstract. We investigate quasilinear systems of parabolic partial differential equations with fully nonlinear boundary conditions on bounded or exterior domains in the setting of Sobolev–Slobodetskii spaces. We establish local wellposedness and study the time and space regularity of the solutions. Our main results concern the asymptotic behavior of the solutions in the vicinity of a hyperbolic equilibrium. In particular, the local stable and unstable manifolds are constructed.

1. Introduction

In this paper we investigate the qualitative properties of a general class of nonlinear parabolic systems by a unified approach. We consider the equations

$$\begin{aligned} \partial_t u(t) + A(u(t))u(t) = F(u(t)), \quad \text{on } \Omega, \ t > 0, \\ B_j(u(t)) = 0, \quad \text{on } \partial\Omega, \ t \geq 0, \ j = 1, \cdots, m, \\ u(0) = u_0, \quad \text{on } \Omega, \end{aligned} \tag{1.1}$$

on a (possibly unbounded) domain Ω with compact boundary $\partial\Omega$, where the solution $u(t,x)$ takes values in a finite dimensional space $E = \mathbb{C}^N$. The main part of the differential equation is given by a linear differential operator $A(u)$ of order $2m$ (with $m \in \mathbb{N}$) whose matrix–valued coefficients depend on the derivatives of u up to order $2m-1$, and F is a general nonlinear reaction term acting on the derivatives of u up to order $2m-1$. Therefore the differential equation is quasilinear. Our analysis focusses on the fully nonlinear boundary conditions

$$[B_j(u)](x) := b(x, u(x), \nabla u(x), \cdots, \nabla^{m_j} u(x)) = 0, \quad x \in \partial\Omega, \ j = 1, \cdots, m,$$

for the partial derivatives of u up to order $m_j \leq 2m-1$. We look for a solution u in the space $\mathbb{E}_1 = L_p([0,T]; W_p^{2m}(\Omega;\mathbb{C}^N)) \cap W_p^1([0,T]; L_p(\Omega;\mathbb{C}^N))$ for a fixed finite exponent $p > n + 2m$. The terms of highest order are thus contained in L_p spaces. Due to known embedding theorems, a function $u \in \mathbb{E}_1$ also belongs to the space

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$C([0,T];BC^{2m-1}(\overline{\Omega};\mathbb{C}^N))$. Hence, the nonlinear terms in (1.1) are continuous in (t,x) up to $t=0$, and the initial condition can be understood in a classical sense.

We require only local smoothness of the coefficients (e.g., the diffusion coefficients are C^1); in particular, there are no growth restrictions. The parabolicity of (1.1) is expressed in our main assumption saying that the linear boundary value problems $(A(v),B_1'(v),\cdots,B_m'(v))$ are normally elliptic and satisfy the Lopatinskii-Shapiro conditions for suitable functions v and the derivatives $B_j'(v)$. (See Section 2 for the precise statements.) These conditions are necessary and sufficient for the regularity properties of the linearization of (1.1), see Theorem 2.2 and (2.27), which are crucial for our approach. In this sense, our hypotheses are optimal. We note that reaction diffusion systems satisfy our assumptions, see [5] and also Section 6.

The initial value u_0 of (1.1) has to fulfill the boundary conditions $B_j(u_0)=0$ by continuity. Moreover, our solution space $\mathbb{E}_1$ is continuously embedded into $C([0,T];X_p)$ for the Slobodetskii space $X_p=W_p^{2m-2m/p}(\Omega;\mathbb{C}^N)$, and X_p is the smallest space with this property. As a result, u_0 must belong to X_p, the solution u of (1.1) is continuous in X_p on $[0,T]$, and the norm of X_p is the natural norm for our work. So we are led to the nonlinear phase space

$$\mathcal{M}=\{u_0\in X_p : B_1(u_0)=0,\cdots,B_m(u_0)=0\},$$

which is a C^1 manifold in X_p. This genuine nonlinear structure has to be respected when solving (1.1) and when studying the properties of the solutions. In fact, many of the difficulties in our analysis arise from the *compatibility conditions* $B_j(u_0)=0$.

We prove local existence and uniqueness of solutions in $\mathbb{E}_1$ for initial values $u_0\in\mathcal{M}$. We further show that the local semiflow on $\mathcal{M}$ solving (1.1) is continuously differentiable with respect to u_0 and that the equation has an additional smoothing effect in so far for $t>0$ the solution $u(t)$ is Hölder continuous of order $1-1/p$ with values in $W_p^{2m}(\Omega;\mathbb{C}^N)$, although $u_0\in X_p$. These results are presented in Theorem 4.2. However, we are mainly interested in the long term behavior of the solutions near an equilibrium $u_*\in W_p^{2m}(\Omega;\mathbb{C}^N)$ of (1.1). To this aim, we consider the derivative A_* of the map $u\mapsto A(u)u-F(u)$ at u_* and introduce the restriction A_0 of A_* to the kernel of the boundary operator $B_*=(B_1'(u_*),\cdots,B_m'(u_*))$. By [14], the operator $-A_0$ generates an analytic semigroup $T(\cdot)$ on $L_p(\Omega;\mathbb{C}^N)$. It turns out that the spectrum of A_0 determines much of the asymptotic behavior of the solutions to (1.1) near u_*. So we show the principle of linearized stability for (1.1) in Proposition 5.1. Assuming that $i\mathbb{R}\subset\rho(A_0)$ (i.e., that u_* is *hyperbolic*), in Theorem 5.2 we then construct the local stable, respectively unstable, manifolds at u_* which are C^1 in X_p and tangent to the stable, respectively unstable, subspaces of the linear operator $-A_0$. We prove that the stable, respectively unstable, manifolds consist precisely of the solutions to (1.1) which exist and stay in a ball in X_p centered at u_* for all $t\geq 0$, respectively for all $t\leq 0$. Moreover, these solutions converge exponentially to u_* in the norm of $W_p^{2m}(\Omega;\mathbb{C}^N)$ as $t\to\infty$, respectively as $t\to-\infty$.

There is a vast literature on the well–posedness of nonlinear parabolic equations which we cannot discuss in detail here. We refer to the recent survey [7] presenting, in particular, the available approaches to the subject. But we want to point out that most of the existing results impose restrictions on the structure of the boundary conditions. Many works deal with reaction diffusion systems of second order and consider conormal boundary conditions plus lower order terms, see e.g. [23], [39]. Other authors consider quasilinear boundary conditions which can be absorbed into the domains of generators $A_0(u)$, see e.g. [1], [3], [5], [8], [10], [11], [33], [37], [40], where additional lower order terms are admitted in some papers. General boundary conditions were studied for a single equation of second order in [9], [22], [28, Chap.XIII], [30, §8.5.3] in the C^α–setting (even for a fully nonlinear differential equation) and in [41] in our setting.

Fully nonlinear boundary conditions appear naturally in the treatment of free boundary problems, see e.g. [9], [19] and the survey [20], and when considering diffusion through interfaces, see e.g. [27]. The results of the present paper do not directly cover such problems, but we think that our methods can be generalized in order to deal with moving boundaries and transmission problems in future work.

Our approach relies on the results from [15] on the property of *maximal regularity of type* L_p for linear inhomogeneous initial boundary value problems, as stated in Theorem 2.2. (We refer to [14], [15], [28], [30] for its prehistory.) This theorem implies that the linearization of (1.1) possesses a solution in $\mathbb{E}_1$ if and only if if the initial value and the inhomogeneities of the linear problem belong to a certain space $\mathcal{D}$ defined (2.19). This space contains precisely the class of data resulting from the linearization of (1.1), see (2.27). The celebrated paper [11] by G. Da Prato and P. Grisvard initiated the approach to fully nonlinear and quasilinear parabolic problems via maximal regularity in a semigroup framework. Besides the L_p–setting, there are several function spaces where one can obtain analogous properties of maximal regularity, see e.g. [6] or [7] for a discussion. We also refer to the monograph [30] devoted to the study and application of maximal regularity in the Hölder setting. We employ the L_p–setting since the L_p norm in the state space is relatively simple and weak, and still the nonlinearities and the initial conditions are understood in a classical sense. One also obtains weaker conditions for the global solvability than in the C^α–setting, cf. Theorem 4.2 and [5], [33]. We note that one cannot treat fully nonlinear differential equations within the L_p–setting.

Our proof of local existence and uniqueness follows the lines of [41]. But we are not aware of any proofs for the smoothing properties shown in Theorem 4.2 for quasilinear equations with fully nonlinear boundary conditions. (See e.g. [3], [8], [33] for earlier results.) Hölder regularity of fully nonlinear problems was studied in [30, §8.5.3]. The principle of linearized stability was established for various classes of nonlinear equations with special boundary conditions in e.g. [17], [21], [25], [29], [30], [32]. Local invariant manifolds for parabolic problems are well understood in the semilinear case, see in particular [26]. G. Da Prato and A. Lunardi constructed local stable, center and unstable manifolds for fully nonlinear problems with linear boundary conditions in a Hölder setting, see [12] and further [25], [30], [31] for related contributions. In [37] local center manifolds were investigated for quasilinear problems with conormal boundary conditions plus lower order terms. We are only aware of one paper, [9], dealing with invariant manifolds for fully nonlinear boundary conditions. There the unstable manifold was constructed for a single second order equation. In the current paper, we construct both stable and unstable manifolds, and the proof of our Theorem 5.2 indicates that the nonlinear restriction expressed by $\mathcal{M}$ enters only in the stable case explicitely. Other locally invariant, in particular center, manifolds will be treated in another paper (in preparation).

We establish both the local regularity and the asymptotic behavior within the same approach. We linearize the equations (1.1) at a given solution u_* (which is a steady state in the construction of the invariant manifolds), leading to the equations

(2.27). The linear regularity result Theorem 2.2 allows to understand (2.27) as a fix point problem in $\mathbb{E}_1$ for the solutions of (1.1). This problem can be solved by means of the implicit function theorem. However, in contrast to previous works one has to take care of the compatibility conditions. Therefore we have to incorporate certain correction terms which guarantee that the compatibility conditions are fulfilled, see (4.5) and (5.3). In this way we prove in Theorem 4.2 our regularity results, using also the scaling technique from [8]. In Theorem 5.2 we solve the fix point equation in spaces of exponentially decaying function on $\mathbb{R}_\pm$; thus obtaining solutions of (1.1) with the asymptotic behavior one expects for the stable and unstable manifolds. An additional effort is needed to show that, in fact, the initial values of the resulting decaying solutions define the local manifolds with the desired properties.

As indicated above, the spectrum of the generator $A_0 = A_*|\ker(B_*)$ determines much of the asymptotic behavior of solutions near the steady state u_*. Observe that A_0 does not directly appear in our problem (1.1) and also not in the construction of its solutions in Section 4. The relationship between A_0 and (1.1) becomes clear by means of an approach frequently used in boundary control theory, see e.g. [16], [36], and also [5, §11], [24], [30, p.200], [37, §8] for related techniques. Adapting this approach to the problem at hands, we derive in Proposition 2.6 a formula for the solutions of the linear problem (2.18) in terms of the semigroup $T(\cdot)$ generated by $-A_0$ and its extrapolation, cf. [6], [18]. Although this formula does not help much in questions of local regularity, it does allow to invoke the exponential dichotomy of $T(\cdot)$ in the study of the asymptotic behavior of the solutions to (1.1), cf. (3.1).

Our setting and the main concepts are described in Section 2, where also some auxiliary results are proved. Based on Theorem 2.2 and Proposition 2.6, we show the maximal regularity of the linear problem on $\mathbb{R}_+$ and $\mathbb{R}_-$ in Propositions 3.1 and 3.2, respectively. The technically most demanding result is Proposition 3.3 which establishes the continuous differentiability of the substitution (or Nemytskii) operators appearing in our fix point problems. Here the main difficulties arise from the (rather unpleasant) fact that the boundary data of the linear problem (2.18) live in the anisotropic Slobodetskii spaces defined in (2.13). The main results on local existence and regularity and on the asymptotic behavior are established in Sections 4 and 5, respectively. In Section 6 we study a reaction diffusion system in order to illustrate the spectral condition $i\mathbb{R} \subset \rho(A_0)$.

Notation. We set $D_k = -i\partial_k = -i\partial/\partial x_k$ and use the multi index notation. The k–tensor of the partial derivatives of order k is denoted by ∇^k, and we let $\underline{\nabla}^k u = (u, \nabla u, \cdots, \nabla^k u)$. For an operator A on a Banach space we write $\mathrm{dom}(A)$, $\ker(A)$, $\mathrm{ran}(A)$, $\sigma(A)$, and $\rho(A)$ for its domain, kernel, range, spectrum, and resolvent set, respectively. $\mathcal{B}(X, Y)$ is the space of bounded linear operators between two Banach spaces X and Y. For an open set U with boundary ∂U, $C^k(U)$ (resp., $BC^k(U)$, $BUC^k(U)$, $C_0^k(U)$) are the spaces of k–times continuously differentiable functions u on U (such that u and its derivatives up to order k are bounded, bounded and uniformly continuous, vanish at ∂U and at infinity (if U is unbounded), respectively), where $BC^k(U)$ is endowed with its canonical norm. For $C^k(\overline{U})$, $BC^k(\overline{U})$, $BUC^k(\overline{U})$, we require in addition that u and its derivatives up to order k have a continuous extension to ∂U. For unbounded U, we write $C_0^k(\overline{U})$ for the space of $u \in C^k(\overline{U})$ such that u and its derivatives up to order k vanish at infinity. By $W_p^k(U)$ we designate the Sobolev spaces, see e.g. [2, Def.3.1]. A generic constant will be denoted by c; by $\varepsilon : \mathbb{R}_+ \to \mathbb{R}_+$ we denote a generic nondecreasing function with $\varepsilon(r) \to 0$ as $r \to 0$. Finally, $J \subset \mathbb{R}$ is a closed interval.

2. Setting and preliminaries

Let $\Omega \subset \mathbb{R}^n$ be an open connected set with a compact boundary $\partial\Omega$ of class C^{2m} and outer unit normal $\nu(x)$, where $m \in \mathbb{N}$. Note that Ω is either bounded or an unbounded exterior domain. Throughout this paper, we fix a finite exponent p with

$$p > n + 2m. \tag{2.1}$$

Let $E = \mathbb{C}^N$ with $\mathcal{B}(E) = \mathbb{C}^{N\times N}$ for some fixed $N \in \mathbb{N}$. For a $\mathbb{C}^N$–valued function $u(t) = u(t,x)$, $t \geq 0$, $x \in \overline{\Omega}$, we investigate the quasilinear initial boundary value problem with fully nonlinear boundary conditions given by

$$\begin{aligned} \partial_t u(t) + A(u(t))u(t) = F(u(t)), &\quad \text{on } \Omega, \ \text{ a.e. } t > 0, \\ B_j(u(t)) = 0, &\quad \text{on } \partial\Omega, \ t \geq 0, \ j \in \{1, \cdots, m\}, \\ u(0) = u_0, &\quad \text{on } \Omega. \end{aligned} \tag{2.2}$$

Here we use the maps

$$\begin{aligned} [A(u)v](x) &= \sum_{|\alpha|=2m} a_\alpha(x, u(x), \nabla u(x), \cdots, \nabla^{2m-1}u(x))\, D^\alpha v(x), \quad x \in \Omega, \\ [F(u)](x) &= f(x, u(x), \nabla u(x), \cdots, \nabla^{2m-1}u(x))), \quad x \in \Omega, \\ [B_j(u)](x) &= b_j(x, u(x), \nabla u(x), \cdots, \nabla^{m_j}u(x)), \quad x \in \partial\Omega, \end{aligned} \tag{2.3}$$

for functions $u \in BC^{2m-1}(\overline{\Omega}; \mathbb{C}^N)$, resp. $u \in C^{m_j}(\overline{\Omega}; \mathbb{C}^N)$ in the last line of (2.3), and $v \in W_p^{2m}(\Omega; \mathbb{C}^N)$, integers $m_j \in \{0, 1, \cdots, 2m-1\}$, and coefficients satisfying

(R) $a_\alpha \in C^1(E \times E^n \times \cdots \times E^{(n^{2m-1})}; BC(\overline{\Omega}; \mathcal{B}(E)))$ for $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = 2m$,
$a_\alpha(x, 0) \longrightarrow a_\alpha(\infty)$ in $\mathcal{B}(E)$ as $x \to \infty$, if Ω is unbounded,
$f \in C^1(E \times E^n \times \cdots \times E^{(n^{2m-1})}; BC(\overline{\Omega}; E))$,
$b_j \in C^{2m+1-m_j}(\partial\Omega \times E \times E^n \times \cdots \times E^{(n^{m_j})}; E)$ for $j \in \{1, \cdots, m\}$.

We set $B = (B_1, \cdots, B_m)$. We point out that, for a fixed $u_0 \in BC^{2m-1}(\overline{\Omega}; \mathbb{C}^N)$, $A(u_0)$ is a linear differential operator of order $2m$ with bounded coefficients; whereas F contains all terms involving only derivatives of order $|\alpha| < 2m$. The boundary term $B_j(u_0)(x)$ is defined in the following way: One computes $\nabla^k u_0$ in Ω, then one takes the trace γ at $\partial\Omega$ and inserts $x \in \partial\Omega$, and finally one applies b_j. Usually we do not use γ explicitly in our notation, in particular if it is applied to a function being continuous up to $\partial\Omega$. We fix a numbering of the components of ∇^k so that a partial derivative $\partial^\beta u_0(x)$ of order $|\beta| = k$ is inserted at a fixed position called $l(\beta, k)$ into the functions a_α, f, and b_j. Given $u_0 \in C^{m_j}(\overline{\Omega}; \mathbb{C}^N)$, we further define

$$\begin{aligned} [B_j'(u_0)v](x) &= (\partial_z b_j)(x, u_0(x), \nabla u_0(x), \cdots, \nabla^{m_j}u_0(x)) \cdot \gamma \underline{\nabla}^{m_j} v(x) \\ &= \sum_{k=0}^{m_j} (\partial_{z_k} b_j)(x, u_0(x), \nabla u_0(x), \cdots, \nabla^{m_j}u_0(x))\, \gamma \nabla^k v(x) \\ &= \sum_{k=0}^{m_j} \sum_{|\beta|=k} i^k\, (\partial_{l(\beta,k)} b_j)(x, u_0(x), \nabla u_0(x), \cdots, \nabla^{m_j}u_0(x))\, \gamma D^\beta v(x) \end{aligned} \tag{2.4}$$

for $x \in \partial\Omega$, $v \in C^{m_j}(\overline{\Omega}; \mathbb{C}^N)$, and $j \in \{1, \cdots, m\}$. Here $\partial_z = (\partial_{z_0}, \cdots, \partial_{z_{m_j}})$ denotes the partial derivatives with respect to the variables $z = (z_0, z_1, \cdots, z_{m_j}) \in$

$E\times E^n\times\cdots\times E^{(n^{m_j})}$ and $\partial_{z_k}b_j(x,z)\in\mathcal{B}(E^{(n^k)},E)$ has the n^k components $\partial_{l(\beta,k)}b_j$. Observe that $B_j'(u_0)$ is a linear differential operator of order m_j with bounded coefficients acting from a space of functions on Ω to a space of functions on $\partial\Omega$. In Corollary 3.5 we show that $B_j'(u_0)$ is in fact the derivative of $u\mapsto B_j(u)$ at $u=u_0$ in a suitable topology. We set $B'(u_0)=(B_1'(u_0),\cdots,B_m'(u_0))$.

The symbols of the principal parts of the linear differential operators are the matrix–valued functions given by

$$\mathcal{A}_\#(x,z,\xi)=\sum_{|\alpha|=2m}a_\alpha(x,z)\,\xi^\alpha,\quad \mathcal{B}_{j\#}(x,z,\xi)=\sum_{|\beta|=m_j}i^{m_j}\,(\partial_{l(\beta,m_j)}b_j)(x,z)\,\xi^\beta$$

for $x\in\overline{\Omega}$, $z\in E\times\cdots\times E^{(n^{2m-1})}$ and $\xi\in\mathbb{R}^n$, resp. $x\in\partial\Omega$, $z\in E\times\cdots\times E^{(n^{m_j})}$ and $\xi\in\mathbb{R}^n$. We further set $\mathcal{A}_\#(\infty,\xi)=\sum_{|\alpha|=2m}a_\alpha(\infty)\,\xi^\alpha$ if Ω is unbounded. One defines the *normal ellipticity* and the *Lopatinskii–Shapiro condition* for $A(u_0)$ and $B'(u_0)$ at a function $u_0\in C_0^{2m-1}(\overline{\Omega};\mathbb{C}^N)$ as follows:

(E) $\sigma(\mathcal{A}_\#(x,\underline{\nabla}^{2m-1}u_0(x),\xi))\subset\{\lambda\in\mathbb{C}:\operatorname{Re}\lambda>0\}=:\mathbb{C}_+$ and (if Ω is unbounded) $\sigma(\mathcal{A}_\#(\infty,\xi))\subset\mathbb{C}_+$, for $x\in\overline{\Omega}$ and $\xi\in\mathbb{R}^n$ with $|\xi|=1$.

(LS) Let $x\in\partial\Omega$, $\xi\in\mathbb{R}^n$, and $\lambda\in\overline{\mathbb{C}_+}$ with $\xi\perp\nu(x)$ and $(\lambda,\xi)\neq(0,0)$. The function $\varphi=0$ is the only solution in $C_0(\mathbb{R}_+;\mathbb{C}^N)$ of the ode system

$$\lambda\varphi(y)+\mathcal{A}_\#(x,\underline{\nabla}^{2m-1}u_0(x),\xi+i\nu(x)\partial_y)\varphi(y)=0,\quad y>0,\tag{2.5}$$

$$\mathcal{B}_{j\#}(x,\underline{\nabla}^{m_j}u_0(x),\xi+i\nu(x)\partial_y)\varphi(0)=0,\quad j\in\{1,\cdots,m\}.\tag{2.6}$$

We refer to [5], [14], [15], and the references therein for more information concerning these conditions. In Section 6 we discuss a second order reaction–diffusion system as an example. We note a perturbation result for (E) and (LS) which was shown in Theorem 2.1 of [5] for the case $m=1$. So we only sketch its proof.

Remark 2.1. Assume that (R) holds and that (E) and (LS) hold for some $u_0\in C_0^{2m-1}(\overline{\Omega};\mathbb{C}^N)$. Take another function $u_1\in C_0^{2m-1}(\overline{\Omega};\mathbb{C}^N)$. Then (E) is valid for u_1 provided that $|u_0-u_1|_{BC^{2m-1}}$ is sufficiently small. We consider the equations in (LS) for a given $u\in C_0^{2m-1}(\overline{\Omega};\mathbb{C}^N)$ (instead of u_0) and for fixed $x\in\partial\Omega$, $\xi\in\mathbb{R}^n$, $\lambda\in\overline{\mathbb{C}_+}$ with $\xi\perp\nu(x)$ and $(\lambda,\xi)\neq(0,0)$. Using (E), we may rewrite the N–dimensional differential equation (2.5) of order $2m$ as an autonomous first order ode of dimension $2mN$ with corresponding N–dimensional boundary conditions $\mathbb{B}_j(u)v^{(j)}(0)=0$, $j\in\{1,\cdots,m\}$, cf. [14, p.73]. The resulting coefficient matrix $\mathbb{A}(u)$ is hyperbolic by [14, Prop.6.1]. Moreover, it can be seen as in the proof of Theorem 2.1 in [5] that $\mathbb{A}(u)$ has mN eigenvalues with negative real parts. Let $P(u)$ be the Riesz projection from $\mathbb{C}^{2mN}$ onto the stable subspace of $\mathbb{A}(u)$. Hence, the equation (2.5) has a mN–dimensional solution space in $C_0(\mathbb{R}_+;\mathbb{C}^N)$ isomorphic to $P(u)\mathbb{C}^{2mN}$. Observe that the Lopatinskii–Shapiro condition is equivalent to the surjectivity of the map $\mathbb{B}(u)P(u):\mathbb{C}^{2mN}\to\mathbb{C}^{mN}$, where $\mathbb{B}(u)=(\mathbb{B}_1(u),\cdots,\mathbb{B}_m(u))$. As a result, if $|u_0-u_1|_{BC^{2m-1}}$ is sufficiently small, then (LS) also holds for u_1. $\Diamond$

In this paper we need (E) and (LS) to obtain the maximal regularity of linearizations of (2.2), see Theorem 2.2 below. To state this result, we have to introduce spaces of functions on Ω, $\partial\Omega$, $J\times\Omega$, and $J\times\partial\Omega$, respectively. We first put

$$X_0=L_p(\Omega;\mathbb{C}^N),\quad X_1=W_p^{2m}(\Omega;\mathbb{C}^N),\quad X_p=W_p^{2m(1-1/p)}(\Omega;\mathbb{C}^N),$$

and denote the norms of these spaces by $|\cdot|_0$, $|\cdot|_1$, and $|\cdot|_p$, respectively. Various equivalent norms of the Slobodetskii spaces W_p^s are discussed in [2, Chap.VII], [38,

§4.4]. We use the 'intrinsic' norm given by

$$|v|_{W_p^s(\Omega)} = |v|_{L_p(\Omega)} + \sum_{|\alpha|=k} [\partial^\alpha v]_{W_p^\sigma(\Omega)}\,, \quad [w]^p_{W_p^\sigma(\Omega)} = \iint_{\Omega^2} \frac{|w(y)-w(x)|^p}{|y-x|^{n+\sigma p}}\,dx\,dy,$$

for $s = k+\sigma$ with $k \in \mathbb{N}_0$ and $\sigma \in (0,1)$, see [2, Thm.7.48], [38, Rem.4.4.1.2]. Occasionally we use without further notice that W_p^s coincides with the real interpolation space $(L_p, W_p^l)_{s/l,p}$ if $l \in \mathbb{N}$ and $s \in (0,l)$ is not an integer. (In our setting this fact can be shown as the results in [38, §4.3.1] using [2, Thm.4.26].) We note that $X_1 \hookrightarrow X_p \hookrightarrow X_0$ and that

$$X_p \hookrightarrow C_0^{2m-1}(\overline{\Omega};\mathbb{C}^N) \tag{2.7}$$

by (2.1) and standard properties of Sobolev spaces, cf. [38, §4.6.1]. Let $I \subset \mathbb{R}$ be an interval (maybe, not closed) containing more than a point. Then we introduce the function spaces

$$\mathbb{E}_0(I) = L_p(I; L_p(\Omega;\mathbb{C}^N)) = L_p(I;X_0),$$
$$\mathbb{E}_1(I) = W_p^1(I; L_p(\Omega;\mathbb{C}^N)) \cap L_p(I; W_p^{2m}(\Omega;\mathbb{C}^N)) = W_p^1(I;X_0) \cap L_p(I;X_1),$$

equipped with the natural norms. Mostly, we deal with closed intervals which are denoted by J instead of I.

We will look for solutions of (2.2) in the space $\mathbb{E}_1([0,T])$. Since we want to insert functions of the class C^{2m-1} into the nonlinearities, the following embedding is crucial for our approach:

$$\mathbb{E}_1(I) \hookrightarrow BUC(I;X_p) \hookrightarrow BUC(I; C_0^{2m-1}(\overline{\Omega};\mathbb{C}^N)), \tag{2.8}$$

see [6, Thm.III.4.10.2] for the first and (2.7) for the second embedding. We denote by $c_0 = c_0(I)$ the maximum of the norms of the first embedding in (2.8) and of $\mathbb{E}_1(I) \hookrightarrow BUC(I; C_0^{2m-1}(\overline{\Omega};\mathbb{C}^N))$. We point out that one can choose the same c_0 for intervals of length greater than a fixed $T_0 > 0$, see [6, Lem.III.4.10.1]. Moreover, one can choose an I–independent constant c_0 for functions vanishing at the left end point of I. (If u is given on $[0,T]$, say, then reflect it at T and extent it by 0 to $[2T,\infty)$. This extension operator is bounded from $\{u \in \mathbb{E}_1([0,T]) : u(0) = 0\}$ to $\mathbb{E}_1(\mathbb{R}_+)$ independently of T.)

We next discuss several mapping properties of traces in time and space. The trace operator at time $t = 0$ is denoted by γ_0. Lemma 3.7 of [15] shows that

$$\gamma_0 \in \mathcal{B}(\mathbb{E}_1([0,1]), X_p) \quad \text{has a bounded right inverse.} \tag{2.9}$$

Recall that the spatial trace operator γ at $\partial\Omega$ induces continuous maps

$$\gamma : W_p^s(\Omega;\mathbb{C}^N) \to W_p^{s-1/p}(\partial\Omega;\mathbb{C}^N) \tag{2.10}$$

for $1/p < s \le 2m$ if $s - 1/p$ is not an integer, and that these maps have bounded right inverses, see [2, Thm.7.53], [38, §4.7.1]. Here the Sobolev–Slobodetskii spaces on $\partial\Omega$ are defined via local charts, see [2, §7.51], [38, Def.3.6.1]. We set

$$Y_0 = L_p(\partial\Omega;\mathbb{C}^N), \quad Y_{j1} = W_p^{2m\kappa_j}(\partial\Omega;\mathbb{C}^N), \quad Y_{jp} = W_p^{2m\kappa_j - 2m/p}(\partial\Omega;\mathbb{C}^N)$$

for $j \in \{1, \cdots, m\}$ and the number

$$\kappa_j = 1 - \frac{m_j}{2m} - \frac{1}{2mp}\,. \tag{2.11}$$

Since $2m\kappa_j = 2m - m_j - 1/p$, (2.10) and (2.1) imply that

$$\gamma\partial^\beta \in \mathcal{B}(X_1, Y_{j1}) \cap \mathcal{B}(X_p, Y_{jp}), \qquad |\beta| \le m_j. \tag{2.12}$$

We let $Y_1 = Y_{11} \times \cdots \times Y_{m1}$ and $Y_p = Y_{1p} \times \cdots \times Y_{mp}$. The boundary data of our linearized equations will be contained in the spaces

$$\begin{aligned} \mathbb{F}_j(J) &= W_p^{\kappa_j}(J; L_p(\partial\Omega;\mathbb{C}^N)) \cap L_p(J; W_p^{2m\kappa_j}(\partial\Omega;\mathbb{C}^N)) \\ &= W_p^{\kappa_j}(J; Y_0) \cap L_p(J; Y_{j1}), \qquad j \in \{1, \cdots, m\}, \end{aligned} \tag{2.13}$$

endowed with their natural norms, where $\mathbb{F}(J) := \mathbb{F}_1(J)\times\cdots\times\mathbb{F}_m(J)$. If the context is clear, we also write $\mathbb{E}_0 = \mathbb{E}_0(\mathbb{R}_\pm)$, $\mathbb{E}_1 = \mathbb{E}_1(\mathbb{R}_\pm)$, and $\mathbb{F} = \mathbb{F}(\mathbb{R}_\pm)$. Moreover,

$$\begin{aligned} &\mathbb{F}_j(J) \hookrightarrow BUC(J; Y_{jp}) \hookrightarrow BUC(J \times \partial\Omega) \qquad \text{and} \\ &\gamma_0 \in \mathcal{B}(\mathbb{F}_j([0,1]), Y_{jp}) \quad \text{has a bounded right inverse.} \end{aligned} \tag{2.14}$$

Here the second embedding follows from Sobolev's embedding theorem using (2.1). For $\partial\Omega = \mathbb{R}^{n-1}$, the first embedding is a consequence of Proposition 3 in [34] applied to $(I-\Delta)^m$. Similarly, Proposition 4 in [34] gives the asserted right inverse of γ_0 in this case. The corresponding assertions for Ω with compact boundary of class C^{2m} can then be deduced via local change of coordinates, cf. the end of Section 3 of [15]. The norms of the embeddings in (2.14) depend on J as described after (2.8). Due to Lemma 3.5 of [15], the spatial trace extends to a continuous operator

$$\gamma : W_p^{1-m_j/2m}(J; X_0) \cap L_p(J; W_p^{2m-m_j}(\Omega;\mathbb{C}^N)) \to \mathbb{F}_j(J), \tag{2.15}$$

with a bounded right inverse. Further, Lemma 3.8 of [15] yields the continuity of

$$\partial^\beta : \mathbb{E}_1(J) \to W_p^{1-k/2m}(J; X_0) \cap L_p(J; W_p^{2m-k}(\Omega;\mathbb{C}^N)), \tag{2.16}$$

for $|\beta| \le k \le 2m$. We note that the cited results from [15] are stated for $J = \mathbb{R}_+$ and $\Omega = \{x \in \mathbb{R}^n : x_n > 0\}$. From these results, the assertions (2.9), (2.15), and (2.16) follow by local change of coordinates in $x \in \overline{\Omega}$ and by reflection and extension in t as indicated above.

We are now in a position to state the crucial existence and maximal regularity theorem for the linear initial boundary value problem associated with (2.2). Fix $T > 0$, $J = [0, T]$, and a function $u_* \in \mathbb{E}_1(J)$. Assume that (R), (E), and (LS) hold at all $u_*(t) \in C_0^{2m-1}(\overline{\Omega};\mathbb{C}^N)$, $t \in J$. The functions $a_\alpha^*(t,x) = a_\alpha(x, \underline{\nabla}^{2m-1}u_*(t,x))$, $|\alpha| = 2m$, belong to $BC(J \times \overline{\Omega}; \mathcal{B}(E))$ and $a_\alpha^*(t,x) \to a_\alpha(\infty)$ as $x \to \infty$ uniformly in $t \in J$, since $u_* \in C(J; C_0^{2m-1}(\overline{\Omega};\mathbb{C}^N))$ due to (2.8). Set $b_{j\beta}^*(t,x) = i^k(\partial_{l(\beta,k)}b_j)(x, \underline{\nabla}^{m_j}u_*(t,x))$ for $k = |\beta| \le m_j$ and $j \in \{1, \cdots, m\}$. (Recall the definition (2.4).) As in the proof of Proposition 3.3 one verifies that $b_{j\beta}^* \in \mathbb{F}_j(J)$. Thus the differential operators

$$\begin{aligned} &A(t) := A(u_*(t)) \in \mathcal{B}(X_1, X_0), \qquad t \in J, \\ &B_{j*}(t) := B_j'(u_*(t))) \in \mathcal{B}(X_1, Y_{j1}) \cap \mathcal{B}(X_p, Y_{jp}), \ \text{(a.e.)}\ t \in J,\ \ j \in \{1, \cdots, m\}, \end{aligned} \tag{2.17}$$

satisfy assumptions (E), (LS), (SD), (SB) from [15]. (The mapping properties of $B_{j*}(t)$ follow from (2.12), $b_{j\beta}^* \in \mathbb{F}_j(J)$, [35, Thm.4.6.4.1], and (2.1). We note that $B_j'(u_*(t))) \in \mathcal{B}(X_1, Y_{j1})$ holds if $b_{j\beta}^*(t) \in Y_{j1}$.) So Theorem 2.1 of [15] yields the following result (taking into account that $\kappa_j > 1/p$ by (2.1)).

Theorem 2.2. *Let $u_* \in \mathbb{E}_1(J)$ for $J = [0, T]$. Assume that (R) holds and that (E) and (LS) hold at all functions $u_*(t) \in C_0^{2m-1}(\overline{\Omega};\mathbb{C}^N)$, $t \in J$. Define $A(t)$ and*

$B_{j*}(t)$ by (2.17) for $t \in J = [0,T]$ and $j \in \{1, \cdots, m\}$. Then there is a unique $v =: S(v_0, g, h) \in \mathbb{E}_1(J)$ satisfying

$$\begin{aligned} \partial_t v(t) + A(t)v(t) = g(t) \qquad &\text{on } \Omega, \quad a.e.\ t > 0, \\ B_{j*}(t)v(t) = h_j(t) \qquad &\text{on } \partial\Omega, \quad t \geq 0, \quad j \in \{1, \cdots, m\}, \\ v(0) = v_0, \qquad &\text{on } \Omega, \end{aligned} \tag{2.18}$$

if and only if

$$(v_0, g, h) \in \mathcal{D}(J) := \{(v_0, g, h) \in X_p \times \mathbb{E}_0(J) \times \mathbb{F}(J) : B_*(0)v_0 = h(0)\}, \tag{2.19}$$

where $h := (h_1, \cdots, h_m)$. In this case, there is a constant $c_1 = c_1(J)$ such that

$$\|v\|_{\mathbb{E}_1(J)} \leq c_1 \left(|v_0|_p + \|g\|_{\mathbb{E}_0(J)} + \|h\|_{\mathbb{F}(J)}\right). \tag{2.20}$$

If the equivalence stated in Theorem 2.2 and estimate (2.20) hold, then we say that the initial boundary value problem (2.18) has *maximal regularity of type L_p on J*. Using extension arguments as above, one can check that $c_1 = c_1(T_0, T_1)$ if $T \in [T_0, T_1]$ and $0 < T_0 < T_1 < \infty$, and that $c_1 = c_1(T_1)$ if $h_j(0) = 0$ for all j. (The continuity of the extension operator from $\mathbb{F}(J)$ to $\mathbb{F}([0, T_1])$ can be shown via interpolation.) We point out that Theorem 2.2 gives *necessary* and *sufficient* conditions for the regularity of data which give rise to a solution of (2.18) in the desired regularity class $\mathbb{E}_1$. This fact forces us to use the spaces X_p and $\mathbb{F}$ if we want to treat (2.2) in an L_p–setting.

Next, we only assume that (R) holds. Let $u_0, v \in BC^{2m-1}(\overline{\Omega}; \mathbb{C}^N)$ and $w \in X_1$. In order to linearize (2.2), we introduce the operators

$$[F'(u_0)v](x) = \sum_{k=0}^{2m-1} \sum_{|\beta|=k} i^k \, (\partial_{l(\beta,k)} f)(x, u_0(x), \nabla u_0(x), \cdots, \nabla^{2m-1} u_0(x)) \, D^\beta v(x),$$

$$\begin{aligned} [A'(u_0)w]v(x) &= A'(u_0)[v,w](x) \\ &= \sum_{|\alpha|=2m} \sum_{k=0}^{2m-1} \sum_{|\beta|=k} (\partial_{l(\beta,k)} a_\alpha)(x, u_0(x), \cdots, \nabla^{2m-1} u_0(x)) \, [\partial^\beta v(x), D^\alpha w(x)] \end{aligned}$$

for $x \in \Omega$, with a similar notation as in (2.4). Note that $\partial_{l(\beta,k)} a_\alpha(x,z) : E^2 \to E$ is bilinear. For fixed $u_0 \in BC^{2m-1}(\overline{\Omega}; \mathbb{C}^N)$ and $w \in X_1$, the maps $F'(u_0)$ and $A'(u_0)w$ are linear differential operators of order $2m-1$. The matrix–valued coefficients of $F'(u_0)$ are bounded due to (R) and $u_0 \in BC^{2m-1}(\overline{\Omega}; \mathbb{C}^N)$. Sobolev's embedding theorem and (2.1) show that $X_p \hookrightarrow W_p^{2m-1}(\Omega; \mathbb{C}^N)$. We can thus consider $F'(u_0)$ as a bounded operator from X_p to X_0. By means of (2.7) and (R), we also obtain that $F' : X_p \to \mathcal{B}(X_p, X_0)$ is continuous and that

$$|F'(u_0)|_{\mathcal{B}(X_p, X_0)} \leq c(r) \qquad \text{for } \; |u_0|_{BC^{2m-1}} \leq r. \tag{2.21}$$

Similarly, the coefficients of $A'(u_0)$ are bounded, so that $[v,w] \mapsto A'(u_0)[v,w]$ is a bilinear map from $X_p \times X_1$ to X_0 with

$$|A'(u_0)[v,w]|_0 \leq c(|u_0|_{BC^{2m-1}}) \, |v|_{BC^{2m-1}} \, |w|_1 \leq c(|u_0|_{BC^{2m-1}}) \, |v|_p \, |w|_1, \tag{2.22}$$

employing again (2.7). Moreover, the map $u_0 \mapsto A'(u_0)$ is continuous from X_p to $\mathcal{B}(X_p, \mathcal{B}(X_1, X_0))$. On the other hand, using (R) and (2.7) one can easily check that there is a nondecreasing function $\varepsilon : \mathbb{R}_+ \to \mathbb{R}_+$ with $\varepsilon(r) \to 0$ as $r \to 0$ and

$$\begin{aligned} |F(u_0 + v) - F(u_0) - F'(u_0)v|_0 &\leq \varepsilon(|v|_p) \, |v|_p, \\ |A(u_0 + v)w - A(u_0)w - [A'(u_0)w]v|_0 &\leq \varepsilon(|v|_p) \, |v|_p \, |w|_1 \end{aligned} \tag{2.23}$$

for $v \in X_p$ and fixed $u_0 \in X_p$ and $w \in X_1$. Here ε depends on a_α, f, and $|u_0|_{BC^{2m-1}}$, but not on v or w. As a result, A' and F' are in fact the Fréchet derivatives of the functions

$$A \in C^1(X_p; \mathcal{B}(X_1, X_0)) \quad \text{and} \quad F \in C^1(X_p; X_0), \tag{2.24}$$

respectively. We also note that A' and F' are uniformly continuous on balls of X_p. We further introduce the nondecreasing function

$$c_{u_0}(r) = \sup\Big\{\|A'(u_0+v)\|_{\mathcal{B}(X_p,\mathcal{B}(X_1,X_0))} : \ |v|_p \le r\Big\}.$$

Employing the identity $[A(u_0+v) - A(u_0)]w = \int_0^1 A'(u_0+\theta v)[v,w]\,d\theta$, we can estimate

$$|[A(u_0+v) - A(u_0)]w|_0 \le c_{u_0}(r)\,|v|_p\,|w|_1 \tag{2.25}$$

for $u_0, v \in X_p$, $w \in X_1$, and $|v|_p \le r$.

We linearize (2.2) at its solution $u_* \in \mathbb{E}_1(J)$ obtaining the linear operators

$$\begin{aligned} A_*(t) &= A(u_*(t)) + A'(u_*(t))u_*(t) - F'(u_*(t)) \in \mathcal{B}(X_1, X_0), \\ B_{j*}(t) &= B_j'(u_*(t)) \in \mathcal{B}(X_p, Y_{jp}) \cap \mathcal{B}(X_1, Y_{j1}), \end{aligned} \tag{2.26}$$

for $t \in J$, cf. (2.17). Set $B_*(t) = (B_{1*}(t), \cdots, B_{m*}(t))$. Suppose that (R) is true and that (E) and (LS) hold for all $u_0 = u_*(t)$, $t \in J$. Then we can apply Theorem 2.1 of [15] also to $A_*(t)$ and $B_*(t))$, $t \in J$, since the lower order terms $A'(u_*(t))u_*(t) - F'(u_*(t))$ do not enter into (E) and (LS) of [15] and their coefficients belong to $L_\infty(J\times\Omega; \mathcal{B}(E)) + L_p(J\times\Omega;\mathcal{B}(E))$. Thus Theorem 2.2 holds for $A_*(t)$ and $B_*(t)$, $t \in J$.

For a given function $u \in \mathbb{E}_1([0,T])$, we set $v(t) = u(t) - u_*(t)$ and $v_0 = u_0 - u_*(0)$. Since u_* solves (2.2), the initial boundary value problem (2.2) for u is equivalent to the problem for v given by

$$\begin{aligned} \partial_t v(t) + A_*(t)v(t) &= G(t, v(t)) \qquad \text{on } \Omega, \quad \text{a.e. } t>0, \\ B_{j*}(t)v(t) &= H_j(t, v(t)) \qquad \text{on } \partial\Omega, \quad t \ge 0, \quad j \in \{1,\cdots,m\}, \\ v(0) &= v_0, \qquad \text{on } \Omega. \end{aligned} \tag{2.27}$$

Here we have used the nonlinear maps G and H defined by

$$\begin{aligned} G(t,v) &= \big(A(u_*(t))v - A(u_*(t)+v)v\big) - \big(A(u_*(t)+v)u_*(t) - A(u_*(t))u_*(t) \\ &\quad - [A'(u_*(t))u_*(t)]v\big) + \big(F(u_*(t)+v) - F(u_*(t)) - F'(u_*(t))v\big), \end{aligned} \tag{2.28}$$

$$H_j(t,v) = B_j'(u_*(t))v - B_j(u_*(t)+v), \quad j \in \{1,\cdots,m\}, \tag{2.29}$$

for a given $u_* \in \mathbb{E}_1(J)$ and all $t \in J$, $v \in X_1$ and $v \in C^{m_j}(\overline{\Omega}; \mathbb{C}^N)$, respectively. As usual, we set $H(t,v) = (H_1(t,v), \cdots, H_m(t,v))$. The mapping properties of G and H will be discussed in the next section. If u_* does not depend on t, then we write $A_* = A_*(t)$, $B_* = B_*(t)$, $G(v) = G(t,v)$, and $H(v) = H(t,v)$.

Definition 2.3. We say that a function u *solves* problem (2.2), (2.18) or (2.27) on a (possibly noncompact) interval I containing 0 if u belongs to $\mathbb{E}_1(J)$ for each compact interval $J \subset I$ and satisfies the respective problem for (a.e.) $t \in I$.

In the remainder of this section we discuss the setting for our investigations of the asymptotic behavior of the nonlinear problem (2.2).

Hypothesis 2.4. (a) Condition (R) holds and (E), (LS) hold at some $u_* \in X_1$.
(b) In addition, u_* is a steady solution of (2.2), i.e.,

$$A(u_*)u_* = F(u_*) \quad \text{on } \Omega, \qquad B(u_*) = 0 \quad \text{on } \partial\Omega.$$

Assuming Hypothesis 2.4(a), we define $A_0 = A_*| \ker(B_*)$, i.e.,

$$A_0 u = A_* u, \qquad u \in \mathrm{dom}(A_0) = \{u \in X_1 : B_{j*}u = 0,\ j = 1, \cdots, m\}. \tag{2.30}$$

The operator $-A_0$ generates an analytic semigroup $T(\cdot)$ in X_0 due to Theorem 8.2 of [14]. We fix a real number μ such that $\mu + A_0$ is invertible.

Proposition 2.5. *(a) Assume that Hypothesis 2.4(a) holds. Take $(\varphi_1, \cdots, \varphi_m) \in Y_1$. Then there is unique solution $u \in X_1$ of the elliptic boundary value problem*

$$\begin{aligned} (\mu + A_*)u &= 0 \qquad \textit{on } \Omega, \\ B_{j*}u &= \varphi_j \qquad \textit{on } \partial\Omega, \quad j \in \{1, \cdots, m\}. \end{aligned} \tag{2.31}$$

Setting $\mathcal{N}_1(\varphi_1, \cdots, \varphi_m) := u$, we further have $\mathcal{N}_1 \in \mathcal{B}(Y_1, X_1)$.

(b) Assume that (R) holds and that (E) and (LS) hold at some $u_0 \in X_p$. Then there exists a bounded right inverse $\mathcal{N}_p : Y_p \to X_p$ of the operator $B'(u_0) : X_p \to Y_p$.

Proof. We first want to show that $B_* : X_1 \to Y_1$ and $B'(u_0) : X_p \to Y_p$ are surjective. First, take $\varphi \in Y_1$ and a smooth scalar function χ with $\chi(0) = 0$ and $\chi(t) = 1$ for $t \geq 1$. Let $h(t, x) = \chi(t)\varphi(x)$, $v_0 = 0$, and $g = 0$. Then there is a solution $v \in \mathbb{E}_1([0, 2])$ of (2.18) for $A(t) = A_*$ and $B_*(t) = B_*$. Taking $t \geq 1$ with $v(t) \in X_1$, we obtain $B_* v(t) = \varphi$ due to (2.18). Second, let $\varphi \in Y_p$. By (2.14), there exists $h \in \mathbb{F}([1, 2])$ such that $h(1) = \varphi$ and $\|h\|_{\mathbb{F}} \leq c\,|\varphi|_p$. Set $h(t) = th(2-t)$ for $t \in [0, 1]$. Then $h \in \mathbb{F}([0, 2])$ and $h(0) = 0$. Similarly, one extends u_0 to a function $u \in \mathbb{E}_1([0, 2])$ such that $u(1) = u_0$ and $u(t) \in X_p$ satisfies (E) and (LS) for $t \in [0, 2]$ (use (2.9), Remark 2.1, and (2.8)). We consider the problem (2.18) with $A(t) = A(u(t))$, $B_*(t) = B'(u(t))$, the above h, $v_0 = 0$, and $g = 0$. Now one obtains as in the first step a function $v(1) \in X_p$ with $B'(u_0)v(1) = \varphi$. Moreover, the map $\mathcal{N}_p : Y_p \to X_p$ given by $\varphi \mapsto v(1)$ is bounded by (2.8) and (2.20).

Finally, we recall that $\mu + A_* : \mathrm{dom}(A_0) \to X_0$ is invertible and $B_* \in \mathcal{B}(X_1, Y_1)$. So we can apply Lemma 1.2 in [24] saying that X_1 is the direct sum of $\mathrm{dom}(A_0)$ and $\ker(\mu + A_*)$ and that the restriction $B_* : \ker(\mu + A_*) \to Y_1$ is an isomorphism. Thus the inverse $\mathcal{N}_1 := [B_*| \ker(\mu + A_*)]^{-1} \in \mathcal{B}(Y_1, X_1)$ solves (2.31). □

We note that for smooth coefficients and $N = 1$ it was shown in [35, Thm.3.5.3] that one can extend $\mathcal{N}_1$ to an operator in $\mathcal{B}(Y_p, X_p)$ still solving (2.31). However, we do not need such a result in this paper.

We can now establish a representation formula of the solution to (2.18) which is crucial for the study of the asymptotic behavior. The next proposition goes back to work in control theory, see e.g. [16] or [36]. For the formulation of the result we have to introduce some more concepts. Let X_{-1} denote the *extrapolation space* for A_0, that is, the completion of X_0 with respect to the norm $|u_0|_{-1} = |(\mu + A_0)^{-1}u_0|_0$, see e.g. [6, §V.1.3], [18, §II.5]. We can extend A_0 to an operator $A_{-1} : X_0 \to X_{-1}$ generating an analytic semigroup $T_{-1}(\cdot)$ on X_{-1} satisfying $T_{-1}(t)|X_0 = T(t)$. The semigroups $T(\cdot)$ and $T_{-1}(\cdot)$ are similar via the isomorphism $\mu + A_{-1} : X_0 \to X_{-1}$. We point out that $A_* u \neq A_{-1}u$ if $u \in X_1 \setminus \mathrm{dom}(A_0)$ due to (2.34) below. We further employ the map

$$\Pi := (\mu + A_{-1})\mathcal{N}_1 \in \mathcal{B}(Y_1, X_{-1}). \tag{2.32}$$

It can be seen that in our situation Π has better mapping properties than in (2.32), but we will not use this fact.

Proposition 2.6. *Assume that Hypothesis 2.4(a) holds and let* $v \in \mathbb{E}_1(J)$, $g \in \mathbb{E}_0(J)$, $h \in L_p(J; Y_1)$, *and* $v_0 \in X_0$ *for* $J = [0, T]$. *Consider the equations*

$$(a) \begin{cases} \dot v(t) + A_* v(t) = g(t), \\ B_* v(t) = h(t), \\ v(0) = v_0, \end{cases} \qquad (b) \begin{cases} \dot v(t) + A_{-1} v(t) = g(t) + (\mu + A_{-1})\mathcal{N}_1 h(t), \\ v(0) = v_0. \end{cases}$$

Then v *satisfies (a) for a.e.* $t \in J$ *if and only if it satisfies (b) for a.e.* $t \in J$. *If the solution exists, it is given by*

$$v(t) = T(t)v_0 + \int_0^t T(t-s)g(s)\,ds + \int_0^t T_{-1}(t-s)\Pi h(s)\,ds, \quad t \in J. \tag{2.33}$$

Proof. Let $u_0 \in X_1$. Observe that $B_*(u_0 - \mathcal{N}_1 B_* u_0) = 0$ by the definition of $\mathcal{N}_1$, and thus $u_0 - \mathcal{N}_1 B_* u_0 \in \operatorname{dom}(A_0)$. Hence, $(\mu + A_*)u_0 = (\mu + A_*)(u_0 - \mathcal{N}_1 B_* u_0) = (\mu + A_0)(u_0 - \mathcal{N}_1 B_* u_0) = (\mu + A_{-1})(u_0 - \mathcal{N}_1 B_* u_0)$, proving that

$$A_{-1}u_0 = A_* u_0 + (\mu + A_{-1})\mathcal{N}_1 B_* u_0 \quad \text{for all} \quad u_0 \in X_1. \tag{2.34}$$

Next, assume that v is a solution of (a). Since $v \in \mathbb{E}_1$ and $\mathcal{N}_1 B_* v = \mathcal{N}_1 h$, we can use (2.34) with $u_0 = v(t)$ to conclude that v solves (b). Conversely, assume that v is a solution of (b). Then $(\mu + A_{-1})(v(t) - \mathcal{N}_1 h(t)) = \mu v(t) - \dot v(t) + g(t)$ belongs to X_0 for a.e. $t \in J$. So we deduce $v(t) - \mathcal{N}_1 h(t) \in \operatorname{dom}(A_0)$, i.e., $B_*(v - \mathcal{N}_1 h) = 0$. This fact implies the second line in (a). To check the first line, we use (2.34) with $u_0 = v(t)$ again. □

Hypothesis 2.7. Assume that Hypothesis 2.4(a) holds and that $i\mathbb{R} \subseteq \rho(A_0)$, where A_0 is given by (2.30).

Under Hypothesis 2.7, the semigroup $T(\cdot)$ has an *exponential dichotomy*, i.e, there exist the (stable) projection $P \in \mathcal{B}(X_0)$ and a dichotomy exponent $\delta_0 > 0$ such that $T(t)P = PT(t)$, $T(t) : \ker(P) \to \ker(P)$ has an inverse denoted by $T_Q(-t)$, and

$$\|T(t)P\|, \|T_Q(-t)Q\| \le ce^{-\delta_0 t} \tag{2.35}$$

for $t \ge 0$, where we set $Q = I - P$. The projection Q maps X_0 to $\operatorname{dom}(A_0) \subseteq X_1$ because Q is the Riesz projection corresponding to the bounded part of $\sigma(-A_0)$ located in the open right half plane. (See [18] or [30].) Since $P = I - Q$, we have

$$P \in \mathcal{B}(X_1, X_1) \cap \mathcal{B}(\operatorname{dom}(A_0), \operatorname{dom}(A_0)) \cap \mathcal{B}(X_p, X_p). \tag{2.36}$$

Since also $i\mathbb{R} \in \rho(A_{-1})$, the extrapolated semigroup $T_{-1}(\cdot)$ has an exponential dichotomy on X_{-1}. Its dichotomy projections P_{-1} and Q_{-1} are extensions of P and Q, respectively. Observe that $Q_{-1} = QQ_{-1} \in \mathcal{B}(X_{-1}, \operatorname{dom}(A_0))$.

3. The main operators

First we want to show the maximal regularity of (2.18) on the interval $J = \mathbb{R}_+$ if Hypothesis 2.7 holds. Given $(w_0, g, h) \in \mathcal{D}(\mathbb{R}_+)$, we define

$$L(w_0, g, h)(t) = T(t)w_0 + \int_0^t T(t-s)Pg(s)\,ds - \int_t^\infty T_Q(t-s)Qg(s)\,ds \tag{3.1}$$

$$+ \int_0^t T_{-1}(t-s)P_{-1}\Pi h(s)\,ds - \int_t^\infty T_{Q,-1}(t-s)Q_{-1}\Pi h(s)\,ds$$

for $t \geq 0$, cf. (2.19) and (2.32). Observe that $T_Q(t-s)Q = QT_Q(t-s)Q$ and that $Q_{-1}\Pi = Q(\mu + A_0)Q\mathcal{N}_1$ is a bounded operator from Y_1 into $\mathrm{dom}(A_0)$. Taking into account (2.35), we see that the Q–integrals converge even in $\mathrm{dom}(A_0)$. We thus omit the index -1 in the last integral. Setting

$$v_0 = w_0 - \int_0^\infty T_Q(-s)Qg(s)\,ds - \int_0^\infty T_Q(-s)Q\Pi h(s)\,ds, \tag{3.2}$$

we obtain

$$L(w_0,g,h)(t) = T(t)v_0 + \int_0^t T(t-s)g(s)\,ds + \int_0^t T_{-1}(t-s)\Pi h(s)\,ds \tag{3.3}$$

for $t \geq 0$. Observe that $v_0 \in X_p$ and $B_*v_0 = B_*w_0 = h(0)$ because of $\mathrm{ran}(Q) \subset \ker(B_*)$ and (2.19). Therefore, due to Proposition 2.6 and Theorem 2.2, the function $L(w_0,g,h) = S(v_0,g,h)$ solves (2.18) on $\mathbb{R}_+$ with $A(t) = A_*$, $B_*(t) = B_*$, and the initial value v_0. We note that w_0 belongs to $\mathrm{ran}(P)$ if and only if

$$w_0 = Pv_0 \quad \text{or, equivalently,} \quad Qv_0 = -\int_0^\infty T_Q(-s)Q\Big(g(s) + \Pi h(s)\Big)ds, \tag{3.4}$$

where v_0 is defined by (3.2).

Proposition 3.1. *Assume that Hypothesis 2.7 holds. Take $g \in \mathbb{E}_0(\mathbb{R}_+)$, $h \in \mathbb{F}(\mathbb{R}_+)$, and $w_0 \in X_p$ with $B_*w_0 = h(0)$. Then $L(w_0,g,h) \in L_p(\mathbb{R}_+;X_0)$ if and only if $w_0 \in \mathrm{ran}(P)$, i.e.* (3.4) *holds. In this case, $L(w_0,g,h) = L(Pv_0,g,h)$ is the unique solution in $\mathbb{E}_1(\mathbb{R}_+)$ of* (2.18) *with $A(t) = A_*$, $B_*(t) = B_*$, and the initial value v_0 given by* (3.2) *and, moreover,*

$$\|L(w_0,g,h)\|_{\mathbb{E}_1(\mathbb{R}_+)} \leq c_1'\,(|v_0|_p + \|g\|_{\mathbb{E}_0(\mathbb{R}_+)} + \|h\|_{\mathbb{F}(\mathbb{R}_+)}). \tag{3.5}$$

Proof. We write $L(w_0,g,h) = T(t)w_0 + I_1 + I_2 + I_3 + I_4$, where I_j are the integrals in (3.1). Using (2.35) for $T_{-1}(t)$, the properties of Q and Proposition 2.5, one deduces that $\|I_2\|_{\mathbb{E}_1(\mathbb{R}_+)} \leq c\,\|g\|_{\mathbb{E}_0(\mathbb{R}_+)}$ and $\|I_4\|_{\mathbb{E}_1(\mathbb{R}_+)} \leq c\,\|h\|_{L_p(\mathbb{R}_+;Y_1)}$. Proposition 2.6, Theorem 2.2, and (3.3) further show that

$$\|L(w_0,g,h)\|_{\mathbb{E}_1([0,2])} \leq c_1\,(|v_0|_p + \|g\|_{\mathbb{E}_0([0,2])} + \|h\|_{\mathbb{F}([0,2])})\,.$$

Choose $\chi \in C^\infty([-1,1];\mathbb{R})$ with $\chi(-1) = 1$ and $\chi = 0$ on $[-1/2,1]$. For $n = 2,3,\dots$, set $\chi_n(s) = \chi(s-n)$ for $s \in [n-1,n+1]$ and $h_n = (1-\chi_n)h|[n-1,n+1]$. For $t \in [n,n+1]$, we can write

$$\begin{aligned} I_3(t) &= P\int_{n-1}^t T_{-1}(t-s)\Pi h_n(s)\,ds \\ &\quad + T(t-n)T_{-1}(\tfrac{1}{2})P_{-1}\int_{n-1}^{n-\frac{1}{2}} T_{-1}(n-\tfrac{1}{2}-s)\chi_n(s)\Pi h(s)\,ds \\ &\quad + T(t-n)T_{-1}(1)\int_0^{n-1} T_{-1}(n-1-s)P_{-1}\Pi h(s)\,ds \\ &=: I_{31}(t) + I_{32}(t) + I_{33}(t). \end{aligned} \tag{3.6}$$

Due to $h_n(n-1) = 0$, Theorem 2.2 combined with Proposition 2.6 and (2.36) yields

$$\|I_{31}\|_{\mathbb{E}_1([n,n+1])} \leq c\,\|h_n\|_{\mathbb{F}([n-1,n+1])} \leq c\,\|h\|_{\mathbb{F}([n-1,n+1])}\,.$$

We can sum the p–th power of this inequality employing

$$\sum_{n=2}^{\infty}[h_j]^p_{W_p^{\kappa_j}([n-1,n+1];Y_0)} = \sum_{n=2}^{\infty}\int_{n-1}^{n+1}\int_{n-1}^{n+1}\frac{|h_j(t)-h_j(s)|^p_{Y_0}}{|t-s|^{1+\kappa_j p}}\,dt\,ds$$
$$\le \sum_{n=2}^{\infty}\int_{n-1}^{n+1}\int_{1}^{\infty}\frac{|h_j(t)-h_j(s)|^p_{Y_0}}{|t-s|^{1+\kappa_j p}}\,dt\,ds \le 2\,[h_j]^p_{W_p^{\kappa_j}(\mathbb{R}_+;Y_0)}\,.$$

Since $T_{-1}(\tau) = T(\tau/2)T_{-1}(\tau/2) : X_{-1} \to \mathrm{dom}(A_0)$ for $\tau > 0$, we further deduce from (2.35) for $T_{-1}(t)$ that

$$\|I_{32}\|_{E_1([n,n+1])} \le c\,\|h\|_{L^p([n-1,n];Y_1)}\,,$$
$$|I_{33}(t)|_1 + |\partial_t I_{33}(t)|_0 \le c\int_0^{n-1} e^{-\delta_0(n-1-s)}|h(s)|_{Y_1}\,ds \le c\int_0^t e^{-\delta_0(t-s)}|h(s)|_{Y_1}\,ds.$$

These estimates imply that $\|I_3\|_{\mathbb{E}_1([2,\infty))} \le c\,\|h\|_{\mathbb{F}(\mathbb{R}_+)}$. In a similar way one can treat I_1. Finally, $t \mapsto T(t)w_0$ belongs to $L_p([2,\infty);X_0)$ if and only if $w_0 \in \mathrm{ran}(P)$. In this case we have $\|T(\cdot)w_0\|_{\mathbb{E}_1([2,\infty))} \le c\,|w_0|_0$. The proposition now follows by combining the above facts. □

We further need a modification of Proposition 3.1 for backward solutions of (2.18) on $\mathbb{R}_-$. Let $v_0 \in X_0$, $g \in \mathbb{E}_0(\mathbb{R}_-)$, and $h \in \mathbb{F}(\mathbb{R}_-)$. Assume that $v \in \mathbb{E}_0(\mathbb{R}_-)$ satisfies $v(0) = v_0$ and

$$v(t) = T(t-\tau)v(\tau) + \int_\tau^t T(t-s)g(s)\,ds + \int_\tau^t T_{-1}(t-s)\Pi h(s)\,ds \tag{3.7}$$

for all $\tau < t \le 0$. One can verify as in (3.6) that $v \in \mathbb{E}_1(J)$ for each interval $J = [a,0] \subset \mathbb{R}_-$ and that v solves (the analogue of) (2.18) on such intervals with the initial value $v(a)$ (using Proposition 2.6 and Theorem 2.2). We rewrite (3.7) as

$$v(t) = T(t-\tau)\Big[Pv(\tau) - \int_{-\infty}^{\tau} T_{-1}(\tau-s)\,P_{-1}(g(s)+\Pi h(s))\,ds\Big] \tag{3.8}$$
$$+ \int_{-\infty}^{t} T_{-1}(t-s)\,P_{-1}(g(s)+\Pi h(s))\,ds$$
$$+ T(t-\tau)Qv(\tau) + \int_\tau^t T(t-s)Q(g(s)+\Pi h(s))\,ds\,,$$

using (2.35). The last line is equal to $Qv(t)$ due to (3.7), so that we derive

$$Pv(t) = T(t-\tau)\Big[Pv(\tau) - \int_{-\infty}^{\tau} T_{-1}(\tau-s)P_{-1}\,(g(s)+\Pi h(s))\,ds\Big]$$
$$+ \int_{-\infty}^{t} T_{-1}(t-s)P_{-1}\,(g(s)+\Pi h(s))\,ds.$$

There is a sequence $\tau_n \to -\infty$ such that $v(\tau_n) \to 0$ in X_0. Letting $\tau = \tau_n \to -\infty$ in the above equation and taking $t = 0$, we thus obtain

$$Pv(t) = \int_{-\infty}^{t} T_{-1}(t-s)P_{-1}\Big(g(s)+\Pi h(s)\Big)\,ds, \tag{3.9}$$

$$Pv_0 = Pv(0) = \int_{-\infty}^{0} T_{-1}(-s)P_{-1}\Big(g(s)+\Pi h(s)\Big)\,ds, \tag{3.10}$$

by means of (2.35). If we first set $t = 0$ in (3.8) and then replace τ by t, we deduce

$$Qv(0) = T_Q(-t)Qv(t) + \int_t^0 T_{-1}(-s)Q\Big(g(s) + \Pi h(s)\Big)\,ds. \tag{3.11}$$

Combining (3.9) and (3.11), we see that $v(t)$ is equal to

$$L^-(v_0, g, h)(t) := T_Q(t)Qv_0 + \int_{-\infty}^t T(t-s)Pg(s)\,ds - \int_t^0 T_Q(t-s)Qg(s)\,ds$$
$$+ \int_{-\infty}^t T_{-1}(t-s)P_{-1}\Pi h(s)\,ds - \int_t^0 T_Q(t-s)Q\Pi h(s)\,ds \tag{3.12}$$

for $t \le 0$. Conversely, if (3.10) holds, then the function $L^-(v_0, g, h)$ satisfies (3.7) and $L^-(v_0, g, h)(0) = v_0$. Therefore $L^-(v_0, g, h)$ is a solution of (2.18) on $\mathbb{R}_-$ with the final value v_0. The following result can now be proved as Proposition 3.1.

Proposition 3.2. *Assume that Hypothesis 2.7 holds. Let $g \in \mathbb{E}_0(\mathbb{R}_-)$, $h \in \mathbb{F}(\mathbb{R}_-)$, and $v_0 \in X_0$. Consider problem* (2.18) *on $\mathbb{R}_-$ with $A(t) = A_*$, $B_*(t) = B_*$, and the final value $v(0) = v_0$. Then there is a solution v of* (2.18) *on $\mathbb{R}_-$ belonging to $L_p(\mathbb{R}_-; X_0)$ if and only if* (3.10) *holds. In this case, $v = L^-(v_0, g, h)$ is the unique solution of* (2.18) *in $\mathbb{E}_1(\mathbb{R}_-)$ with the final value v_0 and*

$$\|L^-(v_0, g, h)\|_{\mathbb{E}_1(\mathbb{R}_-)} \le c_1' \,(|Qv_0|_0 + \|g\|_{\mathbb{E}_0(\mathbb{R}_-)} + \|h\|_{\mathbb{F}(\mathbb{R}_-)}). \tag{3.13}$$

We will apply the above propositions mostly in 'rescaled' versions since we have to work in function spaces on $J = \mathbb{R}_\pm$ with exponential weight. We set $e_\delta(t) = e^{\delta t}$ for $t \in \mathbb{R}$ and $\delta \in \mathbb{R}$, and introduce the spaces

$$\mathbb{E}_k(\mathbb{R}_\pm, \delta) = \{v : e_\delta v \in \mathbb{E}_k(\mathbb{R}_\pm)\} \quad (k = 0, 1), \qquad \mathbb{F}(\mathbb{R}_\pm, \delta) = \{v : e_\delta v \in \mathbb{F}(\mathbb{R}_\pm)\}$$

endowed with the norms

$$\|v\|_{\mathbb{E}_k(\mathbb{R}_\pm,\delta)} = \|e_\delta v\|_{\mathbb{E}_k(\mathbb{R}_\pm)} \quad (k = 0, 1), \qquad \|v\|_{\mathbb{F}(\mathbb{R}_\pm,\delta)} = \|e_\delta v\|_{\mathbb{F}(\mathbb{R}_\pm)}.$$

We also use the analogous norms on compact intervals J. Mostly we deal with the interval $J = \mathbb{R}_+$ and abbreviate $\mathbb{E}_0(\mathbb{R}_+, \delta) = \mathbb{E}_0(\delta)$ etc. Assume that Hypothesis 2.7 and (3.4) hold, and take a solution v of (2.18) with $A(t) = A_*$ and $B_*(t) = B_*$. We define $w(t) = e^{\delta t} v(t)$ for $t \ge 0$, where $|\delta| < \delta_0$ and δ_0 is the exponential dichotomy constant, cf. (2.35). From $v = L(Pv_0, g, h)$ we deduce

$$w = e_\delta L(Pv_0, g, h) = L_\delta(Pv_0, e_\delta g, e_\delta h), \tag{3.14}$$

where L_δ is defined as L but for the generator $-A_0 + \delta$. Replacing $F(u)$ by $F(u) + \delta u$ in (R), we see that $A_0 - \delta$ satisfies Hypothesis 2.7. Thus we can apply Proposition 3.1 to L_δ, so that (3.14) yields

$$\|L(Pv_0, g, h)\|_{\mathbb{E}_1(\delta)} = \|w\|_{\mathbb{E}_1(\mathbb{R}_+)} \le c_2\,(|v_0|_p + \|g\|_{\mathbb{E}_0(\delta)} + \|h\|_{\mathbb{F}(\delta)}). \tag{3.15}$$

We point out that c_2 does not depend on δ with $|\delta| \le \delta_1 < \delta_0$.

We next study the Nemytskii operators $\mathbb{G}$ and $\mathbb{H}$ induced by the maps G and H from (2.28) and (2.29), assuming that (R) holds. For the intervals $\mathbb{R}_\pm$ we take a t–independent function $u_* \in X_1$ with $B(u_*) = 0$. For a compact interval J we take a function $u_* \in \mathbb{E}_1(J)$. For v belonging to $\mathbb{E}_1(\mathbb{R}_\pm, \delta)$ or $\mathbb{E}_1(J)$, respectively, we define $\mathbb{G}(v)(t) = G(t, v(t))$ and $\mathbb{H}_j(v)(t) = H_j(t, v(t))$ for a.e. $t \in J$, setting $\mathbb{H} = (\mathbb{H}_1, \cdots, \mathbb{H}_m)$ as usual. We stress the restrictions on δ in the following result; also, the choice of $+\delta$ corresponds to $\mathbb{R}_+$ while the choice of $-\delta$ corresponds to $\mathbb{R}_-$.

Proposition 3.3. *Assume that (R) holds, and let J be a compact interval.*

(I) Let $\delta \ge 0$. Take $u_* \in X_1$ with $B(u_*) = 0$ for the intervals $\mathbb{R}_\pm$, or respectively take $u_* \in \mathbb{E}_1(J)$ for the compact interval J. Then the following assertions are valid.
(a) We have $\mathbb{G} \in C^1(\mathbb{E}_1(\mathbb{R}_\pm, \pm\delta), \mathbb{E}_0(\mathbb{R}_\pm, \pm\delta))$, respectively $\mathbb{G} \in C^1(\mathbb{E}_1(J), \mathbb{E}_0(J))$. Moreover, $\mathbb{G}(0) = 0$, $\mathbb{G}'(0) = 0$, and

$$\begin{aligned} \mathbb{G}'(v)w = [F'(u_* + v) - F'(u_*)]w + [A(u_*) - A(u_* + v)]w \\ + [A'(u_*)u_* - A'(u_* + v)(u_* + v)]w \end{aligned} \tag{3.16}$$

for $v, w \in \mathbb{E}_1(\pm\delta, \mathbb{R}_\pm)$, respectively $v, w \in \mathbb{E}_1(J)$.
(b) We have $\mathbb{H} \in C^1(\mathbb{E}_1(\mathbb{R}_\pm, \pm\delta), \mathbb{F}(\mathbb{R}_\pm, \pm\delta))$, respectively $\mathbb{H} \in C^1(\mathbb{E}_1(J), \mathbb{F}(J))$. Moreover, $\mathbb{H}'(0) = 0$ and

$$\mathbb{H}'(v)w = [B'(u_*) - B'(u_* + v)]w \tag{3.17}$$

for $v, w \in \mathbb{E}_1(\mathbb{R}_\pm, \pm\delta)$, respectively $v, w \in \mathbb{E}_1(J)$. Finally, $\mathbb{H}(0) = 0$ if and only if $B(u_*(t)) = 0$ for all $t \in J$.

(II) Take an arbitrary $\delta \in \mathbb{R}$ and assume that $u_* \in X_1$ satisfies $B(u_*) = 0$ and that $v \in \mathbb{E}_1(\mathbb{R}_\pm, \delta)$ with $|v(t)|_p \le r$ for $t \in \mathbb{R}_\pm$. Then there is a nondecreasing function $\varepsilon : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\varepsilon(r) \to 0$ as $r \to 0$ and

$$\begin{aligned} \|\mathbb{G}(v)\|_{\mathbb{E}_0(\mathbb{R}_\pm,\delta)} &\le \varepsilon(r)\, \|e_\delta v\|_{L_p(\mathbb{R}_\pm;X_1)}\,, \\ \|\mathbb{H}(v)\|_{\mathbb{F}(\mathbb{R}_\pm,\delta)} &\le \varepsilon(r)\, \|v\|_{\mathbb{E}_1(\mathbb{R}_\pm,\delta)}\,, \\ \|e_\delta \mathbb{H}(v)\|_{L_p(\mathbb{R}_\pm;Y_1)} &\le \varepsilon(r)\, \|e_\delta v\|_{L_p(\mathbb{R}_\pm;X_1)}\,, \end{aligned} \tag{3.18}$$

where ε can be chosen uniformly for δ in compact intervals.

Proof. (1) In the proof we restrict ourselves to the case $J = \mathbb{R}_+$. The other cases can be treated in the same way. Also, the last assertion in (Ib) is an immediate consequence of (2.29). We point out that for $\delta \ge 0$ we have

$$|w(t)|_{BC^{2m-1}} \le c\,|w(t)|_p \le c\,|e^{\delta t} w(t)|_p \le c\,\|w\|_{\mathbb{E}_1(\delta)}\,, \qquad t \ge 0, \tag{3.19}$$

due to (2.7), (2.8), and $\delta t \ge 0$. In the following we always take $\delta \ge 0$ unless we are dealing with part (II).

We define $\mathbb{G}'(v)$ by (3.16) for $v \in \mathbb{E}_1(\delta)$. From (3.19), (2.21), (2.22), (2.23), and (2.25) we deduce that $\mathbb{G}(v) \in \mathbb{E}_0(\delta)$, $\mathbb{G}'(v) \in \mathcal{B}(\mathbb{E}_1(\delta), \mathbb{E}_0(\delta))$ and that the first line of (3.18) holds. Further, $\mathbb{G}'(v)$ is the Fréchet derivative of $\mathbb{G}$ at v due to (3.19), (2.23), (2.25), $\delta t \ge 0$, and the formula

$$\begin{aligned} &G(v + w) - G(v) - G'(v)w \\ &= \big(F(u_* + v + w) - F(u_* + v) - F'(u_* + v)w\big) - \big(A(u_* + v + w) - A(u_* + v)\big)w \\ &\quad - \big(A(u_* + v + w)(u_* + v) - A(u_* + v)(u_* + v) - [A'(u_* + v)(u_* + v)]w\big). \end{aligned}$$

The continuity of $v \mapsto \mathbb{G}'(v)$ follows from (3.19), (2.22), (2.24), and (2.25).

(2) We give the proof of the assertions concerning $\mathbb{H}_j$ for a fixed $j \in \{1, \cdots, m\}$ which will mostly be suppressed from the notation. We fix $v \in \mathbb{E}_1(\delta)$ and take $w \in \mathbb{E}_1(\delta)$ with $\|w\|_{\mathbb{E}_1(\delta)} \le r_0$ for a fixed, but arbitrary $r_0 > 0$. In the following, the constants will depend on v and r_0, but not on w. Define $\mathbb{H}'$ by (3.17). One can verify that $\mathbb{H}(v) \in \mathbb{F}(\delta)$ and $\mathbb{H}'(v) \in \mathcal{B}(\mathbb{E}_1(\delta), \mathbb{F}(\delta))$ by similar, but simpler arguments as used below. In view of (2.4) and (2.29), we can write

$$\begin{aligned} &-[H(t, v(t) + w(t)) - H(t, v(t)) - [\mathbb{H}'(v)w](t)](x) \\ &\quad = [B(u_* + v(t) + w(t)) - B(u_*(t) + v(t)) - B'(u_* + v(t))w(t)](x) \end{aligned}$$

$$
\begin{aligned}
&= b(x, \underline{\nabla}[u_*(x) + v(t,x) + w(t,x)]) - b(x, \underline{\nabla}[u_*(x) + v(t,x)]) \\
&\quad - (\partial_z b)(x, \underline{\nabla}[u_*(x) + v(t,x)]) \cdot \underline{\nabla} w(t,x) \\
&=: h(x, \underline{\nabla}[u_*(x) + v(t,x)], \underline{\nabla} w(t,x))
\end{aligned} \tag{3.20}
$$

where we set $\underline{\nabla} := \underline{\nabla}^{m_j} = (\nabla^0, \nabla^1, \cdots, \nabla^{m_j})$ and ∂_z is the partial derivative of b with respect to the corresponding arguments in $E \times E^n \times \cdots \times E^{(n^{m_j})}$. (Recall that we have suppressed the trace operator in front of all $\underline{\nabla}$ terms.) We set $\xi = \underline{\nabla}[u_*(x) + v(t,x)]$ and $\eta = \underline{\nabla} w(t,x)$ for fixed $x \in \partial\Omega$ and $t \geq 0$. Then we obtain

$$h(x, \xi, \eta) = b(x, \xi + \eta) - b(x, \xi) - (\partial_z b)(x, \xi) \cdot \eta, \tag{3.21}$$

$$\partial_\xi h(t, \xi, \eta) = (\partial_z b)(x, \xi + \eta) - (\partial_z b)(x, \xi) - (\partial_{zz} b)(x, \xi) \cdot \eta, \tag{3.22}$$

$$\partial_\eta h(t, \xi, \eta) = (\partial_z b)(x, \xi + \eta) - (\partial_z b)(x, \xi). \tag{3.23}$$

Assertion (R) and estimate (3.19) yield

$$|h(x, \xi, \eta)|, |\partial_\xi h(x, \xi, \eta)| \leq \varepsilon(|\eta|)\, |\eta|, \qquad |\partial_\eta h(t, \xi, \eta)| \leq c\, |\eta|, \tag{3.24}$$

where c and $\varepsilon(r)$ do not depend on x and are uniform for ξ, η in bounded sets. Using again (3.19) and $\delta t \geq 0$, we derive

$$
\begin{aligned}
&e^{\delta t}\, |H(v(t) + w(t)) - H(v(t)) - [\mathbb{H}'(v) w](t)|_{Y_0} \leq \varepsilon(|w(t)|_{BC^{2m-1}})\, |e^{\delta t} w(t)|_{BC^{2m-1}}\,, \\
&\|e_\delta\, [\mathbb{H}(v + w) - \mathbb{H}(v) - \mathbb{H}'(v) w]\|_{L_p(\mathbb{R}_+; Y_0)} \leq c\, \varepsilon(\|w\|_{\mathbb{E}_1(\delta)})\, \|e_\delta w\|_{L_p(\mathbb{R}_+; X_1)}\,.
\end{aligned} \tag{3.25}
$$

The corresponding inequality for part (II) is shown similarly.

(3) We now consider the estimate involving $W_p^\kappa(\mathbb{R}_+; Y_0)$ for $\kappa = \kappa_j$, cf. (2.11) and (2.13). We fix $x \in \partial\Omega$ and omit it in the notation. Then we can compute

$$
\begin{aligned}
&h(\underline{\nabla}(u_* + v(t)), \underline{\nabla} w(t)) - h(\underline{\nabla}(u_* + v(s)), \underline{\nabla} w(s)) \\
&\quad = \int_0^1 (\partial_\xi h)\big(\underline{\nabla}(u_* + v(s)) + \theta[\underline{\nabla}(u_* + v(t)) - \underline{\nabla}(u_* + v(s))], \underline{\nabla} w(t)\big) d\theta \\
&\qquad\qquad \cdot \underline{\nabla}[u_* + v(t) - (u_* + v(s))] \\
&\qquad + \int_0^1 (\partial_\eta h)\big(\underline{\nabla}(u_* + v(s)), \underline{\nabla} w(s) + \theta \underline{\nabla}(w(t) - w(s))\big) d\theta \cdot \underline{\nabla}(w(t) - w(s))
\end{aligned} \tag{3.26}
$$

for $t, s \geq 0$. Set $\varphi(t) = h(\underline{\nabla}(u_* + v(t)), \underline{\nabla} w(t))$ and $\psi(t) = \underline{\nabla}[u_* + v(t)]$. Then (3.19), (3.26), and (3.24) yield

$$
\begin{aligned}
|\varphi(t) - \varphi(s)|_{Y_0} &\leq \varepsilon(|w(t)|_{BC^{2m-1}})\, |w(t)|_{BC^{2m-1}}\, |\psi(t) - \psi(s)|_{Y_0} \\
&\quad + c\, |w(t)|_{BC^{2m-1}}\, |\underline{\nabla}(w(t) - w(s))|_{Y_0}
\end{aligned} \tag{3.27}
$$

for $t, s \geq 0$. In view of (2.15) and (2.16), the map $\gamma\, \partial^\beta : \mathbb{E}_1(\mathbb{R}_+) \to W_p^\kappa(\mathbb{R}_+; Y_0)$ is continuous for $|\beta| \leq m_j$. Combining this mapping property with (3.19), (3.27), Lemma 3.4 below, (3.25) and $\delta t \geq 0$, we derive

$$
\begin{aligned}
&[e_\delta\, (\mathbb{H}(v + w) - \mathbb{H}(v) - \mathbb{H}'(v) w)]_{W_p^\kappa(\mathbb{R}_+; Y_0)} \\
&\quad \leq c\, \varepsilon(\|w\|_{\mathbb{E}_1(\delta)})\, \|w\|_{\mathbb{E}_1(\delta)} + c\, \varepsilon(\|w\|_{BC(\mathbb{R}_+; X_p)})\, \|w\|_{BC(\mathbb{R}_+; X_p)}\, \|e_\delta \underline{\nabla} v\|_{W_p^\kappa(\mathbb{R}_+; Y_0)} \\
&\qquad + c\, \|w\|_{BC(\mathbb{R}_+; X_p)}\, \|e_\delta \underline{\nabla} w\|_{W_p^\kappa(\mathbb{R}_+; Y_0)} \\
&\quad \leq c\, \varepsilon(\|w\|_{\mathbb{E}_1(\delta)})\, \|w\|_{\mathbb{E}_1(\delta)}\,,
\end{aligned} \tag{3.28}
$$

possibly changing ε. The corresponding estimate for (II) is shown in the same way.

(4) For the study of the space regularity we may restrict ourselves to the case $\Omega = \{x \in \mathbb{R}^n : x_n > 0\}$ and functions with support in the unit ball in $\mathbb{R}^n$. The general

case is then deduced via local change of coordinates, see e.g. [2, §7.51]. We first consider the case of highest order $m_j = 2m-1$, where $\mathbb{F}_j = L_p(\mathbb{R}_+; W^{1-1/p}(\partial\Omega)) \cap W_p^\kappa(\mathbb{R}_+; L^p(\partial\Omega))$. Since $b \in C^2$ by (R), equation (3.21) yields

$$\partial_x h(x,\xi,\eta) = (\partial_x b)(x,\xi+\eta) - (\partial_x b)(x,\xi) - (\partial_z \partial_x b)(x,\xi)\cdot\eta, \tag{3.29}$$

$$|\partial_x h(x,\xi,\eta)| \le \varepsilon(|\eta|)\,|\eta|, \tag{3.30}$$

with c and ε having the same properties as in (3.24). We fix $t \ge 0$ and suppress it from our notation for a moment. Then we calculate

$$\begin{aligned}
&h(y,\underline{\nabla}(u_*(y)+v(y)),\underline{\nabla}w(y)) - h(x,\underline{\nabla}(u_*(x)+v(x)),\underline{\nabla}w(x)) \qquad (3.31)\\
&= \int_0^1 (\partial_x h)\big(x+\theta(y-x),\underline{\nabla}(u_*(y)+v(y)),\underline{\nabla}w(y)\big)d\theta\cdot(y-x)\\
&+ \int_0^1 (\partial_\xi h)\big(x,\underline{\nabla}(u_*(x)+v(x))+\theta[\underline{\nabla}(u_*(y)+v(y))-\underline{\nabla}(u_*(x)+v(x))],\underline{\nabla}w(y)\big)d\theta\\
&\quad\cdot\underline{\nabla}[u_*(y)+v(y)-u_*(x)-v(x)]\\
&+ \int_0^1 (\partial_\eta h)\big(x,\underline{\nabla}(u_*(x)+v(x)),\underline{\nabla}w(x)+\theta\underline{\nabla}(w(y)-w(x))\big)d\theta\cdot\underline{\nabla}(w(y)-w(x))
\end{aligned}$$

for $x,y \in \partial\Omega$. Set $\varphi(t,x) = h(x,\underline{\nabla}(u_*(x)+v(t,x)),\underline{\nabla}w(t,x))$ and $\psi(t,x) = \underline{\nabla}[u_*(x)+v(t,x)]$. Employing only (3.24) and (3.30), we deduce from (3.31) that

$$\begin{aligned}
|\varphi(t,y)-\varphi(t,x)| \le{}& \varepsilon(|w(t)|_{BC^{2m-1}})\,|w(t)|_{BC^{2m-1}}\Big(|y-x|+|\psi(t,y)-\psi(t,x)|\Big)\\
&+ c\,|w(t)|_{BC^{2m-1}}\,|\underline{\nabla}(w(t,y)-w(t,x))| \qquad (3.32)
\end{aligned}$$

for $x,y \in \partial\Omega$. Let K be the unit ball in $\mathbb{R}^{n-1}$. Estimate (3.32) leads to

$$\begin{aligned}
&\int_0^\infty e^{p\delta t}[\varphi(t)]^p_{W_p^{1-1/p}(\partial\Omega)}\,dt = \int_0^\infty \iint_{K^2} e^{p\delta t}\,\frac{|\varphi(t,y)-\varphi(t,x)|^p}{|y-x|^{n-2+p}}\,dx\,dy\,dt\\
&\le c\varepsilon(\|w\|_{BC(\mathbb{R};BC^{2m-1})})^p \int_0^\infty |e^{\delta t}w(t)|_1^p \iint_{K^2} \frac{|y-x|^p + |\underline{\nabla}u_*(y)-\underline{\nabla}u_*(x)|^p}{|y-x|^{n-2+p}}\,dx\,dy\,dt\\
&+ c\varepsilon(\|w\|_{BC(\mathbb{R};BC^{2m-1})})^p\,\|w\|^p_{BC(\mathbb{R};C^{2m-1})} \int_0^\infty e^{p\delta t}\iint_{K^2} \frac{|\underline{\nabla}v(t,y)-\underline{\nabla}v(t,x)|^p}{|y-x|^{n-2+p}}\,dx\,dy\,dt\\
&+ c\,\|w\|^p_{BC(\mathbb{R};BC^{2m-1})} \int_0^\infty \iint_{K^2} e^{p\delta t}\,\frac{|\underline{\nabla}w(t,y)-\underline{\nabla}w(t,x)|^p}{|y-x|^{n-2+p}}\,dx\,dy\,dt\\
&\le c\,\varepsilon(\|w\|_{\mathbb{E}_1(\delta)})^p\,\|w\|^p_{\mathbb{E}_1(\delta)}(1+\|e_\delta v\|^p_{L_p(\mathbb{R}_+;X_1)}) + c\,\|w\|^p_{\mathbb{E}_1(\delta)}\,\|e_\delta w\|^p_{L_p(\mathbb{R}_+;X_1)}
\end{aligned}$$

due to (3.19), Sobolev's embedding theorem, (2.1), (2.10) and the fact that $\delta t \ge 0$. Therefore, changing ε if needed, we arrive at

$$\|e_\delta\,[\mathbb{H}(v+w)-\mathbb{H}(v)-\mathbb{H}'(v)w]\|_{L_p(\mathbb{R}_+;Y_1)} \le c\,\varepsilon(\|w\|_{\mathbb{E}_1(\delta)})\,\|w\|_{\mathbb{E}_1(\delta)}\,. \tag{3.33}$$

The corresponding estimate for the last line in (3.18) is shown in the same way.

(5) Next, we consider the space regularity case for general $m_j \in \{0,\cdots,2m-1\}$. Define $\varphi(x) = \varphi(x,\xi(x),\eta(x)) = h(x,\underline{\nabla}^{m_j}[u_*(x)+v(t,x)],\underline{\nabla}^{m_j}w(t,x))$ with h from (3.20) and a fixed $t \ge 0$. Take a multiindex β with $|\beta| = 2m-1-m_j$. We

want to verify that the function $\partial^\beta\varphi(x)$ is a function of the form $\tilde h(x,\tilde\xi,\tilde\eta)$, where $\tilde\xi = \underline{\nabla}^{2m-1}[u_*(x)+v(t,x)]$ and $\tilde\eta = \underline{\nabla}^{2m-1}w(t,x)$, and that $\tilde h$ satisfies the analogues of (3.24) and (3.30). If this is the case, we can check as in step (4) that (3.33) and the last line in (3.18) also hold for lower order boundary terms. To this aim we claim that $\partial^\gamma\varphi(x)$ with $|\gamma| = l \in \{0,1,\cdots,2m-m_j-1\}$ is a linear combination of functions of the following type

$$\begin{aligned}
&[\psi(x,\xi(x)+\eta(x)) - \psi(x,\xi(x)) - \partial_2\psi(x,\xi(x))\cdot\eta(x)]\,P(\xi(x)),\\
&[\psi(x,\xi(x)+\eta(x)) - \psi(x,\xi(x))]\,P(\xi(x))\,Q_1(\eta(x)),\\
&\psi(x,\xi(x)+\eta(x))\,P(\xi(x))\,Q_2(\eta(x)),
\end{aligned}\tag{3.34}$$

for (differing) functions $\psi \in C^{2m+1-m_j-l}(\partial\Omega\times E\times\cdots\times E^{(n^{m_j})};E)$ and products P and Q_k of partial derivatives $\partial^a\xi(x)$ and $\partial^b\eta(x)$ having order $|a|,|b| \le l+m_j$. The products Q_1, resp. Q_2, contain at least 1, resp. 2, factors $\partial^b\eta(x)$. This assertion is easily checked via induction over l using (R). For $l = 2m-1-m_j$ we thus obtain functions $\psi \in C^2$ and products P, Q_k with factors $\partial_x^\alpha(u_*(x)+v(t,x))$ and $\partial_x^\alpha w(t,x)$ having order $|\alpha| \le 2m-1$. We compute the derivatives with respect to x, $\tilde\xi$, $\tilde\eta$ of the functions in (3.34) as we did in (3.22), (3.23), and (3.29). Taking into account (3.19) and (R), we can then derive (3.24) and (3.30) for $\tilde h(x,\tilde\xi,\tilde\eta)$.

(6) Using similar arguments, one can check the continuity of the map $v \mapsto \mathbb{H}'(v)$ from $\mathbb{E}_1(\delta)$ to $\mathcal{B}(\mathbb{E}_1(\delta),\mathbb{F}(\delta))$. □

Lemma 3.4. *If Z is a Banach space, $\alpha \in (0,1)$, and $\delta \in \mathbb{R}$, then*

$$\begin{aligned}
[e_\delta f]_{W_p^\alpha(\mathbb{R}_+;Z)} &\le c\,\|e_\delta f\|_{L_p(\mathbb{R}_+:Z)} + c\Big[\iint_{|t-s|\le 1} e^{\delta tp}\,\frac{|f(t)-f(s)|_Z^p}{|t-s|^{1+\alpha p}}\,ds\,dt\Big]^{\frac1p}\\
&\le c\,\|e_\delta f\|_{W_p^\alpha(\mathbb{R}_+;Z)}\,.
\end{aligned}$$

Proof. Let $\varphi(\tau) = \tau^{-1-\alpha p}$ for $|\tau| \ge 1$ and $\varphi(\tau) = 0$ for $|\tau| \le 1$. Using Minkowski's and Young's inequalities, we calculate

$$\begin{aligned}
&[e_\delta f]_{W_p^\alpha(\mathbb{R}_+;Z)}\\
&\le \Big[\iint_{|t-s|\ge 1}\frac{|e^{\delta t}f(t)-e^{\delta s}f(s)|_Z^p}{|t-s|^{1+\alpha p}}\,ds\,dt\Big]^{\frac1p} + \Big[\iint_{|t-s|\le 1}\frac{|e^{\delta t}f(t)-e^{\delta s}f(s)|_Z^p}{|t-s|^{1+\alpha p}}\,ds\,dt\Big]^{\frac1p}\\
&\le c\,\|\varphi * e_\delta|f|_Z\|_{L_p(\mathbb{R}_+)} + \Big[\iint_{|t-s|\le 1} e^{\delta tp}\,\frac{|f(t)-f(s)|_Z^p}{|t-s|^{1+\alpha p}}\,ds\,dt\Big]^{\frac1p}\\
&\quad + \Big[\iint_{|t-s|\le 1} e^{p\delta s}|f(s)|^p\,\frac{|e^{\delta(t-s)}-1|_Z^p}{|t-s|^{1+\alpha p}}\,dt\,ds\Big]^{\frac1p}\\
&\le c\,\|e_\delta f\|_{L_p(\mathbb{R}_+;Z)} + c\Big[\iint_{|t-s|\le 1} e^{\delta tp}\frac{|f(t)-f(s)|_Z^p}{|t-s|^{1+\alpha p}}\,ds\,dt\Big]^{\frac1p}.
\end{aligned}$$

The second estimate is shown in a similar way. □

Corollary 3.5. *Assume that (R) holds. Then $u_0 \mapsto B(u_0)$ belongs to $C^1(X_p;Y_p)$ with the derivative $B'(u_0)$ given by (2.4).*

Proof. Let R denote a bounded right inverse of $\gamma_0 \in \mathcal{B}(\mathbb{E}_1([0,1]),X_p)$, see (2.9). Define $\mathbb{H}$ with $u_* = 0$. Then $\Phi := \gamma_0\mathbb{H}R \in C^1(X_p;Y_p)$ and $\Phi'(u_0) = B'(0)u_0 - B'(u_0)$ by Proposition 3.3 and (2.14). Since $B'(0) \in \mathcal{B}(X_p,Y_p)$ by (2.17), the assertion follows. □

We start with the basic existence and uniqueness result for (2.2). For a single second order equation the next proposition (and its proof) is a special case of Theorem 6.1.2 in [41].

Proposition 4.1. *Assume that condition (R) holds and that (E) and (LS) hold at a function $u_0 \in X_p$ satisfying $B(u_0) = 0$. Then there is a number $T = T(u_0) > 0$ such that the problem (2.2) has a unique solution $u \in \mathbb{E}_1([0,T]) \hookrightarrow C([0,T];X_p)$.*

Proof. By (2.9) there exists a function $u_* \in \mathbb{E}_1(\mathbb{R}_+)$ with $u_*(0) = u_0$. (We do not require that u_* solves (2.2).) Remark 2.1 combined with (2.8) gives a number $T_0 > 0$ such that conditions (E) and (LS) for $A(u_*(t))$ and $B'(u_*(t)))$ hold at the function $u_*(t)$ for each $t \in [0,T_0]$. Temporarily we define $H(t,v)$ by (2.29) replacing u_* in this equation by zero. Then we can write $B'(u_*)v - B(v) = \mathbb{H}(v) - \mathbb{H}'(u_*)v$ for $v \in \mathbb{E}_1([0,T_0])$ and the resulting Nemytskii operator. Therefore Proposition 3.3 yields that

$$B'(u_*)v - B(v) \in \mathbb{F}([0,T_0]) \quad \text{for} \quad v \in \mathbb{E}_1([0,T_0]). \tag{4.1}$$

Taking into account (2.8), (2.24), (4.1) and $B(u_0) = 0$, Theorem 2.2 provides us with a solution $w \in \mathbb{E}_1([0,T_0])$ of the linear problem

$$\begin{aligned} \partial_t w(t) + A(u_*(t))w(t) &= F(u_*(t)) \qquad \text{on } \Omega, \quad \text{a.e. } t>0, \\ B'(u_*(t))w(t) &= B'(u_*(t))u_*(t) - B(u_*(t)) \qquad \text{on } \partial\Omega, \quad t \ge 0, \\ w(0) &= u_0, \qquad \text{on } \Omega. \end{aligned} \tag{4.2}$$

We define the space

$$\Sigma(T,\rho) = \{v \in \mathbb{E}_1([0,T]) : v(0) = u_0,\ \|v-w\|_{\mathbb{E}_1([0,T])} \le \rho\}$$

for $\rho > 0$ and $T \in (0,T_0]$. The set $\Sigma(T,\rho)$ is closed in $\mathbb{E}_1([0,T])$. For a given $u \in \Sigma(\rho,T)$, we consider the linear problem

$$\begin{aligned} \partial_t v(t) + A(u_*(t))v(t) &= F(u(t)) + [A(u_*(t)) - A(u(t))]u(t) \qquad \text{on } \Omega, \quad \text{a.e. } t>0, \\ B'(u_*(t))v(t) &= B'(u_*(t))u(t) - B(u(t)) \qquad \text{on } \partial\Omega, \quad t \ge 0, \\ v(0) &= u_0, \qquad \text{on } \Omega. \end{aligned} \tag{4.3}$$

Again, there is a solution $v \in \mathbb{E}_1([0,T])$ of (4.3) thanks to Theorem 2.2, (2.8), (2.24), (4.1), and $B(u_0) = 0$. We define the map $\mathcal{S} : \Sigma(T,\rho) \to \mathbb{E}_1([0,T])$ by setting $\mathcal{S}(u) := v$. Notice that $u \in \Sigma(T,\rho)$ solves (2.2) if and only if $u = \mathcal{S}(u)$.

We want to show that $\mathcal{S}$ is a strict contraction on $\Sigma(T,\rho)$ if $T > 0$ and $\rho > 0$ are small enough. By (4.2) and (4.3), the function $z = \mathcal{S}(u) - w \in \mathbb{E}_1([0,T])$ satisfies

$$\begin{aligned} \partial_t z(t) + A(u_*(t))z(t) &= F(u(t)) - F(u_*(t)) + [A(u_*(t)) - A(u(t))]u(t) =: g(t), \\ B'(u_*(t))z(t) &= B'(u_*(t))(u(t) - u_*(t)) - (B(u(t)) - B(u_*(t))) =: h(t), \\ z(0) &= 0. \end{aligned}$$

Observe that $h(0) = 0$ and $h = \mathbb{H}(u - u_*) - \mathbb{H}(0)$, where $\mathbb{H}$ is defined via (2.29) with u_* from the present proof. Using (2.20), (2.21), (2.25), Proposition 3.3, (2.8) and $u \in \Sigma(\rho,T)$, we estimate

$$\begin{aligned} &\|\mathcal{S}(u) - w\|_{\mathbb{E}_1([0,T])} \le c_1 \left(\|g\|_{\mathbb{E}_0([0,T])} + \|h\|_{\mathbb{F}([0,T])}\right) \\ &\le c\,\|u-u_*\|_{L_p([0,T];X_p)} + c\,\|u-u_*\|_{C([0,T];X_p)}\,\|u\|_{L_p([0,T];X_1)} \\ &\qquad + c\,\varepsilon(\|u-u_*\|_{\mathbb{E}_1([0,T])})\,\|u-u_*\|_{\mathbb{E}_1([0,T])} \end{aligned}$$

$$\leq c\,T^{\frac{1}{p}}(\rho + \|w-u_*\|_{C([0,T];X_p)}) + c\,(\rho + \|w-u_*\|_{C([0,T];X_p)})\,(\rho + \|w\|_{L_p([0,T];X_1)})$$
$$+ c\,\varepsilon(\rho + \|w-u_*\|_{\mathbb{E}_1([0,T])})\,(\rho + \|w-u_*\|_{\mathbb{E}_1([0,T])}).$$

Observe that the constants in the estimate above do not depend on $T \in (0, T_0]$ because of $h(0) = 0$ and $u(0) - w(0) = 0$. Since w and u_* are fixed with $w(0) - u_*(0) = 0$, we may choose sufficiently small $\rho_1 \in (0, \rho_0]$ and $T_1 \in (0, T_0]$ such that $\|\mathcal{S}(u) - w\|_{\mathbb{E}_1([0,T])} \leq \rho$ if $T \in (0, T_1]$ and $\rho \in (0, \rho_1]$. Consequently, $\mathcal{S}$ leaves $\Sigma(T, \rho)$ invariant for $T \in (0, T_1]$ and $\rho \in (0, \rho_1]$. Next, take $u, \overline{u} \in \Sigma(T, \rho)$ and set $v = \mathcal{S}(u)$ and $\overline{v} = \mathcal{S}(\overline{u})$. In view of (4.3), the function $z = v - \overline{v} \in \mathbb{E}_1([0,T])$ fulfills

$$\begin{aligned}
\partial_t z(t) + A(u_*(t))z(t) &= F(u(t)) - F(\overline{u}(t)) + [A(u_*(t)) - A(u(t))](u(t) - \overline{u}(t)) \\
&\qquad - [A(u(t)) - A(\overline{u}(t))]\overline{u}(t), \\
B'(u_*(t))z(t) &= B'(u_*(t))(u(t) - \overline{u}(t)) - (B(u(t)) - B(\overline{u}(t))), \\
z(0) &= 0.
\end{aligned}$$

Due to $\mathbb{H}'(0) = 0$, the right hand side of the second identity is equal to

$$-[\mathbb{H}(\overline{u} - u_*) - \mathbb{H}(u - u_*) - \mathbb{H}'(u - u_*)(\overline{u} - u)] + (\mathbb{H}'(0) - \mathbb{H}'(u - u_*))(\overline{u} - u),$$

where $\mathbb{H}$ is defined with via (2.29). Now we can proceed as above and deduce that $\mathcal{S}$ has the Lipschitz constant $1/2$ on $\Sigma(T, \rho)$ if we decrease T and ρ once more. As a result, we have obtained a local solution u of (2.2) on $[0, T]$.

Assume there is a different solution $\widehat{u}$ of (2.2) on $[0, T]$. Then there are numbers $t_0, t_n \in [0, T)$ such that $t_n \searrow t_0$ as $n \to \infty$, $u(t) = \widehat{u}(t)$ for $t \in [0, t_0]$, and $u(t_n) \neq \widehat{u}(t_n)$. We may apply the above argument with some $T', \rho' > 0$, the initial time t_0, and the initial value $u(t_0) =: u_1 \in X_p$ satisfying $B(u_1) = 0$. This leads to a contradiction establishing the uniqueness assertion. □

We now introduce in a standard way the maximal existence interval for the solution with initial value u_0. Under the assumptions of Proposition 4.1, let $t^+(u_0)$ be the supremum of those $T > 0$ such that (2.2) has a solution $u \in \mathbb{E}_1([0, T])$. Proposition 4.1 implies that $t^+(u_0) > 0$. This solution is unique provided that (E) and (LS) hold at the function $u(t)$ for each $t \in [0, t^+(u_0))$.

Next, we establish our main well–posedness result. It says that (2.2) generates a local semiflow on the nonlinear phase space

$$\mathcal{M} = \{u_0 \in X_p : B(u_0) = 0\}, \tag{4.4}$$

which is a C^1 manifold in X_p due to Corollary 3.5. Moreover, the equation possesses a smoothing effect because of the quasilinear structure. We write tu for the function $v(t) = tu(t)$. For a given $u_0 \in X_p$, we set

$$X_p^0 = \{z_0 \in X_p : B'(u_0)z_0 = 0\}.$$

If $u_0 \in \mathcal{M}$, then X_p^0 is the tangent space of $\mathcal{M}$ at u_0. Finally, if $u_0 \in X_p$ satisfies (E) and (LS), then we define a projection $\mathcal{P} : X_p \to X_p^0$ by $\mathcal{P}v_0 = (I - \mathcal{N}_p B'(u_0))v_0$, using the right inverse $\mathcal{N}_p \in \mathcal{B}(Y_p, X_p)$ of $B'(u_0)$ obtained in Proposition 2.5(b).

Theorem 4.2. *Assume that condition (R) holds and that (E) and (LS) hold at a function $u_0 \in X_p$ satisfying $B(u_0) = 0$. Let $u = u(\cdot; u_0)$ denote the unique solution of (2.2), and let (E) and (LS) hold at the function $u(t; u_0)$ for each $t \in [0, t^+(u_0))$. Let $T \in (0, t^+(u_0))$ and $J = [0, T]$. Then the following assertions are true.*

(a) There is an open ball $B_\rho(u_0)$ in X_p such that there exists a solution $w \in \mathbb{E}_1(J)$ of (2.2) for each initial value $w_0 \in B_\rho(u_0)$ satisfying $B(w_0) = 0$. Moreover, there is

an open ball W^0 in X_p^0 centered at 0 and a map $\Phi \in C^1(W^0;\mathbb{E}_1(J))$ with uniformly bounded derivative and $\Phi(0)=0$ such that $w = u + \Phi(\mathcal{P}(w_0-u_0))$ for $w_0 \in B_\rho(u_0)$ with $B(w_0)=0$.

(b) We have $tu \in W_p^1(J;X_1)\cap W_p^2(J;X_0)$, and thus $u \in C^1((0,T];X_p)\cap C^{2-1/p}((0,T];X_0)\cap C^{1-1/p}((0,T];X_1)$.

(c) Assume in addition that (E) and (LS) hold for all $u_1\in X_p$ with $B(u_1)=0$. If the number $t^+(u_0)$ is finite, then $\|u\|_{\mathbb{E}_1([0,t^+(u_0)))}=\infty$ and $u(t)$ does not converge in X_p as $t\to t^+(u_0)$.

Proof. (a) For the solution $u=u(t;u_0)$ of (2.2) with the given initial value u_0 we define $A_*(t)$, $B_*(t)$, $G(t)$, and $H(t)$ for $t\in J$ as in formulas (2.26), (2.28), and (2.29) but replacing in these formulas $u_*(t)$ by $u(t;u_0)$. Then $w\in\mathbb{E}_1(J)$ solves (2.2) with the initial value $w(0)=w_0\in X_p$ satisfying $B(w_0)=0$ if and only if $v=w-u$ solves (2.27) with the initial value $v_0=w_0-u_0\in X_p$ satisfying $B_*(0)v_0=H(0,v(0))$. We recall that $S:\mathcal{D}(J)\to\mathbb{E}_1(J)$ is the solution operator of (2.18) with $A_*(t)$ and $B_*(t)$ on J given by Theorem 2.2. We introduce the map

$$\mathcal{L}: X_p^0\times\mathbb{E}_1(J)\to\mathbb{E}_1(J);\quad \mathcal{L}(z_0,v)=v-S(z_0+\mathcal{N}_p\gamma_0\mathbb{H}(v),\mathbb{G}(v),\mathbb{H}(v)). \tag{4.5}$$

Observe that $\gamma_0\in\mathcal{B}(\mathbb{F}(J),Y_p)$ by (2.14) and that $\mathbb{H}(0)=B(u)=0$. We further have $B_*(0)(z_0+\mathcal{N}_p\gamma_0\mathbb{H}(v))=H(0,v(0))$, i.e.,

$$\Gamma: X_p^0\times\mathbb{E}_0(J)\times\mathbb{F}(J)\longrightarrow\mathcal{D}(J);\quad \Gamma(z_0,g,h)=(z_0+\mathcal{N}_p\gamma_0 h,g,h)$$

is a bounded linear map, cf. (2.19). Theorem 2.2 and Proposition 3.3 thus imply that $\mathcal{L}(0,0)=0$, $\mathcal{L}\in C^1(X_p^0\times\mathbb{E}_1(J);\mathbb{E}_1(J))$, and $\partial_2\mathcal{L}(0,0)=I$. Therefore the implicit function theorem, see e.g. [13, Cor.15.1], gives a ball $B_{r_0}(0)$ in X_p^0 and a map $\Phi\in C^1(B_{r_0}(0);\mathbb{E}_1(J))$ such that $\Phi(0)=0$ and $\mathcal{L}(z_0,\Phi(z_0))=0$ for $z_0\in B_{r_0}(0)$. This equation, Theorem 2.2, and Proposition 3.3 further yield

$$\Phi'(z_0)=S\Big(I+\mathcal{N}_p\gamma_0\mathbb{H}'(\Phi(z_0))\Phi'(z_0),\mathbb{G}'(\Phi(z_0))\Phi'(z_0),\mathbb{H}'(\Phi(z_0))\Phi'(z_0)\Big),$$
$$\|\Phi'(z_0)\|\le c+c\,(\|\mathbb{G}'(\Phi(z_0))\|+\|\mathbb{H}'(\Phi(z_0))\|)\,\|\Phi'(z_0)\|$$

(with the respective operator norms). Decreasing the radius $r_0>0$, we can make the factor in front of $\|\Phi'(z_0)\|$ on the right hand side smaller than 1/2. So $\Phi'(z_0)$ is uniformly bounded for z_0 in this smaller ball.

If we start with a given function $w_0\in X_p$ satisfying $B(w_0)=0$, then we set $v_0=w_0-u_0\in X_p$ and $z_0=v_0-\mathcal{N}_pH(0,v_0)=v_0-\mathcal{N}_pB'(u_0)v_0=\mathcal{P}v_0$. Hence, $z_0\in X_p^0$ and $|z_0|_p\le c\,|v_0|_p$. So we can fix a number $\rho>0$ such that $|w_0-u_0|_p<\rho$ implies $|z_0|_p<r_0$. Then $v=\Phi(z_0)\in\mathbb{E}_1(J)$ solves (2.27) with the initial value v_0, i.e., $w=v+u$ solves (2.2) with the initial value w_0.

(b) Take numbers $T>0$ and $\epsilon\in(0,1)$ such that u is a solution of (2.2) on $[0,T']$ with $T'=(1+\epsilon)T$. Let $J=[0,T]$, $\lambda\in(1-\epsilon,1+\epsilon)$, and $u_\lambda(t)=u(\lambda t)$. Then $v=u_\lambda$ is the unique solution of the problem

$$\begin{aligned}\partial_t v(t)+\lambda A(v(t))v(t)&=\lambda F(v(t)), &&\text{on }\Omega,\ \text{a.e. } t>0,\\ B(v(t))&=0, &&\text{on }\partial\Omega,\ t\ge 0,\\ v(0)&=u_0, &&\text{on }\Omega,\end{aligned}\tag{4.6}$$

on $[0,\lambda^{-1}T']$. We define $A_*(t)$ and $B_*(t)$ as in part (a), and we temporarily set $G(\lambda,t,v)=-\lambda A(v)v+A_*(t)v+\lambda F(v)$ and $H(t,v)=B_*(t)v-B(v)$. Then (4.6) is equivalent to

$$\begin{aligned} \partial_t v(t) + A_*(t)v(t) &= G(\lambda, t, v(t)), \qquad \text{on } \Omega, \ \text{ a.e. } t>0, \\ B_*(t)v(t) &= H(t, v(t)), \qquad \text{on } \partial\Omega, \ \ t \geq 0, \\ v(0) &= u_0, \qquad \text{on } \Omega. \end{aligned} \tag{4.7}$$

Let $\mathbb{G}(\lambda, \cdot)$ and $\mathbb{H}$ be the Nemytskii operators for $G(\lambda,\cdot)$ and H. As in Proposition 3.3, we see that $\mathbb{G} \in C^1((1-\epsilon, 1+\epsilon) \times \mathbb{E}_1(J); \mathbb{E}_0(J))$ with $\partial_2 \mathbb{G}(1, u) = 0$. Proposition 3.3 implies that $\mathbb{H} \in C^1(\mathbb{E}_1(J); \mathbb{F}(J))$ with $\mathbb{H}'(u) = 0$, cf. (4.1). The function $z_0 = u_0 - \mathcal{N}_p H(0, u_0)$ belongs to X_p^0. Fixing this z_0, we introduce the map

$$\mathcal{L}_0 : (1-\epsilon, 1+\epsilon) \times \mathbb{E}_1(J) \to \mathbb{E}_1(J); \ \mathcal{L}_0(\lambda, v) = v - S(z_0 + \mathcal{N}_p \gamma_0 \mathbb{H}(v), \mathbb{G}(\lambda, v), \mathbb{H}(v)),$$

where S is the solution operator of (2.18) for the operators $A_*(t)$ and $B_*(t)$. Since u solves (2.2), we have $\mathcal{L}_0(1, u) = 0$. As in part (a), we see that $\mathcal{L}_0$ is a C^1-map and $\partial_2 \mathcal{L}_0(1, u) = I$. The implicit function theorem thus yields an $\epsilon' \in (0, \epsilon)$, a ball $\mathbb{B}_{\rho_0}(u)$ in $\mathbb{E}_1(J)$, and a map $\Psi \in C^1((1-\epsilon', 1+\epsilon'); \mathbb{E}_1(J))$ such that $\Psi(1) = u$ and $\Psi(\lambda)$ solves (4.7) with u_0 replaced by $u_0(\lambda) := [\Psi(\lambda)](0)$. We further have

$$\begin{aligned} u_0(\lambda) &= z_0 + \mathcal{N}_p H(0, u_0(\lambda)) = u_0 + \mathcal{N}_p\big(H(0, u_0(\lambda)) - H(0, u_0)\big), \\ u_0(\lambda) - u_0 &= -\mathcal{N}_p\big(B(u_0(\lambda)) - B(u_0) - B'(u_0)(u_0(\lambda) - u_0)\big). \end{aligned}$$

Therefore Proposition 2.5, Corollary 3.5 and (2.8) yield

$$|u_0(\lambda) - u_0|_p \leq c\varepsilon(|u_0(\lambda) - u_0|_p)\, |u_0(\lambda) - u_0|_p \leq c\varepsilon(c\, \|\Psi(\lambda) - \Psi(1)\|_{\mathbb{E}_1})\, |u_0(\lambda) - u_0|_p$$

for constants c and a function ε with $\varepsilon(r) \to 0$ as $r \to 0$ which do not depend on λ. Decreasing $\epsilon' > 0$, we deduce that $u_0(\lambda) = u_0$, and thus $\Psi(\lambda)$ solves (4.6) provided $|\lambda - 1|$ is sufficiently small. So $u_\lambda = \Psi(\lambda)$ by the uniqueness of (4.6).

As a result, $u_\lambda = \Psi(\lambda) \in E_1(J)$ is continuously differentiable in λ with derivative $(\frac{d}{d\lambda} u_\lambda)(t) = t\dot{u}(\lambda t)$. Taking $\lambda = 1$, we deduce that $t\partial_t u \in \mathbb{E}_1(J)$. Consequently, $\partial_t(tu) = t\partial_t u + u \in \mathbb{E}_1(J) \hookrightarrow C(J; X_p)$, and hence $tu \in W_p^2(J; X_0) \cap W_p^1(J; X_1) \cap C^1(J; X_p)$. Assertion (b) now follows from Sobolev's embedding theorem.

(c) Suppose that $t^+(u_0) < \infty$ and $u \in \mathbb{E}_1([0, t^+(u_0)))$. Embedding (2.8) shows that $u(t)$ converges in X_p to some u_1 as $t \to t^+(u_0)$, and so $B(u_1) = 0$ follows from (R). Proposition 4.1 yields a solution $\bar{u}$ of (2.2) on $[t^+(u_0), t^+(u_0) + T_0]$ with the initial value u_1 and some $T_0 > 0$. Thus we obtain a solution $w \in \mathbb{E}_1([0, t^+(u_0) + T_0])$ of (2.2) by setting $w(t) = u(t)$ for $0 \leq t < t^+(u_0)$ and $w(t) = \bar{u}(t)$ for $t^+(u_0) \leq t \leq t^+(u_0) + T_0$. This fact contradicts the definition of $t^+(u_0)$. □

In the next section we need the following quantitative version of Theorem 4.2(b).

Proposition 4.3. *Let Hypothesis 2.4 hold. Take $T > 0$ and $\rho > 0$ from Theorem 4.2(a) for u_* (instead of u_0). Let $u = u(\cdot; u_0)$ solve* (2.2) *on $J = [0, T]$ for the initial value $u_0 \in B_\rho(u_*)$ with $B(u_0) = 0$. Then there exists $\widehat{\rho} \in (0, \rho]$ such that*

$$\|t(u - u_*)\|_{W_p^1(J; X_1)} + \|t(u - u_*)\|_{W_p^2(J; X_0)} \leq c\, |u_0 - u_*|_p$$

if also $|u_0 - u_*|_p < \widehat{\rho}$, with a uniform constant for such u_0.

Proof. Under the conditions of the current proposition, Theorem 4.2(a) yields $\|u - u_*\|_{\mathbb{E}_1(J)} \leq c\rho$. We define A_*, B_*, G, H, and S by (2.26), (2.28), (2.29), and Theorem 2.2 for the given steady state u_*. We further set $v(t) = u(t) - u_*$ and $v_0 = u_0 - u_*$. Then the function $v_\lambda(t) = v(\lambda t)$, $t \in J$, is the unique solution of

$$\partial_t w(t) + A_* w(t) = \lambda G(w(t)) + (1-\lambda) A_* w(t) =: G(\lambda, w(t)), \quad \text{on } \Omega, \ t > 0,$$

$$B_*w(t) = H(w(t)), \quad \text{on } \partial\Omega,\ t>0, \tag{4.8}$$
$$w(0) = v_0\,, \quad \text{on } \Omega,$$

where we take $\lambda \in (1-\epsilon, 1+\epsilon)$ and $\epsilon \in (0,1)$ such that $(1+\epsilon)T < t^+(u_0)$. Let $\mathcal{N}_p$ be the right inverse of $B_* = B'(u_*) \in \mathcal{B}(X_p, Y_p)$. We now proceed as in the proof of Theorem 4.2(b) using the operator

$$\mathcal{L}_0(\lambda, w) = w - S(z_0 + \mathcal{N}_p\gamma_0\mathbb{H}(w), \mathbb{G}(\lambda, w), \mathbb{H}(w))$$

for $\lambda \in (1-\epsilon, 1+\epsilon)$, $w \in \mathbb{E}_1(J)$, and $z_0 = v_0 - \mathcal{N}_p H(v_0)$. As above, we see that $\mathcal{L}_0 \in C^1((1-\epsilon, 1+\epsilon) \times \mathbb{E}_1(J); \mathbb{E}_1(J))$,

$$\mathcal{L}_0(1, v) = 0, \quad \text{and} \quad \partial_2\mathcal{L}_0(1, v) = I - S(\mathcal{N}_p\gamma_0\mathbb{H}'(v), \mathbb{G}'(v), \mathbb{H}'(v)).$$

Possibly after decreasing $\rho > 0$, and thus $\|v\|_{\mathbb{E}_1}$, Theorem 2.2 and Proposition 3.3 imply that $\partial_2\mathcal{L}_0(1, v)$ is invertible in $\mathbb{E}_1(J)$. So the implicit function theorem provides us with a map $\Psi \in C^1((1-\hat\epsilon, 1+\hat\epsilon); \mathbb{E}_1(J))$ such that $\Psi(1) = v$ and $\mathcal{L}_0(\lambda, \Psi(\lambda)) = 0$ for $|1-\lambda| \le \hat\epsilon$ and some $\hat\epsilon \in (0,1)$. We set $v_0(\lambda) = [\Psi(\lambda)](0)$. As in the proof of Theorem 4.2(b) we then obtain

$$\begin{aligned} v_0(\lambda) - v_0 = &-\mathcal{N}_p\big(B(v_0(\lambda)+u_*) - B(v_0+u_*) - B'(v_0+u_*)(v_0(\lambda)-v_0)\big) \\ &+ \mathcal{N}_p\big(B'(u_*) - B'(v_0+u_*)\big)(v_0(\lambda)-v_0), \end{aligned}$$

and we conclude that $v_0(\lambda) = v_0$, and hence $\Psi(\lambda) = v_\lambda$, if $\hat\epsilon > 0$ and $\rho > 0$ are small enough. Again it follows that $t\partial_t v = \Psi'(1) \in \mathbb{E}_1(J)$. We further compute

$$\Psi'(1) = -[\partial_2\mathcal{L}_0(1,v)]^{-1}\partial_1\mathcal{L}_0(1,v) = [\partial_2\mathcal{L}_0(1,v)]^{-1}S(0, G(v) - A_*v, 0).$$

Taking into account $\partial_t(tv) = v + t\partial_t v = v + \Psi'(1)$ and $v = u - u_*$, we arrive at

$$\|\partial_t(t(u-u_*))\|_{\mathbb{E}_1(J)} \le c\,\|u-u_*\|_{\mathbb{E}_1(J)} \le c\,|u_0-u_*|_p\,.$$

where we also used Theorem 2.2, Proposition 3.3, and Theorem 4.2(a). ☐

5. The hyperbolic saddle

In this section we will construct the stable and unstable manifolds for (2.2), which are C^1-submanifolds of the phase space $\mathcal{M}$ defined in (4.4). Let $u_* \in X_1$ be a steady state solution of (2.2) satisfying Hypothesis 2.4. Throughout this section, the maps G and H from (2.28) and (2.29) and the corresponding Nemytskii operators $\mathbb{G}$ and $\mathbb{H}$ are defined for the given u_*. We start with a simpler special case, proving the principle of linearized stability. Let $s(-A_0)$ denote the spectral bound of the generator $-A_0$ of the semigroup $T(\cdot)$ on X_0 introduced in (2.30).

Proposition 5.1. *Assume that Hypothesis 2.4 holds and that $s(-A_0) < -\delta < 0$. Then there exists a constant $\rho > 0$ such that for all $u_0 \in X_p$ with $|u_0 - u_*|_p \le \rho$ and $B(u_0) = 0$ the solution u of (2.2) exists for all $t \ge 0$ and satisfies $|u(t) - u_*|_1 \le ce^{-\delta t}$ for $t \ge 1$ and a constant not depending on t and u_0.*

Proof. Let $\rho > 0$, $v_0 \in X_p$, $|v_0|_p \le \rho$, and $B_*v_0 = H(v_0)$. We set

$$\Sigma(\rho) = \big\{v \in \mathbb{E}_1(\delta) : v(0) = v_0,\ \|v\|_{\mathbb{E}_1(\delta)} \le 2c_2\rho\big\}\,,$$

where c_2 the constant from (3.15) with $P = I$. We define $\mathcal{L}(v) = L(v(0), \mathbb{G}(v), \mathbb{H}(v))$ for $v \in \Sigma(\rho)$, where L is given by (3.1) with $Q = 0$ (and thus $w_0 = v_0$ in (3.2)). Note that $\mathcal{L}v(0) = v_0$, $H(v(0)) = H(v_0) = B_* v(0)$, and $|v(t)|_p \le c_0 \|v\|_{\mathbb{E}_1} \le 2c_0c_2\rho =: r$. Choosing ρ (and thus r) sufficiently small, we deduce from (3.15) and (3.18) that

$$\begin{aligned}\|\mathcal{L}v\|_{\mathbb{E}_1(\delta)} &\le c_2\left(|v_0|_p + \|\mathbb{G}(v)\|_{\mathbb{E}_0(\delta)} + \|\mathbb{H}(v)\|_{\mathbb{F}(\delta)}\right)\\ &\le c_2\rho + 2c_2\,\varepsilon(r)\|v\|_{\mathbb{E}_1(\delta)} \le 2c_2\rho.\end{aligned}$$

Take $v, w \in \Sigma(\rho)$. Since $H(v(0)) - H(w(0)) = 0 = v(0) - w(0)$, the estimate (3.15) and Proposition 3.3 imply that

$$\begin{aligned}\|\mathcal{L}v - \mathcal{L}w\|_{\mathbb{E}_1(\delta)} &\le c_2\left(\|\mathbb{G}(v) - \mathbb{G}(w)\|_{\mathbb{E}_0(\delta)} + \|\mathbb{H}(v) - \mathbb{H}(w)\|_{\mathbb{F}(\delta)}\right)\\ &\le 2c_2\eta(\rho)\,\|v - w\|_{\mathbb{E}_1(\delta)},\end{aligned}$$

where $\eta(\rho)$ is the supremum of $\|\mathbb{G}'(v)\|$ and $\|\mathbb{H}'(v)\|$ over $v \in \Sigma(\rho)$. Since $\eta(\rho) \to 0$ as $\rho \to 0$ by Proposition 3.3, we can decrease $\rho > 0$ once more to establish that $\mathcal{L}$ is a strict contraction on $\Sigma(\rho)$. So we obtain a fix point $v = \mathcal{L}v \in \Sigma(\rho)$, and thus a solution $u = v + u_*$ of (2.2) on $\mathbb{R}_+$ with

$$e^{\delta t}|u(t) - u_*|_p \le \|e_\delta v\|_{BC(\mathbb{R}_+;X_p)} \le c_0\|v\|_{\mathbb{E}_1(\delta)} \le 2c_0c_2\rho$$

for $t \ge 0$ using again (2.8). Proposition 4.3 further yields $|u(t+1) - u_*|_1 \le c\,|u(t) - u_*|_p$ for $t \ge 0$ if we decrease ρ to obtain $r < \widehat{\rho}$. ☐

We now come to the main result of our paper, assuming that $i\mathbb{R} \subset \rho(A_0)$. We recall the notation $X_p^0 = \{z_0 \in X_p : B_* z_0 = 0\}$ and denote by $B_r(u_0)$ and $\mathbb{B}_\rho(u)$ open balls in X_p and $\mathbb{E}_1(\delta)$, respectively. Recall that $\mathcal{M} = \{u_0 \in X_p : B(u_0) = 0\}$ is the solution manifold of (2.2). Observe that the dimension of the unstable manifold constructed below is equal to $\dim \operatorname{ran}(Q)$.

Theorem 5.2. *Assume that Hypotheses 2.4 and 2.7 hold with the dichotomy constant $\delta_0 > 0$. Fix $\delta \in (0, \delta_0)$, and let P and Q denote the stable and unstable projections on X_0 for the semigroup $T(\cdot)$. Then there exist constants $r \ge \rho > 0$ and manifolds $\mathcal{M}_\mathrm{s}$ and $\mathcal{M}_\mathrm{u}$ located in $\mathcal{M} \cap B_\rho(u_*)$ which are C^1 in X_p and tangential to the affine subspaces $u_* + PX_p^0$ and $u_* + QX_0$, respectively, such that for all $u_0 \in \mathcal{M}$ satisfying $|u_0 - u_*|_p < \rho$ the following assertions hold.*

(i) *If $u_0 \in \mathcal{M}_\mathrm{s}$, then the solution $u(t; u_0)$ of* (2.2) *exists and $|u(t; u_0) - u_*|_p \le r$ for all $t \ge 0$. Moreover, $|u(t; u_0) - u_*|_1 \le c\,|u_0 - u_*|_p\, e^{-\delta t}$ for all $t \ge 1$.*

(ii) *If $u_0 \notin \mathcal{M}_\mathrm{s}$, then $|u(t; u_0) - u_*|_p > r$ for some $t > 0$.*

(iii) *If $u_0 \in \mathcal{M}_\mathrm{u}$, then a backward solution $u(t; u_0)$ of* (2.2) *exists for all $t \le 0$, and it is the only backward solution staying in $\overline{B}_r(u_*)$ for all $t \le 0$. Also, $|u(t; u_0) - u_*|_p \le r$ and $|u(t; u_0) - u_*|_1 \le c\,|u_0 - u_*|_0\, e^{\delta t}$ for all $t \le 0$.*

(iv) *If $u_0 \notin \mathcal{M}_\mathrm{u}$, then any backward solution $u(t; u_0)$ of* (2.2) *either ceases to exist or leaves the ball $\overline{B}_r(u_*)$ at some $t < 0$.*

(The constants c do not depend on t or u_0.) As a result, $\mathcal{M}_\mathrm{s}$ (resp., $\mathcal{M}_\mathrm{u}$) is uniquely given as the set of the initial values $u_0 \in \mathcal{M} \cap B_\rho(u_*)$ of global forward (resp., backward) solutions $u(\cdot; u_0)$ with $|u(t; u_0) - u_*|_p \le r$ for all $t \ge 0$ (resp., $t \le 0$). Thus $\mathcal{M}_\mathrm{s}$ and $\mathcal{M}_\mathrm{u}$ are invariant for (2.2) *relative to $B_\rho(u_*)$ in the following sense: Let $u_0 \in \mathcal{M}_\mathrm{s}$ (resp., $u_0 \in \mathcal{M}_\mathrm{u}$), and let $u(\cdot; u_0)$ be a solution of* (2.2) *on $[0, t]$ if $t > 0$ or on $[t, 0]$ if $t < 0$ staying in $B_\rho(u_*)$ (where $u(\cdot; u_0)$ has to be the solution from (iii) if $u_0 \in \mathcal{M}_\mathrm{u}$ and $t < 0$). Then $u(t; u_0)$ belongs to $\mathcal{M}_s$ (resp., $\mathcal{M}_\mathrm{u}$).*

Proof. Construction of the stable manifold $\mathcal{M}_\mathrm{s}$. Observe that (2.8) yields

$$|v(t)|_p \le e^{\delta t}|v(t)|_p \le c_0\,\|v\|_{\mathbb{E}_1(\delta)}, \quad t \ge 0, \tag{5.1}$$

since $\delta t \geq 0$. Recall that $PX_p \subset X_p$ by (2.36). Moreover, due to $P = I - Q$ and $\operatorname{ran}(Q) \subset \operatorname{dom}(A_0)$, we have $PX_p^0 \subset X_p^0$ and thus $PX_p^0 = \operatorname{ran}(P) \cap X_p \cap \ker(B_*)$. Let $\mathcal{N}_p$ be the right inverse of $B_* = B'(u_*) \in \mathcal{B}(X_p, Y_p)$ obtained in Proposition 2.5. Then the operator $\Gamma(z_0, g, h) = (z_0 + P\mathcal{N}_p\gamma_0 h, g, h)$ maps $PX_p^0 \times \mathbb{E}_0(\delta) \times \mathbb{F}(\delta)$ into the space $\mathcal{D}_P(\delta) = \{(v_0, g, h) \in PX_p \times \mathbb{E}_0(\delta) \times \mathbb{F}(\delta) : B_* v_0 = h(0)\}$ by (2.14) and

$$B_* P\mathcal{N}_p = (B_* - B_* Q)\mathcal{N}_p = I \qquad \text{on } \; Y_p\,. \tag{5.2}$$

Note that $\mathcal{D}_P(\delta)$ is a closed subspace of $X_p \times \mathbb{E}_0(\delta) \times \mathbb{F}(\delta)$ thanks to (2.17) and (2.14). Proposition 3.1 and (3.15) say that the linear operator L defined in (3.1) is bounded from $\mathcal{D}_P(\delta)$ to $\mathbb{E}_1(\delta)$. We now introduce the Lyapunov-Perron map

$$\mathcal{L}_{\mathrm{s}} : PX_p^0 \times \mathbb{E}_1(\delta) \to \mathbb{E}_1(\delta); \quad \mathcal{L}_{\mathrm{s}}(z_0, v) = v - L(z_0 + P\mathcal{N}_p\gamma_0\mathbb{H}(v), \mathbb{G}(v), \mathbb{H}(v)). \tag{5.3}$$

Since $\delta > 0$, we may apply Proposition 3.3 to deduce that $\mathcal{L}_{\mathrm{s}} \in C^1(PX_p^0 \times \mathbb{E}_1(\delta); \mathbb{E}_1(\delta))$ and that $\mathcal{L}_{\mathrm{s}}(0,0) = 0$ and $\partial_2\mathcal{L}_{\mathrm{s}}(0,0) = I - L\Gamma(0, \mathbb{G}'(0), \mathbb{H}'(0)) = I$ hold. So the implicit function theorem, see e.g. [13, Cor.15.1], yields numbers $r_0, \rho_0 > 0$ and a C^1-map Φ_{s} from $PX_p^0 \cap B_{\rho_0}(0) \subset X_p$ to $\mathbb{B}_{r_0}(0) \subset \mathbb{E}_1(\delta)$ such that $\Phi_{\mathrm{s}}(0) = 0$ and $\mathcal{L}_{\mathrm{s}}(z_0, \Phi_{\mathrm{s}}(z_0)) = 0$ for each $z_0 \in PX_p^0 \cap B_{\rho_0}(0)$ and, moreover, $v = \Phi_{\mathrm{s}}(z_0)$ is the only solution of the equation $\mathcal{L}_{\mathrm{s}}(z_0, v) = 0$ satisfying $z_0 \in B_{\rho_0}(0)$ and $v \in \mathbb{B}_{r_0}(0)$. Due to Proposition 3.1 and (3.2), the function $v = \Phi_{\mathrm{s}}(z_0)$ solves problem (2.27) with the initial value

$$v_0 := v(0) = z_0 + P\mathcal{N}_p H(v(0)) - \int_0^\infty T_Q(-s)Q\Big(G(v(s)) + \Pi H(v(s))\Big)ds, \tag{5.4}$$

where $v(0) \in X_p$ and $B_* v(0) = H(v(0))$ by (5.2). Therefore the function $u(t; u_0) := v(t) + u_*$ solves (2.2) on $\mathbb{R}_+$ with the initial value $u_0 = v_0 + u_* \in \mathcal{M}$.

In view of decomposition (5.4), we define the map $\phi_{\mathrm{s}} : PX_p^0 \cap B_{\rho_0}(0) \to \operatorname{ran}(Q)$ by the formula

$$\phi_{\mathrm{s}}(z_0) = -\int_0^\infty T_Q(-s)Q\Big(G(\Phi_{\mathrm{s}}(z_0)(s)) + \Pi H(\Phi_{\mathrm{s}}(z_0)(s))\Big)ds, \tag{5.5}$$

and the map $\vartheta_{\mathrm{s}} : PX_p^0 \cap B_{\rho_0}(0) \to PX_p$ by the formula

$$\vartheta_{\mathrm{s}}(z_0) = P\mathcal{N}_p\gamma_0\mathbb{H}(\Phi_{\mathrm{s}}(z_0)). \tag{5.6}$$

So we can introduce the stable manifold

$$\mathcal{M}_{\mathrm{s}} = \big\{u_* + z_0 + \vartheta_{\mathrm{s}}(z_0) + \phi_{\mathrm{s}}(z_0) : z_0 \in PX_p^0,\ |z_0|_p < \rho\big\},$$

where $\rho \in (0, \rho_0]$ is fixed later. We have already checked that $\mathcal{M}_{\mathrm{s}} \subset \mathcal{M}$. The map Φ_s is C^1 from PX_p^0 to $\mathbb{E}_1(\delta)$ so that Proposition 3.3 and the properties of the linear operators in (5.5) and (5.6) show that the maps ϕ_s and ϑ_s are C^1 from PX_P^0 to $\operatorname{dom}(A_0)$ and $PX_p \subset X_p$, respectively. The identities $\phi_s(0) = \vartheta_s(0) = 0$ and $\phi_s'(0) = \vartheta_s'(0) = 0$ follow from $\Phi_s(0) = 0$, $\mathbb{G}(0) = 0$, $\mathbb{G}'(0) = 0$, $\mathbb{H}(0) = 0$, and $\mathbb{H}'(0) = 0$. As a result, $\mathcal{M}_{\mathrm{s}}$ is a C^1 manifold in X_p being tangent to PX_0^p at u_*.

Proof of assertion (*i*). Let $u_0 \in \mathcal{M}_{\mathrm{s}}$, $v_0 = u_0 - u_* = z_0 + \vartheta_s(z_0) + \phi_s(z_0)$, and $v = \Phi_s(z_0)$. As noted above, $u(t; u_0) = v(t) + u_*$ solves (2.2) on $\mathbb{R}_+$ with the initial value u_0. Estimate (5.1) further yields $|u(t; u_0) - u_*|_p \leq c_0\|v\|_{\mathbb{E}_1(\delta)}\, e^{-\delta t}$ for $t \geq 0$. Observe that $z_0 = P(v_0 - \mathcal{N}_p H(v_0)) = P(v_0 - \mathcal{N}_p B_* v_0)$ and thus $|z_0|_p \leq c\,|v_0|_p$ by (2.36), Proposition 2.5, and (2.17). From $\Phi_s(0) = 0$ we infer that

$$\|v\|_{\mathbb{E}_1(\delta)} \leq \|\Phi_s(z_0) - \Phi_s'(0)z_0\|_{\mathbb{E}_1(\delta)} + \|\Phi_s'(0)z_0\|_{\mathbb{E}_1(\delta)} \leq c\,|z_0|_p \leq c'\,|v_0|_p\,. \tag{5.7}$$

If $|v_0|_p < \rho_1$, then the above inequalities yield

$$|u(t;u_0) - u_*|_p \le c_0 c' \, |u_0 - u_*|_p \, e^{-\delta t} \le c_0 c' \rho_1 =: r_1$$

for $t \ge 0$. As in Proposition 5.1 one deduces the exponential estimate in X_1, where one may choose a small ρ_1 so that one can apply Proposition 4.3

Proof of assertion (ii). Take an initial value $u_0 \in \mathcal{M}$ with the corresponding solution $u = u(\cdot; u_0)$ of (2.2), and assume that

$$|u_0 - u_*|_p < \rho \quad \text{ and } \quad |u(t;u_0) - u_*|_p \le r \quad \text{for } \ t \ge 0 \tag{5.8}$$

and some numbers $\rho \in (0, \rho_1]$ and $r \in (0, r_1]$. We want to find sufficiently small $\rho_3 \in (0, \rho_1]$ and $r_3 \in (0, r_1]$ such that (5.8) with $\rho = \rho_3$ and $r = r_3$ implies that $u_0 \in \mathcal{M}_{\rm s}$. We let $v(t) = u(t; u_0) - u_*$ for $t \in \mathbb{R}_+$ so that v solves (2.27) for the initial value $v_0 = u_0 - u_*$ satisfying $B_* v_0 = H(v_0)$. Let us assume for a moment that Claim 5.3 below is true. Then Propositions 3.1 and 3.3 yield $v = L(Pv_0, \mathbb{G}(v), \mathbb{H}(v))$ if $\rho \in (0, \rho_2]$ and $r \in (0, r_2]$. We further set $z_0 = P(v_0 - \mathcal{N}_p H(v_0)) = P(v_0 - \mathcal{N}_p B_* v_0)$. Then $z_0 \in PX_p^0$ and $|z_0|_p \le c\rho$ by Proposition 2.5, (2.17), (2.36), and (5.2). Decreasing ρ if necessary, we thus obtain $|z_0|_p < \rho_0$ and hence there is a zero $w = \Phi(z_0) \in \mathbb{E}_1(\delta)$ of $\mathcal{L}_s$, i.e., $w = L(z_0 + P\mathcal{N}_p H(w(0)), \mathbb{G}(w), \mathbb{H}(w))$ and $w(0) + u_* \in \mathcal{M}_{\rm s}$. Possibly after choosing a smaller $\rho > 0$, we also have $\|w\|_{\mathbb{E}_1(\mathbb{R}_+)} \le r$ due to (5.7). Moreover, $B_*(Pv_0 - z_0 - P\mathcal{N}_p H(w(0))) = H(v(0)) - H(w(0))$ by (5.2). Propositions 3.1 and 3.3 and formulas (3.2) and (2.14) now imply that

$$\begin{aligned}
\|v - w\|_{\mathbb{E}_1} &\le c\big(|P(v(0) - w(0)) + Q(v(0) - w(0))|_p + \|\mathbb{G}(v) - \mathbb{G}(w)\|_{\mathbb{E}_0} \\
&\qquad + \|\mathbb{H}(v) - \mathbb{H}(w)\|_{\mathbb{F}}\big) \\
&\le c\big(|H(v(0)) - H(w(0))|_{Y_p} + \|\mathbb{G}(v) - \mathbb{G}(w)\|_{\mathbb{E}_0} + \|\mathbb{H}(v) - \mathbb{H}(w)\|_{\mathbb{F}}\big) \\
&\le c\eta(r) \, \|v - w\|_{\mathbb{E}_1} \,,
\end{aligned}$$

where $\eta(r)$ is the supremum of $\|\mathbb{G}'(\phi)\|$ in $\mathcal{B}(\mathbb{E}_1, \mathbb{E}_0)$ and $\|\mathbb{H}'(\phi)\|$ in $\mathcal{B}(\mathbb{E}_1, \mathbb{F})$ over ϕ with $\|\phi\|_{\mathbb{E}_1(\mathbb{R}_+)} \le r$. Decreasing $r > 0$ once more in (5.8), we see that $v = w$ and so $u_* + v_0 = u_* + w(0) \in \mathcal{M}_{\rm s}$. Thus we have obtained the desired numbers $\rho_3 \in (0, \rho_1]$ and $r_3 \in (0, r_1]$.

Claim 5.3. There are $\rho_2 \in (0, \rho_1]$ and $r_2 \in (0, r_1]$ such that each solution u of (2.2) satisfying (5.8) for some $\rho \in (0, \rho_2]$ and $r \in (0, r_2]$ already belongs to $\mathbb{E}_1(\mathbb{R}_+)$.

Proof of the claim. We take $\sigma \in (0, \delta]$ and $T \ge 1$, and we set $J = [0, T]$. The constants below do not depend on σ and T, unless explicitly stated. The function $v = u - u_*$ solves (2.27), and thus

$$Pv = T(\cdot)Pv_0 + T(\cdot)P * \mathbb{G}(v) + T_{-1}(\cdot)P * \Pi\mathbb{H}(v)$$

due to (2.33). Employing $B_* v_0 = H(v(0))$ and (2.36), we can argue as in the proof of Proposition 3.1 in order to estimate

$$\|Pv\|_{\mathbb{E}_1(J,-\sigma)} \le c\big(|Pv_0|_p + \|\mathbb{G}(v)\|_{\mathbb{E}_0(J,-\sigma)} + \|\mathbb{H}(v)\|_{\mathbb{F}(J,-\sigma)}\big). \tag{5.9}$$

Using the extension $v(t) = 0$ for $t \ge 2T$ and $v(t) = (2 - t/T)v(2T - t)$ for $T \le t \le 2T$, one obtains the estimates from (3.18) also on J with the weight $e_{-\sigma}$ and a function ε not depending on $T \ge 1$. We then deduce from (5.9), (5.8), (2.36), (3.18) that

$$\|Pv\|_{\mathbb{E}_1(J,-\sigma)} \le c\rho + c\varepsilon(r) \, \|v\|_{\mathbb{E}_1(J,-\sigma)} \le c\rho + c\varepsilon(r) \, \big(\|Pv\|_{\mathbb{E}_1(J,-\sigma)} + \|Qv\|_{\mathbb{E}_1(J,-\sigma)}\big).$$

Since $\varepsilon(r) \to 0$ as $r \to 0$, we can take a small r to infer

$$\|Pv\|_{\mathbb{E}_1(J,-\sigma)} \le c\rho + c\varepsilon(r) \, \|Qv\|_{\mathbb{E}_1(J,-\sigma)}. \tag{5.10}$$

We recall that Q maps X_0 in $\mathrm{dom}(A_0) \subset X_1$ and thus $|Qv(t)|_1 \le cr$ by (5.8), so that $e_{-\sigma}Qv \in L_p(\mathbb{R}_+; X_1)$. Proposition 2.6 further implies that

$$\begin{aligned} e_{-\sigma}Q\dot{v} &= e_{-\sigma}Q(-A_{-1}v + \Pi\mathbb{H}(v)) + e_{-\sigma}Q\mathbb{G}(v) \\ &= -A_0Qe_{-\sigma}v + (\mu + A_0)Q\mathcal{N}_1 e_{-\sigma}\mathbb{H}(v) + Qe_{-\sigma}\mathbb{G}(v). \end{aligned} \tag{5.11}$$

By means of $A_0Q \in \mathcal{B}(X_0)$, $|v(t)|_p \le r$, Proposition 2.5 and (3.18), we estimate

$$\begin{aligned} \|Q\dot{v}\|_{\mathbb{E}_0(J,-\sigma)} &\le c\left(\|e_{-\sigma}v\|_{\mathbb{E}_0(J)} + \|e_{-\sigma}\mathbb{H}(v)\|_{L_p(J;Y_1)} + \|e_{-\sigma}\mathbb{G}(v)\|_{\mathbb{E}_0(J)}\right) \\ &\le c(\sigma)r + c\varepsilon(r)\,\|e_{-\sigma}v\|_{L_p(J;X_1)} \\ &\le c(\sigma)r + c\varepsilon(r)\,\|e_{-\sigma}Pv\|_{L_p(J;X_1)}\,. \end{aligned} \tag{5.12}$$

Inserting this inequality into (5.10) and choosing a small $r > 0$ (not depending on J and σ), we arrive at the inequality $\|Pv\|_{\mathbb{E}_1(J,-\sigma)} \le c\rho + c(\sigma)r$. Hence, $Pv \in \mathbb{E}_1(\mathbb{R}_+, -\sigma)$ and, by (5.12), $Q\dot{v} \in \mathbb{E}_0(\mathbb{R}_+, -\sigma)$. As a result, $v \in \mathbb{E}_1(\mathbb{R}_+, -\sigma)$ if $r \le r_2'$, for a number $r_2' \in (0, r_0]$ independent of σ. Now (5.10) yields

$$\|Pv\|_{\mathbb{E}_1(\mathbb{R}_+,-\sigma)} \le c\rho + c\varepsilon(r)\,\|Qv\|_{\mathbb{E}_1(\mathbb{R}_+,-\sigma)}. \tag{5.13}$$

Observe that the shifted operator $-A_0 - \sigma$ satisfies Hypothesis 2.7. Thus we can transform (2.33) into (3.1) with $w_0 = Pv_0$ from (3.2) (where $g = \mathbb{G}(v)$ and $h = \mathbb{H}(v)$), and hence

$$Qv(t) = -\int_t^\infty T_Q(t-s)Q\big(\mathbb{G}(v(s)) + \Pi\mathbb{H}(v(s))\big)\,ds,$$

thanks to (2.35), (5.8), and (3.18). This formula combined with (2.35), (3.18) and (5.13) leads to the estimates

$$\begin{aligned} \|Qv\|_{\mathbb{E}_1(-\sigma)} &\le c\,\|Qv\|_{\mathbb{E}_0(-\sigma)} + \|Q\dot{v}\|_{\mathbb{E}_0(-\sigma)} \\ &\le c\left(\|Qv\|_{\mathbb{E}_0(-\sigma)} + \|\mathbb{G}(v)\|_{\mathbb{E}_0(-\sigma)} + \|e_{-\sigma}\mathbb{H}(v)\|_{L_p(\mathbb{R}_+;Y_1)}\right) \\ &\le c\left(\|\mathbb{G}(v)\|_{\mathbb{E}_0(-\sigma)} + \|e_{-\sigma}\mathbb{H}(v)\|_{L_p(\mathbb{R}_+;Y_1)}\right) \\ &\le c\varepsilon(r)\,\|e_{-\sigma}Qv\|_{L_p(\mathbb{R}_+;X_1)} + c\varepsilon(r)\,\|e_{-\sigma}Pv\|_{L_p(\mathbb{R}_+;X_1)} \\ &\le c\rho + c\varepsilon(r)\,\|Qv\|_{\mathbb{E}_1(-\sigma)}\,. \end{aligned} \tag{5.14}$$

Taking a small $r > 0$ independent of $\sigma \in (0, \delta]$, we see that $\sup_\sigma \|Qv\|_{\mathbb{E}_1(-\sigma)}$ is finite. Fatou's lemma then yields $Qv \in \mathbb{E}_1(\mathbb{R}_+)$, and so $Pv \in \mathbb{E}_1(\mathbb{R}_+)$ by (5.13). $\Diamond$

Construction of the unstable manifold $\mathcal{M}_{\mathrm{u}}$. The arguments for the unstable part are similar and somewhat simpler, so that we can omit some details. This time we employ the Lyapunov Perron map

$$\mathcal{L}_{\mathrm{u}} : \mathrm{ran}(Q) \times \mathbb{E}_1(\mathbb{R}_-, -\delta) \to \mathbb{E}_1(\mathbb{R}_-, -\delta); \quad \mathcal{L}_{\mathrm{u}}(z_0, v) = v - L^-(z_0, \mathbb{G}(v), \mathbb{H}(v)),$$

cf. (3.12). Propositions 3.2 and 3.3 then imply that $\mathcal{L}_{\mathrm{u}}$ is a C^1 map, $\mathcal{L}_{\mathrm{u}}(0,0) = 0$, and $\partial_2\mathcal{L}_{\mathrm{u}}(0,0) = I$. Hence, by the implicit function theorem, there exist balls $B_{\rho_0'}(0) \cap \mathrm{ran}(Q)$ and $\mathbb{B}_{r_0'}(0) \subseteq \mathbb{E}_1(\mathbb{R}_-, -\delta)$ and a C^1 map $\Phi_{\mathrm{u}} : B_{\rho_0'}(0) \to \mathbb{B}_{r_0'}(0)$ such that $v = \Phi_{\mathrm{u}}(z_0)$ is the unique solution of the equation $\mathcal{L}_{\mathrm{u}}(z_0, v) = 0$ for z_0 and v in these balls. Thus $u = \Phi_{\mathrm{u}}(z_0) + u_*$ is the unique function in $\mathbb{B}_{r_0'}(u_*)$ solving (2.2) on $\mathbb{R}_-$ with the final value $u_0 = v_0 + u_*$, see Proposition 3.2. We further define the map $\phi_{\mathrm{u}} : \mathrm{ran}(Q) \cap B_{\rho_0'}(0) \to PX_p$ by $\phi_{\mathrm{u}}(z_0) = \gamma_0\Phi_{\mathrm{u}}(z_0) - z_0$; that is,

$$\phi_{\mathrm{u}}(z_0) = \int_{-\infty}^0 T_{-1}(-s)P_{-1}\Big(G(\Phi_{\mathrm{u}}(z_0)(s)) + \Pi H(\Phi_{\mathrm{u}}(z_0)(s))\Big)ds.$$

Therefore $v(0) = \Phi_{\rm u}(z_0)(0) = z_0 + \phi_{\rm u}(z_0)$, $\phi_{\rm u}$ is C^1 due to (2.8), $\phi_{\rm u}(0) = 0$, and $\phi_{\rm u}'(0) = 0$. We now introduce the unstable manifold

$$\mathcal{M}_{\rm u} = \{u_* + z_0 + \phi_{\rm u}(z_0) : z_0 \in \operatorname{ran}(Q), |z_0|_p < \rho\}$$

for $\rho \in (0, \rho_0']$ to be fixed later. Clearly, $\mathcal{M}_{\rm u}$ is a C^1 manifold in X_p tangential to $u_* + \operatorname{ran}(Q)$.

Proof of assertion (*iii*). Let $u_0 \in \mathcal{M}_{\rm u}$, $z_0 = Q(u_0 - u_*)$, and $v = \Phi_u(z_0)$. Then $u(t; u_0) = v(t) + u_*$ solves (2.2) on $\mathbb{R}_-$ with the final value u_0. As in part (i), we can deduce that $|u(t; u_0) - u_*|_p \le c\,|u_0 - u_*|_0\, e^{\delta t}$ for $t \le 0$, using (2.8), (5.7), and $Q \in \mathcal{B}(X_0, X_1)$. Proposition 4.3 further yields $|u(t; u_0) - u_*|_1 \le c\,|u(t-1; u_0) - u_*|_p$ for $t \le 0$ (possibly after decreasing ρ). This fact implies assertion (iii) for all numbers $\rho \in (0, \rho_4]$ and $r \in (0, r_4]$ and some $\rho_4 \in (0, \rho_3]$ and $r_4 \in (0, r_3]$.

Proof of assertion (*iv*). Let u be a backward solution of (2.2) on $\mathbb{R}_-$ with $|u(t) - u_*|_p \le r$ for $t \le 0$ and $|u_0 - u_*|_p < \rho$. As in part (*ii*) we have to show that $v = u - u_* \in \mathbb{E}_1(\mathbb{R}_-)$ provided that $r, \rho > 0$ are small enough. We take $0 < \sigma \le \delta$ and $T \le -2$ and set $J = [T + 1, 0]$. In what follows, the constants do not depend on σ and T unless otherwise stated. The formula (2.33) yields

$$Pv(t) = T(t-T)Pv(T) + \int_T^t T(t-s)PG(v(s))\,ds + \int_T^t T_{-1}(t-s)P_{-1}\Pi H(v(s))\,ds$$

for $T \le t \le 0$. Arguing as in (3.6) and using (3.18), we estimate

$$\begin{aligned} \|Pv\|_{\mathbb{E}_1(J,\sigma)} &\le c\big(r + \|\mathbb{G}(v)\|_{\mathbb{E}_0(J,\sigma)} + \|\mathbb{H}(v)\|_{\mathbb{F}(J,\sigma)}\big) \\ &\le cr + c\varepsilon(r)\,\|Pv\|_{\mathbb{E}_1(J,\sigma)} + c\varepsilon(r)\,\|Qv\|_{\mathbb{E}_1(J,\sigma)}\,, \\ \|Pv\|_{\mathbb{E}_1(J,\sigma)} &\le cr + c\varepsilon(r)\,\|Qv\|_{\mathbb{E}_1(J,\sigma)}\,, \end{aligned} \tag{5.15}$$

taking a small r independent of J and σ. We further have $|Qv(t)|_1 \le cr$ for $t \le 0$, and so $e_\sigma Qv \in L_p(\mathbb{R}_-; X_1)$. As in (5.11) and (5.12), one obtains

$$\|e_\sigma Q\dot{v}\|_{L_p(J;X_0)} \le c(\sigma)r + c\varepsilon(r)\,\|e_\sigma Pv\|_{L_p(J;X_1)}\,.$$

So we conclude that $v \in \mathbb{E}_1(\mathbb{R}_-, \sigma)$ if $0 < r \le r_5$ where $0 < r_5 \le r_4$ is sufficiently small and does not depend on σ. Thus we can transform (2.33) into the form (3.12) with Pv_0 from (3.10), and so

$$Qv(t) = T_Q(t)Qv_0 - \int_t^0 T_Q(t-s)Q\big(G(v(s)) + \Pi H(v(s))\big)\,ds,$$

thanks to (2.35), $|v(t)|_p \le r$, and (3.18). We argue as in (5.14) in order to deduce

$$\|Qv\|_{\mathbb{E}_1(\mathbb{R}_-,\sigma)} \le c\rho + cr + c\varepsilon(r)\|Qv\|_{\mathbb{E}_1(\mathbb{R}_-,\sigma)}\,.$$

Taking a small σ–independent $r_6 \in (0, r_5]$, we obtain a σ–independent bound on $\|Qv\|_{\mathbb{E}_1(\mathbb{R}_-,\sigma)}$. So Fatou's lemma yields $Qv \in \mathbb{E}_1(\mathbb{R}_-)$, and (5.15) implies $Pv \in \mathbb{E}_1(\mathbb{R}_+)$. The theorem follows fixing sufficiently small $\rho \in (0, \rho_4]$ and $r \in (0, r_6]$. □

6. A reaction diffusion system

In this section we study a quasilinear reaction diffusion system for two species u_1 and u_2 on a bounded domain $\Omega \subset \mathbb{R}^n$ with C^2 boundary $\partial\Omega$ and outer unit normal ν. The validity of (E) and (LS) was established in [5] for large classes of reaction diffusion systems of second order. Here we concentrate on a simple situation where we can give more explicit criteria for the hyperbolicity condition $i\mathbb{R} \subset \rho(A_0)$ from

Hypothesis 2.7. For the unknown function $u(t,x) = (u_1(t,x), u_2(t,x)) \in \mathbb{R}^2$ we consider the problem

$$\begin{aligned} \partial_t u_i(t,x) - \operatorname{div}[d_i(u(t,x))\nabla u_i(t,x)] &= r_i(u(t,x)), \quad t>0,\ x\in\Omega,\ i=1,2,\\ d_i(u(t,x))\partial_\nu u_i(t,x) - q_i(u_i(t,x)) &= b_i^0(x,u(t,x),\nabla u(t,x)), \quad t\ge 0,\ x\in\partial\Omega,\\ u(0,x) &= u_0(x), \quad x\in\Omega, \end{aligned} \tag{6.1}$$

where $d_i \in C^2(\mathbb{R}^2;\mathbb{R})$, $q_i \in C^2(\mathbb{R};\mathbb{R})$, $r_i \in C^1(\mathbb{R}^2;\mathbb{R})$, and $b_i^0 \in C^2(\partial\Omega\times\mathbb{R}^2\times\mathbb{R}^{2n};\mathbb{R})$ for $i=1,2$. We work with real valued functions in this section, considering the complexification if necessary (in particular when applying the results of the previous sections). We assume that there is a vector $u_* = (u_{*1}, u_{*2}) \in \mathbb{R}^2$ such that

$$d_i(u_*) > 0, \quad r_i(u_*) = q_i(u_*) = b_i^0(x,u_*,0) = 0, \quad \partial_{(2,3)} b_i^0(x,u_*,0) = 0$$

for $i=1,2$ and $x\in\partial\Omega$. Thus the constant function u_* is a steady state solution of (6.1). Moreover, (6.1) contains conormal boundary conditions combined with the nonlinear source terms $q_i(u_i)$ and the additional fully nonlinear perturbations b_i^0 which vanish at the equilibrium. Let $d = \operatorname{diag}(d_1,d_2)$, $r=(r_1,r_2)$, $q=(q_1,q_2)$, $b^0 = (b_1^0, b_2^0)$. Then we can transform (6.1) into the form (2.2) by setting

$$A(u)v = -d(u)\Delta u, \qquad b(u) = d(u)(\nu\cdot\nabla u_1, \nu\cdot\nabla u_2) - q(u) - b^0(\cdot,u,\nabla u),$$

$$F(u) = r(u) + \Big[\sum_{j=1}^{n} (d_i'(u)\cdot\partial_j u)\,\partial_j u_i\Big]_{i=1,2},$$

where $x\cdot y$ denotes the standard scalar product in $\mathbb{R}^2$. Since $\nabla u_* = 0$, we obtain

$$A_* = -d(u_*)\Delta - r'(u_*) \quad \text{and} \quad B_* = d(u_*)\partial_\nu - q'(u_*),$$

cf. (2.26). It is clear that (R) holds. Moreover $A(u_*)$ and $B_* = B'(u_*)$ satisfy (E) and (LS) due to [5, Prop.4.3] (or a straightforward direct calculation). Setting $d_i(u_*) = \delta_i$, $q_i'(u_{*i}) = \beta_i$, and $r'(u_*) = [r_{kl}]$ for $i=1,2$, the operator $A_0 = A_*|\ker(B_*)$ in X_0 is given by

$$-A_0 = \begin{pmatrix} \delta_1\Delta + r_{11} & r_{12} \\ r_{21} & \delta_2\Delta + r_{22}\end{pmatrix}, \qquad \operatorname{dom}(A_0) = \mathcal{D}_1\times\mathcal{D}_2,$$

$$\mathcal{D}_i = \{v\in W_p^2(\Omega) : \partial_\nu v = \beta_i\delta_i^{-1} v\}, \quad i=1,2.$$

We now want to study the spectrum of A_0 in terms of the operators $C_i(\lambda) = \delta_i\Delta + r_{ii} - \lambda$ in X_0 with domain $\mathcal{D}_i$, where $i=1,2$ and $\lambda\in\mathbb{C}$. Since the case $r_{21}=0$ is rather simple we restrict ourselves to the case $r_{21}\neq 0$. Observe that A_0 has compact resolvent. Suppose that λ is an eigenvalue of $-A_0$ with eigenvector $(v_1,v_2)\in\operatorname{dom}(A_0)$. Then we have $v_2\neq 0$, $C_2(\lambda)v_2 = -r_{21}v_1 \in \mathcal{D}_1$, and

$$r_{21}C_1(\lambda)v_1 + r_{21}r_{12}v_2 = 0, \qquad r_{21}C_1(\lambda)v_1 + C_1(\lambda)C_2(\lambda)v_2 = 0.$$

As a result, $C_1(\lambda)C_2(\lambda)v_2 = r_{12}r_{21}v_2$. Conversely, let $v_2\in\operatorname{dom}(C_1(\lambda)C_2(\lambda)) = \{v\in\mathcal{D}_2 : C_2(\lambda)v\in\mathcal{D}_1\}$ be an eigenvector of $C_1(\lambda)C_2(\lambda)$ with the eigenvalue $r_{12}r_{21}$, for some λ. Then we set $v_1 = -r_{21}^{-1}C_2(\lambda)v_2\in\mathcal{D}_1$, obtaining an eigenvector (v_1,v_2) of $-A_0$ for the eigenvalue λ. So we have shown that

$$\sigma(-A_0) = \{\lambda\in\mathbb{C} : r_{12}r_{21}\in\sigma_p(C_1(\lambda)C_2(\lambda))\}.$$

This equation becomes much simpler if we assume in addition that $\mathcal{D}_1 = \mathcal{D}_2 =: \mathcal{D}$. For instance, this equality is true if $q_1'(u_{*1}) = q_2'(u_{*2}) = 0$. Let μ_n, $n\in\mathbb{N}_0$, be the distinct eigenvalues of the Laplacian $\Delta_{\mathcal{D}}$ with the domain $\mathcal{D}$ and set

$$M_n = \begin{pmatrix} \delta_1\mu_n + r_{11} & r_{12} \\ r_{21} & \delta_2\mu_n + r_{22} \end{pmatrix}.$$

Note that the spectrum of A_0 on $X_0 = L_p(\Omega)^2$ does not depend on $p \in (1,\infty)$ since the resolvent is compact. Moreover, $\Delta_{\mathcal{D}}$ is self adjoint on $L_2(\Omega)$, so that μ_n is real, $\mu_n \to -\infty$, and $\mu_{n+1} < \mu_n$. Then one easily obtains that

$$\sigma(-A_0) = \bigcup_{n\in\mathbb{N}_0} \sigma(M_n).$$

In order to satisfy Hypothesis 2.7, we thus have to ensure that none of the matrices M_n, $n \in \mathbb{N}_0$, has an eigenvalue on $i\mathbb{R}$. One obtains a purely imaginary eigenvalue of M_n if and only if either $\det M_n = 0$ for some $n \in \mathbb{N}_0$, or $\operatorname{tr} M_n = 0$ and $\det M_n > 0$ for some $n \in \mathbb{N}_0$. Moreover, there is an eigenvalue of $-A_0$ with strictly positive real part if and only if $s(M_0) > 0$.

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