The domain of the Schrödinger operator $-\Delta + x^2y^2$

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Abstract

We compute the domain of the Schrödinger operator $-\Delta + x^2y^2$ in $L^2(\mathbb{R}^2)$.

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1 Introduction

Let V be a nonnegative potential in $\mathbb{R}^d$ which belongs to $L^2_{loc}(\mathbb{R}^d)$. Then the quadratic form

$$a(u,v) = \int_{\mathbb{R}^d} (\nabla u \cdot \nabla \bar{v} + V u \bar{v})\, dx, \qquad u, v \in H = \{u \in H^1(\mathbb{R}^d) : V^{1/2}u \in L^2(\mathbb{R}^d)\}$$

is closed, symmetric and nonnegative in $L^2(\mathbb{R}^d)$. Therefore a defines a self-adjoint operator $(A, D(A))$ in $L^2(\mathbb{R}^d)$ formally given by $A = -\Delta + V$, see e.g. [2, Chapter 8]. Moreover, A can be described by

$$D(A) = \{u \in H : \exists f \in L^2(\mathbb{R}^d) \text{ s.t. } a(u,v) = \int_{\mathbb{R}^d} f\bar{v}\, dx \quad \forall\, v \in H\}, \qquad Au = f. \tag{1.1}$$

The test function space $C_0^\infty(\mathbb{R}^d)$ is a core for A since $V \in L^2_{loc}(\mathbb{R}^d)$, due to [6, Corollary VII.2.7]. Thus the question arises whether $D(A)$ coincides with the intersection $H^2(\mathbb{R}^d) \cap D(V)$, see [5] where this problem seems to be considered for the first time from the point of view of operator inequalities like 2.1. Here $H^k(\mathbb{R}^d)$ is the usual Sobolev space and $D(V) = \{u \in L^2(\mathbb{R}^d) : Vu \in L^2(\mathbb{R}^d)\}$ is the domain of the multiplication operator $V : u \mapsto Vu$. The equality $D(A) = H^2(\mathbb{R}^d) \cap D(V)$ holds if V satisfies the oscillation condition

$$|\nabla V(x)| \le aV(x)^{3/2} + b \tag{1.2}$$

for $x \in \mathbb{R}^d$ and positive a, b with $a^2 < 2$, see [3] and [4] where also potentials with local singularities are considered. We refer the reader to [1], [10], [11] for results in L^p, $1 < p < \infty$. Examples show that $D(A)$ can be strictly larger than $H^2(\mathbb{R}^d) \cap D(V)$ if (1.2) does not hold, see again [3] and [4] for counterexamples with singular potentials and [10] for smooth potentials. Surprisingly enough the situation is much better in $L^1(\mathbb{R}^d)$ where the domain of $-\Delta + V$ is always the intersection of the domains of $-\Delta$ and of the potential V, [7].

In this note we prove that $D(A) = H^2(\mathbb{R}^2) \cap D(V)$ for the potential $V(x,y) = x^2y^2$ which, as is easy to see, does not satisfy (1.2). The same potential was studied in detail in [12] where the compactness of the resolvent was proved, (see also [9] for a characterization of the discreteness of the spectrum

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for polynomial potentials). Even though Lemma 2.1 can be applied to more general homogenuous potentials, our method depends in an essential way on precise estimates of some constants and we do not see how to extend it to treat other operators.

Notation. The norm of $L^p(\mathbb{R}^d)$ is denoted by $\|\cdot\|_p$. $H^k(\mathbb{R}^d)$ is the Sobolev space of all functions in $L^2(\mathbb{R}^d)$ having weak derivatives in $L^2(\mathbb{R}^d)$ up to the order k. $C_0^\infty(\mathbb{R}^d)$ is the space of test functions.

2 The result

We begin with the following elementary lemma.

Lemma 2.1 *Let $0 \le V \in L^2_{loc}(\mathbb{R})$. Assume that there exists a constant $C > 0$ such that*

$$\|Vu\|_2 \le C \, \| - u'' + Vu\|_2 \tag{2.1}$$

for every $u \in C_0^\infty(\mathbb{R})$. Then the potential $V_\lambda(x) = \lambda^{-2}V(x/\lambda)$ satisfies (2.1) with the same constant C for every $\lambda > 0$.

PROOF. Applying (2.1) to the function $v(x) = u(\lambda x)$, we obtain

$$\int_{\mathbb{R}} |V(x)u(\lambda x)|^2 \, dx \le C^2 \int_{\mathbb{R}} | - \lambda^2 u''(\lambda x) + V(x)u(\lambda x)|^2 \, dx.$$

Setting $y = \lambda x$, this inequality leads to

$$\int_{\mathbb{R}} |V(y/\lambda)u(y)|^2 \, dy \le C^2 \int_{\mathbb{R}} | - \lambda^2 u''(y) + V(y/\lambda)u(y)|^2 \, dy,$$

which implies the assertion. □

In order to compute the domain of $-\Delta + x^2y^2$ we have to estimate the constant C in (2.1) for the potential $V(x) = x^2$.

Proposition 2.2 *The estimate*

$$\|x^2u\|_2 \le C \, \| - u'' + x^2u\|_2$$

holds for every $u \in C_0^\infty(\mathbb{R})$ and a constant $C > 0$ satisfying $C^2 < 2$.

Before proving this proposition, we show how the announced domain characterization follows from Proposition 2.2 and Lemma 2.1.

Theorem 2.3 *The domain of $-\Delta + x^2y^2$ in $L^2(\mathbb{R}^2)$ coincides with $H^2(\mathbb{R}^2) \cap D(V)$.*

PROOF. The representation (1.1) of A implies that $H^2(\mathbb{R}^2) \cap D(V)$ is contained in $D(A)$ and that $Au = -\Delta u + x^2y^2u$ for $u \in H^2(\mathbb{R}^2) \cap D(V)$. Since $C_0^\infty(\mathbb{R}^2)$ is a core for $D(A)$, see [6, Corollary VII.2.7], it suffices to prove that the graph norm and the canonical norm of $H^2(\mathbb{R}^2) \cap D(V)$ are equivalent on $C_0^\infty(\mathbb{R}^2)$. Clearly, $\|u\|_2 + \|Au\|_2 \le \|u\|_{H^2} + \|x^2y^2u\|_2$ for $u \in C_0^\infty(\mathbb{R}^2)$.

Thus it remains to establish the converse inequality. To estimate the H^1-norm of $u \in C_0^\infty(\mathbb{R}^2)$, we note that

$$\int_{\mathbb{R}^2} (u + Au)\bar{u} \, dx \, dy = \int_{\mathbb{R}^2} (|u|^2 + |\nabla u|^2 + x^2y^2|u|^2) \, dx \, dy,$$

Hence, $\|u\|_{H^1} \le c\,(\|u\|_2 + \|Au\|_2)$ for a suitable $c > 0$. We next treat the L^2-norms of the functions x^2y^2u and D^2u. We set $f = -\Delta u + x^2y^2u$ for $u \in C_0^\infty(\mathbb{R}^2)$. Then $-u_{xx} + x^2y^2u = f + u_{yy}$. Fix $y \in \mathbb{R} \setminus \{0\}$. Proposition 2.2 and Lemma 2.1 with $\lambda^4 = y^{-2}$ show that

$$\int_{\mathbb{R}} x^4y^4u(x,y)^2\,dx \le C^2 \int_{\mathbb{R}} |f(x,y) + u_{yy}(x,y)|^2\,dx,$$

where C is the constant from Proposition 2.2. Integrating this estimate with respect to y, we obtain

$$\int_{\mathbb{R}^2} x^4y^4u^2\,dx\,dy \le C^2 \int_{\mathbb{R}^2} |f + u_{yy}|^2\,dx\,dy.$$

In the same way one deduces that

$$\int_{\mathbb{R}^2} x^4y^4u^2\,dx\,dy \le C^2 \int_{\mathbb{R}^2} |f + u_{xx}|^2\,dx\,dy.$$

Summing the last two inequalities and using $f = -\Delta u + x^2y^2u$, we conclude

$$\begin{aligned}\int_{\mathbb{R}^2} x^4y^4u^2\,dx\,dy &\le C^2 \int_{\mathbb{R}^2} (f^2 + f\Delta u + \frac{1}{2}u_{xx}^2 + \frac{1}{2}u_{yy}^2)\,dx\,dy \\ &= C^2 \int_{\mathbb{R}^2} (f^2 - |\Delta u|^2 + x^2y^2u\Delta u + \frac{1}{2}u_{xx}^2 + \frac{1}{2}u_{yy}^2)\,dx\,dy.\end{aligned}$$

On the other hand, we compute

$$\int_{\mathbb{R}^2} |\Delta u|^2\,dx\,dy = \int_{\mathbb{R}^2} (u_{xx}^2 + u_{yy}^2 + 2u_{xy}^2)\,dx\,dy \tag{2.2}$$

integrating by parts twice, which leads to

$$\int_{\mathbb{R}^2} x^4y^4u^2\,dx\,dy \le C^2 \int_{\mathbb{R}^2} (f^2 - \frac{1}{2}\,|\Delta u|^2 + x^2y^2u\Delta u)\,dx\,dy.$$

Young's inequality then implies

$$\begin{gathered}\int_{\mathbb{R}^2} x^4y^4u^2\,dx\,dy \le C^2 \int_{\mathbb{R}^2} (f^2 + \frac{1}{2}x^4y^4u^2)\,dxdy, \\ \|x^2y^2u\|_2^2 \le \frac{C^2}{1 - C^2/2}\|f\|_2^2\,,\end{gathered}$$

since $1 - C^2/2 > 0$ by Proposition 2.2. This estimate and (2.2) further yield

$$\|D^2u\|_2^2 \le C_1\,\|\Delta u\|_2^2 = C_1\,\|x^2y^2u - f\|_2^2 \le C_2\,\|f\|_2^2\,.$$

As a result, $\|u\|_{H^2} + \|x^2y^2u\|_2 \le C_3\,(\|u\|_2 + \|Au\|_2)$ for some constant C_3. □

In order to prove Proposition 2.2 we need some elementary properties of the Hermite functions

$$H_n(x) = \frac{(-1)^n}{\sqrt{2^n n!\sqrt{\pi}}}\,e^{x^2/2}\,\frac{d^n}{dx^n}e^{-x^2} =: \frac{1}{\sqrt{2^n n!\sqrt{\pi}}}\,\psi_n(x), \qquad n \in \mathbb{N}_0\,,$$

for which we refer to [8, §5.6.2]. The Hermite functions are an orthonormal basis of $L^2(\mathbb{R})$ and $-H_n'' + x^2H_n = (2n+1)H_n$. The functions ψ_n satisfy the identity $\psi_{n+1} = 2x\psi_n - 2n\psi_{n-1}$ for $n \in \mathbb{N}_0$, where $\psi_{-1} = 0$. Using this recursion formula, one easily computes the integrals

$$c_{n,m} = \int_{\mathbb{R}} x^2H_n(x)H_m(x)\,dx, \qquad n, m \in \mathbb{N}_0,$$

obtaining

$$
\begin{aligned}
&c_{n,n-2} = \frac{1}{2}\sqrt{n(n-1)} \quad (n \geq 2), \quad c_{n,n} = \frac{1}{2}(2n+1), \quad c_{n,n+2} = \frac{1}{2}\sqrt{(n+2)(n+1)} \\
&c_{n,m} = 0, \qquad \text{if } \ m \neq n, n-2, n+2.
\end{aligned} \tag{2.3}
$$

PROOF OF PROPOSITION 2.2. Let $u \in C_0^\infty(\mathbb{R})$ and expand $f = -u'' + x^2 u$ with respect to the orthonormal basis (H_n), i.e.,

$$
f = \sum_{m=0}^{\infty} \langle f, H_m \rangle \, H_m = \sum_{m=0}^{\infty} f_m H_m
$$

where the brackets denote the inner product of $L^2(\mathbb{R})$ and $f_m = \langle f, H_m \rangle$. Then we obtain

$$
u = \sum_{m=0}^{\infty} (2m+1)^{-1} f_m H_m \qquad \text{and} \qquad x^2 u = \sum_{m=0}^{\infty} (2m+1)^{-1} f_m x^2 H_m .
$$

From the identities (2.3) it follows that

$$
\langle x^2 u, H_n \rangle = \alpha_n f_{n-2} + \frac{1}{2} f_n + \beta_n f_{n+2}
$$

for $n \in \mathbb{N}_0$, where

$$
\alpha_n = \frac{\sqrt{n(n-1)}}{2(2n-3)}, \qquad \beta_n = \frac{\sqrt{(n+2)(n+1)}}{2(2n+5)}, \qquad f_{-2} = f_{-1} = 0.
$$

These equalities yield

$$
x^2 u = \frac{1}{2} f + \sum_{n=0}^{\infty} (\alpha_n f_{n-2} + \beta_n f_{n+2}) H_n =: \frac{1}{2} f + g. \tag{2.4}
$$

We further estimate

$$
\begin{aligned}
\|g\|_2^2 &= \sum_{n=0}^{\infty} (\alpha_n f_{n-2} + \beta_n f_{n+2})^2 \\
&= \alpha_2^2 f_0^2 + \alpha_3^2 f_1^2 + 2\alpha_2 \beta_2 f_0 f_4 + 2\alpha_3 \beta_3 f_1 f_5 + \sum_{n=2}^{\infty} (\alpha_{n+2}^2 + \beta_{n-2}^2) f_n^2 \\
&\qquad + 2 \sum_{n=4}^{\infty} \alpha_n f_{n-2} \, \beta_n f_{n+2} \\
&\leq \alpha_2^2 f_0^2 + \alpha_3^2 f_1^2 + 2\alpha_2 \beta_2 f_0 f_4 + 2\alpha_3 \beta_3 f_1 f_5 + 2 \sum_{n=2}^{\infty} (\alpha_{n+2}^2 + \beta_{n-2}^2) f_n^2
\end{aligned}
$$

using Hölder's and Young's inequalities. Observe that $\alpha_{n+2}^2 + \beta_{n-2}^2 \le 7/50$ for $n \ge 2$. Hence,

$$
\begin{aligned}
\|g\|_2^2 &\le \frac{1}{2} f_0^2 + \frac{1}{6} f_1^2 + \frac{\sqrt{6}}{9} f_0 f_4 + \frac{\sqrt{30}}{33} f_1 f_5 + \frac{14}{50} \sum_{n=2}^{\infty} f_n^2 \\
&\le \left(\frac{1}{2} + \frac{\sqrt{6}}{18}\right) f_0^2 + \left(\frac{1}{6} + \frac{\sqrt{30}}{66}\right) f_1^2 + \frac{14}{50} f_2^2 + \frac{14}{50} f_3^2 + \left(\frac{\sqrt{6}}{18} + \frac{14}{50}\right) f_4^2 \\
&\quad + \left(\frac{\sqrt{30}}{66} + \frac{14}{50}\right) f_5^2 + \frac{14}{50} \sum_{n=6}^{\infty} f_n^2 \\
&\le \left(\frac{1}{2} + \frac{\sqrt{6}}{18}\right) \|f\|_2^2 .
\end{aligned}
$$

Together with (2.4), we conclude

$$
\|x^2 u\|_2 \le \left(\frac{1}{2} + \sqrt{\frac{1}{2} + \frac{\sqrt{6}}{18}}\right) \|f\|_2 =: C \|f\|_2 .
$$

The assertion is established since $C^2 < 2$. □

Remark 2.4 As in the proof of Theorem 2.3 one can establish that $D(-\Delta + bV) = H^2(\mathbb{R}^2) \cap D(V)$ for $b > 0$ and $V(x,y) = x^2 y^2$. But it seems that one cannot treat more general potentials by the method used in this paper.

Remark 2.5 Let $V(x,y) = x^2 y^2$ and $A = -\Delta + V$. The domain characterization in Theorem 2.3 is equivalent to the boundedness of the operator $V(I + A)^{-1}$ in $L^2(\mathbb{R}^2)$. On the other hand, the operator $V(I+A)^{-1}$ is bounded in $L^1(\mathbb{R}^2)$ due to [7]. By interpolation, $V(I+A)^{-1}$ is bounded in $L^p(\mathbb{R}^2)$ for $1 < p < 2$, and thus the domain of A in $L^p(\mathbb{R}^2)$, $1 < p < 2$, is equal to $W^{2,p}(\mathbb{R}^2) \cap D_p(V)$, where $D_p(V) = \{u \in L^p(\mathbb{R}^2) : Vu \in L^p(\mathbb{R}^2)\}$. We do not know whether a similar result is valid for $p > 2$.

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