An Extension of Kato's Stability Condition for Nonautonomous Cauchy Problems

Gregor Nickel and Roland Schnaubelt*

Abstract: An extension of Kato's stability condition for nonautonomous Cauchy problems is presented. It is proved that in the commutative case this condition and a mild regularity assumption imply wellposedness. If one supposes the Kato-stability, then the solutions are given by an integral formula. By means of examples we show that in general these stability conditions cannot be omitted in our results. Moreover, it is seen that the Kato-stability is not necessary for wellposedness.

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1 Introduction

The theory of nonautonomous Cauchy problems

$$(nCP)\begin{cases} \dot{u}(t) = A(t)u(t) \\ u(s) = u_s, \qquad a \le s \le t \le b, \end{cases}$$

for unbounded linear operators on Banach spaces is still in an incomplete state.

However, at least in the so–called hyperbolic case, all results are based on the classical 1970 paper of Kato [7] and his stability condition. In preliminary results (see, e. g., [6]) the operators $A(t)$ were assumed to generate contraction semigroups, thus Kato's stability condition was automatically satisfied (see below). It is then used, e. g., in [3], [2], [8], or [1] in combination with more or less complicated regularity conditions to obtain wellposedness of (nCP). A necessary and sufficient characterisation of wellposedness is still lacking.

In this paper we show that even this stability condition is not necessary for wellposedness (cf. Example 3.4). We introduce a weaker concept of stability which suffices to show wellposedness in the commutative case under rather weak regularity assumptions (Section 2). Here we partly generalize a result for generators of contraction semigroups obtained by Goldstein [5]. In Section 3 we give examples and counterexamples illustrating the various notions and thus, hopefully, contributing to a better understanding of the complicated behaviour of (nCP).

Let us first fix some notations. Throughout, we consider a compact interval $I := [a,b] \subset \mathbb{R}$ and the triangle $D := \{(t,s) \in I^2 : s \le t\}$. On the interval I we consider partitions $P_n := \{t_0^n, t_1^n, \ldots, t_{m(n)}^n\}$ with $a = t_0^n < t_1^n < \ldots < t_{m(n)}^n = b$. A sequence of partitions, denoted by (P_n), converges to 0 if

$$\lim_{n\to\infty} \sup_{1\le j\le m(n)} |t_j^n - t_{j-1}^n| = 0.$$

The Kato–stability can be defined in the following way (cf. [10, p.131]) where products are always taken to be time–ordered.

Definition 1.1 (Kato–tability) *A family $(A(t), D(A(t)))_{t\in I}$ of generators of C_0–semigroups on a Banach space X is called* **Kato–stable***, if there exist constants $M \ge 1$ and $\omega \in \mathbb{R}$ such that*

$$\left\| \prod_{j=0}^{m(n)} e^{s_j A(t_j^n)} \right\| \le M e^{\omega \sum_{j=0}^{m(n)} s_j}$$

for any partition $P_n := \{t_0^n, t_1^n, \ldots, t_{m(n)}^n\}$ and $s_j \ge 0$.

It can easily be verified (cf. [10, Thm. 5.2.2]) that the Kato–stability is equivalent to the condition

$$(\omega, \infty) \subset \rho(A(t)) \quad \text{for } t \in [a, b]$$

and

$$\left\| \prod_{j=0}^{m(n)} R(\lambda, A(t_j^n)) \right\| \leq M(\lambda - \omega)^{-m(n)-1}$$

for $\lambda > \omega$ and for any partition $P_n := \{t_0^n, t_1^n, \dots, t_{m(n)}^n\}$.

Clearly, a family of generators of quasicontractive semigroups with uniform exponential bound is Kato–stable.

Inspired by the explicit formula for the solution of the above Cauchy problem in the commutative (and bounded) case (see Section 2) we now weaken the concept of stability.

Definition 1.2 ((P_n)–stability) *For a given sequence of partitions (P_n) converging to 0 a family $(A(t), D(A(t)))_{t\in I}$ of generators of C_0–semigroups on a Banach space X is called (P_n)–**stable**, if there exists an $M \geq 1$ such that for any subinterval $[s, t] \subseteq I$ we have*

$$\left\| \left(e^{(t-t_l^n)A(t_{l+1}^n)} \right) \left(\prod_{i=k+1}^{l} e^{(t_i^n - t_{i-1}^n)A(t_i^n)} \right) \left(e^{(t_k^n - s)A(t_k^n)} \right) \right\| \leq M$$

for every $n \in \mathbb{N}$, where $\{t_k^n, t_{k+1}^n, \dots, t_l^n\} := [s, t] \cap P_n$.

Stability on every subinterval $[s, t] \subseteq I$ is essential for proving wellposedness and does not follow automatically from stability on the whole interval I (see Section 3, Example 3.3).

Example 3.4 shows that the (P_n)–stability actually depends on the choice of the special sequence of partitions (P_n) and that it is strictly weaker than the Kato–stability. Moreover, in Example 3.2 we present a family which is unstable for any sequence of partitions converging to 0.

Definition 1.3 (Wellposedness) *A family $(U(t, s))_{(t,s)\in D}$ of bounded operators on a Banach space X is an* **strongly continuous evolution family***, if $U(t, r)U(r, s) = U(t, s)$ and $U(s, s) = Id$ for $t \geq r \geq s$ and $t, r, s \in I$ and the mapping $D \ni (t, s) \mapsto U(t, s)$ is strongly continuous. For a given family $(A(t), D(A(t)))_{t\in I}$ of closed, linear operators on a Banach space X the nonautonomous Cauchy problem*

$$(nCP) \begin{cases} \dot{u}(t) = A(t)u(t) \\ u(s) = u_s, \qquad a \leq s \leq t \leq b, \end{cases}$$

is **wellposed** *if there exists a unique strongly continuous evolution family* $(U(t,s))_{(t,s)\in D}$ *on* X *and a dense subspace* $Y \subseteq X$*, such that* $U(t,s)Y \subseteq D(A(t))$ *for every* $(t,s) \in D$ *and* $t \mapsto U(t,s)u_s$ *is a (classical) solution of* (nCP) *for any* $u_s \in Y$*.*

Remark 1.4 *The principle of uniform boundedness implies that a strongly continuous evolution family* $U(t,s)$ *is bounded on the compact set* D*. It is possible to consider unbounded evolution families on unbounded intervals by restricting to arbitrary compact intervals.*

2 A wellposedness theorem for the commutative case

In the case of bounded commuting operators $A(t)$ the solution of the nonautonomous Cauchy problem (nCP) is given by

$$U(t,s)u_s = e^{\int_s^t A(r)\,dr}u_s.$$

The following considerations are inspired by this fact.

We consider a family $(A(t), D(A(t)))_{t\in I}$ of generators of C_0–semigroups $((e^{\tau A(t)})_{\tau\geq 0})_{t\in I}$ on the Banach space X and assume the following commutativity and continuity property.

Assumption 2.1 *The semigroups* $((e^{\tau A(t)})_{\tau\geq 0})_{t\in I}$ *pairwise commute and there exists a space* $Y \subseteq \tilde{Y} := \cap_{t\in I} D(A(t))$ *which is dense in X. Moreover the mapping* $t \mapsto A(t)y$ *is continuous from* I *to* X *for all* $y \in Y$*.*

For the rest of this section we fix a sequence of partitions (P_n) converging to 0. To approximate the solution of (nCP) we consider for any partition $P_n := \{t_0^n, t_1^n, \ldots, t_{m(n)}^n\}$ the piecewise constant family

$$A_n(s) := A(t_k^n) \;\text{ for }\; t_{k-1}^n < s \leq t_k^n \;\text{ and }\; t_k^n \in P_n.$$

Now we define an operator $B_n(t,s) : Y \to X$ by

$$\begin{aligned} B_n(t,s)y &:= \int_s^t A_n(r)y\,dr \\ &= (t_k^n - s)A(t_k^n)y + \sum_{i=k+1}^{l}(t_i^n - t_{i-1}^n)A(t_i^n)y + (t - t_l^n)A(t_{l+1}^n)y \end{aligned}$$

for $t_{k-1}^n < s \leq t_k^n \leq t_l^n \leq t < t_{l+1}^n$. Since the semigroups $((e^{\tau A(t)})_{\tau\geq 0})_{t\in I}$ commute, $B_n(t,s)$ is closable and its closure $\overline{B_n}(t,s)$ is the generator of a C_0–semigroup still commuting with all the semigroups $((e^{\tau A(t)})_{\tau\geq 0})_{t\in I}$ as well as with all the resolvents $R(\lambda, A(s))$ (see [9, A-I.3.8]). Let

$$\begin{aligned} U_n(t,s) &:= e^{\overline{B_n}(t,s)} \\ &= \left(e^{(t-t_l^n)A(t_{l+1}^n)}\right)\left(\prod_{i=k+1}^{l} e^{(t_i^n-t_{i-1}^n)A(t_i^n)}\right)\left(e^{(t_k^n-s)A(t_k^n)}\right). \end{aligned} \tag{2.1}$$

The following stability condition is our main assumption.

Assumption 2.2 ((P_n)–stability) *The family $(A(s))_{s\in I}$ is (P_n)–stable, i. e., there is an $M\geq 1$ such that for any $a\leq s\leq t\leq b$ and for all $n\in\mathbb{N}$ we have*

$$\|U_n(t,s)\| \leq M(t,s) := \sup_n \|U_n(t,s)\| \leq M < \infty.$$

We can now state our main result.

Theorem 2.3 *If the family $(A(s))_{s\in I}$ fulfills the Assumptions 2.1 and 2.2, then there exists a unique strongly continuous evolution family $(U(t,s))_{(t,s)\in D}$ such that*

(a) $U(t,s)Y\subseteq\tilde{Y}$,

(b) the function $t\mapsto U(t,s)y$ *is continuously differentiable for all* $y\in Y$ *and* $\frac{\partial}{\partial t}U(t,s)y = A(t)U(t,s)y$ *for* $t\geq s$,

(c) $\|U(t,s)\|\leq M(t,s)$.

Consequently, (nCP) is wellposed.

Proof. The first part of the proof uses ideas which can be found, e. g., in the proof of [10, Thm. 5.3.1]. Since the operators $U_n(t,s)$ commute with the resolvent $R(\lambda, A(r))$ for all $(t,s)\in D$, $r\in I$, we have $U_n(t,s)Y\subseteq\tilde{Y}$ and $A(r)U_n(t,s)y = U_n(t,s)A(r)y$ for every $y\in Y$, and $(t,s)\in D$, $r\in I$. Moreover, $\frac{\partial}{\partial t}U_n(t,s)y = A(t_{l+1}^n)U_n(t,s)y$ and $\frac{\partial}{\partial s}U_n(t,s)y = -U_n(t,s)A(t_k^n)y$ for $t\neq t_l^n$ and $s\neq t_k^n$. Therefore $\frac{\partial}{\partial r}U_n(r,s)y = U_n(r,s)A_n(r)y$ and $\frac{\partial}{\partial r}U_n(t,r)y = -U_n(t,r)A_n(r)y$ for $r\in(s,t)\setminus P_n$.

Note that all $(U_n(t,s))_{(t,s)\in D}$ are strongly continuous evolution families and that all the mappings $(t,s)\mapsto\frac{\partial}{\partial s}U_n(t,s)y$ as well as $(t,s)\mapsto\frac{\partial}{\partial t}U_n(t,s)y$ are piecewise continuous.

Since the mapping $r\mapsto A(r)y$ is uniformly continuous on the compact interval I for $y\in Y$, we obtain for the piecewise constant functions $(A_n(\cdot)y)$ that

$$\lim_{m,n\to\infty}\sup_{s\in I}\|A_n(s)y - A_m(s)y\| = 0. \tag{2.2}$$

It now follows that

$$\begin{aligned}\|U_n(t,s)y - U_m(t,s)y\| &= \left\| \int_s^t \frac{\partial}{\partial r}(U_n(t,r)U_m(r,s))y\, dr \right\| \\ &\le \int_s^t \|U_n(t,r)U_m(r,s)(A_n(r) - A_m(r))y\|\, dr \\ &\le M^2(b-a) \sup_{r\in I} \|A_n(r)y - A_m(r)y\|.\end{aligned}$$

By equation (2.2) the righthand side converges to 0 as $n, m \to \infty$ independently of $(t,s) \in D$. Therefore we can define

$$U(t,s)y := \lim_{n\to\infty} U_n(t,s)y. \tag{2.3}$$

By the density of $Y \subset X$ and the stability condition 2.2 we extend this mapping by defining

$$U(t,s)x := \lim_{n\to\infty} U_n(t,s)x, \quad x \in X. \tag{2.4}$$

From the properties of the evolution families $(U_n(t,s))$ we obtain a strongly continuous evolution family $(U(t,s))_{(t,s)\in D}$ satisfying $\|U(t,s)\| \le M(t,s)$. In addition, the operators $U(t,s)$ commute with all the resolvents $R(\lambda, A(r))$. This implies $U(t,s)\tilde{Y} \subseteq \tilde{Y}$ and $A(r)U(t,s)y = U(t,s)A(r)y$ for every $y \in \tilde{Y}$.

It remains to show assertion (b). To do this we remark first that for fixed $y \in Y$

$$\begin{aligned}&\lim_{n\to\infty} [A(r)U(t,s)y - A_n(r)U_n(t,s)y] \\ &\quad = \lim_{n\to\infty} [U_n(t,s)(A(r) - A_n(r))y + (U(t,s) - U_n(t,s))A(r)y] = 0 \end{aligned} \tag{2.5}$$

independently of $(t,s) \in D$ and $r \in I$. Taking now $t \notin \Omega := \bigcup_{n\in\mathbb{N}} P_n$, $s < t < b$ and $h \in \mathbb{R}$ with small $|h| > 0$ we obtain

$$\begin{aligned}&\left\| \frac{U_n(t+h,s) - U_n(t,s)}{h} y - \frac{U_m(t+h,s) - U_m(t,s)}{h} y \right\| \\ &= \left\| \frac{1}{h} \int_t^{t+h} [A_n(r)U_n(r,s)y - A_m(r)U_m(r,s)y]\, dr \right\| \\ &\le \sup_{s\le r\le b} \|A_n(r)U_n(r,s)y - A_m(r)U_m(r,s)y\|. \end{aligned} \tag{2.6}$$

Using (2.5) we obtain that for any $\varepsilon > 0$

$$\begin{aligned}&\left\| \frac{U(t+h,s) - U(t,s)}{h} y - \frac{U_n(t+h,s) - U_n(t,s)}{h} y \right\| \\ &\le \sup_{s\le r\le b} \|A(r)U(r,s)y - A_n(r)U_n(r,s)y\| \le \epsilon \end{aligned} \tag{2.7}$$

if $n \geq N(\varepsilon, y)$, $t \notin \Omega$, and $h \in (0, b-t)$.

Consider now the following expression for fixed $b > t > s \geq a$, $y \in Y$, small $|h| > 0$ and $t \notin \Omega$:

$$\begin{aligned}
&\frac{1}{h}[U(t+h,s) - U(t,s)]y - A(t)U(t,s)y \\
&= \underbrace{\frac{1}{h}[U(t+h,s) - U(t,s)]y - \frac{1}{h}[U_n(t+h,s) - U_n(t,s)]y}_{I_n} \\
&\quad + \underbrace{\frac{1}{h}[U_n(t+h,s) - U_n(t,s)]y - A_n(t)U_n(t,s)y}_{II_n} \\
&\quad + \underbrace{[U_n(t,s) - U(t,s)]A_n(t)y}_{III_n} \\
&\quad + \underbrace{U(t,s)[A_n(t) - A(t)]y}_{IV_n}.
\end{aligned}$$

For arbitrary $\varepsilon > 0$ we can — using (2.7) and (2.2) — choose $N_1 = N_1(\varepsilon, y) \in \mathbb{N}$ such that $\|I_n\| \leq \varepsilon$ and $\|IV_n\| \leq \varepsilon$ for every $n \geq N_1$ uniformly for $h \in (0, b-t)$. Since the sequence $x_n := A_n(t)y$ converges to $A(t)y$ and $U_n(t,s)$ converges strongly to $U(t,s)$, we have $III_n \to 0$ as $n \to \infty$. So we can choose $N = N(\varepsilon, y) \in \mathbb{N}$ such that $\|I_n\| + \|III_n\| + \|IV_n\| \leq 3\varepsilon$ for $n \geq N$. For this $N \in \mathbb{N}$ we obtain that $II_N \to 0$ as $h \to 0$. Consequently,

$$\lim_{h \to 0} \left\| \frac{1}{h}[U(t+h,s) - U(t,s)]y - A(t)U(t,s)y \right\| \leq 3\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, the equation $(\frac{\partial}{\partial t})^+ U(t,s)y = A(t)U(t,s)y$ follows for any $t \notin \Omega$.

The continuity of the mapping $(t,s) \mapsto A(t)U(t,s)y$ follows from

$$\begin{aligned}
&A(t')U(t',s')y - A(t)U(t,s)y \\
&= U(t',s')[A(t') - A(t)]y + [U(t',s') - U(t,s)]A(t)y
\end{aligned}$$

and the strong continuity of $U(t,s)$ and $A(t)y$. Thus the continuous function $t \mapsto U(t,s)y$ is differentiable for $t \in I \setminus \Omega$, where $\frac{\partial}{\partial t} U(t,s)y = A(t)U(t,s)y$ for $t \in I \setminus \Omega$. Since Ω is countable and the function $t \mapsto A(t)U(t,s)y$ is continuous, we obtain

$$\frac{\partial}{\partial t} U(t,s)y = A(t)U(t,s)y$$

for all $(t,s) \in D$. Uniqueness of the evolution family follows by a standard argument (cf. [10, p.138]). □

Remark 2.4 *If the family* $(A(t))_{t\in I}$ *is Kato–stable, then the space* X *can be equivalently renormed such that each* $(e^{\tau A(s)})_{\tau\geq 0}$ *becomes a (quasi)contraction semigroup (cf. [10, p.260]). The contractive case was considered by Goldstein [5], where our result is proved even for piecewise continuous functions* $A(\cdot)$*.*

Even more can be said in this case.

Proposition 2.5 *If the family* $(A(t))_{t\in I}$ *is Kato–stable and satisfies Assumption 2.1, then all assertions of Theorem 2.3 hold. Moreover,* $\int_s^t A(r)\,dr$ *(defined pointwise on* Y*) is closable and its closure (we denote it still by* $\int_s^t A(r)dr$*) is a generator. The solution of the Cauchy problem is then given by*

$$U(t,s) = e^{\int_s^t A(r)\,dr} \qquad \text{for } (t,s) \in D. \tag{2.8}$$

Proof. The Kato–stability implies that for an arbitrary sequence of partitions (P_n) all operators $B_n(t,s)$ defined as above are generators of the same type $(M, \omega(t-s))$. In addition, we have by Assumption 2.1

$$B_n(t,s)y \overset{n\to\infty}{\to} \int_s^t A(r)y\,dr \text{ for all } y \in Y.$$

To apply the theorem of Trotter–Kato (compare [10, Thm. 3.4.5]) we have to show that the range of $(\lambda - \int_s^t A(r)\,dr)$ is dense in X. In fact, for $\lambda > \omega$ and $y \in Y$

$$\begin{aligned} &\left\| \left(\lambda - \int_s^t A(r)\,dr\right) R(\lambda, \overline{B_n}(t,s))y - y \right\| \\ \leq \;& \|R(\lambda, \overline{B_n}(t,s))\| \left\| \left(B_n(t,s) - \int_s^t A(r)\,dr\right) y \right\| \end{aligned}$$

which shows convergence to 0 as $n \to \infty$. So the closure of the range contains Y which is dense in X. By the theorem of Trotter–Kato we now obtain that the closure $\overline{B}(t,s)$ of $\int_s^t A(r)\,dr$ is a generator and that $e^{\overline{B_n}(t,s)}x \overset{n\to\infty}{\to} e^{\int_s^t A(r)\,dr}x$. By equation (2.4) we conclude $U(t,s) = e^{\int_s^t A(r)\,dr}$ which gives the solutions of (nCP). □

3 Discussion and examples

In Theorem 2.3 and Proposition 2.5 the main assumptions were the (P_n)–stability and the Kato–stability of the family $(A(s))_{s\in I}$, respectively. In this section we present examples showing that these assumptions cannot be weakened in general. Moreover, a family is constructed which is (P_n)–stable for a certain sequence of partitions (P_n) converging to 0, while it is not Kato–stable and is unstable with respect to another sequence of partitions (Q_n).

In order to achieve this, we first construct uniformly bounded, commuting $\mathcal{C}_0$–semigroups $(e^{\tau A_k})_{\tau\geq 0}$ such that $\|e^{\tau_n A_n}\cdots e^{\tau_1 A_1}\| = M^n$ for $0 < \tau_k < 2a_k$ and a given sequence (a_k), where the operators $e^{\tau_k A_k}$ are translations on a weighted L^1–space.

To be more precise, fix $M > 1$ and $0 < a_k \leq \min\{\frac{1}{2(M-1)}, \frac{1}{M+1}\}$ for $k \in \mathbb{N}$. Set $b_k := 1 - (M-1)a_k$ and $J_k := [0, b_k]$. Then $\max\{\frac{1}{2}, 2a_k\} \leq b_k < 1$. Let $h_k : J_k \to \mathbb{R}$ be defined by

$$h_k(\xi) := \begin{cases} 1 & \text{for } 0 \leq \xi < a_k \text{ or } 2a_k < \xi \leq b_k\,, \\ M & \text{for } a_k \leq \xi \leq 2a_k\,. \end{cases}$$

Set $\mu_k := h_k\lambda$, where λ is the Lebesgue measure on J_k. Then (J_k, μ_k), $k \in \mathbb{N}$, is a probability space. Thus the product space $(\Omega, \nu) := \prod_{k\in\mathbb{N}}(J_k, \mu_k)$ and the space $X := L^1(\Omega, \nu)$ are well-defined. We denote by $\|\cdot\|$ the 1–norm on X and by χ_C the characteristic function of a set C. Note that the space

$$X_0 = \operatorname{lin}\{\chi_C : C = C_1 \times \cdots \times C_n \times J_{n+1} \times \cdots, n \in \mathbb{N},\ C_k \subseteq J_k \text{ measurable}\}$$

is dense in X because $\overline{X_0}$ contains all simple functions in $L^1(\Omega, \nu)$. In addition, if $\varphi_k \in L^1(J_k, \mu_k)$ and $\Omega^{(n)} := \prod_{k=n+1}^{\infty} J_k$, then the product function

$$f := \varphi_1 \cdots \varphi_n \chi_{\Omega^{(n)}} \;:\; (\xi_1, \cdots) \mapsto \varphi_1(\xi_1) \cdots \varphi_n(\xi_n)$$

is an element of X and $\|f\| = \prod_{k=1}^{n} \int_{J_k} |\varphi_k|\, d\mu_k$. Finally, for measurable sets $C_k \subseteq J_k$, $k \in \mathbb{N}$, the set $C := \prod_k C_k$ is ν–measurable and $\nu(C) = \prod_k \mu_k(C_k)$. (See e.g. [11, pp.167] for relevant facts from the integration theory.)

On the space X we define for each $k \in \mathbb{N}$ a nilpotent right–translation semigroup by

$$\left(e^{\tau A_k} f\right)(\xi_1, \cdots) := \begin{cases} f(\xi_1, \cdots, \xi_k - \tau, \xi_{k+1}, \cdots), & \text{if } \xi_k - \tau \in J_k\,, \\ 0, & \text{if } \xi_k - \tau \notin J_k\,. \end{cases}$$

In the following lemma we show that these semigroups are strongly continuous. We denote the generator of $(e^{\tau A_k})_{\tau\geq 0}$ by $(A_k, D(A_k))$. Formally, we have $A_k f = -\frac{\partial}{\partial \xi_k} f$.

Lemma 3.1 *Let X and $((e^{\tau A_k})_{\tau\geq 0})_{k\in\mathbb{N}}$ be defined as above. Then the following assertions hold.*

(a) The semigroups $((e^{\tau A_k})_{\tau\geq 0})_{k\in\mathbb{N}}$ are strongly continuous and commute pairwise. Moreover, $\|e^{\tau A_k}\| = M$ for $0 < \tau < 2a_k$, $\|e^{\tau A_k}\| \leq 1$ for $\tau \geq 2a_k$ and $e^{\tau A_k} = 0$ for $\tau \geq b_k$.

(b) $\|e^{\tau_n A_n}\cdots e^{\tau_1 A_1}\| = M^n$ *for* $0 < \tau_k < 2a_k$, $1 \le k \le n$, *and* $n = 1, 2, \cdots$.

(c) *The space* $Y := \{f \in \cap_{k\in\mathbb{N}} D(A_k) : \sup_{k\in\mathbb{N}} \|A_k f\| < \infty\}$ *endowed with the norm* $\|f\|_Y := \max\{\|f\|, \sup_k \|A_k f\|\}$ *is continuously and densely embedded in* X.

Proof. For $f \in X_0$ one easily shows $\|e^{\tau A_k} f\| \le M\|f\|$ and $\lim_{\tau\to 0} \|e^{\tau A_k} f - f\| = 0$. Therefore the semigroups $((e^{\tau A_k})_{\tau\ge 0})_{k\in\mathbb{N}}$ are strongly continuous. The other assertions in (a) are obvious. In order to prove assertion (b) we consider

$$C_k := \begin{cases} [a_k - \tau_k, a_k], & \text{if } 0 < \tau_k \le a_k\,, \\ [0, 2a_k - \tau_k], & \text{if } a_k < \tau_k < 2a_k \end{cases}$$

for $k = 1, \cdots, n$. Let $C := C_1 \times \cdots \times C_n \times J_{n+1} \times \cdots$. Then we have

$$\|e^{\tau_n A_n}\cdots e^{\tau_1 A_1}\chi_C\| = \prod_{k=1}^{n} \mu_k(C_k + \tau_k) = M^n \prod_{k=1}^{n} \lambda(C_k) = M^n \|\chi_C\|.$$

Thus, (b) holds. It remains to show that the space Y is dense in X.

Let $Y_n := \{f = \varphi_1\cdots\varphi_n\chi_{\Omega^{(n)}} : \varphi_k \in W^{1,1}(J_k, \mu_k),\ \varphi_k(0) = 0,\ 1 \le k \le n\}$ for $n \in \mathbb{N}$. For $k \le n$ a straightforward computation yields

$$Y_n \subseteq D(A_k) \quad \text{and} \quad A_k f = -\varphi_1\cdots\varphi_k'\cdots\varphi_n\chi_{\Omega^{(n)}} = -\frac{\partial}{\partial\xi_k} f.$$

Fix $0 < \varepsilon \le 1$ and $f = \chi_C$, where $C = C_1 \times \cdots \times C_n \times J_{n+1} \times \cdots$ and $C_k \subseteq J_k$ is measurable. Then there is a function $g_n = \varphi_1\cdots\varphi_n\chi_{\Omega^{(n)}} \in Y_n$ such that

$$\|f - g_n\| = \int_{J_1}\cdots\int_{J_n} |\chi_{C_1}\cdots\chi_{C_n} - \varphi_1\cdots\varphi_n|\, d\mu_n\cdots d\mu_1 \le \varepsilon. \tag{3.1}$$

We set $c := \max_{1\le l\le n}\{\int_{J_l} |\varphi_l|\, d\mu_l\,,\ \int_{J_l} |\varphi_l'| d\mu_l\}$. In addition, consider functions $\varphi_l \in W^{1,1}(J_l, \mu_l)$ with $\varphi_l(0) = 0$, $l = n+1, \cdots$, which are equal to 1 on $[\varepsilon 2^{-l+1}, b_l]$ and linear on $[0, \varepsilon 2^{-l+1}]$. Then we have

$$\int_{J_l} \varphi_l\, d\mu_l \le 1, \quad \int_{J_l} (1 - \varphi_l)\, d\mu_l \le \varepsilon\, M 2^{-l} \quad \text{and} \quad \int_{J_l} \varphi_l'\, d\mu_l \le M \tag{3.2}$$

for $l = n+1, \cdots$. Set $g_m := \varphi_1\cdots\varphi_m\chi_{\Omega^{(m)}}$ for $m \ge n$. Then $g_m \in Y_k \subseteq D(A_k)$ and $A_k g_m = -\frac{\partial}{\partial\xi_k} g_m$ for $m \ge k$. From (3.2) we deduce

$$\begin{aligned} \|g_m - g_{m-1}\| &\le \int_{J_m} (1 - \varphi_m)\, d\mu_m \prod_{l=1}^{m-1} \int_{J_l} |\varphi_l| d\mu_l \\ &\le \varepsilon\, M c^n\, 2^{-m} \qquad \text{and} \\ \|A_k g_m - A_k g_{m-1}\| &\le \int_{J_k} |\varphi_k'| d\mu_k \int_{J_m} (1 - \varphi_m)\, d\mu_m \prod_{\substack{l=1\\ l\ne k}}^{m-1} \int_{J_l} |\varphi_l| d\mu_l \\ &\le \varepsilon\, M^2\, c^n\, 2^{-m} \end{aligned} \tag{3.3}$$

for $m \geq N := \max\{n+1, k+1\}$. Hence, $(g_m)_{m\geq N}$ and $(A_k g_m)_{m\geq N}$ are Cauchy sequences. By the closedness of the operator A_k the function $g := \lim_m g_m$ is an element of $D(A_k)$ and $A_k g = \lim_m A_k g_m$, $k \in \mathbb{N}$. Thus, $\|A_k g\| \leq Mc^n$ and $g \in Y$. Finally, (3.1) and (3.3) yield

$$\|f - g\| \leq \|f - g_n\| + \|g_n - g\| \leq (1 + Mc^n)\,\varepsilon.$$

This shows that Y is dense in X. □

We now can construct the announced counterexamples. In the sequel we use of the spaces X, Y and the operators $(A_k, D(A_k))$ and $e^{\tau A_k}$ as defined above.

Example 3.2 *There is a family $(A(s), D(A(s)))_{0\leq s\leq 1}$ of generators of commuting $\mathcal{C}_0$–semigroups $((e^{\tau A(s)})_{\tau\geq 0})_{0\leq s\leq 1}$ on X such that*

(a) $\|e^{\tau A(s)}\| \leq 2$ for all $\tau \geq 0$ and $Y \subseteq \bigcap_{0\leq s\leq 1} D(A(s))$,

(b) $A(\cdot) \in C([0,1], \mathcal{L}(Y, X))$,

(c) the Cauchy problem (nCP) corresponding to $(A(s))_{0\leq s\leq 1}$ is not wellposed and the family $(A(s))_{0\leq s\leq 1}$ is not (P_n)–stable for any sequence of partitions (P_n) converging to 0.

Proof. Set $M := 2$ and $s_k := 1 - 2^{-k}$ for $k = 0, 1, \ldots$. Moreover, let $\tau_k := s_{k-1} + \frac{s_k - s_{k-1}}{2} = s_{k-1} + 2^{-k-1}$ for $k \in \mathbb{N}$. Define functions $\alpha_k \in C[0,1]$, $k \in \mathbb{N}$, such that $\alpha_k(s) = 0$ for $s \notin \,]s_{k-1}, s_k[$, $\alpha_k(\tau_k) = 2^{-k+1}$ and α_k is linear on $[s_{k-1}, \tau_k]$ and $[\tau_k, s_k]$. Set $a_k := \int_{s_{k-1}}^{s_k} \alpha_k(r)\,dr$, $k \in \mathbb{N}$. Then $a_k = 2^{-2k}$. Now, according to the sequence (a_k) we can define the spaces X, Y and the operators $(A_k, D(A_k))$ as above.

Let $A(s) := \alpha_k(s) A_k$ and $D(A(s)) := D(A_k)$ for $s_{k-1} \leq s < s_k$ and $k = 1, 2, \ldots$, and $A(1) := 0$. By Lemma 3.1 the operators $A(s)$ generate commuting $\mathcal{C}_0$–semigroups satisfying (a). Moreover, we have for $s_{k-1} < s, t < s_k$ and $f \in Y$

$$\|A(t)f - A(s)f\| \leq 4\,|t-s|\,\|f\|_Y \quad \text{and} \quad \|A(s)f\| \leq |\alpha_k(s)|\,\|f\|_Y\,.$$

Hence, (b) holds.

For $0 \leq s \leq s_l \leq s_m \leq t < 1$ we set

$$U(t,s) := \left(e^{\int_{s_m}^{t} \alpha_{m+1}(r)dr\, A_{m+1}}\right)\left(e^{a_m A_m}\right) \cdots \left(e^{a_{l+1} A_{l+1}}\right)\left(e^{\int_{s}^{s_l} \alpha_l(r)dr\, A_l}\right). \tag{3.4}$$

This defines a strongly continuous evolution family $(U(t,s))_{0\leq s\leq t<1}$ solving the Cauchy problem (nCP) corresponding to $(A(s))_{0\leq s<1}$. However, Lemma 3.1(b) implies $\|U(s_m, 0)\| = 2^m$. Hence, by Remark 1.4 (nCP) is not wellposed on

$[0,1]$. On the other hand, by assertions (a), (b) and Lemma 3.1(c) the family $(A(s))_{0\le s\le 1}$ satisfies Assumption 2.1. Thus, it follows from Theorem 2.3 that the family $(A(s))_{0\le s\le 1}$ cannot be (P_n)–stable for any sequence of partitions (P_n) converging to 0. □

In the following two examples we use the family $(A(s))_{0\le s\le 1}$ from Example 3.2.

Example 3.3 *Let the family $(B(s), D(B(s)))_{-1\le s\le 1}$ be defined by $B(s) = A_1$ with $D(B(s)) = D(A_1)$ for $s \in [-1, 0[$ and $B(s) = A(s)$ with $D(B(s)) = D(A(s))$ for $s \in [0,1]$. This family is stable on the interval $[-1,1]$ but unstable on the interval $[0,1]$ with respect to any sequence of partitions (P_n) converging to 0.*

Proof. The second claim has been shown in Example 3.2. Moreover, recall that $e^{\tau A_1} = 0$ for $\tau \ge b_1 = \frac{3}{4}$. Hence, for any sequence of partitions (P_n) converging to 0 we can find an index N such that $V_n(1,-1) = 0$ for $n \ge N$, where $V_n(t,s)$ are the products with respect to $B(s)$ and P_n (cf. (2.1)). □

Example 3.4 *Consider the operators $C(s) := 2A(s)$ with $D(C(s)) = D(A(s))$, $0 \le s \le 1$. Then the following assertions hold.*

(a) There are sequences of partitions (P_n) and (Q_n) converging to 0 such that the family $(C(s))_{0\le s\le 1}$ is (P_n)–stable, but neither (Q_n)–stable nor Kato–stable.

(b) The Cauchy problem corresponding to $(C(s))_{0\le s\le 1}$ is wellposed, but the representation formula (2.8) does not hold for $1 = t > s \ge 0$.

Proof. We want to show stability of the products $W_n(t,s)$ corresponding to the family $(C(s))_{0\le s\le 1}$ and the partitions $P_n := \{t_k^n = k\,2^{-n} : k = 0,\cdots,2^n\}$, $n \in \mathbb{N}$ (cf. (2.1)). First, observe that every partition P_n is a refinement of the preceding one and $1 - 2^{-m} = s_m \in P_n$ for $n \ge m$. Moreover, we have $t_{2^n}^n = 1$, $t_{2^n-1}^n = s_n$, $t_{2^n-2}^n = s_{n-1}$ and

$$\sum_{t_k^n \in]s_{l-1}, s_l[} 2^{-n}\,\alpha_l(t_k^n) = 2^{-n}\,2^{-l+1} + 2\sum_{k=1}^{2^{n-l-1}-1} 4\,k\,2^{-2n} = a_l$$

for $n \ge l+1$. This yields

$$\begin{aligned} W_n(1,s_l) &= e^{0\,A_{n+1}}\,e^{0\,A_n}\,e^{2a_{n-1}A_{n-1}}\cdots e^{2a_{l+1}A_{l+1}} \quad \text{and} \\ W_n(s_m,s_l) &= e^{2a_N A_N}\cdots e^{2a_{l+1}A_{l+1}}, \end{aligned}$$

where $N = \max\{m, n-1\}$ and $n \ge l+2$. Thus, $\|W_n(1,s_l)\|$, $\|W_n(s_m,s_l)\| \le 1$ for $n \in \mathbb{N}$ and $m \ge l$. The stability of the products $W_n(t,s)$, $0 \le s \le t \le 1$, follows readily.

Clearly, the family $(C(s))_{0\le s\le 1}$ satisfies Assumption 2.1. Hence, Theorem 2.3 implies wellposedness of the corresponding Cauchy problem. Moreover, the family $(C(s))_{0\le s\le 1}$ cannot be Kato–stable, since then the operators $A(s) = \frac{1}{2}\,C(s)$ would be Kato–stable contradicting Example 3.2.

We now define partitions $Q_n := \{r_k^n : k = 0, \cdots, 2^n\},\ n \in \mathbb{N}$, by

$$r_k^n := t_k^n - 2^{-n-1} \text{ if } t_k^n = \tau_1, \cdots, \tau_{n-1} \text{ and } r_k^n := t_k^n \text{ otherwise.}$$

Let $\tilde{W}_n(t,s)$, $0 \le s \le t \le 1$, be the products with respect to the family $(C(s))_{0\le s\le 1}$ and the partitions Q_n. Then we obtain

$$\sigma_l := \sum_{r_k^n \in]s_{l-1}, s_l[} 2^{-n}\,\alpha_l(r_k^n) = a_l - 3\cdot 2^{-2n}$$

for $n \ge l+1$. Since we have $\tilde{W}_{n+1}(1,0) = e^{2\sigma_n A_n}\ \cdots\ e^{2\sigma_1 A_1}$ for $n \in \mathbb{N}$, Lemma 3.1(b) implies $\|\tilde{W}_{n+1}(1,0)\| = 2^n$. Thus, the family $(C(s))_{0\le s\le 1}$ is not (Q_n)–stable.

It remains to show the second claim in assertion (b). Let $(W(t,s))_{0\le s\le t\le 1}$ be the evolution family corresponding to $(C(s))_{0\le s\le 1}$ which is given as in (3.4) for $t < 1$. Since $\lim_{t\to 1} W(t,0)f = W(1,0)f$, we obtain for $f \in X_0$

$$(W(1,0)f)(\xi_1,\xi_2,\cdots) = \begin{cases} f(\xi_1 - 2a_1, \xi_2 - 2a_2, \cdots), & \text{if } \xi_k - 2a_k \in J_k \text{ for all } k \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Assume now, that there is a space $\mathcal{D} \subseteq \bigcap_{0\le s\le 1} D(A(s))$ such that the closure of the operator $(\int_0^1 C(r)\,dr, \mathcal{D})$ is a generator and $W(1,0) = e^{\int_0^1 C(r)\,dr}$. Then we would have

$$\left(e^{\frac{1}{2}\int_0^1 C(r)\,dr} f\right)(\xi_1,\xi_2,\cdots) = \begin{cases} f(\xi_1 - a_1, \xi_2 - a_2, \ldots), & \text{if } \xi_k - a_k \in J_k \text{ for all } k \in \mathbb{N}, \\ 0, & \text{otherwise} \end{cases}$$

for $f \in X_0$. But this mapping cannot be continuously extended to a bounded operator on X (use again Lemma 3.1(b)). Thus the representation formula (2.8) for $W(1,0)$ does not hold. Clearly, one can show in the same way that this is also true for $W(1,s)$, $0 < s < 1$. □

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