WELL–POSEDNESS AND ASYMPTOTIC BEHAVIOUR OF NON–AUTONOMOUS LINEAR EVOLUTION EQUATIONS

ROLAND SCHNAUBELT

To the memory of Brunello Terreni

Abstract. We review results on the existence and the long term behaviour of non–autonomous linear evolution equations. Emphasis is put on recent results on the asymptotic behaviour using a semigroup approach.

1. Introduction

There is a striking difference between autonomous and non–autonomous linear evolution equations. Autonomous problems are well understood in the framework of strongly continuous operator semigroups and their generalizations. The Hille–Yosida type theorems settle the question of well–posedness to a great extend, many perturbation and approximation results have been established, and for a large class of problems the asymptotic behaviour can be studied on the basis of spectral theory and transform methods. In these and many other areas semigroup theory has reached a considerable degree of maturity, and its applications thrive in plenty of fields.

For non–autonomous problems, however, we do not yet know of a coherent and general theory. We can rely on several more or less independent, quite sophisticated existence theorems due to P. Acquistapace, H. Amann, G. Da Prato, T. Kato, J.L. Lions, B. Terreni, and others. But these facts cannot be combined into a unified approach. Accordingly, other subjects like perturbation or duality can be treated only in a case by case analysis leaving many questions open. The situation becomes much worse if we look at asymptotic properties because spectral and transform theory cannot be applied directly (with the partial exception of time periodic problems). The available results are usually restricted to problems 'close' to an equation with known behaviour (in particular an autonomous one). Any attempt to overcome these short–comings must face the difficulties which can be demonstrated by mostly rather simple examples refuting many natural conjectures. We present in Section 3 such examples concerning the asymptotic behaviour, whereas R. Nagel and G. Nickel discuss in [91] many examples concerning the existence theory, see also Section 2.

This state of affairs motivates a different approach to non–autonomous Cauchy problems based on the idea to transform them into autonomous ones. The standard ODE method in this context leads to a nonlinear semigroup which seems difficult to handle, see [44]. G.R. Sell and others introduced the 'hull' of a given equation (compare Section 5) which allows to treat problems possessing certain almost periodicity and compactness properties

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in the framework of dynamical systems. In the present paper we concentrate on the approach via *evolution semigroups* proposed by J.S. Howland in 1974. Originally devised to study perturbation theory, this concept might help to unify the existence theory of non–autonomous evolution equations, cf. Section 4 and [91]. So far its major merits concern the investigation of asymptotic properties, namely exponential dichotomy, employing the remarkable fact that the spectrum of the evolution semigroup determines the exponential dichotomy of the underlying problem, see Sections 5 and 6.

This survey exposes the above mentioned subjects aiming at a reader with a solid background in evolution equations. We strive for a coherent non–technical description of the results, problems, and methods in the field. The major features of the theorems should become intelligible though we can state only a few of them explicitly.

In Section 2 we give an outline of the existence theory starting with parabolic problems which allow for the most powerful results. Here the efforts of many renowned researchers culminate in the outstanding papers by P. Acquistapace and B. Terreni. On a more general level, T. Kato created his well known theory which partially extends the Hille–Yosida theorem and is directed to hyperbolic equations. These fundamental contributions are complemented by the operator sum method developed by G. Da Prato, P. Grisvard, and J.L. Lions. We then proceed to discuss exponential dichotomy being one of the most basic asymptotic properties of evolution equations. Evolution semigroups are introduced in Section 4, where their impact on the well–posedness of evolution equations is presented. The latter point is closely related to the operator sum method. Most of the known results on exponential dichotomy rely on certain characterizations of this notion treated in Section 5. Here we emphasize evolution semigroups, but we also explain the connection with the more traditional 'Perron-type' characterizations involving the inhomogeneous problem. Moreover, we present a new proof of Datko's famous stability theorem using evolution semigroups. In the last section results on the asymptotic behaviour of evolution equations are discussed concentrating on exponential dichotomy and qualitative properties of the inhomogeneous problem.

We have compiled a large list of references which nevertheless is far from being complete. In order to keep the bibliography within a reasonable size, we cite no paper dealing exclusively with autonomous problems and very few papers treating the question of (exponential) stability alone. For similar reasons we only give a few hints concerning applications to nonlinear problems, and we usually quote just a sample of an author's publications. The monograph [39] is our standard source for semigroup theory, where one can find unexplained notation, concepts, and results on autonomous problems.

2. Well–posedness of evolution equations

In this section we review existence results for the non–autonomous Cauchy problem

$$(CP) \begin{cases} \frac{d}{dt}\, u(t) = A(t)u(t) + f(t), & t \geq s,\ t, s \in J, \\ \quad u(s) = x, \end{cases}$$

on a Banach space X, where $A(t)$ are linear operators on X, $x \in X$, $f \in L^1_{loc}(J, X)$, and $J \subseteq \mathbb{R}$ is a closed interval. The homogeneous problem with $f = 0$ is denoted by $(\mathrm{CP})_0$.

For suitable differential operators $A(t)$ such equations describe, for instance, diffusion processes with time dependent diffusion coefficients or boundary conditions, quantum mechanical systems with time varying potential, or non–autonomous hyperbolic systems of first order. Using standard techniques, one can also transform wave equations or retarded differential equations into an evolution equation of the form (CP). Further, partial differential equations on non–cylindrical domains can be formulated as non–autonomous Cauchy problems, [21], [22]. Evolution equations further arise if one solves semi- and quasilinear equations by means of linearization methods; see [63], [64], [136], [137] for hyperbolic problems and [5], [53], [86], [128], [141] for parabolic problems.

In the literature various types of solutions u to (CP) are used depending on the required differentiability properties of u. We always assume that $u : J_s \to X$ is continuous and satisfies $u(s) = x$, where $J_s = J \cap [s, \infty)$. First of all, one can suppose that $u \in C^1(J_s, X)$, $u(t) \in D(A(t))$, and $u'(t) = A(t)u(t) + f(t)$ for $t \in J_s$. Second, if the data $A(t)$ and $f(t)$ are not continuous in t, it is appropriate to look for functions $u \in W^{1,p}_{loc}(J_s, X)$ satisfying $u(t) \in D(A(t))$ and $u'(t) = A(t)u(t) + f(t)$ for a.e. $t \in J_s$ and some $1 \le p \le \infty$. One can also replace the interval J_s by $J'_s = J \cap (s, \infty)$ in both cases. This is suitable for parabolic problems with initial values x not contained in $D(A(s))$. Further notions of solutions are introduced below. In our definition of well–posedness we employ C^1–solutions for simplicity.

Definition 2.1. *The homogeneous problem (CP)$_0$ is called* well–posed (on spaces Y_s) *if there are dense subspaces Y_s, $s \in J$, of X with $Y_s \subseteq D(A(s))$ such that for each $x \in Y_s$ there is a unique solution $u = u(\cdot\,; s, x) \in C^1(J_s, X)$ of (CP)$_0$ with $u(t) \in Y_t$ for $t \in J_s$ and if $s_n \to s$ and $x_n \to x$ in X for $s_n \in J$, $x_n \in Y_{s_n}$, $x \in Y_s$, then $\hat{u}(t\,; s_n, x_n) \to \hat{u}(t\,; s, x)$ in X uniformly for t in compact subsets of J.*

Here we set $\hat{u}(t\,; r, y) := u(t\,; r, y)$ for $t \ge r$ and $\hat{u}(t\,; r, y) := y$ for $t \le r$ and $y \in Y_r$. In other words, we require that there exists a unique solution of the homogeneous problem for sufficiently many initial values x and that the solutions depend continuously on the initial data, cf. [42, §7.1].

Starting with a well–posed Cauchy problem, we define $U(t, s)x := u(t\,; s, x)$ for $t \ge s$, $t, s \in J$, and $x \in Y_s$. It is not difficult to show that $U(t, s)$ can be extended to a unique bounded linear operator on X (denoted by the same symbol) such that

(E1) $U(t, s) = U(t, r)U(r, s)$, $U(s, s) = I$, and

(E2) $(t, s) \mapsto U(t, s)$ is strongly continuous

for $t \ge r \ge s$ and $t, s, r \in J$, cf. [91]. This motivates the following definition.

Definition 2.2. *A collection $U(\cdot, \cdot) = (U(t, s))_{t \ge s, t, s \in J} \subseteq \mathcal{L}(X)$ satisfying (E1) and (E2) is called an* evolution family. *If (CP)$_0$ is well–posed (on Y_t) with solutions $u = U(\cdot, s)x$, we say that $U(\cdot, \cdot)$* solves *(CP)$_0$ (on Y_t) or that $A(\cdot)$* generates *$U(\cdot, \cdot)$.*

Evolution families are also called evolution systems, evolution operators, evolution processes, propagators, or fundamental solutions in the literature. In contrast to semigroups, it is possible that the mapping $t \mapsto U(t, s)x$ is differentiable only for $x = 0$. This happens if $U(t, s) = \frac{p(t)}{p(s)}$ on $X = \mathbb{C}$ and $p : \mathbb{R} \to [1, 2]$ is continuous and nowhere differentiable. Thus an evolution family need not be generated by an operator family $A(\cdot)$. We refer to [91] for further information on these topics.

If $(\mathrm{CP})_0$ is well–posed on spaces Y_s, then

$$\tfrac{1}{h}(U(t,s+h)x - U(t,s)x) = U(t,s+h)\,\tfrac{1}{h}(x - U(s+h,s)x) \to -U(t,s)A(s)x \qquad (2.1)$$

as $h \searrow 0$ for $t > s$ and $x \in Y_s$. Assume that $u \in C^1(J_s', X)$ solves (CP) for $f \in C(J_s, X)$ and $u(t) \in Y_t$ for $t \in J_s'$. Then (2.1) yields $\frac{\partial^+}{\partial\tau}[U(t,\tau)u(\tau)] = U(t,\tau)f(\tau)$ for $\tau \in (s,t)$, and hence

$$u(t) = U(t,s)x + \int_s^t U(t,\tau)f(\tau)\,d\tau, \qquad t \ge s. \qquad (2.2)$$

For this reason, the function u defined by (2.2) is called the *mild solution* of (CP) for every $x \in X$ and $f \in L^1_{loc}(\mathbb{R}_+, X)$.

We now survey existence theorems for (CP) and the methods used to establish them in the case $J = [0,T]$. Most of the results discussed in paragraph (a) and (b) are well presented in the monographs [6], [42], [86], [104], [133], [134].

(a) The parabolic case. Here one assumes that the operators $A(t)$ generate analytic C_0–semigroups of the same type and that $t \mapsto A(t)$ is regular in a sense specified below. Then there exists an evolution family $U(\cdot,\cdot)$ on X solving $(\mathrm{CP})_0$ on $D(A(t))$ such that $U(t,s)X \subseteq D(A(t))$, $\frac{\partial}{\partial t}U(t,s) = A(t)U(t,s)$ in $\mathcal{L}(X)$, and $\|A(t)U(t,s)\| \le \frac{C}{t-s}$ for $0 \le s < t \le T$. The operators $U(t,s)$ can be constructed as solutions to certain integral equations like

$$U(t,s) = e^{(t-s)A(s)} + \int_s^t U(t,\tau)(A(\tau) - A(s))e^{(\tau-s)A(s)}\,d\tau \qquad (2.3)$$

(in the case $D(A(t)) \equiv D(A(0))$). This is an abstract version of the method of 'freezing coefficients'; see also the survey given in [2].

If the domains $D(A(t))$ do not depend on t, it suffices to suppose that $A(\cdot) : [0,T] \to \mathcal{L}(Y,X)$ is Hölder continuous, where $Y := D(A(0))$ is endowed with the graph norm. The fundamental results in this direction were established by P.E. Sobolovskii, [128], and H. Tanabe, [132], around 1960. A. Lunardi treated equations with non–dense $D(A(t))$ in her book [86]. Most of the above mentioned results still hold if $A(\cdot) \in C([0,T], \mathcal{L}(Y,X))$ and the operators $A(t)$ possess some additional space regularity, [18], [19], [33], [43], [109].

The case of time varying domains was studied by T. Kato and H. Tanabe in [65] imposing (besides another condition) that $A(\cdot)^{-1} \in C^{1+\alpha}([0,T], \mathcal{L}(X))$. Their results were refined by A. Yagi in the seventies, [139]. One can weaken this differentiability assumption to a Hölder condition like (2.4) below if one requires the time independence of the domain of a fractional power of $A(t)$, [61], [129], or of an interpolation space between $D(A(t))$ and X, [3], [6]. The latter situation is treated in detail in H. Amann's monograph [6]. Finally, P. Acquistapace and B. Terreni introduced in [1] and [4] the assumption

$$\|A(t)R(\lambda, A(t))\,(A(t)^{-1} - A(s)^{-1})\| \le L\,|t-s|^{\mu}\,|\lambda|^{-\nu} \qquad (2.4)$$

for $t, s \in J$, $|\arg\lambda| \le \phi$, and constants $\phi \in (\frac{\pi}{2}, \pi)$, $L \ge 0$, and $\mu, \nu \in (0,1]$ with $\mu + \nu > 1$ (without assuming the density of $D(A(t))$ in X), see also A. Yagi's papers [140], [141]. The hypothesis (2.4) is logically independent of those used in [65] and [139], see [4, §7], but in some sense weaker since only a Hölder condition is needed. The approach of P. Acquistapace and B. Terreni is inspired by the work of R. Labbas and B. Terreni on operator sums, see [68], [69], and (c) below, whereas the results of H. Amann and A. Yagi rely on suitable versions of the formula (2.3).

Based on the properties of $U(\cdot,\cdot)$, one can investigate the regularity of the mild solution u to the inhomogeneous problem (CP). One obtains, for instance, $u \in C^1(J_s, X)$ and $u' = A(\cdot)u + f$ if $x \in D(A(s))$ and f is Hölder continuous or bounded with respect to interpolation norms. In [4], [6], [86] it is even shown that u satisfies optimal regularity of Hölder type.

(b) The hyperbolic case. One can partially extend the Hille–Yosida theorem for semigroups to the non–autonomous situation. To that purpose, one assumes that the operators $A(t)$ are densely defined and *stable* in the sense that

$$\|R(\lambda, A(t_n))R(\lambda, A(t_{n-1})) \cdots R(\lambda, A(t_1))\| \le M\,(\lambda - w)^{-n} \tag{2.5}$$

for all $0 \le t_1 \le \cdots \le t_n \le T$, $n \in \mathbb{N}$, $\lambda > w$, and some constants $M \ge 1$ and $w \in \mathbb{R}$. Observe that then $A(t)$ generates a C_0–semigroup by the Hille–Yosida theorem. One further requires the existence of a Banach space Y being continuously and densely embedded in X such that $Y \subseteq D(A(t))$ and the parts of $A(t)$ in Y are also stable for $t \in [0,T]$. If, in addition, $A(\cdot) \in C([0,T], \mathcal{L}(Y,X))$, then T. Kato succeeded to construct an evolution family $U(\cdot,\cdot)$ on X satisfying $\frac{\partial}{\partial t} U(t,s)x_{|t=s} = A(s)x$ for $t \ge s$ and $x \in Y$. He first studied the special case that the operators $A(t)$ have a common domain and generate contraction semigroups in his paper [60] from 1953. The general case was established in 1970, [62]. He introduced a time discretization $A_n(\cdot)$ of $A(\cdot)$ and solved the corresponding Cauchy problem by finite products $U_n(t,s)$ of the given operators $e^{\tau A(r)}$. Then the assumptions allow to prove that $U_n(t,s)$ converges strongly to an operator $U(t,s)$ having the asserted properties.

Imposing an additional regularity hypothesis on $t \mapsto A(t)$, see [62, Thm.6.1], Kato also showed that $U(t,s)Y \subseteq Y$, $U(\cdot,\cdot)$ is strongly continuous on Y, and $\frac{\partial}{\partial t} U(t,s)x = A(t)U(t,s)x$ for $t \ge s$ and $x \in Y$, i.e., $(\mathrm{CP})_0$ is well-posed on Y. (This extra assumption holds in particular if $D(A(t)) \equiv Y$ and $A(\cdot)y \in C^1(J,X)$ for $y \in Y$.) It is then not difficult to verify that the mild solution belongs to $C^1([s,T],X) \cap C([s,T],Y)$ and solves (CP) provided that $f \in C([s,T],Y)$ and $x \in Y$.

These results were extended to strongly continuous $A(\cdot)$ in [66], to strongly measurable $A(\cdot)$ in [58], and to strongly measurable resolvents in [34]. Of course, in the latter two cases (CP) can be solved only for a.e. $t \ge s$. If X is a Hilbert space, results involving conditions on the numerical range of $A(t)$ are proved in [100], [101]. Finally, the case of non–dense domains is investigated in [35], [136], [137], and of so–called C–evolution systems in [135]. Nonlinear versions of Kato's result were shown by M. Crandall and A. Pazy, [29], in 1972 using a resolvent approximation scheme, see also [41].

(c) The operator sum method. In most of the previous two paragraphs, (CP) is treated in two steps. One first constructs the evolution family solving the homogeneous equation and then one establishes the required regularity of the mild solution. There is another approach which directly tackles the inhomogeneous problem. One considers (CP) with $x = 0$ as an equation

$$Gu := -\tfrac{d}{dt}\, u + A(\cdot)u = -f$$

on a function space such as $E = L^p([0,T],X)$ and tries to invert the operator G.* This idea is pursued in the work by G. Da Prato and P. Grisvard both in the hyperbolic and the parabolic case, see in particular [32]. A related approach is developed in [68] and [69].

One says that the problem (CP) has *maximal regularity of type* L^p if the operator G is invertible in $E = L^p([0,T],X)$ on the domain $F = \{f \in W^{1,p}([0,T],X) : f(0) = 0,\ f(t) \in D(A(t))$ for a.e. $t \in [0,T],\ A(\cdot)f(\cdot) \in E\}$. In this case the function $u = -G^{-1}f$ solves (CP) in the $W^{1,p}$–sense. This property holds for many parabolic problems on L^q–spaces (if $1 < p,q < \infty$), see [6], [43], [54], [55], [89], [109], [131], [142], where further references to the autonomous case are given. Following the approach of Da Prato and Grisvard, one can construct the inverse of G by a contour integral involving the resolvents of $\frac{d}{dt}$ and $A(\cdot)$. The main step in the proof is to show that G^{-1} maps into $D(A(\cdot))$.

In the hyperbolic case the operator $G : D(\frac{d}{dt}) \cap D(A(\cdot)) \to E$ is not closed (and thus not invertible) in general. For instance, the closure of $G = -\frac{\partial}{\partial t} + \frac{\partial}{\partial s}$ with $D(G) = W^{1,2}(\mathbb{R}, L^2(\mathbb{R})) \cap L^2(\mathbb{R}, W^{1,2}(\mathbb{R}))$ in $L^2(\mathbb{R}, L^2(\mathbb{R})) \cong L^2(\mathbb{R}^2)$ is the derivative in direction $t = s$ with maximal domain. Nevertheless, it was shown in [34] that $G = -\frac{d}{dt} + A(\cdot)$ has an invertible closure if, for instance, the spaces X and Y from paragraph (b) are reflexive, the operators $A(t)$ and the parts of $A(t)$ in Y are stable, $\|A(t)\|_{\mathcal{L}(Y,X)} \le c$ for $0 \le t \le T$, and $R(\lambda, A(\cdot))$ is strongly measurable in X and Y for $\lambda > w$. Moreover, if $f \in L^p([0,T],Y)$, then $u = -\overline{G}^{-1}f$ belongs to $W^{1,p}([0,T],X) \cap L^\infty([0,T],Y)$, $1 < p < \infty$, and u is the $W^{1,p}$–solution of (CP). An inhomogeneity $f \in L^p([0,T],X)$ can be approximated by functions $f_n \in L^p([0,T],Y)$. The $W^{1,p}$–solutions u_n corresponding to f_n then converge uniformly to $u = -\overline{G}^{-1}f$ so that (CP) can be solved approximately in $W^{1,p}$–sense. This line of research was further pursued in [20] and [35]. We will come back to this approach in Section 4.

J.L. Lions developed the operator sum method in the context of weak solutions in Hilbert spaces, see [80, §3.1, 3.4]. Let $A(t)$ be defined by uniformly bounded and coercive sesquilinear forms a_t on a Hilbert space X with a dense form domain V such that $t \mapsto a_t(x,y)$ is measurable for all $x, y \in V$. Thus, $V \hookrightarrow X \cong X' \hookrightarrow V'$ and $A(t)$ can be extended to a bounded operator from V to V'. Using a duality argument, it is possible to establish the bijectivity of $G : W_0^{1,2}((0,T),V') \cap L^2([0,T],V) \to L^2([0,T],V')$. In other words, (CP) is solved in the larger space V'. Other proofs of this result can be found in [79, §III.1], where a Galerkin scheme is used, and in [133, §5.5], where the theory of paragraph (a) is applied. It is also possible to obtain weak solutions for wave equations using Galerkin's method, see [80, §3.8, 3.9]. We finally mention the recent contributions [59] and [130], where form methods are employed for parabolic problems.

We have recalled a multitude of methods, notions of solutions, and existence results for non–autonomous Cauchy problems which are mostly independent of each other. In the autonomous case the Hille–Yosida type theorems provide a powerful characterization of well–posedness which is embedded in a coherent theory. So far, however, no analogues are known for non–autonomous equations. In fact, there are plenty of examples, discussed in [91], which indicate that it is rather difficult to find necessary and sufficient conditions on the operators $A(t)$ for the well–posedness of $(\mathrm{CP})_0$ (except for special situations, see [23], [24]).

*It is possible to incorporate nontrivial initial values x, cf. [6, §III.1.5], [34], but for simplicity we concentrate on the case $x = 0$.

Perturbation theory is another important method to solve (CP), see e.g. [38], [53], [62], [65], [113], [126]. Here one considers the Cauchy problem

$$u'(t) = [A(t) + B(t)]\, u(t), \quad t \ge s, \qquad u(s) = x, \tag{2.6}$$

where $A(\cdot)$ generates an evolution family $U(\cdot,\cdot)$ and $B(t)$ is 'small compared with $A(t)$'. If $B(\cdot) \in C([0,T], \mathcal{L}_s(X))$, one easily obtains a unique evolution family $U_B(\cdot,\cdot)$ such that

$$U_B(t,s)x = U(t,s)x + \int_s^t U(t,\tau)B(\tau)U_B(\tau,s)x\, d\tau, \quad t \ge s, \tag{2.7}$$

by means of the Dyson–Phillips expansion, see [105]. Then $u = U_B(\cdot,s)x$ can be considered as the mild solution of (2.6), cf. (2.2). However, in general (2.6) does not inherit the well–posedness of $(\mathrm{CP})_0$ as is seen by the next example which is a variant of [105, Ex.6.4].

Example 2.3. On $X = C_0([0,1))$ we define $A\varphi = \varphi'$ with $D(A) = C_0^1([0,1))$ and

$$(B(t)\varphi)\,(\xi) = \begin{cases} \varphi(\xi), & 1 > \xi \ge \max\{0, \frac{1}{2} - t\}, \\ 2(\xi + t)\varphi(\xi), & \frac{1}{2} - t > \xi \ge 0, \end{cases}$$

for $\varphi \in X$. Observe that $B(\cdot) \in C_b(\mathbb{R}_+, \mathcal{L}(X))$. One easily checks that the solution is

$$u(t)\,(\xi) := U_B(t,0)\varphi\,(\xi) = \begin{cases} 0, & \xi + t \ge 1, \\ e^t \varphi(\xi + t), & 1 \le 2(\xi + t) \le 2, \\ e^{2(\xi+t)t}\varphi(\xi + t), & 0 \le 2(\xi + t) \le 1, \end{cases}$$

for $t \ge 0$ and $0 \le \xi < 1$. Clearly, $u \in C^1(\mathbb{R}_+, X)$ if and only if $\varphi \in D(A)$ and $\varphi(\frac{1}{2}) = 0$, so that (2.6) is not well–posed.

3. Exponential dichotomy of evolution families

We now turn our attention to asymptotic properties of evolution families $U(\cdot,\cdot)$ for $J = [a, \infty)$ or $\mathbb{R}$. We first define the *(uniform exponential) growth bound* of $U(\cdot,\cdot)$ by

$$\omega(U) := \inf\{w \in \mathbb{R} : \exists\, M_w \ge 1 \text{ with } \|U(t,s)\| \le M_w\, e^{w(t-s)} \text{ for } t \ge s,\ s \in J\}.$$

The evolution family is called *exponentially bounded* if $\omega(U) < \infty$ and *exponentially stable* if $\omega(U) < 0$. For instance, $U(t,s) = \exp(t^2 - s^2)$ has growth bound $+\infty$ and $V(t,s) = \exp(s^2 - t^2)$ has growth bound $-\infty$.

A semigroup $T(\cdot)$ is exponentially stable if and only if the spectral radius of one operator $T(t)$, $t > 0$, is less than 1. In the non–autonomous case, however, the following examples show that the location of $\sigma(U(t,s))$ has no influence on the asymptotic behaviour of the evolution family $U(\cdot,\cdot)$, in general.

Example 3.1. Let $S(t)$, $t \ge 0$, be the nilpotent right translation on $X = L^1[0,1]$ and $U(t,s) = e^{t^2 - s^2} S(t-s)$. Then $r(U(t,s)) = 0$ for $t > s$ and $U(s+t,s) = 0$ for $t \ge 1$ and $s \in \mathbb{R}$, but $\|U(s + \frac{1}{2}, s)\| = e^{s + 1/4}$ and hence $\omega(U) = +\infty$.

Example 3.2. In Example 3.5 we construct an evolution family such that $r(U(t,s)) = 0$ for $t > s$, but $t \mapsto \|U(t,s)\|$ grows faster than any exponential function as $t \to \infty$.

We add that, due to a standard argument, it is possible to control $\omega(U)$ by means of uniform norm estimates of $U(t,s)$ on strips $0 < t-s \le t_0$, see [30, §III.4] or [39, p.479].

One is further interested in exponential decay and increase on invariant subspaces, see [25], [28], [30], [31], [51], [53], [86], [119]. We write $Q = I - P$ for a projection P.

Definition 3.3. *An evolution family $U(\cdot,\cdot)$ on a Banach space X (with $J = \mathbb{R}$ or $[a,\infty)$) has an* exponential dichotomy *(or is called* hyperbolic*) if there are projections $P(t)$, $t \in J$, and constants $N, \delta > 0$ such that $P(\cdot) \in C_b(J, \mathcal{L}_s(X))$ and, for $t \ge s$, $s \in J$,*

(a) $U(t,s)P(s) = P(t)U(t,s)$,
(b) *the restriction $U_Q(t,s) : Q(s)X \to Q(t)X$ of $U(t,s)$ is invertible (and we set $U_Q(s,t) := U_Q(t,s)^{-1}$),*
(c) $\|U(t,s)P(s)\| \le Ne^{-\delta(t-s)}$ *and* $\|U_Q(s,t)Q(t)\| \le Ne^{-\delta(t-s)}$.

The existence of an exponential dichotomy gives an important insight into the long–term behaviour of an evolution family. It is also used to study the asymptotic properties of (mild) solutions to the inhomogeneous problem (CP), see Section 6. Further, exponential splittings are preserved under small nonlinear perturbations which leads to principles of linearized (in–)stability and to the construction of stable, unstable, and center manifolds, as exposed in the monographs cited before Definition 3.3.

If the hyperbolic evolution family $U(t,s) = T(t-s)$ is given by a semigroup $T(\cdot)$ for $J = \mathbb{R}$, then $P(t)$ does not depend on t by [121, Cor.3.3] so that $T(\cdot)$ is hyperbolic in the usual sense. Recall that a semigroup $T(\cdot)$ is hyperbolic if and only if the unit circle $\mathbb{T}$ belongs to $\rho(T(t_0))$ for some/all $t_0 > 0$. Moreover, the dichotomy projection P then coincides with the *spectral projection*

$$P = \frac{1}{2\pi i}\int_{\mathbb{T}} R(\lambda, T(t_0))\, d\lambda. \tag{3.1}$$

Similar results hold for *periodic* evolution families $U(\cdot,\cdot)$, i.e., $U(t+p, s+p) = U(t,s)$ for $t \ge s$, $t,s \in \mathbb{R}$, and some $p > 0$. In that case, one has the equalities

$$\begin{aligned} &\sigma(U(s+p,s)) = \sigma(U(p,0)) \quad \text{and} \\ &U(t,s) = U(t,t-p)^n U(s+\tau,s) = U(t,t-\tau)U(s+p,s)^n, \end{aligned} \tag{3.2}$$

where $t = s + np + \tau$, $n \in \mathbb{N}_0$, and $\tau \in [0,p)$. A periodic evolution family $U(\cdot,\cdot)$ is hyperbolic if and only if $\sigma(U(p,0)) \cap \mathbb{T} = \emptyset$ because of these relations. The dichotomy projections are then given by

$$P(s) = \frac{1}{2\pi i}\int_{\mathbb{T}} R(\lambda, U(s+p,s))\, d\lambda \tag{3.3}$$

for $s \in \mathbb{R}$, cf. [30], [31], [51], [53], [86].

In the autonomous case the exponential dichotomy of a semigroup $T(\cdot)$ generated by A always implies the spectral condition $\sigma(A) \cap i\mathbb{R} = \emptyset$. In particular, the exponential stability of $T(\cdot)$ yields $s(A) := \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\} < 0$. The converse implications fail for general C_0–semigroups, but can be verified if the spectral mapping theorem

$$\sigma(T(t)) \setminus \{0\} = e^{t\sigma(A)}, \qquad t \ge 0, \tag{3.4}$$

holds. Formula (3.4) is satisfied by eventually norm continuous semigroups, and hence by analytic or eventually compact semigroups. Starting from these results a powerful spectral theory for semigroups has been developed, see [39] and the references therein.

In many cases it is thus possible to characterize the exponential dichotomy of a semigroup by the spectrum of its generator. This is an important fact since in applications usually the generator A is the given object. Unfortunately, in the non–autonomous case there is no hope to relate the location of $\sigma(A(t))$ to the asymptotic behaviour of the evolution family $U(\cdot,\cdot)$ generated by $A(\cdot)$ as is shown by the following examples. We point out that the first one, [28, p.3], [121, Ex.3.4], deals with periodic evolution families $U_k(\cdot,\cdot)$ on $X = \mathbb{C}^2$ satisfying

$$s(A_1(t)) = -1 < \omega(U_1) \quad \text{and} \quad s(A_2(t)) = 1 > \omega(U_2) \quad \text{for } t \in \mathbb{R},$$

whereas for semigroups $T(\cdot)$ with generator A one always has $s(A) \le \omega(T)$.

Example 3.4. Let $A_k(t) = D(-t)A_kD(t)$ for $t \in \mathbb{R}$, where

$$D(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, \qquad A_1 = \begin{pmatrix} -1 & -5 \\ 0 & -1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The operators $A_1(t)$ and $A_2(t)$, $t \in \mathbb{R}$, generate the evolution families

$$U_1(t,s) = D(-t)\ \exp\left[(t-s)\begin{pmatrix} -1 & -4 \\ -1 & -1 \end{pmatrix}\right]\ D(s) \quad \text{and}$$
$$U_2(t,s) = D(-t)\ \exp\left[(t-s)\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\right]\ D(s),$$

respectively. Thus, $\omega(U_1) = 1$ and $\omega(U_2) = 0 = \omega(U_2^{-1})$, but $\sigma(A_1(t)) = \{-1\}$ and $\sigma(A_2(t)) = \{-1, 1\}$ for $t \in \mathbb{R}$. In other words, the exponential stability of $e^{\tau A_1(t)}$ and the exponential dichotomy of $e^{\tau A_2(t)}$ (with constants independent of t) are lost when passing to the non–autonomous problem.

In Example 3.4 we even have $\|R(\lambda, A_k(t))\|_2 = \|R(\lambda, A_k)\|_2$ for $t \in \mathbb{R}$, $\lambda \in \rho(A_k)$, and $k = 1, 2$. This shows that we cannot expect to deduce asymptotic properties of an evolution family from estimates on the resolvent of $A(t)$ along vertical lines as it is possible for a semigroup on a Hilbert space by virtue of the theorem of Gearhart–Howland–Prüss, see e.g. [25, Thm.2.16]. In the following infinite dimensional example from [121, §5] we even have $\omega(U) = +\infty$ and $s(A(t)) = -\infty$ for a.e. $t \ge 0$.

Example 3.5. Let $h(s) = 1$ for $s \in [0, \frac{1}{2})$ and $h(s) = 2$ for $s \in [\frac{1}{2}, \frac{3}{4}]$. Set $\mu = h\,ds$, $(\Omega, \nu) = \bigotimes_{n\in\mathbb{N}}([0, \frac{3}{4}], \mu)$, and $X = L^1(\Omega, \nu)$. We define for each $k \in \mathbb{N}$ and $\tau \ge 0$ the right–translation

$$(R_k(\tau)f)\,(x_1, \cdots) = \begin{cases} f(x_1, \cdots, x_{k-1}, x_k - \tau, x_{k+1}, \cdots), & x_k - \tau \in [0, \frac{3}{4}], \\ 0, & x_k - \tau \notin [0, \frac{3}{4}], \end{cases}$$

for simple functions $f \in X$. In [98, Lemma 2.1] it is shown that these operators can be extended to C_0–semigroups on X with generator A_k such that

$$\begin{aligned} &\|R_k(\tau)\| \le 2 \quad \text{for } \tau \ge 0, \quad R_k(\tau) = 0 \quad \text{for } \tau \ge \tfrac{3}{4}\,, \\ &\|R_n(\tau_n)\cdots R_m(\tau_m)\| = 2^{n-m+1} \quad \text{for } 0 < \tau_l \le \tfrac{1}{4} \text{ and } m \le l \le n, \\ &Y = \{f \in \bigcap_{k\in\mathbb{N}} D(A_k) : \|f\|_Y := \sup_{k\in\mathbb{N}}\{\|f\|_1\,,\ \|A_k f\|_1\} < \infty\} \quad \text{is dense in } X. \end{aligned} \tag{3.5}$$

Take $(t_k) = (0, 1, 2, \frac{5}{2}, 3, \frac{10}{3}, \frac{11}{3}, 4, \dots)$ and choose functions $\alpha_k \in C^1(\mathbb{R})$ satisfying $\alpha_k > 0$ on (t_{k-1}, t_k), $\alpha_k = 0$ on $\mathbb{R}_+ \setminus (t_{k-1}, t_k)$, and $\|\alpha_k\|_\infty, \|\alpha_k'\|_\infty \le \frac{1}{4}$ for $k \in \mathbb{N}$. Then $a_k = \int \alpha_k(t)\,dt \le \frac{1}{4}$. We now define

$$A(t) = \alpha_k(t) A_k \quad \text{with } D(A(t)) = D(A_k) \quad \text{for } t_{k-1} < t < t_k \quad \text{and} \quad A(t_{k-1}) = 0$$

for $k \in \mathbb{N}$. Note that the operators $A(t)$ generate uniformly bounded, commuting, positive semigroups which are nilpotent if $t \ne t_k$, and $A(\cdot) \in C_b^1(\mathbb{R}_+, \mathcal{L}(Y, X))$. Moreover, $A(\cdot)$ generates the evolution family

$$U(t,s) = R_{l+1}\Big(\int_{t_l}^{t} \alpha_{l+1}(r)\,dr\Big)\, R_l(a_l)\ \cdots\ R_{k+1}(a_{k+1})\, R_k\Big(\int_s^{t_k} \alpha_k(r)\,dr\Big), \quad t \ge s \ge 0,$$

where $t_{k-1} < s \le t_k \le t_l \le t < t_{l+1}$. Finally, $\|U(2n, 0)\| \ge 2^{(n^2)}$ by (3.5), but $s(A(t)) = -\infty$ if $t \ne t_k$, $A(t_k) = 0$, and $r(U(t,s)) = 0$ for $t > s \ge 0$.

4. Evolution semigroups

We now present a semigroup approach to non–autonomous Cauchy problems. Let $U(\cdot,\cdot)$ be an exponentially bounded evolution family on a Banach space X and $J \in \{[a,b], [a,\infty), \mathbb{R}\}$. We define operators $T(t)$, $t \ge 0$, by setting

$$(T(t)f)(s) := \begin{cases} U(s, s-t) f(s-t), & s-t, s \in J, \\ 0, & s \in J, s-t \notin J, \end{cases}$$

on the function spaces $E = L^p(J, X)$, $1 \le p < \infty$, or

$$E = C_{00}(J, X) := \begin{cases} C_0(\mathbb{R}, X), & J = \mathbb{R}, \\ \{f \in C_0([a,\infty), X) : f(a) = 0\}, & J = [a, \infty), \\ \{f \in C([a,b], X) : f(a) = 0\}, & J = [a,b], \end{cases}$$

endowed with the usual p–norm, $1 \le p \le \infty$. It is easily verified that $T(\cdot)$ is a strongly continuous semigroup on E with $\omega(T) = \omega(U)$. We call $T(\cdot)$ the *evolution semigroup* on E associated with $U(\cdot,\cdot)$, and denote its generator by G. Evolution semigroups on the above spaces were introduced in the seventies by J.S. Howland, [56], D.E. Evans, [40], and L. Paquet, [103]. Other early contributions are due to H. Neidhardt, [95], and G. Lumer, [82], [83], [84]. Evolution semigroups can also be defined on other Banach function spaces, [112], [120], and on spaces of (almost) periodic functions, [8], [14], [45], [47], [57], [94]. The recent monograph [25] by C. Chicone and Y. Latushkin is devoted to the investigation of evolution semigroups with an emphasis on spectral properties and applications to dynamical systems. In the latter field similar concepts have been developed since J. Mather's paper [88] from 1968, see [25, Chap.6-8].

In this section we indicate how evolution semigroups can be used to solve non–autonomous Cauchy problems. This subject is further treated in [91]. The relationship to the asymptotic behaviour of evolution families is exposed in the next section.

We first give a representation of the generator G in the case that $U(\cdot,\cdot)$ solves a Cauchy problem, see [122, Prop.3.14] which is a refinement of [25, Thm.3.12], [71], [83], [103].

Proposition 4.1. *Let (CP)$_0$ be well-posed on Y_t with an exponentially bounded evolution family $U(\cdot,\cdot)$, where $J \in \{[a,b],[a,\infty),\mathbb{R}\}$. Then $F = \{f \in C^1(J,X) : f(t) \in Y_t$ for $t \in J,\ f, f', A(\cdot)f \in E\}$ is a core of the generator G of the induced evolution semigroup on $E = C_{00}(J,X)$ and $Gf = -f' + A(\cdot)f(\cdot)$ for $f \in F$.*

We remark that an analogous result can be shown for the evolution semigroup on the space $E = L^p(J,X)$ with $F = \{f \in W^{1,p}(J,X) : f(a) = 0,\ f(t) \in Y_t$ for a.e. $t \in J,\ A(\cdot)f \in E\}$. Here one could also consider Cauchy problems being 'well–posed in $W^{1,p}$–sense.'

We now try to establish a converse of Proposition 4.1: Show first that the sum $-\frac{d}{dt}+A(\cdot)$ on E has a closure which generates a semigroup $T(\cdot)$. (The sum is defined on $F = \{f \in D(A(\cdot)) \cap D(\frac{d}{dt}), f(a) = 0\}$ which is assumed to be dense in E.) Then the implication "(c)⇒(a)" of the following characterization of evolution semigroups provides us with an evolution family $U(\cdot,\cdot)$. Finally, solve (CP) using $U(\cdot,\cdot)$. Observe that the first of these three steps is closely connected to the operator sum method discussed in Section 2.

The next theorem is taken from [110, Thm.2.4] and [120, Thm.2.6]. Previous and different versions are contained in [40], [56], [82], [84], [85], [95], [103]. We define for $\varphi : J \to \mathbb{C}$ and $t \geq 0$ the translated function φ_t by $\varphi_t(s) = \varphi(s-t)$ if $s, s-t \in J$ and $\varphi_t(s) = 0$ if $s \in J,\ s-t \notin J$.

Theorem 4.2. *For a C_0–semigroup $T(\cdot)$ on $E = C_{00}(J,X)$ or $L^p(J,X)$, $1 \leq p < \infty$, with generator G, the following assertions are equivalent.*

(a) *$T(\cdot)$ is an evolution semigroup given by an evolution family with index set $J \setminus \inf J$.*
(b) *$T(t)(\varphi f) = \varphi_t T(t) f$ for $f \in E$, $\varphi \in C_b(J)$, $t \geq 0$. $D(G)$ is a dense subset of $C_{00}(J,X)$.*
(c) *For all f contained in a core of G and $\varphi \in C^1(J)$ with $\varphi, \varphi' \in C_{00}(J)$, we have $\varphi f \in D(G)$ and $G(\varphi f) = \varphi G f - \varphi' f$. $D(G)$ is a dense subset of $C_{00}(J,X)$.*

Notice that the second condition in (b) and (c) is trivially satisfied if $E = C_{00}(J,X)$. Instead of this condition it was assumed in [110] and [120] that $R(\lambda, G)$ maps E continuously into $C_{00}(J,X)$ with dense range for some $\lambda \in \rho(G)$, but these two assertions are equivalent due to the closed graph theorem.

These characterizations were already used in the earliest papers on evolution semigroups to deduce perturbation results for evolution families from known perturbation theorems for semigroups. We refer to [82], [83], [93], [113] for bounded perturbations, to [110], [127] for relatively bounded perturbations of Miyadera type, and to [40], [56] for perturbation results in the context of scattering theory. We further observe that

$$u \in D(G),\ Gu = -f \iff u = T(r)u + \int_0^r T(\rho) f \, d\rho, \quad r \geq 0,$$

$$\iff u(t) = U(t,s)u(s) + \int_s^t U(t,\tau) f(\tau) \, d\tau, \quad t \geq s, \qquad (4.1)$$

for $u, f \in E$, compare [11], [73], [96]. For $J = [a, b]$ or $[a, \infty)$, we arrive at

$$u \in D(G),\ Gu = -f \iff u(t) = \int_a^t U(t,\tau) f(\tau)\, d\tau, \quad t \geq a, \tag{4.2}$$

since $u(a) = 0$ for $u \in D(G)$. In view of (2.2), (4.1), and (4.2), the elements of $D(G)$ for $J = [s, b]$ coincide with the mild solutions of (CP) with $x = 0$ and $f \in E$, and $U(\cdot, s)x$ with the mild solution of the homogeneous problem. At first glance, this is just a formal correspondence since we have introduced the term 'mild solution' only for well–posed problems. In the present situation, however, we obtain approximative solutions in C^1– or $W^{1,p}$–sense (depending on the choice of E) since G is the closure of $(-\frac{d}{dt} + A(\cdot), F)$. In fact, if $Gu = -f$ on $[s, b]$ (hence $u(s) = 0$), then there are $u_n \in F$ and $f_n \in E$ such that $u_n \to u$ and $f_n \to f$ in E and $u_n' = A(\cdot)u_n + f_n$. Since $u_n = -G^{-1}f_n$, we obtain that $u_n \to u$ uniformly also if $E = L^p$. Further, for $x \in X$, $s \in (a, b)$, and $n \in \mathbb{N}$, we define $u_n = \alpha_n(\cdot)U(\cdot, s - \frac{1}{n})x$, where $\alpha_n \in C^1([a, b])$ is equal to 1 on $[s, b]$ and vanishes on $[a, s - \frac{1}{n}]$ and $U(t, r) := 0$ for $t < r$. It is easy to check that $u_n \in D(G)$ and $Gu_n = -\alpha_n'(\cdot)U(\cdot, s - \frac{1}{n})x$ on $[a, b]$, see [25, p.64]. By another approximation, we find $v_n \in F$ such that $v_n(t) \to U(t, s)x$ and $v_n' - A(\cdot)v_n \to 0$ on $[s, b]$.

It then remains to determine $D(G)$ or to verify regularity properties of $U(t, s)$ in order to obtain differentiable solutions of (CP). In [85] and [109] we investigated parabolic problems by means of this method. Using maximal regularity of type L^p, we proved that the mild solutions solve (CP) in $W^{1,p}$–sense. We also refer to [90] for a related semigroup approach to parabolic evolution equations.

It is interesting to recapitulate the results by G. Da Prato and M. Iannelli from [34] (see Paragraph (c) of Section 2) from the present point of view. In [34, Cor.1, Prop.4] it was shown that $-\frac{d}{dt} + A(\cdot)$ defined on the dense subspace $F = W_0^{1,p}((0, T], X) \cap L^p([0, T], Y)$ has an invertible closure G in $L^p([0, T], X)$ assuming Kato type conditions. Further, $D(G)$ is dense in $C_{00}([0, T], X)$ due to Proposition 3 and Remark 3 of [34]. Let $A_n(t)$ be the Yosida approximation of $A(t)$. Then $G_n = -\frac{d}{dt} + A_n(\cdot)$ converges strongly to G on F, and the semigroups generated by G_n are uniformly bounded by [34, Prop.1]. So the Trotter–Kato theorem shows that G generates a C_0–semigroup which is an evolution semigroup by Theorem 4.2. One now obtains approximate solutions as indicated above.

5. Characterizations of exponential dichotomy

Evolution semigroups have attracted a lot of interest in recent years since it was discovered that their spectra characterize the exponential dichotomy of evolution families. This is quite remarkable in view of the discouraging examples in Section 3. We first discuss spectral properties of the evolution semigroup $T(\cdot)$ and its generator G on $E = C_{00}(J, X)$ or $E = L^p(J, X)$. If J is compact, then $T(\cdot)$ is nilpotent so that $\sigma(G) = \emptyset$ and $\sigma(T(t)) = \{0\}$ for $t > 0$. This case is of course irrelevant to the long term behaviour of $U(\cdot, \cdot)$. For unbounded J, we have the following result from, e.g., Section 3.2.2 and 3.3.1 of [25].

Theorem 5.1. *Let $U(\cdot, \cdot)$ be an exponentially bounded evolution family on X and $J = \mathbb{R}$ or $\mathbb{R}_+$. Then the associated evolution semigroup $T(\cdot)$ on $E = C_{00}(J, X)$ or $E = L^p(J, X)$, $1 \leq p < \infty$, with generator G has the following properties.*

(a) *The spectral mapping theorem $\sigma(T(t)) \setminus \{0\} = e^{t\sigma(G)}$, $t \geq 0$, holds.*

(b) $\sigma(T(t))$, $t>0$, *is rotationally invariant and* $\sigma(G)$ *is invariant under translations along the imaginary axis. In the case* $J=\mathbb{R}_+$, *the spectrum of* $T(t)$, $t>0$, *is the full disc with center 0 and radius* $e^{t\omega(U)}$.

R. Rau showed part (b) in [116, Prop.2]. He also established (a) on $L^2(\mathbb{R},X)$ for a Hilbert space X using the Gearhart–Howland–Prüss theorem, see [114, Prop.2.2] and also [115, Prop.6]. This idea was previously employed in [74] in a somewhat different context. In the general case, the spectral mapping theorem for $J=\mathbb{R}$ is due to Y. Latushkin and S. Montgomery-Smith who proved it in [70, Thm.3.1] by means of a reduction to the autonomous case. In [112], we proved (a) first directly on $C_0(\mathbb{R},X)$ and then for a large class of Banach function spaces E on $\mathbb{R}$ (containing $L^p(\mathbb{R},X)$) by showing that the spectra of $T(t)$ and G do not depend on the choice of E. Further direct proofs for the spectral mapping theorem are presented in [71] for the C_0–case and in [25, p.83] for the L^p–case. All these proofs of (a) (except for Rau's) are based on explicit formulas for approximate eigenfunctions of G using given approximate eigenfunctions of $I-T(t)$ and the definition of $T(t)$. These constructions can be extended to the halfline case, see [120, Thm.5.3] for $E=C_0$. Theorem 5.1 was also shown independently by A.G. Baskakov in [10] and [11] employing completely different methods indicated below. The importance of Theorem 5.1 relies on the equivalence (a)⇔(b) in the next result, see e.g. [25, §3.2.3].

Theorem 5.2. *Let* $U(\cdot,\cdot)$ *be an exponentially bounded evolution family on* X *and* $J=\mathbb{R}$. *Let* $T(\cdot)$ *be the associated evolution semigroup on* $E=C_0(\mathbb{R},X)$ *or* $E=L^p(\mathbb{R},X)$, $1\le p<\infty$, *with generator* G. *Then the following assertions are equivalent.*

(a) $U(\cdot,\cdot)$ *has an exponential dichotomy with projections* $P(\cdot)$.
(b) $T(\cdot)$ *has an exponential dichotomy with projection* $\mathcal{P}$.
(c) G *is invertible.*

If this is the case, then

$$P(\cdot)=\mathcal{P}=\frac{1}{2\pi i}\int_{\mathbb{T}} R(\lambda,T(t))\,d\lambda,\quad t>0. \tag{5.1}$$

The equivalence of the hyperbolicity of the evolution family and of the induced evolution semigroup on $C_0(\mathbb{R},X)$ was essentially shown by R. Rau in [116, Thm.6], see also [111]. The corresponding result for $L^p(\mathbb{R},X)$ is due to Y. Latushkin, S. Montgomery-Smith, and T. Randolph, see [70, Thm.3.4] and [72, Thm.3.6], who used a rather involved discretization technique. We gave an elementary proof in the spirit of Rau's work in [111]. The most important step in each proof is to establish that the spectral projection $\mathcal{P}$ of the hyperbolic evolution semigroup is a multiplication operator which then gives the dichotomy projections for $U(\cdot,\cdot)$. This fact is a consequence of the algebraic structure of $T(t)$. One can also deduce the L^p–case from Rau's result on $C_0(\mathbb{R},X)$ and the p–independence of the spectra of $T(t)$ and G, see [112]. We further refer to [7], [15], [97] for the case of bounded $A(t)$. Observe that the formula (5.1) gives a substitute for the more direct representations (3.1) and (3.3) valid in the autonomous and periodic case.

In view of (4.1), Theorem 5.2 immediately implies that

$$U(\cdot,\cdot)\text{ is hyperbolic} \iff \forall\, f\in E\ \ \exists!\, u\in E\ \text{ satisfying }\ (4.1) \tag{5.2}$$

for $E = C_0(\mathbb{R}, X)$ or $L^p(\mathbb{R}, X)$, [73]. The equivalence (5.2) also holds for $E = C_b(\mathbb{R}, X)$. This can be seen by reducing it to the C_0–case, [25, p.128]. Moreover, V.V. Zhikov proved it already in 1972 directly by clever manipulations of the equation (4.1), see [76, Chap.10] and also [28, §8], [30, §IV.3]. Yet another proof is presented in [73]. Conversely, A.G. Baskakov deduced Theorem 5.1 and 5.2 for $J = \mathbb{R}$ from the equivalence (5.2) on $E = C_b(\mathbb{R}, X)$, [11].

In the same way, Theorem 5.1 for $J = \mathbb{R}_+$ and (4.2) allow to characterize exponential stability of an evolution family, see [25, Thm.3.26]. We set $(\mathbb{K}f)(t) = \int_0^t U(t,\tau)f(\tau)\, d\tau$ for $t \geq 0$ and $f \in L^1_{loc}(\mathbb{R}_+, X)$.

Corollary 5.3. *An exponentially bounded evolution family $U(\cdot,\cdot)$ on X with $J = \mathbb{R}_+$ is exponentially stable if and only if $\mathbb{K}$ maps E into E if and only if the generator G of the evolution semigroup on E is invertible, where $E = C_{00}(\mathbb{R}_+, X)$ or $L^p(\mathbb{R}_+, X)$, $1 \leq p < \infty$.*

The use of (4.2) in this context goes back to a paper by O. Perron from 1930. For different proofs and similar results (e.g., for $E = C_b(\mathbb{R}_+, X)$) we refer to [10], [17], [30, §III.5], [36, §5], [96]. A related characterization of exponential stability is due to R. Datko, [36, Thm.1, Rem.3], see also [25, Cor.3.24], [30, Thm.III.6.2], and the references therein. Here we give a new proof of Datko's theorem based on the spectral theory of evolution semigroups.

Theorem 5.4. *Let $U(\cdot,\cdot)$ be an exponentially bounded evolution family on X with $J = \mathbb{R}_+$. Then $U(\cdot,\cdot)$ is exponentially stable if and only if for some $1 \leq p < \infty$ and all $x \in X$ and $s \geq 0$ there is a constant M such that*

$$\int_s^\infty \|U(t,s)x\|^p\, dt \leq M^p\, \|x\|^p. \tag{5.3}$$

Proof. The necessity of (5.3) is clear. The integral version of Minkowski's inequality, [52, §202], and (5.3) yield

$$\begin{aligned}\Big(\int_0^\infty \Big\| \int_0^t U(t,s)f(s)\, ds\Big\|^p dt\Big)^{\frac{1}{p}} &\leq \int_0^\infty \Big(\int_s^\infty \|U(t,s)f(s)\|^p\, dt\Big)^{\frac{1}{p}} ds \\ &\leq M \int_0^\infty \|f(s)\|\, ds = M\, \|f\|_1\end{aligned}$$

for $f \in L^1(\mathbb{R}_+, X)$. Thus, $\mathbb{K}$ maps $L^1(\mathbb{R}_+, X)$ into $L^p(\mathbb{R}_+, X)$. If $p = 1$, then Corollary 5.3 implies the assertion. For arbitrary $p \in [1, \infty)$, the equivalence (4.2) shows that the range of the generator G of the evolution semigroup $T(\cdot)$ on $E = L^p(\mathbb{R}_+, X)$ contains $L^p(\mathbb{R}_+, X) \cap L^1(\mathbb{R}_+, X)$ and is therefore dense in E. Suppose that $0 \in A\sigma(G)$. Then $1 \in A\sigma(T(1))$ by the spectral inclusion theorem, see e.g. [39, Thm.IV.3.6]. From this fact one easily derives the existence of a constant $c > 0$ and functions $f_n \in E$ such that $\|f_n\|_p = 1$ and $\|T(t)f_n\|_p \geq c$ for $t \in [0, n]$ and $n \in \mathbb{N}$. Now (5.3) yields

$$\begin{aligned} n\, c^p &\leq \int_0^n \|T(t)f_n\|_p^p\, dt = \int_0^n \int_0^\infty \|U(s+t,s)f_n(s)\|^p\, ds\, dt \\ &\leq \int_0^\infty \int_0^\infty \|U(s+t,s)f_n(s)\|^p\, dt\, ds \leq M^p \int_0^\infty \|f_n(s)\|^p\, ds = M^p\end{aligned}$$

which is impossible. Hence, $0 \notin A\sigma(G)$ and G is invertible. The theorem is now a consequence of Corollary 5.3. □

Exponential dichotomy on $J = \mathbb{R}_+$ can be characterized in the spirit of (5.2) using the equation

$$u(t) = U(t,0)x + \int_0^t U(t,\tau)f(\tau)\,d\tau, \quad t \geq 0, \tag{5.4}$$

for $x \in X$. In this case, however, for a given $f \in E$ there exists more than one $u \in E$ satisfying (5.4) if $U(\cdot,\cdot)$ has an exponential dichotomy with $P(0) \neq 0$. For matrices $A(t)$ and assuming $\omega(U) < \infty$, it was shown in [28, §3] or [30, §IV.3] that exponential dichotomy of $U(\cdot,\cdot)$ is equivalent to the existence of bounded solutions u of (5.4) for each $f \in C_b(\mathbb{R}_+, X)$. This leads to Fredholm properties of the operator $\frac{d}{dt} - A(\cdot)$ which are studied in [15], [16], [102]. These results were generalized to certain parabolic equations in [143]. In [96] we proved similar theorems for general evolution families. We remark that these more complicated characterizations can usually be avoided by extending the given problem on $\mathbb{R}_+$ to $\mathbb{R}$, see [121], [124], [125]. Nevertheless, dichotomies on $\mathbb{R}_+$ and $\mathbb{R}_-$ play an important role in dynamical systems, cf. [77], [87], and the references therein.

We mention three more characterizations of hyperbolicity. First, one can replace in the equivalence (a)⇔(b) of Theorem 5.2 function spaces on $\mathbb{R}$ by sequence spaces over $\mathbb{Z}$, see [26], [53, §7.6], [108], and [11], [25], [71], [72], where this approach is used in the context of evolution semigroups. Sequence spaces over $\mathbb{N}$ are considered in [10] and [16].

For $X = \mathbb{C}^n$ it is known that exponential dichotomy is equivalent to the existence of a bounded, continuously differentiable Hermitian matrix function H such that

$$\tfrac{d}{dt} H(t) + H(t)A(t) + A^*(t)H(t) \leq -I$$

(where we use the order of symmetric matrices), see [28, §7]. In Section 4.4 of [25] these results are extended to unbounded operators $A(t)$ on a Hilbert space X which satisfy Kato type conditions, see also [92], [117]. We note that in [25] the characterization is deduced via the evolution semigroup from an easier accessible result for semigroups.

A third characterization of exponential dichotomy involves the *hull* $\mathcal{A}$ of the operators $A(s)$, i.e., the closure of the translates $\mathbb{R} \ni \tau \mapsto A(t+\tau)$, $t \in \mathbb{R}$, in an appropriate metric space, see [25], [26], [27], [28], [76], [108], [119], and the references therein. This approach requires, besides the existence of $\mathcal{A}$, certain compactness and almost periodicity properties. Then one obtains that the evolution family generated by $A(\cdot)$ is hyperbolic if the equations $u'(t) = B(t)u(t)$, $t \in \mathbb{R}$, do not admit a nontrivial bounded solution for any $B(\cdot) \in \mathcal{A}$, cf. [28], [119]. Due to space limitations we do not treat this approach in detail, see [25, Chap.6–8], [119], and the bibliography therein for further information.

6. Asymptotic behaviour of evolution equations

In this final section we mainly present sufficient conditions on the operators $A(t)$ implying the existence of an exponential dichotomy for the homogeneous evolution equation. Here we concentrate on parabolic problems possibly having a retarded term. We further mention some results on strong stability of evolution families and discuss the impact of exponential dichotomy on the qualitative behaviour of the inhomogeneous problem.

The following collection is guided by the treatment of exponential dichotomy in the books [28] and [53] by W.A. Coppel and D. Henry, which deal with matrices $A(t)$ and 'lower order' perturbations $A(t) = A + B(t)$ of a sectorial operator A, respectively. The basic idea is to find situations where the spectra of $A(t)$ or of a related operator A_0 allow to describe the asymptotic behaviour of the evolution family $U(\cdot,\cdot)$ generated by $A(\cdot)$. Most of these results are based on the theory discussed in the previous section.

The reader should further recall the results for time periodic problems which make use of the monodromy operator $U(p,0)$, compare the references given after (3.3). We add that (exponential) stability can also be tackled by direct estimates or by Lyapunov functions; however we do not give specific references concerning this well–known method.

(a) Robustness. We study the perturbed problem (2.6) assuming that $U(\cdot,\cdot)$ is hyperbolic. If the perturbations $B(t)$ are small in suitable norms, one can expect that the evolution family $U_B(\cdot,\cdot)$ given by (2.7) is also hyperbolic. Unfortunately, one has to impose quite restrictive smallness conditions, but for some classes of finite dimensional problems refinements of the standard bounds are known to be optimal, see [99] and the references therein. Among the vast literature on robustness we mention [26], [28], [30], [71], [72] for the case of bounded perturbations, [27], [53], [78], [108], [121] for unbounded perturbations, and [126] for perturbations $B(t) : \underline{X}_t \to \overline{X}_t$ where $\underline{X}_t \hookrightarrow X \hookrightarrow \overline{X}_t$. As a sample we state a theorem (put in a slightly different setting) which follows immediately from an estimate due to C.J.K. Batty and R. Chill, [12, Thm.4.7], and Theorem 5.2.

Theorem 6.1. *Let $A(t)$ and $B(t)$, $t \in \mathbb{R}$, be operators on X such that $A(t) - w$ and $B(t)-w$ are sectorial of the same type and satisfy* (2.4) *for some $w \geq 0$. Assume that the evolution family $U(\cdot,\cdot)$ generated by $A(\cdot)$ has an exponential dichotomy with projections $P_U(\cdot)$. Then there is a number $\varepsilon > 0$ such that*

$$q := \sup_{t\in\mathbb{R}} \|R(w, A(t)) - R(w, B(t))\| \leq \varepsilon$$

implies that the evolution family $V(\cdot,\cdot)$ generated by $B(\cdot)$ has an exponential dichotomy with projections $P_V(\cdot)$. Moreover, $\dim P_U(t)X = \dim P_V(t)X$ *and* $\dim(I - P_U(t))X = \dim(I - P_V(t))X$.

Proof. Let $T_U(\cdot)$ and $T_V(\cdot)$ be the evolution semigroups on $C_0(\mathbb{R}, X)$ induced by $U(\cdot,\cdot)$ and $V(\cdot,\cdot)$, respectively. In view of Theorem 5.2 one has to show that

$$\|T_U(1) - T_V(1)\| = \sup_{t\in\mathbb{R}} \|U(t+1,t) - V(t+1,t)\| =: \delta$$

is smaller than a certain number $\delta_0 > 0$ depending on the dichotomy constants of $U(\cdot,\cdot)$, see [124, Prop.2.3]. So the assertion is a consequence of Theorem 4.7 of [12] which says that $\delta \leq cq^\eta$ for some $\eta > 0$ and a constant c independent of q. □

Looking at the results used in the above proof, one sees that ε only depends on the constants in (2.4), w, the type of $A(t) - w$ and $B(t) - w$, and the dichotomy constants of $U(\cdot,\cdot)$. Moreover, the operators $A(t)$ and $B(t)$ need not be densely defined.

Instead of $T_U(1) - T_V(1)$, one can also try to estimate the difference $G_U - G_V$ of the corresponding generators. This approach leads to better numerical values of ε but does not seem to work in the same generality as the above theorem, see [121].

Analogous results for functional differential equations are treated in [77] for $X = \mathbb{C}^n$ and in [47], [122], [125] for the infinite dimensional case, where [125] deals with perturbations of the type $B(t)\phi = \phi(-\tau(t))$. In [47] and [125] we employed a generalized 'characteristic equation' which determines the exponential dichotomy of the delay problem. This characteristic equation was derived in [47] from Theorem 5.2. We also refer to [37], [49], [50], and the references therein for further results in the context of (exponential) stability.

Exponential dichotomy is also inherited in some cases if the perturbation is 'integrally small', i.e., a suitable norm of $\int_s^{s+t} B(\tau)\,d\tau$ is small uniformly for $s \geq 0$ and $t \in [0, t_0]$, see [28, §5], [53, §7.6].

(b) Asymptotically autonomous equations. Assume that $A(t)$ tends in a suitable sense to an operator A as $t \to \infty$. If A generates a hyperbolic semigroup e^{tA} with projection P, one can expect that $U(\cdot,\cdot)$ has an exponential dichotomy on a halfline $[a, \infty)$. It is not difficult to reduce this problem to case (a), see [76, p.180] or [124, (3.2)]. Such inheritance properties were established in [76, Chap.10] for bounded $A(t)$, in [77], [87] for ordinary delay equations (see also [51, §6.6.3], [106], [107] for the case of a dominant eigenvalue of the autonomous problem), in [12], [124] for parabolic problems (see also [48], [133, §5.8] for the case of exponential stability), and in [122], [125] for retarded parabolic equations. Further, the operators $U(s+t, s)$ and the dichotomy projections $P(s)$ tend strongly to e^{tA} and P as $s \to \infty$, and $P(s)$ and $I - P(s)$ inherit the rank of P and $I - P$, respectively, due to [124], [125]. In view of (6.1) below, these facts imply the convergence of the mild solution of (CP) if $f(t) \to f_\infty$, see [48], [124], [125], [133, §5.8]. In [12] also asymptotically periodic problems and the almost periodicity of $U(\cdot,\cdot)$ were studied.

(c) Slowly oscillating coefficients. Example 3.4 shows that the hyperbolicity of the semigroups $e^{\tau A(t)}$ does not imply the exponential dichotomy of $U(\cdot,\cdot)$, in general. Nevertheless, $U(\cdot,\cdot)$ is hyperbolic if in addition the Hölder constant of $t \mapsto A(t)$ is small enough (in an appropriate sense). For bounded $A(t)$ this has been established in [9], [28, Prop.6.1], and for ordinary delay equations in [81], see also [51, §12.7]. Extending the approach of [9], we studied parabolic problems in [121] and [123]. There we showed the left– and right–invertibility of the generator G of the induced evolution semigroup making heavy use of parabolic regularity and inter/extrapolation spaces. Thus $U(\cdot,\cdot)$ has an exponential dichotomy, and one can also see that its dichotomy projections have the same rank as those of $e^{\tau A(r)}$.

(d) Rapidly oscillating coefficients. Suppose that $\lim_{T\to\infty} \frac{1}{T} \int_t^{t+T} A(s)\,ds = A$ (in a suitable sense) uniformly for $t \geq 0$ and that $e^{\tau A}$ is hyperbolic. Then it is known in some cases that $A(\omega t)$ generates an hyperbolic evolution family for large ω. This is shown for matrices $A(t)$ in [28, Prop.5.3], for bounded operators in [76, Chap.11], and for certain classes of parabolic problems in [53, §7.6], [75], [76, Chap.11]; see also [51, §12.4]. We note that in [28] and [53] the problem is reduced to the robustness of exponential dichotomy under integrally small perturbations.

(e) Strong stability. The Arendt–Batty–Lyubich–Vũ theorem shows that a bounded semigroup $T(\cdot)$ with generator A tends strongly to 0 if $\sigma(A) \cap i\mathbb{R}$ is countable and $P\sigma(A') \cap i\mathbb{R}$ is empty, see e.g. [39, Thm.V.2.21]. This result cannot be applied to evolution semigroups since $\sigma(G)$ consists of vertical strips. However, C.J.K. Batty, R. Chill, and Y. Tomilov could characterize strong stability of a bounded evolution family (i.e.,

$U(t,s)x \to 0$ as $t \to \infty$ for each $s \geq 0$ and $x \in X$) by the density of the range of G on $L^1(\mathbb{R}_+, X)$, see [13, Thm.2.2] for this and related facts.

In the time periodic case the asymptotic behaviour of $U(\cdot, s)x$ is essentially determined by the operator $U(s+p, s)$ due to (3.2). This allows to apply discrete time versions of the Arendt–Batty–Lyubich–Vũ theorem and its relatives to deduce strong stability or almost periodicity of $U(\cdot, s)x$ and of the solution of the inhomogeneous problem, see [14], [57], [118], [138].

(f) Inhomogeneous problems. Observe that due to (5.2) the exponential dichotomy of $U(\cdot, \cdot)$ implies the strong convergence to 0 of the mild solution u to (CP) with $f \in C_0(\mathbb{R}, X)$. This follows in a more direct way also from the formula

$$u(t) = \int_{-\infty}^{t} U(t,s)P(s)f(s)\,ds - \int_{t}^{\infty} U_Q(t,s)Q(s)f(s)\,ds, \quad t \in \mathbb{R}, \tag{6.1}$$

which is an easy consequence of (4.1) (see e.g. [25, p.108]). If $U(\cdot, \cdot)$ is periodic, one can further prove the equivalence (5.2) for the space $E = AP(\mathbb{R}, X)$ of almost periodic functions, see [94] and also [57]. As a result, the almost periodicity of the inhomogeneity f is inherited by the mild solution u if $U(\cdot, \cdot)$ is periodic and hyperbolic. Delay equations were treated in an analogous way in [47] by means of evolution semigroups and in [46], [51, Chap.6], [125] using variants of (6.1).

Related results (also for non hyperbolic $U(\cdot, \cdot)$) are shown by differing methods in [14], [28, Chap.8], [31, Chap.6], [53, §7.6], [57], [67], [76, Chap.8], [86, §6.3], [138].

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R. Schnaubelt, FB Mathematik und Informatik, Martin–Luther–Universität, Theodor–Lieser–Str. 5, 06120 Halle, Germany.

E-mail address: `roland@euler.mathematik.uni-halle.de`