INTERPOLATION, EMBEDDINGS AND TRACES OF ANISOTROPIC FRACTIONAL SOBOLEV SPACES WITH TEMPORAL WEIGHTS

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ABSTRACT. We investigate the properties of a class of weighted vector-valued L_p-spaces and the corresponding (an)isotropic Sobolev-Slobodetskii spaces. These spaces arise naturally in the context of maximal L_p-regularity for parabolic initial-boundary value problems. Our main tools are operators with a bounded $\mathcal{H}^\infty$-calculus, interpolation theory, and operator sums.

1. INTRODUCTION AND PRELIMINARIES

In this paper we investigate a class of anisotropic fractional Sobolev spaces on space-time with weights in the time variable. We treat in particular interpolation results for such spaces, Sobolev type embeddings as well as temporal and spatial trace theorems in a systematic, operator theoretic way. Our choice of both the class of spaces and the studied properties is motivated by applications to quasilinear parabolic evolution equations with nonlinear boundary conditions which will be treated in subsequent papers focussing on the longterm behavior, cf. [24]. These papers will be based on linearization; i.e., one writes the nonlinear equation as a linear initial-boundary value problem whose inhomogeneities contain the nonlinear terms. Since the underlying problem is quasilinear, the linear and the nonlinear parts will be of the same order so that it is crucial to have sharp results for the linear part giving optimal regularity. In the companion paper [25] we establish the necessary theory for linear inhomogeneous initial-boundary value problems, based on the present study of the underlying function spaces.

The main focus of this work is the presence of the temporal weights. To explain their role, we first recall some features of the known theory for the unweigthed case by means of an example. For a domain $\Omega \subset \mathbb{R}^n$ with smooth compact boundary $\partial\Omega$ and given functions f, g and u_0, we consider the heat equation

$$\begin{aligned}
\partial_t u(t,x) - \Delta u(t,x) &= f(t,x), & x &\in \Omega, & t &> 0, \\
\partial_\nu u(t,x) &= g(t,x), & x &\in \partial\Omega, & t &> 0, \\
u(0,x) &= u_0(x), & x &\in \Omega,
\end{aligned} \qquad (1.1)$$

with an inhomogeneous Neumann boundary condition. We want to work in a framework where the boundary conditions can be understood in the sense of traces, and not just weakly. So we choose an L_p setting and require that the solution u belongs to the space

$$\mathbb{E}_1 := W_p^1\big(J; L_p(\Omega)\big) \cap L_p\big(J; W_p^2(\Omega)\big)$$

for some $p \in (1, \infty)$ and a finite interval $J = (0, T)$. To obtain the optimal spaces for the data, one needs sharp trace results for the space $\mathbb{E}_1$. It is known that the Neumann trace g of $u \in \mathbb{E}_1$ belongs to the anisotropic Slobodetskii space

$$\mathbb{F}_1 = W_p^{1/2 - 1/2p}\big(J; L_p(\partial\Omega)\big) \cap L_p\big(J; W_p^{1 - 1/p}(\partial\Omega)\big)$$

and that the trace u_0 of u at time $t = 0$ belongs to $W_p^{2-2/p}(\Omega)$. Observe that $\mathbb{F}_1$ still retains some time regularity! Moreover, one has $f \in L_p\big(J; L_p(\Omega)\big)$ and the compatibility condition $\partial_\nu u_0 = g|_{t=0}$ should hold at time $t = 0$ if g has a trace at $t = 0$, which happens for $p > 3$. In fact, problem (1.1) has a unique solution $u \in \mathbb{E}_1$ if and only if these conditions hold. We refer to e.g. [6] for these and much more general results. Further, the trace space $W_p^{2-2/p}(\Omega)$ is the smallest space in which all solutions of (1.1) in $\mathbb{E}_1$ are continuous on $[0, T]$, and the corresponding nonlinear problems share this property. Correspondingly, the (local) semiflows of solutions live in $W_p^{2-2/p}(\Omega)$ and the norm of this space is the right one to describe their properties. For instance, it gives the blow-up condition, see e.g. [22]. One often takes a large p to simplify the treatment of the nonlinearities (say, $p > n + 2$). Thus the norm of $W_p^{2-2/p}(\Omega)$ is far away from the norms one can control by the usual a priori estimates, such as the norms of W_2^1 or L_∞. This unpleasant regularity gap is even more significant in the Hölder setting.

There is a known technique to reduce the necessary regularity for initial conditions of evolution equations. One studies the problem in the weighted spaces

$$L_{p,\mu}(\mathbb{R}_+; E) := \big\{ u : \mathbb{R}_+ \to E : t^{1-\mu} u \in L_p(\mathbb{R}_+; E) \big\},$$

where E is a Banach space and $\mu \in (1/p, 1]$. The corresponding weighted Sobolev spaces are defined similarly, and the fractional Sobolev spaces $W_{p,\mu}^s(\mathbb{R}_+; E)$ and $H_{p,\mu}^s(\mathbb{R}_+; E)$ are introduced via real and complex interpolation, respectively. Observe that for $\mu = 1$ we recover the unweigthed case and that the spaces become larger if we decrease μ. So this type of weights allows for functions being more singular at $t = 0$. (We note that the weight $t^{p(1-\mu)}$ belongs to the class A_p used in harmonic analysis, see e.g. [30].)

This approach has been carried out in an L_p setting by Prüss & Simonett in [26] for problems with homogeneous boundary conditions (i.e., $g = 0$ in (1.1)). In this case one can work entirely in the framework of semigroup theory. In fact, for the generator $-A$ of an exponentially stable semigroup one looks for a solution u in

$$\mathbb{E}_{1,\mu} := W_{p,\mu}^1(\mathbb{R}_+; E) \cap L_{p,\mu}(\mathbb{R}_+; D(A))$$

for the evolution equation

$$u'(t) + Au(t) = f(t), \quad t > 0, \qquad u(0) = u_0. \tag{1.2}$$

Clearly, it must hold that $f \in L_{p,\mu}(\mathbb{R}_+; E)$. It is known that the initial value u_0 has to belong to the real interpolation space $(E, D(A))_{\mu-1/p,p}$, see e.g. Theorem 1.14.5 in [31]. The main result in [26] now says that one has a unique solution $u \in \mathbb{E}_{1,\mu}$ of (1.2) for all $f \in L_{p,\mu}(\mathbb{R}_+; E)$ and $u_0 \in (E, D(A))_{\mu-1/p,p}$ if (and only if) A has maximal L_p-regularity; i.e., (1.2) with $u_0 = 0$ has a solution $u \in \mathbb{E}_1$ for each $f \in L_p(\mathbb{R}_+; E)$. The latter property is well understood, see [20] and the references therein. As a result, one can reduce the needed initial regularity for (1.2) almost to the base space, say, $E = L_p(\Omega)$. We further point out that the temporal weights give the scale of phase spaces $(E, D(A))_{\mu-1/p,p}$, which are compactly embedded into each other in many cases. Precisely this point was used in the very recent paper [17] to establish attractivity of equilibria of quasilinear evolution equations based on the results of [26], see also [18].

Unfortunately, it seems to be impossible obtain sharp (Sobolev) regularity for inhomogeneous boundary value problems such as (1.1) in a pure semigroup framework. Instead one has to restrict to a PDE setting and reduce equations like (1.1) to model problems on full- and half-spaces by means of local charts and the freezing of coefficients. Nevertheless we see below that semigroup and operator theory can still play a crucial role in the proofs. In this way, vector-valued parabolic initial-boundary value problems with inhomogeneous boundary conditions were treated by Denk, Hieber, Prüss & Zacher in the papers [5], [6] and [7] without temporal weights. In the present and the following paper [25] we extend their results to the case of the weights $t^{p(1-\mu)}$, focussing here on the function spaces themselves. Our setting also covers linearizations of problems arising from free boundary problems, see e.g. [12] and [18].

In Section 2 we start with basic properties of the weighted spaces. The crucial point is that the operator $-\partial_t$ is maximally accretive and has a bounded $\mathcal{H}^\infty$-calculus on $L_{p,\mu}(\mathbb{R}_+; E)$ (if E is of class $\mathcal{HT}$), see Lemma 2.6 and Theorem 2.7. Using a theorem by Yagi, one can conclude that the spaces $H^s_{p,\mu}(\mathbb{R}_+; E)$ are the domains of fractional powers of $1-\partial_t$, see Proposition 2.9. This fact also allows us to establish the natural interpolation properties for the scales of $W^s_{p,\mu}$ and $H^s_{p,\mu}$ in Lemma 2.8. These results are complemented by several propositions on extension, density, intrinsic norms, Sobolev embeddings and a Poincaré inequality. The proofs rely on interpolation theory, Hardy's inequality and results by Grisvard, [13].

In view of the above considerations it is clear that we have to study the mapping properties of temporal and spatial derivatives and traces in anisotropic spaces like $\mathbb{E}_{1,\mu}$. Here spaces like

$$W^{s+\alpha}_{p,\mu}\big(\mathbb{R}_+; W^r_p(\Omega; E)\big) \cap W^s_{p,\mu}\big(\mathbb{R}_+; W^{r+\beta}_p(\Omega; E)\big), \qquad s, r, \alpha, \beta \geq 0,$$

naturally appear, as well as analogous Bessel potential spaces H. Of course, one also needs Sobolev type embeddings. By localization and local charts, we reduce these results to the model cases of full- and half-spaces. There we use that the related H spaces are the domains of the operator $L = (1 - \partial_t)^\alpha + (1 - \Delta)^{\beta/2}$ having bounded imaginary powers, see Lemma 3.1. Then Sobolevskii's mixed derivative theorem and interpolation arguments imply the fundamental embedding result Proposition 3.2 which allows to interpolate between time and space regularity. We can conclude the desired mapping properties of spatial derivatives in Lemma 3.4. Another crucial ingredient is Lemma 4.1 on time traces of semigroup orbits in weighted spaces. Employing these tools, we then establish our main Theorem 4.2 on the time traces of the anisotropic fractional spaces. Similarly, we deduce the final Theorem 4.5 on spatial traces. It turns out that our main results are natural extensions of the unweighted case $\mu = 1$, and thus there is no disadvantage when working in a weighted framework in the context of parabolic problems.

The basic strategy of our proofs of the trace theorems is similar to that in e.g. Section 3 of [6] where the unweighted case was treated, see also [12] or [32]. The more recent paper [4] gives a comprehensive account of unweighted anisotropic spaces. However, for the weighted case we first had to establish the underlying theory contained in Sections 2 and 3 and in Lemma 4.1. As a fundamental difference to the unweighted case, our weighted spaces are not invariant under the right shift and have no suitable extension to the full time interval $\mathbb{R}$. This behavior reflects the desired fact that our weights vanish at $t = 0$ and allow for stronger singularities at $t = 0$. Moreover, some proofs in the literature cannot simply be generalized to the weighted case (e.g., those of the temporal trace theorems in [12] or [32] which use fractional evolution equations).

We also point out that the multiplication operator $\Phi_\mu u = t^{1-\mu} u$ does *not* give an isomorphism from $W^1_{p,\mu}(\mathbb{R}_+; E)$ to $W^1_p(\mathbb{R}_+; E)$, since $(\Phi_\mu \mathbb{1})' \notin L_p(0,1)$. As a consequence, one cannot simply reduce the results to the unweighted case by isomorphy. For other types of weights this approach would be possible, for instance for exponential weights $e^{-p\rho t}$ which do not change the behavior at $t = 0$. This was done in the recent paper [8] where interpolation and trace theorems are shown for the exponentially weighted spaces. (Of course, this paper focusses on different results.) Isotropic weighted spaces have been studied in detail in the literature, see e.g. [13], [19] and [31].

In this work we strive for a self contained and systematic treatment of the results needed for evolution equations. This should be of some interest also in the known unweighted case $\mu = 1$. To keep the length within reasonable bounds, we omitted some arguments or calculations which are either routine or similar to parallel cases discussed here or in the literature. Most of the omitted details are presented in the Ph.D. thesis [24] to which we then refer.

We now indicate some concepts and results from operator theory that we use frequently. Details can be found at the references given below. The real and complex interpolation functors $(\cdot,\cdot)_{\theta,q}$ and $[\cdot,\cdot]_\theta$, respectively, are a fundamental tool in our investigations. For the relevant theory the reader is referred to [23] and [31]. If A is a sectorial operator on a Banach space E, $\theta \in (0,1)$ and $q \in [1,\infty]$, we set $D_A(\theta,p) := \big(E, D(A)\big)_{\theta,q}$. The theory of strongly continuous semigroups is presented in e.g. [11]. Throughout we assume that the Banach space E is of class $\mathcal{HT}$ (or, equivalently, is a UMD space), which means that the Hilbert transform is bounded on $L_2(\mathbb{R}; E)$, see Sections III.4.3–5 of [2]. We note that Hilbert spaces are of class $\mathcal{HT}$, as well as the reflexive Lebesgue and (fractional) Sobolev spaces. The definition and properties of the bounded $\mathcal{H}^\infty$-calculus of a sectorial operator A are discussed in [5] or [20]. However, in the sequel we almost never use the calculus itself, but rather some theorems requiring it as an assumption. An important consequence of the bounded $\mathcal{H}^\infty$-calculus is that A admits bounded imaginary powers, see e.g. Sections 2.3 and 2.4 of [5] or Section III.4.7 of [2]. Finally, we recall the Dore-Venni theorem from [10] in a version of [27] and combined with the mixed derivative theorem due to Sobolevskii, [29].

Proposition 1.1. *Let E be a Banach space of class $\mathcal{HT}$ and suppose that the operators A, B on E are resolvent commuting and admit bounded imaginary powers with power angles satisfying $\theta_A + \theta_B < \pi$. Assume further that A or B is invertible. Then $A + B$ is invertible, admits bounded imaginary powers with angle not larger than $\theta_A + \theta_B$ and the operator $A^\alpha B^{1-\alpha}(A+B)^{-1}$ is bounded on E for all $\alpha \in [0,1]$.*

We write $a \lesssim b$ for some quantities a, b if there is a generic positive constant C with $a \le Cb$. If X, Y are Banach spaces, $\mathcal{B}(X,Y)$ is the space of bounded linear operators between them, and we put $\mathcal{B}(X) := \mathcal{B}(X,X)$.

2. Basic properties of the weighted spaces

Throughout, we consider a time interval $J = \mathbb{R}_+ := (0,\infty)$ or $J = (0,T)$ for some $T > 0$ and a Banach space E of class $\mathcal{HT}$, and we let

$$p \in (1,\infty), \qquad \mu \in (1/p, 1].$$

We study the Banach space $L_{p,\mu}(J;E) := \{u : J \to E : t^{1-\mu}u \in L_p(J;E)\}$ with the norm

$$|u|_{L_{p,\mu}(J;E)} := |t^{1-\mu}u|_{L_p(J;E)} = \left(\int_J t^{p(1-\mu)}|u(t)|_E^p \, \mathrm{d}t\right)^{1/p}.$$

Occasionally, we use these spaces also with μ replaced by some $\alpha > 1/p$. It clearly holds

$$L_p(0,T;E) \hookrightarrow L_{p,\mu}(0,T;E) \quad \text{and} \quad L_{p,\mu}(0,T;E) \hookrightarrow L_p(\tau,T;E)$$

for all $\tau \in (0,T)$, but $L_p(\mathbb{R}_+;E) \nsubseteq L_{p,\mu}(\mathbb{R}_+;E)$ for $\mu \in (1/p,1)$. For $k \in \mathbb{N}_0$, we define the corresponding weighted Sobolev space

$$W_{p,\mu}^k(J;E) = H_{p,\mu}^k(J;E) := \left\{u \in W_{1,\mathrm{loc}}^k(J;E) \ : \ u^{(j)} \in L_{p,\mu}(J;E), \ \ j \in \{0,...,k\}\right\}$$

(where $W_{p,\mu}^0 = H_{p,\mu}^0 = L_{p,\mu}$ by definition), which is a Banach space endowed with the norm

$$|u|_{W_{p,\mu}^k(J;E)} = |u|_{H_{p,\mu}^k(J;E)} := \left(\sum_{j=0}^k |u^{(j)}|_{L_{p,\mu}(J;E)}^p\right)^{1/p}.$$

For $s \in \mathbb{R}_+\backslash\mathbb{N}$, we write $s = [s] + s_*$ with $[s] \in \mathbb{N}_0$ and $s_* \in (0,1)$. The weighted Slobodetskii and Bessel potential spaces

$$W_{p,\mu}^s(J;E) := \left(W_{p,\mu}^{[s]}(J;E), W_{p,\mu}^{[s]+1}(J;E)\right)_{s_*,p}, \quad H_{p,\mu}^s(J;E) := \left[W_{p,\mu}^{[s]}(J;E), W_{p,\mu}^{[s]+1}(J;E)\right]_{s_*}$$

are introduced by means of real and complex interpolation, respectively. In view of Theorem 4.2/2 of [28] and Satz 3.21 of [33], this definition is consistent with the unweighted case; i.e., we have $W_p^s = W_{p,1}^s$ and $H_p^s = H_{p,1}^s$ for all $s \geq 0$. The general properties of real and complex interpolation spaces (see [23] or [31]) imply that one has the scale of dense embeddings

$$W_{p,\mu}^{s_1} \overset{d}{\hookrightarrow} H_{p,\mu}^{s_2} \overset{d}{\hookrightarrow} W_{p,\mu}^{s_3} \overset{d}{\hookrightarrow} H_{p,\mu}^{s_4}, \qquad s_1 > s_2 > s_3 > s_4 \geq 0, \tag{2.1}$$

for all fixed $p \in (1,\infty)$ and $\mu \in (1/p,1]$. In the sequel we will often use that

$$\mathcal{B}\left(W_{p,\mu}^k(J;E)\right) \cap \mathcal{B}\left(W_{p,\mu}^{k+1}(J;E)\right) \hookrightarrow \mathcal{B}\left(W_{p,\mu}^s(J;E)\right) \cap \mathcal{B}\left(H_{p,\mu}^s(J;E)\right),$$

where $k \in \mathbb{N}_0$ and $s \in (k, k+1)$, to deduce assertions for exponents $s \geq 0$ from the integer case. Before giving further definitions, we state a first basic property which can easily be proved using Hölder's inequality.

Lemma 2.1. *Let $J = (0,T)$ be finite or infinite, $p \in (1,\infty)$ and $\mu \in (1/p,1]$. We then have*

$$L_{p,\mu}(J;E) \hookrightarrow L_{q,\mathrm{loc}}(\overline{J};E) \qquad \text{for all} \ \ 1 \leq q < \frac{1}{1 - \mu + 1/p} \leq p. \qquad \blacksquare$$

It thus holds $W_{p,\mu}^k(J;E) \hookrightarrow W_{1,\mathrm{loc}}^k(\overline{J};E)$ for all $k \in \mathbb{N}$ so that the trace $u \mapsto u^{(j)}(0)$ is continuous from $W_{p,\mu}^k(J;E)$ to E for all $j \in \{0,...,k-1\}$. So we can define

$${}_0W_{p,\mu}^k(J;E) = {}_0H_{p,\mu}^k(J;E) := \left\{u \in W_{p,\mu}^k(J;E) \ : \ u^{(j)}(0) = 0, \ \ j \in \{0,...,k-1\}\right\}$$

for $k \in \mathbb{N}$ which are Banach spaces with the norm of $W_{p,\mu}^k$. For convenience we further set

$${}_0W_{p,\mu}^0(J;E) = {}_0H_{p,\mu}^0(J;E) := L_{p,\mu}(J;E).$$

For a number $s = [s] + s_* \in \mathbb{R}_+\backslash\mathbb{N}$, we again define the corresponding fractional order spaces by interpolation; i.e., we put

$${}_0W_{p,\mu}^s(J;E) := \left({}_0W_{p,\mu}^{[s]}(J;E), {}_0W_{p,\mu}^{[s]+1}(J;E)\right)_{s_*,p},$$

$$_0H_{p,\mu}^s(J;E) := \left[{}_0W_{p,\mu}^{[s]}(J;E), {}_0W_{p,\mu}^{[s]+1}(J;E)\right]_{s_*},$$

obtaining as before a scale of function spaces

$$_0W_{p,\mu}^{s_1} \overset{d}{\hookrightarrow} {}_0H_{p,\mu}^{s_2} \overset{d}{\hookrightarrow} {}_0W_{p,\mu}^{s_3} \overset{d}{\hookrightarrow} {}_0H_{p,\mu}^{s_4}, \qquad s_1 > s_2 > s_3 > s_4 \geq 0. \tag{2.2}$$

We further have

$$_0W_{p,\mu}^s(J;E) \hookrightarrow W_{p,\mu}^s(J;E) \quad \text{and} \quad {}_0H_{p,\mu}^s(J;E) \hookrightarrow H_{p,\mu}^s(J;E) \qquad \text{for all} \;\; s \geq 0.$$

Hardy's inequality is a fundamental tool for the investigation of the $L_{p,\mu}$-spaces. It states that

$$\int_0^\infty \left(t^{-\alpha} \int_0^t \varphi(\tau)\,\mathrm{d}\tau\right)^p \mathrm{d}t \leq \frac{1}{(\alpha - 1/p)^p} \int_0^\infty (t^{1-\alpha}\varphi(t))^p \,\mathrm{d}t \tag{2.3}$$

for all $\alpha \in (1/p, \infty)$ and measurable nonnegative functions $\varphi : \mathbb{R}_+ \to \mathbb{R}$, see Theorem 330 in [15]. For the spaces with vanishing initial values we deduce the following weighted inequality.

Lemma 2.2. *Let $J = (0,T)$ be finite or infinite, $p \in (1,\infty)$, $\mu \in (1/p, 1]$ and $s \geq 0$. Then*

$$\int_J t^{p(1-\mu-s)}|u(t)|_E^p \,\mathrm{d}t \leq C_{p,\mu,s}\,|u|_{{}_0W_{p,\mu}^s(J;E)}^p \qquad \text{if} \;\; u \in {}_0W_{p,\mu}^s(J;E),$$

and this remains true if one replaces ${}_0W_{p,\mu}^s(J;E)$ by ${}_0H_{p,\mu}^s(J;E)$.

Proof. The case $s = 0$ is trivial and the case $s = 1$ is a consequence of Hardy's inequality with $\varphi = |u'|_E$. The assertion for $s = k \in \mathbb{N}_0$ follows by induction, so that we have

$$_0W_{p,\mu}^k(J;E) \hookrightarrow L_p(J, t^{p(1-\mu-k)}\,\mathrm{d}t; E). \tag{2.4}$$

Theorem 1.18.5 of [31] yields that

$$\left(L_p(J, t^{p(1-\mu-k)}\,\mathrm{d}t; E), L_p(J, t^{p(1-\mu-(k+1))}\,\mathrm{d}t; E)\right)_{\theta,p} = L_p(J, t^{p(1-\mu-\theta k)}\,\mathrm{d}t; E) \tag{2.5}$$

for all $\theta \in (0,1)$ and that (2.5) remains true if one replaces $(\cdot,\cdot)_{\theta,p}$ by $[\cdot,\cdot]_\theta$. Interpolation of the embedding (2.4) now implies the assertions for all $s \geq 0$. ∎

The above inequalities can be used to show that the multiplication operator $\Phi_\mu u = t^{1-\mu}u$ is an isomorphism from the weighted to the unweighted spaces, provided one restricts to vanishing initial values. Without this restriction, Φ_μ does not map properly since $(\Phi_\mu \mathbb{1})' \notin L_p$.

Lemma 2.3. *Let $J = (0,T)$ be finite or infinite, $p \in (1,\infty)$, $\mu \in (1/p, 1]$ and $s \geq 0$. Then the map Φ_μ, given by*

$$(\Phi_\mu u)(t) := t^{1-\mu}u(t),$$

induces an isomorphism from ${}_0W_{p,\mu}^s(J;E)$ to ${}_0W_p^s(J;E)$ and from ${}_0H_{p,\mu}^s(J;E)$ to ${}_0H_p^s(J;E)$. The inverse Φ_μ^{-1} is given by $(\Phi_\mu^{-1}u)(t) = t^{-(1-\mu)}u(t)$. ∎

The lemma is shown for $s \in \{0,1\}$ in Proposition 2.2 of [26]. The case of other integers can be treated similarly, see Lemma 1.1.3 of [24]. The general case then follows by interpolation. We use the above lemma to prove basic density results for the weighted spaces.

Lemma 2.4. *Let $J = (0,T)$ be finite or infinite, $p \in (1,\infty)$, $\mu \in (1/p, 1]$ and $s \geq 0$. Then the space $C_c^\infty(\overline{J}\backslash\{0\}; E)$ is dense in ${}_0W_{p,\mu}^s(J;E)$ and ${}_0H_{p,\mu}^s(J;E)$, and $C_c^\infty(\overline{J}; E)$ is dense in $W_{p,\mu}^s(J;E)$ and $H_{p,\mu}^s(J;E)$.*

Proof. Due to (2.1) and (2.2) we only have to consider the case $s = k \in \mathbb{N}_0$. The preceding lemma shows that $\Phi_\mu u$ belongs to $_0 W_p^k(J;E)$ for every $u \in {}_0 W_{p,\mu}^k(J;E)$. As in Theorem 2.9.1 of [31] for the scalar-valued case, one sees that $C_c^\infty(\overline{J}\backslash\{0\};E)$ is dense in $_0 W_p^k(J;E)$ for $k \in \mathbb{N}_0$. For any given $\varepsilon > 0$ there thus exists a function $\psi \in C_c^\infty(\overline{J}\backslash\{0\};E)$ with $|\Phi_\mu u - \psi|_{W_p^k(J;E)} < \varepsilon$. Consequently, $|u - \Phi_\mu^{-1}\psi|_{W_{p,\mu}^k(J;E)} \lesssim \varepsilon$, and so $C_c^\infty(\overline{J}\backslash\{0\};E)$ is dense in $_0 W_{p,\mu}^k(J;E)$.

For $u \in W_{p,\mu}^k(J;E)$, we choose $\psi_1 \in C_c^\infty(\overline{J};E)$ with $\psi_1^{(j)}(0) = u^{(j)}(0)$ for $j \in \{0, ..., k-1\}$. Since $u - \psi_1 \in {}_0 W_{p,\mu}^k(J;E)$, by the above considerations there is $\psi_2 \in C_c^\infty(\overline{J} \backslash \{0\};E)$ being ε-close to $u - \psi_1$ in $_0 W_{p,\mu}^k(J;E)$ Hence, $\psi_1 + \psi_2 \in C_c^\infty(\overline{J};E)$ is ε-close to u in $W_{p,\mu}^k(J;E)$. ∎

We next construct extension operators to $\mathbb{R}_+$ for the weighted spaces on finite intervals; i.e., continuous right-inverses of the restriction operator. For vanishing initial values one can achieve an extension whose norm is bounded independently of the length of the underlying interval. This fact is crucial for the treatment of nonlinear evolution equations. For simplicity we consider only $s \in [0,2]$ in this case.

Lemma 2.5. *Let $J = (0,T)$ be finite and $0 < 1/p < \mu \leq 1$. Given $k \in \mathbb{N}$, there is an extension operator $\mathcal{E}_J$ from J to $\mathbb{R}_+$ with*

$$\mathcal{E}_J \in \mathcal{B}\big(W_{p,\mu}^s(J;E), W_{p,\mu}^s(\mathbb{R}_+;E)\big) \cap \mathcal{B}\big(H_{p,\mu}^s(J;E), H_{p,\mu}^s(\mathbb{R}_+;E)\big) \quad \text{for all } s \in [0,k].$$

Here we can replace W by $_0 W$ and H by $_0 H$. There is further an extension operator $\mathcal{E}_J^0$ from J to $\mathbb{R}_+$ with

$$\mathcal{E}_J^0 \in \mathcal{B}\big(_0 W_{p,\mu}^s(J;E), {}_0 W_{p,\mu}^s(\mathbb{R}_+;E)\big) \cap \mathcal{B}\big(_0 H_{p,\mu}^s(J;E), {}_0 H_{p,\mu}^s(\mathbb{R}_+;E)\big) \quad \text{for all } s \in [0,2],$$

whose operator norm is independent of T. Moreover, $\mathcal{E}_J, \mathcal{E}_J^0 \in \mathcal{B}\big(L_\infty(J;E), L_\infty(\mathbb{R}_+;E)\big)$ with operator norms independent of T.

Proof. Let $k \in \mathbb{N}$. We extend $u \in W_{p,\mu}^k(0,T;X)$ to $\mathcal{E}u \in W_{p,\mu}^k(0,T_k;X)$ with $T_k = T + T/(2k+2)$ in the same way as in, e.g., Theorem 5.19 of [1]. In particular, $\mathcal{E}u(t)$ only depends on u on $(T/2,T)$ if $t \geq T$. Take a function $\varphi \in C^\infty(\mathbb{R}_+)$ which is equal to 1 on $(0,T)$ and has its support in $(0,T_k)$, and set $\mathcal{E}_J := \varphi\mathcal{E}$ (with trivial extension for $t \geq T_k$). It is easy to check that $\mathcal{E}_J$ is bounded from $W_{p,\mu}^k(0,T;X)$ to $W_{p,\mu}^k(\mathbb{R}_+;E)$, and the general case follows from interpolation. For $k \in \{0,1,2\}$ and a function $u \in {}_0 W_{p,\mu}^k(J;E)$ we define $\mathcal{E}_J^0$ by the formula

$$(\mathcal{E}_J^0 u)(t) := \begin{cases} u(t), & t \in (0,T), \\ 3(\psi^{1-\mu}u)(2T-t)\mathbb{1}_{[T,2T]}(t) - 2(\psi^{1-\mu}u)(3T-2t)\mathbb{1}_{[T,\frac{3}{2}T]}(t), & t \geq T, \end{cases}$$

where $\psi(\tau) = \frac{2T\tau-\tau^2}{T^2}$ and $\mathbb{1}_I$ is the characteristic function of an interval I. Using Lemma 2.2, it can be checked by a direct calculation that $\mathcal{E}_J^0$ is bounded on $_0 W_{p,\mu}^k(J;E)$ for all $k \in \{0,1,2\}$, see Lemma 1.1.5 in [24]. Interpolation yields the general case. The explicit representations of $\mathcal{E}_J$ and $\mathcal{E}_J^0$ show that these operators admit an L_∞-estimate independent of T. ∎

We now investigate the realization of the derivative ∂_t and its fractional powers on the weighted spaces. The properties of these and similar operators are fundamental for all our further considerations. We start with a generation result.

Lemma 2.6. *For $p \in (1,\infty)$, $\mu \in (1/p, 1]$ and $s \geq 0$, the family of left translations $\{\Lambda_t^E\}_{t\geq 0}$, given by*

$$(\Lambda_t^E u)(\tau) := u(\tau + t), \qquad \tau \geq 0,$$

forms a strongly continuous contraction semigroup on $W_{p,\mu}^s(\mathbb{R}_+; E)$ and on $H_{p,\mu}^s(\mathbb{R}_+; E)$. Its generator is the derivative ∂_t with domain $W_{p,\mu}^{s+1}(\mathbb{R}_+; E)$ and $H_{p,\mu}^{s+1}(\mathbb{R}_+; E)$, respectively.

Proof. (I) We write $\Lambda_t = \Lambda_t^E$ for simplicity. For each $t_0 \geq 0$ the operator Λ_{t_0} maps $L_{p,\mu}(\mathbb{R}_+; E)$ into itself and is contractive because of

$$|\Lambda_{t_0} u|_{L_{p,\mu}(\mathbb{R}_+;E)}^p \leq \int_0^\infty (\tau + t_0)^{p(1-\mu)} |u(\tau + t_0)|_E^p \, \mathrm{d}\tau \leq |u|_{L_{p,\mu}(\mathbb{R}_+;E)}^p.$$

Analoguously one shows that Λ_{t_0} is a contractive map on $W_{p,\mu}^k(\mathbb{R}_+; E)$ for all $k \in \mathbb{N}$. By interpolation, this fact carries over to $W_{p,\mu}^s(\mathbb{R}_+; E)$ and $H_{p,\mu}^s(\mathbb{R}_+; E)$ for all $s \geq 0$. It is further clear that $\{\Lambda_t\}_{t\geq 0}$ forms a semigroup of operators on these spaces. Lemma 2.4 says that $\mathcal{D} = C_c^\infty([0,\infty); E)$ is dense in all the spaces above. The left translations act strongly continuous on $\mathcal{D}$ with the sup-norm, and they thus form C_0-semigroups on $W_{p,\mu}^s(\mathbb{R}_+; E)$ and $H_{p,\mu}^s(\mathbb{R}_+; E)$.

(II) We denote the generator of $\{\Lambda_t\}$ on $W_{p,\mu}^s(\mathbb{R}_+; E)$ by A. We first let $s = k \in \mathbb{N}_0$. To show $A \subseteq \partial_t$, we take $u \in D(A) \subset W_{p,\mu}^k(\mathbb{R}_+; E)$. Then $u^{(j)} \in L_{1,\mathrm{loc}}([0,\infty); E)$ for all $j \in \{0, \cdots, k\}$ by Lemma 2.1, and for $a, b \in \mathbb{R}_+$ with $a < b$ it holds

$$\int_a^b \frac{1}{h} \big(u^{(j)}(\tau + h) - u^{(k)}(\tau)\big) \, \mathrm{d}\tau = \frac{1}{h} \int_b^{b+h} u^{(j)}(\tau) \, \mathrm{d}\tau - \frac{1}{h} \int_a^{a+h} u^{(j)}(\tau) \, \mathrm{d}\tau.$$

As $h \to 0$, the right hand side converges to $u^{(j)}(b) - u^{(j)}(a)$ for almost all $a, b \in \mathbb{R}_+$. The integrand on the left hand side tends to $Au^{(j)}$ in $L_{p,\mu}(\mathbb{R}_+; E)$, and thus in $L_1(a, b; E)$. Hence, the left hand side converges to $\int_a^b Au^{(j)}(\tau) \, \mathrm{d}\tau$. We infer that $u \in W_{1,\mathrm{loc}}^{k+1}(\mathbb{R}_+; E)$, with $u^{(j+1)} = Au^{(j)}$, so that $D(A) \subset W_{p,\mu}^{k+1}(\mathbb{R}_+; E)$ and $\partial_t|_{D(A)} = A$.

Since A generates a contraction semigroup, we know that $1 - A$ is invertible. It is further easy to see that $1 - \partial_t$ is injective on $W_{p,\mu}^{k+1}(\mathbb{R}_+; E)$. Consequently, $1 - \partial_t = 1 - A$, which yields $\partial_t = A$. This fact implies that $1 - \partial_t$ is an isomorphism from $W_{p,\mu}^{k+1}(\mathbb{R}_+; E)$ to $W_{p,\mu}^k(\mathbb{R}_+; E)$. By interpolation, we can replace here $k \in \mathbb{N}_0$ by any $s \geq 0$, and W by H.

Finally, let $s \in (k, k+1)$. The first part and Proposition II.2.3 of [11] imply that $D(A) = \{u \in W_{p,\mu}^{k+1}(\mathbb{R}_+; E) : u' \in W_{p,\mu}^s(\mathbb{R}_+; E)\}$ and $Au = u'$. Using that $1 - \partial_t$ is an isomorphism from $W_{p,\mu}^{s+1}(\mathbb{R}_+; E)$ to $W_{p,\mu}^s(\mathbb{R}_+; E)$, we conclude that $D(A) = W_{p,\mu}^{s+1}(\mathbb{R}_+; E)$. The same arguments work with W replaced by H. ∎

Since $W_{p,\mu}^1(\mathbb{R}_+; E)$ is the domain of the generator of $\{\Lambda_t^E\}_{t\geq 0}$ by the above lemma, standard interpolation theory yields

$$|u|_{W_{p,\mu}^s(\mathbb{R}_+;E)}^p \sim |u|_{L_{p,\mu}(\mathbb{R}_+;E)}^p + \int_0^\infty t^{-sp-1} |\Lambda_t^E u - u|_{L_{p,\mu}(\mathbb{R}_+;E)}^p \, \mathrm{d}t$$

for $s \in (0, 1)$, see Proposition 5.7 of [23]. Substitution and localization then imply that

$$|u|_{W_{p,\mu}^s(J;E)} \sim |u|_{L_{p,\mu}(J;E)} + [u]_{W_{p,\mu}^s(J;E)} \qquad \text{with}$$

$$[u]_{W_{p,\mu}^s(J;E)}^p := \int_0^T \int_0^t \tau^{p(1-\mu)} \frac{|u(t) - u(\tau)|_E^p}{(t-\tau)^{1+sp}} \, \mathrm{d}\tau \, \mathrm{d}t \tag{2.6}$$

for finite or infinite $J = (0, T)$, $s \in (0, 1)$ and $u \in W_{p,\mu}^s(J; E)$, see Proposition 1.1.13 of [24].

Using the generation property of ∂_t and the transference principle, we show that $-\partial_t$ with domain $W_{p,\mu}^1(\mathbb{R}_+; E)$ possesses a bounded $\mathcal{H}^\infty$-calculus on $L_{p,\mu}(\mathbb{R}_+; E)$. Here it is for the first time essential that E is of class $\mathcal{HT}$. As shown in [26], the realization of ∂_t with domain

$_0W_{p,\mu}^1(\mathbb{R}_+; E)$ also admits a bounded $\mathcal{H}^\infty$-calculus on $L_{p,\mu}(\mathbb{R}_+; E)$ (and it is in particular sectorial), although $-\partial_t$ with domain $_0W_{p,\mu}^1(\mathbb{R}_+; E)$ does not generate a semigroup on $L_{p,\mu}(\mathbb{R}_+; E)$.

Theorem 2.7. *Let $0 < 1/p < \mu \le 1$. Then the operators ∂_t with domain $_0W_{p,\mu}^1(\mathbb{R}_+; E)$ and $-\partial_t$ with domain $W_{p,\mu}^1(\mathbb{R}_+; E)$ possess a bounded $\mathcal{H}^\infty$-calculus on $L_{p,\mu}(\mathbb{R}_+; E)$ with $\mathcal{H}^\infty$-angle $\pi/2$.*

Proof. The assertion on ∂_t is proved in Theorem 4.5 of [26]. To treat the operator $-\partial_t$, we employ the vector-valued transference principle from [16]. We first introduce vector-valued extensions of operators. Let (Ω, ν) be a measure space and S be a bounded, positive operator on $L_p(\Omega, \nu)$. For simple functions u of the form $u = \sum_{i=1}^N u_i x_i$ with $x_i \in E$ and simple $u_i : \Omega \to \mathbb{C}$, we define $S^E u(\cdot) := \sum_{i=1}^N (Su_i)(\cdot)x_i$. The operator S^E extends uniquely to $L_p(\Omega, \nu; E)$ with $\|S^E\|_{\mathcal{B}(L_p(\Omega,\nu;E))} = \|S\|_{\mathcal{B}(L_p(\Omega,\nu))}$, see Lemma 10.14 of [20].

We consider the left translation Λ_t^E, $t \ge 0$, on $L_{p,\mu}(\mathbb{R}_+; E) = L_p(\mathbb{R}_+, t^{p(1-\mu)}\,\mathrm{d}t; E)$. Clearly, $\Lambda_t^E = (\Lambda_t^{\mathbb{C}})^E$ is the vector-valued extension of the scalar left translation $\Lambda_t^{\mathbb{C}}$. Due to Lemma 2.6, the family $\{\Lambda_t^{\mathbb{C}}\}_{t\ge0}$ forms a C_0-semigroup of positive contractions on $L_{p,\mu}(\mathbb{R}_+)$, and ∂_t with domain $W_{p,\mu}^1(\mathbb{R}_+; E)$ generates the vector-valued extension $\{\Lambda_t^E\}_{t\ge0}$ on $L_{p,\mu}(\mathbb{R}_+; E)$. Moreover, ∂_t is injective on this space. Theorem 6 of [16] now yields that $-\partial_t$ admits a bounded $\mathcal{H}^\infty$-calculus with angle $\pi/2$. ∎

Theorem 2.7 implies that $1+\partial_t : {}_0W_{p,\mu}^1(\mathbb{R}_+; E) \to L_{p,\mu}(\mathbb{R}_+; E)$ is isomorphic. The restriction of this operator to finite J is still injective, and its bounded inverse is given by the restriction of the inverse for $J = \mathbb{R}_+$. By induction and interpolation we deduce that $1+\partial_t : {}_0W_{p,\mu}^{s+1}(J; E) \to {}_0W_{p,\mu}^s(J; E)$ is isomorphic for all $s \ge 0$ and J. For $s = [s] + s_*$ with $[s] \in \mathbb{N}_0$ and $s_* \in [0, 1)$, we thus obtain the important (and expected) equality

$$_0W_{p,\mu}^s(J; E) = \big\{ u \in {}_0W_{p,\mu}^{[s]}(J; E) \,:\, u^{([s])} \in {}_0W_{p,\mu}^{s_*}(J; E) \big\}, \tag{2.7}$$

where the natural norms are equivalent with constants independent of J. In Lemma 2.6 we have seen that $1 - \partial_t$ is an isomorphism from $W_{p,\mu}^{s+1}(\mathbb{R}_+; E)$ to $W_{p,\mu}^s(\mathbb{R}_+; E)$ for all $s \ge 0$. For $J = \mathbb{R}_+$, it follows that

$$W_{p,\mu}^s(J; E) = \big\{ u \in W_{p,\mu}^{[s]}(J; E) \,:\, u^{([s])} \in W_{p,\mu}^{s_*}(J; E) \big\}. \tag{2.8}$$

By means of extension and restriction, one can extend this identity to finite J. (Observe that the formula for $\mathcal{E}_J$ given in the proof of Lemma 2.5 imply that if u is contained the space on right hand side, then its extension belongs to the analogous space with $J = \mathbb{R}_+$.) These characterizations remain valid if one replaces the W-spaces by the H-spaces. We next prove general interpolation properties of the weighted spaces.

Lemma 2.8. *Let $J = (0, T)$ be finite or infinite, $p \in (1, \infty)$, $\mu \in (1/p, 1]$, $0 \le s_1 < s_2$ and $\theta \in (0, 1)$. Set $s = (1-\theta)s_1 + \theta s_2$. Then it holds*

$$\big[H_{p,\mu}^{s_1}(J; E), H_{p,\mu}^{s_2}(J; E) \big]_\theta = H_{p,\mu}^s(J; E). \tag{2.9}$$

If also $s \notin \mathbb{N}$, we have

$$\big(H_{p,\mu}^{s_1}(J; E), H_{p,\mu}^{s_2}(J; E) \big)_{\theta,p} = W_{p,\mu}^s(J; E). \tag{2.10}$$

Moreover, if $s_1, s_2, s \notin \mathbb{N}_0$ then

$$\big[W_{p,\mu}^{s_1}(J; E), W_{p,\mu}^{s_2}(J; E) \big]_\theta = W_{p,\mu}^s(J; E), \quad \big(W_{p,\mu}^{s_1}(J; E), W_{p,\mu}^{s_2}(J; E) \big)_{\theta,p} = W_{p,\mu}^s(J; E). \tag{2.11}$$

If $F \overset{d}{\hookrightarrow} E$ is a Banach space of class $\mathcal{HT}$, then it holds for $\tau \geq 0$ that

$$\big(H_{p,\mu}^{\tau}(J;E), H_{p,\mu}^{\tau}(J;F)\big)_{\theta,p} = H_{p,\mu}^{\tau}\big(J;(E,F)_{\theta,p}\big), \tag{2.12}$$

$$\big[H_{p,\mu}^{\tau}(J;E), H_{p,\mu}^{\tau}(J;F)\big]_{\theta} = H_{p,\mu}^{\tau}\big(J;[E,F]_{\theta}\big). \tag{2.13}$$

All these assertions remain true if one replaces the W- and H-spaces by $_0W$- and $_0H$-spaces, respectively, where the constants of the norm equivalences do not depend on T if $s_2 \leq 2$.

Proof. (I) First, let $J = \mathbb{R}_+$. Throughout, we consider the operator $A = 1 - \partial_t$ on $L_{p,\mu}(\mathbb{R}_+;Z)$ with $D(A) = W_{p,\mu}^1(\mathbb{R}_+;Z)$ for any Banach space Z of class $\mathcal{HT}$, and we put $X := L_{p,\mu}(\mathbb{R}_+;E)$. By Theorem 2.7, the operator A possesses a bounded $\mathcal{H}^\infty$-calculus on X so that A admits bounded imaginary powers (see (2.15) of [5]). A theorem by Yagi now implies that $D(A^\alpha) = [X, D(A)]_\alpha = H_{p,\mu}^\alpha(\mathbb{R}_+;E)$ for all $\alpha \in (0,1)$, see Theorem 1.15.3 of [31]. For $\alpha \geq 1$, formula (2.8) further yields

$$D(A^\alpha) = \{u \in D(A^{[\alpha]}) \ : \ A^{[\alpha]}u \in D(A^{\alpha-[\alpha]})\}$$
$$= \{u \in H_{p,\mu}^{[\alpha]}(\mathbb{R}_+;E) \ : \ u^{([\alpha])} \in H_{p,\mu}^{\alpha-[\alpha]}(\mathbb{R}_+;E)\} = H_{p,\mu}^\alpha(\mathbb{R}_+;E). \tag{2.14}$$

Employing this equation and again Yagi's theorem, we obtain (2.9) by computing

$$\big[H_{p,\mu}^{s_1}(\mathbb{R}_+;E), H_{p,\mu}^{s_2}(\mathbb{R}_+;E)\big]_\theta = \big[D(A^{s_1}), D(A^{s_2})\big]_\theta = D(A^s) = H_{p,\mu}^s(\mathbb{R}_+;E).$$

(II) We continue with (2.10). The operator A^{s_1} induces an isomorphism

$$\big(H_{p,\mu}^{s_1}(J;E), H_{p,\mu}^{s_2}(J;E)\big)_{\theta,p} = \big(D(A^{s_1}), D(A^{s_2})\big)_{\theta,p} \to \big(X, D(A^\tau)\big)_{\theta,p},$$

where $\tau = s_2 - s_1$. We first assume that $\theta\tau \notin \mathbb{N}_0$, and put $k = [\theta\tau] \in \mathbb{N}_0$. It follows from reiteration (see e.g. Theorem 1.10.3.2 of [31]) that

$$\big(X, D(A^\tau)\big)_{\theta,p} = \big(D(A^k), D(A^\tau)\big)_{\sigma,p},$$

with $\sigma = \frac{\tau\theta-k}{\tau-k} \in (0,1)$, and that the operator A^k induces an isomorphism

$$\big(D(A^k), D(A^\tau)\big)_{\sigma,p} \to \big(X, D(A^{\tau-k})\big)_{\sigma,p} = \big(X, D(A)\big)_{\sigma(\tau-k),p} = W_{p,\mu}^{\sigma(\tau-k)}(\mathbb{R}_+;E),$$

where $\sigma(\tau-k) = \tau\theta - k \in (0,1)$. On the other hand, $A^{-(s_1+k)}$ induces an isomorphism

$$W_{p,\mu}^{\tau\theta-k}(\mathbb{R}_+;E) \to W_{p,\mu}^{(s_2-s_1)\theta+s_1}(\mathbb{R}_+;E) = W_{p,\mu}^s(\mathbb{R}_+;E).$$

Hence, (2.10) holds if $\theta(s_2 - s_1) \notin \mathbb{N}_0$.

(III) Next we derive (2.11). We take an integer $k > s_2$. We can write $W_{p,\mu}^{s_j}(\mathbb{R}_+;E) = \big(X, H_{p,\mu}^k(\mathbb{R}_+;E)\big)_{s_j/k,p}$ thanks to part (II) and $k \cdot s_j/k = s_j \notin \mathbb{N}_0$. Now reiteration yields

$$\big(W_{p,\mu}^{s_1}(J;E), W_{p,\mu}^{s_2}(J;E)\big)_{\theta,p} = \big(X, H_{p,\mu}^k(\mathbb{R}_+;E)\big)_{s/k,p} = W_{p,\mu}^s(\mathbb{R}_+;E),$$

using again part (II) and that $k \cdot s/k \notin \mathbb{N}_0$. The other equation in (2.11) is shown similarly, employing Remark 1.10.3.2 of [31]. The remaining case $\theta(s_2 - s_1) \in \mathbb{N}_0$ in Step II can now be treated by an reiteration argument with exponents $\theta \pm \varepsilon$.

(IV) Now let $F \overset{d}{\hookrightarrow} E$ be a Banach space of class $\mathcal{HT}$ and $\tau \geq 0$. Due to (2.14), the operator A^τ is an isomorphism

$$\big(H_{p,\mu}^{\tau}(\mathbb{R}_+;E), H_{p,\mu}^{\tau}(\mathbb{R}_+;F)\big)_{\theta,p} \to \big(L_{p,\mu}(\mathbb{R}_+;E), L_{p,\mu}(\mathbb{R}_+;F)\big)_{\theta,p}.$$

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Theorem 1.18.4 of [31] says that the latter space equals $L_{p,\mu}\big(\mathbb{R}_+; (E,F)_{\theta,p}\big)$. Since $A^{-\tau}$ maps this space isomorphically to $H_{p,\mu}^\tau\big(\mathbb{R}_+; (E,F)_{\theta,p}\big)$, we have shown (2.12). The formula (2.13) is shown in the same way.

(V) Replacing the operator $A = 1 - \partial_t$ by $A_0 := 1 + \partial_t$, the same arguments as above show the asserted equalities for the $_0W$- and the $_0H$-spaces. This finishes the case $J = \mathbb{R}_+$. The case of a finite interval is then deduced from the half-line case, using the extension operators $\mathcal{E}_J$ and $\mathcal{E}_J^0$ from Lemma 2.5 and Theorem 1.2.4 of [31] (see also Section I.2.3 of [2]). The dependence of the norm equivalence constants on the length of J carries over from the properties of the extension operators. ∎

We now extend Theorem 2.7 to fractional powers of the derivative acting on the W- and H-scales. To this aim, we note that if a sectorial operator A has a bounded $\mathcal{H}^\infty$-calculus with $\mathcal{H}^\infty$-angle $\phi \in [0,\pi)$ and if $\alpha \in (0, \pi/\phi)$, then A^α has the same property with angle less or equal $\alpha\phi$. This fact easily follows from the properties of the calculus, see Lemma A.3.5 in [24].

Proposition 2.9. *Let $0 < 1/p < \mu \le 1$, $s \ge 0$, $\alpha \in (0,2)$ and $\omega > 0$. Then the operators*

$$(\omega - \partial_t)^\alpha \quad on \quad H_{p,\mu}^s(\mathbb{R}_+; E) \qquad with \ domain \quad H_{p,\mu}^{s+\alpha}(\mathbb{R}_+; E),$$

$$(\omega - \partial_t)^\alpha \quad on \quad W_{p,\mu}^s(\mathbb{R}_+; E) \qquad with \ domain \quad W_{p,\mu}^{s+\alpha}(\mathbb{R}_+; E), \qquad s, s+\alpha \notin \mathbb{N}_0,$$

$$(\omega + \partial_t)^\alpha \quad on \quad {}_0H_{p,\mu}^s(\mathbb{R}_+; E) \qquad with \ domain \quad {}_0H_{p,\mu}^{s+\alpha}(\mathbb{R}_+; E),$$

$$(\omega + \partial_t)^\alpha \quad on \quad {}_0W_{p,\mu}^s(\mathbb{R}_+; E) \qquad with \ domain \quad {}_0W_{p,\mu}^{s+\alpha}(\mathbb{R}_+; E), \qquad s, s+\alpha \notin \mathbb{N}_0,$$

are invertible and possess a bounded $\mathcal{H}^\infty$-calculus with $\mathcal{H}^\infty$-angle less or equal $\alpha\pi/2$.

Proof. We first consider the case $s = 0$. Proposition 2.11 of [5] and Theorem 2.7 imply that the realization of $\omega - \partial_t$ on $L_{p,\mu}(\mathbb{R}_+; E)$ with domain $W_{p,\mu}^1(\mathbb{R}_+; E)$ admits a bounded $\mathcal{H}^\infty$-calculus with $\mathcal{H}^\infty$-angle equal to $\pi/2$. As noted above, also $(\omega - \partial_t)^\alpha$ possesses a bounded $\mathcal{H}^\infty$-calculus with $\mathcal{H}^\infty$-angle less or equal $\alpha\pi/2$, because of $\alpha \in (0,2)$. As in (2.14) we see that then $D((\omega - \partial_t)^\alpha) = H_{p,\mu}^\alpha(\mathbb{R}_+; E)$ holds for all $\alpha \ge 0$.

Since $\omega - \partial_t$ is invertible and $H_{p,\mu}^{\tau+s}(\mathbb{R}_+; E) = D((\omega - A)^{\tau+s})$ holds for all $s, \tau \ge 0$, the operator $(\omega - \partial_t)^s$ induces an isomorphism from $H_{p,\mu}^{\tau+s}(\mathbb{R}_+; E)$ to $H_{p,\mu}^\tau(\mathbb{R}_+; E)$. Using that $(\omega - \partial_t)^{-s}$ and $(\omega - \partial_t)^\alpha$ commute on $H_{p,\mu}^\alpha(\mathbb{R}_+; E)$, we derive from Proposition 2.11 of [5] that $(\omega - \partial_t)^\alpha$ has a bounded $\mathcal{H}^\infty$-calculus on $H_{p,\mu}^s(\mathbb{R}_+; E)$ with angle not larger than $\alpha\pi/2$, and that its domain equals $H_{p,\mu}^{s+\alpha}(\mathbb{R}_+; E)$.

Let $s, s+\alpha \notin \mathbb{N}_0$. Interpolation of the H-case and Lemma 2.8 show that $(\omega - \partial_t)^\alpha$ with domain $W_{p,\mu}^{s+\alpha}(\mathbb{R}_+; E)$ has a bounded $\mathcal{H}^\infty$-calculus on $W_{p,\mu}^s(\mathbb{R}_+; E)$ with $\mathcal{H}^\infty$-angle less or equal $\alpha\pi/2$. The operator $\omega + \partial_t$ can be treated in the same way. ∎

The $_0W_{p,\mu}^s$-spaces can be characterized in terms of the kernel of the temporal trace, as the next proposition shows. These results are mainly due to Grisvard [13]. Observe that the limit number for the existence of a trace is $s = 1 - \mu + 1/p$. Therefore, if μ runs through the interval $(1/p, 1]$ this limit number runs through the interval $[1/p, 1)$. Of course, for $\mu = 1$ the limit number $s = 1/p$ for the unweighted case is recovered. We do not treat the limit cases $s = k + 1 - \mu + 1/p$ with $k \in \mathbb{N}_0$, see Remarque 4.2 of [13] and Remark 3.6.3.2 of [31] for a discussion. We also do not consider the corresponding characterizations of the $_0H_{p,\mu}^s$-spaces, which are not needed below. They should be correct, but it seems that their proofs require a much greater effort.

Proposition 2.10. *Let $J = (0, T)$ be finite or infinite, $p \in (1, \infty)$ and $\mu \in (1/p, 1]$. Then for $0 < s < 1 - \mu + 1/p$ it holds*

$$C_c^\infty(\overline{J}\backslash\{0\}; E) \xrightarrow{d} W_{p,\mu}^s(J; E) \quad \text{and} \quad W_{p,\mu}^s(J; E) = {}_0W_{p,\mu}^s(J; E). \tag{2.15}$$

For $k + 1 - \mu + 1/p < s < k + 1 + (1 - \mu + 1/p)$ with $k \in \mathbb{N}_0$ we have

$$W_{p,\mu}^s(J; E) \hookrightarrow BUC^k(\overline{J}; E), \tag{2.16}$$

where here one may replace $W_{p,\mu}^s$ by $H_{p,\mu}^s$, and moreover

$${}_0W_{p,\mu}^s(J; E) = \left\{ u \in W_{p,\mu}^s(J; E) \; : \; u^{(j)}(0) = 0, \; j \in \{0, ..., k\} \right\}. \tag{2.17}$$

If in addition $s \in [0, 2]$, then the embedding constants for

$${}_0W_{p,\mu}^s(J; E) \hookrightarrow BUC^k(\overline{J}; E) \quad \text{and} \quad {}_0H_{p,\mu}^s(J; E) \hookrightarrow BUC^k(\overline{J}; E)$$

do not depend on J.

Proof. In [13] much of the proposition is proved for the W-spaces in the scalar-valued case $E = \mathbb{C}$ with $J = \mathbb{R}_+$. An inspection of the proofs given there shows that they only make use of basic facts, interpolation theory and the scalar versions of Lemmas 2.4 and 2.6 above. Thus the results of [13] carry over to the case of a general Banach space E. Moreover, the case of a finite interval is obtained from the half-line case by extension and restriction. One can replace W by H as asserted because of (2.1) and (2.2).

Assertion (2.15) is shown in Théorème 2.1 and Théorème 4.1 of [13]. The embedding (2.16) is proved for $k = 0$ in Théorème 5.2 of [13], and the general case $k \in \mathbb{N}$ is an immediate consequence in view of (2.8). For $s \leq 1$, identity (2.17) is shown in Proposition 1.2 and Théorème 4.1 of [13]. For integers $s = k + 1$ this equality holds by definition. The general case then follows from (2.7), (2.15) and (2.17) for $s - [s]$, as well as (2.8). ∎

We next establish embeddings of Sobolev type for $W_{p,\mu}^s$ and $H_{p,\mu}^s$.

Proposition 2.11. *Let $J = (0, T)$ be finite, $1 < p < q < \infty$, $\mu > 1/p$ and $s > \tau \geq 0$. Then*

$$W_{p,\mu}^s(J; E) \hookrightarrow W_{q,\mu}^\tau(J; E) \qquad \text{holds if} \quad s - (1 - \mu + 1/p) > \tau - \frac{p(1 - \mu + 1/p)}{q}, \tag{2.18}$$

$$W_{p,\mu}^s(J; E) \hookrightarrow W_q^\tau(J; E) \qquad \text{holds if} \quad s - (1 - \mu + 1/p) > \tau - 1/q. \tag{2.19}$$

These embeddings remain true if one replaces the W-spaces by the H-, the ${}_0W$- and the ${}_0H$-spaces, respectively. In the two latter cases, restricting to $s \in [0, 2]$, for given $T_0 > 0$ the embeddings hold with a uniform constant for all $0 < T \leq T_0$.

Proof. Throughout this proof, let $T_0 > 0$ be given. Since the inequality signs in (2.18) and (2.19) are strict, we may assume that $s \notin \mathbb{N}$. Again we only have to consider the W-case due to (2.1) and (2.2).

(I) We prove (2.18) for $\tau = 0$. For $s > 1 - \mu + 1/p$, Proposition 2.10 yields

$$W_{p,\mu}^s(J; E) \hookrightarrow L_\infty(J; E) \hookrightarrow L_{q,\mu}(J; E) \qquad \text{for all} \;\; q \in (p, \infty),$$

with the asserted behaviour of the embedding constant in the ${}_0W$-case. If $s \leq 1 - \mu + 1/p$, take $q \in (p, \infty)$ as in (2.18). Choose $\eta \in (1 - \mu + 1/p, 1)$ and $r \in (p, \infty)$ such that $\frac{1}{q} \leq \frac{1 - s/\eta}{p} + \frac{s/\eta}{r}$.

We have already shown that $W_{p,\mu}^\eta(J;E) \hookrightarrow L_{r,\mu}(J;E)$. Lemma 2.8 and Theorem 1.18.5 of [31] then imply

$$W_{p,\mu}^s(J;E) = \left(L_{p,\mu}(J;E), W_{p,\mu}^\eta(J;E)\right)_{s/\eta,p} \hookrightarrow \left(L_{p,\mu}(J;E), L_{r,\mu}(J;E)\right)_{s/\eta,p} \hookrightarrow L_{q,\mu}(J;E).$$

In the $_0W$-case, all embedding constants are uniform in $T \le T_0$ for $s \in [0,2]$.

(II) To prove (2.18) for $\tau > 0$, we set $\alpha = (1-p/q)(1-\mu+1/p) > 0$ for an exponent $q > p$ as in (2.18). We fix some $\varepsilon > 0$ and $k \in \mathbb{N}_0$ such that $s - \varepsilon > \tau + \alpha > 0$, $k + \alpha < s - \varepsilon < k + 1 + \alpha$, and $\kappa := k + \alpha + \varepsilon$ is not an integer. Lemma 2.8 now yields

$$W_{p,\mu}^s(J;E) = \left(W_{p,\mu}^\kappa(J;E), W_{p,\mu}^{\kappa+1}(J;E)\right)_{s-\kappa,p},$$

Using (2.8) and (2.18) with $\tau = 0$, we obtain

$$W_{p,\mu}^\kappa(J;E) \hookrightarrow W_{q,\mu}^k(J;E) \quad \text{and} \quad W_{p,\mu}^{\kappa+1}(J;E) \hookrightarrow W_{q,\mu}^{k+1}(J;E).$$

Since $(\cdot,\cdot)_{\theta,p} \hookrightarrow (\cdot,\cdot)_{\theta,q}$ for $\theta \in (0,1)$, it follows

$$W_{p,\mu}^s(J;E) \hookrightarrow \left(W_{q,\mu}^k(J;E), W_{q,\mu}^{k+1}(J;E)\right)_{s-\kappa,q} = W_{q,\mu}^{k+s-\kappa}(J;E) \hookrightarrow W_{q,\mu}^\tau(J;E).$$

(III) For $_0W$-spaces, the dependence on T for $s \in [0,2]$ carries over from Lemma 2.8 and (2.18) with $\tau = 0$. The embedding (2.19) is shown similarly employing Lemma 2.1, see Proposition 1.1.12 in [24]. ∎

We finish this section with Poincaré's inequality in the weighted spaces.

Lemma 2.12. *Let $J = (0,T)$ be finite, $p \in (1,\infty)$, $\mu \in (1/p, 1]$ and $s \in [0,1)$. Then it holds*

$$|u|_{L_{p,\mu}(J;E)} \lesssim T\, |u'|_{L_{p,\mu}(J;E)}, \qquad |u|_{_0W_{p,\mu}^s(J;E)} + |u|_{_0H_{p,\mu}^s(J;E)} \lesssim T^{1-s}\, |u|_{W_{p,\mu}^1(J;E)},$$

for all $u \in {_0W_{p,\mu}^1}(J;E)$.

Proof. Let $u \in {_0W_{p,\mu}^1}(J;E)$. Using Hölder's inequality, we estimate

$$t^{p(1-\mu)}|u(t)|_E^p \le t^{p(1-\mu)} \left(\int_0^T s^{-(1-\mu)} s^{1-\mu}|u'(s)|_E \,\mathrm{d}s\right)^p \lesssim t^{p(1-\mu)}\, T^{(1-p'(1-\mu))p/p'} |u'|_{L_{p,\mu}(J;E)}^p$$

for $t \in J$. The first assertion now follows by integration over J. For $s \in [0,1)$ we have

$$|u|_{_0W_{p,\mu}^s(J;E)} + |u|_{_0H_{p,\mu}^s(J;E)} \lesssim |u|_{_0W_{p,\mu}^1(J;E)}^s |u|_{L_{p,\mu}(J;E)}^{1-s}$$

by interpolation, implying the second asserted inequality. ∎

3. Weighted anisotropic spaces

Let again E be a Banach space of class $\mathcal{HT}$, let $J = (0,T)$ be finite or infinite, and let further $\Omega \subset \mathbb{R}^n$ be a domain with compact smooth boundary $\partial\Omega$, or $\Omega \in \{\mathbb{R}^n, \mathbb{R}_+^n\}$. For $p, q \in (1,\infty)$ and $r > 0$ we denote by

$$H_p^r(\Omega;E), \qquad W_p^r(\Omega;E), \qquad B_{p,q}^r(\Omega;E),$$

the E-valued Sobolev, Slobodetskii and Besov spaces, respectively. If $\Omega = \mathbb{R}^n$ we also use this spaces with $r \le 0$. For the definitions and properties of these spaces we refer to [3], [28] or [33]. The scalar-valued case is treated extensively in [31]. In particular, we have

$$B_{p,p}^r(\Omega;E) = W_p^r(\Omega;E) \qquad \text{for all } p \in [1,\infty), \quad r \notin \mathbb{N}_0.$$

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The corresponding spaces over the boundary $\partial\Omega$ of Ω are defined via local charts as in Definition 3.6.1 of [31], for instance.

We mainly investigate weighted anisotropic spaces, i.e., intersections of spaces of the form

$$H^s_{p,\mu}\big(J; H^r_p(\Omega; E)\big), \quad W^s_{p,\mu}\big(J; W^r_p(\Omega; E)\big), \quad H^s_{p,\mu}\big(J; W^r_p(\Omega; E)\big), \quad W^s_{p,\mu}\big(J; W^r_p(\Omega; E)\big), \quad (3.1)$$

where $s, r \geq 0$. In what follows we refer to $t \in J$ as time and to $x \in \overline{\Omega}$ as space variables.

We start with two important tools. First, given $k \in \mathbb{N}$, there is an extension operator $\mathcal{E}_\Omega$ to $\mathbb{R}^n$ for functions defined on Ω (i.e., we have $(\mathcal{E}_\Omega u)|_\Omega = u$) satisfying

$$\mathcal{E}_\Omega \in \mathcal{B}\big(B^r_{p,q}(\Omega; E), B^r_{p,q}(\mathbb{R}^n; E)\big) \cap \mathcal{B}\big(H^r_p(\Omega; E), H^r_p(\mathbb{R}^n; E)\big) \quad (3.2)$$

for all $p, q \in (1, \infty)$ and $r \in [0, k]$. For integer $r \in [0, k]$, the constructions in the proofs of Theorems 5.21 and 5.22 [1] for the scalar-valued spaces literally carries over to the vector-valued case. The general case $r \in [0, k]$ can be treated via interpolation. Applying $\mathcal{E}_\Omega$ pointwise almost everywhere in time and using again interpolation, we obtain a spatial extension operator for the anisotropic spaces, again denoted by $\mathcal{E}_\Omega$:

$$\mathcal{E}_\Omega \in \mathcal{B}\big(H^s_{p,\mu}(J; H^r_p(\Omega; E)), H^s_{p,\mu}(J; H^r_p(\mathbb{R}^n; E))\big), \qquad s \geq 0, \quad r \in [0, k]. \quad (3.3)$$

Of course, here a H-space may be replaced by a W-space at the first or the second or at both positions, and this remains true for the $_0H^s_{p,\mu}$- and the $_0W^s_{p,\mu}$-spaces with respect to time.

Second, we consider operators build from the time derivative and the Laplacian whose domains are given by intersections of the spaces in (3.1) with $\Omega = \mathbb{R}^n$ and $J = \mathbb{R}_+$. This fact will allow us to study the anisotropic spaces by means of these operators.

Lemma 3.1. *Let E be a Banach space of class $\mathcal{HT}$, let $p \in (1, \infty)$, $\mu \in (1/p, 1]$, $s, r \geq 0$, $\alpha \in (0, 2)$ and $\beta > 0$, let $\omega, \omega' \geq 0$ satisfy $\omega + \omega' \neq 0$ and set*

$$H^s_{p,\mu}(H^r_p) := H^s_{p,\mu}\big(\mathbb{R}_+; H^r_p(\mathbb{R}^n; E)\big), \qquad {}_0H^s_{p,\mu}(H^r_p) := {}_0H^s_{p,\mu}\big(\mathbb{R}_+; H^r_p(\mathbb{R}^n; E)\big),$$

and analogously for the other types of spaces in (3.1). Let Δ_n be the Laplacian on $\mathbb{R}^n$. Then the following holds true.

    a) *The pointwise realization of $(\omega - \Delta_n)^{\beta/2}$ on any of the spaces*

$$\begin{aligned}
H^s_{p,\mu}(H^r_p), &\quad \text{with domain} \quad H^s_{p,\mu}(H^{r+\beta}_p), \\
H^s_{p,\mu}(W^r_p), &\quad \text{with domain} \quad H^s_{p,\mu}(W^{r+\beta}_p), \quad r, r+\beta \notin \mathbb{N}_0, \\
W^s_{p,\mu}(H^r_p), &\quad \text{with domain} \quad W^s_{p,\mu}(H^{r+\beta}_p), \\
W^s_{p,\mu}(W^r_p), &\quad \text{with domain} \quad W^s_{p,\mu}(W^{r+\beta}_p), \quad r, r+\beta \notin \mathbb{N}_0,
\end{aligned}$$

    admits a bounded $\mathcal{H}^\infty$-calculus with $\mathcal{H}^\infty$-angle equal to zero, and it is invertible if $\omega > 0$. Here one can replace the $H^s_{p,\mu}, W^s_{p,\mu}$-spaces by the $_0H^s_{p,\mu}, {}_0W^s_{p,\mu}$-spaces, respectively.

    b) *The operator $L := (\omega' - \partial_t)^\alpha + (\omega - \Delta_n)^{\beta/2}$ acting on any of the spaces*

$$\begin{aligned}
H^s_{p,\mu}(H^r_p), &\quad \text{with domain} \quad H^{s+\alpha}_{p,\mu}(H^\beta_p) \cap H^s_{p,\mu}(H^{r+\beta}_p), \\
H^s_{p,\mu}(W^r_p), &\quad \text{with domain} \quad H^{s+\alpha}_{p,\mu}(W^\beta_p) \cap H^s_{p,\mu}(W^{r+\beta}_p), \quad r, r+\beta \notin \mathbb{N}_0, \\
W^s_{p,\mu}(H^r_p), &\quad \text{with domain} \quad W^{s+\alpha}_{p,\mu}(H^\beta_p) \cap W^s_{p,\mu}(H^{r+\beta}_p), \quad s, s+\alpha \notin \mathbb{N}_0, \\
W^s_{p,\mu}(W^r_p), &\quad \text{with domain} \quad W^{s+\alpha}_{p,\mu}(W^\beta_p) \cap W^s_{p,\mu}(W^{r+\beta}_p), \quad s, s+\alpha, r, r+\beta \notin \mathbb{N}_0,
\end{aligned}$$

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is invertible and admits bounded imaginary powers with power angle not larger than $\alpha\pi/2$. This remains true for the operator $L_0 := (\omega' + \partial_t)^\alpha + (\omega - \Delta_n)^{\beta/2}$ if one replaces the $H^s_{p,\mu}, W^s_{p,\mu}$-spaces by the $_0H^s_{p,\mu}, {}_0W^s_{p,\mu}$-spaces, respectively.

c) For $\tau \in (0,1]$ it holds

$$D(L^\tau) = D((\omega' - \partial_t)^{\alpha\tau}) \cap D((\omega - \Delta_n)^{\beta\tau/2}),$$

$$D_L(\tau, p) = D_{(\omega'-\partial_t)^\alpha}(\tau, p) \cap D_{(\omega-\Delta_n)^{\beta/2}}(\tau, p),$$

and this remains true if one replaces L by L_0 and $\omega' - \partial_t$ by $\omega' + \partial_t$.

Proof. Since E is of class $\mathcal{HT}$, the operator $-\Delta_n$ with domain $H^2_p(\mathbb{R}^n; E)$ possesses a bounded $\mathcal{H}^\infty$-calculus on $L_p(\mathbb{R}^n; E)$ with $\mathcal{H}^\infty$-angle equal to zero, see Theorem 5.5 of [5]. As in Proposition 2.9, the fractional power $(\omega - \Delta_n)^{\beta/2}$ with domain $H^\beta_p(\mathbb{R}^n; E)$ has the same property. If $\omega > 0$, it is invertible in $L_p(\mathbb{R}^n; E)$.

Let $r \geq 0$. Using $(\omega - \Delta_n)^{r/2}$ as an isomorphism between $H^r_p(\mathbb{R}^n; E)$ and $L_p(\mathbb{R}^n; E)$, it then follows from Proposition 2.11 of [5] that $(\omega - \Delta_n)^{\beta/2}$ with domain $H^{r+\beta}_p(\mathbb{R}^n; E)$ possesses a bounded $\mathcal{H}^\infty$-calculus on $H^r_p(\mathbb{R}^n; E)$ with $\mathcal{H}^\infty$-angle equal to zero. By interpolation, this fact remains true if one considers $(\omega - \Delta_n)^{\beta/2}$ with domain $W^{r+\beta}_p(\mathbb{R}^n; E)$ in $W^r_p(\mathbb{R}^n; E)$, provided $r, r + \beta \notin \mathbb{N}_0$. It is straightforward to see that these properties carry over to the pointwise realizations on the anisotropic spaces, cf. Lemma A.3.6 in [24].

Since all the spaces under consideration are of class $\mathcal{HT}$, Proposition 2.9 implies that $(\omega' - \partial_t)^\alpha$ with domain $H^{s+\alpha}_{p,\mu}(H^r_p)$ admits a bounded $\mathcal{H}^\infty$-calculus on $H^s_{p,\mu}(H^r_p)$ with $\mathcal{H}^\infty$-angle less or equal $\alpha\pi/2$, and on the corresponding spaces where H is replaced by W, with the asserted exceptions. Using these facts and that the resolvents of $(\omega' - \partial_t)^\alpha$ and $(\omega - \Delta_n)^{\beta/2}$ commute, the assertions for L are a consequence of Proposition 1.1. The same arguments show the assertions for L_0. Finally, c) is a consequence of Lemma 9.5 in [12] and Teorema 5 in [14] (see also Lemma 3.1 in [6]). ∎

With the help of the operators from Lemma 3.1 we establish fundamental embeddings for the anisotropic spaces with regularity exponents (s, r) for time and space. Roughly speaking, we prove that the regularities $(s + \alpha, r)$ and $(s, r + \beta)$ imply all regularities on the line segment from $(s + \alpha, r)$ to $(s, r + \beta)$.

Proposition 3.2. Let $J = (0, T)$ be finite or infinite, $0 < 1/p < \mu \leq 1$, and let $\Omega \subset \mathbb{R}^n$ be a domain with compact smooth boundary $\partial\Omega$, or $\Omega \in \{\mathbb{R}^n, \mathbb{R}^n_+\}$. Let further

$$s, r \geq 0, \qquad \alpha \in (0, 2), \qquad \beta > 0, \qquad \sigma \in [0, 1],$$

and set $H^s_{p,\mu}(H^r_p) := H^s_{p,\mu}\big(J; H^r_p(\Omega; E)\big)$, and analogously for the other anisotropic spaces. Then it holds

$$H^{s+\alpha}_{p,\mu}(H^r_p) \cap H^s_{p,\mu}(H^{r+\beta}_p) \hookrightarrow H^{s+\sigma\alpha}_{p,\mu}(H^{r+(1-\sigma)\beta}_p), \tag{3.4}$$

and moreover each of the spaces

$$H^{s+\alpha}_{p,\mu}(W^r_p) \cap H^s_{p,\mu}(W^{r+\beta}_p), \quad W^{s+\alpha}_{p,\mu}(H^r_p) \cap W^s_{p,\mu}(H^{r+\beta}_p), \quad W^{s+\alpha}_{p,\mu}(H^r_p) \cap H^s_{p,\mu}(W^{r+\beta}_p),$$

is continuously embedded into

$$W^{s+\sigma\alpha}_{p,\mu}(H^{r+(1-\sigma)\beta}_p) \cap H^{s+\sigma\alpha}_{p,\mu}(W^{r+(1-\sigma)\beta}_p),$$

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provided all the occurring $W_{p,\mu}$- and W_p-spaces have a noninteger order of differentiability. Finally, assuming all orders of differentiability to be noninteger, we have

$$W_{p,\mu}^{s+\alpha}(W_p^r) \cap W_{p,\mu}^s(W_p^{r+\beta}) \hookrightarrow W_{p,\mu}^{s+\sigma\alpha}(W_p^{r+(1-\sigma)\beta}). \tag{3.5}$$

These embeddings remain true if one replaces Ω by its boundary $\partial\Omega$. They are also valid if one replaces all $H_{p,\mu}$-, $W_{p,\mu}$- spaces by ${}_0H_{p,\mu}$-, ${}_0W_{p,\mu}$-spaces, respectively. Restricting in the latter case to $s + \alpha \le 2$, the embedding constants do not depend on the length of J.

Proof. Using extensions and restrictions, and employing that the spaces over $\partial\Omega$ are defined via local charts, it suffices to consider the case $J \times \Omega = \mathbb{R}_+ \times \mathbb{R}^n$. The dependence of the embedding constants on J carries over from the properties of the extension operators.

(I) For (3.4) we consider the operators $(1 - \partial_t)^\alpha$ and $(1 - \Delta_n)^{\beta/2}$ on $H_{p,\mu}^s(H_p^r)$, which were treated in Proposition 2.9 and Lemma 3.1. Note that we have to restrict to $\alpha \in (0, 2)$ to obtain sectoriality of $(1 - \partial_t)^\alpha$. Due to the invertibility of these operators and the description of their domains, the expression

$$|(1 - \partial_t)^{\alpha\sigma}(1 - \Delta_n)^{\beta(1-\sigma)/2} \cdot |_{H_{p,\mu}^s(H_p^r)}$$

is an equivalent norm on $H_{p,\mu}^{s+\sigma\alpha}(H_p^{r+(1-\sigma)\beta})$ for all $\sigma \in (0, 1)$. Since the sum L of these operators is invertible by Lemma 3.1, we obtain the equivalent norm

$$|((1 - \partial_t)^\alpha + (1 - \Delta_n)^{\beta/2}) \cdot |_{H_{p,\mu}^s(H_p^r)}$$

on $D(L) = H_{p,\mu}^{s+\alpha}(H_p^r) \cap H_{p,\mu}^s(H_p^{r+\beta})$. Hence (3.4) follows directly from Proposition 1.1. The same arguments show that the embeddings

$$H_{p,\mu}^{s+\alpha}(W_p^r) \cap H_{p,\mu}^s(W_p^{r+\beta}) \hookrightarrow H_{p,\mu}^{s+\sigma\alpha}(W_p^{r+(1-\sigma)\beta}),$$
$$W_{p,\mu}^{s+\alpha}(H_p^r) \cap W_{p,\mu}^s(H_p^{r+\beta}) \hookrightarrow W_{p,\mu}^{s+\sigma\alpha}(H_p^{r+(1-\sigma)\beta}),$$

and (3.5) hold, with the asserted exceptions.

(II) We derive the remaining embeddings from (3.4) by suitable interpolation arguments, which were indicated in Remark 5.3 of [12] in a more specific situation. We concentrate on the case $W_{p,\mu}^{s+\alpha}(H_p^r) \cap H_{p,\mu}^s(W_p^{r+\beta})$, the other assertions are obtained similarly, see Proposition 1.3.12 of [24]. For $s + \alpha, r + \beta \notin \mathbb{N}$ and sufficiently small $\varepsilon > 0$, we apply the real interpolation functor $(\cdot, \cdot)_{1/2,p}$ to the embeddings

$$H_{p,\mu}^{s+\alpha(1\pm\varepsilon/\beta)}(H_p^r) \cap H_{p,\mu}^s(H_p^{r+\beta\pm\varepsilon}) \hookrightarrow H_{p,\mu}^{s+\sigma\alpha}(H_p^{r+(1-\sigma)\beta\pm\varepsilon}),$$
$$H_{p,\mu}^{s+\alpha(1\pm\varepsilon/\beta)}(H_p^r) \cap H_{p,\mu}^s(H_p^{r+\beta\pm\varepsilon}) \hookrightarrow H_{p,\mu}^{s+\alpha(\sigma\pm\varepsilon/\beta)}(H_p^{r+(1-\sigma)\beta}).$$

By Lemma 2.8 the terms in the first and in the second line on the right-hand side interpolate to $H_{p,\mu}^{s+\sigma\alpha}(W_p^{r+(1-\sigma)\beta})$ and $W_{p,\mu}^{s+\sigma\alpha}(H_p^{r+(1-\sigma)\beta})$, respectively. To interpolate the left-hand side, we consider the operator

$$L = (1 - \partial_t)^{\alpha(1+\varepsilon/\beta)} + (1 - \Delta)^{(\beta+\varepsilon)/2},$$

on $H_{p,\mu}^s(H_p^r)$ with domain $D(L) = H_{p,\mu}^{s+\alpha(1+\varepsilon/\beta)}(H_p^r) \cap H_{p,\mu}^s(H_p^{r+\beta+\varepsilon})$. Lemma 3.1 and Proposition 2.9 imply

$$D(L^{(\beta-\varepsilon)/(\beta+\varepsilon)}) = H_{p,\mu}^{s+\alpha(1-\varepsilon/\beta)}(H_p^r) \cap H_{p,\mu}^s(H_p^{r+\beta-\varepsilon}),$$

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and reiteration yields

$$\big(D(L^{(\beta-\varepsilon)/(\beta+\varepsilon)}), D(L)\big)_{1/2,p} = D_L\big((1 + (\beta - \varepsilon)/(\beta + \varepsilon))/2, p\big).$$

Using again Lemma 3.1 and Proposition 2.9, we see that the latter space equals $W_{p,\mu}^{s+\alpha}(H_p^r) \cap H_{p,\mu}^s(W_p^{r+\beta})$, as required.

(III) Starting in Step I with $(1 + \partial_t)^\alpha$ instead of $(1 - \partial_t)^\alpha$, the same arguments as above show that the asserted embeddings are also true for the $_0H_{p,\mu}$- and $_0W_{p,\mu}$-spaces. ∎

Remark 3.3. The proof shows that in the embeddings where only Proposition 1.1 was used, the orders of integrability in space and time do not have to coincide. In fact, the assertions of Lemma 3.1 remain true for the extension of the Laplacian on $H_q^r(\mathbb{R}^n; E)$ to $H_{p,\mu}^s\big(J; H_q^r(\mathbb{R}^n; E)\big)$, where $p, q \in (1, \infty)$ and $\mu \in (1/p, 1]$. As in Step I of the proof above, we then derive $H_{p,\mu}^{s+\alpha}(H_q^r) \cap H_{p,\mu}^s(H_q^{r+\beta}) \hookrightarrow H_{p,\mu}^{s+\sigma\alpha}(H_q^{r+(1-\sigma)\beta})$. ∎

A typical application of Proposition 3.2 is the following result on the mapping behavior of the spatial derivative on anisotropic H-spaces.

Lemma 3.4. *Let E be a Banach space of class $\mathcal{HT}$, let $J = (0, T)$ be finite or finite, and let $\Omega \subset \mathbb{R}^n$ be a domain with compact smooth boundary, or $\Omega \in \{\mathbb{R}_+^n, \mathbb{R}^n\}$. Let further*

$$s \geq 0, \qquad r \in [0, 1), \qquad \alpha \in (0, 2), \qquad \beta \geq 1, \qquad 0 < 1/p < \mu \leq 1.$$

Then the pointwise realization of ∂_{x_i}, $i \in \{1, ..., n\}$, is a continuous map

$$H_{p,\mu}^{s+\alpha}(H_p^r) \cap H_{p,\mu}^s(H_p^{r+\beta}) \to H_{p,\mu}^{s+\alpha-\alpha/\beta}(H_p^r) \cap H_{p,\mu}^s(H_p^{r+\beta-1}).$$

Its operator norm is independent of T if we restrict to $s + \alpha \leq 2$ and to $_0H_{p,\mu}$-spaces.

Proof. By extension and restriction it suffices to consider the case $J \times \Omega = \mathbb{R}_+ \times \mathbb{R}^n$. Clearly the operator ∂_{x_i} maps continuously $H_{p,\mu}^{s+\alpha}(H_p^r) \cap H_{p,\mu}^s(H_p^{r+\beta}) \to H_{p,\mu}^s(H_p^{r+\beta-1})$. Proposition 3.2 further yields the embedding

$$H_{p,\mu}^{s+\alpha}(H_p^r) \cap H_{p,\mu}^s(H_p^{r+\beta}) \hookrightarrow H_{p,\mu}^{s+\alpha-\alpha/\beta}(H_p^{r+1}).$$

Hence, ∂_{x_i} is also a continuous map from $H_{p,\mu}^{s+\alpha}(H_p^r) \cap H_{p,\mu}^s(H_p^{r+\beta})$ to $H_{p,\mu}^{s+\alpha-\alpha/\beta}(H_p^r)$. ∎

4. The temporal and the spatial trace

We first consider the temporal trace on anisotropic spaces. Using integration by parts, one can see that the representation

$$u(0) = (2-\mu)\Big(\sigma^{-(2-\mu)} \int_0^\sigma \tau^{1-\mu} u(\tau)\, \mathrm{d}\tau - (2-\mu) \int_0^\sigma t^{-(3-\mu)} \int_0^t \tau^{1-\mu}(u(t) - u(\tau))\, \mathrm{d}\tau\, \mathrm{d}t\Big) \quad (4.1)$$

holds true for all $\sigma > 0$, Banach spaces X and $u \in W_{1,\mathrm{loc}}^1([0, \infty); X)$. Let $-A$ generate an exponentially stable analytic C_0-semigroup on X and $\theta \in (0, 1)$. Then the norm of the space $D_A(\theta, p)$ is equivalent to the norm given by

$$|x|_{D_A(\theta,p),*}^p = \int_0^\infty \sigma^{p(1-\theta)} |Ae^{-\sigma A} x|_X^p \frac{\mathrm{d}\sigma}{\sigma}, \tag{4.2}$$

see Theorem 1.14.5 of [31]. The formula (4.1) is the key to the following abstract trace theorem.

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Lemma 4.1. *Let X be a Banach space, $p \in (1, \infty)$, $\mu \in (1/p, 1]$, and let the operator A on X with domain $D(A)$ be invertible and admit bounded imaginary powers with power angle strictly smaller than $\pi/2$. Let $s \in (0, 1 - \mu + 1/p)$ and $\alpha > 0$ satisfy $s + \alpha \in (1 - \mu + 1/p, 1)$. Then the temporal trace tr_0, i.e., $\mathrm{tr}_0 u = u|_{t=0}$, maps continuously*

$$W_{p,\mu}^{s+\alpha}\big(\mathbb{R}_+; D(A^s)\big) \cap W_{p,\mu}^s\big(\mathbb{R}_+; D(A^{s+\alpha})\big) \to D_A\big(2s + \alpha - (1 - \mu + 1/p), p\big). \qquad (4.3)$$

Moreover, tr_0 is for $\alpha \in (1 - \mu + 1/p, 1]$ continuous

$$W_{p,\mu}^\alpha\big(\mathbb{R}_+; X\big) \cap L_{p,\mu}\big(\mathbb{R}_+; D_A(\alpha, p)\big) \to D_A\big(\alpha - (1 - \mu + 1/p), p\big), \qquad (4.4)$$

and for $s \in (0, 1 - \mu + 1/p)$ it is continuous

$$W_{p,\mu}^1\big(\mathbb{R}_+; D_A(s, p)\big) \cap W_{p,\mu}^s\big(\mathbb{R}_+; D(A)\big) \to D_A\big(1 + s - (1 - \mu + 1/p), p\big). \qquad (4.5)$$

Proof. Lemma 11 and 12 in [9] establish the variants of the embeddings (4.4) and (4.5) in the case without weight. The proofs for $\mu \in (1/p, 1)$ are similar, using (4.1), the representation (2.6) of the weighted Slobodetskii seminorm and Lemma 2.2, and therefore we omit them. Instead, we concentrate on (4.3), taking u from the function space on the left-hand side there and assuming that $s \in (0, 1 - \mu + 1/p)$, $\alpha > 0$ and $s + \alpha \in (1 - \mu + 1/p, 1)$. From (4.1) and (4.2) we deduce

$$|u(0)|_{D_A(2s+\alpha-(1-\mu+1/p),p),*}^p = \int_0^\infty \sigma^{p(1-(2s+\alpha-(1-\mu+1/p)))}|Ae^{-\sigma A}u(0)|_X^p \sigma^{-1}\,\mathrm{d}\sigma$$

$$\lesssim \int_0^\infty \sigma^{-p(2s+\alpha)}\Big(\int_0^\sigma \tau^{1-\mu}|Ae^{-\sigma A}u(\tau)|\,\mathrm{d}\tau\Big)^p\,\mathrm{d}\sigma \qquad (4.6)$$

$$+ \int_0^\infty \Big(\sigma^{2-\mu-(2s+\alpha)}\int_0^\sigma t^{-(3-\mu)}\int_0^t \tau^{1-\mu}|Ae^{-\sigma A}(u(t)-u(\tau))|\,\mathrm{d}\tau\,\mathrm{d}t\Big)^p\,\mathrm{d}\sigma.$$

Recall that

$$|Ae^{-\sigma A}x|_X \lesssim \sigma^{-1+\theta}|A^\theta x|_X \qquad (4.7)$$

holds for all $\theta \in (0, 1)$ and $x \in X$. Employing Hölder's inequality, (4.7), (2.6), Lemma 2.2 and Proposition 2.10, we estimate the first summand in (4.6) by

$$\int_0^\infty \sigma^{-p(2s+\alpha)}\Big(\int_0^\sigma \tau^{1-\mu}|Ae^{-\sigma A}u(\tau)|\,\mathrm{d}\tau\Big)^p\,\mathrm{d}\sigma$$

$$\leq \int_0^\infty \int_0^\sigma \tau^{p(1-\mu)}|Ae^{-\sigma A}u(\tau)|^p \sigma^{p-1}\sigma^{-p(2s+\alpha)}\,\mathrm{d}\tau\,\mathrm{d}\sigma$$

$$\lesssim \int_0^\infty \int_0^\sigma \tau^{p(1-\mu)}|A^{s+\alpha}u(\tau)|^p \sigma^{-(1+ps)}\,\mathrm{d}\tau\,\mathrm{d}\sigma$$

$$\lesssim \int_0^\infty \Big(\int_0^\sigma \tau^{p(1-\mu)}|(u(\sigma)-u(\tau))|_{D(A^{s+\alpha})}^p (\sigma-\tau)^{-(1+ps)}\,\mathrm{d}\tau + \sigma^{p(1-\mu-s)}|u(\sigma)|_{D(A^{s+\alpha})}^p\Big)\,\mathrm{d}\sigma$$

$$\lesssim |u|_{W_{p,\mu}^s(\mathbb{R}_+;D(A^{s+\alpha}))}^p.$$

We further use (4.7), Hardy's inequality (2.3), Hölder's inequality and (2.6) to control the second summand in (4.6) by

$$\int_0^\infty \Big(\sigma^{2-\mu-(2s+\alpha)}\int_0^\sigma t^{-(3-\mu)}\int_0^t \tau^{1-\mu}|Ae^{-\sigma A}(u(t)-u(\tau))|\,\mathrm{d}\tau\,\mathrm{d}t\Big)^p\,\mathrm{d}\sigma$$

$$\lesssim \int_0^\infty \Big(\sigma^{-(s+\alpha-(1-\mu+1/p))}\int_0^\sigma t^{-1}\Big(t^{-(2-\mu)}\int_0^t \tau^{1-\mu}|u(t)-u(\tau)|_{D(A^s)}\,\mathrm{d}\tau\Big)\,\mathrm{d}t\Big)^p\sigma^{-1}\,\mathrm{d}\sigma$$

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$$\lesssim \int_0^\infty \sigma^{-p(s+\alpha-(1-\mu+1/p))}\sigma^{-p(2-\mu)}\Big(\int_0^\sigma \tau^{1-\mu}|u(\sigma)-u(\tau)|_{D(A^s)}\,\mathrm{d}\tau\Big)^p\sigma^{-1}\,\mathrm{d}\sigma$$

$$\leq \int_0^\infty \int_0^\sigma \tau^{p(1-\mu)}|u(\sigma)-u(\tau)|_{D(A^s)}^p\sigma^{-(1+p(s+\alpha))}\,\mathrm{d}\tau\,\mathrm{d}\sigma \leq [u]_{W_{p,\mu}^{s+\alpha}(\mathbb{R}_+;D(A^s))}^p.$$

Thus (4.3) holds. ∎

From the above lemma we deduce a general trace theorem in the time variable for the weighted anisotropic spaces.

Theorem 4.2. *Let E be a Banach space of class $\mathcal{HT}$, let $J = (0,T)$ be finite or infinite, and let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, or $\Omega \in \{\mathbb{R}^n, \mathbb{R}_+^n\}$. Assume that $r \geq 0$, $\beta > 0$, $0 < 1/p < \mu \leq 1$ and that $k \in \mathbb{N}_0$, $s \geq 0$ and $\alpha \in (0,2)$ satisfy*

$$s < k+1-\mu+1/p < s+\alpha.$$

Set $H_{p,\mu}^s(W_p^r) := H_{p,\mu}^s(J; W_p^r(\Omega; E))$, and analogously for the other anisotropic spaces. The order of differentiability of each occurring $W_{p,\mu}$- and W_p-spaces is required to be noninteger. Then each of the spaces

$$H_{p,\mu}^{s+\alpha}(W_p^r) \cap H_{p,\mu}^s(W_p^{r+\beta}), \quad W_{p,\mu}^{s+\alpha}(H_p^r) \cap W_{p,\mu}^s(H_p^{r+\beta}), \quad W_{p,\mu}^{s+\alpha}(H_p^r) \cap H_{p,\mu}^s(W_p^{r+\beta}), \quad (4.8)$$

is continuously embedded into

$$BUC^k\big(\overline{J}, B_{p,p}^{r+\beta(1+(s-(k+1-\mu+1/p))/\alpha)}(\Omega; E)\big). \tag{4.9}$$

Moreover, for $\alpha \leq 1$ it holds

$$W_{p,\mu}^\alpha(W_p^r) \cap L_{p,\mu}(W_p^{r+\beta}) \hookrightarrow BUC\big(\overline{J}, B_{p,p}^{r+\beta(1-(1-\mu+1/p)/\alpha)}(\Omega; E)\big), \tag{4.10}$$

$$W_{p,\mu}^1(W_p^r) \cap W_{p,\mu}^s(W_p^{r+\beta}) \hookrightarrow BUC\big(\overline{J}, B_{p,p}^{r+\beta(\mu-1/p)/(1-s)}(\Omega; E)\big). \tag{4.11}$$

All these embeddings remain true if one replaces Ω by its boundary $\partial\Omega$ or the $H_{p,\mu}$ and $W_{p,\mu}$-spaces by $_0H_{p,\mu}$ and $_0W_{p,\mu}$-spaces, respectively. In the latter case, the embedding constants are independent of T if $s+\alpha \leq 2$.

Proof. (I) We first indicate how to reduce the assertions to some basic cases. Again we may assume that $J \times \Omega = \mathbb{R}_+ \times \mathbb{R}^n$. The left translations form a contractive C_0-semigroup on the five spaces under consideration, due to Lemma 2.6. Hence, one can see as in Proposition III.1.4.2 of [2] that the asserted embeddings hold if we can show that the temporal trace operator $\mathrm{tr}_0 u = u|_{t=0}$ maps these function spaces continuously into

$$Y := B_{p,p}^{r+\beta(1+(s-(k+1-\mu+1/p))/\alpha)}(\mathbb{R}^n; E),$$

where one has to set $s = k = 0$ for (4.10) as well as $k = 0$ and $\alpha = 1-s$ for (4.11). We may assume that

$$s > k-\mu+1/p.$$

In fact, if this not the case, we take $\sigma \in (0,1)$ such that $s+\sigma\alpha > k-\mu+1/p$. Proposition 3.2 then allows to embed, say, $Z := W_{p,\mu}^{s+\alpha}(H_p^r) \cap H_{p,\mu}^s(W_p^{r+\beta})$ into $W_{p,\mu}^{s+\sigma\alpha}(H_p^{r+(1-\sigma)\beta})$. For the space $Z_1 := W_{p,\mu}^{s+\alpha}(H_p^r) \cap W_{p,\mu}^{s+\sigma\alpha}(H_p^{r+(1-\sigma)\beta})$ we then have the embedding into (4.9) proved below, which is precisely the assertion for Z.

Moreover, we only have to consider the case $k = 0$ for the spaces in (4.8). In fact, if $s \geq k$ we can simply apply ∂_t^k to these spaces and then use the case $k = 0$. If $s \in (k-\alpha, k)$, then we

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first use Proposition 3.2 and embed, say, $Z = W^{s+\alpha}_{p,\mu}(H^r_p) \cap H^s_{p,\mu}(W^{r+\beta}_p)$ into $W^{k+\varepsilon}_{p,\mu}(H^{r+(1-\sigma)\beta}_p)$ with $\sigma = (k+\varepsilon-s)/\alpha$, for some $\varepsilon \in (0, s+\alpha-k)$. Now, ∂^k_t maps Z into $W^{s+\alpha-k}_{p,\mu}(H^r_p) \cap W^{\varepsilon}_{p,\mu}(H^{r+(1-\sigma)\beta}_p)$ and the trace result for $k=0$ will imply the assertion. Similarly, if we have $s+\alpha \in [1, 2-\mu+1/p]$, we can embed, say, Z into $Z_2 := W^{s+\alpha}_{p,\mu}(H^r_p) \cap W^{s+\gamma}_{p,\mu}(H^{r+(1-\sigma)\beta}_p)$, where $\sigma = \gamma/\alpha$ and $s+\gamma \in (s,1)$. The trace result for Z_2 will again imply the assertion for Z.

Finally, Proposition 3.2 yields the embeddings

$$H^{s+\alpha}_{p,\mu}(W^r_p) \cap H^s_{p,\mu}(W^{r+\beta}_p) \hookrightarrow W^{s+(1-\varepsilon)\alpha}_{p,\mu}(H^{r+\varepsilon\beta}_p) \cap W^{s+\varepsilon\alpha}_{p,\mu}(H^{r+(1-\varepsilon)\beta}_p),$$

$$W^{s+\alpha}_{p,\mu}(H^r_p) \cap H^s_{p,\mu}(W^{r+\beta}_p) \hookrightarrow W^{s+(1-\varepsilon)\alpha}_{p,\mu}(H^{r+\varepsilon\beta}_p) \cap W^{s+\varepsilon\alpha}_{p,\mu}(H^{r+(1-\varepsilon)\beta}_p)$$

where $\varepsilon > 0$ is chosen such that the exponents of the W-spaces are noninteger. If we can show the assertion for the traces on the right hand side, then we also obtain the asserted embeddings for the spaces on left hand side. So besides (4.10) and (4.11), it remains to consider the space

$$W^{s+\alpha}_{p,\mu}(H^r_p) \cap W^s_{p,\mu}(H^{r+\beta}_p) \qquad \text{for} \;\; k=0 \;\; \text{and} \;\; s+\alpha < 1. \tag{4.12}$$

(II) We first show the trace result for the case (4.12). To that purpose, we apply (4.3) with

$$X = H^{r-s\beta/\alpha}_p, \qquad A = (1-\Delta_n)^{\beta/2\alpha}, \qquad D(A) = H^{r+(1-s)\beta/\alpha}_p,$$

such that $D(A^s) = H^r_p$ and $D(A^{s+\alpha}) = H^{r+\beta}_p$. (Observe that $r - s\beta/\alpha < 0$ is possible.) Hence tr_0 is continuous

$$W^{s+\alpha}_{p,\mu}(H^r_p) \cap W^s_{p,\mu}(H^{r+\beta}_p) \to D_A\big(2s+\alpha-(1-\mu+1/p),p\big) = B^{r+\beta(1+(s-(1-\mu+1/p))/\alpha)}_{p,p},$$

as asserted. To obtain (4.10), we use (4.4) applied to

$$X = B^r_{p,p}, \qquad A = (1-\Delta_n)^{\beta/2\alpha}, \qquad D(A) = B^{r+\beta/\alpha}_{p,p},$$

giving $D_A(\alpha-(1-\mu+1/p),p) = B^{r+\beta(1-(1-\mu+1/p)/\alpha)}_{p,p}$. Finally, we deduce (4.11) from (4.5) with

$$X = B^{r-s\beta/(1-s)}_{p,p}, \qquad A = (1-\Delta_n)^{\beta/2(1-s)}, \qquad D(A) = W^{r+\beta}_p,$$

so that $D_A(1+s-(1-\mu+1/p),p) = B^{r+\beta(\mu-1/p)/(1-s)}_{p,p}$ in this case. ∎

Arguing as in the proof of Theorem 4.5 in [8], one should be able to show that the temporal trace is surjective for all of the spaces under consideration in the Theorem 4.2. At this point we only consider a right-inverse for an important special case related to maximal $L_{p,\mu}$-regularity.

Lemma 4.3. *Let E be a Banach space of class $\mathcal{HT}$, $p \in (1,\infty)$, $\mu \in (1/p, 1]$, $m \in \mathbb{N}$, and let $\Omega \subset \mathbb{R}^n$ be a domain with compact smooth boundary $\partial\Omega$, or $\Omega \in \{\mathbb{R}^n, \mathbb{R}^n_+\}$. Then there exists a continuous right-inverse*

$$S : B^{2m(\mu-1/p)}_{p,p}(\Omega; E) \to W^1_{p,\mu}\big(\mathbb{R}_+; L_p(\Omega; E)\big) \cap L_{p,\mu}\big(\mathbb{R}_+; W^{2m}_p(\Omega; E)\big)$$

of tr_0 which is given by

$$Su_0(t) = \mathcal{R}_\Omega e^{-t(1-\Delta_n)^m} \mathcal{E}_\Omega u_0, \qquad t > 0.$$

Here, $\mathcal{E}_\Omega$ is the extension to $\mathbb{R}^n$ from (3.2) and $\mathcal{R}_\Omega$ denotes the restriction from $\mathbb{R}^n$ to Ω.

Proof. The operator $A = (1 - \Delta_n)^m$ with domain $W_p^{2m}(\mathbb{R}^n; E)$ generates an exponentially stable analytic C_0-semigroup on $X = L_p(\mathbb{R}^n; E)$. As in the proof of Lemma 3.1, we obtain $D(A^\beta) = H_p^{2m\beta}(\mathbb{R}^n; E)$ for $\beta \in (0,1)$ so that $D_A(\mu - 1/p, p) = B_{p,p}^{2m(\mu-1/p)}(\mathbb{R}^n; E)$ by reiteration. Theorem 1.14.5 of [31] further implies that

$$|e^{-\cdot A}x|_{W_{p,\mu}^1(\mathbb{R}_+; X) \cap L_{p,\mu}(\mathbb{R}_+; D(A))} \lesssim |x|_{D_A(\mu-1/p,p)} \qquad \text{for all} \ \ x \in D_A(\mu - 1/p, p).$$

Using also (3.2), the assertion now follows. ∎

We now specialize to weighted anisotropic spaces of the form

$$H_{p,\mu}^{s,2ms}(J \times \Omega; E) := H_{p,\mu}^s(J; L_p(\Omega; E)) \cap L_{p,\mu}(J; H_p^{2ms}(\Omega; E)), \qquad (4.13)$$

where $m \in \mathbb{N}$ and $s \in (0,1]$ satisfy $2ms \in \mathbb{N}$, and to the corresponding spaces where H is replaced by W. We are interested in the behavior of the outer normal derivative and more general boundary operators on the maximal $L_{p,\mu}$-regularity class. In view of Lemma 3.4, we thus have to investigate the properties of the spatial trace tr_Ω, i.e.,

$$\mathrm{tr}_\Omega u := u|_{\partial\Omega},$$

on spaces like (4.13). We first give a heuristic derivation of the range space of tr_Ω.

It is known that tr_Ω, which is originally only defined on $C_c^\infty(\mathbb{R}^n; E)$, extends uniquely to a continuous map

$$H_p^{2ms}(\Omega; E) \to W_p^{2ms-1/p}(\partial\Omega; E). \qquad (4.14)$$

This can be seen as in Theorems 2.9.3 and 4.7.1 of [31] for the scalar-valued case. Applied pointwise almost everywhere in time, tr_Ω then extends to a continuous map

$$L_{p,\mu}(J; H_p^{2ms}(\Omega; E)) \to L_{p,\mu}(J; W_p^{2ms-1/p}(\partial\Omega; E)).$$

Observe that Proposition 3.2 yields $H_{p,\mu}^{s,2ms}(J \times \Omega; E) \hookrightarrow H_{p,\mu}^{s-1/2mp}(J; H_p^{1/p}(\Omega; E))$. Although tr_Ω is not bounded from $H_p^{1/p}(\Omega; E)$ to $L_p(\partial\Omega; E)$, this suggests that tr_Ω maps $H_{p,\mu}^{s,2ms}(J \times \Omega; E)$ also in a space like $H_{p,\mu}^{s-1/2mp}(J; L_p(\partial\Omega; E))$.

To give a rigorous proof of the correct result, let us first assume that

$$J \times \Omega = \mathbb{R}_+ \times \mathbb{R}_+^n,$$

and describe an alternative representation of $\mathrm{tr}_{\mathbb{R}_+^n}$. In the sequel we write

$$x = (x', y) \in \mathbb{R}_+^n, \qquad x' \in \mathbb{R}^{n-1}, \quad y \in \mathbb{R}_+.$$

Considering a function $u = u(t, x', y)$ on $\mathbb{R}_+ \times \mathbb{R}_+^n$ as a function of $y \in \mathbb{R}_+$ with values in the functions of $(t, x') \in \mathbb{R}_+ \times \mathbb{R}^{n-1}$, Fubini's theorem yields the embedding

$$\iota : L_{p,\mu}(\mathbb{R}_+; H_p^{2ms}(\mathbb{R}_+^n; E)) \hookrightarrow H_p^{2ms}(\mathbb{R}_+; L_{p,\mu}(\mathbb{R}_+; L_p(\mathbb{R}^{n-1}; E))).$$

Since $2ms \geq 1$, the trace $\mathrm{tr}_0 := \mathrm{tr}_{y=0}$ thus acts on $L_{p,\mu}(\mathbb{R}_+; H_p^{2ms}(\mathbb{R}_+^n; E))$ via $\mathrm{tr}_0 \circ \iota$ and maps this space continuously into $L_{p,\mu}(\mathbb{R}_+; L_p(\mathbb{R}^{n-1}; E))$. In a straightforward way, one can check that the set $\mathrm{Step}(\mathbb{R}_+; C_c^\infty(\mathbb{R}^n; E))$, consisting of step functions of the form

$$\phi = \sum_{i=1}^l \alpha_i(\cdot)\phi_i, \qquad \alpha_i \in C_c(\mathbb{R}_+), \quad \phi_i \in C_c^\infty(\mathbb{R}^n; E), \quad l \in \mathbb{N},$$

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is dense in $L_{p,\mu}\big(J; H_p^{2m}(\mathbb{R}_+^n; E)\big)$, see Lemma 1.3.10 of [24]. For $\phi \in \mathrm{Step}\big(\mathbb{R}_+; C_c^\infty(\mathbb{R}^n; E)\big)$ it clearly holds $\mathrm{tr}_{\mathbb{R}_+^n} \phi = (\mathrm{tr}_0 \circ \iota)\phi$. We therefore obtain that

$$\mathrm{tr}_{\mathbb{R}_+^n} = \mathrm{tr}_0 \circ \iota \qquad \text{on } L_{p,\mu}\big(\mathbb{R}_+; H_p^{2ms}(\mathbb{R}_+^n; E)\big). \tag{4.15}$$

This representation allows to prove the temporal regularity for spatial traces of functions in $H_{p,\mu}^{s,2ms}\big(\mathbb{R}_+ \times \mathbb{R}_+^n; E\big)$ as suggested above. Our proof is inspired by Lemma 3.5 of [6].

Lemma 4.4. *Let E be a Banach space of class $\mathcal{HT}$, $0 < 1/p < \mu \leq 1$, and let $m \in \mathbb{N}$ and $s \in (0,1]$ satisfy $2ms \in \mathbb{N}$. Then the trace $\mathrm{tr}_{\mathbb{R}_+^n}$ maps continuously*

$$H_{p,\mu}^{s,2ms}(\mathbb{R}_+ \times \mathbb{R}_+^n; E) \to W_{p,\mu}^{s-1/(2mp),2ms-1/p}(\mathbb{R}_+ \times \mathbb{R}^{n-1}; E).$$

It is further surjective and has a continuous right-inverse.

Proof. Throughout this proof we set $\mathbb{X} := L_{p,\mu}\big(\mathbb{R}_+; L_p(\mathbb{R}^{n-1}; E)\big)$.

   (I) Considering a function in $L_{p,\mu}\big(\mathbb{R}_+; H_p^{2ms}(\mathbb{R}_+^n; E)\big)$ as a function of $y \in \mathbb{R}_+$ taking values in the functions of $(t, x') \in \mathbb{R}_+ \times \mathbb{R}^{n-1}$, we obtain from Fubini's theorem that

$$L_{p,\mu}\big(\mathbb{R}_+; H_p^{2ms}(\mathbb{R}_+^n; E)\big) \hookrightarrow H_p^{2ms}(\mathbb{R}_+; \mathbb{X}).$$

Moreover, it follows from $H_p^{2ms}(\mathbb{R}_+^n; E) \hookrightarrow L_p\big(\mathbb{R}_+; H_p^{2ms}(\mathbb{R}^{n-1}; E)\big)$ that

$$L_{p,\mu}\big(\mathbb{R}_+; H_p^{2ms}(\mathbb{R}_+^n; E)\big) \hookrightarrow L_p\big(\mathbb{R}_+; L_{p,\mu}(\mathbb{R}_+; H_p^{2ms}(\mathbb{R}^{n-1}; E))\big).$$

Fubini's theorem and interpolation further yield

$$H_{p,\mu}^s\big(\mathbb{R}_+; L_p(\mathbb{R}_+^n; E)\big) \hookrightarrow L_p\big(\mathbb{R}_+; H_{p,\mu}^s(\mathbb{R}_+; L_p(\mathbb{R}^{n-1}; E))\big).$$

By Lemma 3.1, the operator $L = 1 - \partial_t + (-\Delta_{n-1})^m$ with domain $D(L) = H_{p,\mu}^{1,2m}(\mathbb{R}_+ \times \mathbb{R}_+^n; E)$ is invertible on $\mathbb{X}$ and admits bounded imaginary powers with power angle not exceeding $\pi/2$. Hence for $\tau \in (0,1]$ its power L^τ has bounded imaginary powers with angle not larger than $\tau\pi/2$, and it holds

$$D(L^\tau) = H_{p,\mu}^\tau\big(\mathbb{R}_+; L_p(\mathbb{R}^{n-1}; E)\big) \cap L_{p,\mu}\big(\mathbb{R}_+; H_p^{2m\tau}(\mathbb{R}^{n-1}; E)\big), \tag{4.16}$$

again by Lemma 3.1. Therefore we obtain the embedding

$$\widetilde{\iota} : H_{p,\mu}^{s,2ms}(\mathbb{R}_+ \times \mathbb{R}_+^n; E) \hookrightarrow H_p^{2ms}(\mathbb{R}_+; \mathbb{X}) \cap L_p(\mathbb{R}_+; D(L^s)),$$

and equation (4.15) implies that $\mathrm{tr}_{\mathbb{R}_+^n} = \mathrm{tr}_0 \circ \widetilde{\iota}$.

   (II) We now claim that the space $H_p^{2ms}(\mathbb{R}; \mathbb{X}) \cap L_p(\mathbb{R}; D(L^s))$ embeds continuously into $H_p^1(\mathbb{R}; D(L^{s-1/2m}))$. To see this, we consider the realization of the operators $A = (1 - \partial_y^2)^{sm}$ and $B = L^s$ on $L_p(\mathbb{R}; \mathbb{X})$ with domains

$$D(A) = H_p^{2ms}(\mathbb{R}; \mathbb{X}) \qquad \text{and} \qquad D(B) = L_p(\mathbb{R}; D(L^s)),$$

respectively. These operators are invertible and possess bounded imaginary powers with power angles equal to zero and $s\pi/2$, respectively, see e.g. Lemma 3.1. Moreover, A and B are resolvent commuting on step functions in $L_p(\mathbb{R}; \mathbb{X})$, which carries over to $L_p(\mathbb{R}; \mathbb{X})$ by density. Thus Proposition 1.1 shows that the operator $A + B$ is invertible on $L_p(\mathbb{R}; \mathbb{X})$ with domain

$$D(A + B) = H_p^{2ms}(\mathbb{R}; \mathbb{X}) \cap L_p(\mathbb{R}; D(L^s)).$$

Since A, B and $A + B$ are invertible, $|A^{1/2ms}B^{1-1/2ms} \cdot |_{L_p(\mathbb{R};\mathbb{X})}$ and $|(A + B) \cdot |_{L_p(\mathbb{R};\mathbb{X})}$ are equivalent norms on $D(A^{1/2ms}B^{1-1/2ms}) = H_p^1\big(\mathbb{R}; D(L^{s-1/2m})\big)$ and $D(A + B)$, respectively. Now, Proposition 1.1 implies the asserted embedding.

(III) It follows from extension and restriction that also

$$H_p^{2ms}(\mathbb{R}_+;\mathbb{X}) \cap L_p(\mathbb{R}_+; D(L^s)) \hookrightarrow H_p^1(\mathbb{R}_+; D(L^{s-1/2m})),$$

which implies that $L^{s-1/2m}$ maps continuously

$$H_p^{2ms}(\mathbb{R}_+;\mathbb{X}) \cap L_p(\mathbb{R}_+; D(L^s)) \to H_p^1(\mathbb{R}_+;\mathbb{X}) \cap L_p(\mathbb{R}_+; D(L^{1/2m})).$$

Note that $L^{1/2m}$ is sectorial of angle at most $\pi/4m < \pi/2$, and thus $-L^{1/2m}$ generates an exponentially stable analytic C_0-semigroup on $\mathbb{X}$. Due to Theorem III.4.10.2 in [2] we have

$$H_p^1(\mathbb{R}_+;\mathbb{X}) \cap L_p(\mathbb{R}_+; D(L^{1/2m})) \hookrightarrow BUC\big([0,\infty); D_{L^{1/2m}}(1 - 1/p, p)\big),$$

and from the reiteration theorem we infer

$$D_{L^{1/2m}}(1 - 1/p, p) = D_L\big((1 - 1/p)/2m, p\big).$$

(IV) We now write

$$\mathrm{tr}_{\mathbb{R}_+^n} = \mathrm{tr}_0\, L^{-(s-1/2m)}\, L^{s-1/2m}\widetilde{\iota},$$

where $L^{s-1/2m}$ and its inverse are applied pointwise. By the above considerations, the operator $L^{s-1/2m}\widetilde{\iota}$ is continuous

$$H_{p,\mu}^{s,2ms}(\mathbb{R}_+ \times \mathbb{R}_+^n; E) \to BUC\big([0,\infty); D_L((1 - 1/p)/2m, p)\big).$$

Clearly, tr_0 and $L^{-(s-1/2m)}$ commute on $BUC\big([0,\infty); D_L((1 - 1/p)/2m, p)\big)$. By reiteration $L^{-(s-1/2m)}$ maps $D_L((1 - 1/p)/2m, p)$ continuously into

$$D_L(s - 1/2mp, p) = W_{p,\mu}^{s-1/(2mp),2ms-1/p}(\mathbb{R}_+ \times \mathbb{R}^{n-1}; E),$$

where we use Lemma 3.1. Hence, $\mathrm{tr}_{\mathbb{R}_+^n}$ maps as asserted.

(V) A continuous right-inverse of $\mathrm{tr}_{\mathbb{R}_+^n}$ is defined by

$$e^{-yL^{1/2m}}g(t, x'), \qquad t > 0, \quad (x', y) \in \mathbb{R}_+^n.$$

for $g \in W_{p,\mu}^{s-1/(2mp),2ms-1/p}(\mathbb{R}_+ \times \mathbb{R}^{n-1}; E)$. To see this, observe that

$$W_{p,\mu}^{s-1/(2mp),2ms-1/p}(\mathbb{R}_+ \times \mathbb{R}^{n-1}; E) = D_{L^{1/2m}}(2ms - 1/p, p)$$

by reiteration. For $f \in D_{L^{1/2m}}(2ms-1/p, p)$ and $k = 2ms-1$, we have that $(-\partial_y)^k e^{-yL^{1/2m}}f = e^{-yL^{1/2m}}L^{k/2m}f$ and $L^{k/2m}f \in D_{L^{1/2m}}(1 - 1/p, p)$. Using e.g. Theorem 1.14.5 of [31], we then derive that $e^{-\cdot L^{1/2m}}$ maps $D_{L^{1/2m}}(2ms - 1/p, p)$ continuously into $\mathbb{Y} := H_p^{2ms}(\mathbb{R}_+;\mathbb{X}) \cap L_p(\mathbb{R}_+; D(L^s))$. Therefore (4.16) yields

$$\mathbb{Y} \hookrightarrow L_p\big(\mathbb{R}_+; H_{p,\mu}^s(\mathbb{R}_+; L_p(\mathbb{R}^{n-1}; E))\big) \hookrightarrow H_{p,\mu}^s\big(\mathbb{R}_+; L_p(\mathbb{R}_+^n; E)\big).$$

Moreover,

$$\begin{aligned}
\mathbb{Y} &\hookrightarrow H_p^{2ms}\big(\mathbb{R}_+; L_{p,\mu}(\mathbb{R}_+; L_p(\mathbb{R}^{n-1}; E))\big) \cap L_p\big(\mathbb{R}_+, L_{p,\mu}(\mathbb{R}_+, H_p^{2ms}(\mathbb{R}^{n-1}; E))\big) \\
&\hookrightarrow L_{p,\mu}\big(\mathbb{R}_+; H_p^{2ms}(\mathbb{R}_+^n; E)\big),
\end{aligned}$$

where the latter embedding follows again from Fubini's theorem. This shows the asserted continuity of the right-inverse. $\blacksquare$

The mapping properties of the spatial trace on a domain are now obtained from the half-space case via localization.

Theorem 4.5. *Let E be a Banach space of class $\mathcal{HT}$, $J = (0, T)$ be a finite or infinite, $0 < 1/p < \mu \leq 1$, and let $m \in \mathbb{N}$ and $s \in (0, 1]$ satisfy $2ms \in \mathbb{N}$. Assume that $\Omega \subset \mathbb{R}^n$ is a domain with compact smooth boundary, or $\Omega \in \{\mathbb{R}^n, \mathbb{R}^n_+\}$. Then the trace tr_Ω is continuous*

$$H^{s,2ms}_{p,\mu}(J \times \Omega; E) \to W^{s-1/2mp, 2ms-1/p}_{p,\mu}(J \times \partial\Omega; E).$$

It is further surjective and has a continuous right-inverse. The operator norm of tr_Ω on $_0H^{s,2ms}_{p,\mu}(J \times \Omega; E)$ is independent of the length of J.

Proof. Using the extension operators $\mathcal{E}_J$ and $\mathcal{E}^0_J$ from Lemma 2.5, it suffices to consider the case $J = \mathbb{R}_+$. We describe $\partial\Omega$ by a finite number of charts (U_i, φ_i) and a partition of unity $\{\psi_i\}$ subordinate to the cover $\bigcup_i U_i$. We further denote by Φ_i the push-forward with respect to φ_i; i.e., $\Phi_i u = u \circ \varphi_i^{-1}$. Then for $\phi \in \mathrm{Step}\big(\mathbb{R}_+; C_c^\infty(\mathbb{R}^n; E)\big)$ it holds

$$\mathrm{tr}_\Omega \phi = \sum_i \Phi_i^{-1}\big(\mathrm{tr}_{\mathbb{R}^n_+} \Phi_i(\psi_i \phi)\big) \qquad \text{on } \partial\Omega. \tag{4.17}$$

By restriction to $\Omega \cap U_i$ and trivial extension from $\mathbb{R}^n_+ \cap \varphi_i(U_i)$ to $\mathbb{R}^n_+$, for each i we obtain that $\Phi_i(\psi_i \cdot)$ maps continuously

$$H^{s,2ms}_{p,\mu}(\mathbb{R}_+ \times \Omega; E) \to H^{s,2ms}_{p,\mu}(\mathbb{R}_+ \times \mathbb{R}^n_+; E).$$

Applying Lemma 4.4 and restricting back to $\mathbb{R}^n_+ \cap \varphi_i(U_i)$ yields that $\Phi_i^{-1}\mathrm{tr}_{\mathbb{R}^n_+}$ maps the latter space continuously into $W^{s-1/2mp, 2ms-1/p}_{p,\mu}(\mathbb{R}_+ \times \partial\Omega; E)$. Thus the map $\sum_i \Phi_i^{-1}\big(\mathrm{tr}_{\mathbb{R}^n_+} \Phi_i(\psi_i \cdot)\big)$ is continuous

$$H^{s,2ms}_{p,\mu}(\mathbb{R}_+ \times \Omega; E) \to W^{s-1/2mp, 2ms-1/p}_{p,\mu}(\mathbb{R}_+ \times \partial\Omega; E).$$

Since $\mathrm{Step}\big(\mathbb{R}_+; C_c^\infty(\mathbb{R}^n; E)\big)$ is dense in $L_{p,\mu}\big(\mathbb{R}_+; H_p^{2ms}(\Omega; E)\big)$, the representation (4.17) holds for all elements of this space, and in particular for all functions from $H^{s,2ms}_{p,\mu}(J \times \Omega; E)$. This shows that tr_Ω is continuous as asserted. Similiar localization arguments also reduce the existence of a continuous right-inverse to the half-space case treated in Lemma 4.4. ∎

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