Lecture Notes
Evolution Equations

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These lecture notes are based on my course from winter semester 2023/24, though there are minor corrections and improvements as well as small changes in the numbering of equations. Typically, the proofs and calculations in the notes are a bit shorter than those given in class. The drawings and many additional oral remarks from the lectures are omitted here. On the other hand, the notes contain very few proofs (of peripheral statements) and a short additional chapter not presented during the course. Occasionally I use the notation and definitions of my lecture notes Analysis 1–4 and Functional Analysis without further notice.

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CHAPTER 1

Strongly continuous semigroups and their generators

Throughout, X and Y are non-zero Banach spaces over the field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, where we mostly write $\|\cdot\|$ instead of $\|\cdot\|_X$ etc. for their norms. The space of all bounded linear maps $T : X \to Y$ is denoted by $\mathcal{B}(X, Y)$ and endowed with the operator norm $\|T\|_{\mathcal{B}(X,Y)} = \|T\| = \sup_{x \neq 0} \|Tx\|/\|x\|$. We abbreviate $\mathcal{B}(X) = \mathcal{B}(X, X)$ and $X^* = \mathcal{B}(X, \mathbb{F})$, where $x^* \in X^*$ acts as $\langle x, x^* \rangle$ on X. Further, I is the identity map on X. Let $A : \mathrm{D}(A) \to Y$ be linear with domain $\mathrm{D}(A)$ being a linear subspace of X. Then $\mathrm{R}(A) = A\mathrm{D}(A)$ denotes the range of A and $\mathrm{N}(A) = \{x \in \mathrm{D}(A) \,|\, Ax = 0\}$ its kernel. For $\omega \in \mathbb{R}$, we set

$$\mathbb{R}_{\geq 0} = [0, \infty), \quad \mathbb{R}_+ = (0, \infty), \quad \mathbb{R}_{\leq 0} = (-\infty, 0], \quad \mathbb{R}_- = (-\infty, 0),$$
$$\mathbb{F}_\omega = \{\lambda \in \mathbb{F} \,|\, \mathrm{Re}\,\lambda > \omega\}, \quad \mathbb{C}_+ = \mathbb{C}_0, \quad \mathbb{C}_- = \{\lambda \in \mathbb{C} \,|\, \mathrm{Re}\,\lambda < 0\},$$
$$\omega_+ = \max\{\omega, 0\}, \quad \omega_- = \max\{-\omega, 0\}.$$

In this course we study linear evolution equations such as

$$u'(t) = Au(t), \quad t \geq 0, \qquad u(0) = u_0, \tag{EE}$$

on a *state space* X for given linear operators A and initial values $u_0 \in \mathrm{D}(A)$. (For a moment we assume that A is closed and densely defined.) We are looking for the *state* $u(t) \in X$ describing the system governed by A at time $t \geq 0$. A reasonable description of the system requires a unique *solution* u of (EE) that continuously depends on u_0. In this case (EE) is called *wellposed*, cf. Definitions 1.9 and 2.1. We will show in Section 2.1 that wellposedness is equivalent to the fact that A *generates* a C_0*-semigroup* $T(\cdot)$ which yields the solutions via $u(t) = T(t)u_0$. In the next section we will define and investigate these concepts, before we characterize generators in Sections 1.2 and 1.3. In the final section the theory is then applied to operators like the Laplacian. Three intermezzi present basic notions and facts on closed and closable operators, spectral theory, and Sobolev spaces, mostly taken from the lecture notes [**27**].

1.1. Basic concepts and properties

We introduce the fundamental notions of these lectures.

Definition 1.1. *A map* $T(\cdot) : \mathbb{R}_{\geq 0} \to \mathcal{B}(X)$ *is called a* strongly continuous operator semigroup *or just* C_0-semigroup *if it satisfies*

(a) $T(0) = I$ *and* $T(t + s) = T(t)T(s)$ *for all* $t, s \in \mathbb{R}_{\geq 0}$,

(b) *for each* $x \in X$ *the* orbit $T(\cdot)x : \mathbb{R}_{\geq 0} \to X;\ t \mapsto T(t)x$, *is continuous.*

Here, (a) is the semigroup property *and (b) the* strong continuity *of* $T(\cdot)$.

The generator A of $T(\cdot)$ *is given by*

$$\mathrm{D}(A) = \big\{x \in X \,\big|\, \lim_{t\to 0,\, t\in\mathbb{R}_{\geq 0}\setminus\{0\}} \tfrac{1}{t}(T(t)x - x) \;\text{ exists in } X\big\},$$
$$Ax = \lim_{t\to 0,\, t\in\mathbb{R}_{\geq 0}\setminus\{0\}} \tfrac{1}{t}(T(t)x - x) \qquad \text{for } \; x \in \mathrm{D}(A).$$

Replacing $\mathbb{R}_{\geq 0}$ *by* $\mathbb{R}$, *one obtains the concept of a* C_0-group *with* generator A.

Observe that the domain $\mathrm{D}(A)$ of the generator is defined in a 'maximal' way, in the sense that it contains all vectors whose orbits are differentiable at $t = 0$. In view of the introductory remarks, typically the generator is the given object and $T(\cdot)$ describes the unknown solution. We will first study basic properties of C_0-semigroups, starting with simple observations.

Remark 1.2. a) Let A generate a C_0-semigroup or a C_0-group. Then its domain $\mathrm{D}(A)$ is a linear subspace and A is a linear map.

b) Let $(T(t))_{t\in\mathbb{R}}$ be a C_0-group with generator A. Then its restriction $(T(t))_{t\geq 0}$ is a C_0-semigroup whose generator extends[1] A. (Actually these two operators coincide by Theorem 1.29.)

c) Let $T(\cdot) : \mathbb{R}_{\geq 0} \to \mathcal{B}(X)$ be a semigroup. We then have

$$T(t)T(s) = T(t+s) = T(s+t) = T(s)T(t),$$
$$T(nt) = T\big(\textstyle\sum_{j=1}^{n} t\big) = \prod_{j=1}^{n} T(t) = T(t)^n$$

for all $t, s \geq 0$ and $n \in \mathbb{N}$.

Let $T(\cdot) : \mathbb{R} \to \mathcal{B}(X)$ be a group. Then the above properties are valid for all $s, t \in \mathbb{R}$, and hence $T(t)T(-t) = T(0) = I = T(-t)T(t)$. There thus exists the inverse $T(t)^{-1} = T(-t)$ for every $t \in \mathbb{R}$. ♢

We next construct a C_0-group with a bounded generator, which is actually differentiable in operator norm. Conversely, an exercise shows that a C_0-semigroup with $T(t) \to I$ in $\mathcal{B}(X)$ as $t \to 0^+$ must have a bounded generator.

Example 1.3. Let $A \in \mathcal{B}(X)$ and $b > 0$. For $t \in \mathbb{F}$ with $|t| \leq b$, the numbers $\big\|\frac{t^n}{n!} A^n\big\| \leq \frac{(b\|A\|)^n}{n!}$ are summable in $n \in \mathbb{N}_0$. As in Lemma 4.23 of [**24**], the operator-valued exponential series

$$T(t) = \mathrm{e}^{tA} := \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n, \qquad t \in \mathbb{F},$$

thus converges in $\mathcal{B}(X)$ uniformly for $|t| \leq b$. In the same way one sees that

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_{n=0}^{N} \frac{t^n}{n!} A^n = \sum_{n=1}^{N} \frac{t^{n-1}}{(n-1)!} A^n = A \sum_{k=0}^{N-1} \frac{t^k}{k!} A^k$$

tends to $A\mathrm{e}^{tA}$ in $\mathcal{B}(X)$ as $N \to \infty$ locally uniformly in $t \in \mathbb{F}$. As in Analysis 1 or 4 one then shows that the map $\mathbb{F} \to \mathcal{B}(X); t \mapsto \mathrm{e}^{tA}$, is continuously differentiable with derivative $A\mathrm{e}^{tA}$. Moreover, $(\mathrm{e}^{tA})_{t\in\mathbb{F}}$ is a group (where one replaces $\mathbb{R}_{\geq 0}$ by $\mathbb{C}$ in Definition 1.1 (a) if $\mathbb{F} = \mathbb{C}$).

The case of a matrix A on $X = \mathbb{C}^m$ was treated in Section 4.5 of [**25**]. ♢

[1] This concept is defined before Lemma 1.23.

In the next lemma the exponential boundedness of a semigroup follows from a mild extra assumption. This assumption is satisfied if $\|T(t)\|$ is uniformly bounded on an interval $[0,b]$ with $b>0$ or if $T(\cdot)$ is strongly continuous. (We need both cases below.)

LEMMA 1.4. *Let $T(\cdot):\mathbb{R}_{\ge 0}\to\mathcal{B}(X)$ satisfy condition (a) in Definition 1.1 as well as $\limsup_{t\to 0}\|T(t)x\|<\infty$ for each $x\in X$. Then there are constants $M\ge 1$ and $\omega\in\mathbb{R}$ such that $\|T(t)\|\le M\mathrm{e}^{\omega t}$ for all $t\ge 0$.*

PROOF. 1) We first claim that there are constants $c\ge 1$ and $t_0>0$ with $\|T(t)\|\le c$ for all $t\in[0,t_0]$. To show this claim, we suppose that there is a null sequence (t_n) in $\mathbb{R}_{\ge 0}$ such that $\lim_{n\to\infty}\|T(t_n)\|=\infty$. The principle of uniform boundedness (Theorem 4.4 in [**24**]) then yields a vector $x\in X$ with $\sup_n\|T(t_n)x\|=\infty$. There thus exists a subsequence satisfying $\|T(t_{n_j})x\|\to\infty$ as $j\to\infty$. This fact contradicts the assumption, and so the claim is true.

2) Let $t\ge 0$. Then there are numbers $n\in\mathbb{N}_0$ and $\tau\in[0,t_0)$ with $t=nt_0+\tau$. Take $\omega=t_0^{-1}\ln\|T(t_0)\|$ if $T(t_0)\neq 0$ and any $\omega<0$ otherwise. Set $M=c\mathrm{e}^{\omega-t_0}$. Using Remark 1.2, we estimate

$$\|T(t)\|=\|T(\tau)T(t_0)^n\|\le c\|T(t_0)\|^n\le c\mathrm{e}^{nt_0\omega}=c\mathrm{e}^{t\omega}\mathrm{e}^{-\tau\omega}\le M\mathrm{e}^{\omega t}. \qquad \square$$

The above considerations lead to the following concept, which is discussed below and will be explored more thoroughly in Section 4.1.

DEFINITION 1.5. *Let $T(\cdot)$ be a C_0-semigroup with generator A. The quantity*

$$\omega_0(T)=\omega_0(A):=\inf\big\{\omega\in\mathbb{R}\,\big|\,\exists\,M_\omega\ge 1\;\forall\,t\ge 0:\|T(t)\|\le M_\omega\mathrm{e}^{\omega t}\big\}\in[-\infty,\infty)$$

is called its (exponential) growth bound. *If $\sup_{t\ge 0}\|T(t)\|<\infty$, then $T(\cdot)$ is* bounded. *(Similarly one defines $\omega_0(f)\in[-\infty,+\infty]$ for any map $f:\mathbb{R}_{\ge 0}\to Y$.)*

REMARK 1.6. Let $T(\cdot)$ be a C_0-semigroup.

a) Lemma 1.4 implies that $\omega_0(T)<\infty$.

b) There are C_0-semigroups with $\omega_0(T)=-\infty$, see Example 1.8.

c) In general the infimum in Definition 1.5 is not a minimum. For instance, let $X=\mathbb{F}^2$ be endowed with the 1-norm $|\cdot|_1$ and $A=\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$. We then have $T(t)=\mathrm{e}^{tA}=\left(\begin{smallmatrix}1&t\\0&1\end{smallmatrix}\right)$ and $\|T(t)\|=1+t$ for $t\ge 0$. As a result, the number

$$M_\varepsilon:=\sup_{t\ge 0}\mathrm{e}^{-\varepsilon t}\|T(t)\|=\sup_{t\ge 0}(1+t)\mathrm{e}^{-\varepsilon t}=\varepsilon^{-1}\mathrm{e}^{\varepsilon-1}$$

tends to infinity as $\varepsilon\to 0^+$, where $\varepsilon\in(0,1]$.

d) Let $X=\mathbb{C}^m$ and $A\in\mathbb{C}^{m\times m}$. Satz 4.22 and Theorem 6.3 of [**25**] imply

$$\omega_0(A)=\mathrm{s}(A):=\max\big\{\operatorname{Re}\lambda_j\,\big|\,\lambda_1,\dots,\lambda_k\text{ are the eigenvalues of }A\big\}.$$

This result can be generalized to bounded A if $\dim X=\infty$, cf. Example 5.4 of [**27**]. Every generator satisfies $\omega_0(A)\ge\mathrm{s}(A)$ by Proposition 1.20. However, the converse inequality is much more important since A is the given object and $T(\cdot)x$ the unknown solution. In Chapter 4 we will discuss this point in detail.

Similarly, a semigroup $(\mathrm{e}^{tA})_{t\ge 0}$ on $X=\mathbb{C}^m$ is bounded if and only if $\mathrm{s}(A)\le 0$ and all eigenvalues of A on $\mathrm{i}\mathbb{R}$ are semi-simple. This indicates that boundedness of C_0-semigroups is a more subtle property. $\Diamond$

The next auxiliary result will often be used to check strong continuity.

Lemma 1.7. *Let $T(\cdot) : \mathbb{R}_{\geq 0} \to \mathcal{B}(X)$ be a map satisfying condition (a) in Definition 1.1. Then the following assertions are equivalent.*

a) $T(\cdot)$ is strongly continuous (and thus a C_0-semigroup).

b) $T(t)x \to x$ in X as $t \to 0^+$ for all $x \in X$.

c) There are numbers $c, t_0 > 0$ and a dense subspace D of X such that $\|T(t)\| \leq c$ and $T(t)x \to x$ in X as $t \to 0^+$ for all $t \in [0, t_0]$ and $x \in D$.

For groups one has analogous equivalences.

Proof. Assertion c) follows from a) because of Lemma 1.4, and b) from c) by Lemma 4.10 in [**24**].

Let statement b) be true. Take $x \in X$ and $t > 0$. For $h > 0$, the semigroup property and b) imply the limit

$$\|T(t+h)x - T(t)x\| = \|T(t)(T(h)x - x)\| \leq \|T(t)\| \, \|T(h)x - x\| \longrightarrow 0$$

as $h \to 0^+$. Let $h \in (-t, 0)$. Lemma 1.4 yields the bound

$$\|T(t+h)\| \leq M e^{\omega(t+h)} \leq M e^{\omega_+ t}$$

for some constants $M \geq 1$ and $\omega \in \mathbb{R}$. We then derive

$$\|T(t+h)x - T(t)x\| \leq \|T(t+h)\| \, \|x - T(-h)x\| \leq M e^{\omega_+ t} \|x - T(-h)x\| \to 0$$

as $h \to 0^-$, so that a) is true. The addendum is shown similarly. ☐

In the above lemma the implication 'c) ⇒ a)' can fail if one omits the boundedness assumption, cf. Exercise I.5.9(4) in [**7**].

We now examine translation semigroups, which are easy to grasp and still illustrate many of the basic features of C_0-semigroups. Another important class of simple examples are multiplication semigroups as discussed in the exercises.

We recall that $\operatorname{supp} f$ is the support of a function $f : M \to Y$ on a metric space M; i.e., the closure in M of the set $\{s \in M \mid f(s) \neq 0\}$,

Example 1.8. a) Let $X = C_0(\mathbb{R}) \coloneqq \{f \in C(\mathbb{R}) \mid f(s) \to 0 \text{ as } |s| \to \infty\}$ be endowed with $\|\cdot\|_\infty$, which is a Banach space by Example 1.14 in [**24**]. Take $f \in X$ and $t, r, s \in \mathbb{R}$. We define the translations

$$(T(t)f)(s) = f(s+t).$$

They shift the graph of f to the left if $t > 0$, since $(T(t)f)(s)$ is equals the value of f at $s + t > s$. Clearly, $T(0) = I$ and $T(t)$ is a linear isometry on X so that $\|T(t)\| = 1$. We further obtain $T(t)T(r) = T(t+r)$ noting

$$\big(T(t)T(r)f\big)(s) = \big(T(r)f\big)(s+t) = f(r+s+t) = \big(T(t+r)f\big)(s).$$

We claim that $C_c(\mathbb{R}) \coloneqq \{f \in C(\mathbb{R}) \mid \operatorname{supp}(f) \text{ is compact}\}$ is dense in $C_0(\mathbb{R})$. Indeed, let $f \in C_0(\mathbb{R})$ and choose cut-off functions $\varphi_n \in C_c(\mathbb{R})$ satisfying $\varphi_n = 1$ on $[-n, n]$ and $0 \leq \varphi_n \leq 1$. Then the maps $\varphi_n f$ belong to $C_c(\mathbb{R})$ and

$$\|f - \varphi_n f\|_\infty \leq \sup_{|s| \geq n} |(1 - \varphi_n(s)) f(s)| \leq \sup_{|s| \geq n} |f(s)|$$

tends to 0 as $n \to \infty$.

Pick $f \in C_c(\mathbb{R})$ and a number $a > 0$ with $\operatorname{supp} f \subseteq [-a, a]$. Let $t \in [-1, 1]$. If $|s| > a + 1$, we have $|s + t| > a$ and thus $f(s + t) = 0$; i.e., $\operatorname{supp} T(t)f$ is contained in $[-a - 1, a + 1]$. It follows

$$\|T(t)f - f\|_\infty \leq \sup_{|s|\leq a+1} |f(s+t) - f(s)| \longrightarrow 0$$

as $t \to 0$, since f is uniformly continuous on $[-a - 1, a + 1]$. Lemma 1.7 then implies that $T(\cdot)$ is a C_0-group.

Similarly, one shows that $T(\cdot)$ is an (isometric) C_0-group on $X = L^p(\mathbb{R})$ with $1 \leq p < \infty$, see Example 4.12 in [**24**].

In contrast to these results, $T(\cdot)$ is not strongly continuous on $X = L^\infty(\mathbb{R})$. Indeed, consider $f = \mathbb{1}_{[0,1]}$ and observe that

$$T(t)f(s) = \mathbb{1}_{[0,1]}(s+t) = \left\{ \begin{matrix} 1, & s + t \in [0, 1] \\ 0, & \text{else} \end{matrix} \right\} = \mathbb{1}_{[-t,1-t]}(s)$$

for $s, t \in \mathbb{R}$. We thus have $\|T(t)f - f\|_\infty = 1$ for every $t \neq 0$.

In addition, $T(\cdot)$ is not continuous as a $\mathcal{B}(X)$-valued function for $X = C_0(\mathbb{R})$ (and neither for $X = L^p(\mathbb{R})$ by Example 4.12 in [**24**]). In fact, take functions $f_n \in C_c(\mathbb{R})$ with $0 \leq f_n \leq 1$, $f_n(n) = 1$, and $\operatorname{supp} f_n \subseteq \big(n - \frac{1}{n}, n + \frac{1}{n}\big)$ for $n \in \mathbb{N}$. We then obtain $\|f_n\|_\infty = 1$ and

$$\|T(\tfrac{1}{n}) - I\| \geq \|T(\tfrac{1}{n})f_n - f_n\|_\infty \geq |f_n(n + \tfrac{1}{n}) - f_n(n)| = 1 \qquad \text{for all } \ n \in \mathbb{N}.$$

b) For an interval that is bounded from above, one has to prescribe the behavior of the left translation at the right boundary point. Here we simply prescribe the value 0. We work on the Banach space $X = C_0([0, 1)) := \{f \in C([0, 1)) \mid \lim_{s\to1} f(s) = 0\}$ with $\|\cdot\|_\infty$, see Example 1.14 in [**24**]. Let $t, r \geq 0$, $f \in X$, and $s \in [0, 1)$. We define

$$(T(t)f)(s) := \begin{cases} f(s+t), & s + t < 1, \\ 0, & s + t \geq 1. \end{cases}$$

Since $f(s + t) \to 0$ as $s \to 1 - t$ if $t < 1$, the function $T(t)f$ belongs to X. Clearly, $T(t)$ is linear on X and $\|T(t)\| \leq 1$. We stress that $T(t) = 0$ whenever $t \geq 1$. (One says that $T(\cdot)$ is *nilpotent*.) As a consequence, $\omega_0(T) = -\infty$ and $T(\cdot)$ cannot be extended a group in view of Remark 1.2. We next compute

$$\begin{aligned} \big(T(t)T(r)f\big)(s) &= \begin{cases} (T(r)f)(s+t), & s < 1 - t, \\ 0, & s \geq 1 - t, \end{cases} \\ &= \begin{cases} f(s+t+r), & s < 1 - t, \quad s + t < 1 - r, \\ 0, & \text{else}, \end{cases} \\ &= (T(t+r)f)(s). \end{aligned}$$

Hence, $T(\cdot)$ is a semigroup.

As in part a) or in Example 1.19 of [**24**], one sees that

$$C_c([0, 1)) := \{f \in C([0, 1)) \mid \exists\, b_f \in (0, 1) : \operatorname{supp} f \subseteq [0, b_f]\}$$

is a dense subspace of X. For $f \in C_c([0,1))$ and $t \in (0, 1-b_f)$ we compute

$$T(t)f(s) - f(s) = \begin{cases} f(s+t) - f(s), & \text{if } s \in [0, 1-t), \\ 0, & \text{if } s \in [1-t, 1) \subseteq [b_f, 1), \end{cases}$$

and deduce $\lim_{t\to 0} \|T(t)f - f\|_\infty = 0$ using the uniform continuity of f. According to Lemma 1.7, $T(\cdot)$ is a C_0-semigroup on X. $\Diamond$

We introduce a solution concept for the problem (EE). Different ones will be discussed in Section 2.2. Let $u : J \to X$, $t \in J$, and $J \subseteq \mathbb{R}$ be an interval. The derivative of u is defined by $u'(t) = \lim_{h\to 0} \frac{1}{h}(u(t+h) - u(t))$, if the limit exists.

Definition 1.9. *Let A be a linear operator on X with domain $\mathrm{D}(A)$ and let $x \in \mathrm{D}(A)$. A function $u : \mathbb{R}_{\geq 0} \to X$* solves *the homogeneous* evolution equation *(or* Cauchy problem*)*

$$u'(t) = Au(t), \quad t \geq 0, \qquad u(0) = x, \tag{1.1}$$

if u belongs to $C^1(\mathbb{R}_{\geq 0}, X)$ and satisfies $u(t) \in \mathrm{D}(A)$ and (1.1) for all $t \geq 0$.

We next show the fundamental regularity properties of C_0-semigroups. Recall that the generator's domain $\mathrm{D}(A)$ was 'maximally' defined as the set of all initial values for which the orbit is differentiable at $t = 0$. We now use the semigroup law to transfer this property to later times. The crucial invariance of the domain under the semigroup then directly follows from its definition.

Proposition 1.10. *Let A generate the C_0-semigroup $T(\cdot)$ and $x \in \mathrm{D}(A)$. Then $T(t)x$ belongs to $\mathrm{D}(A)$, the orbit $T(\cdot)x$ to $C^1(\mathbb{R}_{\geq 0}, X)$, and we have*

$$\tfrac{\mathrm{d}}{\mathrm{d}t} T(t)x = AT(t)(x) = T(t)Ax \qquad \text{for all } \ t \geq 0.$$

Moreover, the function $u = T(\cdot)x$ is the only solution of (1.1).

Proof. 1) Let $t > 0$, $h > 0$, and $x \in \mathrm{D}(A)$. Remark 1.2 and the continuity of $T(t)$ then imply the convergence

$$\tfrac{1}{h}\big(T(h) - I\big)T(t)x = T(t)\tfrac{1}{h}\big(T(h)x - x\big) \ \longrightarrow \ T(t)Ax$$

as $h \to 0$. By Definition 1.1 of the generator, the vector $T(t)x$ thus belongs to $\mathrm{D}(A)$ and satisfies $AT(t)x = T(t)Ax$. Next, let $0 < h < t$. We then compute

$$\tfrac{1}{-h}\big(T(t-h)x - T(t)x\big) = T(t-h)\tfrac{1}{h}\big(T(h)x - x\big) \ \longrightarrow \ T(t)Ax$$

as $h \to 0$, by means of Lemma 1.12 below (with $S(\tau,\sigma) = T(\tau - \sigma)$). Together we have shown that the orbit $u = T(\cdot)x$ has the derivative $AT(\cdot)x$. Since $T(\cdot)Ax$ is continuous, u is contained in $C^1(\mathbb{R}_{\geq 0}, X)$. Summing up, u solves (1.1).

2) Let also v solve (1.1). We show $v = u$ by a standard trick. Take $t > 0$ and set $w(s) = T(t-s)v(s)$ for $s \in [0,t]$. Let $h \in [-s, t-s] \setminus \{0\}$. We write

$$\tfrac{1}{h}(w(s+h) - w(s)) = T(t-s-h)\tfrac{1}{h}(v(s+h) - v(s)) - \tfrac{1}{-h}(T(t-s-h) - T(t-s))v(s).$$

Using $v \in C^1$, Lemma 1.12, $v(s) \in \mathrm{D}(A)$ and the first step, we infer that w is differentiable with derivative

$$w'(s) = T(t-s)v'(s) - T(t-s)Av(s) = 0,$$

where the last equality follows from (1.1) for v. One directly infers that the scalar function $\langle w(\cdot), x^*\rangle$ is differentiable with vanishing derivative for each $x^* \in X^*$. It is thus constant, which leads to the equality

$$\langle T(t)x, x^*\rangle = \langle w(0), x^*\rangle = \langle w(t), x^*\rangle = \langle v(t), x^*\rangle$$

for all $t > 0$. The Hahn–Banach theorem now yields $T(\cdot)x = v$ as asserted, see Corollary 5.10 of [**24**]. □

REMARK 1.11. Let $f \in C_0(\mathbb{R}) \setminus C^1(\mathbb{R})$. Then the orbit $T(\cdot)f = f(\cdot + t)$ of the translation semigroup on $C_0(\mathbb{R})$ is not differentiable (cf. Example 1.8). ♢

The following simple lemma is used in the above proof and also later on.

LEMMA 1.12. *Let $D = \{(\tau, \sigma) \,|\, a \le \sigma \le \tau \le b\}$ for some $a < b$ in $\mathbb{R}$, $S : D \to \mathcal{B}(X)$ be strongly continuous, and f be contained in $C([a, b], X)$. Then the function $g : D \to X$; $g(\tau, \sigma) = S(\tau, \sigma)f(\sigma)$, is also continuous.*

PROOF. Observe that $\sup_{(\tau,\sigma)\in D} \|S(\tau, \sigma)x\| < \infty$ for every $x \in X$ by continuity. The uniform boundedness principle thus says that $c := \sup_D \|S(\tau, \sigma)\|$ is finite. For $(t, s), (\tau, \sigma) \in D$ we then obtain

$$\|S(t, s)f(s) - S(\tau, \sigma)f(\sigma)\| \le \|(S(t, s) - S(\tau, \sigma))f(s)\| + c\|f(s) - f(\sigma)\|.$$

The right-hand side of this inequality tends to 0 as $(\tau, \sigma) \to (t, s)$. □

REMARK 1.13. Let $x_n \to x$ in X and $T_n \to T$ strongly in $\mathcal{B}(X, Y)$. As in the proof of Lemma 1.12 one then shows that $T_n x_n \to Tx$ in Y as $n \to \infty$. ♢

Intermezzo 1: Closed operators, spectrum, and X-valued Riemann integrals. As noted above, generators of C_0-semigroups are unbounded unless the semigroup is continuous in $\mathcal{B}(X)$. However, we will see in Proposition 1.19 that they still respect limits to some extent. We introduce the relevant concepts here. See Chapter 1 in [**27**] for more details.

Let $\mathrm{D}(A) \subseteq X$ be a linear subspace and $A : \mathrm{D}(A) \to X$ be linear. (One could also take $Y \neq X$ as range space.) We often endow $\mathrm{D}(A)$ with the *graph norm* $\|x\|_A := \|x\| + \|Ax\|$, writing $[\mathrm{D}(A)]$, X_1^A, or X_1 for $(\mathrm{D}(A), \|\cdot\|_A)$ and also $\|x\|_1$ for $\|x\|_A$. Observe that $[\mathrm{D}(A)]$ is a normed vector space and that A is an element of $\mathcal{B}([\mathrm{D}(A)], X)$. Also, a map $f \in C([a, b], X)$ belongs to $C([a, b], [\mathrm{D}(A)])$ if and only if f takes values in $\mathrm{D}(A)$ and $Af : [a, b] \to X$ is continuous.

The operator A is called *closed* if for every sequence (x_n) in $\mathrm{D}(A)$ possessing the limits

$$\lim_{n\to\infty} x_n = x \quad \text{and} \quad \lim_{n\to\infty} Ax_n = y \qquad \text{in } X,$$

we obtain

$$x \in \mathrm{D}(A) \quad \text{and} \quad Ax = y.$$

We start with prototypical examples.

EXAMPLE 1.14. a) Every operator $A \in \mathcal{B}(X)$ with $\mathrm{D}(A) = X$ is closed, since here $Ax_n \to Ax$ if $x_n \to x$ in X as $n \to \infty$.

b) Let $X = C([0, 1])$ and $Af = f'$ with $\mathrm{D}(A) = C^1([0, 1])$. Take a sequence (f_n) in $\mathrm{D}(A)$ such that (f_n) and (f_n') tend in X to f and g, respectively. By Analysis 1, the limit f belongs to $C^1([0, 1])$ and satisfies $f' = g$; i.e., A is closed.

Next, consider the map $A_0 f = f'$ with $\mathrm{D}(A_0) = \{f \in C^1([0,1]) \,|\, f'(0) = 0\}$. Take (f_n) in $\mathrm{D}(A)$ such that $f_n \to f$ and $f_n' \to g$ in X as $n \to \infty$. We again obtain $f \in C^1([0,1])$ and $f' = g$. It further follows $f'(0) = g(0) = \lim_{n\to\infty} f_n'(0) = 0$, so that also A_0 is closed. $\Diamond$

Before we discuss basic properties of closed operators, we define the Riemann integral for X-valued functions. Let $a < b$ be real numbers. A (tagged) partition Z of $[a,b]$ is a finite set of numbers $a = t_0 < t_1 < \ldots < t_m = b$ together with 'tags' $\tau_k \in [t_{k-1}, t_k]$ for all $k \in \{1, \ldots, m\}$. Set $\delta(Z) = \max_{k\in\{1,\ldots,m\}}(t_k - t_{k-1})$. For a map $f \in C([a,b], X)$ and a partition Z, the *Riemann sum* is given by

$$S(f,Z) = \sum_{k=1}^{m} f(\tau_k)(t_k - t_{k-1}) \ \in \ X.$$

As for real-valued f it can be shown that for any sequence (Z_n) of (tagged) partitions with $\lim_{n\to\infty} \delta(Z_n) = 0$ the sequence $(S(f, Z_n))_n$ converges in X and that the limit J does not depend on the choice of such (Z_n). We then say that $S(f,Z)$ tends in X to J as $\delta(Z) \to 0$. The *Riemann integral* is now defined by

$$\int_a^b f(t)\,\mathrm{d}t = \lim_{\delta(Z)\to 0} S(f,Z).$$

We also set $\int_b^a f(t)\,\mathrm{d}t = -\int_a^b f(t)\,\mathrm{d}t$. Like in the real-valued case, one shows the basic properties the integral (except for monotony), e.g., linearity, additivity and validity of the standard estimate. Moreover, the same definition and results work for piecewise continuous functions. The fundamental theorem of calculus and a result on dominated convergence are shown in the next remark.

Remark 1.15. For a linear operator A in X the following assertions hold.

a) The operator A is closed if and only if its *graph* $\mathrm{Gr}(A) = \{(x, Ax) \,|\, x \in \mathrm{D}(A)\}$ is closed in $X \times X$ (endowed with the sum norm) if and only if $\mathrm{D}(A)$ is a Banach space with respect to the graph norm $\|\cdot\|_A$.

b) If A is closed with $\mathrm{D}(A) = X$, then A is continuous (*closed graph theorem*).

c) Let A be injective. Set $\mathrm{D}(A^{-1}) := \mathrm{R}(A) = \{Ax \,|\, x \in \mathrm{D}(A)\}$. Then A is closed if and only if A^{-1} is closed.

d) Let A be closed and $f \in C([a,b], [\mathrm{D}(A)])$. We then have

$$\int_a^b f(t)\,\mathrm{d}t \ \in \ \mathrm{D}(A) \qquad \text{and} \qquad A\int_a^b f(t)\,\mathrm{d}t = \int_a^b Af(t)\,\mathrm{d}t.$$

An analogous result is valid for piecewise continuous f and Af. So we can commute the Riemann integral and bounded linear operators, since $[\mathrm{D}(A)]$ is just X (with an equivalent norm) if $A \in \mathcal{B}(X)$.

e) Let $f_n, f : [a,b] \to X$ be (piecewise) continuous for $n \in \mathbb{N}$ such that $f_n(s) \to f(s)$ in X as $n \to \infty$ for each $s \in [a,b]$ and $\|f_n(\cdot)\| \le \varphi$ for a map $\varphi \in L^1(a,b)$ and all $n \in \mathbb{N}$. Then there exists the limit

$$\lim_{n\to\infty} \int_a^b f_n(s)\,\mathrm{d}s = \int_a^b f(s)\,\mathrm{d}s.$$

The assumptions are satisfied if $f_n \to f$ uniformly as $n \to \infty$.

f) For $f \in C([a,b],X)$, the function

$$[a,b] \to X;\ t \mapsto \int_a^t f(s)\,\mathrm{d}s,$$

is continuously differentiable with derivative

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^t f(s)\,\mathrm{d}s = f(t) \tag{1.2}$$

for each $t \in [a,b]$. For $g \in C^1([a,b],X)$, we have

$$\int_a^b g'(s)\,\mathrm{d}s = g(b) - g(a). \tag{1.3}$$

g) Let $J \subseteq \mathbb{R}$ be an interval. Take a sequence (f_n) in $C^1(J,X)$ and maps $f, g \in C(J,X)$ such that $f_n \to f$ and $f_n' \to g$ uniformly on J as $n \to \infty$. We then obtain $f \in C^1(J,X)$ and $f' = g$.

Proof. Parts a) and c) are shown in Lemma 1.4 of [**27**], and b) is established in Theorem 1.5 of [**27**].

For d), let f be as in the statement. Note that for each partition Z of $[a,b]$ the Riemann sum $S(f,Z)$ belongs to $\mathrm{D}(A)$. Since Af is continuous, we obtain

$$AS(f,Z) = \sum_{k=1}^{m}(Af)(\tau_k)(t_k - t_{k-1}) = S(Af,Z) \ \longrightarrow\ \int_a^b Af(t)\,\mathrm{d}t$$

as $\delta(Z) \to 0$. Claim d) now follows from the closedness of A.

Dominated convergence with majorant $\|f\|_\infty \mathbb{1} + \varphi$ yields assertion e) because

$$\Big\| \int_a^b f(s)\,\mathrm{d}s - \int_a^b f_n(s)\,\mathrm{d}s \Big\| \le \int_a^b \|f(s) - f_n(s)\|\,\mathrm{d}s.$$

For f), take $t \in [a,b]$ and $h \neq 0$ such that $t+h \in [a,b]$. We can then estimate

$$\begin{aligned} \Big\| \frac{1}{h}\Big(\int_a^{t+h} f(s)\,\mathrm{d}s - \int_a^t f(s)\,\mathrm{d}s \Big) - f(t) \Big\| &= \Big\| \frac{1}{h} \int_t^{t+h} (f(s) - f(t))\,\mathrm{d}s \Big\| \\ &\le \sup_{|s-t|\le h} \|f(s) - f(t)\| \ \longrightarrow\ 0 \end{aligned} \tag{1.4}$$

as $h \to 0$. So we have shown (1.2). In the proof of Proposition 1.10 we have seen that a function in $C^1([a,b])$ is constant if its derivative vanishes. Equation (1.3) can thus be deduced from (1.2) as in Analysis 2.

Let f_n, f, and g be as in part g). Take $a \in J$. Formula (1.3) says that

$$f_n(t) = f_n(a) + \int_a^t f_n'(s)\,\mathrm{d}s$$

for all $t \in J$. Letting $n \to 0$, from e) we deduce

$$f(t) = f(a) + \int_a^t g(s)\,\mathrm{d}s$$

for all $t \in J$. Due to (1.2), the map f belongs $C^1(J,X)$ and satisfies $f' = g$. □

For a closed operator A we define its *resolvent set*

$$\rho(A) = \big\{\lambda \in \mathbb{F} \,\big|\, \lambda I - A : \mathrm{D}(A) \to X \text{ is bijective}\big\}.$$

If $\lambda \in \rho(A)$, we write $R(\lambda, A)$ for $(\lambda I - A)^{-1}$ and call it *resolvent*.[2] The *spectrum of A* is the set

$$\sigma(A) = \mathbb{F} \setminus \rho(A).$$

The *point spectrum*

$$\sigma_{\mathrm{p}}(A) = \big\{\lambda \in \mathbb{F} \,\big|\, \exists\, v \in \mathrm{D}(A) \setminus \{0\} \text{ with } Av = \lambda v\big\}$$

is a subset of $\sigma(A)$ which can be empty if $\dim X = \infty$, see Example 1.25 in [**27**]. We discuss basic properties of spectrum and resolvent which will be used throughout these lectures.

Remark 1.16. a) Let A be closed and $\lambda \in \rho(A)$. It is easy to check that also the operator $\lambda I - A$ is closed (see Corollary 1.8 in [**27**]), and hence $R(\lambda, A)$ is closed by Remark 1.15 c). Assertion d) of that remark then shows the boundedness of $R(\lambda, A)$.

b) Let A be a linear operator and $\lambda \in \mathbb{F}$ such that $\lambda I - A : \mathrm{D}(A) \to X$ is bijective with bounded inverse. Then $(\lambda I - A)^{-1}$ is closed, so that Remark 1.15 c) implies the closedness of A. In particular, λ belongs to $\rho(A)$.

c) We list several important statements of Theorem 1.13 in [**27**]. The set $\rho(A)$ is open and so $\sigma(A)$ is closed. More precisely, for $\lambda \in \rho(A)$ all μ with $|\mu - \lambda| < 1/\|R(\lambda, A)\|$ are also contained in $\rho(A)$ and we have the power series

$$R(\mu, A) = \sum\nolimits_{n=0}^{\infty} (\lambda - \mu)^n R(\lambda, A)^{n+1}. \tag{1.5}$$

This series converges absolutely in $\mathcal{B}(X, [\mathrm{D}(A)])$ and uniformly for μ with $|\mu - \lambda| \le \delta/\|R(\lambda, A)\|$ and $\delta \in (0, 1)$, where one also obtains the inequality $\|R(\mu, A)\| \le \|R(\lambda, A)\|/(1 - \delta)$. The resolvent has the derivatives

$$\big(\tfrac{\mathrm{d}}{\mathrm{d}\lambda}\big)^n R(\lambda, A) = (-1)^n n! R(\lambda, A)^{n+1} \tag{1.6}$$

for all $\lambda \in \rho(A)$ and $n \in \mathbb{N}_0$. It further fulfills the *resolvent equation*

$$R(\mu, A) - R(\lambda, A) = (\lambda - \mu) R(\lambda, A) R(\mu, A) = (\lambda - \mu) R(\mu, A) R(\lambda, A), \tag{1.7}$$

for $\lambda, \mu \in \rho(A)$, and we have

$$\|R(\lambda, A)\| \ge \mathrm{d}(\lambda, \sigma(A))^{-1}. \tag{1.8}$$

d) Let $T \in \mathcal{B}(X)$ and $\mathbb{F} = \mathbb{C}$. By Theorem 1.16 of [**27**], the spectrum $\sigma(T)$ is even compact and always non-empty, and the *spectral radius* of T is given by

$$\mathrm{r}(T) \coloneqq \max\big\{|\lambda| \,\big|\, \lambda \in \sigma(A)\big\} = \inf_{n \in \mathbb{N}} \|T^n\|^{\frac{1}{n}} = \lim_{n \to \infty} \|T^n\|^{\frac{1}{n}}.$$

e) Example 1.21 provides closed operators A with $\sigma(A) = \emptyset$ or $\sigma(A) = \mathbb{F}$. $\Diamond$

This ends the intermezzo, and we come back to the investigation of C_0-semigroups. We first note a simple *rescaling* lemma which is often used to simplify the reasoning.

[2] Usually one takes $\mathbb{F} = \mathbb{C}$ in spectral theory, but many facts also hold for $\mathbb{F} = \mathbb{R}$. Sometimes real scalars are more convenient, and so we treat both fields if it is feasible. In examples we often restrict to $\mathbb{F} = \mathbb{C}$, as already real matrices may only have non-real eigenvalues.

Lemma 1.17. *Let $T(\cdot)$ be a C_0-semigroup with generator A, $\lambda \in \mathbb{F}$, and $a > 0$. Set $S(t) = \mathrm{e}^{\lambda t} T(at)$ for $t \geq 0$. Then $S(\cdot)$ is a C_0-semigroup and has the generator $B = \lambda I + aA$ with $\mathrm{D}(B) = \mathrm{D}(A)$.*

Proof. Let $t, s > 0$. We compute $S(t+s) = \mathrm{e}^{\lambda t}\mathrm{e}^{\lambda s} T(at)T(as) = S(t)S(s)$. The strong continuity of $S(\cdot)$ and the identity $S(0) = I$ are clear. Let B be the generator of $S(\cdot)$. Because of

$$\tfrac{1}{t}(S(t)x - x) = a\mathrm{e}^{\lambda t} \tfrac{1}{at}(T(at)x - x) + \tfrac{1}{t}(\mathrm{e}^{\lambda t} - 1)x,$$

x belongs to $\mathrm{D}(B)$ if and only if $x \in \mathrm{D}(A)$, and we then have $Bx = aAx + \lambda x$. □

Below we will derive key features of generators, which are consequences of the next *fundamental lemma.*

Lemma 1.18. *Let $T(\cdot)$ be a C_0-semigroup with generator A, $\lambda \in \mathbb{F}$, $t > 0$, and $x \in X$. Then the integral $\int_0^t \mathrm{e}^{-\lambda s} T(s)x \,\mathrm{d}s$ belongs to $\mathrm{D}(A)$ and satisfies*

$$\mathrm{e}^{-\lambda t} T(t)x - x = (A - \lambda I) \int_0^t \mathrm{e}^{-\lambda s} T(s)x \,\mathrm{d}s. \tag{1.9}$$

Furthermore, for $x \in \mathrm{D}(A)$ we have

$$\mathrm{e}^{-\lambda t} T(t)x - x = \int_0^t \mathrm{e}^{-\lambda s} T(s)(A - \lambda I)x \,\mathrm{d}s. \tag{1.10}$$

Proof. We only consider $\lambda = 0$ since the general case then follows by means of Lemma 1.17. For $h > 0$ and $t > 0$ we compute

$$\begin{aligned}
\frac{1}{h}(T(h) - I) \int_0^t T(s)x \,\mathrm{d}s &= \frac{1}{h}\Big(\int_0^t T(s+h)x \,\mathrm{d}s - \int_0^t T(s)x \,\mathrm{d}s \Big) \\
&= \frac{1}{h}\Big(\int_h^{t+h} T(r)x \,\mathrm{d}r - \int_0^t T(s)x \,\mathrm{d}s \Big) \\
&= \frac{1}{h} \int_t^{t+h} T(s)x \,\mathrm{d}s - \frac{1}{h} \int_0^h T(s)x \,\mathrm{d}s, \qquad (1.11)
\end{aligned}$$

where we substituted $r = s + h$. The last line tends to $T(t)x - x$ as $h \to 0$ due to the continuity of the orbits and (1.4). By the definition of the generator, this means that $\int_0^t T(s)x \,\mathrm{d}s$ is an element of $\mathrm{D}(A)$ and (1.9) holds.

Let $x \in \mathrm{D}(A)$. Proposition 1.10 then shows that $T(\cdot)x$ belongs to $C^1(\mathbb{R}_{\geq 0}, X)$ with derivative $\frac{\mathrm{d}}{\mathrm{d}t} T(\cdot)x = T(\cdot)Ax$. Hence, formula (1.10) follows from (1.3). □

We can now show basic properties of generators. Recall that they commute with their semigroup by Proposition 1.10.

Proposition 1.19. *Let A generate a C_0-semigroup $T(\cdot)$. Then A is closed and densely defined. Moreover, $T(\cdot)$ is the only C_0-semigroup generated by A. If $\lambda \in \rho(A)$, then we have $R(\lambda, A)T(t) = T(t)R(\lambda, A)$ for all $t \geq 0$.*

Proof. 1) To show closedness, we take a sequence (x_n) in $\mathrm{D}(A)$ with limit x in X such that (Ax_n) converges to some y in X. Equation (1.10) yields

$$\frac{1}{t}(T(t)x_n - x_n) = \frac{1}{t} \int_0^t T(s)Ax_n \,\mathrm{d}s$$

for all $n \in \mathbb{N}$ and $t > 0$. Letting $n \to \infty$, we infer

$$\frac{1}{t}(T(t)x - x) = \frac{1}{t}\int_0^t T(s)y\,\mathrm{d}s$$

by means of Remark 1.15 e). Because of (1.4), the right-hand side tends to y as $t \to 0$. This exactly means that x belongs $\mathrm{D}(A)$ and $Ax = y$; i.e., A is closed.

2) Let $x \in X$. For $n \in \mathbb{N}$, we define the vector

$$x_n = n\int_0^{\frac{1}{n}} T(s)x\,\mathrm{d}s$$

which belongs to $\mathrm{D}(A)$ by Lemma 1.18. Formula (1.4) shows that (x_n) tends to x, and hence the domain $\mathrm{D}(A)$ is dense in X.

3) Let A generate another C_0-semigroup $S(\cdot)$. The function $S(\cdot)x$ then solves (1.1) for each $x \in \mathrm{D}(A)$ by Proposition 1.10. The uniqueness statement in this result thus implies that $T(t)x = S(t)x$ for all $t \geq 0$ and $x \in \mathrm{D}(A)$. Since these operators are bounded, step 2) leads to $T(\cdot) = S(\cdot)$ as desired.

4) Let $\lambda \in \rho(A)$, $t \geq 0$, and $x \in X$. Proposition 1.10 implies the identity

$$R(\lambda, A)T(t)x = R(\lambda, A)T(t)(\lambda I - A)R(\lambda, A)x = T(t)R(\lambda, A)x. \qquad \square$$

We next derive important information about spectrum and resolvent of generators. Actually we show a bit more than needed later on.

PROPOSITION 1.20. *Let A generate the C_0-semigroup $T(\cdot)$ and $\lambda \in \mathbb{F}$. Then the following assertions hold.*

a) If the improper integral

$$R(\lambda)x := \int_0^\infty \mathrm{e}^{-\lambda s} T(s)x\,\mathrm{d}s := \lim_{t\to\infty}\int_0^t \mathrm{e}^{-\lambda s} T(s)x\,\mathrm{d}s$$

exists in X for all $x \in X$, then λ belongs to $\rho(A)$ and $R(\lambda) = R(\lambda, A)$.

b) The integral in a) exists even absolutely for all $x \in X$ if $\operatorname{Re}\lambda > \omega_0(T)$. Hence, the spectral bound (of A)

$$\mathrm{s}(A) := \sup\left\{\operatorname{Re}\lambda \,\middle|\, \lambda \in \sigma(A)\right\} \tag{1.12}$$

is less or equal than $\omega_0(T)$.

c) Let $M \geq 1$ and $\omega \in \mathbb{R}$ with $\|T(t)\| \leq M\mathrm{e}^{\omega t}$ for all $t \geq 0$. Take $n \in \mathbb{N}$ and $\lambda \in \mathbb{F}_\omega$ (i.e., $\operatorname{Re}\lambda > \omega$). We then have

$$\|R(\lambda, A)^n\| \leq \frac{M}{(\operatorname{Re}\lambda - \omega)^n}.$$

We recall from Definition 1.5 and Lemma 1.4 that the exponent ω in part c) has to satisfy $\omega \geq \omega_0(T)$ and that any number $\omega \in (\omega_0(T), \infty)$ fulfills the conditions in c).

The integral in part a) is called the *Laplace transform* of $T(\cdot)x$. It can be used for an alternative approach to the theory of C_0-semigroups (and their generalizations), cf. [**3**]. In Chapter 4 we will study whether the equality $\mathrm{s}(A) = \omega_0(T)$ can be shown in b). This property would allow to control the growth (or decay) of the semigroup in terms of the given object A.

Proof of Proposition 1.20. a) Let $h > 0$ and $x \in X$. By Lemma 1.17, we have the C_0-semigroup $T_\lambda(\cdot) = (\mathrm{e}^{-\lambda s} T(s))_{s \ge 0}$ with generator $A - \lambda I$ on the domain $\mathrm{D}(A)$. Equation (1.11) yields

$$\begin{aligned} \frac{1}{h}(T_\lambda(h) - I) R(\lambda) x &= \lim_{t\to\infty} \frac{1}{h}(T_\lambda(h) - I) \int_0^t T_\lambda(s) x \,\mathrm{d}s \\ &= \lim_{t\to\infty} \frac{1}{h} \int_t^{t+h} T_\lambda(s) x \,\mathrm{d}s - \frac{1}{h} \int_0^h T_\lambda(s) x \,\mathrm{d}s \\ &= -\frac{1}{h} \int_0^h T_\lambda(s) x \,\mathrm{d}s, \end{aligned}$$

due to the convergence of $\int_0^\infty T_\lambda(s) x \,\mathrm{d}s$. The right-hand side tends to $-x$ as $h \to 0$ by (1.4), so that $R(\lambda)x$ belongs to $\mathrm{D}(A - \lambda I) = \mathrm{D}(A)$ and satisfies $(\lambda I - A) R(\lambda) x = x$.

Let $x \in \mathrm{D}(A)$. Proposition 1.10 says that $T(s)Ax = AT(s)x$ for $s \ge 0$, and A is closed due to Proposition 1.19. Using also Remark 1.15 d), we deduce

$$\begin{aligned} R(\lambda)(\lambda I - A)x &= \lim_{t\to\infty} \int_0^t \mathrm{e}^{-\lambda s} T(s)(\lambda I - A) x \,\mathrm{d}s = \lim_{t\to\infty} (\lambda I - A) \int_0^t \mathrm{e}^{-\lambda s} T(s) x \,\mathrm{d}s \\ &= (\lambda I - A) \lim_{t\to\infty} \int_0^t \mathrm{e}^{-\lambda s} T(s) x \,\mathrm{d}s = (\lambda I - A) R(\lambda) x = x. \end{aligned}$$

Hence, part a) is shown.

b) Let $x \in X$. Fix a number $\omega \in (\omega_0(T), \operatorname{Re}\lambda)$. It follows $\|\mathrm{e}^{-\lambda s} T(s) x\| \le M \mathrm{e}^{(\omega - \operatorname{Re}\lambda)s}$ for some $M \ge 1$ and all $s \ge 0$. For $0 < a < b$ we can thus estimate

$$\Big\| \int_0^b T_\lambda(s) x \,\mathrm{d}s - \int_0^a T_\lambda(s) x \,\mathrm{d}s \Big\| \le \int_a^b \|T_\lambda(s) x\| \,\mathrm{d}s \le M \|x\| \int_a^b \mathrm{e}^{(\omega - \operatorname{Re}\lambda)s} \,\mathrm{d}s \;\to\; 0$$

as $a, b \to \infty$. Consequently, $\int_0^t T_\lambda(s) x \,\mathrm{d}s$ converges (absolutely) in X as $t \to \infty$ for all $x \in X$, and thus assertion b) follows from a).

c) Let $n \in \mathbb{N}$, $x \in X$, and $t \ge 0$. Arguing as in Analysis 2, one can differentiate

$$\Big(\frac{\mathrm{d}}{\mathrm{d}\lambda}\Big)^{n-1} \int_0^t \mathrm{e}^{-\lambda s} T(s) x \,\mathrm{d}s = \int_0^t (-1)^{n-1} s^{n-1} \mathrm{e}^{-\lambda s} T(s) x \,\mathrm{d}s.$$

As in step b), the integrals converge as $t \to \infty$ uniformly for $\operatorname{Re}\lambda \ge \omega + \varepsilon$ and any $\varepsilon > 0$. Hence, part b), (1.6) and a variant of Remark 1.15 g) imply

$$\begin{aligned} R(\lambda, A)^n x &= \frac{(-1)^{n-1}}{(n-1)!} \Big(\frac{\mathrm{d}}{\mathrm{d}\lambda}\Big)^{n-1} \lim_{t\to\infty} \int_0^t \mathrm{e}^{-\lambda s} T(s) x \,\mathrm{d}s \\ &= \lim_{t\to\infty} \frac{1}{(n-1)!} \int_0^t s^{n-1} T_\lambda(s) x \,\mathrm{d}s = \frac{1}{(n-1)!} \int_0^\infty s^{n-1} \mathrm{e}^{-\lambda s} T(s) x \,\mathrm{d}s. \end{aligned}$$

Computing an elementary integral, one can now estimate

$$\|R(\lambda, A)^n x\| \le \frac{M\|x\|}{(n-1)!} \int_0^\infty s^{n-1} \mathrm{e}^{(\omega - \operatorname{Re}\lambda)s} \,\mathrm{d}s = \frac{M}{(\operatorname{Re}\lambda - \omega)^n} \|x\|$$

for all $\operatorname{Re}\lambda > \omega$ since ε is arbitrary. □

We calculate the generators of the translation semigroups from Example 1.8 and discuss their spectra. They turn out to be the first derivative endowed with appropriate domains. We also use the above necessary conditions to show that on certain domains the first derivative fails to be a generator.

EXAMPLE 1.21. a) Let $T(t)f = f(\cdot + t)$ be the translation group on $X = C_0(\mathbb{R})$. We compute the generator A and its spectrum.

1) Below we use that a function $g \in X$ is uniformly continuous since $C_c(\mathbb{R})$ is dense in X and uniform continuity is preserved by uniform limits.

For $f \in \mathrm{D}(A)$, $t \neq 0$ and $s \in \mathbb{R}$, there exist the pointwise limits

$$Af(s) = \lim_{t\to 0} \tfrac{1}{t}(T(t)f(s) - f(s)) = \lim_{t\to 0} \tfrac{1}{t}(f(s+t) - f(s)) = f'(s)$$

so that f is differentiable with $f' = Af \in C_0(\mathbb{R})$. We have shown the inclusion

$$\mathrm{D}(A) \subseteq C_0^1(\mathbb{R}) := \{f \in C^1(\mathbb{R}) \mid f, f' \in X\}.$$

Conversely, let $f \in C_0^1(\mathbb{R})$. For $s \in \mathbb{R}$, we compute

$$\begin{aligned}\left|\tfrac{1}{t}(T(t)f(s) - f(s)) - f'(s)\right| &= \left|\tfrac{1}{t}(f(s+t) - f(s)) - f'(s)\right| \\ &= \left|\frac{1}{t}\int_0^t (f'(s+\tau) - f'(s))\,\mathrm{d}\tau\right| \\ &\leq \sup_{0\leq|\tau|\leq|t|} |f'(s+\tau) - f'(s)|.\end{aligned}$$

The right-hand side tends to 0 as $t \to 0$ uniformly in $s \in \mathbb{R}$ since $f' \in C_0(\mathbb{R})$ is uniformly continuous. This means that f belongs to $\mathrm{D}(A)$, and so we obtain $A = \frac{\mathrm{d}}{\mathrm{d}s}$ with 'maximal domain' $\mathrm{D}(A) = C_0^1(\mathbb{R})$.

2) For the spectrum, we let $\mathbb{F} = \mathbb{C}$. In Theorem 1.29 we will see that A generates the C_0-semigroup $(T(t))_{t\geq 0}$ and $-A$ is the generator of $(S(t))_{t\geq 0} = (T(-t))_{t\geq 0}$. Proposition 1.20 yields the inqualities $\mathrm{s}(A) \leq \omega_0(A) = 0$ and $\mathrm{s}(-A) \leq 0$. Observing $-(\lambda I - (-A)) = -\lambda I - A$, we conclude $\sigma(-A) = -\sigma(A)$ as well as $-R(\lambda, -A) = R(-\lambda, A)$. So we have proven the inclusion $\sigma(A) \subseteq \mathrm{i}\mathbb{R}$.

To show the converse, let $\lambda \in \mathbb{C}_+$, $f \in X$, and $s \in \mathbb{R}$. Since all of the following limits exist with respect to the supremum norm in s, Proposition 1.20 yields

$$\begin{aligned}(R(\lambda, A)f)(s) &= \Big(\lim_{b\to\infty}\int_0^b \mathrm{e}^{-\lambda t}T(t)f\,\mathrm{d}t\Big)(s) = \lim_{b\to\infty}\int_0^b \mathrm{e}^{-\lambda t}(T(t)f)(s)\,\mathrm{d}t \\ &= \lim_{b\to\infty}\int_0^b \mathrm{e}^{-\lambda t}f(t+s)\,\mathrm{d}t = \lim_{b\to\infty}\int_s^{b+s} \mathrm{e}^{\lambda(s-\tau)}f(\tau)\,\mathrm{d}\tau \\ &= \int_s^\infty \mathrm{e}^{\lambda(s-\tau)}f(\tau)\,\mathrm{d}\tau.\end{aligned}$$

We pick functions $\varphi_n \in C_c(\mathbb{R})$ with $0 \leq \varphi_n \leq 1$ and $\varphi_n = 1$ on $[0, n]$ for $n \in \mathbb{N}$, and set $\alpha = \operatorname{Re}\lambda > 0$, $\beta = \operatorname{Im}\lambda$, as well as $f_n(\tau) = \mathrm{e}^{\mathrm{i}\beta\tau}\varphi_n(\tau)$. Since $\|f_n\|_\infty = 1$, the above formula leads to the lower bound

$$\begin{aligned}\|R(\lambda, A)\| \geq \|R(\lambda, A)f_n\|_\infty \geq |R(\lambda, A)f_n(0)| &= \Big|\int_0^\infty \mathrm{e}^{-\alpha\tau}\mathrm{e}^{-\mathrm{i}\beta\tau}f_n(\tau)\,\mathrm{d}\tau\Big| \\ = \int_0^\infty \mathrm{e}^{-\alpha\tau}\varphi_n(\tau)\,\mathrm{d}\tau \geq \int_0^n \mathrm{e}^{-\alpha\tau}\,\mathrm{d}\tau &= \frac{1 - \mathrm{e}^{-\alpha n}}{\alpha}.\end{aligned}$$

Letting $n \to \infty$, we arrive at $\|R(\lambda, A)\| \geq \frac{1}{\operatorname{Re}\lambda}$. Proposition 1.20 then yields the equality $\|R(\lambda, A)\| = \frac{1}{\operatorname{Re}\lambda}$ (take $M = 1$, $\omega = 0$, and $n = 1$ there). If $\mathrm{i}\beta$ belonged to $\rho(A)$ for some $\beta \in \mathbb{R}$, then we would infer

$$\tfrac{1}{\alpha} = \|R(\alpha + \mathrm{i}\beta, A)\| \to \|R(\mathrm{i}\beta, A)\|$$

as $\alpha \to 0$, which is impossible. We thus obtain $\sigma(A) = \mathrm{i}\mathbb{R}$.

b) We treat the nilpotent left translation semigroup on $X = C_0([0, 1))$; i.e.,

$$(T(t)f)(s) = \begin{cases} f(s+t), & s+t < 1, \\ 0, & s+t \geq 1, \end{cases}$$

for $f \in X$, $t \geq 0$ and $s \in [0, 1)$. Let A be its generator. Take $f \in \mathrm{D}(A)$. As in part a), one shows that the right derivative $\frac{\mathrm{d}^+}{\mathrm{d}s} f$ exists and $\frac{\mathrm{d}^+}{\mathrm{d}s} f = Af$. (Here we can only consider $t \to 0^+$.) However, since f and Af are continuous, Corollary 2.1.2 of [22] says that $f \in C^1([0, 1))$, and so we have the inclusion

$$\mathrm{D}(A) \subseteq C_0^1([0,1)) := \big\{ f \in C^1([0,1)) \,\big|\, f, f' \in X \big\}$$

as well as $Af = f'$. Let $f \in C_0^1([0, 1))$ and note that its 0-extension $\tilde{f}$ to $\mathbb{R}_{\geq 0}$ belongs to $C_0^1(\mathbb{R}_{\geq 0})$ and has compact support. As in part a), it follows

$$\begin{aligned} \frac{1}{t}(T(t)f(s) - f(s)) &= \begin{cases} \frac{1}{t}(f(s+t) - f(s)), & 0 \leq s < 1-t, \\ -\frac{1}{t} f(s), & 1-t \leq s < 1, \end{cases} \\ &= \tfrac{1}{t}(\tilde{f}(s+t) - \tilde{f}(s)) \;\longrightarrow\; \tilde{f}'(s) = f'(s) \end{aligned}$$

as $t \to 0^+$ uniformly in $s \in [0, 1)$, since $\tilde{f}'$ is uniformly continuous. Hence, $\mathrm{D}(A) = C_0^1([0, 1))$ and $Af = f'$. Because of $\omega_0(A) = -\infty$, Proposition 1.20 yields $\sigma(A) = \emptyset$ and $\rho(A) = \mathbb{F}$.

c) The operator $Af = f'$ with $\mathrm{D}(A) = C^1([0, 1])$ on $X = C([0, 1])$ has the spectrum $\sigma(A) = \mathbb{F}$. In fact, for each $\lambda \in \mathbb{F}$ the function $t \mapsto e_\lambda(t) := \mathrm{e}^{\lambda t}$ belongs to $\mathrm{D}(A)$ with $Ae_\lambda = \lambda e_\lambda$ so that even $\lambda \in \sigma_p(A)$. Hence, A is not a generator in view of Proposition 1.20.

d) Let $X = C_0(\mathbb{R}_{\leq 0}) := \{f \in C(\mathbb{R}_{\leq 0}) \,|\, f(s) \to 0 \text{ as } s \to -\infty\}$ and $A = \frac{\mathrm{d}}{\mathrm{d}s}$ with $\mathrm{D}(A) = C_0^1(\mathbb{R}_{\leq 0}) := \{f \in C^1(\mathbb{R}_{\leq 0}) \,|\, f, f' \in X\}$. Then A is not a generator. Indeed, for all $\lambda \in \mathbb{F}_+$ we have $e_\lambda \in \mathrm{D}(A)$ and $Ae_\lambda = \lambda e_\lambda$ so that $\lambda \in \sigma(A)$, violating $\mathrm{s}(A) < \infty$ in Proposition 1.20.

e) On $X = C([0, 1])$ the map $A = \frac{\mathrm{d}}{\mathrm{d}s}$ with $\mathrm{D}(A) = \{f \in C^1([0, 1]) \,|\, f(1) = 0\}$ is not a generator as $\overline{\mathrm{D}(A)} = \{f \in X \,|\, f(1) = 0\} \neq X$, cf. Proposition 1.19. $\Diamond$

We stress that in parts c) and d) one does not impose conditions at the upper boundary of the spatial interval, as needed for a left translation, in contrast to the example in b). This lack of boundary conditions leads to spectral properties of A ruling out that it is a generator. We will come back to this point in Example 1.36.

1.2. Characterization of generators

Proposition 1.19 and 1.20 contain necessary conditions to be a generator. In this section we want to show their sufficiency. This is the content of the Hille–Yosida Theorem 1.26 which is the core of the theory of C_0-semigroups. Our approach is based on the so-called *Yosida approximations* which are defined by

$$A_\lambda := \lambda A R(\lambda, A) = \lambda^2 R(\lambda, A) - \lambda I \ \in \ \mathcal{B}(X). \tag{1.13}$$

for $\lambda \in \rho(A)$. Here we note the basic identities

$$AR(\lambda, A) = \lambda R(\lambda, A) - I \quad \text{ and } \quad AR(\lambda, A)x = R(\lambda, A)Ax \tag{1.14}$$

for $x \in \mathrm{D}(A)$. The next lemma is stated in somewhat greater generality than needed later on. In view of Proposition 1.19 and 1.20, for a generator A it says that the *bounded* operators A_λ approximate A strongly on $\mathrm{D}(A)$ as $\lambda \to \infty$.

Lemma 1.22. *Let A be a closed operator satisfying $(\omega, \infty) \subseteq \rho(A)$ and $\|R(\lambda, A)\| \le \frac{M}{\lambda-\omega}$ for some $M \ge 1$ and $\omega \in \mathbb{R}$ and all $\lambda > \omega$. As $\lambda \to \infty$, we then have*

$$\forall\, x \in \overline{\mathrm{D}(A)}: \quad \lambda R(\lambda, A)x \to x,$$

$$\forall\, y \in \mathrm{D}(A) \text{ with } Ay \in \overline{\mathrm{D}(A)}: \quad \lambda A R(\lambda, A)y \to Ay.$$

Proof. Let $x \in \mathrm{D}(A)$ and $\lambda \ge \omega + 1$. The assumption and (1.14) yield

$$\|\lambda R(\lambda, A)\| \le \frac{M|\lambda|}{\lambda - \omega} \le M \max\{|\omega + 1|, 1\},$$

$$\|\lambda R(\lambda, A)x - x\| = \|R(\lambda, A)Ax\| \le \frac{M}{\lambda - \omega}\|Ax\| \ \longrightarrow \ 0, \quad \lambda \to \infty.$$

By density, the first assertion follows. Taking $x = Ay$ and using (1.14), one then deduces the second assertion from the first one. □

For linear operators A, B on X we write $A \subseteq B$ if $\mathrm{Gr}(A) \subseteq \mathrm{Gr}(B)$; i.e., if $\mathrm{D}(A) \subseteq \mathrm{D}(B)$ and $Ax = Bx$ for all $x \in \mathrm{D}(A)$. In this case we call B an *extension* of A. Equality of A and B is then often shown by means of the next observation, requiring that $\mathrm{D}(A)$ is not 'too small' and $\mathrm{D}(B)$ is not 'too large.'

Lemma 1.23. *Let A and B be linear operators with $A \subseteq B$ such that A is surjective and B is injective. We then have $A = B$. In particular, A and B are equal if they satisfy $A \subseteq B$ and $\rho(A) \cap \rho(B) \ne \emptyset$.*

Proof. We have to prove the inclusion $\mathrm{D}(B) \subseteq \mathrm{D}(A)$. Let $x \in \mathrm{D}(B)$. By the assumptions, there is a vector $y \in \mathrm{D}(A)$ with $Bx = Ay = By$. Since B is injective, we obtain $x = y$ so that x belongs to $\mathrm{D}(A)$.

Let $\lambda \in \rho(A) \cap \rho(B)$. The first part then shows the equality $\lambda I - A = \lambda I - B$, and hence $A = B$. □

We introduce a class of C_0-semigroups which is easier to handle in many respects, cf. Theorem 1.39.

Definition 1.24. *Let $\omega \in \mathbb{R}$. An ω-*contraction semigroup* is a C_0-semigroup $T(\cdot)$ satisfying $\|T(t)\| \le \mathrm{e}^{\omega t}$ for all $t \ge 0$. Such a semigroup is also said to be* quasi-contractive*. If $\omega = 0$, we call $T(\cdot)$ a* contraction semigroup*.*

We first discuss this concept and its relation to the exponential bound from Lemma 1.4, also noting the dependence on the choice of the norm on X.

Remark 1.25. a) Let $T(\cdot)$ be a contraction semigroup. Then the norm of the orbit $t \mapsto T(x)x$ is non-increasing since

$$\|T(t)x\| = \|T(t-s)T(s)x\| \le \|T(s)x\|$$

for $x \in X$ and $t \ge s \ge 0$. This fact is important since often $\|x\|$ is related significant quantities in applications, e.g., the energy of the state x.

b) Let $A \in \mathcal{B}(X)$. Estimating the power series in Example 1.3, we derive $\|e^{tA}\| \le e^{t\|A\|}$; i.e., A generates a $\|A\|$-contractive semigroup. However, its growth bound $\omega_0(A)$ is possibly much smaller than $\|A\|$ by Remark 1.6 d).

c) There are unbounded generators A of a C_0-semigroup having norms $\|T(t)\| \ge M$ for all $t > 0$ and some $M > 1$. Hence, they cannot be ω-contractive for any $\omega \in \mathbb{R}$. As an example, let $X = C_0(\mathbb{R})$ be endowed with the norm

$$\|f\| = \max\left\{ \sup_{s\ge 0} |f(s)|, M \sup_{s<0} |f(s)|\right\}$$

for some $M > 1$, which is equivalent to the supremum norm. The translations $T(t)f = f(\cdot + t)$ thus yield a C_0-semigroup on $(X, \|\cdot\|)$. Take any $t > 0$. Choose a function $f \in C_0(\mathbb{R})$ with $\|f\|_\infty = 1$ and $\operatorname{supp} f \subseteq (0, t)$. We then obtain $\|f\| = 1$, $\operatorname{supp} T(t)f \subseteq (-t, 0)$, and so

$$\|T(t)\| \ge \|T(t)f\| = M \sup_{-t\le s\le 0} |f(s+t)| = M.$$

Since $\|T(t)\| \le M$, we actually have $\|T(t)\| = M$ for all $t > 0$.

d) However, for each C_0-semigroup $T(\cdot)$ on a Banach space X one can find an equivalent norm on X for which $T(\cdot)$ becomes ω-contractive. Indeed, take numbers $M \ge 1$ and $\omega \in \mathbb{R}$ such that $\|T(t)\| \le M e^{\omega t}$ for all $t \ge 0$. We set

$$|||x||| = \sup_{s\ge 0} e^{-\omega s}\|T(s)x\|$$

for $x \in X$, which defines an equivalent norm since $\|x\| \le |||x||| \le M\|x\|$. We further obtain

$$|||e^{-\omega t}T(t)x||| = \sup_{s\ge 0} e^{-\omega(s+t)}\|T(s+t)x\| \le |||x|||$$

so that $T(\cdot)$ is ω-contractive for this norm. However, this renorming can destroy additional properties as the Hilbert space structure, and in general one cannot do it for two C_0-semigroups at the same time. See Remark I.2.19 in [**10**]. ♢

The following major theorem characterizes the generators of C_0-semigroups. It was shown in the contraction case independently by *Hille* and *Yosida* in 1948. Yosida's proof extends very easily to the general case and is presented below. As we see in Theorem 2.2, the generator property of A is equivalent to 'wellposedness' of (1.1). In other words, the Hille–Yosida Theorem describes the class of operators for which (1.1) is solvable in a reasonable sense. It is thus the fundament of the theory of linear evolution equations, which is actually concerned with many topics beyond wellposedness – below we treat regularity, perturbation, approximation, and long-time behavior, for instance.

THEOREM 1.26. *Let $M \geq 1$ and $\omega \in \mathbb{R}$. A linear operator A generates a C_0-semigroup $T(\cdot)$ on X satisfying $\|T(t)\| \leq M e^{\omega t}$ for all $t \geq 0$ if and only if*

$$\begin{aligned} &A \text{ is closed}, \quad \overline{\mathrm{D}(A)} = X, \quad (\omega, \infty) \subseteq \rho(A), \\ &\forall\, n \in \mathbb{N},\ \lambda > \omega: \quad \|R(\lambda, A)^n\| \leq \frac{M}{(\lambda - \omega)^n}. \end{aligned} \tag{1.15}$$

In this case, if $\mathbb{F} = \mathbb{C}$ one even has $\mathbb{C}_\omega = \{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda > \omega\} \subseteq \rho(A)$ and

$$\forall\, n \in \mathbb{N},\ \lambda \in \mathbb{C}_\omega: \quad \|R(\lambda, A)^n\| \leq \frac{M}{(\operatorname{Re} \lambda - \omega)^n}. \tag{1.16}$$

The operator A generates an ω-contraction semigroup if and only if

$$A \text{ closed}, \ \overline{\mathrm{D}(A)} = X, \ (\omega, \infty) \subseteq \rho(A), \ \forall\, \lambda > \omega: \|R(\lambda, A)\| \leq \frac{1}{\lambda - \omega}. \tag{1.17}$$

In this case (1.16) is true with $M = 1$, if $\mathbb{F} = \mathbb{C}$.

In applications it is of course much easier check the assumptions in the quasi-contractive case. Based on the above result, Theorem 1.39 will provide another, even more convenient characterization of generators in this case. In the exercises we discuss a variant without the assumption of a dense domain.

PROOF OF THEOREM 1.26. It is clear that (1.17) yields (1.15) for $M = 1$. Propositions 1.19 and 1.20 imply (1.16) and the necessity of (1.15), respectively (1.17). If (1.15) is true, then the shifted operator $A - \omega I$ satisfies (1.15) with '$\omega = 0$.' Below we show that $A - \omega I$ generates a bounded semigroup. Lemma 1.17 then yields the assertion.

To establish the sufficiency of (1.15), we use the (bounded) Yosida approximations $A_n = n^2 R(n, A) - nI$ which tend to A strongly on $\mathrm{D}(A)$ as $n \to \infty$ and generate semigroups $(e^{tA_n})_{t \geq 0}$. We first prove that the latter converge to a C_0-semigroup $T(\cdot)$. In a second step we show that $T(\cdot)$ is generated by A.

1) Let (1.15) be true with $\omega = 0$. Take $n, m \in \mathbb{N}$ and $t \geq 0$. Employing Lemma 1.17, the powers series representation of e^{tA_n} in Example 1.3 and (1.15), we estimate

$$\begin{aligned} \|e^{tA_n}\| &= \|e^{-nt} e^{n^2 R(n,A) t}\| \leq e^{-tn} \sum_{j=0}^{\infty} \frac{(nt)^j\, n^j \|R(n, A)^j\|}{j!} \leq M e^{-nt} \sum_{j=0}^{\infty} \frac{(nt)^j}{j!} \\ &= M. \end{aligned} \tag{1.18}$$

We further have $A_n A_m = A_m A_n$ and hence the core commutativity

$$A_n e^{tA_m} = A_n \sum_{j=0}^{\infty} \frac{t^j}{j!} A_m^j = \sum_{j=0}^{\infty} \frac{t^j}{j!} A_m^j A_n = e^{tA_m} A_n.$$

Take $t_0 > 0$, $y \in \mathrm{D}(A)$, and $t \in [0, t_0]$. Using (1.3), from the above equation we infer

$$e^{tA_n} y - e^{tA_m} y = \int_0^t \frac{\mathrm{d}}{\mathrm{d}s} e^{(t-s)A_m} e^{sA_n} y \,\mathrm{d}s = \int_0^t e^{(t-s)A_m} e^{sA_n} (A_n - A_m) y \,\mathrm{d}s.$$

Estimate (1.18) and Lemma 1.22 then lead to the limit

$$\|e^{tA_n} y - e^{tA_m} y\| \leq t_0 M^2 \|A_n y - A_m y\| \ \longrightarrow\ 0 \tag{1.19}$$

as $n, m \to \infty$. Because of the density of $\mathrm{D}(A)$ and the bound (1.18), we can apply Lemma 4.10 of [**24**]. Since $t_0 > 0$ is arbitrary, this lemma yields operators $T(t)$ in $\mathcal{B}(X)$ such that $\mathrm{e}^{tA_n}x \to T(t)x$ as $n \to \infty$ and $\|T(t)\| \le M$ for all $t \ge 0$ and $x \in X$. We also obtain $T(0) = I$ and

$$T(t+s)x = \lim_{n\to\infty} \mathrm{e}^{(t+s)A_n}x = \lim_{n\to\infty} \mathrm{e}^{tA_n}\mathrm{e}^{sA_n}x = T(t)T(s)x$$

for all $t, s \ge 0$, using Remark 1.13. Letting $m \to \infty$ in (1.19), we further deduce

$$\|\mathrm{e}^{tA_n}y - T(t)y\| \le t_0 M^2 \,\|A_n y - Ay\|$$

for all $t \in [0, t_0]$. This means that $\mathrm{e}^{tA_n}y$ converges to $T(t)y$ uniformly for $t \in [0, t_0]$, and hence $T(\cdot)y$ is continuous for all $y \in \mathrm{D}(A)$. Lemma 1.7 and the density of $\mathrm{D}(A)$ now imply that $T(\cdot)$ is a (bounded) C_0-semigroup.

2) Let B generate $T(\cdot)$. We have $\mathbb{R}_+ \subseteq \rho(A) \cap \rho(B)$ by Proposition 1.20 and the assumptions. In view of Lemma 1.23 it thus remains to show $A \subseteq B$ (or $B \subseteq A$, but more is known about A). For $t > 0$ and $y \in \mathrm{D}(A)$, Lemmas 1.18 and 1.22, Remarks 1.13 and 1.15 e), as well as estimate (1.18) yield

$$\frac{1}{t}(T(t)y - y) = \lim_{n\to\infty} \frac{1}{t}(\mathrm{e}^{tA_n}y - y) = \lim_{n\to\infty} \frac{1}{t}\int_0^t \mathrm{e}^{sA_n}A_n y \,\mathrm{d}s = \frac{1}{t}\int_0^t T(s)Ay \,\mathrm{d}s.$$

As $t \to 0$, from (1.4) we conclude that $y \in \mathrm{D}(B)$ and $By = Ay$; i.e., $A \subseteq B$. □

We illustrate the above theorem by some examples. Applications to more complicated partial differential operators will be discussed in Section 1.4.

Example 1.27. a) Let $X = C_0(\mathbb{R}_{\le 0})$ and $A = -\frac{\mathrm{d}}{\mathrm{d}s}$ with $\mathrm{D}(A) = C_0^1(\mathbb{R}_{\le 0})$, cf. Example 1.21. Then A generates the C_0-semigroup given by $T(t)f = f(\cdot - t)$ for $t \ge 0$ and $f \in X$. It has the spectrum $\sigma(A) = \overline{\mathbb{C}_-}$, where we let $\mathbb{F} = \mathbb{C}$.

Proof. We first check in several steps the conditions (1.17).

1) Let $f \in X$ and $\varepsilon > 0$. We extend f to a function $\tilde f \in C_0(\mathbb{R})$. As in Example 1.8 one finds a map $\tilde g \in C_c(\mathbb{R})$ with $\|\tilde f - \tilde g\|_\infty \le \varepsilon$. By the proof of Proposition 4.13 in [**24**] there is function $\tilde h \in C_c^\infty(\mathbb{R})$ with $\|\tilde g - \tilde h\|_\infty \le \varepsilon$. The restriction h of $\tilde h$ to $\mathbb{R}_{\le 0}$ thus belongs to $\mathrm{D}(A)$ and satisfies $\|f - h\|_\infty \le 2\varepsilon$, so that A is densely defined.

2) Let the sequence (u_n) in $\mathrm{D}(A)$ tend in X to a function u, and (Au_n) to some f in X. The map u is thus differentiable with $-u' = f \in X$. As a result, u is contained in $\mathrm{D}(A)$ and satisfies $Au = f$; i.e., A is closed.

3) Let $f \in X$ and $\lambda > 0$. To show the bijectivity of $\lambda I - A$, we note that a function u belongs to $\mathrm{D}(A)$ and solves $\lambda u - Au = f$ if and only if

$$u' = -\lambda u + f \quad \text{and} \quad u \in C^1(\mathbb{R}_{\le 0}) \cap X$$

(using that $u' \in X$ by the formulas in display). This condition is equivalent to

$$u \in C^1(\mathbb{R}_{\le 0}) \cap X, \quad \forall\, t_0 \le s \le 0: \quad u(s) = \mathrm{e}^{-\lambda(s-t_0)}u(t_0) + \int_{t_0}^s \mathrm{e}^{-\lambda(s-\tau)} f(\tau)\,\mathrm{d}\tau.$$

Since u and f are bounded and $\lambda > 0$, here one can let $t_0 \to -\infty$ and derive

$$u(s) = \int_{-\infty}^s \mathrm{e}^{-\lambda(s-\tau)} f(\tau)\,\mathrm{d}\tau =: R(\lambda)f(s) \quad \text{for all } s \le 0, \qquad \lim_{s\to-\infty} u(s) = 0.$$

Conversely, if the function $v := R(\lambda)f$ belongs to X, a direct calculation shows that it is an element of $\mathrm{D}(A)$ and satisfies $\lambda v - Av = f$.

We now show $R(\lambda)f \in X$, where the continuity is clear. Let $\varepsilon > 0$. There is a number $s_\varepsilon \le 0$ with $|f(\tau)| \le \varepsilon$ for all $\tau \le s_\varepsilon$. For $s \le s_\varepsilon$ we then estimate

$$|R(\lambda)f(s)| \le \int_{-\infty}^{s} \mathrm{e}^{-\lambda(s-\tau)}|f(\tau)|\,\mathrm{d}\tau \le \varepsilon \int_0^\infty \mathrm{e}^{-\lambda r}\,\mathrm{d}r = \frac{\varepsilon}{\lambda},$$

substituting $r = s - \tau$. As a result, $R(\lambda)f(s)$ tends to 0 as $s \to -\infty$, and so λ is contained in $\rho(A)$ and $R(\lambda) = R(\lambda, A)$.

4) Employing the above formula for the resolvent, we calculate

$$\|R(\lambda, A)f\|_\infty \le \sup_{s\le 0} \int_{-\infty}^{s} \mathrm{e}^{-\lambda(s-\tau)}\|f\|_\infty\,\mathrm{d}\tau = \|f\|_\infty \int_0^\infty \mathrm{e}^{-\lambda r}\,\mathrm{d}r = \frac{\|f\|_\infty}{\lambda}$$

for all $f \in X$ and $\lambda > 0$. Theorem 1.26 now implies that A generates a contraction semigroup $T(\cdot)$. In particular, $\sigma(A)$ is contained in $\overline{\mathbb{C}_-}$. For $\lambda \in \mathbb{C}_-$, the function $e_{-\lambda}$ belongs to $\mathrm{D}(A)$ and satisfies $Ae_{-\lambda} = -e'_{-\lambda} = \lambda e_{-\lambda}$ so that $\mathbb{C}_- \subseteq \sigma(A)$. The closedness of $\sigma(A)$ then implies the second assertion.

5) To determine $T(\cdot)$, we take $\varphi \in \mathrm{D}(A)$. We set $u(t,s) = (u(t))(s) = (T(t)\varphi)(s)$ and for $t \ge 0$ and $s \le 0$. By Proposition 1.10, the function u belongs to $C^1(\mathbb{R}_{\ge 0}, X) \cap C(\mathbb{R}_{\ge 0}, [\mathrm{D}(A)])$ and solves the problem

$$\begin{aligned} \partial_t u(t,s) &= -\partial_s u(t,s), \quad t \ge 0,\ s \le 0,\\ u(0,s) &= \varphi(s), \quad s \le 0. \end{aligned}$$

(Note that X includes the 'boundary condition' $u(t,s) \to 0$ as $s \to -\infty$.) It is straighforward to see that via $v(t,s) = \varphi(s-t)$ one defines another solution in the same function spaces. The uniqueness statement in Proposition 1.10 then yields $u = v$ and hence $T(t)\varphi = \varphi(\cdot - t)$ for all $t \ge 0$. This equation holds for all $\varphi \in X$ by approximation. □

b) We present an operator A that satisfies (1.15) for $n = 1$ and some $M > 1$, but is not a generator. So one cannot omit the powers n in (1.15).

Let $X = C_0(\mathbb{R})^2$ with $\mathbb{F} = \mathbb{C}$, $\|(f,g)\| = \max\{\|f\|_\infty, \|g\|_\infty\}$, $m(s) = \mathrm{i}s$, and

$$A\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} mu + mv \\ mv \end{pmatrix} = \begin{pmatrix} m & m \\ 0 & m \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

for $(u,v) \in \mathrm{D}(A) = \{(u,v) \in X \mid (mu, mv) \in X\}$.

Since $C_c(\mathbb{R}) \times C_c(\mathbb{R}) \subseteq \mathrm{D}(A)$, the domain $\mathrm{D}(A)$ is dense in X. Take (u_n, v_n) in $\mathrm{D}(A)$ such that $(u_n, v_n) \to (u,v)$ and $A(u_n, v_n) \to (f,g)$ in X as $n \to \infty$. By pointwise limits, we infer that $mu + mv = f$ and $mv = g \in C_0(\mathbb{R})$, so that also $mu \in C_0(\mathbb{R})$. As a result, the vector (u,v) belongs to $\mathrm{D}(A)$ and A is closed.

Let $\lambda \in \mathbb{C}_+$. Since $1/(\lambda - m)$ and $m/(\lambda - m)$ are bounded, the operator

$$R(\lambda) = \begin{pmatrix} \frac{1}{\lambda - m} & \frac{m}{(\lambda-m)^2} \\ 0 & \frac{1}{\lambda - m} \end{pmatrix}$$

maps X into $\mathrm{D}(A)$. We further compute

$$(\lambda I - A)R(\lambda) = \begin{pmatrix} \lambda - m & -m \\ 0 & \lambda - m \end{pmatrix}\begin{pmatrix} \frac{1}{\lambda - m} & \frac{m}{(\lambda-m)^2} \\ 0 & \frac{1}{\lambda - m} \end{pmatrix} = I,$$

and similarly $R(\lambda)(\lambda w - Aw) = w$ for $w \in \mathrm{D}(A)$. So we have shown that $\mathbb{C}_+ \subseteq \rho(A)$ and $R(\lambda) = R(\lambda, A)$.

For $\lambda > 0$ and $\|(f,g)\| \le 1$ we next estimate

$$\begin{aligned}\Big\|R(\lambda, A)\begin{pmatrix} f \\ g \end{pmatrix}\Big\| &\le \max\Big\{\Big\|\frac{f}{\lambda - m}\Big\|_\infty + \Big\|\frac{mg}{(\lambda - m)^2}\Big\|_\infty, \Big\|\frac{g}{\lambda - m}\Big\|_\infty\Big\} \\ &\le \sup_{s\in\mathbb{R}}\Big(\frac{1}{|\lambda - \mathrm{i}s|} + \frac{|s|}{|\lambda - \mathrm{i}s|^2}\Big) \le \frac{1}{\lambda} + \sup_{s\in\mathbb{R}}\frac{|s|}{\lambda^2 + s^2} \\ &= \frac{3/2}{\lambda}.\end{aligned}$$

On the other hand, for $a > 0$ and $n \in \mathbb{N}$ we choose $g_n \in C_0(\mathbb{R})$ such that $g_n(n) = 1$ and $\|g_n\|_\infty = 1$. It then follows

$$\begin{aligned}\|R(a + \mathrm{i}n, A)\| &\ge \Big\|R(a + \mathrm{i}n, A)\begin{pmatrix} 0 \\ g_n \end{pmatrix}\Big\| \ge \Big\|\frac{m}{(a + \mathrm{i}n - m)^2}g_n\Big\|_\infty \\ &\ge \Big|\frac{\mathrm{i}n}{(a + \mathrm{i}n - \mathrm{i}n)^2}g_n(n)\Big| = \frac{n}{a^2}.\end{aligned}$$

The resolvent $R(\lambda, A)$ is thus unbounded on every imaginary line $\operatorname{Re}\lambda = a$, violating Proposition 1.20 c); i.e., A does not generate a C_0-semigroup.

There are operators satisfying even $\|R(\lambda, A)\| \le \frac{c}{\operatorname{Re}(\lambda)}$ for some $c > 1$ and all $\lambda \in \mathbb{C}_+$ which fail to be a generator (see Example 2 in §12.4 of [**12**]). ◊

We now turn our attention to the generation of groups. We will reduce this question to the semigroup case, using the following simple fact.

Lemma 1.28. *Let $T(\cdot)$ be a C_0-semigroup and $t_0 > 0$ such that $T(t_0)$ is invertible. Then $T(\cdot)$ can be extended to a C_0-group $(T(t))_{t\in\mathbb{R}}$.*

Proof. Take constants $M \ge 1$ and $\omega \in \mathbb{R}$ with $\|T(t)\| \le M\mathrm{e}^{\omega t}$ for all $t \ge 0$. Set $c = \|T(t_0)^{-1}\|$. Let $0 \le t \le t_0$. We then compute

$$\begin{aligned}T(t_0) &= T(t_0 - t)T(t) = T(t)T(t_0 - t), \\ I &= T(t_0)^{-1}T(t_0 - t)T(t) = T(t)T(t_0 - t)T(t_0)^{-1}.\end{aligned}$$

The operator $T(t)$ thus has the inverse $T(t_0)^{-1}T(t_0 - t)$ with norm less than or equal to $M_1 \coloneqq cM\mathrm{e}^{\omega_+ t_0}$. Next, let $t = nt_0 + \tau$ for some $n \in \mathbb{N}$ and $\tau \in [0, t_0)$. In this case $T(t) = T(\tau)T(t_0)^n$ is invertible with $T(t)^{-1} = T(t_0)^{-n}T(\tau)^{-1}$.

We now define $T(t) \coloneqq T(-t)^{-1}$ for $t \le 0$. This definition gives a group, since for $t, s \ge 0$ we can calculate

$$\begin{aligned}T(-t)T(-s) &= T(t)^{-1}T(s)^{-1} = (T(s)T(t))^{-1} = T(s + t)^{-1} = T(-s - t), \\ T(-t)T(s) &= T(t)^{-1}T(t)T(s - t) = T(s - t) \qquad \text{for } s \ge t, \\ T(-t)T(s) &= (T(s)T(t - s))^{-1}T(s) = T(t - s)^{-1}T(s)^{-1}T(s) \\ &= T(t - s)^{-1} = T(s - t) \qquad \text{for } t \ge s;\end{aligned}$$

and similarly for $T(s)T(-t)$. Let $t \in [0, t_0]$ and $x \in X$. We then obtain

$$\|T(-t)x - x\| = \|T(-t)(x - T(t)x)\| \le M_1\|x - T(t)x\| \to 0$$

as $t \to 0$. So $(T(t))_{t\in\mathbb{R}}$ is a C_0-group by Lemma 1.7. □

We next characterize generators of C_0-groups in the same way as in Theorem 1.26, by requiring resolvent bounds also for negative λ. Moreover, A generates $(T(t))_{t\in\mathbb{R}}$ if and only if A and $-A$ generate $(T(t))_{t\geq 0}$ and $(T(-t))_{t\geq 0}$, respectively. By rescaling, one can shift the growth bound to $\|T(t)\| \leq M\mathrm{e}^{\omega_+ t}$ for $t \geq 0$ and $\|T(t)\| \leq M\mathrm{e}^{\omega_- t}$ for $t \leq 0$, and any given $\omega_- \leq \omega_+$ in $\mathbb{R}$.

Theorem 1.29. *Let A be a linear operator, $M \geq 1$, and $\omega \geq 0$. The following assertions are equivalent.*

a) A generates a C_0-group $(T(t))_{t\in\mathbb{R}}$ with $\|T(t)\| \leq M\mathrm{e}^{\omega|t|}$ for all $t \in \mathbb{R}$.

b) A generates a C_0-semigroup $(T_+(t))_{t\geq 0}$, and $-A$ with $\mathrm{D}(-A) := \mathrm{D}(A)$ generates a C_0-semigroup $(T_-(t))_{t\geq 0}$ with $\|T_\pm(t)\| \leq M\mathrm{e}^{\omega t}$ for all $t \geq 0$.

c) A is closed, $\overline{\mathrm{D}(A)} = X$, and for all $\lambda \in \mathbb{R}$ with $|\lambda| > \omega$ we have $\lambda \in \rho(A)$ and $\|(|\lambda| - \omega)^n R(\lambda, A)^n\| \leq M$ for all $n \in \mathbb{N}$.

If one (and thus all) of these conditions is (are) fulfilled, one has $T_+(t) = T(t)$ and $T_-(t) = T(-t)$ for every $t \geq 0$. Moreover, in part c) one can then replace '$\lambda \in \mathbb{R}$' by '$\lambda \in \mathbb{C}$' and '$|\lambda|$' by '$|\mathrm{Re}\,\lambda|$' (provided that $\mathbb{F} = \mathbb{C}$).

Proof. 1) We first deduce statement b) from a). Assuming a), we set $T_+(t) = T(t)$ and $T_-(t) = T(-t)$ for each $t \geq 0$. Recall from Remark 1.2 that $T(-t) = T(t)^{-1}$. It is easy to check that one thus obtains two C_0-semigroups with growth bounds $\|T_\pm(t)\| \leq M\mathrm{e}^{\omega t}$. We denote their generators by $A_\pm$.

For $x \in \mathrm{D}(A)$, there exists $\frac{\mathrm{d}}{\mathrm{d}t}T(0)x = Ax$ implying $A \subseteq A_+$ and $A \subseteq -A_-$. To show the inverse inclusion, let $x \in \mathrm{D}(A_-)$ and $t > 0$. We then compute

$$\begin{aligned} \tfrac{1}{t}(T(t)x - x) &= -T(t)\tfrac{1}{t}(T_-(t)x - x) \longrightarrow -A_-x, \\ \tfrac{1}{-t}(T(-t)x - x) &= -\tfrac{1}{t}(T_-(t)x - x) \longrightarrow -A_-x \end{aligned}$$

as $t \to 0$, so that $x \in \mathrm{D}(A)$ and hence $A = -A_-$. One proves $A = A_+$ similarly. Therefore, property b) and the first adddendum are true.

2) Let b) be valid. For $\lambda > \omega$, assertion c) follows from Theorem 1.26. For $\lambda < -\omega$, we use that $\sigma(A) = -\sigma(-A)$ with $R(\lambda, A) = -R(-\lambda, -A)$, cf. Example 1.21 a). Theorem 1.26 thus also yields the estimate in part c) for $\lambda < -\omega$ since here $-\lambda = |\lambda|$. The second addendum is shown in the same way.

3) We assume claim c) and derive statement a). Theorem 1.26 implies that A generates a C_0-semigroup $(T_+(t))_{t\geq 0}$ and $-A$ generates a C_0-semigroup $(T_-(t))_{t\geq 0}$ (arguing for $-A$ as in the previous step). Let $x \in \mathrm{D}(A) = \mathrm{D}(-A)$ and $t \geq s \geq 0$. Proposition 1.10 and its proof imply

$$\tfrac{\mathrm{d}}{\mathrm{d}s}T_+(s)T_-(s)x = T_+(s)AT_-(s)x + T_+(s)(-A)T_-(s)x = 0$$

and then $T_+(t)T_-(t)x = x$. Analogously, one obtains $T_-(t)T_+(t)x = x$. It follows that $I = T_+(t)T_-(t) = T_-(t)T_+(t)$ since $\mathrm{D}(A)$ is dense. By Lemma 1.28, $T_+(\cdot)$ can thus be extended to a C_0-group. Let B be its generator. We have $B \subseteq A$ by definition and $\mathrm{s}(B) < \infty$ by step 1) and Proposition 1.20. Condition c) and Lemma 1.23 then yield $A = B$ and hence assertion a). □

1.3. Dissipative operators

Even in the contraction case, the Hille-Yosida Theorem 1.26 poses the difficult task to show a resolvent estimate with constant 1 for all $\lambda > 0$. In this section we prove the Lumer-Phillips Theorem 1.39 which reduces this task to checking the *dissipativity* and a certain range condition of A. The former property can often be verified by direct computations, and for the latter there are powerful (also functional analytic) tools to solve the occuring equations. Below these matters will be illustrated by the first derivative again, more involved applications will be treated in the following section.

We start with an auxiliary notion. The *duality set* $J(x)$ of a vector $x \in X$ is defined by

$$J(x) = \{x^* \in X^* \,|\, \langle x, x^* \rangle = \|x\|^2, \ \|x\| = \|x^*\| \},$$

where $\langle x, x^* \rangle = x^*(x)$ for all $x \in X$ and $x^* \in X^*$. The Hahn–Banach theorem ensures that $J(x) \neq \emptyset$, cf. Corollary 5.10 in [**24**]. In standard function spaces one can compute elements in the duality set explicitely.

EXAMPLE 1.30. a) Let X be a Hilbert space with scalar product $(\cdot|\cdot)$. By Riesz' Theorem 3.10 in [**24**], for each functional $y^* \in X^*$ there is a unique vector $y \in X$ satisfying $\langle z, y^* \rangle = (z|y)$ for all $z \in X$, and one has $\|y\| = \|y^*\|$. As a result, $y^* \in J(x)$ is equivalent to $\|x\| = \|y\|$ and $(x|y) = \|x\|^2$, or to $\|x\| = \|y\|$ and $(x|y) = \|x\|\|y\|$. These conditions are valid if and only if $y = \alpha x$ for some $\alpha \in \mathbb{F}$ with $|\alpha| = 1$ (due to the characterization of equality in the Cauchy–Schwarz inequality). Inserting this expression in $(x|y) = \|x\|^2$, we see that $x = y$. The converse implication is clear. Consequently, $J(x) = \{\varphi_x\}$ for the functional given by $\varphi_x(z) = (z|x)$.

b) Let $X = L^p(\mu)$ for an exponent $p \in [1, \infty)$ and a measure space $(S, \mathcal{A}, \mu)$, which has to be σ-finite if $p = 1$. We identify X^* with $L^{p'}(\mu)$ via the usual duality pairing $\langle f, g \rangle_{L^p \times L^{p'}} = \int fg \, \mathrm{d}\mu$, where $p' = \frac{p}{p-1}$ for $p > 1$ and $1' = \infty$, see Theorem 5.4 in [**24**]. Take $f \in X \setminus \{0\}$. We set

$$g = \|f\|_p^{2-p} \, \overline{f} |f|^{p-2}$$

writing $\frac{0}{0} := 0$. For $p = 1$, we have $\|g\|_\infty = \|f\|_1$. For $p > 1$, we compute

$$\|g\|_{p'} = \|f\|_p^{2-p} \Big(\int_S |f|^{(p-1)\cdot \frac{p}{p-1}} \, \mathrm{d}\mu \Big)^{\frac{p-1}{p}} = \|f\|_p^{2-p} \, \|f\|_p^{p-1} = \|f\|_p.$$

Since also

$$\langle f, g \rangle = \|f\|_p^{2-p} \int_S f \, \overline{f} |f|^{p-2} \, \mathrm{d}\mu = \|f\|_p^{2-p} \, \|f\|_p^p = \|f\|_p^2 \,,$$

we obtain $g \in J(f)$. It follows from an exercise that $J(f) = \{g\}$ if $p \in (1, \infty)$. Note that $g = \overline{f}$ for $p = 2$ which corresponds to part a).

c) Let $\emptyset \neq U \subseteq \mathbb{R}^m$ be open and $E = C_0(U)$ with

$$C_0(U) := \{ f \in C(U) \,|\, f(x) \to 0 \text{ as } x \to \partial U \text{ and as } |x| \to \infty \text{ for unbounded } U \},$$

which is a Banach space for the supremum norm. For $f \in E$ there is a point $x_0 \in U$ with $|f(x_0)| = \|f\|_\infty$. Set $\varphi(g) = \overline{f(x_0)} g(x_0)$ for $g \in E$; i.e., $\varphi = \overline{f(x_0)} \delta_{x_0}$.

As in Example 2.8 of [**24**] one checks that $\varphi \in E^*$ with $\|\varphi\| = |f(x_0)| = \|f\|_\infty$. We clearly have $\varphi(f) = |f(x_0)|^2 = \|f\|_\infty^2$. Hence, φ belongs to $J(f)$. The same construction works on $E = C(K)$ for a compact metric space K. $\Diamond$

We now state the core concept of this section.

Definition 1.31. *A linear operator A is called* dissipative *if for each vector $x \in \mathrm{D}(A)$ there is a functional $x^* \in J(x)$ such that $\operatorname{Re}\langle Ax, x^*\rangle \le 0$. The operator A is called* accretive *if $-A$ is dissipative.*

The next fundamental characterization provides the link between dissipativity and the resolvent condition (1.17) in the Hille–Yosida theorem. We also show that a generator of a contraction semigroup is dissipative in a somewhat stronger sense, which will be used in Theorem 3.8.

Proposition 1.32. *A linear operator A is dissipative if and only if it satisfies $\|\lambda x - Ax\| \ge \lambda\|x\|$ for all $\lambda > 0$ and $x \in \mathrm{D}(A)$. If A generates a contraction semigroup, then we have $\operatorname{Re}\langle Ax, x^*\rangle \le 0$ for every $x \in \mathrm{D}(A)$ and all $x^* \in J(x)$.*

Proof. 1) Let A generate the contraction semigroup $T(\cdot)$. Take $x \in \mathrm{D}(A)$ and $x^* \in J(x)$. Using $x^* \in J(x)$ and the contractivity, we estimate

$$\begin{aligned}\operatorname{Re}\langle Ax, x^*\rangle &= \lim_{t\to 0^+} \operatorname{Re}\left\langle \tfrac{1}{t}(T(t)x - x), x^*\right\rangle = \lim_{t\to 0^+} \tfrac{1}{t}\big(\operatorname{Re}\langle T(t)x, x^*\rangle - \|x\|^2\big)\\ &\le \limsup_{t\to 0^+} \tfrac{1}{t}(\|x\|\,\|x^*\| - \|x\|^2) = 0.\end{aligned}$$

2) Let A be dissipative. Take $x \in \mathrm{D}(A)$ and $\lambda > 0$. There thus exists a functional $x^* \in J(x)$ with $\operatorname{Re}\langle Ax, x^*\rangle \le 0$. These facts imply the inequalities

$$\lambda\|x\|^2 \le \operatorname{Re}(\lambda\langle x, x^*\rangle) - \operatorname{Re}\langle Ax, x^*\rangle \le |\langle \lambda x - Ax, x^*\rangle| \le \|\lambda x - Ax\|\|x^*\|.$$

Since $\|x\| = \|x^*\|$, it follows $\lambda\|x\| \le \|\lambda x - Ax\|$.

3) Conversely, let $\|\lambda x - Ax\| \ge \lambda\|x\|$ be true for all $\lambda > 0$ and $x \in \mathrm{D}(A)$. If $x = 0$ we can take $x^* = 0$ in the definition of dissipativity. Otherwise, we replace x by $\|x\|^{-1}x$, and will thus assume that $\|x\| = 1$.

Take $y_\lambda^* \in J(\lambda x - Ax)$. This functional is not zero since $\|y_\lambda^*\| = \|\lambda x - Ax\| \ge \lambda\|x\| = \lambda > 0$ by the assumptions. We now set $x_\lambda^* = \|y_\lambda^*\|^{-1} y_\lambda^*$ and note that $\|x_\lambda^*\| = 1$. Using the above properties, we deduce

$$\begin{aligned}\lambda \le \|\lambda x - Ax\| &= \frac{1}{\|y_\lambda^*\|}\langle \lambda x - Ax, y_\lambda^*\rangle = \operatorname{Re}\langle \lambda x - Ax, x_\lambda^*\rangle\\ &= \lambda \operatorname{Re}\langle x, x_\lambda^*\rangle - \operatorname{Re}\langle Ax, x_\lambda^*\rangle \le \min\{\lambda - \operatorname{Re}\langle Ax, x_\lambda^*\rangle,\ \lambda\operatorname{Re}\langle x, x_\lambda^*\rangle + \|Ax\|\}.\end{aligned}$$

This inequality implies the core bounds

$$\operatorname{Re}\langle Ax, x_\lambda^*\rangle \le 0 \qquad \text{and} \qquad 1 - \tfrac{1}{\lambda}\|Ax\| \le \operatorname{Re}\langle x, x_\lambda^*\rangle.$$

Let $\tilde{x}_\lambda^*$ be the restriction of x_λ^* to the space $E = \operatorname{lin}\{x, Ax\}$ equipped with the norm of X. Because of $\|\tilde{x}_\lambda^*\| \le \|x_\lambda^*\| = 1$, the Bolzano–Weierstraß theorem yields a sequence (λ_j) in $\mathbb{R}_+$ and a vector $y^* \in E^*$ such that $\lambda_j \to \infty$ and $\tilde{x}_{\lambda_j}^* \to y^*$ as $j \to \infty$. These limits lead to

$$\|y^*\| \le 1, \qquad \operatorname{Re}\langle Ax, y^*\rangle \le 0 \qquad \text{and} \qquad 1 \le \operatorname{Re}\langle x, y^*\rangle.$$

The Hahn–Banach theorem allows us to extend y^* to a functional $x^* \in X^*$ with $\|x^*\| = \|y^*\| \leq 1$. It then satisfies $\operatorname{Re}\langle Ax, x^*\rangle \leq 0$ and

$$1 \leq \operatorname{Re}\langle x, x^*\rangle \leq |\langle x, x^*\rangle| \leq \|x^*\| \leq 1$$

since $\|x\| = 1$. So we have equalities in the above formula, which means that $\|x^*\| = 1 = \|x\|$ and $\langle x, x^*\rangle = 1 = \|x\|^2$; i.e., $x^* \in J(x)$ and A is dissipative. □

The dissipativity of differential operators A heavily depends on the boundary conditions, as we now discuss for first-order operators on an interval.

EXAMPLE 1.33. a) Let $X = C_0(\mathbb{R})$, $b, c \in C_b(\mathbb{R})$ be real-valued, and $Au = bu' + cu$ with $\mathrm{D}(A) = C_0^1(\mathbb{R})$. Take $u \in \mathrm{D}(A)$ and some $s_0 \in \mathbb{R}$ with $|u(s_0)| = \|u\|_\infty$. Then $\varphi = \overline{u}(s_0)\delta_{s_0}$ belongs to $J(u)$ by Example 1.30. We compute

$$\begin{aligned} r &:= \operatorname{Re}\langle Au - \|c_+\|_\infty u, \varphi\rangle = b(s_0)\operatorname{Re}(u'(s_0)\overline{u}(s_0)) + (c(s_0) - \|c_+\|_\infty)\,|u(s_0)|^2 \\ &\leq b(s_0)\operatorname{Re}(u'(s_0)\overline{u}(s_0)). \end{aligned}$$

We set $h(s) = \operatorname{Re}(\overline{u}(s_0)u(s))$ for $s \in \mathbb{R}$. Clearly, $h \in C_0^1(\mathbb{R})$ is real-valued and

$$|u(s_0)|^2 = h(s_0) \leq \|h\|_\infty \leq |u(s_0)|\,\|u\|_\infty = |u(s_0)|^2,$$

so that h attains its maximum at s_0. It follows $h'(s_0) = 0$ implying $r \leq 0$. This means that $A - \|c_+\|_\infty I$ is dissipative.

b) Let $X = C([0,1])$, $b, c \in X$ be real-valued, $b(0) \geq 0$ for simplicity, and $A_j = bu' + cu$ with $\mathrm{D}(A_j) = \{u \in C^1([0,1]) \,|\, u'(j) = 0\}$ for $j \in \{0,1\}$. Then $A_1 - \|c_+\|_\infty I$ is dissipative. If $b(1) \leq 0$, also $A_0 - \|c_+\|_\infty I$ is dissipative. On the other hand, if $b(1) > 0$ the operator $A_0 - \omega I$ does not generate a contraction semigroup for any $\omega \in \mathbb{R}$.[3]

PROOF. For $u \in \mathrm{D}(A_j)$, we use the functional $\varphi(v) = \overline{u}(s_0)v(s_0)$ on X, where $|u(s_0)| = \|u\|_\infty$ for some $s_0 \in [0,1]$. We also set $h(s) = \operatorname{Re}(\overline{u}(s_0)u(s))$ for $s \in [0,1]$. As in a), one sees that φ belongs to $J(u)$, $h \in C^1([0,1])$ attains its maximum at s_0, and

$$r := \operatorname{Re}\langle A_j u - \|c_+\|_\infty u, \varphi\rangle \leq b(s_0)\operatorname{Re}(u'(s_0)\overline{u}(s_0)) = b(s_0)h'(s_0).$$

If $s_0 \in (0,1)$, this inequality again yields $r \leq 0$. Similarly, for $s_0 = 0$ we obtain

$$h'(0) = \lim_{s\to 0^+} \tfrac{1}{s}(h(s) - h(0)) \leq 0$$

since $h(0)$ is a maximum of h. Using $b(0) \geq 0$, we infer $r \leq 0$.

Finally, let $s_0 = 1$. In this case the above argument leads to $h'(1) \geq 0$. We first look at $j = 1$. Here we have the boundary condition $u'(1) = 0$ and thus $h'(1) = 0$. It follows that $r \leq b(1)h'(1) = 0$ and so $A_1 - \|c_+\|_\infty I$ is dissipative.

Let $j = 0$. For $b(1) \leq 0$, we derive $r \leq b(1)h'(1) \leq 0$ so that $A_0 - \|c_+\|_\infty I$ is dissipative in this case. Next, let $b(1) > 0$. Fix $\omega \in \mathbb{R}$. Choose a real-valued function $u \in \mathrm{D}(A_0)$ with maximum $u(1) = 1$ and $u'(1) > (\omega - c(1))/b(1)$. Since then $\varphi = \delta_1$, we obtain the inequality

$$\operatorname{Re}\langle A_0 u - \omega u, \varphi\rangle = b(1)u'(1) + c(1) - \omega > 0.$$

Hence, $A_0 - \omega I$ cannot generate a contraction semigroup by Proposition 1.32. (Note that we did not show that $\operatorname{Re}\langle A_0 u - \omega u, \psi\rangle > 0$ for *all* $\psi \in J(u)$.)

[3] Using a more sophisticated construction, one can show that it is not dissipative.

□

c) Let $X = L^2(\mathbb{R})$ and $A = \frac{\mathrm{d}}{\mathrm{d}s}$ with $\mathrm{D}(A) = C_c^1(\mathbb{R})$. For $u \in \mathrm{D}(A)$, Example 1.30 yields $\{\overline{u}\} = J(u)$. Integrating by parts, we compute

$$2\,\mathrm{Re}\langle Au, \overline{u}\rangle = \langle Au, \overline{u}\rangle + \overline{\langle Au, \overline{u}\rangle} = \int_{\mathbb{R}} u'\overline{u}\,\mathrm{d}s + \int_{\mathbb{R}} \overline{u}'u\,\mathrm{d}s = 0;$$

i.e., A is dissipative (but not closed by Example 1.42). Then above formula also implies the dissipativity of $-A$.

d) Let $X = L^2(0,1)$, $A_j = \frac{\mathrm{d}}{\mathrm{d}s}$, and $\mathrm{D}(A_j) = \{u \in C^1([0,1]) \,|\, u(j) = 0\}$ for $j \in \{0,1\}$. For $u \in \mathrm{D}(A_j)$ we again have $\{\overline{u}\} = J(u)$ and obtain

$$2\,\mathrm{Re}\langle Au, \overline{u}\rangle = \int_0^1 u'\overline{u}\,\mathrm{d}s + \int_0^1 \overline{u}'u\,\mathrm{d}s = u\overline{u}\big|_0^1 = |u(1)|^2 - |u(0)|^2.$$

It follows that A_1 is dissipative. However, $A_0 - \omega I$ is not dissipative for any $\omega \in \mathbb{R}$, since we can find a map u in $\mathrm{D}(A_0)$ satisfying $|u(1)|^2 > 2\omega\|u\|_2^2$ and so

$$\mathrm{Re}\langle A_0u - \omega u, \overline{u}\rangle = \tfrac{1}{2}|u(1)|^2 - \omega\|u\|_2^2 > 0. \qquad \Diamond$$

Examples c) and d) can be extended to L^p with $p \in [1,\infty)$, cf. Example 1.48. Above we have encountered rather natural dissipative, but non-closed operators. To treat such operators, we introduce a concept extending closedeness.

Intermezzo 2: Closable operators.

Definition 1.34. *A linear operator A is called* closable *if it possesses a closed extension B.*

Note that a closed operator is closable since $A \subseteq A$. We first characterize closability and construct the *closure* $\overline{A}$ of a closable operator A, which is the smallest closed extension of A.

Lemma 1.35. *For a linear operator A, the following statements are equivalent.*

a) The operator A is closable.

b) Let (x_n) be a sequence in $\mathrm{D}(A)$ with $x_n \to 0$ and $Ax_n \to y$ in X as $n \to \infty$. Then $y = 0$.

c) In the set $\mathrm{D}(\overline{A}) = \{x \in X \,|\, \exists\,(x_n) \text{ in } \mathrm{D}(A),\ y \in X : x_n \to x,\ Ax_n \to y,\ n \to \infty\}$ the vector y is uniquely determined by x. Letting $\overline{A} : \mathrm{D}(\overline{A}) \to X$; $\overline{A}x = y$, one thus defines a map. The operator $\overline{A}$ is linear, closed, and extends A.

Let one and hence all of the properties a)–c) are valid. Then $\mathrm{Gr}(\overline{A}) = \overline{\mathrm{Gr}(A)}$, $\mathrm{D}(A)$ is dense in $[\mathrm{D}(\overline{A})]$, and we have $\overline{A} \subseteq B$ for every closed operator $B \supseteq A$.

Proof. Part c) clearly implies a). Let a) be true and B be a closed extension of A. Take (x_n) as in statement b). Then the vectors $Ax_n = Bx_n$ tend to $y = B0 = 0$ since B is closed.

We assume that property b) holds. Let (x_n) and (z_n) be sequences in $\mathrm{D}(A)$ with limit x in X such that (Ax_n) converges to y and (Az_n) to w in X. Then $(x_n - z_n)$ is a null sequence in X with $A(x_n - z_n) = Ax_n - Az_n \to y - w$ as $n \to \infty$. Part b) thus implies $y = w$, so that $\overline{A}$ is a mapping. One easily verifies that $\overline{A}$ is linear and that $\mathrm{Gr}(\overline{A}) = \overline{\mathrm{Gr}(A)}$, which shows the first part of the addendum. Hence, $\overline{A}$ is closed due to Remark 1.15 and $\overline{A}$ extends A. Therefore assertion c) is shown.

Let B be another closed extension of A. We then have $\mathrm{Gr}(A) \subseteq \mathrm{Gr}(B)$ and so $\mathrm{Gr}(\overline{A}) = \overline{\mathrm{Gr}(A)} \subseteq \mathrm{Gr}(B)$ because of the closedness of B. In particular, B extends $\overline{A}$. The density assertion is an immediate consequence of $\overline{\mathrm{Gr}(A)} = \mathrm{Gr}(\overline{A})$ and the definition of the graph norm. □

As consequence of this lemma, a linear operator is closed if and only if it is its own closure. We illustrate the concepts of extension and closure by the first derivative, again stressing the role of the boundary conditions.

EXAMPLE 1.36. a) Let $X = L^1(0,1)$ and $Af = f(0)\mathbb{1}$ with $\mathrm{D}(A) = C([0,1])$. This operator is not closable. In fact, the functions $f_n \in \mathrm{D}(A)$ given by $f_n(s) = \max\{1 - ns, 0\}$ satisfy $\|f_n\|_1 = \frac{1}{2n} \to 0$ as $n \to \infty$, but $Af_n = \mathbb{1}$ for all $n \in \mathbb{N}$, contradicting Lemma 1.35 b).

b) Let $X = C([0,1])$ and $A_0 u = u'$ with $\mathrm{D}(A_0) = C_c^1(0,1) \coloneqq C_c^1((0,1))$, as well as $Au = u'$ with $\mathrm{D}(A) = C_0^1(0,1) \coloneqq C_0^1((0,1))$. As in Example 1.14 we see that A is closed. Hence, A_0 is closable and $\overline{A_0} \subseteq A$ since $A_0 \subseteq A$. To check equality, let $u \in C_0^1(0,1)$. Take $\varphi_n \in C_c^1(0,1)$ such that $\varphi = 1$ on $[\frac{1}{n}, 1 - \frac{1}{n}]$, $0 \le \varphi_n \le 1$ and $\|\varphi_n'\|_\infty \le cn$ for some $c > 0$ and all $n \in \mathbb{N}$ with $n \ge 2$. (For instance, one can take

$$\varphi_n(s) = \begin{cases} 0, & 0 < s < \frac{1}{4n}, \\ 8n^2\big(s - \frac{1}{4n}\big)^2, & \frac{1}{4n} \le s \le \frac{1}{2n}, \\ 1 - 8n^2\big(\frac{3}{4n} - s\big)^2, & \frac{1}{2n} \le s \le \frac{3}{4n}, \\ 1, & \frac{3}{4n} < s \le \frac{1}{2}, \\ \varphi_n(1-s), & \frac{1}{2} < s < 1, \end{cases}$$

where $c = 4$.) Then the function $u_n = \varphi_n u$ belongs to $\mathrm{D}(A_0)$, and we have

$$\|u_n - u\|_\infty = \sup_{s \in [0,\frac{1}{n}] \cup [1-\frac{1}{n},1]} |(\varphi_n(s) - 1)u(s)| \le \sup_{s \in [0,\frac{1}{n}] \cup [1-\frac{1}{n},1]} |u(s)| \longrightarrow 0,$$

$$\|\varphi_n u' - u'\|_\infty \le \sup_{s \in [0,\frac{1}{n}] \cup [1-\frac{1}{n},1]} |(\varphi_n(s) - 1)u'(s)| \longrightarrow 0$$

as $n \to \infty$ since $u, u' \in C_0(0,1)$. We further obtain

$$\begin{aligned} \|\varphi_n' u\|_\infty &\le \sup_{s \in [0,\frac{1}{n}]} |\varphi_n'(s) u(s)| + \sup_{s \in [1-\frac{1}{n},1]} |\varphi_n'(s) u(s)| \\ &\le \sup_{s \in [0,\frac{1}{n}]} cn \Big| \int_0^s u'(\tau)\,\mathrm{d}\tau \Big| + \sup_{s \in [1-\frac{1}{n},1]} cn \Big| \int_s^1 u'(\tau)\,\mathrm{d}\tau \Big| \\ &\le cn \int_0^{\frac{1}{n}} |u'(\tau)|\,\mathrm{d}\tau + cn \int_{1-\frac{1}{n}}^1 |u'(\tau)|\,\mathrm{d}\tau \longrightarrow 0 \end{aligned}$$

as $n \to \infty$, because of (1.4) and $u' \in C_0(0,1)$. Hence, $A_0(\varphi_n u) = \varphi_n' u + \varphi_n u'$ converges to $Au = u'$. This means that $A \subseteq \overline{A_0}$ and thus $\overline{A_0} = A$. In particular A_0 is not closed and thus fails to be a generator.

We discuss further closed extensions of A_0 given by $A_j u = u'$ for $j \in \{1, 2, 3\}$.

1) Let $\mathrm{D}(A_1) = \{u \in C^1([0,1]) \,|\, u'(1) = 0\}$. By an exercise, A_1 generates a C_0-semigroup on X and $\sigma(A_1) = \{0\}$. Observe that A_1 is a strict extension of A. Lemma 1.23 thus implies that $\rho(A) \cap \rho(A_1) = \emptyset$ and hence $\mathbb{F} \setminus \{0\} \subseteq \sigma(A)$. (Actually, we have $\sigma(A) = \mathbb{F}$ since $\mathbb{1} \notin A\mathrm{D}(A)$.) As a result, A is not generator – it has too many boundary conditions, namely four instead of one as in $\mathrm{D}(A_1)$.

2) Let $\mathrm{D}(A_3) = C^1([0,1])$. Example 1.21 says that $\sigma(A_3) = \mathbb{F}$. So A_3 is not a generator because it has not enough boundary conditions, namely none. We have $A \subsetneq A_1 \subsetneq A_3$.

3) Let $\mathrm{D}(A_2) = \{u \in C^1([0,1]) \,|\, u(1) = 0\}$. Also A_2 is 'sandwiched' between A and A_3; i.e., $A \subsetneq A_2 \subsetneq A_3$, but A_1 and A_2 are not comparable. The operator A_2 is not a generator as its domain is not dense, see Example 1.21.

Summing up, the 'minimal' operator A and the 'maximal' operator A_3 do not generate C_0-semigroups. Between them there are various, partly noncomparable operators (so–called 'realizations' of $\frac{\mathrm{d}}{\mathrm{d}s}$) which may or may not be generators. Their domains are often determined by boundary conditions. ◊

We come back to the study of C_0-semigroups. Below we use closures in a generation result, but at first we establish sufficient conditions for a subspace D to be dense in $\mathrm{D}(A)$ in the graph norm. Such a subspace is called *core* of a closed operator A. Observe that D is a core if and only if A is the closure of the restriction $A{\restriction_D}$. In Example 1.36 b) the set $C_c^1(0,1)$ is a core for A. One can often extend properties from cores to the full domain, see e.g. Proposition 1.38 c).

It is often difficult to decide whether a subspace D is a core of an operator A. The next result gives a convenient sufficient condition involving the semigroup.

Proposition 1.37. *Let A generate the C_0-semigroup $T(\cdot)$ on X. Let D be a linear subspace of* $\mathrm{D}(A)$ *which is dense in X and* invariant *under the semigroup; i.e., $T(t)D \subseteq D$ for all $t \geq 0$. Then D is dense in* $[\mathrm{D}(A)]$.

Proof. Set $C = \sup_{0\le t\le 1} \|T(t)\| < \infty$. Let $x \in \mathrm{D}(A)$. The map $T(\cdot)x : \mathbb{R}_{\geq 0} \to [\mathrm{D}(A)]$ is continuous by Proposition 1.10. Take $\varepsilon > 0$. There is a time $\tau = \tau(\varepsilon, x) \in (0,1]$ with $\|T(t)x - x\|_A \leq \varepsilon$ for all $t \in [0,\tau]$. It follows

$$\Big\|\frac{1}{\tau}\int_0^\tau T(t)x\,\mathrm{d}t - x\Big\|_A \leq \frac{1}{\tau}\int_0^\tau \|T(t)x - x\|_A\,\mathrm{d}t \leq \varepsilon.$$

Using the density of D in X, we find a vector $y \in D$ with

$$\|x - y\| \leq \Big(C + \frac{C+1}{\tau}\Big)^{-1}\varepsilon.$$

Let $\tilde{D}$ be the closure of D in $[\mathrm{D}(A)]$. We want to replace y by a vector z in $\tilde{D}$ that is close to x for $\|\cdot\|_A$. To this aim, we set

$$z = \frac{1}{\tau}\int_0^\tau T(t)y\,\mathrm{d}t.$$

The integrand $T(t)y$ takes values in D by assumption, and as above it is continuous in $[\mathrm{D}(A)]$. In view of the definition of the integral, z thus belongs to $\tilde{D}$. The previous inequalities and Lemma 1.18 imply the bound

$$\|x - z\|_A \leq \Big\|x - \frac{1}{\tau}\int_0^\tau T(t)x\,\mathrm{d}t\Big\|_A + \frac{1}{\tau}\Big\|\int_0^\tau T(t)(x-y)\mathrm{d}t\Big\| + \frac{1}{\tau}\Big\|A\int_0^\tau T(t)(x-y)\mathrm{d}t\Big\|$$

$$\leq \varepsilon + \frac{C}{\tau}\int_0^\tau \|x-y\|\,\mathrm{d}t + \frac{1}{\tau}\,\|(T(\tau)-I)(x-y)\|$$
$$\leq \varepsilon + \Big(C + \frac{C+1}{\tau}\Big)\|x-y\| \leq 2\varepsilon.$$

Finally, there is a vector $w \in D$ with $\|z-w\|_A \leq \varepsilon$, and hence $\|x-w\|_A \leq 3\varepsilon$. □

The next result shows further important properties of dissipative operators following from the characterization in Proposition 1.32. In particular, the Hille–Yosida estimate (1.17) is reduced to a range condition, and a densely defined, dissipative operator has a dissipative closure.

Proposition 1.38. *Let A be dissipative. Then the following assertions hold.*

a) Let $\lambda > 0$. Then the operator $\lambda I - A$ is injective and for $y \in \mathrm{R}(\lambda I - A) = (\lambda I - A)(\mathrm{D}(A))$ we have $\|(\lambda I - A)^{-1}y\| \leq \frac{1}{\lambda}\|y\|$.

b) Let $\lambda_0 I - A$ be surjective for some $\lambda_0 > 0$. Then A is closed, $\mathbb{R}_+ \subseteq \rho(A)$, and $\|R(\lambda, A)\| \leq \frac{1}{\lambda}$ for all $\lambda > 0$.

c) Let $\mathrm{D}(A)$ be dense in X. Then A is closable and $\overline{A}$ is also dissipative.

Proof. Assertion a) immediately follows from Proposition 1.32, using that $y = \lambda x - Ax$ for some $x \in \mathrm{D}(A)$.

Let the assumptions in b) hold. Part a) then implies that $\lambda_0 I - A$ has an inverse with norm less than or equal to $\frac{1}{\lambda_0}$. In particular, A is closed by Remark 1.16 b). Let $\lambda \in (0, 2\lambda_0)$. Since $|\lambda - \lambda_0| < \lambda_0 \leq \|R(\lambda_0, A)\|^{-1}$, Remark 1.16 c) shows that λ belongs to $\rho(A)$. Step a) also yields the estimate $\|R(\lambda, A)\| \leq \frac{1}{\lambda}$. In view of (1.8) also $2\lambda_0$ is contained in $\rho(A)$ and thus $\|R(2\lambda_0, A)\| \leq \frac{1}{2\lambda_0}$. We can now iterate this argument, deriving assertion b).

c) Assume that $\mathrm{D}(A)$ is dense in X. To check the closability of A, we choose a sequence (x_n) in $\mathrm{D}(A)$ with limit 0 in X such that (Ax_n) converges in X to some $y \in X$. By density, there are vectors y_k in $\mathrm{D}(A)$ tending to y in X as $k \to \infty$. Take $\lambda > 0$ and $n, k \in \mathbb{N}$. Proposition 1.32 implies the lower bound

$$\|\lambda^2 x_n - \lambda A x_n + \lambda y_k - A y_k\| = \|(\lambda I - A)(\lambda x_n + y_k)\| \geq \lambda \|\lambda x_n + y_k\|.$$

Letting $n \to \infty$, we deduce $\|-\lambda y + \lambda y_k - A y_k\| \geq \lambda \|y_k\|$ and thus

$$\|-y + y_k - \lambda^{-1} A y_k\| \geq \|y_k\|.$$

As $\lambda \to \infty$, it follows that $\|-y + y_k\| \geq \|y_k\|$. Taking the limit $k \to \infty$, we conclude $y = 0$. Due to Lemma 1.35, the operator A is closable.

Let $x \in \mathrm{D}(\overline{A})$. Then there are vectors $z_n \in \mathrm{D}(A)$ with $z_n \to x$ and $Az_n \to \overline{A}x$ in X as $n \to \infty$. Using Proposition 1.32, we now infer the estimate

$$\|\lambda x - \overline{A}x\| = \lim_{n\to\infty} \|\lambda z_n - A z_n\| \geq \lim_{n\to\infty} \lambda \|z_n\| = \lambda \|x\|,$$

and thus the dissipativity of $\overline{A}$. □

The following theorem by *Lumer* and *Phillips* from 1961 is the most important result to verify the generator property in concrete cases (besides Theorem 2.25 below). To show that an operator A (or its closure) generates a contraction semigroup, one only has to establish the density of $\mathrm{D}(A)$, the dissipativity of A, and that $\lambda_0 I - A$ is surjective (or has dense range) for some

$\lambda_0 > 0$. The first two properties can often be checked by direct computations using the given information on A. The range condition usually is harder to show. One has to solve the 'stationary problem'

$$\exists\, u \in \mathrm{D}(A): \quad \lambda_0 u - Au = f$$

at least for f from a dense set of 'good' vectors. (We thus reduce the investigation of the dynamical problem (1.1) to a stationary one.) Based on our preparations, the Lumer–Phillips theorem can easily be deduced from the contraction case of the Hille–Yosida Theorem 1.26. In Example 1.49 we will see that one cannot omit the range conditions in parts a) or b).

THEOREM 1.39. *Let A be a linear and densely defined operator. The following assertions hold.*

a) Let A be dissipative and $\lambda_0 > 0$ such that $\lambda_0 I - A$ has dense range. Then $\overline{A}$ generates a contraction semigroup.

b) Let A be dissipative and $\lambda_0 > 0$ such that $\lambda_0 I - A$ is surjective. Then A generates a contraction semigroup.

c) Let A generate a contraction semigroup. Then A is dissipative, $\mathbb{F}_+ \subseteq \rho(A)$, and $\|R(\lambda, A)\| \le 1/\operatorname{Re}(\lambda)$ for $\lambda \in \mathbb{F}_+$.

One can replace 'contraction' by 'ω-contraction' and A by $A - \omega I$ for $\omega \in \mathbb{R}$.

Operators satisfying the assumptions in assertion b) are called *maximally dissipative* or *m-dissipative*. (Such maps cannot have non-trivial dissipative extensions because of Lemma 1.23 and Proposition 1.38 a).) If a closed operator A satisfies the hypotheses of part a), then A generates a contraction semigroup since $A = \overline{A}$. This variant of the result is often very useful in applications. For the addendum, one can easily check that $A - \omega I$ has the closure $\overline{A} - \omega I$.

PROOF OF THEOREM 1.39. Let the conditions in a) be true. Proposition 1.38 then tells us that A possesses a dissipative closure $\overline{A}$. Let $y \in X$. By assumption, there are vectors $x_n \in \mathrm{D}(A)$ such that the images $y_n = \lambda_0 x_n - Ax_n$ tend to y in X as $n \to \infty$. The dissipativity of A yields the inequality

$$\lambda_0 \|x_n - x_m\| \le \|(\lambda_0 - A)(x_n - x_m)\| = \|y_n - y_m\|$$

for all $n, m \in \mathbb{N}$ thanks to Proposition 1.32. This means that (x_n) has a limit x in X, and hence the vectors $\overline{A}x_n = Ax_n = \lambda_0 x_n - y_n$ tend to $\lambda_0 x - y$ as $n \to \infty$. Since $\overline{A}$ is closed, x belongs to $\mathrm{D}(\overline{A})$ and satisfies $\overline{A}x = \lambda_0 x - y$ so that $\lambda_0 I - \overline{A}$ is surjective. Proposition 1.38 b) and Theorem 1.26 now imply assertion a).

Part b) follows directly from Proposition 1.38 b) and Theorem 1.26. Claim c) is a consequence of Propositions 1.32 and 1.20. Finally, a rescaling argument based on Lemma 1.17 yields the addendum. □

We will reformulate the range condition in the Lumer–Phillips theorem using duality. To this aim, we recall the following concept from the lecture Spectral Theory. For a densely defined linear operator A, we define its *adjoint* A^* by

$$A^* x^* = y^* \qquad \text{for all} \quad x^* \in \mathrm{D}(A^*), \text{ where} \tag{1.20}$$
$$\mathrm{D}(A^*) = \big\{ x^* \in X^* \,\big|\, \exists\, y^* \in X^* \;\forall\, x \in \mathrm{D}(A): \; \langle Ax, x^* \rangle = \langle x, y^* \rangle \big\}.$$

This means that $\langle Ax, x^* \rangle = \langle x, A^*x^* \rangle$ for all $x \in \mathrm{D}(A)$ and $x^* \in \mathrm{D}(A^*)$. Recall from Remark 1.23 in [**27**] that A^* is a closed linear operator. The domain $\mathrm{D}(A^*)$ in (1.20) is defined in a 'maximal way' which is convenient for the theory, but for concrete operators it is often very difficult to calculate $\mathrm{D}(A^*)$ explicitly. The next result replaces the range condition by the injectivity of $\lambda_0 I - A^*$ (or the dissipativity of A^*), cf. Theorem 1.24 in [**27**]. In Example 1.49 we present a closed and densely defined dissipative operator having a non-dissipative adjoint.

Corollary 1.40. *Let A be linear and densely defined. If A is dissipative and $\lambda_0 I - A^*$ be injective for some $\lambda_0 > 0$, then $\overline{A}$ generates a contraction semigroup. Moreover, $\lambda I - A^*$ is injective for each $\lambda > 0$ if A^* is dissipative.*

Proof. The addendum follows from Proposition 1.38. Let $\lambda_0 I - A^*$ be injective. Take a functional $x^* \in X^*$ such that $\langle \lambda_0 x - Ax, x^* \rangle = 0$ for all $x \in \mathrm{D}(A)$. From (1.20) we infer that x^* belongs to $\mathrm{D}(A^*)$ and satisfies $A^*x^* = \lambda_0 x^*$, and thus $x^* = 0$. The Hahn–Banach theorem now implies the density of $\mathrm{R}(\lambda_0 I - A)$, see Corollary 5.13 in [**24**]. Theorem 1.39 then yields the assertion. □

Examples 1.33 c) and d) indicate that integration by parts is a very convenient tool to check dissipativity for differential operators in an L^2-context. To tackle such problems, we briefly discuss concepts and basic facts from Section 4.2 of [**24**] and also from Chapter 3 of [**27**], where the topic is treated in much greater detail. The material below is needed in many of our examples.

Intermezzo 3: Weak derivatives and Sobolev spaces. Let $\emptyset \neq G \subseteq \mathbb{R}^m$ be open, $k \in \mathbb{N}$, $j \in \{1, \dots, m\}$, and $p \in [1, \infty]$. A function $u \in L^p(G)$ has a *weak derivative* in $L^p(G)$ with respect to the jth coordinate if there is a map $v \in L^p(G)$ satisfying

$$\int_G u \partial_j \varphi \,\mathrm{d}x = -\int_G v \varphi \,\mathrm{d}x$$

for all $\varphi \in C_c^\infty(G)$. The function v is uniquely determined a.e. by Lemma 4.15 in [**24**]. We set $\partial_j u \coloneqq v$ in the above situation. The *Sobolev space*

$$W^{1,p}(G) \coloneqq \big\{ u \in L^p(G) \,\big|\, \forall\, j \in \{1, \dots, m\} \;\exists\, \partial_j u \in L^p(G) \big\}.$$

is a Banach space when endowed with the norm

$$\|u\|_{1,p} = \begin{cases} \Big(\|u\|_p^p + \sum_{j=1}^m \|\partial_j u\|_p^p \Big)^{\frac{1}{p}}, & p < \infty, \\ \max_{j \in \{1,\dots,m\}} \{ \|u\|_\infty, \|\partial_j u\|_\infty \}, & p = \infty, \end{cases}$$

see Proposition 4.19 of [**24**]. (As usual we identify functions which are equal almost everywhere.) Hence, by definition weak derivatives can be integrated by parts against'test functions' $\varphi \in C_c^\infty(G)$. This norm is equivalent to the one given by $\|u\|_p + \sum_{j=1}^m \|\partial_j u\|_p$ due to Remark 4.16 in [**24**]. Analogously one defines the Sobolov spaces $W^{k,p}(G)$ and higher-order weak derivatives $\partial^\alpha = \partial_1^{\alpha_1} \dots \partial_m^{\alpha_m}$ for $\alpha \in \mathbb{N}_0^m$ and $|\alpha| = \alpha_1 + \dots + \alpha_m \leq k$. We put $\partial^0 u = u$. One often writes $H^k(G)$ instead of $W^{k,2}(G)$ which is a Hilbert space. We summarize properties of Sobolev spaces and weak derivatives that are needed later on.

Remark 1.41. a) Let $u \in C^k(G)$ such that u and all its derivatives up to order k are contained in $L^p(G)$. Then u belongs to $W^{k,p}(G)$ and its classical and weak derivatives coincide by Remark 4.16 of [**24**].

b) Let $u, u_n, v \in L^p(G)$ and $\alpha \in \mathbb{N}_0^m$ such that $u_n \to u$ and $\partial^\alpha u_n \to v$ in $L^p(G)$ as $n \to \infty$. Then u possesses the weak derivative $\partial^\alpha u = v$ as shown in Lemma 4.17 in [**24**] or Lemma 3.16 in [**27**]. In other words, the operator ∂^α with (maximal) domain $\{u \in L^p(G) \,|\, \exists\, \partial^\alpha u \in L^p(G)\}$ is closed in $L^p(G)$.

c) Let $p < \infty$. Theorem 3.27 of [**27**] says that $C_c^\infty(\mathbb{R}^m)$ is dense in $W^{k,p}(\mathbb{R}^m)$ and that $C^\infty(G) \cap W^{k,p}(G)$ is dense in $W^{k,p}(G)$. (See also Theorem 4.21 of [**24**] for the first result.)

d) Let $-\infty \le a < b \le \infty$, $J = (a, b)$, and $u \in L^p(J)$. Then the function u belongs to $W^{1,p}(J) =: W^{1,p}(a, b)$ if and only if (a representative of) u is continuous and there is a map $v \in L^p(J)$ satisfying

$$u(t) = u(s) + \int_s^t v(\tau)\,\mathrm{d}\tau \qquad \text{for all } \; t, s \in J. \tag{1.21}$$

We then have $u' = \partial u := \partial_1 u = v$, and u possesses a continuous extension to a (or b) if $a > -\infty$ (or $b < \infty$). Moreover, $\|u\|_\infty \le c\|u\|_{1,p}$ for all $u \in W^{1,p}(J)$ and a constant $c > 0$, so that we have a linear continuous injection (*embedding*) $W^{1,p}(a, b) \hookrightarrow C_b(\overline{J})$. See Theorems 3.22 and 3.31 as well as Remark 3.33 in [**27**]. (One can also directly show the embedding starting from (1.21).)

As an example, take a function $u \in C_c(\mathbb{R})$ whose restrictions u^+ and u^- to $\mathbb{R}_{\ge 0}$ and $\mathbb{R}_{\le 0}$, respectively, are continuously differentiable. The map u then belongs to $W^{1,p}(\mathbb{R})$ for all $p \in [1, \infty]$ and its derivative is given by $(u^\pm)'$ on $\mathbb{R}_\pm$ by Example 4.18 of [**24**], where one also finds a multidimensional example.

e) Let $u \in W^{1,p}(G)$ and $v \in W^{1,p'}(G)$ with $\frac{1}{p} + \frac{1}{p'} = 1$. Proposition 4.20 of [**24**] yields that uv is an element of $W^{1,1}(G)$ and satisfies the product rule $\partial_j(uv) = u\partial_j v + v\partial_j u$. Analogous results hold for higher-order derivatives.

f) Let G have a C^1-boundary ∂G or G be bounded with a Lipschitz boundary $\partial G \in C^{1-}$. See the beginning of Section 3.3 in [**27**] for these concepts. By the Trace Theorem 3.38 in [**27**], the map $W^{1,p}(G) \cap C(\overline{G}) \to L^p(\partial G, \mathrm{d}\sigma);\, u \mapsto u{\restriction}_{\partial G}$, has a continuous extension $\mathrm{tr} : W^{1,p}(G) \to L^p(\partial G, \mathrm{d}\sigma)$ called the *trace operator*. Its kernel is the closure $W_0^{1,p}(G)$ of the test functions $C_c^\infty(G)$ in $W^{1,p}(G)$. If $\mathrm{tr}\, u = 0$, one says that u vanishes on ∂G 'in the sense of trace.'

Let G have a bounded Lipschitz boundary, $f \in W^{1,p}(G)^m$, and $u \in W^{1,p'}(G)$. The Divergence Theorem 3.41 in [**27**] then yields

$$\int_G u \,\mathrm{div}\, f \,\mathrm{d}x = -\int_G f \cdot \nabla u \,\mathrm{d}x + \int_{\partial G} \mathrm{tr}(u)\, \nu \cdot \mathrm{tr}(f)\,\mathrm{d}\sigma. \tag{1.22}$$

Here ν is the unit outer normal and the dot denotes the scalar product in $\mathbb{R}^m$. We usually omit the trace operator in the boundary integral. If $G = \mathbb{R}^m$ the formula is true without the boundary integral. ♢

Coming back to semigroups, we illustrate the above concepts by a simple example concerning generation properties of $\frac{\mathrm{d}}{\mathrm{d}s}$ in $L^2(\mathbb{R})$.

Example 1.42. Let $X = L^2(\mathbb{R})$ and $A = \frac{\mathrm{d}}{\mathrm{d}s}$ with $\mathrm{D}(A) = C_c^1(\mathbb{R})$.

1) The operators $\pm A$ are densely defined and dissipative by Example 1.33. Proposition 1.38 then yields their closability and the dissipativity of their closures, where $-A$ has the closure $-\overline{A}$. We next prove $\overline{A} = (\partial, W^{1,2}(\mathbb{R}))$.

For each $u \in \mathrm{D}(\overline{A})$ there are functions $u_n \in C_c^1(\mathbb{R})$ such that $u_n \to u$ and $u_n' = Au_n \to \overline{A}u$ in $L^2(\mathbb{R})$ as $n \to \infty$. In view of Remark 1.41 b), the map u thus belongs to $W^{1,2}(\mathbb{R})$ and $\overline{A}u = \partial u$; i.e., $\overline{A} \subseteq (\partial, W^{1,2}(\mathbb{R}))$. For the converse, take $u \in W^{1,2}(\mathbb{R})$. Remark 1.41 c) then provides a sequence (u_n) in $C_c^1(\mathbb{R})$ with limit u in $W^{1,2}(\mathbb{R})$. Hence, $u_n \to u$ and $u_n' \to \partial u$ in $L^2(\mathbb{R})$ so that u is an element of $\mathrm{D}(\overline{A})$.

2) We compute $\overline{A}^*$. Let $u, v \in W^{1,2}(\mathbb{R})$. Formula (1.22) then yields

$$\langle \overline{A}u, v\rangle = \int_{\mathbb{R}} \partial u\, v \,\mathrm{d}s = -\int_{\mathbb{R}} u \partial v \,\mathrm{d}s = \langle u, -\partial v\rangle,$$

so that $(-\partial, W^{1,2}(\mathbb{R}))$ is a restriction of $\overline{A}^*$, see (1.20). Conversely, let $v \in \mathrm{D}(\overline{A}^*)$. The functions v and $\overline{A}^*v$ thus belong to $L^2(\mathbb{R})$ and satisfy

$$\int_{\mathbb{R}} u\overline{A}^* v \,\mathrm{d}s = \langle u, \overline{A}^* v\rangle = \langle Au, v\rangle = \int_{\mathbb{R}} u' v \,\mathrm{d}s$$

for all $u \in C_c^\infty(\mathbb{R}) \subseteq \mathrm{D}(A) \subseteq \mathrm{D}(\overline{A})$, which means that $v \in W^{1,2}(\mathbb{R})$ and $\overline{A}^*v = -\partial v = -\overline{A}v$. We have shown $\overline{A}^* = -\overline{A}$. Corollary 1.40 then implies that $\pm\overline{A}$ generate contraction semigroups.

3) To determine these semigroups, we recall from Example 1.8 that the translation group $T(t)f = f(\cdot + t)$ on X has a generator B. For $f \in \mathrm{D}(A)$ the functions $w(t) = \frac{1}{t}(T(t)f - f)$ converge uniformly to f' as $t \to 0^+$. Moreover, the supports $\operatorname{supp} w(t)$ are contained in the bounded set $\operatorname{supp} f + [-1, 0]$ for all $0 \le t \le 1$, so that $w(t)$ tends to f' in X. We obtain $A \subseteq B$ and so $\overline{A} \subseteq B$. Lemma 1.23 and Theorem1.29 now yield $\overline{A} = B$ and hence $\overline{A}$ generates $T(\cdot)$. $\Diamond$

We conclude this section with a discussion of isometric groups.

Corollary 1.43. *Let A be linear. The following statements are equivalent.*

a) The operator A generates an isometric C_0-group $T(\cdot)$; i.e., $\|T(t)x\| = \|x\|$ for all $x \in X$ and $t \in \mathbb{R}$.

b) The operator A is closed, densely defined, $\pm A$ are dissipative, and $\lambda_0 I \pm A$ are surjective for some $\lambda_0 > 0$.

c) The operator A is closed, densely defined, $\mathbb{R} \setminus \{0\}$ belongs to $\rho(A)$, and $\|R(\lambda, A)\| \le \frac{1}{|\lambda|}$ for all $\lambda \in \mathbb{R} \setminus \{0\}$.

If $\mathbb{F} = \mathbb{C}$, one can also replace in c) the set $\mathbb{R} \setminus \{0\}$ by $\mathbb{C} \setminus \mathrm{i}\mathbb{R}$ and $|\lambda|$ by $|\mathrm{Re}\,\lambda|$.

Proof. The Lumer-Phillips Theorem 1.39 says that b) holds if and only if A and $-A$ generate contraction semigroups. Theorem 1.29 thus implies the equivalence of assertions b) and c), the addendum, and that b) is true if and only if A generates a contractive C_0-group $T(\cdot)$. It remains to show that a contractive C_0-group $T(\cdot)$ is already isometric. Indeed, in this case we have

$$\|T(t)x\| \le \|x\| = \|T(-t)T(t)x\| \le \|T(-t)\|\,\|T(t)x\| \le \|T(t)x\|$$

for all $x \in X$ and $t \in \mathbb{R}$, so that $T(t)$ is isometric. $\square$

We want to show an important variant of the above corollary on Hilbert spaces which requires a few more concepts from [**27**]. Let X be a Hilbert space. For a linear operator on X with dense domain we define the *Hilbert space adjoint* A' of A as in (1.20) replacing the duality pairing $\langle x, x^*\rangle$ by the inner product $(x|y)$. A linear operator A on X is called *symmetric* if

$$\forall\, x, y \in \mathrm{D}(A): \quad (Ax|y) = (x|Ay),$$

which means that $A \subseteq A'$ if $\mathrm{D}(A)$ is dense. If A is densely defined, we say that it is *self-adjoint* if $A = A'$; i.e., if A is symmetric and

$$\begin{aligned}\mathrm{D}(A) = \mathrm{D}(A') &= \big\{ y \in X \,\big|\, \exists\, z \in X\ \forall\, x \in \mathrm{D}(A) : (Ax|y) = (x|z) \big\} \\ &= \big\{ y \in X \,\big|\, (\mathrm{D}(A), \|\cdot\|) \to \mathbb{F};\ x \mapsto (Ax|y), \ \text{ is continuous} \big\}.\end{aligned}$$

(The last equality is a consequence of Riesz' representation Theorem 3.10 in [**24**].) A densely defined, linear operator A is called *skew-adjoint* if $A = -A'$ which is equivalent to the self-adjointness of $\mathrm{i}A$, if $\mathbb{F} = \mathbb{C}$. Finally, $T \in \mathcal{B}(X)$ is *unitary* if it has the inverse $T^{-1} = T'$.

We recall a very useful criterion from Theorem 4.7 of [**27**]. Let $\mathbb{F} = \mathbb{C}$. A symmetric, densely defined, closed operator A is self-adjoint if and only if its spectrum $\sigma(A)$ belongs to $\mathbb{R}$, which in turn follows from $\rho(A) \cap \mathbb{R} \neq \emptyset$.

As in Remark 1.23 in [**27**] one can check that A' is a closed linear map. Hence, every densely defined, symmetric operator is closable with $\overline{A} \subseteq A'$ (cf. Lemma 1.35) and each self-adjoint operator is closed. Let A be symmetric and densely defined. Take $u, v \in \mathrm{D}(\overline{A})$. There are sequences (u_n) and (v_n) in $\mathrm{D}(A)$ with limits u and v in X, respectively, such that $Au_n \to \overline{A}u$ and $Av_n \to \overline{A}v$ in X as $n \to \infty$. We then compute

$$(\overline{A}u|v) = \lim_{n\to\infty} (Au_n|v_n) = \lim_{n\to\infty} (u_n|Av_n) = (u|\overline{A}v),$$

so that also the closure $\overline{A}$ is symmetric.

There are densely defined, symmetric, closed operators that are not self-adjoint. (By Example 4.8 of [**27**] this is the case for $A = \mathrm{i}\partial$ with $\mathrm{D}(A) = \{u \in W^{1,2}(\mathbb{R}_+) \,|\, u(0) = 0\}$ on $X = L^2(\mathbb{R}_+)$. Here one has $\mathrm{D}(A') = W^{1,2}(\mathbb{R}_+)$.)

The next result due to *Stone* from 1930 belongs to the mathematical foundations of quantum mechanics.

THEOREM 1.44. *Let X be a Hilbert space and A be a linear operator on X with a dense domain. Then A generates a C_0-group of unitary operators if and only if A is skew-adjoint.*

PROOF. 1) Let $A' = -A$. Hence, A is closed. For $x \in \mathrm{D}(A)$, we have $J(x) = \{\varphi_x\}$ with $\varphi_x = (\cdot|x)$ by Example 1.30. We thus obtain

$$2\,\mathrm{Re}\langle Ax, \varphi_x\rangle = (Ax|x) + \overline{(Ax|x)} = (x|-Ax) + (x|Ax) = 0.$$

Therefore A, $A' = -A$, and $(-A)' = A$ are dissipative.

From Corollary 1.40 we then deduce that A and $-A$ generate contraction semigroups. Corollary 1.43 now shows that A generates a C_0-group $T(\cdot)$ of invertible isometries. Hence, each $T(t)$ is unitary by Proposition 5.52 in [**24**].

2) Let A generate a unitary C_0-group $T(\cdot)$. We infer $T(t)' = T(t)^{-1} = T(-t)$ for all $t \in \mathbb{R}$ by Remark 1.2, and so $T(\cdot)'$ is a unitary C_0-group with the generator $-A$. For $x, y \in \mathrm{D}(A)$ we thus obtain

$$(Ax|y) = \lim_{t\to 0} \big(\tfrac{1}{t}(T(t)x - x)\big|y\big) = \lim_{t\to 0} \big(x\big|\tfrac{1}{t}(T(t)'y - y)\big) = (x|-Ay).$$

This means that $-A \subseteq A'$. We further know from Theorem 1.29 that $\sigma(A)$ and $\sigma(-A)$ are contained in $\mathrm{i}\mathbb{R}$ (in $\{0\}$ if $\mathbb{F} = \mathbb{R}$). Equation (4.3) in [**27**] then yields $\sigma(A') = \overline{\sigma(A)} \subseteq \mathrm{i}\mathbb{R}$. The assertion $-A = A'$ now follows from Lemma 1.23. $\Box$

1.4. The Laplacian and related operators

In this section we discuss generation and related properties of the Laplacian

$$\Delta = \partial_1^2 + \cdots + \partial_m^2 = \operatorname{div} \nabla$$

in various settings, where we partly treat more general operators. To apply the Lumer–Phillips Theorem 1.39, we have to check three conditions. The *density* of the domain often follows from basic results on function spaces. With the right tools one can usually verify *dissipativity* in a straightforward way (imposing appropriate boundary conditions). For the *range condition* one has to solve the ‘elliptic problem’ $u - \Delta u = f$ plus boundary conditions for given f.

Using differing methods, this will be done first on $\mathbb{R}^m$, then on intervals, and finally on bounded domains imposing Dirichlet boundary conditions. As we will see in the next chapter, these results will allow us to solve diffusion equations, actually with improved regularity. We will further use the Dirichlet–Laplacian in the wave equation, cf. Example 1.55. We strive for a self-contained presentation (employing the lectures Functional Analysis and Spectral Theory), but for certain additional facts we have to cite standard results from the theory of partial differential equations.

A) The Laplacian on $\mathbb{R}^m$. Since the Laplacian has constant coefficients, on the full space $\mathbb{R}^m$ the Fourier transform is a very powerful tool to check the range condition. We first recall relevant results from Spectral Theory, taken from Sections 3.1 and 3.2 of [**27**]. Let $\mathbb{F} = \mathbb{C}$. For a function $f \in L^1(\mathbb{R}^m)$ we define its *Fourier transform*

$$(\mathcal{F}f)(\xi) = \hat{f}(\xi) := \frac{1}{(2\pi)^{\frac{m}{2}}} \int_{\mathbb{R}^m} \mathrm{e}^{-\mathrm{i}\xi\cdot x} f(x)\,\mathrm{d}x, \qquad \xi \in \mathbb{R}^m,$$

where we put $\xi \cdot x = \sum_{j=1}^m \xi_j x_j$. This formula clearly defines a function $\mathcal{F}f : \mathbb{R}^m \to \mathbb{C}^m$ which is bounded by $(2\pi)^{-m/2}\|f\|_1$. Actually, $\mathcal{F}f$ belongs to $C_0(\mathbb{R}^m)$ by Corollary 3.8 in [**27**]. For further investigations the *Schwartz space*

$$\mathcal{S}_m = \big\{ f \in C^\infty(\mathbb{R}^m) \,\big|\, \forall\, k \in \mathrm{N}_0,\ \alpha \in \mathbb{N}_0^m :\ p_{k,\alpha}(f) := \sup_{x\in\mathbb{R}^m} |x|_2^k\, |\partial^\alpha f(x)| < \infty \big\}.$$

turns out to be very useful.

By Remark 3.6 of [**27**] the family of seminorms $\{p_{k,\alpha} \,|\, k \in \mathbb{N}_0,\ \alpha \in \mathbb{N}_0^m\}$ yields a complete metric on $\mathcal{S}_m$. The space $C_c^\infty(\mathbb{R}^m)$ and also the Gaussian $\gamma(x) = \mathrm{e}^{-\frac{1}{2}|x|_2^2}$ are contained in $\mathcal{S}_m$. Proposition 3.10 of [**27**] shows that the restriction $\mathcal{F} : \mathcal{S}_m \to \mathcal{S}_m$ is bijective and continuous with the continuous inverse given by

$$\mathcal{F}^{-1}g(y) = (\mathcal{F}g)(-y) = \frac{1}{(2\pi)^{\frac{m}{2}}} \int_{\mathbb{R}^m} \mathrm{e}^{\mathrm{i}y\cdot\xi} g(\xi)\,\mathrm{d}\xi, \qquad y \in \mathbb{R}^m,$$

for $g \in \mathcal{S}_m$. In our context the core fact is *Plancherel's theorem*, which says that one can extend $\mathcal{F} : \mathcal{S}_m \to \mathcal{S}_m$ to a unitary map $\mathcal{F}_2 : L^2(\mathbb{R}^m) \to L^2(\mathbb{R}^m)$ satisfying $\mathcal{F}_2 f = \mathcal{F} f$ for $f \in L^2(\mathbb{R}^m) \cap L^1(\mathbb{R}^m)$, see Theorem 3.11 in [**27**]. We stress that $\mathcal{F}_2 f$ is **not** given by the above integral formula if $f \in L^2(\mathbb{R}^m)$ is not integrable; but we still write $\mathcal{F}$ instead of $\mathcal{F}_2$ and $\hat{f}$ instead of $\mathcal{F}_2 f$. The inversion formula $\mathcal{F}^{-1}g(y) = \mathcal{F}g(-y)$ for $y \in \mathbb{R}^m$ extends to $g \in L^2(\mathbb{R}^m)$.

To apply the Fourier transform to differential operators, one needs the following properties. Lemma 3.7 of [**27**] yields the differentiation formulas

$$\mathcal{F}(\partial^\alpha u) = \mathrm{i}^{|\alpha|}\xi^\alpha \mathcal{F}u \qquad \text{and} \qquad \partial^\alpha \mathcal{F}u = (-\mathrm{i})^{|\alpha|}\mathcal{F}(x^\alpha u) \tag{1.23}$$

for $u \in \mathcal{S}_m$ and $\alpha \in \mathbb{N}_0^m$, where we write ξ^α for the map $\xi \mapsto \xi^\alpha = \xi_1^{\alpha_1} \dots \xi_m^{\alpha_m}$ and so on. Due to Theorem 3.25 in [**27**], we have the crucial description

$$W^{k,2}(\mathbb{R}^m) = \big\{ u \in L^2(\mathbb{R}^m) \,\big|\, |\xi|_2^k\, \hat{u} \in L^2(\mathbb{R}^m) \big\} \tag{1.24}$$

with equivalent norms $\|u\|_{k,2} \eqsim \|u\|_2 + \||\xi|_2^k\, \hat{u}\|_2$ for $k \in \mathbb{N}_0$, and also that the first part of (1.23) is true for $u \in W^{|\alpha|,2}(\mathbb{R}^m)$.

To check the range condition for the Laplacian on $\mathbb{R}^m$, we take $f \in L^2(\mathbb{R}^m)$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}_{\le 0}$. We look for a function $u \in W^{2,2}(\mathbb{R}^m)$ satisfying $\lambda u - \Delta u = f$. Because of formula (1.23), such a solution fulfills the problem

$$\hat{f} = \lambda \hat{u} - \sum_{k=1}^m \mathrm{i}^2 \xi_k^2\, \hat{u} = (\lambda + |\xi|_2^2)\hat{u}.$$

The unique map solving this equation is given by $\hat{u} = (\lambda + |\xi|_2^2)^{-1}\hat{f}$, which is an element of $L^2(\mathbb{R}^m)$ by (1.26) below and since $\hat{f} \in L^2(\mathbb{R}^m)$. We now *define*

$$u := R(\lambda)f = \mathcal{F}^{-1}\Big(\frac{\hat{f}}{\lambda + |\xi|_2^2}\Big). \tag{1.25}$$

Since $\mathcal{F}$ is bijective on $L^2(\mathbb{R}^m)$, this function belongs to $L^2(\mathbb{R}^m)$. Based on these observations we can now establish our first generation result for the Laplacian.

Example 1.45. Let $E = L^2(\mathbb{R}^m)$ with $\mathbb{F} = \mathbb{C}$, $A = \Delta$, and $\mathrm{D}(A) = W^{2,2}(\mathbb{R}^m)$. The operator A is self-adjoint and generates a contraction semigroup on E. Moreover, its graph norm is equivalent to that of $W^{2,2}(\mathbb{R}^m)$.

Proof. The asserted norm equivalence follows from (1.24) and Plancherel's theorem because of $\mathcal{F}(\Delta u) = -|\xi|_2^2\, \hat{u}$ for $u \in W^{2,2}(\mathbb{R}^m)$, see (1.23). The domain $\mathrm{D}(A)$ is dense in E since it contains $C_c^\infty(\mathbb{R}^m)$, see Proposition 4.13 of [**24**].

Let $f \in E$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}_{\le 0}$. To check the range condition, we estimate

$$\Big|\frac{\hat{f}}{\lambda + |\xi|_2^2}\Big| \le c_\lambda |\hat{f}| \qquad \text{with} \quad c_\lambda := \begin{cases} \frac{1}{|\lambda|}, & \operatorname{Re}\lambda \ge 0, \\ \frac{1}{|\operatorname{Im}\lambda|}, & \operatorname{Re}\lambda < 0. \end{cases} \tag{1.26}$$

Since $\hat{f} \in E$ by Plancherel's theorem, the term in parentheses in (1.25) thus belongs to E. Using Plancherel once more, we introduce $u = R(\lambda)f \in E$ as in (1.25) and estimate

$$\|u\|_2 = \|\hat{u}\|_2 \le c_\lambda \|f\|_2; \qquad \text{i.e.,} \quad \|R(\lambda)\|_{\mathcal{B}(E)} \le c_\lambda. \tag{1.27}$$

We further compute

$$|\xi|_2^2 \, |\hat{u}| = \Big| \frac{|\xi|_2^2 \pm \lambda}{\lambda + |\xi|_2^2} \Big| |\hat{f}| \le (1 + |\lambda| c_\lambda) |\hat{f}|.$$

Equation (1.24) now implies that u belongs to $W^{2,2}(\mathbb{R}^m)$ with norm $\|u\|_{2,2} \le (1 + |\lambda| c_\lambda)\|f\|_2$. Therefore $R(\lambda)$ maps E continuously into $W^{2,2}(\mathbb{R}^m)$. From the first part of (1.23) and (1.25) we then deduce

$$\mathcal{F}(\lambda u - \Delta u) = (\lambda + |\xi|_2^2)\hat{u} = \hat{f},$$

obtaining $\lambda u - \Delta u = f$ in E by the bijectivity of $\mathcal{F}$.

This means that $\lambda I - A$ is bijective with the bounded inverse $R(\lambda)$. Hence, A is closed by Remark 1.16. Moreover, the spectrum $\sigma(A)$ is contained in $\mathbb{R}_{\le 0}$,[4] and inequality (1.27) implies the Hille–Yosida estimate for $\lambda > 0$. As a result, E generates a contraction semigroup on A by Theorem 1.26.

Let $u, v \in W^{2,2}(\mathbb{R}^m)$. Gauß' formula (1.22) and $\Delta = \operatorname{div} \nabla$ yield

$$(Au|v) = \int_{\mathbb{R}^m} \operatorname{div}(\nabla u)\overline{v} \,\mathrm{d}x = -\int_{\mathbb{R}^m} \nabla u \cdot \nabla \overline{v} \,\mathrm{d}x = \int_{\mathbb{R}^m} u \operatorname{div}(\nabla \overline{v}) \,\mathrm{d}x = (u|Av),$$

so that A is symmetric. Since $\sigma(A) \subseteq \mathbb{R}_{\le 0}$, the self-adjointness of A finally follows from Theorem 4.7 of [**27**]. □

We stress that the above norm equivalence says that one can bound in $L^2(\mathbb{R}^m)$ each derivative of $u \in \mathrm{D}(A)$ up to order 2 just by u and the sum Δu of unmixed second derivatives. In particular, if $m \ge 2$ the possible cancellations in Δu do not play a role! On $C_0(\mathbb{R}^m)$ the situation is quite different. Here we use of the version of the Lumer–Phillips theorem involving the closure. With the available tools we can compute its domain only for $m = 1$, see the comments below.

Example 1.46. Let $E = C_0(\mathbb{R}^m)$, $\mathrm{D}(A_0) = \{u \in C^2(\mathbb{R}^m) \,|\, u, \Delta u \in E\}$, and $A_0 = \Delta$. The operator A_0 has a closure A that generates a contraction semigroup on E. If $m = 1$, we have $Au = u''$ and $\mathrm{D}(A) = \mathrm{D}(A_0) = C_0^2(\mathbb{R}) := \{u \in C^2(\mathbb{R}) \,|\, u, u', u'' \in E\}$.

Proof. 1) The domain of A_0 is dense in E because of $C_c^\infty(\mathbb{R}^m) \subseteq \mathrm{D}(A_0)$, cf. the proof of Proposition 4.13 in [**24**]. Let $u \in \mathrm{D}(A_0)$. Example 1.30 says that the functional $\varphi = \overline{u(x_0)}\delta_{x_0}$ belongs to $J(u)$, where $x_0 \in \mathbb{R}^m$ satisfies $|u(x_0)| = \|u\|_\infty$. Setting $h = \mathrm{Re}(\overline{u(x_0)}u) \in \mathrm{D}(A_0)$, we obtain

$$\mathrm{Re}\langle A_0 u, \varphi\rangle = \mathrm{Re}(\overline{u(x_0)}\Delta u(x_0)) = \Delta h(x_0).$$

As in Example 1.33 we see that $h(x_0)$ is a maximum of h. By Analysis 2, the matrix $D^2 h(x_0)$ is thus negative semidefinite and hence $\Delta h(x_0) = \mathrm{tr}(D^2 h(x_0)) \le 0$; i.e., A_0 is dissipative.

[4] Actually we have the equality $\sigma(A) = \mathbb{R}_{\le 0}$ by Example 3.47 in [**27**].

Let $f \in \mathcal{S}_m$ and define $u = R(1)f$ by (1.25). Lemma 3.7 in [**27**] implies that u is an element of $\mathcal{S}_m \subseteq \mathrm{D}(A_0)$. As seen in the previous proof we have $u - \Delta u = f$, so that $\mathrm{R}(I - A_0)$ contains the dense subspace $\mathcal{S}_m$. The first assertion now follows from the Lumer–Phillips Theorem 1.39.

2) Let $m = 1$ and $u \in \mathrm{D}(A)$. Because of $A = \overline{A_0}$ there are functions $u_n \in \mathrm{D}(A_0)$ with $u_n \to u$ and $u_n'' \to Au$ in E as $n \to \infty$. We further need to control the first derivative. For later use, the argument is presented in somewhat greater generality. We look at an interval J of length $|J| > 0$, a function $v \in C^2(J)$ with bounded v and v'', $\delta \in (0, |J|)$, and points $r, s \in J$ with $\delta < s - r < 2\delta$. Taylor's theorem provides a number $\sigma \in (r, s)$ such that

$$\begin{aligned} v(s) &= v(r) + v'(r)(s-r) + \tfrac{1}{2} v''(\sigma)(s-r)^2, \\ v'(r) &= \frac{v(s) - v(r)}{s - r} - \tfrac{1}{2} v''(\sigma)(s-r). \end{aligned}$$

The last equation yields

$$\begin{aligned} |v'(r)| &\le \tfrac{2}{\delta} \max_{\tau \in [r, r+2\delta]} |v(\tau)| + \delta \max_{\tau \in [r, r+2\delta]} |v''(\tau)|, \\ \|v'\|_\infty &\le \tfrac{2}{\delta} \|v\|_\infty + \delta \|v''\|_\infty \,. \end{aligned} \tag{1.28}$$

Inserting $v = u_n$ into (1.28), we infer that u_n' is an element of E. With $v = u_n - u_m$, it also follows that u_n' tends in E to a map f. As a result, u belongs to $C^1(\mathbb{R})$ with $u' = f \in E$. The limit $(u_n')' \to Au$ in E then leads to $u \in C_0^2(\mathbb{R})$ and $Au = u''$. $\square$

For $m \ge 2$ the domain $\mathrm{D}(A)$ is not $C_0^2(\mathbb{R}^m)$ in Example 1.46. To make this fact plausible, we look at the function

$$\tilde{u}(x, y) = \begin{cases} (x^2 - y^2) \ln(x^2 + y^2), & (x, y) \ne (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$$

By a straightforward computation, the second derivative

$$\partial_x^2 \tilde{u}(x, y) = 2 \ln(x^2 + y^2) + \frac{4x^2}{x^2 + y^2} + \frac{(6x^2 - 2y^2)(x^2 + y^2) - 4x^2(x^2 - y^2)}{(x^2 + y^2)^2}$$

is unbounded on $B(0, 1)$, but the functions $\tilde{u}$, $\nabla \tilde{u}$, and $\Delta \tilde{u}(x, y) = 8 \frac{x^2 - y^2}{x^2 + y^2}$ are bounded on $B(0, 2)$. To deal with larger (x, y), we simply take a smooth map φ with $\operatorname{supp} \varphi \subseteq B(0, 2)$ which is equal to 1 on $B(0, 1)$. Then the functions $u = \varphi \tilde{u}$ and $\Delta u = \varphi \Delta \tilde{u} + 2 \nabla \varphi \cdot \nabla \tilde{u} + \tilde{u} \Delta \varphi$ are bounded and have compact support on $\mathbb{R}^m$, but u does not belong to $W^{2,\infty}(\mathbb{R}^m)$. (One can construct an analogous example in $E = C_0(\mathbb{R}^m)$ instead of $L^\infty(\mathbb{R}^m)$ using ln ln.)

With much more effort, Corollary 3.1.9 in [**18**] shows that the operator $A_1 = \Delta$ with domain

$$\mathrm{D}(A_1) = \big\{ u \in E \,\big|\, \forall\, p \in (1, \infty), r > 0 : u \in W^{2,p}(B(0, r)),\ \Delta u \in E \big\}$$

is closed in E and that $\rho(A_1)$ contains a halfline (ω, ∞). Since $\mathrm{D}(A_0) \subseteq \mathrm{D}(A_1)$, we first obtain $A = \overline{A_0} \subseteq A_1$, and then $A = A_1$ by Lemma 1.23.

B) The second derivative on $(0, 1)$**.** On an interval the equation $\lambda u - \Delta u = f$ with boundary conditions becomes an ordinary boundary value problem. In [**27**] we solved such problems explicitly and obtained concrete formulas for the resolvent. We only look at Dirichlet conditions $u(0) = 1 = u(1)$, others can be treated similarly (see the exercises). We start with the sup-norm case.

Example 1.47. Let $E = C_0(0, 1)$, $\mathrm{D}(A) = \{u \in C^2(0, 1) \,|\, u, u'' \in E\}$, and $Au = u''$. Then the operator A generates a contraction semigroup on E, and its graph norm is equivalent to that of $C^2([0, 1])$.

Proof. The equivalence of the norms can be deduced from (1.28), which is also true with intervals $[r - 2\delta, r] \subseteq (0, 1)$. Let $f \in E$. Take $\varepsilon > 0$. As in Example 1.8 we find a map $\tilde{f} \in C_c(0, 1)$ with $\|f - \tilde{f}\|_\infty \le \varepsilon$. Moreover, proceeding as in the proof of Proposition 4.13 in [**24**] one constructs a function $g \in C_c^\infty(0, 1) \subseteq \mathrm{D}(A)$ satisfying $\|\tilde{f} - g\|_\infty \le \varepsilon$. Hence, A is densely defined. The dissipativity of A is shown as in Example 1.46, where the argument s_0 of the maximum of $|u|$ belongs to $(0, 1)$ since the cases $s_0 \in \{0, 1\}$ are excluded by the boundary conditions unless $u = 0$.

Let $f \in E$. If $\mathbb{F} = \mathbb{C}$, take $\lambda \in \mathbb{C} \setminus \mathbb{R}_{\le 0}$ and $\sqrt{\lambda} =: \mu \in \mathbb{C}_+$. If $\mathbb{F} = \mathbb{R}$, let $\lambda > 0$ and $\mu = \sqrt{\lambda} > 0$. Set

$$\begin{pmatrix} a(f, \mu) \\ b(f, \mu) \end{pmatrix} = \frac{1}{2\mu(\mathrm{e}^{-\mu} - \mathrm{e}^{\mu})} \begin{pmatrix} \mathrm{e}^{-\mu} \int_0^1 (\mathrm{e}^{\mu\tau} - \mathrm{e}^{-\mu\tau}) f(\tau)\,\mathrm{d}\tau \\ \int_0^1 (\mathrm{e}^{\mu}\mathrm{e}^{-\mu\tau} - \mathrm{e}^{-\mu}\mathrm{e}^{\mu\tau}) f(\tau)\,\mathrm{d}\tau \end{pmatrix}.$$

In Example 2.16 of [**27**] it is shown that the map $u : [0, 1] \to \mathbb{F}$;

$$u(s) = a(f, \mu)\mathrm{e}^{\mu s} + b(f, \mu)\mathrm{e}^{-\mu s} + \frac{1}{2\mu} \int_0^1 \mathrm{e}^{-\mu|s-\tau|} f(\tau)\,\mathrm{d}\tau, \tag{1.29}$$

belongs to $C^2([0, 1])$ and satisfies $\lambda u - u'' = f$ as well as $u(0) = 0 = u(1)$. Hence, u an element of $\mathrm{D}(A)$ and $\lambda I - A$ is surjective. The Lumer–Phillips Theorem 1.39 now implies that A is closed and generates a contraction semigroup on E. Moreover, (1.29) gives the resolvent via $R(\lambda, A)f := u$. □

We next show the analogous result for $L^p(0, 1)$. Here we check dissipativity on L^p also for $p \ne 2$.

Example 1.48. Let $E = L^p(0, 1)$, $1 \le p < \infty$, $Au = \partial^2 u$, and

$$\mathrm{D}(A) = \{u \in W^{2,p}(0, 1) \,|\, u(0) = u(1) = 0\} = W^{2,p}(0, 1) \cap W_0^{1,p}(0, 1).$$

(Remark 1.41 yields $W^{1,p}(0, 1) \hookrightarrow C([0, 1])$.) The operator A generates a contraction semigroup on E and its graph norm is equivalent to $\|\cdot\|_{2,p}$.

Proof. The last assertion follows from Proposition 3.37 of [**27**], cf. (1.28). The domain $\mathrm{D}(A)$ is dense due to Proposition 4.13 in [**24**] since it contains $C_c^\infty(0, 1)$. One can extend the operator $R(\lambda, A)$ from (1.29) to a map $R(\lambda)$ on $E = L^p(0, 1)$ for $\lambda \in \mathbb{F} \setminus \mathbb{R}_{\le 0}$ where $\mu = \sqrt{\lambda} \in \mathbb{F}_+$. Omitting a factor, we rewrite the last summand of (1.29) as

$$v(s) := \int_0^1 \mathrm{e}^{-\mu|s-\tau|} f(\tau)\,\mathrm{d}\tau = \mathrm{e}^{-\mu s} \int_0^s \mathrm{e}^{\mu\tau} f(\tau)\,\mathrm{d}\tau + \mathrm{e}^{\mu s} \int_s^1 \mathrm{e}^{-\mu\tau} f(\tau)\,\mathrm{d}\tau$$

for $f \in E$ and $s \in [0,1]$. Using (1.21), we can now differentiate

$$v'(s) = -\mu e^{-\mu s} \int_0^s e^{\mu\tau} f(\tau)\,d\tau + f(s) + \mu e^{\mu s} \int_s^1 e^{-\mu\tau} f(\tau)\,d\tau - f(s).$$

Since the terms $\pm f(s)$ cancel, v belongs to $C^1([0,1])$. Similarly, the weak derivative $\partial^2 v \in L^p(0,1)$ exists and satisfies $\lambda v - \partial^2 v = 2\mu f$. The other two summands u_j in (1.29) are smooth and fulfill $\lambda u_j = u_j''$. The boundary conditions $u(0) = 0 = u(1)$ are shown as in Example 2.16 of [**27**] (where $f \in C([0,1])$). Summing up, $u = R(\lambda)f$ is an element of $\mathrm{D}(A)$ and solves $\lambda u - Au = f$.

To apply the Lumer–Phillips theorem, it remains to check the dissipativity. To avoid certain technical problems we restrict ourselves to $p \in [2,\infty)$, see Example 2.30. for the case $p \in [1,2)$. Let $u \in \mathrm{D}(A) \setminus \{0\}$. We set $w = |u|^{p-2}\,\overline{u}$. Then $\tilde{w} := \|u\|_p^{2-p} w$ belongs to $J(u)$ by Example 1.30. Note that $w(0) = 0 = w(1)$ by the boundary conditions. Remark 1.41 yields the embedding $W^{2,p}(0,1) \hookrightarrow C^1([0,1])$ so that w is contained in $C^1([0,1])$ since $p \geq 2$. We then compute

$$\begin{aligned} w' &= \tfrac{d}{ds}\big((u\overline{u})^{\frac{p-2}{2}}\,\overline{u}\big) = |u|^{p-4}\,|\overline{u}|^2\,\overline{u}' + \tfrac{p-2}{2}(|u|^2)^{\frac{p-2}{2}-1}\,(u'\overline{u} + u\overline{u}')\overline{u} \\ &= |u|^{p-4}\,(|\overline{u}|^2\overline{u}' + (p-2)\,\mathrm{Re}(\overline{u}u')\overline{u}). \end{aligned}$$

Formula (1.22) and $w(0) = 0 = w(1)$ now imply

$$\begin{aligned} \mathrm{Re}\langle Au, w\rangle &= \mathrm{Re}\int_0^1 \partial^2 u\, w\, ds = -\int_0^1 \mathrm{Re}(u'w')\,ds + \mathrm{Re}(u'w)\big|_0^1 \\ &= -\int_0^1 |u|^{p-4}\,\big(|\overline{u}u'|^2 + (p-2)(\mathrm{Re}(\overline{u}u'))^2\big)\,ds \\ &= -\int_0^1 |u|^{p-4}\,\big((\mathrm{Im}(\overline{u}u'))^2 + (p-1)(\mathrm{Re}(\overline{u}u'))^2\big)\,ds \leq 0, \end{aligned}$$

and hence $\mathrm{Re}\langle Au, \tilde{w}\rangle \leq 0$. Theorem 1.39 now implies the assertion, and $R(\lambda)$ is the resolvent of A. □

We add an example where A is dissipative, but not a generator, and A^* is not dissipative, cf. Corollary 1.40. This can happen since we here impose too many (four) boundary conditions instead of two (for two derivatives) as above.

Example 1.49. Let $E = L^2(0,1)$, $Au = \partial^2 u$, and

$$\mathrm{D}(A) = \{u \in W^{2,2}(0,1) \,|\, u(0) = u'(0) = u(1) = u'(1) = 0\} = W_0^{2,2}(0,1).$$

(The last space is the closure of $C_c^\infty(0,1)$ in $W^{2,2}(0,1)$; the final equality follows as in Remark 1.41.) Then A is closed, densely defined, dissipative, and symmetric, but not a generator and not self-adjoint, and A^* is not dissipative.

Proof. The density of $\mathrm{D}(A)$ follows again from Proposition 4.13 in [**24**]. To check closedness, take maps $u_n \in \mathrm{D}(A)$ such that $u_n \to u$ and $u_n'' \to v$ in E as $n \to \infty$. Proposition 3.37 in [**27**] then shows that also (u_n') converges in E, cf. (1.28). From Remark 1.41 we now deduce that u belongs to $W^{2,2}(0,1)$ and $u_n \to u$ in $W^{2,2}(0,1)$. The boundary conditions for u_n transfer to u via the limits of (u_n) and (u_n') since $W^{1,2}(0,1) \hookrightarrow C([0,1])$ by Remark 1.41. Hence, u belongs to $\mathrm{D}(A)$ and A is closed.

Let $u \in \mathrm{D}(A)$ and $v \in W^{2,2}(0,1)$. Using integration by parts (1.22) and the boundary conditions of u, we compute

$$(Au|v) = \int_0^1 \partial^2 u\,\overline{v}\,\mathrm{d}s = -\int_0^1 u'\overline{v}'\,\mathrm{d}s + u'\overline{v}\big|_0^1 = \int_0^1 u\partial^2\overline{v}\,\mathrm{d}s - u\overline{v}'\big|_0^1 = (u|\partial^2 v).$$

Hence, A is symmetric (take $v \in \mathrm{D}(A)$) and dissipative (take $v = u$). Moreover, the operator ∂^2 with domain $W^{2,2}(0,1)$ is a restriction of A' and also of A^*.

Let $v \in \mathrm{D}(A^*)$. As in Example 1.42 one can see that $A^*v \in E$ is the second weak derivative of $v \in E$. Proposition 3.37 in [**27**] thus implies that v belongs to $W^{2,2}(0,1)$. It follows $A^* = \partial^2$ with $\mathrm{D}(A^*) = W^{2,2}(0,1) \neq D(A)$. Here we can replace A^* by A'. Hence, A is not self-adjoint.

Since $\partial^2 \mathrm{e}^{\mu s} = \lambda \mathrm{e}^{\mu s}$ for $\mu = \sqrt{\lambda}$ and $\lambda \in \mathbb{F} \setminus \mathbb{R}_{\leq 0}$, the operator $\lambda I - A^*$ is not injective. As a result, A^* is not a generator and not dissipative in view of Propositions 1.20 and 1.38. Moreover, the spectrum of A contains $\mathbb{F}\setminus\mathbb{R}_{\leq 0}$ since $\sigma(A) = \sigma(A^*)$ by Theorem 1.24 of [**27**]. In particular, A is not a generator. □

C) Operators in L^2 defined by sesquilinear forms. In many applications one looks at the Laplacian or related 'elliptic operators in divergence form' on a domain in $\mathbb{R}^m$. In an L^2-context we can show generation properties of these operators, though it is not possible to describe their domains precisely by our means. (This point is discussed below.) We restrict ourselves again to Dirichlet boundary conditions for simplicity; others are treated in the exercises. Most of the results are presented for a larger class of operators (defined by sesquilinear forms) since the analysis is almost the same as for the Laplacian itself. The main tool is the *Lax–Milgram lemma* which is a core consequence of Riesz' representation of Hilbert space duals.

THEOREM 1.50. *Let Y be a Hilbert space and $\underline{a} : Y \times Y \to \mathbb{F}$ be a sesquilinear map which is* bounded *and* strictly accretive*; i.e.,*

$$|\underline{a}(x,y)| \leq C\|x\|\|y\| \qquad \text{and} \qquad \operatorname{Re}\underline{a}(y,y) \geq \eta\|y\|^2$$

for all $x, y \in Y$ and some constants $C, \eta > 0$. Then for each functional $\psi \in Y^*$ there is a unique vector $z \in Y$ satisfying $\underline{a}(y,z) = \psi(y)$ for all $y \in Y$. The map $Y^* \to Y$; $\psi \mapsto z$, is antilinear and bounded.

PROOF. Let $y \in Y$. The map $\varphi_y := \underline{a}(\cdot, y)$ belongs to Y^* with $\|\varphi_y\| \leq C\|y\|$. Riesz' Theorem 3.10 in [**24**] yields a unique element Sy of Y satisfying $\varphi_y = (\cdot|Sy)$ and $\|Sy\| = \|\varphi_y\| \leq C\|y\|$. Moreover, $S : Y \to Y$ is linear since both maps $y \mapsto \varphi_y$ and $\varphi_y \mapsto Sy$ are antilinear in y. We next estimate

$$\eta\|y\|^2 \leq \operatorname{Re}\underline{a}(y,y) = \operatorname{Re}(y|Sy) \leq |(y|Sy)| \leq \|y\|\|Sy\|,$$

and hence $\|Sy\| \geq \eta\|y\|$ for every $y \in Y$. As a consequence, S is bounded, injective and has a closed range $\mathrm{R}(S)$ by Remark 2.11 in [**24**]. For a vector $y \perp \mathrm{R}(S)$ we also obtain

$$0 = (y|Sy) = \operatorname{Re}(y|Sy) = \operatorname{Re}\underline{a}(y,y) \geq \eta\|y\|^2$$

and hence $y = 0$. It follows that $Y = \overline{\mathrm{R}(S)} = \mathrm{R}(S)$ by Theorem 3.8 in [**24**] (or a corollary to Hahn–Banach), and so S is invertible with $\|S^{-1}\| \leq \frac{1}{\eta}$.

Let $\psi \in Y^*$. There is a unique vector $v \in Y$ such that $\psi = (\cdot|v)$ thanks to Riesz' theorem. The above construction implies the identity

$$\psi(y) = (y|v) = (y|SS^{-1}v) = \underline{a}(y, S^{-1}v)$$

for all $y \in Y$. We set $z = S^{-1}v = S^{-1}\Phi_Y^{-1}\psi$, where $\Phi_Y : Y \to Y^*$ denotes the antilinear isomorphism from Riesz' theorem.

For uniqueness, let also $w \in Y$ satisfy $\underline{a}(y, w) = \psi(y)$ for all $y \in Y$. Taking $y = z - w$, we infer $0 = \underline{a}(z - w, z - w) \geq \eta\|z - w\|$ and thus $w = z$. □

In the typical applications of Theorem 1.50, Y is a subspace of $W^{1,2}(G)$ for an open set $G \subseteq \mathbb{R}^m$ (say, with a regular boundary), where we focus on $Y = W_0^{1,2}(G)$. One is then mainly interested in properties related to the L^2-norm, so that one also looks at $L^2(G)$. We note that $W_0^{1,2}(G)$ is densely embedded in $L^2(G)$. We extend the above setting to the cover this framework.

Let $\underline{a} : Y \times Y \to \mathbb{F}$ be as given in Theorem 1.50. We also assume that Y is densely embedded in a Hilbert space X by $J_Y : Y \to X$. (Often we omit J_Y in our notation, in our examples it is just the inclusion.) By means of the isometric, antilinear Riesz' isomorphism $\Phi = \Phi_X : X \to X^*$ from Theorem 3.10 in [**24**], we identify X and X^* most of the time. Proposition 5.46 in [**24**] yields the dense embedding $J_Y^* : X^* \hookrightarrow Y^*$. It follows

$$Y \hookrightarrow X \cong X^* \hookrightarrow Y^*, \tag{1.30}$$

where we have $\langle y, J_Y^*\Phi x\rangle_Y = (J_Y y|x)_X = (y|x)_X$ for $x \in X$ and $y \in Y$. We stress that we **not** identify Y with Y^* since this would require the Riesz isomorphism Φ_Y, which is quite **different** from Φ_X in our examples.

To associate an operator in X with $\underline{a}$, we define

$$\mathrm{D}(A) = \big\{x \in Y \,\big|\, \exists\, c > 0\ \forall\, y \in Y : |\underline{a}(y, x)| \leq c\|y\|_X\big\}. \tag{1.31}$$

Since Y is dense in X, we can extend $-\underline{a}(\cdot, x)$ to an element φ_x of X^*. Thanks to Riesz' theorem and (1.30), it can be represented by a unique element $Ax \in X$ in the sense that

$$\forall y \in Y : \quad -\underline{a}(y, x) = (y|Ax)_X = \langle y, J_Y^*\Phi Ax\rangle_Y = \langle y, Ax\rangle_Y. \tag{1.32}$$

In the last equality we consider $Ax \in X$ as element in Y^*, as one usually does. Formula (1.32) determines Ax uniquely by the density of Y. Moreover, A is linear as in the proof of Theorem 1.50. We further need the *adjoint form*

$$\underline{a}' : Y \times Y \to \mathbb{F}; \quad \underline{a}'(y, z) = \overline{\underline{a}(z, y)}.$$

Note that $\underline{a}'$ is also sesquilinear, bounded, and strictly accretive. We call $\underline{a}$ *symmetric* if $\underline{a} = \underline{a}'$. We can now show that A has very convenient properties.

THEOREM 1.51. *Let X and Y be Hilbert spaces with an embedding $J_Y : Y \hookrightarrow X$ having dense range and norm κ. Assume that $\underline{a} : Y \times Y \to \mathbb{F}$ is sesquilinear, bounded, and strictly accretive. Define A by* (1.31) *and* (1.32)*. Then A generates an ω-contraction semigroup on X with $\omega := -\eta\kappa^{-2}$. In particular, $\mathrm{s}(A) \leq \omega < 0$ and A is invertible. The adjoint A' of A is given by the form $\underline{a}'$ as in* (1.31) *and* (1.32)*, so that A is self-adjoint if $\underline{a}$ is symmetric.*

PROOF. 1) Let $x \in \mathrm{D}(A)$. The definition (1.32) and the assumptions imply

$$\operatorname{Re}(Ax|x)_X = \operatorname{Re}(x|Ax)_X = -\operatorname{Re}\underline{a}(x,x) \le -\eta\|x\|_Y^2 \le -\eta\kappa^{-2}\|x\|_X^2 \le 0,$$

and thus the dissipativity of A and $A_\omega := A + \eta\kappa^{-2}I$. As $A = A_\omega - \eta\kappa^{-2}I$, Proposition 1.38 a) yields the injectivity of A.

Take $z \in X$ and set $\psi = -(\cdot|z)_X \in X^* \hookrightarrow Y^*$. Theorem 1.50 thus provides an element $x \in Y$ with $\underline{a}(y,x) = -(y|z)_X$ for all $y \in Y$. Since then $|\underline{a}(y,x)| \le \|z\|_X\|y\|_X$ by Cauchy–Schwarz, x belongs to $\mathrm{D}(A)$. Moreover, we have $(y|z)_X = (y|Ax)_X$ due to (1.32) and so $Ax = z$ because of the density of Y. Summing up, $A = A_\omega - \omega I$ is bijective.

To check the density of $\mathrm{D}(A)$ in X, let $z \in X$ be orthogonal to $\mathrm{D}(A)$ and set $y = A^{-1}z \in \mathrm{D}(A)$. Strict accretivity then leads to

$$0 = (y|Ay)_X = -\underline{a}(y,y) = -\operatorname{Re}\underline{a}(y,y) \le -\eta\|y\|_Y^2,$$

so that $y = 0$ and $z = 0$. Hence, $\mathrm{D}(A)$ is dense in X by Theorem 3.8 in [**24**]. Theorem 1.39 now shows that A generates an ω-contraction semigroup on X. In particular, $\mathrm{s}(A) \le \omega < 0$ and A is invertible.

2) We next compute A'. Let $A^\dagger$ be the operator associated with $\underline{a}'$. Take $x \in \mathrm{D}(A)$ and $y \in \mathrm{D}(A^\dagger)$. The definitions imply

$$(Ax|y)_X = \overline{(y|Ax)_X} = -\overline{\underline{a}(y,x)} = -\underline{a}'(x,y) = (x|A^\dagger y)_X;$$

i.e., $A^\dagger \subseteq A'$. Both operators are invertible by part 1) and also (4.3) of [**27**]. Lemma 1.23 now yields $A^\dagger = A'$. □

Chapter VI of [**14**] or Chapter 1 of [**21**] provide more general versions of this result and many related facts, see also the exercises and Section 2.3.

The next straightforward application is a version of Example 5.11 of [**27**]. Here we need Poincaré's inequality. For every bounded open set $G \subseteq \mathbb{R}^m$ and $p \in [1,\infty)$, there is a constant $c = c(G,p) > 0$ such that

$$\|u\|_p \le c\,\||\nabla u|_2\|_p = c\Big(\int_G \Big(\sum_{j=1}^m |\partial_j u|^2\Big)^{\frac{p}{2}}\,\mathrm{d}x\Big)^{\frac{1}{p}} \tag{1.33}$$

for all $u \in W_0^{1,p}(G)$, see Theorem 3.36 in [**27**]. Note $\||\nabla u|_2\|_2^2 = \sum_j \int |\partial_j u|^2\,\mathrm{d}x$.

EXAMPLE 1.52. Let $G \subseteq \mathrm{R}^m$ be open and bounded, and the coefficients $a_{jk} \in L^\infty(G,\mathbb{F})$ for $j,k \in \{1,\dots,m\}$ be *strictly elliptic*; i.e.,

$$\operatorname{Re}\sum\nolimits_{j,k=1}^m a_{jk}(x)\overline{z_j}z_k \ge \eta|z|_2^2 \tag{1.34}$$

for some $\eta > 0$, all $z \in \mathbb{F}^m$, and a.e. $x \in G$. We write $a = (a_{jk})_{j,k}$. Let $E = L^2(G)$ and $V = W_0^{1,2}(G)$, where $\|\cdot\| = \|\cdot\|_2$ and V is equipped the norm $\|v\|_V = \||\nabla v|_2\|_2$ which is equivalent to the usual one by (1.33). We define

$$\underline{a}: V \times V \to \mathbb{F}; \quad \underline{a}(v,w) = \sum\nolimits_{j,k=1}^m \int_G \partial_j v\,\overline{a_{jk}}\partial_k\overline{w}\,\mathrm{d}x.$$

Then the conditions of Theorem 1.51 are satisfied with $C = \||a|_2\|_\infty$, η and $\kappa = c(G,2)$, where $|a(x)|_2$ is the operator norm for $|z|_2$ in $\mathbb{F}^m$.

So $\underline{a}$ induces an invertible generator A of a contraction semigroup on E, and A is self-adjoint if a is Hermitian. After complex conjugation, for $f \in E$ the function $u \in \mathrm{D}(A)$ with $Au = f$ is given by

$$\int_G \overline{v} f \,\mathrm{d}x = -\sum\nolimits_{j,k=1}^{m} \int_G \partial_j \overline{v}\, a_{jk} \partial_k u \,\mathrm{d}x, \tag{1.35}$$

for all $v \in V$, where $\mathrm{D}(A)$ is the set of $u \in V$ such that the right-hand side is bounded by $c\|v\|_2$ for some $c = c(a, u)$ and all $v \in V$. ♢

One calls $u \in V$ satisfying (1.35) a 'weak solution' of $Au = f$. To obtain a better understanding of this equation, we impose stronger conditions on G and a. In particular, let $a_{jk} \in W^{1,\infty}(G)$ for $j, k \in \{1, \dots, m\}$. We then define an 'elliptic' operator in 'divergence form' with 'Dirichlet boundary conditions' via

$$A_0 u = \operatorname{div}(a\nabla u) = \sum\nolimits_{j,k=1}^{m} \partial_j (a_{jk} \partial_k u), \qquad u \in \mathrm{D}(A_0) = W^{2,2}(G) \cap W_0^{1,2}(G).$$

(Operators in non-divergence form and with lower-order terms can then be treated by perturbation arguments, if the coefficients are regular enough, cf. Example 3.11.) One sets $W^{-1,2}(G) = W_0^{1,2}(G)^*$ if $W_0^{1,2}(G)$ is equipped with the full norm $\|\cdot\|_{1,2}$. We write V^* instead since we use the equivalent norm $\|\cdot\|_V$ on $V = W_0^{1,2}(G)$. The second part of the next result is also true in the framework of Theorem 1.51, cf. Section 1.4.2 in [**21**].

Example 1.53. In addition to the hypotheses of Example 1.52, we assume that $a_{jk} \in W^{1,\infty}(G)$ for all $j, k \in \{1, \dots, m\}$ and that ∂G is Lipschitz. Integration by parts via (1.22) then yields $(v|A_0 u)_{L^2} = -\underline{a}(v, u) = (v|Au)_{L^2}$ for all $v \in V$ and $u \in \mathrm{D}(A_0)$. By density of V, it follows $A_0 u = Au$ and hence $A_0 \subseteq A$. In particular, A_0 is dissipative.

Observe that $\mathrm{D}(A)$ is dense in V because of $C_c^\infty(G) \subseteq \mathrm{D}(A_0) \subseteq \mathrm{D}(A)$. For $u \in \mathrm{D}(A)$, the definition (1.32) yields

$$\|Au\|_{V^*} = \sup_{\|v\|_V \le 1} |\langle v, Au\rangle_V| = \sup_{\|v\|_V \le 1} |\underline{a}(v, u)| \le C\|u\|_V.$$

Therefore we can extend A to a bounded operator $\tilde{A} : V \to V^*$, which is the 'weak extension' of A. Its range contains $L^2(G)$ and is thus dense in V^*. We further obtain

$$\eta\|u\|_V^2 \le |\underline{a}(u, u)| = |\langle u, Au\rangle_V| \le \|u\|_V \|Au\|_{V^*}, \qquad \eta\|u\|_V \le \|Au\|_{V^*}.$$

By density, the last inequality can be extended to $\|\tilde{A}u\|_{V^*} \ge \eta\|u\|_V$ for all $u \in V$. Corollary 4.31 in [**24**] then implies the invertibility of $\tilde{A}$. ♢

The equality $A_0 = A$ is not true in the above example, in general. If $\partial G \in C^2$ and $a_{jk} = a_{kj} \in C^1(\overline{G}, \mathbb{R})$, Theorem 6.3.4 of [**8**] shows that $A_0 = A$ and that the graph norm of A and $\|\cdot\|_{2,2}$ are equivalent. The proof uses PDE methods.

D) The Dirichlet–Laplacian and the wave equation. Examples 1.52 and 1.53 can be applied to the case of $a = I$ yielding the Dirichlet–Laplacian Δ_D. For later reference, we restate the results.

Example 1.54. Let $G \subseteq \mathbb{R}^m$ be open and bounded with Lipschitz boundary ∂G, $E = L^2(G)$, and $A_0 = \Delta$ with $\mathrm{D}(A_0) = W^{2,2}(G) \cap W_0^{1,2}(G)$. Then A_0 is densely defined, symmetric, and dissipative. The operator A_0 has an extension Δ_D which is self-adjoint, invertible and generates an ω-contraction semigroup, where $\omega = -c(G,2)^{-2} < 0$ is given by (1.33). Moreover, $[\mathrm{D}(\Delta_D)]$ is densely embedded in $W_0^{1,2}(G)$. The domain $\mathrm{D}(\Delta_D)$ contains all maps $u \in W_0^{1,2}(G)$ for which there is a function $f =: \Delta_D u$ in $L^2(G)$ such that

$$\forall\, v \in W_0^{1,2}(G): \quad (v|\Delta_D u)_{L^2(G)} = -\int_G \nabla v \cdot \nabla \overline{u} \,\mathrm{d}x.$$

Observe that it enough to consider here real-valued $v \in W_0^{1,2}(G)$. Then, for $u \in \mathrm{D}(\Delta_D)$ the above definition yields $\operatorname{Re} u \in \mathrm{D}(\Delta_D)$ and $\Delta_D(\operatorname{Re} u) = \operatorname{Re} \Delta_D u$.

The operator Δ_D has an invertible bounded extension $\tilde{\Delta}_D : W_0^{1,2}(G) \to W^{-1,2}(G)$ (the *weak Dirichlet–Laplacian*) which acts as

$$\forall\, u, v \in W_0^{1,2}(G): \quad \langle v, \tilde{\Delta}_D u\rangle_{W_0^{1,2}(G)} = -\int_G \nabla v \cdot \nabla u \,\mathrm{d}x.$$

To see that $\tilde{\Delta}_D^{-1}$ extends Δ_D^{-1}, take $\varphi \in L^2(G) \hookrightarrow W^{-1,2}(G)$. The maps $\tilde{v} = \tilde{\Delta}_D^{-1}\varphi \in W_0^{1,2}(G)$ and $v = \Delta_D^{-1}\varphi \in \mathrm{D}(\Delta_D)$ both satisfy $\tilde{\Delta}_D \tilde{v} = \varphi$ and $\tilde{\Delta}_D v = \Delta_D v = \varphi$, so that $\tilde{v} = v$ as $\tilde{\Delta}_D$ is injective. ◊

The next operator will be used to solve the wave equation as explained in Example 2.4. We again write V for $W_0^{1,2}(G)$ equipped with the norm $\||\nabla v|_2\|_2$.

Example 1.55. Let $G \subseteq \mathbb{R}^m$ be open and bounded with Lipschitz boundary ∂G and Δ_D be given on $L^2(G)$ by Example 1.54. Set $E = V \times L^2(G)$, $\mathrm{D}(A) = \mathrm{D}(\Delta_D) \times V$, and

$$A = \begin{pmatrix} 0 & I \\ \Delta_D & 0 \end{pmatrix}.$$

Then A is skew-adjoint, and thus generates a unitary C_0-group on E due to Stone's Theorem 1.44. Moreover, $\mathrm{D}(A)$ and $\mathrm{D}(\Delta_D) \times V$ have equivalent norms.

Proof. Let (u_1, v_1) and (u_2, v_2) belong to $\mathrm{D}(A)$. We compute

$$\begin{aligned}\big(A\tbinom{u_1}{v_1}\big|\tbinom{u_2}{v_2}\big)_E &= \big(\tbinom{v_1}{\Delta_D u_1}\big|\tbinom{u_2}{v_2}\big)_E = \int_G \Big(\nabla v_1 \cdot \nabla \overline{u}_2 + \Delta_D u_1 \overline{v}_2\Big)\,\mathrm{d}x \\ &= -\int_G \Big(v_1 \Delta_D \overline{u}_2 + \nabla u_1 \cdot \nabla \overline{v}_2\Big)\,\mathrm{d}x = -\big(\tbinom{u_1}{v_1}\big|\tbinom{v_2}{\Delta_D u_2}\big)_E,\end{aligned}$$

using the scalar product of V and the definition of Δ_D. We thus arrive at $\big(A\tbinom{u_1}{v_1}\big|\tbinom{u_2}{v_2}\big)_E = \big(\tbinom{u_1}{v_1}\big|-A\tbinom{u_2}{v_2}\big)_E$. Hence, $-A$ is a restriction of A'. We define

$$R = \begin{pmatrix} 0 & \Delta_D^{-1} \\ I & 0 \end{pmatrix} : E \to \mathrm{D}(\Delta_D) \times V = \mathrm{D}(A),$$

where Δ_D^{-1} exists thanks to Example 1.54. It is easy to see that $AR = I$ and $RAw = w$ for every $w \in \mathrm{D}(A)$. Hence, A is invertible and thus also $-A$ and A', see (4.3) in [**27**]. Lemma 1.23 then yields that $-A = A'$.

The last assertion can be checked using (1.33) and the definition of Δ_D. □

CHAPTER 2

The evolution equation and regularity

In the first two sections we discuss the solvability properties of (also inhomogeneous) evolution equations. A class of more regular C_0-semigroups and the corresponding Cauchy problems will be investigated in the last section.

2.1. Wellposedness and the inhomogeneous problem

In this section we come back to the relationship between generation properties of A and the solvability of the corresponding differential equation. In a second part we treat inhomogeneous problems in which one adds a given input function to the evolution equation.

Let A be a closed operator on X and $x \in \mathrm{D}(A)$. We study the (homogeneous) evolution equation (or Cauchy problem)

$$u'(t) = Au(t), \quad t \geq 0, \qquad u(0) = x. \tag{2.1}$$

Recall from Definition 1.9 that a *(classical) solution* of (2.1) is a function $u \in C^1(\mathbb{R}_{\geq 0}, X)$ taking values in $\mathrm{D}(A)$ and satisfying (2.1) for all $t \geq 0$. Observe that then Au belongs to $C(\mathbb{R}_{\geq 0}, X)$ and thus u to $C(\mathbb{R}_{\geq 0}, [\mathrm{D}(A)])$.

Let the states $u(t) \in X$ describe a physical system at time time $t \geq 0$ whose properties are encoded in the operator A and its domain. We then want to predict the future behavior of the system by means of (2.1). To this aim, we need solutions for 'many' initial values x. Moreover, the solutions have to be uniquely determined by x since otherwise we do not really predict the behavior. In addition, one will know the initial value only approximately, so that for a reasonable prediction the solutions should not vary too much under small changes of the data.[1] The next definition makes these requirements precise.

Definition 2.1. *Let A be closed. The Cauchy problem* (2.1) *is called* wellposed *if* $\mathrm{D}(A)$ *is dense in X, if for each $x \in \mathrm{D}(A)$ there is a unique solution $u = u(\cdot\,; x)$ of* (2.1)*, and if the solutions depend continuously on the data; i.e.,*

$$\forall\, b > 0: \quad (\mathrm{D}(A), \|\cdot\|_X) \to C([0,b], X); \quad x \mapsto u(\cdot\,; x), \quad \textit{is continuous.} \tag{2.2}$$

The next theorem says that for closed A the wellposedness of (2.1) and the generation property of A are equivalent. This fact justifies the definitions made at the beginning of Chapter 1.

Theorem 2.2. *Let A be a closed operator. It generates a C_0-semigroup $T(\cdot)$ if and only if* (2.1) *is wellposed. In this case, the function $u = T(\cdot)x$ solves* (2.1) *for each given $x \in \mathrm{D}(A)$.*

[1] Actually, the same applies to the dependence on the operator A, but this will be discussed in Section 3.2.

Proof. 1) Let A generate $T(\cdot)$ and $x \in \mathrm{D}(A)$. Then $T(\cdot)x$ is the unique solution of (2.1) by Proposition 1.10. Condition (2.2) follows from the exponential boundedness of $T(\cdot)$ proven in Lemma 1.4.

2) Conversely, let (2.1) be wellposed.

i) We define the map $T(t) : \mathrm{D}(A) \to \mathrm{D}(A)$ by $T(t)x = u(t;x)$ for $x \in \mathrm{D}(A)$ and $t \geq 0$ using uniqueness. Clearly, $T(0) = I$ and $T(\cdot)x : \mathbb{R}_{\geq 0} \to (\mathrm{D}(A), \|\cdot\|_X)$ is continuous. For $x, y \in \mathrm{D}(A)$ and $\alpha, \beta \in \mathbb{F}$, the function $\alpha u(\cdot\,;x) + \beta u(\cdot\,;y)$ solves (2.1) with initial value $\alpha x + \beta y$ since A is linear. Hence, $T(t)$ is linear for every $t \geq 0$. Let $t, s \geq 0$ and $x \in \mathrm{D}(A)$. Then $u(s;x)$ belongs to $\mathrm{D}(A)$ so that $v(t) := T(t)u(s;x) = T(t)T(s)x$ for $t \geq 0$ is the unique solution of (2.1) with initial value $u(s;x)$. On the other hand, $u(t+s;x) = T(t+s)x$ for $t \geq 0$ also solves this problem. Uniqueness then shows that $T(t)T(s)x = T(t+s)x$.

ii) For each $b > 0$ there is a constant $c(b) > 0$ such that $\|T(t)x\| \leq c(b)\,\|x\|$ for all $x \in \mathrm{D}(A)$ and $t \in [0,b]$. In fact, if this claim was wrong, there would exist a time $b > 0$, a sequence (x_n) in $\mathrm{D}(A)$, and times $t_n \in [0,b]$ such that $\|x_n\| = 1$ and $0 < \|T(t_n)x_n\| =: c_n \to \infty$ as $n \to \infty$. Set $y_n = \frac{1}{c_n}x_n \in \mathrm{D}(A)$ for every $n \in \mathbb{N}$. The initial values y_n then tend to 0 as $n \to \infty$, but the norms $\|u(t_n;y_n)\| = \frac{1}{c_n}\|T(t_n)x_n\| = 1$ do not converge to 0. This contradicts assumption (2.2), and thus $T(\cdot)$ is locally uniformly bounded.

Lemma 2.13 of [**24**] now allows us to extend each single map $T(t)$ to a bounded linear operator on $\overline{\mathrm{D}(A)} = X$ (also denoted by $T(t)$) having the same operator norm. Moreover, Lemma 4.10 in [**24**] yields the strong continuity of the family $(T(t))_{t\geq 0}$. The semigroup law extends from $\mathrm{D}(A)$ to X by approximation, so that $T(\cdot)$ is a C_0-semigroup.

iii) Let B be the generator of $T(\cdot)$. We have $A \subseteq B$ since $T(\cdot)$ solves (2.1). Because $\mathrm{D}(A)$ is dense in X and $T(t)\mathrm{D}(A) \subseteq \mathrm{D}(A)$ for all $t \geq 0$, Proposition 1.37 shows that $\mathrm{D}(A)$ is dense in $[\mathrm{D}(B)]$. So for each $x \in \mathrm{D}(B)$ there are vectors x_n in $\mathrm{D}(A)$ such that $x_n \to x$ and $Ax_n = Bx_n \to Bx$ in X as $n \to \infty$. The closedness of A now implies that $x \in \mathrm{D}(A)$, and thus $A = B$. □

We discuss variants of the above result.

Remark 2.3. a) One cannot drop condition (2.2) in Theorem 2.2: Let B be a closed, densely defined, unbounded operator in a Banach space Y. Set $X = Y \times Y$ and $A = \left(\begin{smallmatrix} 0 & B \\ 0 & 0 \end{smallmatrix}\right)$ with $\mathrm{D}(A) = Y \times \mathrm{D}(B)$. Observe that A is closed and $\mathrm{D}(A)$ is dense in X. For $(x,y) \in \mathrm{D}(A)$ one has the unique solution $u(t) = (x + tBy, y)$ of (2.1) with $u(0) = (x,y)$. But for $t > 0$ one cannot continuously extend $T(t) : (x,y) \mapsto u(t)$ to a map on X since $T(t)(0,y) = (tBy, y)$.

b) Let A be closed. By Proposition II.6.6 in [**7**], problem (2.1) has a unique solution for A and each $x \in \mathrm{D}(A)$ if and only if the operator A_1 on $X_1 = [\mathrm{D}(A)]$ given by $A_1x = Ax$ with $\mathrm{D}(A_1) = \{x \in X_1 \mid Ax \in X_1\}$ generates a C_0-semigroup on X_1. Moreover, if $\rho(A) \neq \emptyset$ and (2.1) has a unique solution for each $x \in \mathrm{D}(A)$, then A is a generator on X, see Theorem II.6.7 in [**7**]. ◊

We now use Example 1.55 to solve the wave equation. For simplicity, we restrict ourselves to the time interval $\mathbb{R}_{\geq 0}$ though one could actually treat $\mathbb{R}$, thus solving the problem backward in time starting from $t = 0$.

EXAMPLE 2.4. Let $G \subseteq \mathbb{R}^m$ be open and bounded with $\partial G \in C^{1-}$. We study the wave equation with Dirichlet boundary conditions

$$\begin{aligned} \partial_t^2 u(t,x) &= \Delta u(t,x), \qquad t \geq 0,\ x \in G, \\ u(t,x) &= 0, \qquad t \geq 0,\ x \in \partial G, \\ u(0,x) &= u_0(x), \quad \partial_t u(0,x) = u_1(x), \qquad x \in G, \end{aligned} \tag{2.3}$$

for given functions (u_0, u_1). Let Δ_D on $L^2(G)$ be given by Example 1.54. We take $u_0 \in \mathrm{D}(\Delta_D)$ and $u_1 \in V = W_0^{1,2}(G)$, using the norm $\|v\|_V = \||\nabla v|_2\|_2$.

We interpret the partial differential equation (2.3) as the second-order evolution equation

$$u''(t) = \Delta_D u(t), \quad t \geq 0, \qquad u(0) = u_0, \quad u'(0) = u_1, \tag{2.4}$$

in $L^2(G)$. Here we look for solutions u in $C^2(\mathbb{R}_{\geq 0}, L^2(G)) \cap C^1(\mathbb{R}_{\geq 0}, V) \cap C(\mathbb{R}_{\geq 0}, [\mathrm{D}(\Delta_D)])$. In particular, the boundary condition in (2.3) is understood in the sense of trace $u(t) \in W_0^{1,2}(G)$ and the Laplacian in the form sense of Example 1.54. To obtain a first-order evolution equation, we set

$$E = V \times L^2(G), \qquad \mathrm{D}(A) = \mathrm{D}(\Delta_D) \times V, \qquad \text{and} \qquad A = \begin{pmatrix} 0 & I \\ \Delta_D & 0 \end{pmatrix}.$$

From Example 1.55 we know that A generates a unitary C_0-group $T(\cdot)$ on E.

We claim that (2.4) has a solution u if and only if the problem (2.1) on E for this A and the initial value $w^0 := (u_0, u_1) \in \mathrm{D}(A)$ has a solution $w = (w_1, w_2)$, which is then given by $w = (u, u')$.

To show the claim, let w solve (2.1) for A. The function $u := w_1$ then belongs to $C^1(\mathbb{R}_{\geq 0}, V) \cap C(\mathbb{R}_{\geq 0}, [\mathrm{D}(\Delta_D)])$ with $u(0) = u_0$ and w_2 to $C^1(\mathbb{R}_{\geq 0}, L^2(G)) \cap C(\mathbb{R}_{\geq 0}, V)$. Equation (2.1) for A also yields that $u' = w_1' = w_2$ so that u is an element of $C^2(\mathbb{R}_{\geq 0}, L^2(G))$ with $u'(0) = u_1$ and it satisfies $u'' = w_2' = \Delta_D w_1 = \Delta_D u$ as required. Conversely, let u solve (2.4). We then set $w = (u, u')$. This map is contained in $C(\mathbb{R}_{\geq 0}, [\mathrm{D}(A)]) \cap C^1(\mathbb{R}_{\geq 0}, E)$ and it fulfills $w(0) = w^0$ as well as $w' = (u', u'') = (u', \Delta_D u) = Aw$.

Thus, there is a unique solution u of (2.4) for $(u_0, u_1) \in \mathrm{D}(A)$. It has constant 'energy' $\frac{1}{2}\int_G (|\nabla u(t)|_2^2 + |\partial_t u(t)|^2)\mathrm{d}x = \frac{1}{2}\|w(t)\|_E^2 = \frac{1}{2}\|w^0\|_E^2$ by unitarity. ◊

Inhomogeneous evolution equations. To equation (2.1) we now add a given function f, which can model a force in a wave equation or a source-sink term in a diffusion problem. We take a time interval $J \subseteq \mathbb{R}$ with $\inf J = 0$ and set $J' = J \cup \{0\}$. This general class of J is useful for Section 2.3 and for applications to nonlinear problems, cf. [**26**].

Let A generate the C_0-semigroup $T(\cdot)$, $u_0 \in X$, and $f \in C(J', X)$.[2] We study the *inhomogeneous evolution equation*

$$u'(t) = Au(t) + f(t), \quad t \in J, \qquad u(0) = u_0. \tag{2.5}$$

Our first solution concept is similar to the homogeneous case in Definition 1.9, where we require continuity of u at $t = 0$ because of the initial condition.

DEFINITION 2.5. *A map $u : J' \to X$ is a* (classical) solution *of* (2.5) *on J if u belongs to $C^1(J, X) \cap C(J', X)$, $u(t) \in \mathrm{D}(A)$ for $t \in J$, and u satisfies* (2.5).

[2] In the lectures a somewhat more general setting was treated.

Again, a solution is contained in $C(J, [\mathrm{D}(A)])$. We first show uniqueness of such solutions and that they are given by *Duhamel's formula* (2.6).

PROPOSITION 2.6. *Let A generate the C_0-semigroup $T(\cdot)$, $u_0 \in X$, and $f \in C(J', X)$. If u solves* (2.5) *on J, then it is given by*

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s)\,\mathrm{d}s, \qquad t \in J'. \tag{2.6}$$

In particular, solutions of (2.5) *are unique. If $T(\cdot)$ is a C_0-group, we can take any interval J of positive length that contains 0.*

PROOF. Let $t \in J$ and set $v(s) = T(t-s)u(s)$ for $0 \le s \le t$, where u solves (2.5) on J. We focus on the case $0 \notin J$. As in the proof of Proposition 1.10 and using (2.5), one shows that v is continuously differentiable with derivative

$$v'(s) = T(t-s)u'(s) - T(t-s)Au(s) = T(t-s)f(s)$$

for all $0 < s \le t$. Let $\varepsilon \in (0, t)$. By integration we deduce

$$\int_\varepsilon^t T(t-s)f(s)\,\mathrm{d}s = v(t) - v(\varepsilon) = u(t) - T(t-\varepsilon)u(\varepsilon).$$

Since the integrand is continuous on J', we can let $\varepsilon \to 0$ in the above integral. Lemma 1.12 and (2.5) further imply that $T(t-\varepsilon)u(\varepsilon) \to T(t)u_0$. The last claim is shown in the same way. □

Note that Duhamel's formula (2.6) defines a function u for all $x \in X$ and $f \in C(J', X)$. One can thus ask whether u still solves the equation (2.5) for such data. In the present setting, this is not true in general as the next example shows, but we continue to discuss this point in the following section.

EXAMPLE 2.7. Let $X = C_0(\mathbb{R})$, $A = \frac{\mathrm{d}}{\mathrm{d}s}$ with $\mathrm{D}(A) = C_0^1(\mathbb{R})$, and $\varphi \in X \setminus C^1(\mathbb{R})$. The operator A generates the C_0-group $T(\cdot)$ on X given by $T(t)g = g(\cdot + t)$, see Example 1.21. The function $T(t)\varphi$ then does not belong to $\mathrm{D}(A)$, for each $t \ge 0$, and at some $t_0 \in \mathbb{R}$ the map $t \mapsto (T(t)\varphi)(0) = \varphi(t)$ is not differentiable, cf. Remark 1.11. Define $f \in C(\mathbb{R}_{\ge 0}, X)$ by $f(t) = T(t)\varphi$ and let $u_0 = 0$. Formula (2.6) then yields

$$u(t) = \int_0^t T(t-r)T(r)\varphi\,\mathrm{d}r = tT(t)\varphi, \qquad t \ge 0.$$

So u does not solve (2.5) as $u(t) \notin \mathrm{D}(A)$ and u is not differentiable for $t > 0$. ◊

We now show criteria on f implying that Duhamel's formula (2.6) provides a solution of (2.5). We start with the core step that says that time and 'space' regularity are equivalent. As in Proposition 1.10, for instance, we heavily rely on the Definition 1.1 of generators.

LEMMA 2.8. *Let A generate the C_0-semigroup $T(\cdot)$, $u_0 \in \mathrm{D}(A)$, and $f \in C(J', X)$. Set $v(t) = \int_0^t T(t-s)f(s)\,\mathrm{d}s$ for $t \in J'$. Then the following assertions are equivalent.*

a) $v \in C^1(J, X)$.

b) $v(t) \in \mathrm{D}(A)$ for all $t \in J$ and $Av \in C(J, X)$.

In this case, (2.6) *gives the unique solution of* (2.5) *on J. If* (2.5) *has a solution on J, then properties a) and b) are true.*

PROOF. 1) By Proposition 1.10, the orbit $T(\cdot)u_0$ belongs to $C^1(\mathbb{R}_{\geq 0}, X) \cap C(\mathbb{R}_{\geq 0}, [\mathrm{D}(A)])$ with derivative $\frac{\mathrm{d}}{\mathrm{d}t}T(t)u_0 = AT(t)u_0$ for all $t \geq 0$, since $u_0 \in \mathrm{D}(A)$. Let u solve (2.5). We then deduce $v = u - T(\cdot)x$ from Proposition 2.6, so that v satisfies properties a) and b). Proposition 2.6 yields uniqueness.

2) Let a) or b) be valid. It remains to show that v solves (2.5) with $u_0 = 0$, since then u defined by (2.6) is a solution of (2.5) for the given initial value u_0. Recalling Lemma 1.4, we first note that $\|v(t)\| \leq M\mathrm{e}^{\omega_+ t}\int_0^t \|f(s)\|\,\mathrm{d}s$ tends to 0 as $t \to 0$ by the boundeness of f near 0. It is then easy to check the continuity of $v : J' \to X$, e.g., using Remark 1.15 e).

We next fix $t \in J$ and take $h \neq 0$ such that $t + h \in J$. We compute

$$\begin{aligned} D_1(h) &:= \frac{1}{h}(T(h) - I)v(t) = \frac{1}{h}(v(t+h) - v(t)) - \frac{1}{h}\int_t^{t+h} T(t+h-s)f(s)\,\mathrm{d}s \\ &=: D_2(h) - I(h). \end{aligned}$$

Since $f \in C(J', X)$, it follows

$$\begin{aligned} \|I(h) - f(t)\| &= \Big\|\frac{1}{h}\int_t^{t+h} \Big(T(t+h-s)f(s) - f(t)\Big)\,\mathrm{d}s\Big\| \\ &\leq \frac{|h|}{|h|}\max_{|s-t|\leq|h|}\|T(t+h-s)f(s) - f(t)\| \;\longrightarrow\; 0 \end{aligned}$$

as $h \to 0$, thanks to Lemma 1.12. As a result, $D_1(h)$ converges if and only if $D_2(h)$ converges as $h \to 0$. The convergence of D_1 means that $v(t) \in \mathrm{D}(A)$ and $D_1(h) \to Av(t)$ as $h \to 0$, and that of D_2 is equivalent to the differentiability of v at t with $D_2(h) \to v'(t)$ as $h \to 0$. We further obtain that $Av(t) = v'(t) - f(t)$; i.e., v satisfies the differential equation in (2.5) for this t. For each $t \in J$ the properties a) and b) imply the convergence of D_2 and D_1, respectively, and hence the function v solves (2.5) with $u_0 = 0$. □

The next theorem is the fundamental existence result for the inhomogeneous evolution equation (2.5). For simplicity, we restrict ourselves to the case $0 \in J$.

THEOREM 2.9. *Let A generate the C_0-semigroup $T(\cdot)$, $u_0 \in \mathrm{D}(A)$, and $0 \in J$. Assume either that $f \in C^1(J, X)$ or that $f \in C(J, [\mathrm{D}(A)])$. Then the function u given by* (2.6) *is the unique solution of* (2.5) *on J.*

PROOF. Proposition 2.6 yields uniqueness. Let $f \in C^1(J, X)$ and $t \in J$. Writing $v(t) = \int_0^t T(s)f(t-s)\,\mathrm{d}s$, we see that v has the continuous derivative

$$v'(t) = T(t)f(0) + \int_0^t T(s)f'(t-s)\,\mathrm{d}s$$

as in Analysis 2 or Remark 1.15. Hence, property a) in Lemma 2.8 is satisfied.

Let $f \in C(J, [\mathrm{D}(A)])$. Proposition 1.10 and Lemma 1.12 imply that the vector $T(t-s)f(s)$ belongs to $\mathrm{D}(A)$ and the map $(t,s) \mapsto AT(t-s)f(s) = T(t-s)Af(s)$ is continuous in X for $s \leq t$ in J. Remark 1.15 d) yields that $v(t)$ belongs to $\mathrm{D}(A)$ and $Av(t) = \int_0^t T(t-s)Af(s)\,\mathrm{d}s$. By means of Remark 1.15 e), one then checks that Av is an element of $C(J, X)$, and so statement b) of Lemma 2.8 is fulfilled. The theorem now follows from Lemma 2.8. □

Variants for more regular solutions are discussed in the exercises. We apply the above result to the wave equation with a given force.[3]

Example 2.10. In the setting of Example 2.4, we consider the inhomogeneous wave equation

$$\begin{aligned} \partial_t^2 u(t,x) &= \Delta u(t,x) + g(t,x), \qquad t \ge 0,\ x \in G, \\ u(t,x) &= 0, \qquad t \ge 0,\ x \in \partial G, \\ u(0,x) &= u_0(x), \quad \partial_t u(0,x) = u_1(x), \qquad x \in G, \end{aligned} \tag{2.7}$$

for given $u_0 \in \mathrm{D}(\Delta_D)$, $u_1 \in V = W_0^{1,2}(G)$ and $g \in C(\mathbb{R}_{\ge 0}, L^2(G))$, where we set $g(t,x) = (g(t))(x)$ for all $t \ge 0$ and almost every $x \in G$. As in Example 2.4 we write these equations as

$$u''(t) = \Delta_D u(t) + g(t), \quad t \ge 0, \qquad u(0) = u_0, \quad u'(0) = u_1, \tag{2.8}$$

and look for solutions u in $C^2(\mathbb{R}_{\ge 0}, L^2(G)) \cap C^1(\mathbb{R}_{\ge 0}, V) \cap C(\mathbb{R}_{\ge 0}, [\mathrm{D}(\Delta_D)])$.

Again the second-order problem is equivalent to the first-order problem

$$w'(t) = A(t)w(t) + f(t), \quad t \ge 0, \qquad w^0 = (u_0, u_1),$$

on $E = V \times L^2(G)$ with $w = (u, u')$,

$$A = \begin{pmatrix} 0 & I \\ \Delta_D & 0 \end{pmatrix} \quad \text{on} \quad \mathrm{D}(A) = \mathrm{D}(\Delta_D) \times V, \quad \text{and} \quad f = \begin{pmatrix} 0 \\ g \end{pmatrix}.$$

In view of Theorem 2.9 and Example 1.55, we then obtain a unique solution u of (2.8) if either g belongs to $C^1(\mathbb{R}_{\ge 0}, L^2(G))$ and thus f to $C^1(\mathbb{R}_{\ge 0}, E))$, or g is contained in $C(\mathbb{R}_{\ge 0}, V)$ and thus f in $C(\mathbb{R}_{\ge 0}, [\mathrm{D}(A)])$.

Note that g is only required to possess one derivative, but u has two. One gains this derivative because of the structure of E, $\mathrm{D}(A)$, and f. ♢

2.2. Mild solution and extrapolation

So far we have considered solutions of (2.1) or (2.5) taking values in $\mathrm{D}(A)$, which is surely a natural choice. However, in many situations one wants to admit solutions and initial values in X. For instance, in the wave equation from Examples 2.4 and 2.10 the squared norm of the state space E is (up to factors) equal to the physical energy, and it is often desirable only to require that the solutions have finite energy. We first introduce a concept motivated by Proposition 2.6. It plays an important role for certain nonlinear evolution equations, see [**26**]. Let J be an interval with $\inf J = 0$ and $J' = J \cup \{0\}$.

Definition 2.11. *Let A generate the C_0-semigroup $T(\cdot)$, $u_0 \in X$, and $f \in C(J', X)$. The function $u \in C(J', X)$ given by*

$$u(t) = T(t)u_0 + \int_0^t T(t-s) f(s)\,\mathrm{d}s, \qquad t \in J',$$

is called mild solution *(on J') of (2.5).*

[3]The map g in (2.7) corresponds to a force if the vibrating object has mass density 1.

The continuity of the mild solution and $u(0) = u_0$ were noted in the proof of Lemma 2.8. The above definition has the obvious draw-back that one does not directly see the connection to A and to (2.5). For $f = 0$, Lemma 1.18 suggests the following notion which involves A explicitly.

Definition 2.12. *Let A be a closed operator, $u_0 \in X$, $0 \in J$, and $f \in C(J, X)$. A function $u \in C(J, X)$ is called an* integrated solution *(on J) of* (2.5) *if the integral $\int_0^t u(s)\,\mathrm{d}s$ belongs to $\mathrm{D}(A)$ and satisfies*

$$u(t) = u_0 + A\int_0^t u(s)\,\mathrm{d}s + \int_0^t f(s)\,\mathrm{d}s \qquad \text{for all } \ t \in J. \tag{2.9}$$

The questions arise whether integrated solutions are unique, how they relate to mild ones, and whether they solve a differential equation. At least, the function $t \mapsto \int_0^t u(s)\,\mathrm{d}s$ is differentiable, though in X instead of $[\mathrm{D}(A)]$. Moreover, for mild solutions it is not clear at all how to differentiate $t \mapsto T(t-s)f(s)$. The key idea to solve these problems is to enlarge the state space X suitably.

By Section 2.2 D) of [**24**], each normed vector space Y possesses its completion $\tilde{Y}$. It is a Banach space such that there is an isometry $J : Y \to \tilde{Y}$ with dense range. This property determines $\tilde{Y}$ uniquely up to isometric isomorphisms. Note that Y equipped with an equivalent norm again yields $\tilde{Y}$ (with an equivalent norm).

Definition 2.13. *Let A be a closed operator with $\mu \in \rho(A)$. We define the* extrapolated norm $\|x\|_{-1}^A = \|x\|_{-1} = \|R(\mu, A)x\|$ *for $x \in X$ and the* extrapolation space $X_{-1} = X_{-1}^A$ *as the completion of $(X, \|\cdot\|_{-1})$.*

We identify X with a dense subspace of X_{-1}. Note that $\|\cdot\|_{-1}$ a coarser norm on X than the original one (which is not complete if A is unbounded). Moreover, the norm $\|\cdot\|_{-1}$ does not depend on the choice of $\mu \in \rho(A)$, up to equivalence: Let $\lambda \in \rho(A)\setminus\{\mu\}$. Using the resolvent equation (1.7), we compute

$$\begin{aligned} \|R(\lambda, A)x\| &\le \|R(\mu, A)x\| + |\mu - \lambda|\,\|R(\lambda, A)R(\mu, A)x\| \\ &\le (1 + |\mu - \lambda|\,\|R(\lambda, A)\|)\,\|R(\mu, A)x\|, \end{aligned} \tag{2.10}$$

and one can interchange λ and μ here.

By means of Lemma 2.13 in [**24**] and density, one can extend an operator $S \in \mathcal{B}(X)$ to X_{-1}^A if (and only if) it satisfies $\|R(\mu, A)Sx\|_X \le c\|R(\mu, A)x\|_X$ for some $c > 0$ and all $x \in X$.

In Example 2.17 we compute X_{-1} in one case. But actually one can quite often use X_{-1} to 'legalize illegal computations' without knowing a precise description of it. The next result shows that we can extend the C_0-semigroup generated by A to X_{-1}^A keeping many of its properties.

Proposition 2.14. *Let A generate the C_0-semigroup $T(\cdot)$ on X, $t \ge 0$, and $\mu, \lambda \in \rho(A)$. Then the operators $T(t)$ have bounded extensions $T_{-1}(t)$ to $X_{-1} = X_{-1}^A$, forming a C_0-semigroup on X_{-1}. It is generated by the extension $A_{-1} \in \mathcal{B}(X, X_{-1})$ of A, where $\mathrm{D}(A_{-1}) = X$, and $\|\cdot\|_X$ is equivalent to the graph norm of A_{-1}. Moreover, the resolvent $R(\lambda, A)$ has an extension in $\mathcal{B}(X_{-1}, X)$ which is the resolvent of A_{-1}. The maps $R \coloneqq R(\mu, A_{-1}) : X_{-1} \to X$ and $R^{-1} =$*

$\mu I - A_{-1} : X \to X_{-1}$ are isometric isomorphisms satisfying $A = RA_{-1}R^{-1}$ on $\mathrm{D}(A)$*, so that $\sigma(A) = \sigma(A_{-1})$. Analogous facts are true for $R(\lambda, A)$ and $T(t)$.*

Proof. 1) Let $\lambda \in \rho(A)$ and $x \in X$. By estimate (2.10) we have $\|R(\lambda, A)x\| \le c_\lambda \|x\|_{-1}$ for a constant c_λ. Because X is dense in X_{-1}, we can extend $R(\lambda, A)$ to a map R_λ in $\mathcal{B}(X_{-1}, X)$ using Lemma 2.13 in [**24**]. We note that R_μ is an isometry. For $x \in \mathrm{D}(A)$ we have

$$\|Ax\|_{-1} = \|(A - \mu I + \mu I)R(\mu, A)x\|_X \le (1 + |\mu|\, \|R(\mu, A)\|)\|x\|,$$

so that A has an extension $A_{-1} \in \mathcal{B}(X, X_{-1})$. The identity $I_X = (\lambda I_X - A)R(\lambda, A)$ on X can thus be extended to $I_{X_{-1}} = (\lambda I_{X_{-1}} - A_{-1})R_\lambda$ on X_{-1}, and analogously one obtains $I_X = R_\lambda(\lambda I_{X_{-1}} - A_{-1})$ on X. This means that $\lambda \in \rho(A_{-1})$ and $R_\lambda = R(\lambda, A_{-1})$. (Note that A_{-1} is closed in X_{-1} as $R_\lambda \in \mathcal{B}(X_{-1})$.) We next compute

$$R(\mu, A_{-1})A_{-1}(\mu I - A)x = A_{-1}R(\mu, A)(\mu I - A)x = Ax$$

for $x \in \mathrm{D}(A)$. It follows that $\sigma(A) = \sigma(A_{-1})$ since $R(\lambda I - A_{-1})R^{-1} = \lambda I - A$ on $\mathrm{D}(A)$. Using $X \hookrightarrow X_{-1}$, we show the asserted norm equivalence by

$$\|x\|_{A_{-1}} = \|x\|_{-1} + \|A_{-1}x\|_{-1} \le c\|x\|_X + \|A_{-1}\|\,\|x\|_X\,,$$
$$\|x\|_X = \|RR^{-1}x\|_X = \|\mu x - A_{-1}x\|_{-1} \le \max\{|\mu|, 1\}\,\|x\|_{A_{-1}}\,.$$

2) It is easy to see that $A_{-1} = R^{-1}AR$ with $\mathrm{D}(A_{-1}) = X$ generates the C_0-semigroup on X_{-1} given by $T_{-1}(t) := R^{-1}T(t)R$ for $t \ge 0$, cf. Paragraph II.2.1 in [**7**]. This semigroup extends $T(\cdot)$ because of $T_{-1}(t)x = (\mu I - A)T(t)R(\mu, A)x = T(t)x$ for $x \in X$. The other assertions are shown similarly. □

Part 1) of the proof also works if one only assumes that A is closed and densely defined with $\mu \in \rho(A)$. Using these concepts and results, we can now easily show that mild and integrated solutions coincide and that they are just the unique (classical) solutions in X_{-1} of the extrapolated problem

$$u'(t) = A_{-1}u(t) + f(t), \quad t \in J, \qquad u(0) = u_0 \in X. \tag{2.11}$$

Proposition 2.15. *Let A generate the C_0-semigroup $T(\cdot)$ on X, $u_0 \in X$, $0 \in J$, and $f \in C(J, X)$. Then the mild solution $u \in C(J, X)$ given by (2.6) also belongs to $C^1(J, X_{-1})$ and u is the (classical) solution of (2.11) in X_{-1}. It is also the unique integrated solution of (2.5) in the sense of (2.9).*

Proof. The first assertion follows from Theorem 2.9 and Proposition 2.14 using the conditions on u_0 and f, as well as $X = \mathrm{D}(A_{-1})$ and $T_{-1}(t){\restriction}_X = T(t)$.

Let $u \in C(J, X)$ be the (unique) solution of (2.11). Integrating this differential equation, we derive the identity

$$u(t) - u_0 = \int_0^t A_{-1}u(s)\,\mathrm{d}s + \int_0^t f(s)\,\mathrm{d}s$$

for $t \in J$. We can take $A_{-1} \in \mathcal{B}(X, X_{-1})$ out of the integral, resulting in

$$(A_{-1} - \mu I)\int_0^t u(s)\,\mathrm{d}s = u(t) - u_0 - \mu\int_0^t u(s)\,\mathrm{d}s - \int_0^t f(s)\,\mathrm{d}s.$$

Since the right-hand side belongs to X and $R(\mu, A_{-1})$ extends $R(\mu, A)$, the integral $\int_0^t u(s)\,\mathrm{d}s$ thus belongs to $\mathrm{D}(A)$ and u is an integrated solution of (2.5).

Let $u \in C(J, X)$ be an integrated solution of (2.5). As $A_{-1} \in \mathcal{B}(X, X_{-1})$, we can differentiate $t \mapsto A \int_0^t u(s)\,\mathrm{d}s$ in X_{-1} with derivative $A_{-1}u(t)$. Equation (2.9) then implies that u is contained in $C^1(J, X_{-1})$ and solves (2.11). □

For any Banach space X we have the isometry

$$J_X : X \to X^{**}; \quad J_X(x) = \langle x, \cdot\,\rangle_{X \times X^*},$$

see Proposition 5.24 of [**24**]. The space X is called reflexive if J_X is surjective. By Example 5.27 in [**24**], a Hilbert space X is reflexive (with $J_X = \Phi_{X^*}\Phi_X$ for the Riesz isomorphisms) and also $L^p(\mu)$ for $p \in (1, \infty)$. In the reflexive case one can describe the extrapolation by duality in a convenient way.

Proposition 2.16. *Let A be closed with $\mu \in \rho(A)$ and dense domain. Then there is an isomorphism $\Psi : [\mathrm{D}(A^*)] \to (X_{-1})^*$ satisfying $(\Psi x^*)(x) = \langle x, x^*\rangle_{X \times X^*}$ for $x \in X \hookrightarrow X_{-1}$ and $x^* \in \mathrm{D}(A^*)$. Let X be reflexive, in addition. We then have an isomorphism $\Phi : X_{-1} \to [\mathrm{D}(A^*)]^*$ extending $J_X : X \to X^{**}$.*

Proof. Replacing $A - \mu I$ by A we can restrict ourselves to the case $\mu = 0$. Let $x^* \in \mathrm{D}(A^*)$. For $x_{-1} \in X_{-1}$ we set

$$(\Psi x^*)(x_{-1}) = \langle (A_{-1})^{-1}x_{-1}, A^*x^*\rangle_{X \times X^*}.$$

We first observe that

$$|(\Psi x^*)(x_{-1})| \le \|(A_{-1})^{-1}x_{-1}\|_X \, \|A^*x^*\|_{X^*} \le \|x_{-1}\|_{X_{-1}} \, \|x^*\|_{A^*}\,,$$

so that Ψx^* belongs to $(X_{-1})^*$ with norm less or equal $\|x^*\|_{A^*}$ and hence $\Psi : [\mathrm{D}(A^*)] \to (X_{-1})^*$ is a linear contraction. This map acts as $(\Psi x^*)(x) = \langle x, x^*\rangle_{X \times X^*}$ for $x \in X$ since $(A_{-1})^{-1}$ extends A^{-1} on X.

To show surjectivity, we take $\varphi \in (X_{-1})^*$. Let $x \in X$. We then estimate

$$|\varphi(A_{-1}x)| \le \|\varphi\|_{(X_{-1})^*} \, \|A_{-1}x\|_{X_{-1}} = \|\varphi\|_{(X_{-1})^*} \, \|x\|_X\,,$$

and hence $\varphi \circ A_{-1}$ is contained in X^*. There thus exists an element y^* of X^* such that $\varphi(A_{-1}x) = \langle x, y^*\rangle_X$ for all $x \in X$ and $\|y^*\|_{X^*} \le \|\varphi\|_{(X_{-1})^*}$. We set $x^* = (A^*)^{-1}y^* \in \mathrm{D}(A^*)$ recalling that $\sigma(A^*) = \sigma(A)$ by Theorem 1.24 of [**27**]. It follows $A^*x^* = y^*$ and

$$\|x^*\|_{A^*} = \|(A^*)^{-1}A^*x^*\|_{X^*} + \|A^*x^*\|_{X^*} \le c\|y^*\|_{X^*} \le c\|\varphi\|_{(X_{-1})^*}\,.$$

Moreover, the definitions of Ψ and y^* yield

$$(\Psi x^*)(x_{-1}) = \langle (A_{-1})^{-1}x_{-1}, A^*x^*\rangle_{X \times X^*} = \varphi(A_{-1}(A_{-1})^{-1}x_{-1}) = \varphi(x_{-1})$$

for all $x_{-1} \in X_{-1}$; i.e., $\varphi = \Psi x^*$ and Ψ is surjective. It is also injective with a bounded inverse by the above lower estimate, and thus Ψ is invertible.

Let X be reflexive so that also the isomorphic space X_{-1} is reflexive, see Corollary 5.51 in [**24**]. We then define the isomorphism $\Phi = \Psi^* J_{X_{-1}} : X_{-1} \to [\mathrm{D}(A^*)]^*$. For $x \in X$ and $x^* \in \mathrm{D}(A^*)$ we compute

$$\langle x^*, \Phi x\rangle_{\mathrm{D}(A^*)} = \langle \Psi x^*, J_{X_{-1}}x\rangle_{(X_{-1})^*} = \langle x, \Psi x^*\rangle_{X_{-1}} = \langle x, x^*\rangle_X = \langle x^*, J_X x\rangle_{X^*}\,,$$

using the above properties. This shows the last assertion. □

By extrapolation we now obtain solutions u of the wave equation (2.8) such that $(u(t), u'(t))$ only take values in the space $W_0^{1,2}(G) \times L^2(G)$ of finite energy.

Example 2.17. As in Examples 2.4 and 2.10 we study the wave equation

$$u''(t) = \tilde{\Delta}_D u(t) + g(t), \quad t \in J, \qquad u(0) = u_0, \quad u'(0) = u_1, \tag{2.12}$$

now with the invertible extension $\tilde{\Delta}_D : W_0^{1,2}(G) \to W^{-1,2}(G) = W_0^{1,2}(G)^*$ of $\Delta_D : \mathrm{D}(\Delta_D) \to L^2(G)$ from Example 1.54 and with data $w^0 = (u_0, u_1) \in E = W_0^{1,2}(G) \times L^2(G)$ and $g \in C(J, L^2(G))$, where $0 \in J$. We look for solutions in $Z := C(J, W_0^{1,2}(G)) \cap C^1(J, L^2(G)) \cap C^2(J, W^{-1,2}(G))$ and set again

$$A = \begin{pmatrix} 0 & I \\ \Delta_D & 0 \end{pmatrix} \quad \text{on} \quad \mathrm{D}(A) = \mathrm{D}(\Delta_D) \times W_0^{1,2}(G), \qquad \text{and} \qquad f = \begin{pmatrix} 0 \\ g \end{pmatrix}.$$

Example 1.55 provides the inverse

$$A^{-1} = \begin{pmatrix} 0 & \Delta_D^{-1} \\ I & 0 \end{pmatrix} : E \to \mathrm{D}(A).$$

To compute E_{-1} we recall from Example 1.54 that $\tilde{\Delta}_D^{-1}$ extends Δ_D^{-1}. Set $F = L^2(G) \times W^{-1,2}(G)$. For $(u,v) \in E$ we compute[4]

$$\|(u,v)\|_{E_{-1}} = \|A^{-1}(u,v)\|_E = \|(\tilde{\Delta}_D^{-1} v, u)\|_E \eqsim \|(u,v)\|_F \,.$$

As noted before Definition 2.13, it follows that E_{-1} is also the completion of $(E, \|\cdot\|_F)$. Since $I : (E, \|\cdot\|_F) \to F$ is an isometry with dense range, the space F is isomorphic to E_{-1} by Proposition 2.21 of [**24**]. Identifying F and E_{-1}, the extension of A to E is given by

$$A_{-1} = \begin{pmatrix} 0 & I \\ \tilde{\Delta}_D & 0 \end{pmatrix} : E \to L^2(G) \times W^{-1,2}(G).$$

It generates a C_0-semigroup on F by Proposition 2.14. Theorem 2.9 thus yields a unique solution w of (2.11) in F for our data. As in Examples 2.4 and 2.10, one now obtains a unique solution $u \in Z$ of (2.12) given by $w = (u, u')$. ◊

2.3. Analytic semigroups and sectorial operators

So far we have treated C_0-semigroups and groups without requiring further properties of them. However, both from the view point of applications and from a more theoretical perspective, it is natural and rewarding to study classes of C_0-semigroups with specific properties. (In [**7**] such questions are treated in detail.) For instance, compact semigroup or resolvent operators often occur in concrete problems, and they have special properties, of course. If the Banach space X carries an order structure (e.g., $X = L^p(\mu)$ or $X = C_0(G)$), then 'positive' semigroups preserving the order are important, and they are used to describe diffusion or transport phenomena. Occasionally we will come back to positivity and (more rarely) to compactness later in the course.

Another possible property of C_0-semigroups $T(\cdot)$ is the improved regularity of the map $\mathbb{R}_+ \ni t \mapsto T(t)$ beyond strong continuity.[5] In this section we study the strongest case in this context, namely analyticity of the map $\mathbb{R}_+ \to \mathcal{B}(X)$; $t \mapsto T(t)$. This class turns out to be of great importance in applications to diffusion

[4] The symbol $\eqsim$ means that $c\|(\tilde{\Delta}_D^{-1} v, u)\|_E \le \|(u,v)\|_F \le C\|(\tilde{\Delta}_D^{-1} v, u)\|_E$ for all (u,v) and some constants $c, C > 0$.

[5] Recall from the exercises that the generator A is bounded if $T(t) \to I$ in $\mathcal{B}(X)$ as $t \to 0$.

problems, for instance. We first introduce and discuss a class of operators which is crucial to determine the generators of such 'analytic semigroups.'

In this section, let $\mathbb{F} = \mathbb{C}$. For $\phi \in (0, \pi]$, we write

$$\Sigma_\phi = \{\lambda \in \mathbb{C} \setminus \{0\} \,|\, |\arg \lambda| < \phi\}$$

for the open *sector* with (half) opening angle ϕ. Observe that $\Sigma_{\pi/2} = \mathbb{C}_+$ is the open right halfplane and $\Sigma_\pi = \mathbb{C} \setminus \mathbb{R}_{\le 0}$ is the plane with cut $\mathbb{R}_{\le 0}$.

Definition 2.18. *A closed operator A is called* sectorial (of type (K, ϕ)) *if for some constants $\phi \in (0, \pi)$ and $K > 0$ the sector Σ_ϕ belongs to $\rho(A)$ and the resolvent satisfies the inequality*

$$\|R(\lambda, A)\| \le \frac{K}{|\lambda|} \qquad \textit{for all} \quad \lambda \in \Sigma_\phi. \tag{2.13}$$

The supremum $\varphi(A) = \varphi \in (0, \pi]$ of all such ϕ is called the angle of A.

Often we will look at maps A such that the shifted operator $A - \omega I$ is sectorial for some $\omega \in \mathbb{R}$, which can be treated by rescaling arguments. Clearly, if A is sectorial with angle φ, then it has type (K_ϕ, ϕ) for all $\phi \in (0, \varphi)$. Typically, K_ϕ explodes as $\phi \to \varphi$ as we will see below in several examples.[6] We also note that several variants of the above concepts are used in literature; e.g., many authors consider operators whose resolvent set contains a sector opening to the left.

We first discuss a few relatively simple examples which are typical nevertheless, starting with the arguably 'nicest' class of operators.

Example 2.19. Let X be a Hilbert space and A be densely defined and self-adjoint on X satisfying $\sigma(A) \subseteq \mathbb{R}_{\le 0}$. Then A is sectorial of angle π.

Proof. Let $\phi \in (\frac{\pi}{2}, \pi)$ and $\lambda \in \Sigma_\phi \subseteq \rho(A)$. Since $R(\lambda, A)' = R(\overline{\lambda}, A)$ by (4.3) in [**27**], the operator $R(\lambda, A)$ is normal. Propositions 4.3 and 1.20 of [**27**] and the assumption then yield

$$\|R(\lambda, A)\| = \mathrm{r}(R(\lambda, A)) = \frac{1}{\mathrm{d}(\lambda, \sigma(A))} \le \frac{1}{\mathrm{d}(\lambda, \mathbb{R}_{\le 0})} = \begin{cases} \frac{1}{|\lambda|}, & \operatorname{Re} \lambda \ge 0, \\ \frac{1}{|\operatorname{Im} \lambda|}, & \operatorname{Re} \lambda < 0. \end{cases}$$

If $\operatorname{Re} \lambda < 0$, we can write $\lambda = |\lambda| \, \mathrm{e}^{\pm \mathrm{i} \theta}$ for some $\theta \in (\frac{\pi}{2}, \phi)$. Elementary properties of sine thus imply $|\operatorname{Im} \lambda| = |\lambda| \sin \theta > |\lambda| \sin \phi > 0$, and hence

$$\|R(\lambda, A)\| \le \frac{\frac{1}{\sin \phi}}{|\lambda|} =: \frac{K_\phi}{|\lambda|} \qquad \text{for all} \quad \lambda \in \Sigma_\phi. \qquad \square$$

By Example 1.45 we can apply the above result to $A = \Delta$ with $\mathrm{D}(A) = W^{2,2}(\mathbb{R}^m)$ in $L^2(\mathbb{R}^m)$, and by Example 1.54 to $A = \Delta_D$ in $L^2(G)$ for an open and bounded set $G \subseteq \mathbb{R}^m$ with $\partial G \in C^{1-}$. We next treat the second derivative in sup-norm with Dirichlet boundary conditions. Note that the operator A below is not densely defined and that $\mathrm{D}(A_0)$ requires more boundary conditions.

Example 2.20. Let $X = C([0,1])$, $A = \frac{\mathrm{d}^2}{\mathrm{d}s^2}$, and $\mathrm{D}(A) = \{u \in C^2([0,1]) \,|\, u(0) = u(1) = 0\}$. The closure of $\mathrm{D}(A)$ is $X_0 = C_0(0,1)$. We set $A_0 u = u''$ for $u \in$

[6] If (2.13) holds for $\phi = \pi$, one can deduce $A = 0$ from Theorem 2.25 and results in [**27**].

$\mathrm{D}(A_0) = \{u \in \mathrm{D}(A) \,|\, u'' \in X_0\}$. The operators A in X and A_0 in X_0 are sectorial of angle π. Moreover, A_0 is densely defined by Example 1.47.

Proof. The closure of $\mathrm{D}(A)$ can be determined as in Example 1.47. We only treat A, as A_0 is handled similarly. Let $f \in X$ and $\lambda \in \Sigma_\pi$. Note that $\lambda = \mu^2$ for $\mu = \sqrt{\lambda} \in \mathbb{C}_+$. As in Example 1.47 one can check that $\lambda \in \rho(A)$ and

$$R(\lambda, A)f(s) = a(f,\mu)\mathrm{e}^{\mu s} + b(f,\mu)\mathrm{e}^{-\mu s} + \frac{1}{2\mu}\int_0^1 \mathrm{e}^{-\mu|s-\tau|} f(\tau)\,\mathrm{d}\tau$$

for $s \in [0,1]$ and the coefficients

$$\begin{pmatrix} a(f,\mu) \\ b(f,\mu) \end{pmatrix} = \frac{1}{2\mu(\mathrm{e}^{-\mu} - \mathrm{e}^{\mu})} \begin{pmatrix} \mathrm{e}^{-\mu}\int_0^1 (\mathrm{e}^{\mu\tau} - \mathrm{e}^{-\mu\tau}) f(\tau)\,\mathrm{d}\tau \\ \int_0^1 (\mathrm{e}^{\mu}\mathrm{e}^{-\mu\tau} - \mathrm{e}^{-\mu}\mathrm{e}^{\mu\tau}) f(\tau)\,\mathrm{d}\tau \end{pmatrix}.$$

Fix $\phi \in (\frac{\pi}{2}, \pi)$. Take $\lambda \in \Sigma_\phi$ and hence $\mu \in \Sigma_{\phi/2}$. Set $\theta = \arg\mu$. It follows $0 \le |\theta| < \frac{\phi}{2}$ and $\operatorname{Re}\mu = |\mu|\cos\theta \ge |\mu|\cos\frac{\phi}{2}$. So we can estimate

$$\begin{aligned}
\|R(\lambda, A)f\|_\infty &\le |a(f,\mu)|\,\mathrm{e}^{\operatorname{Re}\mu} + |b(f,\mu)| + \frac{\|f\|_\infty}{2|\mu|} \sup_{s\in[0,1]} \int_{s-1}^{s} \mathrm{e}^{-\operatorname{Re}\mu\,|r|}\,\mathrm{d}r \\
&\le \frac{\|f\|_\infty}{2|\mu|\,(\mathrm{e}^{\operatorname{Re}\mu} - \mathrm{e}^{-\operatorname{Re}\mu})} \bigg(\int_0^1 \big(\mathrm{e}^{\operatorname{Re}\mu\tau} + \mathrm{e}^{-\operatorname{Re}\mu\tau}\big)\,\mathrm{d}\tau \\
&\qquad + \int_0^1 \big(\mathrm{e}^{\operatorname{Re}\mu}\mathrm{e}^{-\operatorname{Re}\mu\tau} + \mathrm{e}^{-\operatorname{Re}\mu}\mathrm{e}^{\operatorname{Re}\mu\tau}\big)\,\mathrm{d}\tau \bigg) + \frac{\|f\|_\infty}{|\mu|\operatorname{Re}\mu} \\
&= \frac{\|f\|_\infty}{2|\mu|\operatorname{Re}\mu\,(\mathrm{e}^{\operatorname{Re}\mu} - \mathrm{e}^{-\operatorname{Re}\mu})} \big((\mathrm{e}^{\operatorname{Re}\mu} - 1 + 1 - \mathrm{e}^{-\operatorname{Re}\mu}) \\
&\qquad + \mathrm{e}^{\operatorname{Re}\mu}(1 - \mathrm{e}^{-\operatorname{Re}\mu}) + \mathrm{e}^{-\operatorname{Re}\mu}(\mathrm{e}^{\operatorname{Re}\mu} - 1) \big) + \frac{\|f\|_\infty}{|\mu|\operatorname{Re}\mu} \\
&= \frac{2\|f\|_\infty}{|\mu|\operatorname{Re}\mu} \le \frac{\frac{2}{\cos(\phi/2)}}{|\lambda|}\,\|f\|_\infty.
\end{aligned}$$

□

For the first derivative we obtain a significantly smaller sectoriality angle.

Example 2.21. Let $X = C_0(\mathbb{R})$ and $Au = u'$ for $\mathrm{D}(A) = C_0^1(\mathbb{R})$. Then A is sectorial of angle $\frac{\pi}{2}$. (The analogous result for $X = L^p(\mathbb{R})$ is shown in Example 5.10 of [**27**].)

Proof. By Example 1.21, we have $\sigma(A) = \mathrm{i}\mathbb{R}$ and $\|R(\lambda, A)\| = 1/\operatorname{Re}\lambda$ for $\lambda \in \mathbb{C}_+$. Take $\phi \in (0, \pi/2)$. Let $\lambda \in \Sigma_\phi$. We obtain $\operatorname{Re}\lambda \ge |\lambda|\cos\phi$ and hence

$$\|R(\lambda, A)\| \le \frac{\frac{1}{\cos\phi}}{|\lambda|},$$

which shows sectoriality of angle greater or equal $\pi/2$. (The same argument works for every generator of a bounded C_0-semigroup.) Since $\mathrm{i}\mathbb{R} \subseteq \sigma(A)$ the angle cannot be greater that $\pi/2$. □

To study analytic semigroups, we need a bit of complex analysis in Banach spaces Y. (See also Section 5.1 of [**27**].) Let $J \subseteq \mathbb{R}$ be a closed interval and

$\gamma \in C(J,\mathbb{C})$ be piecewise C^1. If $J = [a,b]$ and $\gamma(a) = \gamma(b)$, the curve γ is called *closed*. Set $\Gamma = \gamma(J)$. For $f \in C(J,Y)$ we introduce the complex curve integral

$$\int_\gamma f \,\mathrm{d}z = \int_J f(\gamma(s))\gamma'(s)\,\mathrm{d}s.$$

If J is not compact, above it is assumed that the right-hand side exists as an improper Riemann integral in Y. As in the proof of Proposition 1.20, one obtains existence if the function $\|f\circ\gamma\|\,|\gamma'|$ is integrable on J. We also write[7]

$$\int_\gamma f\,|\mathrm{d}\lambda| = \int_J f(\gamma(s))|\gamma'(s)|\,\mathrm{d}s.$$

The curve integral possesses the usual properties.

Let $U \subseteq \mathbb{C}$ be open and starshaped, $f : U \to Y$ be (complex) differentiable, γ be closed, $\Gamma \subseteq U$, and $z \in U \setminus \Gamma$. We then have Cauchy's theorem and formula

$$\int_\gamma f(w)\,\mathrm{d}w = 0, \tag{2.14}$$

$$\mathrm{n}(\gamma,z)f(z) = \frac{1}{2\pi\mathrm{i}}\int_\gamma \frac{f(w)}{w-z}\,\mathrm{d}w, \qquad \text{where} \quad \mathrm{n}(\gamma,z) := \frac{1}{2\pi\mathrm{i}}\int_\gamma \frac{\mathrm{d}w}{w-z} \tag{2.15}$$

is the winding number. In fact, Theorems 2.6 and 2.8 in [**25**] show these equations with f replaced by $\langle f, y^*\rangle$ for each $y^* \in Y^*$. We hence obtain $\big\langle \int_\Gamma f\,\mathrm{d}z, y^*\big\rangle = 0$ for every $y^* \in Y^*$, implying (2.14) by Corollary 5.10 in [**24**]. Formula (2.15) is established similiarly. If $Y = \mathbb{C}$, identity (2.15) yields

$$\mathrm{e}^{ta} = \frac{1}{2\pi\mathrm{i}}\int_{\partial B(a,1)} \mathrm{e}^{\lambda t}(\lambda - a)^{-1}\,\mathrm{d}\lambda \qquad \text{for } a \in \mathbb{C},\ t > 0.$$

We want to imitate this formula for sectorial A. To this aim, we need a curve Γ encircling the (typically unbounded) spectrum of A counter clockwise. This curve has to be contained in Σ_ϕ for some $\phi < \varphi(A)$ in order to use the resolvent estimate (2.13), so that it has to be unbounded. Because of the exponential function, the real part of $\lambda \in \Gamma$ has to tend to $-\infty$ to guarantee the convergence of the integral. We thus assume that A is sectorial with angle φ larger than $\pi/2$. For given numbers $R > r > 0$ and $\theta \in (\pi/2, \varphi)$ we define the paths

$$\begin{aligned}
\Gamma^1 &= \Gamma^1(r,\theta) = \big\{\lambda = \gamma_1(s) = -s\mathrm{e}^{-\mathrm{i}\theta} \,\big|\, s \in (-\infty, -r]\big\},\\
\Gamma^2 &= \Gamma^2(r,\theta) = \big\{\lambda = \gamma_2(\alpha) = r\mathrm{e}^{\mathrm{i}\alpha} \,\big|\, \alpha \in [-\theta, \theta]\big\},\\
\Gamma^3 &= \Gamma^3(r,\theta) = \big\{\lambda = \gamma_3(s) = s\mathrm{e}^{\mathrm{i}\theta} \,\big|\, s \in [r, \infty)\big\},\\
\Gamma &= \Gamma(r,\theta) = \Gamma^1 \cup \Gamma^2 \cup \Gamma^3, \qquad \Gamma_R = \Gamma \cap \overline{B}(0,R).
\end{aligned} \tag{2.16}$$

We write $\int_\Gamma$ instead of $\int_\gamma$ since the maps γ_j are injective. We first show that the relevant integral exists in $\mathcal{B}(X)$.

Lemma 2.22. *Let A be sectorial of type (K,ϕ) with $\phi > \frac{\pi}{2}$, $t > 0$, $\theta_0 \in (\frac{\pi}{2}, \phi)$, $\theta \in [\theta_0, \phi)$, $r > 0$, and $\Gamma = \Gamma(r,\theta)$ be defined by (2.16). Then the integral*

$$\mathrm{e}^{tA} := \frac{1}{2\pi\mathrm{i}}\int_\Gamma \mathrm{e}^{t\lambda}R(\lambda,A)\,\mathrm{d}\lambda = \lim_{R\to\infty}\frac{1}{2\pi\mathrm{i}}\int_{\Gamma_R} \mathrm{e}^{t\lambda}R(\lambda,A)\,\mathrm{d}\lambda \tag{2.17}$$

[7] This notation was not used in the lectures.

converges absolutely in $\mathcal{B}(X)$*. The operator* $\mathrm{e}^{tA} \in \mathcal{B}(X)$ *does not depend on the choice of* $r > 0$ *and* $\theta \in (\frac{\pi}{2}, \phi)$*. We further have* $\|\mathrm{e}^{tA}\| \le M$ *for all* $t > 0$ *and a constant* $M = M(K, \theta_0) > 0$*.*

Proof. Since $\|R(\lambda, A)\| \le \frac{K}{|\lambda|}$ on Γ by (2.13), we can estimate

$$\begin{aligned}
\int_{\Gamma_R} \|\mathrm{e}^{t\lambda} R(\lambda, A)\| |\mathrm{d}\lambda| &\le K \int_r^R \frac{\exp(ts \operatorname{Re} \mathrm{e}^{-\mathrm{i}\theta})}{|s\mathrm{e}^{-\mathrm{i}\theta}|} |\mathrm{e}^{-\mathrm{i}\theta}| \mathrm{d}s + K \int_r^R \frac{\exp(ts \operatorname{Re} \mathrm{e}^{\mathrm{i}\theta})}{|s\mathrm{e}^{\mathrm{i}\theta}|} |\mathrm{e}^{\mathrm{i}\theta}| \mathrm{d}s \\
&\quad + K \int_{-\theta}^{\theta} \frac{\exp(tr \operatorname{Re} \mathrm{e}^{\mathrm{i}\alpha})}{|r\mathrm{e}^{\mathrm{i}\alpha}|} |\mathrm{i}r\mathrm{e}^{\mathrm{i}\alpha}| \, \mathrm{d}\alpha \\
&\le 2K \int_r^\infty \frac{\mathrm{e}^{ts \cos\theta}}{s} \, \mathrm{d}s + K \int_{-\theta}^{\theta} \mathrm{e}^{tr \cos\alpha} \, \mathrm{d}\alpha \\
&\le K \Big(2 \int_{rt\,|\cos\theta|}^\infty \frac{\mathrm{e}^{-\sigma}}{\sigma} (-t\cos\theta) \frac{\mathrm{d}\sigma}{-t\cos\theta} + 2\theta \mathrm{e}^{tr} \Big) \\
&\le K \Big(2 \int_{rt\,|\cos\theta_0|}^\infty \frac{\mathrm{e}^{-\sigma}}{\sigma} \, \mathrm{d}\sigma + 2\pi \mathrm{e}^{tr} \Big) =: K c(t, r, \theta_0)
\end{aligned}$$

for all $R > r$ and $t > 0$, where we substituted $\sigma = -st\cos\theta$ and used that $\cos\theta \le \cos\theta_0 < 0$. The limit in (2.17) thus exists absolutely in $\mathcal{B}(X)$ by the majorant criterium, and $\|\mathrm{e}^{tA}\| \le Kc(t, r, \theta_0)$. If we take $r = 1/t$, then $c(t, t^{-1}, \theta_0) =: c(\theta_0)$ does not depend on $t > 0$.

So it remains to check that the integral in (2.17) is independent of $r > 0$ and $\theta \in (\frac{\pi}{2}, \phi)$. To this aim, we define $\Gamma' = \Gamma(r', \theta')$ for some $r' > 0$ and $\theta' \in (\frac{\pi}{2}, \phi)$, where we may assume that $\theta' \ge \theta$. We further set $\Gamma'_R = \Gamma' \cap \overline{B}(0, R)$ and choose $R > \max\{r, r'\}$. Let C_R^+ and C_R^- be the circle arcs from the endpoints of Γ_R to that of Γ'_R in $\{\lambda \in \mathbb{C} \mid \operatorname{Im}\lambda > 0\}$ and $\{\lambda \in \mathbb{C} \mid \operatorname{Im}\lambda < 0\}$, respectively. (If $\theta = \theta'$, then $C_R^\pm$ contain just one point.) Then $S_R = \Gamma_R \cup C_R^+ \cup (-\Gamma'_R) \cup (-C_R^-)$ is a closed curve in the starshaped domain Σ_ϕ. So (2.14) shows that

$$\int_{S_R} \mathrm{e}^{t\lambda} R(\lambda, A) \, \mathrm{d}\lambda = 0.$$

We further estimate

$$\Big\| \int_{C_R^+} \mathrm{e}^{t\lambda} R(\lambda, A) \, \mathrm{d}\lambda \Big\| \le \int_\theta^{\theta'} \mathrm{e}^{tR \operatorname{Re} \mathrm{e}^{\mathrm{i}\alpha}} \frac{K}{|R\mathrm{e}^{\mathrm{i}\alpha}|} |\mathrm{i}R\mathrm{e}^{\mathrm{i}\alpha}| \, \mathrm{d}\alpha \le K(\theta' - \theta) \mathrm{e}^{tR\cos\theta} \to 0,$$

as $R \to \infty$ since $\cos\alpha \le \cos\theta < 0$, and analogously for C_R^-. So we conclude

$$\begin{aligned}
\int_\Gamma \mathrm{e}^{t\lambda} R(\lambda, A) \, \mathrm{d}\lambda &= \lim_{R\to\infty} \int_{\Gamma_R} \mathrm{e}^{t\lambda} R(\lambda, A) \, \mathrm{d}\lambda = \lim_{R\to\infty} \int_{\Gamma'_R} \mathrm{e}^{t\lambda} R(\lambda, A) \, \mathrm{d}\lambda \\
&= \int_{\Gamma'} \mathrm{e}^{t\lambda} R(\lambda, A) \, \mathrm{d}\lambda. \qquad \square
\end{aligned}$$

We next establish some of the fundamental properties of the operators e^{tA}. We stress that we do not assume that A is densely defined here. In part c) one sees the impact of a dense domain. A typical example for a sectorial operator with non-dense domain is the Dirichlet–Laplacian in supremum norm, unless one includes the Dirichlet boundary condition into the state space X. See Example 2.20 and also Chapter 3 of [**18**].

THEOREM 2.23. *Let A be sectorial of angle $\varphi > \frac{\pi}{2}$. Define e^{tA} as in (2.17) for $t > 0$, and set $\mathrm{e}^{0A} = I$. Then the following assertions hold.*

a) $\mathrm{e}^{tA}\mathrm{e}^{\tau A} = \mathrm{e}^{\tau A}\mathrm{e}^{tA} = \mathrm{e}^{(t+\tau)A}$ for all $t, \tau \geq 0$.

b) The map $t \mapsto \mathrm{e}^{tA}$ belongs to $C^1(\mathbb{R}_+, \mathcal{B}(X))$. Moreover, $\mathrm{e}^{tA}X \subseteq \mathrm{D}(A)$, $\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{tA} = A\mathrm{e}^{tA}$ and $\|A\mathrm{e}^{tA}\| \leq C/t$ for a constant $C > 0$ and all $t > 0$. We also have $A\mathrm{e}^{tA}x = \mathrm{e}^{tA}Ax$ for all $x \in \mathrm{D}(A)$ and $t \geq 0$.

c) Let $x \in X$. Then $\mathrm{e}^{tA}x$ converges as $t \to 0$ in X if and only if x is contained in $\overline{\mathrm{D}(A)}$. In this case, $\mathrm{e}^{tA}x$ tends to x as $t \to 0$.

d) Let $\mathrm{D}(A)$ be dense. Then $(\mathrm{e}^{tA})_{\geq 0}$ is a C_0-semigroup with generator A.

PROOF. Let $r > 0$ and $\frac{\pi}{2} < \theta < \phi < \varphi$ and set $\Gamma = \Gamma(r, \theta)$ as in (2.16).

a) We proceed as for the holomorphic functional calculus in Theorem 5.1 of [**27**]. Let $t, \tau > 0$. We use that e^{tA} does not depend on the choice of r and θ by Lemma 2.22. Take $r < r'$ and $\frac{\pi}{2} < \theta' < \theta$. Set $\Gamma' = \Gamma(r', \theta')$. Employing the resolvent equation (1.7) and Fubini's theorem, we compute

$$\begin{aligned}\mathrm{e}^{tA}\mathrm{e}^{\tau A} &= \frac{1}{(2\pi\mathrm{i})^2}\int_\Gamma \mathrm{e}^{t\lambda}\int_{\Gamma'} \mathrm{e}^{\tau\mu} R(\lambda, A)R(\mu, A)\,\mathrm{d}\mu\,\mathrm{d}\lambda \\ &= \frac{1}{2\pi\mathrm{i}}\int_\Gamma \mathrm{e}^{t\lambda}R(\lambda, A)\frac{1}{2\pi\mathrm{i}}\int_{\Gamma'}\frac{\mathrm{e}^{\tau\mu}}{\mu-\lambda}\,\mathrm{d}\mu\,\mathrm{d}\lambda \\ &\quad + \frac{1}{2\pi\mathrm{i}}\int_{\Gamma'}\mathrm{e}^{\tau\mu}R(\mu, A)\frac{1}{2\pi\mathrm{i}}\int_\Gamma \frac{\mathrm{e}^{t\lambda}}{\lambda-\mu}\,\mathrm{d}\lambda\,\mathrm{d}\mu.\end{aligned}$$

One shows Fubini in this context by inserting the parametrizations and applying a functional $\Phi \in \mathcal{B}(X)^*$. The integrability in the parameters (s, s') etc. is checked as in Lemma 2.22 or below.

Fix $\lambda \in \Gamma$ and take $R' > \max\{r, r', |\lambda|\}$. We set $C'_{R'} = \{z = R'\mathrm{e}^{\mathrm{i}\alpha} \,|\, \alpha \in [\theta', 2\pi - \theta']\}$ and $S'_{R'} = \Gamma'_{R'} \cup C'_{R'}$. Cauchy's formula (2.15) yields

$$\frac{1}{2\pi\mathrm{i}}\int_{S'_{R'}}\frac{\mathrm{e}^{\tau\mu}}{\mu-\lambda}\,\mathrm{d}\mu = \mathrm{e}^{\tau\lambda}$$

since $\mathrm{n}(S'_{R'}, \lambda) = 1$. As in Lemma 2.22, we further compute

$$\begin{aligned}&\int_{\Gamma'_{R'}}\frac{\mathrm{e}^{\tau\mu}}{\mu-\lambda}\,\mathrm{d}\mu \longrightarrow \int_{\Gamma'}\frac{\mathrm{e}^{\tau\mu}}{\mu-\lambda}\,\mathrm{d}\mu \qquad \text{and} \\ &\Big|\int_{C'_{R'}}\frac{\mathrm{e}^{\tau\mu}}{\mu-\lambda}\,\mathrm{d}\mu\Big| \leq 2\pi R' \sup_{\mu\in C'_{R'}}\frac{\mathrm{e}^{\tau\,\mathrm{Re}\,\mu}}{|\mu-\lambda|} \leq \mathrm{e}^{\tau R'\cos\theta'}\frac{2\pi R'}{R'-|\lambda|} \longrightarrow 0\end{aligned}$$

as $R' \to \infty$, using that $\cos\alpha \leq \cos\theta' < 0$. It follows

$$\mathrm{e}^{\tau\lambda} = \frac{1}{2\pi\mathrm{i}}\int_{\Gamma'}\frac{\mathrm{e}^{\tau\mu}}{\mu-\lambda}\,\mathrm{d}\mu.$$

Next, fix $\mu \in \Gamma'$ and take $R > r$. Set $C_R = \{z = R\mathrm{e}^{\mathrm{i}\alpha} \,|\, \alpha \in [\theta, 2\pi - \theta]\}$ and $S_R = \Gamma_R \cup C_R$. We now have $\mathrm{n}(S_R, \mu) = 0$ and derive as above

$$\int_\Gamma \frac{\mathrm{e}^{t\lambda}}{\lambda - \mu}\,\mathrm{d}\lambda = 0.$$

The above equalities imply that

$$\mathrm{e}^{tA}\mathrm{e}^{\tau A} = \frac{1}{2\pi\mathrm{i}}\int_\Gamma \mathrm{e}^{t\lambda}\mathrm{e}^{\tau\lambda}R(\lambda, A)\,\mathrm{d}\lambda = \mathrm{e}^{(t+\tau)A} = \mathrm{e}^{(\tau+t)A} = \mathrm{e}^{\tau A}\mathrm{e}^{tA}.$$

b) Let $t > 0$ and $R > r$. Since $\lambda \mapsto R(\lambda, A)$ is continuous in $\mathcal{B}(X, [\mathrm{D}(A)])$, also the integral

$$T_R(t) = \int_{\Gamma_R} \mathrm{e}^{t\lambda}R(\lambda, A)\,\mathrm{d}\lambda$$

belongs to $\mathcal{B}(X, [\mathrm{D}(A)])$. Recall from (2.17) that $T_R(t)$ tends to $2\pi\mathrm{i}\mathrm{e}^{tA}$ in $\mathcal{B}(X)$ as $R \to \infty$. We further compute

$$AT_R(t) = \int_{\Gamma_R} \mathrm{e}^{t\lambda}AR(\lambda, A)\,\mathrm{d}\lambda \; = \int_{\Gamma_R} \mathrm{e}^{t\lambda}\lambda R(\lambda, A)\,\mathrm{d}\lambda - \int_{\Gamma_R} \mathrm{e}^{t\lambda}\,\mathrm{d}\lambda\, I.$$

Take the circle arc C_R from step a). Using Cauchy's theorem (2.14), one shows as above that

$$\Big|\int_{\Gamma_R} \mathrm{e}^{t\lambda}\mathrm{d}\lambda\Big| = \Big|-\int_{C_R} \mathrm{e}^{t\lambda}\mathrm{d}\lambda\Big| \le 2\pi R \sup_{\alpha\in[\theta,2\pi-\theta]} \mathrm{e}^{tR\cos\alpha} \le 2\pi R\mathrm{e}^{tR\cos\theta} \;\to\; 0, \quad R \to \infty.$$

Following the proof of Lemma 2.22 (with $r = 1/t$ and $\|\lambda R(\lambda, A)\| \le K$ by (2.13)), we then estimate

$$\begin{aligned}\int_{\Gamma_R} \|\lambda\mathrm{e}^{t\lambda}R(\lambda, A)\|\,|\mathrm{d}\lambda| &\le 2K\int_r^\infty \mathrm{e}^{ts\cos\theta}\,\mathrm{d}s + K\int_{-\theta}^{\theta} r\mathrm{e}^{\cos\alpha}\,\mathrm{d}\alpha\\ &\le \frac{2K}{t|\cos\theta|} + \frac{2\theta\mathrm{e}K}{t} =: \frac{C'}{t}\,.\end{aligned}$$

Hence, $AT_R(t)$ converges to $\int_\Gamma \lambda\mathrm{e}^{t\lambda}R(\lambda, A)\,\mathrm{d}\lambda$ in $\mathcal{B}(X)$ as $R \to \infty$. Since A is closed, the operator e^{tA} maps X into $\mathrm{D}(A)$ with

$$A\mathrm{e}^{tA} = \frac{1}{2\pi\mathrm{i}}\int_\Gamma \lambda\mathrm{e}^{t\lambda}R(\lambda, A)\,\mathrm{d}\lambda, \tag{2.18}$$

and it satisfies $\|A\mathrm{e}^{tA}\| \le \frac{C'}{2\pi t}$ for all $t > 0$.

Observe that $T_R(\cdot)$ belongs to $C^1(\mathbb{R}_{\ge 0}, \mathcal{B}(X, [\mathrm{D}(A)]))$ with derivative

$$\tfrac{\mathrm{d}}{\mathrm{d}t}T_R(t) = \int_{\Gamma_R} \lambda\mathrm{e}^{t\lambda}R(\lambda, A)\,\mathrm{d}\lambda$$

for $t \ge 0$. Let $\varepsilon > 0$ and $t \ge \varepsilon$. In a similar way as above, one sees that

$$\Big\|\int_{\Gamma\setminus\Gamma_R} \lambda\mathrm{e}^{t\lambda}R(\lambda, A)\,\mathrm{d}\lambda\Big\| \le 2K\int_R^\infty \mathrm{e}^{ts\cos\theta}\,\mathrm{d}s \le \frac{2K}{\varepsilon|\cos\theta|}\mathrm{e}^{R\varepsilon\cos\theta} \;\longrightarrow\; 0, \quad R \to \infty.$$

Therefore $\frac{\mathrm{d}}{\mathrm{d}t}T_R(t)$ tends to $A\mathrm{e}^{tA}$ in $\mathcal{B}(X)$ uniformly for $t \geq \varepsilon$, see (2.18). Using also Remark 1.15 g), we infer that $t \mapsto \mathrm{e}^{tA} \in \mathcal{B}(X)$ is continuously differentiable for $t > 0$ with $\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{tA} = A\mathrm{e}^{tA}$. For $x \in \mathrm{D}(A)$, we further obtain

$$A\mathrm{e}^{tA}x = \lim_{R\to\infty} \frac{1}{2\pi\mathrm{i}} \int_{\Gamma_R} \mathrm{e}^{t\lambda} R(\lambda, A)Ax \,\mathrm{d}\lambda = \mathrm{e}^{tA}Ax.$$

c) Let $x \in \mathrm{D}(A)$, $R > r$, and $t > 0$. Take the curve C_R from part a). As in step a), Cauchy's formula (2.15) yields

$$\frac{1}{2\pi\mathrm{i}} \int_\Gamma \frac{\mathrm{e}^{t\lambda}}{\lambda} \,\mathrm{d}\lambda = \lim_{R\to\infty} \frac{1}{2\pi\mathrm{i}} \int_{\Gamma_R \cup C_R} \frac{\mathrm{e}^{t\lambda}}{\lambda - 0} \,\mathrm{d}\lambda = 1$$

Noting that $\lambda R(\lambda, A)x - x = R(\lambda, A)Ax$, we conclude

$$\mathrm{e}^{tA}x - x = \frac{1}{2\pi\mathrm{i}} \int_\Gamma \mathrm{e}^{t\lambda} \Big(R(\lambda, A) - \frac{1}{\lambda}\Big) x \,\mathrm{d}\lambda = \frac{1}{2\pi\mathrm{i}} \int_\Gamma \frac{\mathrm{e}^{t\lambda}}{\lambda} R(\lambda, A)Ax \,\mathrm{d}\lambda.$$

Due to (2.13) the right integrand is bounded by $c|\lambda|^{-2}\|Ax\|$ on Γ for all $t \in (0, 1]$. Lebesgue's convergence theorem then implies the existence of the limit

$$\lim_{t\to 0} \mathrm{e}^{tA}x - x = \frac{1}{2\pi\mathrm{i}} \int_\Gamma \frac{1}{\lambda} R(\lambda, A)Ax \,\mathrm{d}\lambda =: z.$$

Let $K_R = \{R\mathrm{e}^{\mathrm{i}\alpha} \mid -\theta \leq \alpha \leq \theta\}$. Cauchy's theorem (2.14) shows that

$$\int_{\Gamma_R \cup (-K_R)} \frac{1}{\lambda} R(\lambda, A)Ax \,\mathrm{d}\lambda = 0.$$

Since also

$$\Big\| \int_{K_R} \frac{1}{\lambda} R(\lambda, A)Ax \,\mathrm{d}\lambda \Big\| \leq \frac{2\pi RK}{R^2} \|Ax\| \longrightarrow 0$$

as $R \to \infty$, we arrive at $z = 0$. Because of the uniform boundedness of e^{tA}, it follows that $\mathrm{e}^{tA}x \to x$ as $t \to 0$ for all $x \in \overline{\mathrm{D}(A)}$.

Conversely, let $\mathrm{e}^{tA}x \to y$ as $t \to 0$. Then y is contained in $\overline{\mathrm{D}(A)}$ by part b). The previous step and the last claim in b) imply that $R(1, A)\mathrm{e}^{tA}x = \mathrm{e}^{tA}R(1, A)x$ tends to $R(1, A)x \in \underline{\mathrm{D}(A)}$ as $t \to 0$. We thus obtain $R(1, A)y = R(1, A)x$, and so $x = y$ belongs to $\overline{\mathrm{D}(A)}$.

d) Let $\mathrm{D}(A)$ be dense. The above results then show that $(\mathrm{e}^{tA})_{\geq 0}$ is a bounded C_0-semigroup. Let B be its generator. To check that $A = B$, take $x \in \mathrm{D}(A)$. For $t > s > 0$, assertion b) leads to

$$\mathrm{e}^{tA}x - \mathrm{e}^{sA}x = \int_s^t \mathrm{e}^{\tau A}Ax \,\mathrm{d}\tau.$$

Since the semigroup is strongly continuous, we can let $s \to 0$ resulting in

$$\frac{1}{t}\big(\mathrm{e}^{tA}x - x\big) = \frac{1}{t} \int_0^t \mathrm{e}^{\tau A}Ax \,\mathrm{d}\tau.$$

The right-hand side tends to Ax as $t \to 0$ by strong continuity; i.e., $A \subseteq B$. As the spectra of A and B are contained in $\overline{\mathbb{C}_-}$, Lemma 1.23 yields $A = B$. □

Below we prove a converse to the above theorem and study further regularity properties of e^{tA}, assuming that $\mathrm{D}(A)$ is dense for simplicity. There are variants of the following results without the density of the domain, see Section 2.1 of [18]. We first introduce a basic concept.

Definition 2.24. *Let $\zeta \in (0, \pi/2]$. An* analytic C_0-semigroup *on Σ_ζ is a family of operators $\{T(z) \,|\, z \in \Sigma_\zeta \cup \{0\}\}$ such that*

(a) $T(0) = I$ and $T(w)T(z) = T(w+z)$ for all $z, w \in \Sigma_\zeta$;
(b) the map $T : \Sigma_\zeta \to \mathcal{B}(X)$; $z \mapsto T(z)$, is (complex) differentiable;
(c) $T(z)x \to x$ in X as $z \to 0$ in $\Sigma_{\zeta'}$ for all $x \in X$ and each $\zeta' \in (0, \zeta)$.

The generator *of $T(\cdot)$ is defined as the generator of the C_0-semigroup $(T(t))_{t\ge 0}$, and the* angle *$\psi \in (0, \pi/2]$ of $T(\cdot)$ is the supremum of possible ζ. If $\|T(z)\|$ is bounded for $z \in \Sigma_{\zeta'}$ and each $\zeta' \in (0, \zeta)$, then $T(\cdot)$ is called* bounded.

We now establish the fundamental characterization theorem of bounded analytic C_0-semigroups which goes back to *Hille* in 1948. Basically it says that a densely defined operator A generates such a semigroup if and only if A is sectorial of angle greater than $\pi/2$. Moreover, it gives two useful characterizations of sectoriality and describes the class of bounded analytic C_0-semigroups in a different, very convenient way. For $n \in \mathbb{N}$ with $n \ge 2$ we inductively define the powers of a linear operator A by

$$\mathrm{D}(A^n) = \big\{x \in \mathrm{D}(A^{n-1}) \,\big|\, A^{n-1}x \in \mathrm{D}(A)\big\} \quad \text{and} \quad A^n x = A(A^{n-1}x).$$

Theorem 2.25. *Let A be a closed linear operator on X. Then the following assertions are equivalent.*

a) A is densely defined and sectorial of angle $\varphi > \pi/2$.

b) A is densely defined, $\mathbb{C}_+ \subseteq \rho(A)$, and there is a constant $C > 0$ such that $\|R(\lambda, A)\| \le C/|\lambda|$ for all $\lambda \in \mathbb{C}_+$.

c) For some $\vartheta \in (0, \pi/2)$, the maps $\mathrm{e}^{\pm \mathrm{i}\vartheta}A$ generate bounded C_0-semigroups.

d) A generates a bounded C_0-semigroup $(T(t))_{t\ge 0}$ such that $T(t)X \subseteq \mathrm{D}(A)$ and $\|AT(t)\| \le M_1/t$ for all $t > 0$ and a constant $M_1 > 0$.

e) A generates a bounded analytic C_0-semigroup with angle $\psi \in (0, \pi/2]$.

If this is the case, $T(t)$ is given by (2.17), and we have $T(t)X \subseteq \mathrm{D}(A^n)$, $\|A^nT(t)\| \le M_n t^{-n}$, $T(\cdot) \in C^\infty(\mathbb{R}_+, \mathcal{B}(X))$, and $\frac{\mathrm{d}^n}{\mathrm{d}t^n}T(t) = A^nT(t)$ for all $t > 0$, $n \in \mathbb{N}$, and some constants $M_n > 0$.

Proof. We prove the chain of implications *e) ⇒ c) ⇒ b) ⇒ a) ⇒ d) ⇒ e)* going from analyticity to sectoriality and back via claim d) using Theorem 2.23.

1) Let part e) be true. Take $\vartheta \in (0, \psi)$. The maps $T(\mathrm{e}^{\pm \mathrm{i}\vartheta}t)$ for $t \ge 0$ then yield two bounded C_0-semigroups $T_{\pm\vartheta}$ with generators $A_{\pm\vartheta}$. Using Proposition 1.20, the path $S = \{z = s\mathrm{e}^{\mathrm{i}\vartheta} | s \ge 0\}$ and Cauchy's theorem (2.14), we obtain

$$\begin{aligned} R(1, A)x &= \int_0^\infty \mathrm{e}^{-t}T(t)x \,\mathrm{d}t = \int_S \mathrm{e}^{-z}T(z)x \,\mathrm{d}z = \mathrm{e}^{\mathrm{i}\vartheta} \int_0^\infty \mathrm{e}^{-s\mathrm{e}^{\mathrm{i}\vartheta}} T_\vartheta(s)x \,\mathrm{d}s \\ &= \mathrm{e}^{\mathrm{i}\vartheta} R(\mathrm{e}^{\mathrm{i}\vartheta}, A_\vartheta)x = R(1, \mathrm{e}^{-\mathrm{i}\vartheta}A_\vartheta)x \end{aligned}$$

for $x \in X$, and hence $A_\vartheta = \mathrm{e}^{\mathrm{i}\vartheta}A$ with domain $\mathrm{D}(A)$. Similarly, one treats $-\vartheta$.

2) We assume property c) and that the semigroups generated by $e^{\pm i\vartheta}A$ are bounded by M. Proposition 1.19 shows the density of $D(A)$. Because of Proposition 1.20 in [27] and Proposition 1.20, condition c) first yields that $\rho(A) = e^{\mp i\vartheta}\rho(e^{\pm i\vartheta}A) \supseteq e^{\mp i\vartheta}\mathbb{C}_+$ and hence $\rho(A) \supseteq \Sigma_{\frac{\pi}{2}+\vartheta} \supseteq \mathbb{C}_+$. To check the resolvent estimate in b), we write $\lambda = re^{i\alpha}$ with $r > 0$ and $0 \le \alpha < \frac{\pi}{2}$. Set $m = \min\{\sin\vartheta, \cos\vartheta\}$. It follows

$$\|R(\lambda, A)\| = \|e^{-i\vartheta}R(e^{-i\vartheta}\lambda, e^{-i\vartheta}A)\| \le \frac{M}{r \operatorname{Re} e^{i(\alpha-\vartheta)}} \le \frac{M/m}{|\lambda|}$$

since $\cos(\alpha-\vartheta) \ge \cos(\frac{\pi}{2}-\vartheta) = \sin\vartheta$ if $\alpha \ge \vartheta$ and $\cos(\alpha-\vartheta) \ge \cos\vartheta$ otherwise. The case $\alpha \in (-\pi/2, 0)$ can similarly be treated using $e^{i\vartheta}A$. Hence, b) is valid.

3) Let assertion b) be true. If a point is with $s \in \mathbb{R} \setminus \{0\}$ was contained in $\sigma(A)$, then $\|R(is + r, A)\|$ would explode as $r \to 0^+$ by (1.8), contradicting the assumption. This means that $i\mathbb{R} \setminus \{0\} \subseteq \rho(A)$, and we infer the bound $\|R(is, A)\| \le C/|s|$ for $s \in \mathbb{R} \setminus \{0\}$ by continuity. Take $q \in (0, 1)$ and $\lambda = r + is$ with $s \ne 0$ and $|r| < q|s|/C$. Set $\theta = \arctan(q/C) > 0$ and note that $\lambda \in \Sigma_{\frac{\pi}{2}+\theta}$. Remark 1.16 c) then shows that λ belongs to $\rho(A)$ and the inequality

$$\|R(\lambda, A)\| \le \frac{C/(1-q)}{|s|} \le \frac{\frac{C}{(1-q)\cos\theta}}{|\lambda|}.$$

Combined with condition b), we obtain the sectoriality of A with angle $\varphi > \pi/2$.

4) The implication 'a) $\Rightarrow$ d)' was proven in Theorem 2.23 together with $T(\cdot) \in C^1(\mathbb{R}_+, \mathcal{B}(X))$ and $\frac{d}{dt}T(t) = AT(t)$ for $t > 0$, where $T(t)$ is given by (2.17).

5) Let statement d) be valid. Let $t > 0$ and $n \in \mathbb{N}$. Since $AT(t) = T(t - t/n)AT(t/n)$, we obtain that $T(t)X \subseteq D(A^2)$, and then $T(t)X \subseteq D(A^n)$ and $A^nT(t) = (AT(\frac{t}{n}))^n$ by iteration. Condition d) now implies the bound $\|A^nT(t)\| \le (M_1 n)^n t^{-n}$.

Observe that $e^n = \sum_{k=0}^{\infty} \frac{n^k}{k!} \ge \frac{n^n}{n!}$. Let $q \in (0, 1)$. We take numbers $z \in \mathbb{C}_+$ with $|z - t| < \frac{qt}{eM_1}$ for some $t > 0$. The power series

$$T(z) = \sum_{n=0}^{\infty} \frac{(z-t)^n}{n!} A^nT(t)$$

at t converges absolutely in $\mathcal{B}(X)$ and uniformly for the above z because of

$$\sum_{n=0}^{\infty} \frac{|z-t|^n}{n!} \frac{M_1^n n^n}{t^n} \le \sum_{n=0}^{\infty} \Big(\frac{qt}{eM_1}\Big)^n \frac{M_1^n e^n}{t^n} = \frac{1}{1-q}.$$

This works for $z \in \Sigma_{\zeta(q)}$ with $\zeta(q) := \arctan \frac{q}{eM_1}$ since taking $t = \operatorname{Re} z$ for such z one obtains $|\operatorname{Im} z| = |z - t| < \frac{qt}{eM_1} = \operatorname{Re}(z) \tan\zeta(q)$. So we have extended $T(\cdot)$ to a differentiable map $T : \Sigma_{\zeta(q)} \to \mathcal{B}(X)$ bounded by $1/(1-q)$.

Let $\Phi \in \mathcal{B}(X)^*$. For fixed $t \ge 0$, we note that the holomorphic functions $z \mapsto \langle T(z)T(t), \Phi\rangle$ and $z \mapsto \langle T(z+t), \Phi\rangle$ coincide for $z \in \mathbb{R}_+$. Consequently, they are the same for all $z \in \Sigma_{\zeta(q)}$ thanks to the Identity Theorem 2.21 in [25]. The Hahn–Banach theorem now yields that $T(z)T(t) = T(z+t)$ for all $z \in \Sigma_{\zeta(q)}$. In the same way one can replace here $t > 0$ by any $w \in \Sigma_{\zeta(q)}$.

It remains to check the strong continuity as $z \to 0$. Let $z \in \Sigma_{\zeta(q)}$, $x \in X$, and $\varepsilon > 0$. We fix $h > 0$ such that $\|T(h)x - x\| < \varepsilon$. Using the boundedness and the continuity of $T(\cdot)$ on $\Sigma_{\zeta(q)}$, we estimate

$$\begin{aligned}\|T(z)x - x\| &\leq \|T(z)\| \, \|x - T(h)x\| + \|T(z+h)x - T(h)x\| + \|T(h)x - x\| \\ &\leq \tfrac{\varepsilon}{1-q} + \|T(z+h) - T(h)\| \, \|x\| + \varepsilon,\end{aligned}$$

$$\varlimsup_{z\to 0} \|T(z)x - x\| \leq \big(1 + \tfrac{1}{1-q}\big)\varepsilon.$$

As a result, $T(z)x \to x$ as $z \to 0$ in $\Sigma_{\zeta(q)}$ and claim e) is proved with $\psi \geq \zeta(1)$.

6) The first three assertions in the addenddum were shown in steps 4) and 5). In step 4) we have also seen that $T(\cdot) \in C^1(\mathbb{R}_+, \mathcal{B}(X))$ and $\frac{\mathrm{d}}{\mathrm{d}t}T(t) = AT(t)$ for $t > 0$. Writing $A^{n-1}T(t) = T(t-\delta)A^{n-1}T(\delta)$ for some $\delta \in (0,t)$ and $n \in \mathbb{N}$, an induction yields that $T(\cdot)$ belongs to $C^n(\mathbb{R}_+, \mathcal{B}(X))$ with $\frac{\mathrm{d}^n}{\mathrm{d}t^n}T(t) = A^nT(t)$. □

We collect additional information concerning the above theorem.

Remark 2.26. a) Let $\omega \in \mathbb{R}$ and A be closed. By rescaling one sees that A generates an analytic C_0-semigroup $(T(z))_{z\in\Sigma_\zeta\cup\{0\}}$ for some $\zeta > 0$ such that $\mathrm{e}^{-\omega z}T(z)$ is bounded on all smaller sectors if and only if A is densely defined and $A - \omega I$ is sectorial of angle greater than $\pi/2$, cf. Section 2.1 in [**18**].

b) Let A be sectorial of angle $\varphi > \frac{\pi}{2}$ and $z \in \Sigma_{\varphi-\pi/2}$. Taking $\theta \in (\frac{\pi}{2} + |\arg z|, \varphi)$ for Γ, in (2.17) one can replace $t > 0$ by z obtaining an analytic C_0-semigroup on $\Sigma_{\varphi-\pi/2}$. See Proposition II.4.3 of [**7**] or Proposition 2.1.1 of [**18**]. This means that in Theorem 2.25 the angle ψ of the semigroup is at least $\varphi - \frac{\pi}{2}$. On the other hand, a variant of step 2) in the above proof shows that $\varphi \geq \psi + \frac{\pi}{2}$. We thus derive the equality $\varphi = \psi + \frac{\pi}{2}$ for the angles.
In a similar way, one can check that ψ is the supremum of all ϑ for which statement c) of Theorem 2.25 is true.

c) In view of property a) or d) in Theorem 2.25, the translations $T(t)f = f(\cdot + t)$ cannot be extended to an analytic C_0-semigroup on $C_0(\mathbb{R})$, cf. Example 2.21. The same is true for every C_0-group $T(t)$ with an unbounded generator since $T(t) : X \to X$ then is a bijection. ◊

In the next result we combine Theorem 2.25 c) with the Lumer–Phillips Theorem 1.39 to obtain a very convenient sufficient condition for the generation of a bounded analytic C_0-semigroup. In this case it is actually contractive on a sector. We note that the corresponding angle can be smaller than the angle ψ of analyticity, and that there exist bounded analytic C_0-semigroups which are contractive only on $\mathbb{R}_+$ or not even there. (There are examples on $X = \mathbb{C}^2$.)

Corollary 2.27. *Let A be densely defined and dissipative. Assume that there are numbers $\lambda_0 > 0$ such that $\lambda_0 I - A$ is surjective and $\vartheta \in (0, \pi/2)$ such that also the operators $\mathrm{e}^{\pm\mathrm{i}\vartheta}A$ are dissipative. Then A generates a bounded analytic C_0-semigroup $T(\cdot)$ of angle $\psi \geq \vartheta$ with $\|T(z)\| \leq 1$ for $z \in \Sigma_\vartheta$.*

Proof. Theorem 1.39 implies that A is closed and $\mathbb{C}_+ \subseteq \rho(A)$. The operators $I - \mathrm{e}^{\pm\mathrm{i}\vartheta}A = \mathrm{e}^{\pm\mathrm{i}\vartheta}(\mathrm{e}^{\mp\mathrm{i}\vartheta}I - A)$ are thus surjective, and hence $\mathrm{e}^{\pm\mathrm{i}\vartheta}A$ generate contraction semigroups again by Theorem 1.39. The first assertion now follows from Theorem 2.25 and Remark 2.26.

To show contractivity, we take $x \in \mathrm{D}(A)$ and $x^* \in J(x)$. Set $\xi = \langle Ax, x^*\rangle$. By dissipativity, the numbers $\mathrm{e}^{\pm \mathrm{i}\vartheta}\xi$ belong to $\overline{\mathbb{C}_-}$ and hence ξ to $\mathbb{C}\setminus\Sigma_{\vartheta+\pi/2}$. It follows that $\mathrm{Re}(\mathrm{e}^{\pm\mathrm{i}\alpha}\xi) \le 0$ for all $\alpha \in [0,\vartheta)$, and so also the operators $\mathrm{e}^{\pm\mathrm{i}\alpha}A$ are dissipative. As above we see that they generate contraction semigroups, which are given by $T(\mathrm{e}^{\pm\mathrm{i}\alpha}t)$ for $t \ge 0$ due to step 1) in the proof of Theorem 2.25. □

We first apply Corollary 2.27 in a mildly improved version of Example 2.19.

Corollary 2.28. *Let X be a Hilbert space and A be densely defined, self-adjoint and dissipative; i.e., $(Ax|x) \le 0$ for $x \in \mathrm{D}(A)$ by Lemma 4.6 of* [**27**] *and Example 1.30.*[8] *Then $\sigma(A) \subseteq \mathbb{R}_{\le 0}$ and A generates a contractive analytic C_0-semigroup of angle $\frac{\pi}{2}$.*

Proof. Let $\lambda > 0$ and $x \in \mathrm{D}(A)$. Dissipativity and Proposition 1.32 yield the lower bound $\|\lambda x - Ax\| \ge \lambda\|x\|$. Since A is self-adjoint, this bound shows that λ is an element of $\rho(A)$ due to Theorem 4.7 d) of [**27**]. The spectrum of $A = A'$ is real by the same theorem, and hence $\sigma(A)$ is contained in $\mathbb{R}_{\le 0}$.

For $\vartheta \in (0,\pi/2)$ the number $(\mathrm{e}^{\pm\mathrm{i}\vartheta}Ax|x)$ belongs to $\overline{\mathbb{C}_-}$ as $(Ax|x) \le 0$. The operators $\mathrm{e}^{\pm\mathrm{i}\vartheta}A$ are thus dissipative, too. Taking the supremum over $\vartheta < \pi/2$, the second assertion follows from Corollary 2.27. □

We next discuss prototypical generators of (contractive) analytic C_0-semigroups starting with operators A on $L^2(G)$ given by strictly accretive forms as in Theorem 1.51. The corollary can directly applied to Example 1.53, i.e., to elliptic operators in divergence-form with Dirichlet boundary conditions. (Neumann boundary conditions can be treated in a similar way.)

Corollary 2.29. *Let X and Y be Hilbert spaces with an embedding $J_Y : Y \hookrightarrow X$ having dense range and norm κ. Assume that $\underline{a} : Y \times Y \to \mathbb{C}$ is sesquilinear, bounded, and strictly accretive with constants $C, \eta > 0$ as in Theorem 1.50. Define A by* (1.31) *and* (1.32)*. Then A generates an analytic C_0-semigroup which is contractive on the sector Σ_ϑ for $\vartheta := \arctan(\eta/C) \in (0, \frac{\pi}{2})$.*

Proof. Theorem 1.51 says that A generates an ω-contraction semigroup on X with $\omega = -\eta\kappa^{-2}$. By Corollary 2.27 and Theorem 1.39, it remains to check the dissipativity of $\mathrm{e}^{\pm\mathrm{i}\vartheta}A$. Let $u \in D(A)$. Using (1.32), we compute

$$\begin{aligned}\mathrm{Re}(\mathrm{e}^{\pm\mathrm{i}\vartheta}Au|u) &= \mathrm{Re}(u|\mathrm{e}^{\pm\mathrm{i}\vartheta}Au) = \mathrm{Re}\left(u\middle|A(\mathrm{e}^{\pm\mathrm{i}\vartheta}u)\right) = -\,\mathrm{Re}\left(\mathrm{e}^{\mp\mathrm{i}\vartheta}\underline{a}(u,u)\right)\\ &= -\cos(\vartheta)\,\mathrm{Re}\,\underline{a}(u,u) \mp \sin(\vartheta)\,\mathrm{Im}\,\underline{a}(u,u).\end{aligned}$$

The first summand in the last line is negative (for $u \neq 0$). Concerning the second one, the properties of the form imply the inequality

$$|\mathrm{Im}\,\underline{a}(u,u)| \le |\underline{a}(u,u)| \le C\|u\|_Y^2 \le C\eta^{-1}\,\mathrm{Re}\,\underline{a}(u,u).$$

Since $\sin(\vartheta) = \cos(\vartheta)\tan(\vartheta) = \eta C^{-1}\cos(\vartheta)$, we conclude

$$\mathrm{Re}(\mathrm{e}^{\pm\mathrm{i}\vartheta}Au|u) \le (-\cos(\vartheta) + C\eta^{-1}\sin(\vartheta))\,\mathrm{Re}\,\underline{a}(u,u) = 0.$$ □

The Dirichlet–Laplacian Δ_D on $L^2(G)$ is a special case of the above results by Example 1.54. We now treat the space $L^p(G)$ for $p \in (1,\infty)$ and $p \neq 2$ with considerably more effort in a direct way, assuming a bit more regularity of ∂G.

[8] One calls such operators *non-positive (definite)* and writes $A = A' \le 0$.

Example 2.30. Let $p \in (1,\infty)$ and $A = \Delta$ for $E = L^p(\mathbb{R}^m)$ and $\mathrm{D}(A) = W^{2,p}(\mathbb{R}^m)$ or for $E = L^p(G)$ and $\mathrm{D}(A) = W^{2,p}(G) \cap W_0^{1,p}(G)$, assuming that $G \subseteq \mathbb{R}^m$ is open and bounded with $\partial G \in C^2$. Then A generates a bounded analytic C_0-semigroup on E which is contractive on Σ_{κ_p} for

$$\kappa_p = \frac{\pi}{2} - \arctan \frac{|p-2|}{2\sqrt{p-1}} = \operatorname{arccot} \frac{|p-2|}{2\sqrt{p-1}} \in \Big(0, \frac{\pi}{2}\Big].$$

Moreover, the graph norm of A and $\|\cdot\|_{2,p}$ are equivalent. In particular, for $p = 2$ the semigroup has angle $\pi/2$. Here one can allow for open and bounded $G \subseteq \mathbb{R}^m$ with $\partial G \in C^{1-}$ if one replaces A by Δ_D from Example 1.54.

Proof. 1) Once we know that A is closed, the graph norm of A and $\|\cdot\|_{2,p}$ are complete on $\mathrm{D}(A)$. As the latter norm is stronger, Corollary 4.29 in [**24**] (to the open mapping theorem) implies their equivalence.

For $p = 2$ and Δ_D the result follows from Corollary 2.28 since then A is self-adjoint and dissipative by Examples 1.45 and 1.54. For the other cases we use Corollary 2.27, also allowing for $G = \mathbb{R}^m$. The domain $\mathrm{D}(A)$ is dense by Proposition 4.13 in [**24**]. Theorem 9.15 in [**9**] for $G \neq \mathbb{R}^m$ and Theorem 4.3.8 in [**15**] for $G = \mathbb{R}^m$ show that $I - A$ is surjective.[9] Below we check that the operators A and $\mathrm{e}^{\pm \mathrm{i}\kappa_p} A$ are dissipative.

2) Let $u \in \mathrm{D}(A) \setminus \{0\}$. We define $u^* = |u|^{p-2}\overline{u}$, where we set $u^*(x) = 0$ if $u(x) = 0$. Recall that $\|u\|_p^{2-p} u^* \in J(u)$ by Example 1.30.

First, take $p \geq 2$. Assume for a moment that $u \in C^1(G)$ so that $u^* \in C^1(G)$. (Here and below it is crucial that $p \geq 2$.) Using $u^* = (u\overline{u})^{\frac{p}{2}-1}\overline{u}$, we compute

$$\begin{aligned}\partial_k u^* &= |u|^{p-2}\partial_k \overline{u} + \tfrac{p-2}{2}(u\overline{u})^{\frac{p}{2}-2}(\overline{u}\partial_k u + u \partial_k \overline{u})\overline{u} \\ &= |u|^{p-4}\big(\overline{u}u\partial_k\overline{u} + (p-2)\overline{u}\operatorname{Re}(\overline{u}\partial_k u)\big) =: v.\end{aligned}$$

for $k \in \{1, \dots, m\}$. The function v is bounded in $L^{p'}$ by $c\|\partial_k u\|_p \|u\|_p^{p-2}$ due to Hölder's inequality with $\frac{1}{p'} = \frac{1}{p} + \frac{p-2}{p}$.

To treat the given map $u \in \mathrm{D}(A)$, we approximate it in $W_0^{1,p}(G)$ by $u_n \in C_c^\infty(G)$ using Remark 1.41. Passing to a subsequence, we can assume that u_n tends to u a.e. and that $|u_n| \leq g$ for a fixed function $g \in L^p(G)$. Dominated convergence then implies that $|u_n|^{p-2}$ converges to $|u|^{p-2}$ in $L^{p/(p-2)}(G)$, and analogously for the other factors without derivatives. Using Hölder again, we can thus extend the above formula for $\partial_k u^*$ to $u \in \mathrm{D}(A)$ (actually to $u \in W_0^{1,p}(G)$), showing that u^* belongs to $W_0^{1,p'}(G)$. It follows

$$\begin{aligned}\partial_k u \partial_k u^* &= |u|^{p-4}\big(|\overline{u}\partial_k u|^2 + (p-2)(\operatorname{Re}(\overline{u}\partial_k u))^2 + \mathrm{i}(p-2)\operatorname{Im}(\overline{u}\partial_k u)\operatorname{Re}(\overline{u}\partial_k u)\big) \\ &= |u|^{p-4}\big((p-1)(\operatorname{Re}(\overline{u}\partial_k u))^2 + (\operatorname{Im}(\overline{u}\partial_k u))^2 + \mathrm{i}(p-2)\operatorname{Im}(\overline{u}\partial_k u)\operatorname{Re}(\overline{u}\partial_k u)\big).\end{aligned}$$

Since $u^* \in W_0^{1,p'}(G)$, formula (1.22) then yields

$$\langle \Delta u, u^* \rangle = -\int_G \nabla u \cdot \nabla u^* \,\mathrm{d}x = -\int_G |u|^{p-4}\big((p-1)|\operatorname{Re}(\overline{u}\nabla u)|^2 + |\operatorname{Im}(\overline{u}\nabla u)|^2\big)$$

[9] These results are based on harmonic analysis for $G = \mathbb{R}^m$ and $p \neq 2$ (e.g., the so-called Calderón–Zygmund theory) and also on PDE methods for bounded G.

$$+\,\mathrm{i}(p-2)|u|^{p-4}\,\mathrm{Im}(\overline{u}\nabla u)\cdot\mathrm{Re}(\overline{u}\nabla u)\big)\,\mathrm{d}x,$$

and $A=\Delta$ is dissipative. The inequalities of Hölder and Young further imply

$$\begin{aligned}
|\mathrm{Im}\langle\Delta u,u^*\rangle| &\le |p-2|\int_G |u|^{\frac{p}{2}-2}|\,\mathrm{Re}(\overline{u}\nabla u)|\;|u|^{\frac{p}{2}-2}|\mathrm{Im}(\overline{u}\nabla u)|\,\mathrm{d}x\\
&\le |p-2|\Big[\sqrt{p-1}\int_G |u|^{p-4}\,|\mathrm{Re}(\overline{u}\nabla u)|^2\,\mathrm{d}x\Big]^{\frac12}\Big[\frac{1}{\sqrt{p-1}}\int_G |u|^{p-4}\,|\mathrm{Im}(\overline{u}\nabla u)|^2\,\mathrm{d}x\Big]^{\frac12}\\
&\le \frac{|p-2|\sqrt{p-1}}{2}\int_G |u|^{p-4}\,|\mathrm{Re}(\overline{u}\nabla u)|^2\,\mathrm{d}x+\frac{|p-2|}{2\sqrt{p-1}}\int_G |u|^{p-4}\,|\mathrm{Im}(\overline{u}\nabla u)|^2\,\mathrm{d}x\\
&= -\frac{|p-2|}{2\sqrt{p-1}}\,\mathrm{Re}\langle\Delta u,u^*\rangle.
\end{aligned}$$

We set $z=-\langle\Delta u,u^*\rangle\in\overline{\mathbb{C}_+}$. The above inequality shows that $z=0$ if $\mathrm{Re}\,z=0$. For $\mathrm{Re}\,z>0$, we derive the inequality

$$|\arg z| = \arctan\frac{|\mathrm{Im}\,z|}{\mathrm{Re}\,z}\le\arctan\frac{|p-2|}{2\sqrt{p-1}}=\frac{\pi}{2}-\kappa_p\,.$$

It follows $|\arg(\mathrm{e}^{\pm\mathrm{i}\kappa_p}z)|\le\frac{\pi}{2}$ and the dissipativity of the operators $\mathrm{e}^{\pm\mathrm{i}\kappa_p}A$. Corollary 2.27 thus implies the assertion for $p\ge 2$.

3) Next, let $p\in(1,2)$.[10] We have to approximate $u\in\mathrm{D}(A)\setminus\{0\}$ in $W^{2,p}(G)$ by more regular functions. If $G=\mathbb{R}^m$, we know from Remark 1.41 that $C_c^\infty(\mathbb{R}^m)$ is dense in $W^{2,p}(\mathbb{R}^m)$. For bounded G, we look at $f:=u-Au\in L^p(G)$. Take $q\in(\max\{m,2\},\infty)$. Proposition 4.13 of [**24**] yields functions $f_n\in C_c^\infty(G)$ tending to f in $L^p(G)$ as $n\to\infty$. By step 2), the maps $u_n=(I-A)^{-1}f_n$ belong to $W^{2,q}(G)\cap W_0^{1,q}(G)$ for $n\in\mathbb{N}$, and also to $C^1(\overline{G})$ due to Sobolev's embedding Theorem 3.31 in [**27**]. Lemma 9.17 of [**9**] shows that (u_n) converges to u in $W^{2,p}(G)$. We now drop the index n temporarily and assume that u is contained in $C_c^\infty(\mathbb{R}^m)$, respectively in $C^1(\overline{G})\cap W^{2,p}(G)\cap W_0^{1,p}(G)$.

To avoid singularities at zeros of u, we further replace u^* by $u_\varepsilon^\circ=u_\varepsilon^{p-2}\overline{u}$ with $u_\varepsilon:=\sqrt{\varepsilon+|u|^2}\ge\max\{|u|,\sqrt{\varepsilon}\}$ for $\varepsilon>0$. The function u_ε° belongs to $C_c^\infty(\mathbb{R}^m)$, respectively to $C^1(\overline{G})\cap C_0(G)$. As in step 2), we first calculate

$$\begin{aligned}
\partial_k u_\varepsilon^\circ &= u_\varepsilon^{p-4}\big(\varepsilon\partial_k\overline{u}+\overline{u}u\partial_k\overline{u}+(p-2)\,\mathrm{Re}(\overline{u}\partial_k u)\overline{u}\big),\\
\partial_k u\,\partial_k u_\varepsilon^\circ &= u_\varepsilon^{p-4}\big(\varepsilon|\partial_k u|^2+(p-1)(\mathrm{Re}(\overline{u}\partial_k u))^2+(\mathrm{Im}(\overline{u}\partial_k u))^2\\
&\qquad+\mathrm{i}(p-2)\,\mathrm{Im}(\overline{u}\partial_k u)\,\mathrm{Re}(\overline{u}\partial_k u)\big).
\end{aligned}$$

Arguing as above, we obtain the inequalities

$$\begin{aligned}
-\,\mathrm{Re}\langle\Delta u,u_\varepsilon^\circ\rangle &= \int_G u_\varepsilon^{p-4}\big(\varepsilon|\nabla u|^2+(p-1)|\mathrm{Re}(\overline{u}\nabla u)|^2+|\mathrm{Im}(\overline{u}\nabla u)|^2\big)\,\mathrm{d}x\\
&\ge \int_G u_\varepsilon^{p-4}\big((p-1)|\mathrm{Re}(\overline{u}\nabla u)|^2+|\mathrm{Im}(\overline{u}\nabla u)|^2\big)\,\mathrm{d}x,\\
|\mathrm{Im}\langle\Delta u,u_\varepsilon^\circ\rangle| &\le \frac{|p-2|}{2\sqrt{p-1}}\int_G u_\varepsilon^{p-4}\big((p-1)|\mathrm{Re}(\overline{u}\nabla u)|^2+|\mathrm{Im}(\overline{u}\nabla u)|^2\big)\,\mathrm{d}x
\end{aligned}$$

[10]In the lectures this part of the proof was omitted.

$$\leq -\frac{|p-2|}{2\sqrt{p-1}}\,\mathrm{Re}\langle \Delta u, u_\varepsilon^\circ\rangle,$$

Observe that $|u_\varepsilon^\circ| \leq |u|^{p-1} \in L^{p'}(G)$ since $p < 2$ and that u_ε° converges pointwise to u^* as $\varepsilon \to 0$, where $u_\varepsilon^\circ(x) = 0 = u^*(x)$ if $u(x) = 0$. So u_ε° tends to u^* in $L^{p'}(G)$ by dominated convergence. It follows

$$|\mathrm{Im}\langle \Delta u, u^*\rangle| \leq -\frac{|p-2|}{2\sqrt{p-1}}\,\mathrm{Re}\langle \Delta u, u^*\rangle.$$

So far the maps $u = u_n$ belong to $C_c^\infty(\mathbb{R}^m)$, respectively to $C^1(\overline{G}) \cap W^{2,p}(G) \cap W_0^{1,p}(G)$, and converge to a given u in $[\mathrm{D}(A)]$. Passing to a subsequence, (u_n) tends to u pointwise a.e. and $|u_n| \leq h$ for some $h \in L^p$ and all $n \in \mathbb{N}$. We infer that $u_n^* \to u^*$ a.e. and $|u_n^*| \leq h^{p-1} \in L^{p'}$. The functions u_n^* thus converge to u^* in $L^{p'}$. Hence, the inequality in display is true for all $u \in \mathrm{D}(A)$. We now proceed as for $p \geq 2$ and conclude that $\mathrm{e}^{\pm \mathrm{i}\vartheta}\Delta$ are dissipative for $0 \leq \vartheta \leq \kappa_p$. The assertion for $p < 2$ then also follows from Corollary 2.27. □

For more general generation result we refer to [**22**], [**28**] and Chapter 3 of [**18**], where the latter focusses on the sup-norm setting. The case $p = 1$ is treated in [**29**]. These works make heavy use of results and methods from partial differential equations. In Example 1.53 we studied certain elliptic operators in divergence form on $L^2(G)$ in a self-contained way using functional analytic methods, though without computing the domain explicitly. This approach can be extended to more general operators and with more effort to $L^p(G)$, see [**21**].

Inhomogeneous evolution equations. If A generates an analytic C_0-semigroup, we next show that also the inhomogeneous problem (2.5) exhibits better regularity properties than in the general case. So the mild solution just misses differentiability in X for continuous inhomogeneities f, and it is differentiable if f possesses very little extra regularity.

Let $x \in X$, $b > 0$, $f \in C([0,b], X)$ and $A - \omega I$ be densely defined and sectorial of angle $\varphi > \frac{\pi}{2}$ for some $\omega \in \mathbb{R}$. We study the inhomogeneous evolution equation

$$u'(t) = Au(t) + f(t), \quad t \in (0,b] =: J, \qquad u(0) = x. \tag{2.19}$$

It has the mild solution

$$u(t) = T(t)x + \int_0^t T(t-s)f(s)\,\mathrm{d}s =: T(t)x + v(t), \qquad t \in [0,b], \tag{2.20}$$

where A generates the analytic C_0-semigroup $T(\cdot)$. By Definition 2.5, a solution of (2.19) on J is a map $u \in C(\overline{J}, X) \cap C^1(J, X)$ with $u(t) \in \mathrm{D}(A)$ for all $t \in J$ which satisfies (2.19). We need the *Hölder space* $C^\alpha([a,b], X)$ with exponent $\alpha \in (0,1)$. It contains all functions $u \in C([a,b], X)$ fulfilling

$$[u]_\alpha := \sup_{a \leq s < t \leq b} \frac{\|u(t) - u(s)\|}{(t-s)^\alpha} < \infty,$$

and it becomes a Banach space when endowed with the norm

$$\|u\|_\alpha := \|u\|_\infty + [u]_\alpha.$$

For $0 < \alpha < \beta < 1$ we have the embeddings

$$C^1([a,b],X) \hookrightarrow C^\beta([a,b],X) \hookrightarrow C^\alpha([a,b],X) \hookrightarrow C([a,b],X). \tag{2.21}$$

We now establish the results indicated above.

THEOREM 2.31. *Let $x \in X$, $b > 0$, $f \in C([0,b],X)$, and $A - \omega I$ be densely defined and sectorial of angle $\varphi > \frac{\pi}{2}$ for some $\omega \in \mathbb{R}$. Then the mild solution u of (2.19) satisfies the following assertions.*

a) We have $u \in C^\beta([\varepsilon,b],X)$ for all $\beta \in (0,1)$ and $\varepsilon \in (0,b)$. If also $x \in \mathrm{D}(A)$, we can even take $\varepsilon = 0$ here.

b) If $f \in C^\alpha([0,b],X)$ for some $\alpha \in (0,1)$, then u solves (2.19) on $(0,b]$. If also $x \in \mathrm{D}(A)$, then u solves (2.19) on $[0,b]$.

REMARK 2.32. For $\alpha = 0$ and $x = 0$, Theorem 2.31 b) is wrong due to Example 4.1.7 in [**18**]. One thus needs a bit of extra regularity of f. Much more detailed and deeper information on the regularity of u can be found in Chapter 4 of [**18**], where also 'spatial regularity' is studied (and not only time regularity as above), see also the exercises and Chapter 3 of [**26**]. ◊

PROOF OF THEOREM 2.31. By Theorem 2.25 and Remark 2.26, the orbit $T(\cdot)x$ solves (2.19) with $f = 0$ on $\mathbb{R}_+$ if $x \in X$ and on $\mathbb{R}_{\geq 0}$ if $x \in \mathrm{D}(A)$. In particular, $T(\cdot)x$ belongs to the space $C^1([\varepsilon,b],X)$ for all $\varepsilon > 0$ (and for $\varepsilon = 0$ if $x \in \mathrm{D}(A)$). In view of (2.21), it thus remains to study the map v from (2.20). Theorem 2.25 and Remark 2.26 yield constants $c_j = c_j(b)$ with $j \in \{0,1\}$ such that $\|T(t)\| \leq c_0$ and $\|tAT(t)\| \leq c_1$ for $t \in (0,b]$.

To show assertion a), let $0 \leq s < t \leq b$. We first note that $\|v\|_\infty \leq c_0 b \|f\|_\infty$. The increment of v is split into the terms

$$v(t) - v(s) = \int_s^t T(t-\tau)f(\tau)\,\mathrm{d}\tau + \int_0^s (T(t-\tau) - T(s-\tau))f(\tau)\,\mathrm{d}\tau =: I_1 + I_2.$$

It follows

$$\|I_1\| \leq c_0(t-s)\,\|f\|_\infty \leq c_0 b^{1-\beta}(t-s)^\beta \|f\|_\infty.$$

For $t > s > \tau \geq 0$, we further compute

$$T(t-\tau) - T(s-\tau) = (T(t-s) - I)T(s-\tau) = \int_0^{t-s} T(\sigma)AT(s-\tau)\,\mathrm{d}\sigma,$$

using that $T(s-\tau)X \subseteq \mathrm{D}(A)$ by Theorem 2.25. This formula leads to the inequality

$$\|T(t-\tau) - T(s-\tau)\| \leq \frac{c_0 c_1 (t-s)}{(s-\tau)},$$

which is not good enough as the denominator is not integrable in $\tau < s$. Since it also gives more than the needed factor $|t-s|^\beta$, we only apply the above estimate to a fraction of the integrand in I_2, obtaining

$$\begin{aligned} \|I_2\| &\leq \int_0^s \|T(t-\tau) - T(s-\tau)\|^\beta \, \|T(t-\tau) - T(s-\tau)\|^{1-\beta} \, \|f(\tau)\|\,\mathrm{d}\tau \\ &\leq \int_0^s c_0^\beta c_1^\beta \, \frac{(t-s)^\beta}{(s-\tau)^\beta} \, (2c_0)^{1-\beta}\,\mathrm{d}\tau \, \|f\|_\infty \leq \frac{2^{1-\beta} c_0 c_1^\beta b^{1-\beta}}{1-\beta} \, \|f\|_\infty \, (t-s)^\beta. \end{aligned}$$

Hence, v belongs to $C^\beta([0,b],X)$ and there is a constant $c = c(\beta,b,c_0,c_1)$ such that $\|v\|_{C^\beta} \le c\|f\|_\infty$. (Observe that c explodes as $\beta \to 1$.)

We now treat part b). In view of Lemma 2.8 (with $u_0 = 0$), it suffices to show that $v \in C([0,b],[\mathrm{D}(A)])$. Let $t \in [0,b]$. To use the Hölder continuity of f, we insert in the Duhamel integral the constant vector $f(t)$ and obtain

$$v(t) = \int_0^t T(t-s)(f(s)-f(t))\,\mathrm{d}s + \int_0^t T(\tau)f(t)\,\mathrm{d}\tau =: v_1(t) + v_2(t),$$

substituting $\tau = t-s$. As in Lemma 2.8, one checks the continuity of the maps $v_1, v_2 : [0,b] \to X$. By Lemmas 1.18 and 1.12, the function v_2 takes values in $\mathrm{D}(A)$ and $t \mapsto Av_2(t) = T(t)f(t) - f(t)$ is continuous in X.

For $0 < \varepsilon < \varepsilon_0 \le t \le b$, Theorem 2.25 implies that the truncated integral

$$v_{1,\varepsilon}(t) := \int_0^{t-\varepsilon} T(t-s)(f(s)-f(t))\,\mathrm{d}s = T(\varepsilon)\int_0^{t-\varepsilon} T(t-\varepsilon-s)(f(s)-f(t))\,\mathrm{d}s$$

is an element of $\mathrm{D}(A)$ and that $Av_{1,\varepsilon} \in C([\varepsilon,b],X)$. Moreover, $v_{1,\varepsilon}(t)$ tends to $v_1(t)$ as $\varepsilon \to 0$, and from $AT(\varepsilon) \in \mathcal{B}(X)$ we infer

$$Av_{1,\varepsilon}(t) = AT(\varepsilon)\int_0^{t-\varepsilon} T(t-\varepsilon-s)(f(s)-f(t))\,\mathrm{d}s = \int_0^{t-\varepsilon} AT(t-s)(f(s)-f(t))\,\mathrm{d}s.$$

Next, let $0 < \varepsilon < \eta < \varepsilon_0 \le t$. It follows

$$Av_{1,\varepsilon}(t) - Av_{1,\eta}(t) = \int_{t-\eta}^{t-\varepsilon} AT(t-s)(f(s)-f(t))\,\mathrm{d}s.$$

From Theorem 2.25 we then deduce that

$$\begin{aligned}\|Av_{1,\varepsilon}(t) - Av_{1,\eta}(t)\| &\le c_1\int_{t-\eta}^{t-\varepsilon} (t-s)^{-1}(t-s)^\alpha [f]_\alpha\,\mathrm{d}s\\ &= \frac{c_1}{\alpha}[f]_\alpha (t-s)^\alpha\Big|_{t-\varepsilon}^{t-\eta} = \frac{c_1}{\alpha}[f]_\alpha(\eta^\alpha - \varepsilon^\alpha).\end{aligned}$$

Hence, $Av_{1,\varepsilon}$ converges in $C([\varepsilon_0,b],X)$ as $\varepsilon \to 0$. Since A is closed, the vector $v_1(t)$ is contained in $\mathrm{D}(A)$ and $(Av_{1,\varepsilon})_\varepsilon$ has the limit Av_1 in $C([\varepsilon_0,b],X)$ for all $\varepsilon_0 > 0$, so that v_1 belongs to $C((0,b],[\mathrm{D}(A)])$. Finally, $v_1(0) = 0 \in \mathrm{D}(A)$ and

$$\|Av_1(t)\| = \lim_{\varepsilon\to 0}\|Av_{1,\varepsilon}(t)\| \le \overline{\lim_{\varepsilon\to 0}}\int_0^{t-\varepsilon} c_1(t-s)^{-1}[f]_\alpha(t-s)^\alpha\,\mathrm{d}s \le \frac{c_1}{\alpha}[f]_\alpha t^\alpha$$

tends to 0 as $t \to 0$. We conclude that $Av \in C([0,b],X)$ as required. □

The following example is a straightforward consequence of our results.

EXAMPLE 2.33. Let $G \subseteq \mathbb{R}^m$ be bounded and open with $\partial G \in C^{1-}$, $u_0 \in E = L^2(G)$, and $f \in C^\alpha([0,b],E)$ for some $\alpha \in (0,1)$. Theorem 2.31 and Example 2.30 then yield a unique solution u in $C^1((0,b],E) \cap C((0,b],[\mathrm{D}(\Delta_D)]) \cap C([0,b],E)$ of the inhomogeneous *diffusion equation*

$$u'(t) = \Delta_D u(t) + f(t), \quad 0 < t \le b, \qquad u(0) = u_0, \tag{2.22}$$

where Δ_D is the Dirichlet–Laplacian from Example 1.54 with $\mathrm{D}(\Delta_D) \subseteq W_0^{1,2}(G)$.

Let u_0 and f be real-valued, in addition. In view of Example 1.54, then also $\operatorname{Re} u$ solves (2.22) and thus $u = \operatorname{Re} u$ by uniqueness. We conclude that $T(\cdot)$ leaves invariant the closed (real) subspace $E_\mathbb{R} = L^2(G,\mathbb{R})$ of E. As in an exercise one sees that $T(\cdot)$ induces a C_0-semigroup on $E_\mathbb{R}$ which is generated by the restriction of Δ_D to $\mathrm{D}(\Delta_D) \cap E_\mathbb{R}$. Moreover, this semgroup satisfies the assertions of Theorem 2.25 except for the extension to complex 'times' $z \in \Sigma_\zeta$.

Next, we also assume that $\partial G \in C^2$ so that $\mathrm{D}(A) = W^{2,2}(G) \cap W_0^{1,2}(G)$ as noted in Example 2.30. Set $f(t,x) = (f(t))(x)$ for all $0 < t \le b$ and almost every $x \in G$, and analogously for u. Then we can interpret (2.22) more concretely as the partial differential equation

$$\begin{aligned}\partial_t u(t,x) &= \Delta u(t,x) + f(t,x), \qquad t \in (0,b],\ x \in G,\\ u(t,x) &= 0, \qquad t \in (0,b],\ x \in \partial G,\\ u(0,x) &= u_0(x), \qquad x \in G.\end{aligned}$$

In general, here the first and third equality hold for a.e. $x \in G$ and the second one in the sense of trace. If $m \le 3$, then $W^{2,2}(G) \hookrightarrow C(\overline{G})$ by Sobolev's embedding Theorem 3.31 in [**27**], and thus the boundary condition is satisfied pointwise. The solutions become more regular if we improve the regularity of u_0, f and ∂G, see Section 5 of [**18**]. $\Diamond$

CHAPTER 3

Perturbation and approximation

So far we have only looked at one given generator A. In this chapter we add another operator to A or we approximate it. Both procedures are of great importance both from a theoretical perspective and for applications.

3.1. Perturbation of generators

Let A generate a C_0-semigroup $T(\cdot)$ and B be linear. We study the question whether '$A+B$' generates a C_0-semigroup $S(\cdot)$, and then also whether $S(\cdot)$ inherits properties of $T(\cdot)$. Positive results in this direction will allow us to transfer our knowledge about A to larger classes of operators. In this setting one faces two basic problems.

First, how one defines '$A+B$' if $\mathrm{D}(A)\cap\mathrm{D}(B)$ is 'small' (e.g., equal to $\{0\}$ as in Example III.5.10 in [7])? In this section we only treat the basic case that $\mathrm{D}(A)\subseteq\mathrm{D}(B)$. We then put $\mathrm{D}(A+B)=\mathrm{D}(A)$.

Second, if B with $\mathrm{D}(B)\supseteq\mathrm{D}(A)$ is 'too large', it can happen that $A+B$ fails to be a generator. For instance, let A be a generator whose spectrum is unbounded to the left (e.g., $\mathrm{d}/\mathrm{d}s$ on $C_0(\mathbb{R}_{\le 0})$ with $\mathrm{D}(A)=C_0^1(\mathbb{R}_{\le 0})$ or Δ on $L^2(\mathbb{R}^m)$ as in Example 1.27, resp. 1.45), and $B=-(1+\delta)A$ for any $\delta>0$. The sum $A+B=-\delta A$ then has the spectral bound $\mathrm{s}(A+B)=\infty$ and hence $A+B$ is not a generator by Proposition 1.20. Below we restrict ourselves to 'small' perturbations B employing the following important concept.

Definition 3.1. *Let A and B be linear operators on X with $\mathrm{D}(A)\subseteq\mathrm{D}(B)$. Then B is called* A-bounded *(or* relatively bounded*) with constants $a,b\ge 0$ if*

$$\forall\, y\in\mathrm{D}(A): \qquad \|By\|\le a\|Ay\|+b\|y\|. \tag{3.1}$$

In this case we set $\mathrm{D}(A+B)=\mathrm{D}(A)$ (unless something else is specified). The A-bound of B *is the infimum of the numbers $a\ge 0$ for which (3.1) is valid with some $b=b(a)\ge 0$.*

Observe that B is A-bounded if and only B belongs to $\mathcal{B}([\mathrm{D}(A)],X)$. For later use, we derive a quantitative version of this equivalence involving the resolvent.

Let A be closed with $\lambda\in\rho(A)$ and B be linear with $\mathrm{D}(A)\subseteq\mathrm{D}(B)$. First assume that $\gamma:=\|BR(\lambda,A)\|$ is finite. Let $y\in\mathrm{D}(A)$. We compute

$$\|By\|=\|BR(\lambda,A)(\lambda y-Ay)\|\le\gamma\|Ay\|+\gamma|\lambda|\,\|y\|, \tag{3.2}$$

which is (3.1) with $a=\gamma$. Conversely, let B be A-bounded and $x\in X$. Using (3.1) and $AR(\lambda,A)=\lambda R(\lambda,A)-I$, we see that $BR(\lambda,A)\in\mathcal{B}(X)$ estimating

$$\begin{aligned}\|BR(\lambda,A)x\|&\le a\|AR(\lambda,A)x\|+b\|R(\lambda,A)x\|\\&\le\big(a|\lambda|\,\|R(\lambda,A)\|+a+b\|R(\lambda,A)\|\big)\|x\|.\end{aligned} \tag{3.3}$$

The next result says that B is A-bounded if it is of 'lower order', cf. the exercises.

LEMMA 3.2. *Let A and B be linear operators satisfying* $\mathrm{D}(A) \subseteq \mathrm{D}(B)$ *and*

$$\|By\| \le c\|Ay\|^{\alpha} \, \|y\|^{1-\alpha}$$

for all $y \in \mathrm{D}(A)$ and some constants $c \ge 0$ and $\alpha \in [0,1)$. Then the map B has the A-bound 0. In the assumption one can replace $\|Ay\|$ by $\|y\|_A$.

PROOF. As the case $\alpha = 0$ is clear, we let $\alpha \in (0,1)$. Recall Young's inequality $ab \le a^p/p + b^{p'}/p'$ from Analysis 1, where $a, b \ge 0$, $p \in (1,\infty)$ and $p' = \frac{p}{p-1}$. Taking $p = \frac{1}{\alpha} > 1$ and $p' = \frac{1}{1-\alpha}$, for $y \in \mathrm{D}(A)$ and $\varepsilon > 0$ we compute

$$\|By\| \le \varepsilon\|Ay\|^{\alpha} \, c\varepsilon^{-1}\|y\|^{1-\alpha} \le \alpha\varepsilon^{\frac{1}{\alpha}}\|Ay\| + c^{\frac{1}{1-\alpha}}(1-\alpha)\varepsilon^{-\frac{1}{1-\alpha}}\|y\|.$$

If one replaces $\|Ay\|$ by $\|y\|_A$, one only obtains an extra summand $\alpha\varepsilon^{\frac{1}{\alpha}}\|y\|$. □

Our arguments are based on the next perturbation result for the resolvent.

LEMMA 3.3. *Let A be closed with $\lambda \in \rho(A)$ and B be linear with* $\mathrm{D}(A) \subseteq \mathrm{D}(B)$ *and $\|BR(\lambda, A)\| < 1$. Then the sum $A + B$ with* $\mathrm{D}(A+B) = \mathrm{D}(A)$ *is closed, λ is contained in $\rho(A+B)$, and the resolvent satisfies*

$$R(\lambda, A+B) = R(\lambda, A)\sum_{n=0}^{\infty}(BR(\lambda, A))^n = R(\lambda, A)(I - BR(\lambda, A))^{-1},$$

$$\|R(\lambda, A+B)\| \le \frac{\|R(\lambda, A)\|}{1 - \|BR(\lambda, A)\|}.$$

Moreover, the graph norms of A and $A+B$ on $\mathrm{D}(A)$ *are equivalent.*

PROOF. In view of Theorem 1.27 in [**27**], we only have to show the last assertion. Note that $\|(I - BR(\lambda, A))^{-1}\| \le 1/(1-q)$ with $q = \|BR(\lambda, A)\|$ by Proposition 4.24 in [**24**]. For $y \in \mathrm{D}(A)$ we estimate

$$\begin{aligned}
\|y\|_A &= \|y\| + \|AR(\lambda, A+B)(\lambda I - A - B)y\| \\
&= \|y\| + \|AR(\lambda, A)(I - BR(\lambda, A))^{-1}(\lambda y - (A+B)y)\| \\
&\le \|y\| + (|\lambda|\,\|R(\lambda, A)\| + 1)\tfrac{1}{1-q}(|\lambda|\,\|y\| + \|(A+B)y\|) \le c\|y\|_{A+B}
\end{aligned}$$

for a constant $c > 0$. The converse inequality is proven similarly. □

We start with the *bounded perturbation theorem* due to *Phillips* (1953) which is the prototype for the results in this section. It characterizes the perturbed semigroup $S(\cdot)$ in terms of an integral equation and describes it by a series expansion, both only involving $T(\cdot)$ and B. These formulas allow us to transfer certain properties from $T(\cdot)$ to $S(\cdot)$, see Example 3.6, the exercises or Section III.1 in [**7**]. The proof is based on the Hille–Yosida Theorem 1.26, where we use an equivalent norm to avoid the powers in the estimate (1.15).

THEOREM 3.4. *Let A generate a C_0-semigroup $T(\cdot)$ satisfying $\|T(t)\| \le M\mathrm{e}^{\omega t}$ for all $t \ge 0$ and constants $M \ge 1$ and $\omega \in \mathbb{R}$. Let $B \in \mathcal{B}(X)$. Then the sum $A + B$ with* $\mathrm{D}(A+B) = \mathrm{D}(A)$ *generates the C_0-semigroup $S(\cdot)$ which fulfills*

$$\|S(t)\| \le M\mathrm{e}^{(\omega + M\|\mathcal{B}\|)t}, \tag{3.4}$$

$$S(t)x = T(t)x + \int_0^t T(t-s)BS(s)x\,\mathrm{d}s, \tag{3.5}$$

$$S(t)x = T(t)x + \int_0^t S(t-s)BT(s)x\,\mathrm{d}s, \tag{3.6}$$

$$S(t) = \sum_{n=0}^{\infty} S_n(t), \quad S_0(t) := T(t), \quad S_{n+1}(t)x := \int_0^t T(t-s)BS_n(s)x\,\mathrm{d}s, \tag{3.7}$$

for all $t \geq 0$*,* $x \in X$*, and* $n \in \mathbb{N}_0$*. The* Dyson-Phillips series *in* (3.7) *converges in* $\mathcal{B}(X)$ *absolutely and uniformly on compact subsets of* $\mathbb{R}_{\geq 0}$*. The operator family* $(S(t))_{t\geq 0}$ *is the only strongly continuous family of operators solving* (3.5)*. The graph norms of* A *and* $A+B$ *on* $\mathrm{D}(A)$ *are equivalent.*

Proof. 1) Observe that $A+B$ is densely defined. The operator $A-\omega I$ generates the C_0-semigroup $\tilde{T}(\cdot) = (\mathrm{e}^{-\omega t}T(t))_{t\geq 0}$ by Lemma 1.17. As in Remark 1.25 we define the norm

$$|||x||| = \sup_{s\geq 0} \|\mathrm{e}^{-\omega s}T(s)x\|$$

on X satisfying $\|x\| \leq |||x||| \leq M\|x\|$ for $x \in X$ and for which $\tilde{T}(\cdot)$ becomes contractive. (We also denote the induced operator norm by triple bars.) For $x \in X$, we estimate

$$|||Bx||| \leq M\|Bx\| \leq M\|B\|\,\|x\| \leq M\|B\|\,|||x|||\,.$$

Take $\lambda > M\|B\| \geq |||B|||$. The Hille–Yosida estimate (1.17) thus implies the inequality $|||BR(\lambda, A-\omega I)||| \leq |||B|||\,/\lambda < 1$. From Lemma 3.3 we then deduce that λ belongs to $\rho(A+B-\omega I)$, the bound

$$|||R(\lambda, A+B-\omega I)||| \leq \frac{\lambda^{-1}}{1-\lambda^{-1}\,|||B|||} = \frac{1}{\lambda - |||B|||},$$

and the equivalence of the graph norms. The Hille–Yosida Theorem 1.26 now shows that $A+B-\omega I$ generates a C_0-semigroup $\tilde{S}(\cdot)$ on $(X, |||\cdot|||)$ with $|||\tilde{S}(t)||| \leq \mathrm{e}^{|||B|||t} \leq \mathrm{e}^{M\|B\|t}$ for all $t \geq 0$. Finally, by Lemma 1.17 the sum $A+B$ generates the semigroup given by $S(t) = \mathrm{e}^{\omega t}\tilde{S}(t)$ fulfilling

$$\|S(t)x\| \leq |||S(t)x||| \leq \mathrm{e}^{\omega t}\mathrm{e}^{M\|B\|t}\,|||\,x||| \leq M\mathrm{e}^{(\omega+M\|B\|)t}\|x\|$$

for all $t \geq 0$ and $x \in X$, as asserted.

2) We next prove (3.5), (3.6) and uniqueness, treating $BS(\cdot)x$ or $-BT(\cdot)x$ as inhomogeneities. For $x \in \mathrm{D}(A)$, the function $u = S(\cdot)x$ solves the problem

$$u'(t) = (A+B)u(t) = Au(t) + f(t), \quad t \geq 0, \qquad u(0) = x,$$

where $f := BS(\cdot)x : \mathbb{R}_{\geq 0} \to X$ is continuous. Proposition 2.6 then shows that u is given by

$$S(t)x = u(t) = T(t)x + \int_0^t T(t-s)f(s)\,\mathrm{d}s = T(t)x + \int_0^t T(t-s)BS(s)x\,\mathrm{d}s$$

for $t \geq 0$. We derive (3.5) for all $x \in X$ by approximation since $\mathrm{D}(A)$ is dense in X and all operators (in particular B) are bounded uniformly in $s \in [0,t]$.

To show (3.6), we perturb $A+B$ by $-B \in \mathcal{B}(X)$. Indeed, $v = T(\cdot)x$ satisfies

$$v'(t) = (A+B)v(t) - Bv(t), \quad t \geq 0, \qquad v(0) = x \in \mathrm{D}(A),$$

and we can proceed as for (3.5).

Let $U(\cdot)$ be another strongly continuous solution of (3.5). For $x \in X$, $t \geq 0$, $t_0 > 0$ and $t \in [0, t_0]$, we estimate

$$\begin{aligned}\|S(t)x - U(t)x\| &= \Big\| \int_0^t T(t-s)B\big(S(s)x - U(s)x\big)\,\mathrm{d}s \Big\| \\ &\leq M\mathrm{e}^{\omega_+ t_0}\|B\| \int_0^t \|S(s)x - U(s)x\|\,\mathrm{d}s.\end{aligned}$$

Gronwall's inequality from Lemma 4.5 in [**25**] then yields that $S(t)x - U(t)x = 0$, and hence $U(\cdot) = S(\cdot)$.

3) Let $t \geq 0$ and $x \in X$. Concerning (3.7), we note that $S_1(\cdot)$ is strongly continuous and satisfies

$$\|S_1(t)x\| \leq \int_0^t M\mathrm{e}^{\omega(t-s)}\|B\|M\mathrm{e}^{\omega s}\|x\|\,\mathrm{d}s = M^2 t\mathrm{e}^{\omega t}\|B\|\|x\|.$$

Inductively one further deduces the strong continuity of $S_n(\cdot)$ and the inequality

$$\|S_n(t)\| \leq \frac{M^{n+1}\|B\|^n}{n!}\, t^n \mathrm{e}^{\omega t}$$

for all $n \in \mathbb{N}$. By the majorant criterion the series in (3.7) thus converges in $\mathcal{B}(X)$ to some $R(t)$ uniformly on compact subsets of $\mathbb{R}_{\geq 0}$. Hence, $R(\cdot)$ is strongly continuous and fulfills

$$\begin{aligned}\int_0^t T(t-s)BR(s)x\,\mathrm{d}s &= \sum_{n=0}^{\infty}\int_0^t T(t-s)BS_n(s)x\,\mathrm{d}s = \sum_{n=0}^{\infty} S_{n+1}(t)x = \sum_{j=1}^{\infty} S_j(t)x \\ &= R(t)x - T(t)x.\end{aligned}$$

The uniqueness of (3.5) finally implies $R(t) = S(t)$. □

Using also $-A$, we extend the above result to the group case.

Corollary 3.5. *Let A generate the C_0-group $T(\cdot)$ satisfying $\|T(t)\| \leq M\mathrm{e}^{\omega|t|}$ for all $t \in \mathbb{R}$ and constants $M \geq 1$ and $\omega \geq 0$. Let $B \in \mathcal{B}(X)$. Then the sum $A+B$ with $\mathrm{D}(A+B) = \mathrm{D}(A)$ generates the C_0-group $S(\cdot)$ which fulfills $\|S(t)\| \leq M\mathrm{e}^{(\omega+M\|B\|)|t|}$ and (3.5)–(3.7) for all $t \in \mathbb{R}$.*

Proof. Theorem 1.29 says that $\pm A$ are generators of C_0-semigroups with $\|T_\pm(t)\| \leq M\mathrm{e}^{\omega t}$ for $t \geq 0$. From Theorem 3.4 we then infer that $A+B$ and $-(A+B)$ generate C_0-semigroups $S_\pm(\cdot)$ with $\|S_\pm(t)\| \leq M\mathrm{e}^{(\omega+M\|B\|)t}$. By Theorem 1.29, the sum $A+B$ is a generator of a C_0-group $S(\cdot)$ with the asserted bound. Formulas (3.5)–(3.7) for $t \in \mathbb{R}$ are shown as in the previous proof. □

If a model involves the mass density of a substance, it is natural to require that a non-negative initial function leads to a non-negative solution. We will come back to this issue at the end the chapter. Here we first show such behavior is inherited under suitable perturbations.

Example 3.6. Let $E = C_0(U)$ or $E = L^p(\mu)$ for an open set $U \subseteq \mathbb{R}^m$, respectively for a measure space $(S, \mathcal{A}, \mu)$ and $1 \le p < \infty$. We set $E_+ = \{f \in E \,|\, f \ge 0\}$. Let $T(\cdot)$ be a C_0-semigroup on E with generator A such that $T(t)f \ge 0$ for all $f \in E_+$ and $t \ge 0$. We call such operators or semigroups *positive*.[1] We look at two classes of perturbations.

a) Let also $B \in \mathcal{B}(E)$ be positive. Take $f \in E_+$. The function $T(t-s)BT(s)f$ is then non-negative for each $s \in [0,t]$. Since E_+ is closed in E, we infer that $S_1(t)f \ge 0$ and, by induction, that all terms $S_n(t)f$ in the Dyson-Phillips series (3.7) belong to E_+. So the semigroup $S(\cdot)$ generated by $A+B$ is positive and satisfies $S(t) \ge T(t) = S_0(t)$; i.e., $S(t)f \ge T(t)f$ for all $f \in E_+$.

b) Let $Bf = bf$ for a map $b \in C_b(U, \mathbb{R})$ if $E = C_0(U)$, resp. $b \in L^\infty(\mu, \mathbb{R})$ if $E = L^p(\mu)$. For all $f \in E_+$ we have $(B + \|b_-\|_\infty I)f \ge b_+ f \ge 0$; i.e., $B_0 := B + \|b_-\|_\infty I$ is positive. By part a), $A + B_0$ is the generator of a positive C_0-semigroup $\tilde{S}(\cdot) \ge T(\cdot)$ and so $A + B = A + B_0 - \|b_-\|_\infty I$ generates the positive C_0-semigroup $S(\cdot)$ given by $S(t) = \mathrm{e}^{-\|b_-\|_\infty t}\tilde{S}(t) \ge \mathrm{e}^{-\|b_-\|_\infty t}T(t)$ for $t \ge 0$.

As a simple example, we take $U = S = \mathbb{R}$, $\mu = \lambda$, and $A = \frac{\mathrm{d}}{\mathrm{d}s}$ with $\mathrm{D}(A) = C_0^1(\mathbb{R})$ if $E = C_0(U)$, resp. $\mathrm{D}(A) = W^{1,p}(\mathbb{R})$ if $E = L^p(\mu)$. Because A generates the positive translation semigroup on E, the operator $Cu = u' + bu$ with $\mathrm{D}(C) = \mathrm{D}(A)$ also generates a positive C_0-semigroup. ◊

We next use Corollary 3.5 to treat a damped or excited wave equation.

Example 3.7. Let $G \subseteq \mathbb{R}^m$ be bounded and open with a C^{1-}-boundary and Δ_D be the Dirichlet–Laplacian on $L^2(G)$ given by Example 1.54. We set $E = Y \times L^2(G)$, where $Y = W_0^{1,2}(G)$ is endowed with the norm $\|v\|_Y = \||\nabla v|_2\|_2$. As in Example 1.55 we define the operator

$$A = \begin{pmatrix} 0 & I \\ \Delta_D & 0 \end{pmatrix} \qquad \text{with } \mathrm{D}(A) = \mathrm{D}(\Delta_D) \times Y$$

on E. It is skew-adjoint and thus generates a unitary C_0-group $T(\cdot)$. We further let $b \in L^\infty(G, \mathbb{R})$ and introduce the bounded operator $B = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$ on X.

1) By Corollary 3.5, the sum $A + B$ with domain $\mathrm{D}(A)$ generates the C_0-group $S(\cdot)$ on E bounded by $\mathrm{e}^{\|b\|_\infty |t|}$. Let $w_0 = (u_0, u_1) \in \mathrm{D}(A)$. Following Example 2.4, we can show that $(u, u') = S(\cdot)w_0$ yields the unique solution u in $C^2(\mathbb{R}_{\ge 0}, L^2(G)) \cap C^1(\mathbb{R}_{\ge 0}, Y) \cap C(\mathbb{R}_{\ge 0}, [\mathrm{D}(\Delta_D)])$ of the perturbed wave equation

$$u''(t) = \Delta_D u(t) + bu'(t), \quad t \ge 0, \qquad u(0) = u_0, \quad u'(0) = u_1. \tag{3.8}$$

We look at the energy $e(t) = \frac{1}{2}\int_G (|\partial_t u(t)|^2 + |\nabla u(t)|_2^2)\,\mathrm{d}x = \frac{1}{2}\|S(t)w_0\|_E^2$ for $t \ge 0$. As in Analysis 2 one defines derivatives in Banach spaces and shows their basic properties as the chain rule, see [1]. In particular, the map $\Phi(x) = \|x\|^2$ on a Hilbert space X has the derivative given by $[\Phi'(x)](y) = 2\,\mathrm{Re}(y|x)$ for $x, y \in X$. Using also the definition of Δ_D and (3.8), we obtain the derivative

$$\begin{aligned} e'(\tau) &= \mathrm{Re}\int_G \big(\partial_t^2 u(\tau)\partial_t \overline{u}(\tau) + \nabla\partial_t u(\tau)\cdot\nabla\overline{u}(\tau)\big)\,\mathrm{d}x \\ &= \mathrm{Re}\int_G \big(\partial_t^2 u(\tau) - \Delta_D u(\tau)\big)\partial_t\overline{u}(\tau)\,\mathrm{d}x = \int_G b|\partial_t u(\tau)|^2\,\mathrm{d}x \end{aligned}$$

[1] We note that these concepts do not fit to our usual notation such as $\mathbb{R}_+ = (0, \infty)$ for the set of positive real numbers. A function $f \ge 0$ is still called non-negative.

for $\tau \geq 0$. (Note that $\operatorname{Re} z = \operatorname{Re} \overline{z}$. The above calculation simplifies a bit if $\mathbb{F} = \mathbb{R}$.) Integration in time now yields the *energy equality*

$$e(t) = e(0) + \int_0^t \int_G b(x)|\partial_t u(\tau, x)|^2 \,\mathrm{d}x \,\mathrm{d}\tau, \qquad t \geq 0. \tag{3.9}$$

The term bu' thus acts as a damping if $b \leq 0$, and as an excitation if $b \geq 0$.

2) As in Example 2.17 we want to admit data $w_0 \in E$. To determine the extrapolation space E_{-1}^{A+B} for $A+B$, we fix $\lambda > 3\|b\|_\infty$ and take $w \in E$. Lemma 3.3 then yields the bound $\|R(\lambda, A+B)w\|_E \leq \frac{3}{2}\|R(\lambda, A)w\|_E$ since $\|BR(\lambda, A)\| \leq \lambda^{-1}\|b\|_\infty$. Conversely, from (3.4) and the Hille–Yosida estimate (1.17) we infer $\|BR(\lambda, A+B)\| \leq \|b\|_\infty/(\lambda - \|b\|_\infty) \leq \frac{1}{2}$. Writing $R(\lambda, A) = R(\lambda, A+B-B)$, Lemma 3.3 then leads to the ineqality $\|R(\lambda, A)w\|_E \leq 2\|R(\lambda, A+B)w\|_E$. These expressions thus define equivalent norms on E, which are also equivalent to $w \mapsto \|A^{-1}w\|_E$ by (2.10). From Example 2.17 we now infer that E_{-1}^{A+B} is isomorphic to $F = L^2(G) \times W^{-1,2}(G) \cong E_{-1}^A$, where the isomorphisms extend the identity on E. Moreover, by approximation we obtain

$$(A+B)_{-1} = \begin{pmatrix} 0 & I \\ \tilde{\Delta}_D & b \end{pmatrix} : E \longrightarrow F \cong E_{-1}^{A+B}$$

for the extension $\tilde{\Delta}_D : Y \to W^{-1,2}(G)$ from Example 1.54.

Let $w_0 \in E$. As in Example 2.4 and Example 2.17, we obtain a unique solution u of (3.8) with $\tilde{\Delta}_D$ in $C^2(\mathbb{R}_{\geq 0}, W^{-1,2}(G)) \cap C^1(\mathbb{R}_{\geq 0}, L^2(G)) \cap C(\mathbb{R}_{\geq 0}, Y)$. Since $S(t)$ is uniformly bounded on E, we can extend (3.9) to $(u(t), u'(t)) = S(t)w_0$. In particular, $S(\cdot)$ is contractive if $b \leq 0$. ♢

We now turn our attention to unbounded perturbations B of a generator A. As noted above, we should impose a smallness assumption on B. We restrict ourselves to two very useful theorems for contraction and analytic semigroups, employing the simpler characterizations of the generation properties available here.[2] We start with the *dissipative perturbation theorem.*

THEOREM 3.8. *Let A generate the contraction semigroup $T(\cdot)$ and B be dissipative. Assume that B is A-bounded with constant $a < 1$ in (3.1). Then $A + B$ with $\mathrm{D}(A+B) = \mathrm{D}(A)$ generates a contraction semigroup $S(\cdot)$ which also satisfies formulas (3.5) and (3.6) for all $x \in \mathrm{D}(A)$. The graph norms of A and $A+B$ on $\mathrm{D}(A)$ are equivalent.*

PROOF. 1) Observe that $A + B$ is densely defined and that we have $\operatorname{Re}\langle Ax, x^*\rangle \leq 0$ for all $x \in \mathrm{D}(A)$ and $x^* \in J(x)$ due to Proposition 1.32. Since B is dissipative, for each $x \in \mathrm{D}(A)$ there is a functional $y^* \in J(x)$ satisfying $\operatorname{Re}\langle Bx, y^*\rangle \leq 0$. Hence, $\operatorname{Re}\langle Ax + Bx, y^*\rangle \leq 0$ and $A+B$ is dissipative. The assumption provides constants $a \in [0,1)$ and $b \geq 0$ with $\|Bx\| \leq a\|Ax\| + b\|x\|$ for all $x \in \mathrm{D}(A)$. First, assume that $a < \frac{1}{2}$. Fix $\lambda_0 > \frac{b}{1-2a} \geq 0$. Inequality (3.3) and the Hille–Yosida estimate (1.17) then yield

$$\|BR(\lambda_0, A)\| \leq a\lambda_0\|R(\lambda_0, A)\| + a + b\|R(\lambda_0, A)\| \leq a + a + b\lambda_0^{-1} < 1.$$

[2]In Section III.3 of [7] one can find results for general generators A based on the fixed point equation (3.5) for $S(\cdot)$.

Lemma 3.3 now implies that $A+B$ is closed, its graph norm is equivalent to $\|\cdot\|_A$, and λ_0 belongs to $\rho(A+B)$. The sum $A+B$ thus generates a contraction semigroup by the Lumer–Phillips Theorem 1.39.

2) Let $a \in [\frac{1}{2}, 1)$. We take $k \in \mathbb{N}$ with $k > \frac{2a}{1-a}$. Then $\frac{1}{k}B$ is dissipative and A-bounded with a constant $a' := \frac{a}{k} < \frac{1-a}{2} < \frac{1}{2}$. By step 1), the sum $C_1 := A + \frac{1}{k}B$ generates a contraction semigroup and fulfills $\|\cdot\|_A \eqsim \|\cdot\|_{C_1}$. We inductively assume that $C_j := A + \frac{j}{k}B$ is a generator of a contraction semigroup and that $\|\cdot\|_A \eqsim \|\cdot\|_{C_j}$ for some $j \in \{1, \dots, k-1\}$. It follows

$$\begin{aligned} \|By\| &\le a\|Ay\| + b\|y\| \le a\|C_jy\| + \frac{aj}{k}\|By\| + b\|y\|, \\ (1-a)\|By\| &\le \Big(1 - \frac{aj}{k}\Big)\|By\| \le a\|C_jy\| + b\|y\|, \\ \big\|\tfrac{1}{k}By\big\| &\le \frac{a}{k(1-a)}\|C_jy\| + \frac{b}{k(1-a)}\|y\| \end{aligned}$$

for all $y \in \mathrm{D}(A)$. Since $\tilde{a} := \frac{a}{k(1-a)} < \frac{1}{2}$, step 1) shows that the operator $C_j + \frac{1}{k}B = C_{j+1}$ generates a contraction semigroup and that its graph norm is equivalent to $\|\cdot\|_{C_j}$, and thus to $\|\cdot\|_A$ by the induction hypothesis. By iteration, $C_k = A + B$ is a generator of a contraction semigroup and $\|\cdot\|_A \eqsim \|\cdot\|_{A+B}$.

The last assertion can be shown as in Theorem 3.4, using that $\pm B \in \mathcal{B}([\mathrm{D}(A)], X)$. But note that it is not clear that (3.5) and (3.6) hold for all $x \in X$ by approximation since B may be unbounded. $\square$

The above iteration scheme relies on the fact that the perturbed operators C_j are all dissipative and thus have the same constant 1 in the Hille–Yosida estimate. If X is reflexive and one has $a = 1$ in the above theorem, the closure of $A+B$ generates a contraction semigroup by Corollary III.2.9 in [**7**].

We now use Theorem 3.8 to solve the Schrödinger equation for the Coulomb potential, see also Example 4.23 in [**27**].

Example 3.9. Let $E = L^2(\mathbb{R}^3)$ with $\mathbb{F} = \mathbb{C}$ and $A = \mathrm{i}\Delta$ with $\mathrm{D}(A) = W^{2,2}(\mathbb{R}^3)$. Example 1.45 implies that A is skew-adjoint, and so it generates a unitary C_0-group $T(\cdot)$ by Stone's Theorem 1.44. We further set $Bv(x) = \mathrm{i}b|x|_2^{-1}v(x) =: -\mathrm{i}V(x)v(x)$ for some $b \in \mathbb{R}$, where $V(0) := 0$.

Sobolev's Theorem 3.31 in [**27**] yields the embedding $W^{2,2}(\mathbb{R}^3) \hookrightarrow C_0(\mathbb{R}^3)$. Let $\varepsilon > 0$ and $v \in W^{2,2}(\mathbb{R}^3)$. Using also polar coordinates, we then estimate

$$\begin{aligned} \|Bv\|_2^2 &= b^2 \int_{B(0,\varepsilon)} \frac{|v(x)|^2}{|x|_2^2}\,\mathrm{d}x + b^2 \int_{\mathbb{R}^3 \setminus B(0,\varepsilon)} \frac{|v(x)|^2}{|x|_2^2}\,\mathrm{d}x \\ &\le 4\pi b^2 \int_0^\varepsilon \frac{\|v\|_\infty^2}{r^2} r^2\,\mathrm{d}r + \frac{b^2}{\varepsilon^2}\int_{\mathbb{R}^3 \setminus B(0,\varepsilon)} |v(x)|^2\,\mathrm{d}x \\ &\le 4\pi b^2 C_{\mathrm{Sob}}\varepsilon \|v\|_{2,2}^2 + b^2\varepsilon^{-2}\|v\|_2^2. \end{aligned}$$

Since the graph norm of A is equivalent to $\|\cdot\|_{2,2}$ by Example 1.45, we conclude that B has the A-bound 0. Further, $\pm B$ is dissipative since

$$\mathrm{Re}(\pm Bv|v) = \pm\,\mathrm{Re}\Big(\mathrm{i}b\int_{\mathbb{R}^3} \frac{|v(x)|^2}{|x|_2^2}\,\mathrm{d}x\Big) = 0.$$

Theorem 3.8 thus says that $A+B$ and $-(A+B)$ generate a contraction semigroup. From Corollary 1.43 we infer that $A+B$ is the generator of the isometric group $S(\cdot)$, which is unitary by Proposition 5.52 of [**24**]. The function $u = S(\cdot)u_0$ then solves the Schrödinger equation

$$u'(t) = \mathrm{i}\Delta u(t) + \mathrm{i}b|x|_2^{-2}u(t), \quad t \in \mathbb{R}, \qquad (\iff \quad \mathrm{i}u'(t) = -(\Delta - V)u(t),)$$
$$u(0) = u_0.$$

Let $\|u_0\|_2 = 1$ so that $\|u(t)\|_2 = 1$ by unitarity. For a suitable constant $b > 0$ and appropriate units, the integral $\int_G |u(t,x)|^2 \,\mathrm{d}x$ is the probability that the electron in the hydrogen atom stays in the (Borel) set $G \subseteq \mathbb{R}^3$ at time $t \in \mathbb{R}$.$\Diamond$

We show the core *sectorial perturbation theorem.* Here the perturbed operator keeps the angle ϕ in the sectoriality estimate, but has an increased shift ω.

Theorem 3.10. *Let $\mathbb{F} = \mathbb{C}$ and A be closed. Assume there are constants $\omega \geq 0$, $K > 0$, and $\phi \in (0,\pi)$ such that $\omega + \Sigma_\phi \subseteq \rho(A)$ and*

$$\forall\, \lambda \in \Sigma_\phi : \qquad \|R(\lambda + \omega, A)\| \leq \tfrac{K}{|\lambda|}.$$

Let B be A-bounded with constant $a \in \big[0, \frac{1}{K+1}\big)$ in (3.1). Then there is a number $\omega' \geq \omega$ such that $A + B - \omega' I$ is sectorial of type (K', ϕ) for some $K' > K$, and we have $[\mathrm{D}(A+B)] = [\mathrm{D}(A)]$ with equivalent norms.

Let $\phi > \pi/2$ and $\mathrm{D}(A)$ be dense. Then the sum $A+B$ generates an analytic C_0-semigroup, which also satisfies formulas (3.5) and (3.6) for all $x \in \mathrm{D}(A)$.

Proof. Let $a, b \geq 0$ as in (3.1). Fix $q \in (a(K+1), 1)$ and set $r = \frac{K(a\omega+b)}{q-a(K+1)} \geq 0$. Take $\lambda \in \Sigma_\phi \setminus B(0,r)$ with $B(0,0) := \emptyset$ and $x \in X$. (Note that $b = 0 = a\omega$ if $r = 0$.) Using (3.1), the assumption and $|\lambda| \geq r$, we estimate

$$\begin{aligned}
\|BR(\lambda, A - \omega I)x\| &\leq a\|AR(\lambda+\omega, A)x\| + b\|R(\lambda+\omega, A)x\| \\
&\leq a\|(\lambda+\omega)R(\lambda+\omega, A)x\| + a\|x\| + \frac{bK}{|\lambda|}\|x\| \\
&\leq a\Big(\frac{K(|\lambda|+\omega)}{|\lambda|} + 1\Big)\|x\| + \frac{bK}{|\lambda|}\|x\| \\
&\leq a(K+1)\|x\| + (q - a(K+1))\|x\| = q\|x\|.
\end{aligned}$$

Lemma 3.3 thus implies that $\lambda \in \rho(A+B-\omega I)$, $\|\cdot\|_{A+B} \eqsim \|\cdot\|_A$, and

$$\|R(\lambda, A+B-\omega I)\| \leq \frac{\|R(\lambda+\omega, A)\|}{1-q} \leq \frac{K/(1-q)}{|\lambda|}$$

for all $\lambda \in \Sigma_\phi \setminus B(0,r)$. To deal with $\lambda \in B(0,r)$, we shift Σ_ϕ by $\gamma = r$ if $\phi \leq \frac{\pi}{2}$ and by $\gamma = r/\sin\phi > r$ if $\phi > \frac{\pi}{2}$. Because of the inclusion $\gamma + \Sigma_\phi \subseteq \Sigma_\phi \setminus B(0,r)$, the above estimate in display yields the inequality

$$\|R(\mu, A+B-(\omega+\gamma)I)\| = \|R(\mu+\gamma, A+B-\omega I)\| \leq \frac{K/(1-q)}{|\mu+\gamma|} \leq \frac{K'}{|\mu|}$$

for all $\mu \in \Sigma_\phi$, with $K' = \frac{K}{1-q}$ if $\phi \leq \pi/2$ and $K' = \frac{K}{(1-q)\sin\phi}$ if $\phi > \pi/2$. Here we use that $|1+\gamma\mu^{-1}|$ is larger than the distance between -1 and Σ_ϕ which is 1, resp. $\sin\phi$. Setting $\omega' = \gamma + \omega$, we arrive at the first assertion. The second one follows from Theorem 2.25 and Remark 2.26, and the proof of Theorem 3.4. $\square$

The following example contains several important techniques which often occur in applications to partial differential equations. It says that first-order perturbations B have Δ_D-bound 0 if the coefficients are not too bad.

EXAMPLE 3.11. Let $G \subseteq \mathbb{R}^m$ be bounded and open with a C^2-boundary, $p \in (1,\infty)$, $E = L^p(G)$ with $\mathbb{F} = \mathbb{C}$, $A = \Delta$ with $\mathrm{D}(A) = W^{2,p}(G) \cap W_0^{1,p}(G)$. By Example 2.30, the operator A is sectorial with angle $\varphi > \frac{\pi}{2}$ and its graph norm is equivalent to $\|\cdot\|_{2,p}$. Theorem 3.31 of [27] yields the Sobolev embedding $W^{2,p}(G) \hookrightarrow W^{1,q_1}(G) \cap L^{q_0}(G)$ where $q_k \in (p,\infty)$ is arbitrary if $p = \frac{m}{2-k}$ and

$$q_k := \begin{cases} \infty, & p > \frac{m}{2-k}, \\ \frac{mp}{m-(2-k)p}, & p < \frac{m}{2-k}, \end{cases}$$

for $k \in \{0,1\}$. (One has $W^{2,p}(G) \hookrightarrow W^{k,q}(G)$ if $q > p$, $2 - \frac{m}{p} \notin \mathbb{Z}$, and $2 - \frac{m}{p} \geq k - \frac{m}{q}$.) Note that $q_k > p$. We choose a number $\theta \in (0,1)$ close to 1 and introduce the exponents $\tilde{q}_k \in (p, q_k)$ and $r_k \in (p,\infty)$ by

$$\frac{1}{\tilde{q}_k} = \frac{1-\theta}{p} + \frac{\theta}{q_k} \qquad \text{and} \qquad \frac{1}{r_k} = \frac{1}{p} - \frac{1}{\tilde{q}_k}.$$

Let $v \in \mathrm{D}(A)$. (Actually $v \in W^{2,p}(G)$ is enough.) For given coefficients $b \in L^{r_1}(G)^m$ and $b_0 \in L^{r_0}(G)$, the operator B is defined by

$$Bv = b \cdot \nabla v + b_0 v = b_0 v + \sum\nolimits_{j=1}^m b_j \partial_j v.$$

Using the above definitions and Hölder's inequality, to show (3.1) we compute

$$\begin{aligned} \|Bv\|_p &\leq \||b|_{r_1}\|_{r_1} \||\nabla v|_{\tilde{q}_1}\|_{\tilde{q}_1} + \|b_0\|_{r_0} \|v\|_{\tilde{q}_0} \\ &\leq \|b\|_{r_1} \|v\|_{1,p}^{1-\theta} \|v\|_{1,q_1}^{\theta} + \|b_0\|_{r_0} \|v\|_p^{1-\theta} \|v\|_{q_0}^{\theta}. \end{aligned}$$

Proposition 3.37 of [27] yields constants $c, \varepsilon_0 > 0$ such that

$$\|v\|_{1,p} \leq \varepsilon \|v\|_{2,p} + c\varepsilon^{-1} \|v\|_p$$

for all $\varepsilon \in (0, \varepsilon_0]$. Further, note that $(a+b)^{1-\theta} \leq a^{1-\theta} + b^{1-\theta}$ for $a, b \geq 0$. Sobolev's embedding, the equivalence of $\|\cdot\|_A$ and $\|\cdot\|_{2,p}$, and the elementary Young inequality then imply

$$\begin{aligned} \|Bv\|_p &\leq c(b)\Big(\varepsilon^{1-\theta} \|v\|_{2,p}^{1-\theta} \|v\|_{2,p}^{\theta} + \varepsilon^{-1} \|v\|_p^{1-\theta} \varepsilon^{\theta} \|v\|_{2,p}^{\theta} + \varepsilon^{-1} \|v\|_p^{1-\theta} \varepsilon \|v\|_{2,p}^{\theta}\Big) \\ &\leq \hat{c}(b)\Big(\varepsilon^{1-\theta} \|v\|_A + 2(1-\theta)\varepsilon^{\frac{-1}{1-\theta}} \|v\|_p + \theta\varepsilon \|v\|_A + \theta \varepsilon^{\frac{1}{\theta}} \|v\|_A\Big) \end{aligned}$$

for constants $c(b), \hat{c}(b) > 0$ depending on $\|b\|_{r_1}$ and $\|b_0\|_{r_0}$, but not on ε. The operator $B : \mathrm{D}(A) \to L^p(G)$ thus has A-bound 0. By Theorem 3.10, the sum $A + B$ with $\mathrm{D}(A)$ then generates an analytic C_0-semigroup on $L^p(G)$. ◊

3.2. The Trotter–Kato theorems

In applications one often knows the parameters in a problem only approximately since they rely on measurements. As in the case of inital values one can then argue that the solution should depend continuously on the parameters. In other words, let A_n and A generate C_0-semigroups $T_n(\cdot)$ and $T(\cdot)$ for $n \in \mathbb{N}$, and assume that '$A_n \to A$' as $n \to \infty$ in some sense. Do we obtain '$T_n(t) \to T(t)$'? Moreover, can we omit (or modify) the assumption that A is a generator?

These questions also occur if one wants to regularize a problem in order to 'legalize' certain calculations, and also in numerical analysis where the operators A_n are matrices on subspaces of finite dimensions $m_n \to \infty$ (if $\dim X = \infty$).

In the easiest case one has $\mathrm{D}(A_n) = \mathrm{D}(A)$ and each difference $A_n - A$ posesses a bounded extension B_n tending to 0 in operator norm as $n \to \infty$. (For instance, take $A_n u = (\Delta_D + V_n)u$ in $L^2(G)$ with $V_n \to V$ in $L^\infty(G)$ and $A = \Delta_D + V$.) We then have $c := \sup_{n\in\mathbb{N}} \|B_n\| < \infty$ and $\|T(t)\| \le M e^{\omega t}$ for all $t \ge 0$ and some $M \ge 1$ and $\omega \in \mathbb{R}$. The perturbation formula (3.5) and estimate (3.4) yield

$$\|T_n(t)x - T(t)x\| = \Big\| \int_0^t T(t-s)B_n T_n(s)x \,\mathrm{d}s \Big\|$$
$$\le M^2 \|B_n\| \int_0^t e^{\omega(t-s)} e^{(\omega + cM)s} \|x\| \,\mathrm{d}s \le c(t_0)\|B_n\| \, \|x\|$$

for all $x \in X$, $t \in [0, t_0]$, $t_0 > 0$, and a constant depending on t_0. This means that $T_n(t)$ tends to $T(t)$ in $\mathcal{B}(X)$ locally uniformly in t if $\|A_n - A\| \to 0$, $n \to \infty$.

We give a typical example for which the question cannot be settled just by the bounded perturbation Theorem 3.4.

Example 3.12. Let $G \subseteq \mathbb{R}^m$ be open and bounded with a C^{1-}-boundary, $E = L^2(G)$ with $\mathbb{F} = \mathbb{C}$, Δ_D be the Dirichlet–Laplacian in E from Example 1.54, and $n \in \mathbb{N}_0$. Recall that Δ_D is invertible and generates a contraction semigroup on E. Let $a_n \in L^\infty(G)$ satisfy $\frac{1}{\delta} \ge a_n(x) \ge \delta > 0$ and $a_n(x) \to a_0(x)$ as $n \to \infty$ for a.e. $x \in G$ and a constant δ.

We define $A_n = a_n \Delta_D$ on the dense domain $\mathrm{D}(A_n) = \mathrm{D}(\Delta_D)$. To treat the multiplicative perturbation A_n of Δ_D, we use the weighted scalar products

$$(f|g)_n = \int_G \frac{1}{a_n} f\overline{g} \,\mathrm{d}x$$

for $f, g \in E$. The induced norm satisfies $\delta\|f\|_{L^2}^2 \le \|f\|_n^2 \le \delta^{-1}\|f\|_{L^2}^2$. Let $v \in \mathrm{D}(\Delta_D)$. We compute

$$\mathrm{Re}(A_n v|v)_n = \mathrm{Re} \int_G \frac{a_n}{a_n} \Delta_D v \,\overline{v} \,\mathrm{d}x = \mathrm{Re}(\Delta_D v|v)_{L^2} \le 0,$$

so that A_n is dissipative with respect to $\|\cdot\|_n$. The same arguments works for the operators $e^{\pm i\vartheta} A_n$ and all $\vartheta \in (0, \frac{\pi}{2}]$, based on the proof of Corollary 2.28.

To check the range condition, take $f \in E$. Since $a_n \Delta_D v = f$ is equivalent to $v = \Delta_D^{-1}(a_n^{-1} f)$, the operator A_n is invertible in E and hence in $(E, \|\cdot\|_n)$. As $\rho(A_n)$ is open, also $\lambda_0 I - A_n$ is invertible for small $\lambda_0 > 0$. By Corollary 2.27, each A_n thus generates an analytic C_0-semigroup $T_n(\cdot)$ which is contractive for $z \in \mathbb{C}_+$ with respect to $\|\cdot\|_n$. For $z \in \mathbb{C}_+$, $f \in E$ and $n \in \mathbb{N}_0$, we then obtain the uniform bound

$$\|T_n(z)f\|_{L^2} \le \delta^{-1/2}\|T_n(z)f\|_n \le \delta^{-1/2}\|f\|_n \le \delta^{-1}\|f\|_{L^2}.$$

Observe that $A_n v$ tends to $A_0 v$ pointwise a.e. as $n \to \infty$ and moreover $|A_n v| \le \delta^{-1}|\Delta_D v|$. Dominated convergence then yields the limit $A_n v \to A_0 v$ in E for each $v \in \mathrm{D}(\Delta_D)$. Does $T_n(T)$ tends to $T_0(t)$ strongly? ◊

The next example indicates that one needs a uniform bound on the semigroups $T_n(\cdot)$ to obtain a general result.

Example 3.13. Let $X = \ell^2$ with $\mathbb{F} = \mathbb{C}$, $n \in \mathbb{N}$, $A((x_k)_k) = (\mathrm{i}kx_k)_k$ with $\mathrm{D}(A) = \{x \in \ell^2 \,|\, (kx_k)_k \in \ell^2\}$ and $A_n((x_k)_k) = (\mathrm{i}kx_k + \delta_{k,n}kx_k)_k$ with $\mathrm{D}(A_n) = \mathrm{D}(A)$ for the Kronecker delta $\delta_{k,n}$. As in the exercises, one sees that the multiplication operators A and A_n generate the C_0-semigroup on X given by $T(t)x = (\mathrm{e}^{\mathrm{i}kt}x_k)_k$ and $T_n(t)x = (\mathrm{e}^{\mathrm{i}kt}\mathrm{e}^{k\delta_{k,n}t}x_k)_k$, respectively. For $x \in \mathrm{D}(A)$ the distance $\|A_nx - Ax\|_2 = |nx_n| = |(Ax)_n|$ tends to 0 as $n \to \infty$; i.e.; A_n converges on the common domain strongly to A. On the other hand, we have

$$\|T_n(t)\| \geq \|T_n(t)e_n\|_2 = |\mathrm{e}^{\mathrm{i}nt}\mathrm{e}^{nt}| = \mathrm{e}^{nt} \;\longrightarrow\; \infty$$

as $n \to \infty$ for each $t > 0$. So $T_n(t)$ cannot converge strongly, since strong convergence would imply uniform boundedness of $\{T_n(t) \,|\, n \in \mathbb{N}\}$. ◊

The *first Trotter–Kato theorem* from 1958/59 shows that the convergence of resolvents and semigroups are equivalent and that these properties follow from the convergence of the generators, provided that the C_0-semigroups $T_n(\cdot)$ are exponentially bounded uniformly in n. The implications b) ⇔ c) ⇐ d) are proven by arguments typical for the theory.

Theorem 3.14. *Let A_n and A generate C_0-semigroups $T_n(\cdot)$ and $T(\cdot)$, respectively, which satisfy $\|T_n(t)\|, \|T(t)\| \leq M\mathrm{e}^{\omega t}$ for all $t \geq 0$ and $n \in \mathbb{N}$, as well as some $M \geq 1$ and $\omega \in \mathbb{R}$. Let D be a core of $\mathrm{D}(A)$. Then the implications a) ⇒ b) ⇔ c) ⇔ d) hold among the following claims, with limits as $n \to \infty$ in X.*

a) For each $n \in \mathbb{N}$ we have $D \subseteq \mathrm{D}(A_n)$, and $A_ny \to Ay$ for all $y \in D$.

b) For all $y \in D$ and $n \in \mathbb{N}$ there are $y_n \in \mathrm{D}(A_n)$ with $y_n \to y$ and $A_ny_n \to Ay$.

c) For some $\lambda \in \mathbb{F}_\omega$ we have $R(\lambda, A_n)x \to R(\lambda, A)x$ for all $x \in X$.

d) For each $t \geq 0$ we have $T_n(t)x \to T(t)x$ for all $x \in X$.

If c) or d) are true, then c) is valid for all $\lambda \in \mathbb{F}_\omega = \omega + \mathbb{F}_+$ and the limit in d) is uniform on all compact subsets of $\mathbb{R}_{\geq 0}$.

Proof. The implication from a) to b) is trivial (take $y_n = y$). Let statement b) be true. Pick any $\lambda \in \mathbb{F}_\omega$. Since $\lambda I - A : [\mathrm{D}(A)] \to X$ is an isomorphism, the set $(\lambda I - A)D$ is dense in X. The Hille–Yosida estimate (1.15) and the assumption yield the uniform bound $\|R(\lambda, A_n)\| \leq \frac{M}{\mathrm{Re}\,\lambda - \omega}$ for all $n \in \mathbb{N}$. By Lemma 4.10 of [**24**] we thus have to show property c) only for all $x = \lambda y - Ay$ with $y \in D$. Let $y \in D$. Due to condition b), there are vectors $y_n \in \mathrm{D}(A_n)$ such that $y_n \to y$ and $A_ny_n \to Ay$ in X as $n \to \infty$. These limits imply

$$x_n := \lambda y_n - A_ny_n \;\longrightarrow\; x = \lambda y - Ay.$$

Estimating

$$\begin{aligned}\|R(\lambda, A_n)x - R(\lambda, A)x\| &\leq \|R(\lambda, A_n)(x - x_n)\| + \|R(\lambda, A_n)x_n - R(\lambda, A)x\| \\ &\leq \tfrac{M}{\mathrm{Re}\,\lambda - \omega}\|x - x_n\| + \|y_n - y\| \;\longrightarrow\; 0, \quad n \to \infty,\end{aligned}$$

we conclude assertion c) for all $\lambda \in \mathbb{F}_\omega$.

Next, let property c) be valid for some $\lambda \in \mathbb{F}_\omega$. Let $y \in D$. We set $x = \lambda y - Ay$ and $y_n = R(\lambda, A_n)x \in \mathrm{D}(A_n)$. It follows that $y_n \to y$ and

$$A_n y_n = \lambda R(\lambda, A_n)x - x \;\longrightarrow\; \lambda R(\lambda, A)x - x = \lambda y - x = Ay$$

as $n \to \infty$; i.e., claim b) holds.

We assume condition d). Take $x \in X$ and $\lambda \in \mathbb{F}_\omega$. Proposition 1.20 yields

$$\|R(\lambda, A)x - R(\lambda, A_n)x\| \le \int_0^\infty \mathrm{e}^{-\operatorname{Re}\lambda t}\|T(t)x - T_n(t)x\|\,\mathrm{d}t.$$

The integrand is bounded by $2M\|x\|\mathrm{e}^{(\omega-\operatorname{Re}\lambda)t}$ and tends to 0 for each $t \ge 0$ as $n \to \infty$. Part c) now follows from dominated convergence, for all $\lambda \in \mathbb{F}_\omega$.

Finally, let again c) be true for some $\lambda \in \mathbb{F}_\omega$. Take $x \in X$, $t_0 > 0$, $t \in [0, t_0]$, and $\varepsilon > 0$. Since $\mathrm{D}(A)$ is dense, there is a vector $y \in \mathrm{D}(A)$ with $\|x - y\| \le \varepsilon$. Set $z = \lambda y - Ay \in X$. We then compute

$$\begin{aligned}\|T_n(t)x - T(t)x\| &\le \|T_n(t)\|\,\|x - y\| + \|T_n(t)y - T(t)y\| + \|T(t)\|\,\|y - x\| \\ &\le 2M\mathrm{e}^{\omega_+ t_0}\varepsilon + \|(T_n(t) - T(t))R(\lambda, A)z\|.\end{aligned}$$

Commuting resolvents and semigroups, the last term is split in the three terms

$$\begin{aligned}\|(T_n(t) - T(t))R(\lambda, A)z\| &\le \|T_n(t)(R(\lambda, A)z - R(\lambda, A_n)z)\| \\ &\quad + \|R(\lambda, A_n)(T_n(t)z - T(t)z)\| \\ &\quad + \|(R(\lambda, A_n) - R(\lambda, A))T(t)z\| \\ &=: d_{1,n}(t) + d_{2,n}(t) + d_{3,n}(t).\end{aligned}$$

Because of c), the summand $d_{1,n}(t) \le M\mathrm{e}^{\omega_+ t_0}\|R(\lambda, A)z - R(\lambda, A_n)z\|$ tends 0 uniformly for $t \in [0, t_0]$ as $n \to \infty$. Since the set $\{T(t)z \,|\, t \in [0, t_0]\}$ is compact, the same holds for $d_{3,n}$ by an exercise in Functional Analysis.

It remains to show this convergence for $d_{2,n}$. As above we find an element $w \in X$ satisfying $\|z - R(\lambda, A)w\| \le \varepsilon$. Inserting $v := R(\lambda, A)w$, we compute

$$\begin{aligned}d_{2,n}(t) &\le \|R(\lambda, A_n)(T_n(t) - T(t))(z - R(\lambda, A)w)\| + \|R(\lambda, A_n)(T_n(t) - T(t))v\| \\ &\le \tfrac{M}{\operatorname{Re}\lambda - \omega} 2M\mathrm{e}^{\omega_+ t_0}\varepsilon + \|[T_n(t)R(\lambda, A_n) - R(\lambda, A_n)T(t)]R(\lambda, A)w\|.\end{aligned}$$

We denote the last summand by $\hat{d}_{2,n}(t)$. To dominate also this term, we write

$$\begin{aligned}\hat{d}_{2,n}(t) &= \Big\| -\int_0^t \partial_s[T_n(t-s)R(\lambda, A_n)T(s)R(\lambda, A)w]\,\mathrm{d}s \Big\| \\ &= \Big\| \int_0^t \big(T_n(t-s)A_nR(\lambda, A_n)T(s)R(\lambda, A)w \\ &\qquad - T_n(t-s)R(\lambda, A_n)T(s)AR(\lambda, A)w\big)\,\mathrm{d}s \Big\| \\ &= \Big\| \int_0^t T_n(t-s)[R(\lambda, A_n) - R(\lambda, A)]T(s)w\,\mathrm{d}s \Big\| \\ &\le M\mathrm{e}^{\omega_+ t_0} t_0 \sup_{s\in[0,t_0]} \|[R(\lambda, A_n) - R(\lambda, A)]T(s)w\|.\end{aligned}$$

The right-hand side converges to 0 uniformly for $t \in [0, t_0]$ as $n \to \infty$, again due to c) and the compactness of $\{T(s)w \mid s \in [0, t_0]\}$. Combining these estimates, we derive assertion d) with local uniform convergence. □

EXAMPLE 3.15. In the setting of Example 3.13, the above theorem implies that the C_0-semigroup generated by $A_n = a_n \Delta_D$ converges strongly on $L^2(G)$ to the one generated by $A = a\Delta_D$. Here we have $\mathrm{D}(\Delta_D) = D = \mathrm{D}(A) = \mathrm{D}(A_n)$, $\omega = 0$, and $M = \delta^{-1}$. ◊

In Theorem 3.14 we have assumed that the limit operator A is a generator. We want to replace this assumption by a range condition as in the Lumer–Phillips theorem. In the main step of our argument we start with strongly converging resolvents and have to show that the limit operators form again the resolvent of a map (which then turns out to be a generator thanks to the Hille–Yosida theorem). In this step we employ the next concept (imitating (1.7)) and discuss some properties which are of independent interest.

DEFINITION 3.16. *Let $\emptyset \neq \Lambda \subseteq \mathbb{F}$. A set $\{R(\lambda) \mid \lambda \in \Lambda\}$ in $\mathcal{B}(X)$ is called* pseudo-resolvent *if it satisfies*

$$R(\lambda) - R(\mu) = (\mu - \lambda) R(\lambda) R(\mu) \qquad \text{for all } \lambda, \mu \in \Lambda. \tag{3.10}$$

We first show that pseudo-resolvents occur as strong limits of resolvents, which only have to converge for one point λ_0.

LEMMA 3.17. *Let $R(\lambda, A_n)$ be resolvents with $\|R(\lambda, A_n)\| \le \frac{M}{\operatorname{Re}\lambda - \omega}$ for all $n \in \mathbb{N}$ and $\lambda \in \mathbb{F}_\omega \subseteq \rho(A_n)$ and some $\omega \in \mathbb{R}$ and $M > 0$. Let $(R(\lambda_0, A_n))_n$ tend strongly to an operator $R(\lambda_0) \in \mathcal{B}(X)$ for some $\lambda_0 \in \mathbb{F}_\omega$. Then for each $\lambda \in \mathbb{F}_\omega$ the maps $R(\lambda, A_n)$ converge strongly to a pseudo-resolvent $\{R(\lambda) \mid \lambda \in \mathbb{F}_\omega\}$ as $n \to \infty$.*

PROOF. We show the strong convergence for all $\lambda \in \mathbb{F}_\omega$ below. Then the resolvent equation (1.7) for $R(\lambda, A_n)$ and $\lambda \in \mathbb{F}_\omega \subseteq \rho(A_n)$ implies (3.10) with $\Lambda = \mathbb{F}_\omega$ in the strong limit. Let $\mu \in \mathbb{F}_\omega$. Remark 1.16 yields the expansion

$$R(\lambda, A_n) = \sum_{k=0}^{\infty} (\mu - \lambda)^k R(\mu, A_n)^{k+1}$$

for all $\lambda \in \mathbb{F}_\omega$ with $|\mu - \lambda| \le \frac{\operatorname{Re}\mu - \omega}{2M} \le \frac{1}{2} \|R(\mu, A_n)\|^{-1}$. If $R(\mu, A_n)$ converges strongly as $n \to \infty$, then also the partial sums of the above series have strong limits. The norms of the remainder terms

$$\sum_{k=K+1}^{\infty} (\mu - \lambda)^k R(\mu, A_n)^{k+1}$$

are bounded by $c \sum_{k=K+1}^{\infty} 2^{-k} = c 2^{-K}$ with $c = M/(\operatorname{Re}\mu - \omega)$, which tends to 0 as $K \to \infty$ independently of n. As a result, the operator $R(\lambda, A_n)$ converges strongly as $n \to \infty$ for $\lambda \in \overline{B}(\mu, \frac{1}{2M}(\operatorname{Re}\mu - \omega))$. The radii of these balls are greater than $\delta/(2M) > 0$ for all $\mu \in \mathbb{F}_{\omega+\delta}$ and each $\delta > 0$. Take $\lambda \in \mathbb{F}_\omega$. Choose $\delta \in (0, \operatorname{Re}\lambda_0 - \omega)$ with $\lambda \in \mathbb{F}_{\omega+\delta}$. Starting from λ_0, we can thus show the strong convergence of $(R(\lambda, A_n))_n$ by a finite iteration. □

We note that in Lemma 3.17 the limits $R(\lambda)$ do not need to be injective, which would be necssary to form a resolvent. For instance, the bounded generators $A_n = -nI$ satisfy $\|e^{tA_n}\| = e^{-nt} \le 1$ for all $t \ge 0$ and $n \in \mathbb{N}$, and their resolvents $R(\lambda, A_n) = \frac{1}{\lambda+n} I$ tend to $0 = R(\lambda)$ as $n \to \infty$ for all $\lambda \in \mathbb{F}_+$. Before we deal with this problem, we derive important properties of pseudo-resolvents.

LEMMA 3.18. *For a pseudo-resolvent* $\{R(\lambda) \,|\, \lambda \in \Lambda\}$ *and all* $\lambda, \mu \in \Lambda$*, we have*
a) $R(\lambda)R(\mu) = R(\mu)R(\lambda)$,
b) $\mathrm{N}(R(\lambda)) = \mathrm{N}(R(\mu))$,
c) $R(\lambda)X = R(\mu)X$.

PROOF. Interchanging λ and μ, equation (3.10) implies assertion a). These facts further yield the formulas

$$R(\lambda) = (I + (\mu - \lambda)R(\lambda))R(\mu) = R(\mu)(I + (\mu - \lambda)R(\lambda)),$$

which lead to the inclusions $\mathrm{N}(R(\mu)) \subseteq \mathrm{N}(R(\lambda))$ and $R(\lambda)X \subseteq R(\mu)X$. The converse inclusions are shown analogously. □

We now establish sufficient conditions for a pseudo-resolvent to be a resolvent.

LEMMA 3.19. *Let* $\{R(\lambda) \,|\, \lambda \in \Lambda\}$ *be a pseudo-resolvent.*

a) Let $R(\lambda_0)$ *be injective for some* $\lambda_0 \in \Lambda$*. Then there is a closed operator* A *domain* $\mathrm{D}(A) = R(\lambda_0)X$ *such that* $\Lambda \subseteq \rho(A)$ *and* $R(\lambda) = R(\lambda, A)$ *for all* $\lambda \in \Lambda$*. Hence,* A *is densely defined if* $R(\lambda_0)$ *has dense range.*

b) Let $R(\mu)$ *have dense range for some* $\mu \in \Lambda$ *and let there be* $\lambda_j \in \Lambda$ *with* $|\lambda_j| \to \infty$ *as* $j \to \infty$ *such that* $\|\lambda_j R(\lambda_j)\| \le M$ *for all* $j \in \mathbb{N}$ *and some constant* $M > 0$*. Then* $R(\lambda)$ *is injective for all* $\lambda \in \Lambda$ *(and thus a resolvent by part a)).*

PROOF. a) The assumption allows us to define the closed operator $A = \lambda_0 I - R(\lambda_0)^{-1}$ with domain $\mathrm{D}(A) = R(\lambda_0)X$. It satisfies the equations

$$(\lambda_0 I - A)R(\lambda_0) = R(\lambda_0)^{-1}R(\lambda_0) = I, \quad R(\lambda_0)(\lambda_0 y - Ay) = R(\lambda_0)R(\lambda_0)^{-1}y = y$$

for all $y \in \mathrm{D}(A)$, so that $\lambda_0 \in \rho(A)$ and $R(\lambda_0) = R(\lambda_0, A)$. Lemma 3.18 shows that $R(\lambda)X = \mathrm{D}(A)$ for all $\lambda \in \Lambda$. Using this fact and (3.10), we compute

$$\begin{aligned}(\lambda I - A)R(\lambda) &= [(\lambda - \lambda_0)I + (\lambda_0 I - A)]R(\lambda_0)[I - (\lambda - \lambda_0)R(\lambda)] \\ &= I + (\lambda - \lambda_0)(R(\lambda_0)[I - (\lambda - \lambda_0)R(\lambda)] - R(\lambda)) = I,\end{aligned}$$

and similarly $R(\lambda)(\lambda y - Ay) = y$ for $y \in \mathrm{D}(A)$. Assertion a) is thus proved.

b) We have $\lambda_j \ne \mu$ and $\lambda_j \ne 0$ for all sufficiently large $j \in \mathbb{N}$. Equation (3.10) and the assumptions then yield the limit

$$\begin{aligned}\|(\lambda_j R(\lambda_j) - I)R(\mu)\| &= \Big\|\frac{\lambda_j}{\mu - \lambda_j}(R(\lambda_j) - R(\mu)) - R(\mu)\Big\| \\ &= \Big\|\frac{\lambda_j}{\mu - \lambda_j}R(\lambda_j) - \frac{\mu}{\mu - \lambda_j}R(\mu)\Big\| \\ &\le \frac{M + \|\mu R(\mu)\|}{|\mu - \lambda_j|} \longrightarrow 0\end{aligned}$$

as $j \to \infty$. Since the set $R(\mu)X$ is dense and the operators $\lambda_j R(\lambda_j)$ are uniformly bounded, it follows that $\lambda_j R(\lambda_j)x \to x$ as $j \to \infty$ for all $x \in X$. If we had $R(\lambda)x = 0$ for some $x \in X$, then $R(\lambda_j)x = 0$ for all j by Lemma 3.18. As a result, $R(\lambda)$ is injective for every $\lambda \in \Lambda$. □

With these preparations we can now show the *second Trotter–Kato theorem* which adds a generation result to the first one, imposing range conditions.

THEOREM 3.20. *Let A_n generate C_0-semigroups $T_n(\cdot)$ such that $\|T_n(t)\| \le M\mathrm{e}^{\omega t}$ for all $t \ge 0$ and $n \in \mathbb{N}$ and some constants $M \ge 1$ and $\omega \in \mathbb{R}$. We then obtain the implications a) ⇒ b) ⇔ c) among the following statements.*

a) There exists a densely defined operator A_0 such that $\mathrm{D}(A_0) \subseteq \mathrm{D}(A_n)$ for all $n \in \mathbb{N}$ and $A_n y \to A_0 y$ as $n \to \infty$ for all $y \in \mathrm{D}(A_0)$, and the range $(\lambda_0 I - A_0)\mathrm{D}(A_0)$ is dense in X for some $\lambda_0 \in \mathbb{F}_\omega$.

b) For some $\lambda_0 \in \mathbb{F}_\omega$ the operators $R(\lambda_0, A_n)$ converge strongly to a map $R \in \mathcal{B}(X)$ with dense range.

c) There is a C_0-semigroup $T(\cdot)$ with generator A such that $T_n(t)$ converges strongly to $T(t)$ for all $t \ge 0$ as $n \to \infty$.

If property b) is true, then $R = R(\lambda_0, A)$. If part a) holds, then $A = \overline{A_0}$. The semigroups $T_n(\cdot)$ and $T(\cdot)$ satisfy the assertions of Theorem 3.14 if we assume conditions a), b) or c).

PROOF. The implication 'c) ⇒ b)' with $R = R(\lambda_0, A)$ is a consequence of Theorem 3.14 since $\|T(t)\| \le M\mathrm{e}^{\omega t}$ follows from the assumptions.

Let statement a) be true. Take $y \in \mathrm{D}(A_0)$ and set $x = \lambda_0 y - A_0 y$. Using the assumption and the Hille–Yosida estimate (1.15), we compute

$$\begin{aligned} \|R(\lambda_0, A_n)x - y\| &= \|R(\lambda_0, A_n)\big((\lambda_0 y - A_0 y) - (\lambda_0 I - A_n)y\big)\| \\ &\le \frac{M}{\operatorname{Re}\lambda_0 - \omega}\|A_0 y - A_n y\| \;\longrightarrow\; 0 \end{aligned}$$

as $n \to \infty$. Since the range $(\lambda_0 I - A_0)\mathrm{D}(A_0)$ is dense and $R(\lambda_0, A_n)$ is uniformly bounded, the resolvents $R(\lambda_0, A_n)$ thus converge strongly to a map $R \in \mathcal{B}(X)$. The range of R contains the dense set $\mathrm{D}(A_0)$; so that claim b) is shown.

Assume condition b). By assumption, the resolvents satisfy the Hille–Yosida estimate (1.15) with uniform constants. Hence, Lemma 3.17 shows that the operators $R(\lambda, A_n)$ converge strongly to a pseudo-resolvent $\{R(\lambda) \,|\, \lambda \in \mathbb{F}_\omega\}$ as $n \to \infty$, which also fulfills $\|R(\lambda)\| \le M(\operatorname{Re}\lambda_0 - \omega)$ as a strong limit. Moreover, $R(\lambda_0) = R$ has dense range by b). Lemma 3.19 thus provides a closed operator A with dense domain $R(\lambda_0)X$ such that $R(\lambda) = R(\lambda, A)$ for all $\lambda \in \mathbb{F}_\omega \subseteq \rho(A)$. Also the products $(\lambda - \omega)^k R(\lambda, A_n)^k$ tend to $(\lambda - \omega)^k R(\lambda)^k$ strongly for all $k \in \mathbb{N}$ and $\lambda \in \mathbb{F}_\omega$ as $n \to \infty$, so that A satisfies (1.15). From the Hille–Yosida Theorem 1.26 we then infer that A generates a C_0-semigroup $T(\cdot)$. Theorem 3.14 now yields statement c) and the last addendum.

Finally, we have to show that A_0 has the closure A if property a) is true. Let $y \in \mathrm{D}(A_0)$. Assertions a) and b) yield

$$y = \lim_{n\to\infty} R(\lambda_0, A_n)(\lambda_0 y - A_n y) = R(\lambda_0, A)(\lambda_0 y - A_0 y),$$

so that $Ay = A_0 y$ and $A_0 \subseteq A$. Therefore, A_0 possesses the closure $\overline{A_0} \subseteq A$.

On the other hand, the range $(\lambda_0 I - \overline{A_0})\mathrm{D}(\overline{A_0})$ is dense in X since it contains the set $(\lambda_0 I - A_0)\mathrm{D}(A_0)$. Let $y \in \mathrm{D}(\overline{A_0})$. There exist vectors $y_k \in \mathrm{D}(A_0)$ such that $y_k \to y$ and $A_0 y_k \to \overline{A_0} y$ in X as $k \to \infty$. Above we have checked the equality $y_k = R(\lambda_0, A)(\lambda_0 y_k - A_0 y_k)$ which tends to $y = R(\lambda_0, A)(\lambda_0 y - \overline{A_0} y)$. Hence, $\|y\|$ is bounded by a constant times $\|\lambda_0 y - \overline{A_0} y\|$. Proposition 1.19 of [**27**] then implies that the range $(\lambda_0 I - \overline{A_0})\mathrm{D}(\overline{A_0})$ is closed, and so $\lambda_0 I - \overline{A_0}$ is surjective. Because of $\lambda_0 \in \rho(A)$, Lemma 1.23 yields the quality $\overline{A_0} = A$. $\square$

3.3. Approximation formulas

Based on the Trotter–Kato theorems, we now discuss further approximation results for C_0-semigroups. We start with an auxiliary fact.

LEMMA 3.21. *Let $S \in \mathcal{B}(X)$ satisfy $\|S^n\| \le M$ for all $n \in \mathbb{N}$ and some $M > 0$. We then obtain*

$$\|\mathrm{e}^{n(S-I)}x - S^n x\| \le M\sqrt{n}\|Sx - x\| \qquad \text{for all } \ n \in \mathbb{N},\ x \in X.$$

PROOF. For $n, m, l \in \mathbb{N}$ with $m > l$ and $x \in X$, we first compute

$$\mathrm{e}^{n(S-I)} - S^n = \mathrm{e}^{-n}\sum_{j=0}^{\infty}\frac{n^j}{j!}S^j - \sum_{j=0}^{\infty}\frac{n^j}{j!}\mathrm{e}^{-n}S^n = \mathrm{e}^{-n}\sum_{j=0}^{\infty}\frac{n^j}{j!}(S^j - S^n),$$

$$\|S^m x - S^l x\| = \Big\|\sum_{j=l}^{m-1} S^j(S-I)x\Big\| \le M(m-l)\|Sx - x\|.$$

Using Hölder's inequality and elementary series, we then estimate

$$\begin{aligned}\|\mathrm{e}^{n(S-I)x} - S^n x\| &\le M\mathrm{e}^{-n}\|Sx - x\|\sum_{j=0}^{\infty}\sqrt{\tfrac{n^j}{j!}}\sqrt{\tfrac{n^j}{j!}}\,|n-j| \\ &\le M\mathrm{e}^{-n}\|Sx - x\|\Big(\sum_{j=0}^{\infty}\frac{n^j}{j!}\Big)^{\frac12}\Big(\sum_{j=0}^{\infty}\frac{n^j}{j!}(n-j)^2\Big)^{\frac12} \\ &\le M\mathrm{e}^{-n}\|Sx - x\|\,\mathrm{e}^{\frac{n}{2}}\sqrt{n}\mathrm{e}^{\frac{n}{2}} = M\sqrt{n}\,\|Sx - x\|.\end{aligned}$$

$\square$

We next show the *Lax–Chernoff product formula* which is the core of this section. It was proved by Lax and Richtmyer in 1957 without its generation part, which was added by Chernoff in 1972 (who also discussed further variants of the result). The theorem says that

consistency and stability imply convergence,

which is a fundamental principle in numerical analysis. In this context one has to combine it with finite dimensional approximations, cf. Section 3.6 of [**22**]. In the exercises we treat convergence rates for vectors x in suitable subspaces.

THEOREM 3.22. *Let $V : \mathbb{R}_{\ge 0} \to \mathcal{B}(X)$ be a function such that $V(0) = I$ and $\|V(t)^k\| \le M\mathrm{e}^{k\omega t}$ for all $t \ge 0$ and $k \in \mathbb{N}$ and some $\omega \in \mathbb{R}$ and $M \ge 1$. Assume that the limit $A_0 y := \lim_{t\to 0}\frac{1}{t}(V(t)x - x)$ exists for all y in a dense subspace $\mathrm{D}(A_0)$. Let the range $(\lambda I - A_0)\mathrm{D}(A_0)$ be dense in X for some $\lambda \in \mathbb{F}_\omega$. Then A_0 is closable and its closure A generates the C_0-semigroup $T(\cdot)$. The products $V(\frac{t}{n})^n$ strongly converge to $T(t)$ locally uniformly in $t \ge 0$ as $n \to \infty$.*

Proof. By rescaling, we may assume that $\omega = 0$. For $s > 0$ we define the bounded operator $A_s = \frac{1}{s}(V(s) - I)$ on X. The assumptions imply that $A_s y \to A_0 y$ for all $y \in \mathrm{D}(A_0)$ as $s \to 0$ and that

$$\|\mathrm{e}^{tA_s}\| = \mathrm{e}^{-\frac{t}{s}} \|\mathrm{e}^{\frac{t}{s}V(s)}\| \le \mathrm{e}^{-\frac{t}{s}} \sum_{k=0}^{\infty} \frac{t^k}{s^k k!} \|V(s)^k\| \le \mathrm{e}^{-\frac{t}{s}} \mathrm{e}^{\frac{t}{s}} M = M$$

for all $t \ge 0$. Theorem 3.20 thus shows that A_0 has a closure A which generates the C_0-semigroup $T(\cdot)$ and that for each null sequence (s_n) the operators $\mathrm{e}^{tA_{s_n}}$ strongly tend to $T(t)$ as $n \to \infty$, uniformly for $t \in [0, t_0]$ and every $t_0 > 0$.

We claim that also $\mathrm{e}^{tA_{t/n}}$ strongly converges to $T(t)$ locally uniformly in t as $n \to \infty$. If the claim was wrong, there would exist a vector $x \in X$ and times $t_n \in [0, t_0]$ for some $t_0 > 0$ such that

$$\inf_{n \in \mathbb{N}} \|\mathrm{e}^{t_n A_{t_n/n}} x - T(t_n) x\| > 0.$$

Since $s_n := t_n/n \to 0$ as $n \to \infty$, we obtain a contradiction.

Let $x \in X$, $\varepsilon > 0$, $t_0 > 0$, and $t \in [0, t_0]$. Choose $y \in \mathrm{D}(A_0)$ with $\|x - y\| \le \varepsilon$. Lemma 3.21 then yields

$$\begin{aligned}
&\|\mathrm{e}^{tA_{t/n}} x - V(t/n)^n x\| \\
&\quad \le \|\mathrm{e}^{tA_{t/n}}\| \, \|x - y\| + \|\mathrm{e}^{n(V(t/n)-I)} y - V(t/n)^n y\| + \|V(t/n)^n\| \, \|x - y\| \\
&\quad \le 2M\varepsilon + M\sqrt{n} \|V(t/n)y - y\| = 2M\varepsilon + \tfrac{tM}{\sqrt{n}} \|A_{t/n} y\| \\
&\quad \le 2M\varepsilon + \tfrac{t_0 M}{\sqrt{n}} \sup_{0 \le s \le t_0} \|A_s y\|.
\end{aligned}$$

The right-hand side tends to $2M\varepsilon$ as $n \to \infty$, so that

$$V(t/n)^n = \mathrm{e}^{tA_{t/n}} + V(t/n)^n - \mathrm{e}^{tA_{t/n}}$$

strongly converges to $T(t)$ locally uniformly in t. □

We add two special cases of the above general approximation result. (More examples are discussed in the exercises.) The first one is the *Lie–Trotter product formula*, shown by Trotter 1959 in a more direct way. It is of great importance in numerical analysis for problems where one can compute approximations of $T(\cdot)$ and $S(\cdot)$ in an efficient way, cf. the exercises. It also provides a generation result for sums of operators without a smallness condition, but involving a closure. Note that the assumptions after (3.11) are satisfied if we know that (the closure of) C is a generator.

Corollary 3.23. *Assume that A and B generate C_0-semigroups $T(\cdot)$ and $S(\cdot)$, respectively, subject to the stability bound*

$$\big\| \big(T\big(\tfrac{t}{n}\big) S\big(\tfrac{t}{n}\big)\big)^n \big\| \le M \mathrm{e}^{\omega t} \tag{3.11}$$

for all $n \in \mathbb{N}$ and $t \ge 0$ and some $M \ge 1$ and $\omega \in \mathbb{R}$. Let $D := \mathrm{D}(A) \cap \mathrm{D}(B)$ and $(\lambda I - (A+B))D$ be dense in X for some $\lambda \in \mathbb{F}_\omega$. Then the sum $C = A+B$ on $\mathrm{D}(C) = D$ has a closure $\overline{C}$ which generates a C_0-semigroup $U(\cdot)$ given by

$$U(t)x = \lim_{n \to \infty} \big(T\big(\tfrac{t}{n}\big) S\big(\tfrac{t}{n}\big)\big)^n x$$

uniformly on all compact subsets of $\mathbb{R}_{\ge 0}$ and for all $x \in X$.

Proof. Define $V(t) = T(t)S(t)$ for $t \geq 0$. For $x \in D$, the vectors

$$\tfrac{1}{t}(V(t)x - x) = T(t)\tfrac{1}{t}(S(t)x - x) + \tfrac{1}{t}(T(t)x - x)$$

converge to $Bx + Ax$ as $t \to 0^+$. The result now follows from Theorem 3.22. □

The stability condition (3.11) holds if both semigroups are $\frac{\omega}{2}$-contractive. In general, one cannot find an equivalent norm for which *both* semigroups become quasi-contractive, cf. Remark 1.25. In fact, there are generators A and B such that $\overline{A+B}$ exists and generates a C_0-semigroup, but (3.11) is violated, and thus the Lie–Trotter product formula fails, see [**16**].

The Lie–Trotter formula can be used to give an alternative proof of the positivity assertion in Example 3.6. It also yields a rigorous mathematical interpretation for the 'Feynman path integral formula' in quantum mechanics for the Schrödinger group $\mathrm{e}^{\mathrm{i}t(\Delta - V)}$, see Paragraph 8.13 in [**10**].

By Proposition 1.20, the resolvent of the generator is the Laplace transform

$$\mathcal{L}(T(\cdot)x)(\lambda) = \int_0^\infty \mathrm{e}^{-\lambda t} T(t)x \,\mathrm{d}t = R(\lambda, A)x, \qquad \operatorname{Re}\lambda > \omega_0(A). \tag{3.12}$$

of the semigroup. In the next corollary we invert this transformation for semigroup orbits, approximating $T(t)$ by powers of the resolvent. In numerics the resulting formula is called 'implicit Euler scheme.' These two fundamental formulas often allow us to transfer properties from the resolvent to the semigroup and back, see e.g. Corollary 3.25. This is an important fact since the resolvent is closely related to the generator, which is usually the given object in applications. We use this link in Example 3.26.

Corollary 3.24. *Let A generate the C_0-semigroup $T(\cdot)$. We then have*

$$T(t)x = \lim_{n\to\infty} \big(\tfrac{n}{t} R\big(\tfrac{n}{t}, A\big)\big)^n x = \lim_{n\to\infty} \big(I - \tfrac{t}{n}A\big)^{-n} x$$

uniformly on all compact subsets of $\mathbb{R}_{\geq 0}$ and for all $x \in X$.

Proof. Take $M \geq 1$ and $\omega > 0$ with $\|T(t)\| \leq M\mathrm{e}^{\omega t}$ for all $t \geq 0$. Set $\delta = \frac{1}{\omega(\omega+1)}$. We then define $V(0) = I$, $V(t) = \frac{1}{t}R(\frac{1}{t}, A)$ for $0 < t \leq \delta$, and $V(t) = 0$ for $t > \delta$. The Hille–Yosida estimate (1.15) yields

$$\|V(t)^n\| = t^{-n}\|R(\tfrac{1}{t}, A)^n\| \leq \frac{M}{t^n(t^{-1} - \omega)^n} = \frac{M}{(1 - \omega t)^n} \leq M\mathrm{e}^{n(1+\omega)t}$$

for $0 < t \leq \delta < \frac{1}{\omega}$ by the choice of δ. From Lemma 1.22 we deduce the limit

$$\tfrac{1}{t}(V(t)x - x) = \tfrac{1}{t}\big(\tfrac{1}{t}R\big(\tfrac{1}{t}, A\big)x - x\big) = \tfrac{1}{t}R\big(\tfrac{1}{t}, A\big)Ax \;\longrightarrow\; Ax$$

as $t \to 0$ for all $x \in \mathrm{D}(A)$. Theorem 3.22 implies the assertion. □

We note that one can show the resolvent approximation directly without involving Chernoff's product formula, see Theorem 1.8.3 in [**22**]. In the next result we use notions introduced in Example 3.6.

Corollary 3.25. *Let $U \subseteq \mathbb{R}^m$ be open and $E = C_0(U)$ or let $(S, \mathcal{A}, \mu)$ be a measure space and $E = L^p(\mu)$ for some $1 \leq p < \infty$. We assume that A generates a C_0-semigroup $T(\cdot)$ on E. Then $T(t)$ is positive for all $t \geq 0$ if and only if $R(\lambda, A)$ is positive for all $\lambda \geq \omega$ and some $\omega > \omega_0(A)$.*

Proof. Let the resolvent be positive for $\lambda > \omega$ and take $t > 0$. For all $f \in E_+$ and large $n \in \mathbb{N}$, then the functions $(\frac{n}{t}R(\frac{n}{t}, A))^n f$ are non-negative and hence their limit $T(t)f$ also belongs to E_+. (Here we use Corollary 3.24.) For $\lambda > \omega_0(A)$, the converse follows in a similiar way from formula (3.12). □

Employing the above result and the 'weak maximum principle', we show that the Dirichlet–Laplacian generates a positive semigroup.

Example 3.26. Let $G \subseteq \mathbb{R}^m$ be open and bounded with a C^2-boundary, $1 < p < \infty$, $E_p = L^p(G)$, and $A_p = \Delta$ with $\mathrm{D}(A_p) = W^{2,p}(G) \cap W_0^{1,p}(G)$. These operators generate bounded analytic C_0-semigroups $T_p(\cdot)$ on E_p, see Example 2.30. We want to prove their positivity.

Let $\lambda > 0$, $1 < p < q < \infty$, and $f \in C_0(G)$. Note that $\mathbb{C}_+ \subseteq \rho(A_p)$ by Proposition 1.20 and that $C_0(G)$ is densely embedded into E_r for $r \in (1, \infty)$. Set $u = R(\lambda, A_q)f \in \mathrm{D}(A_q)$. Then $\lambda u - \Delta u = f$ on G and u also belongs to $\mathrm{D}(A_p)$, since G is bounded. It follows $u = R(\lambda, A_p)f$ as $\lambda \in \rho(A_p)$. This means that $u \in \bigcap_{1<r<\infty} D(A_r)$ and that $R(\lambda, A_q)$ is the restriction of $R(\lambda, A_p)$, by density. Hence, u and $\Delta u = \lambda u - f$ are contained in $C_0(G)$ by the Sobolev embedding $W^{2,q}(G) \hookrightarrow C(\overline{G})$ for $q > \frac{m}{2}$, see Theorem 3.31 in [**27**].

Let also $f \geq 0$. We show that $u \geq 0$. First, $u = R(\lambda, A_2)f$ is real-valued by Example 2.33 and formula (3.12). Suppose there was a point $x_0 \in G$ such that $u(x_0) < 0$. Since $u = 0$ on ∂G and $\overline{G}$ is compact, the function u must have a minimum $u(x_1) < 0$ for some $x_1 \in G$. Proposition 3.1.10 in [**18**] thus yields $\Delta u(x_1) \geq 0$, implying that $f(x_1) = \lambda u(x_1) - \Delta u(x_1) < 0$ which is impossible. Hence, $u = R(\lambda, A_p)f$ is non-negative.

Since $C_0(G)$ is dense in E_p and the map $v \mapsto v_+$ is Lipschitz on E_p, we obtain the positivity of $R(\lambda, A_p)$ by approximation. Corollary 3.25 then shows the positivity of $T_p(t)$ for all $t \geq 0$ and $p \in (1, \infty)$. ♢

Similarly one can treat the case $E = C_0(G)$ starting from the sectoriality result Corollary 3.1.21 in [**18**]. In [**21**] one finds criteria for positivity of semigroups on $L^2(G)$ generated by sesquilinear forms.

CHAPTER 4

Long-time behavior

This chapter is devoted to the long-term behavior of C_0-semigroups focusing on exponential stability and dichotomy. We want to derive these basic properties from conditions on the spectrum and the resolvent of the (given) generator. In this chapter we take $\mathbb{F} = \mathbb{C}$, unless $\mathbb{F} = \mathbb{R}$ is also admitted explicitely.

4.1. Exponential stability and dichotomy

We first introduce the most basic property concerning the long-time behavior. This concept and Definition 4.8 also make sense if $\mathbb{F} = \mathbb{R}$.

DEFINITION 4.1. *A C_0-semigroup $T(\cdot)$ is called* (uniformly) exponentially stable *if there exist constants $M, \varepsilon > 0$ such that*

$$\|T(t)\| \le M\mathrm{e}^{-\varepsilon t} \qquad \textit{for all} \;\; t \ge 0.$$

The above concept can be reformulated as $\omega_0(T) = \omega(A) < 0$, see Definition 1.5, or equivalently as $\|T(t)x\| \le M\mathrm{e}^{-\varepsilon t}\|x\|$ for all $x \in X$ and $t \ge 0$.

Let A generate $T(\cdot)$ and $\varepsilon > 0$. Observe that we have $\|T(t)\| \le \mathrm{e}^{-\varepsilon t}$ for all $t \ge 0$ if and only if $A + \varepsilon I$ is dissipative by the Lumer–Phillips Theorem 1.39. Though this is a rather special situation, it covers the important case of the Dirichlet–Laplacian Δ_D on $L^2(G)$ for a bounded domain, see Example 1.54.

We first characterize exponential stability by properties of the semigroup itself. To this aim, we recall from Theorem 1.16 in [**27**] that an operator $T \in \mathcal{B}(X)$ has a spectral radius satisfying

$$\mathrm{r}(T) = \max\big\{|\lambda| \,\big|\, \lambda \in \sigma(T)\big\} = \lim\nolimits_{n\to\infty} \|T^n\|^{\frac{1}{n}} = \inf\nolimits_{n\in\mathbb{N}} \|T^n\|^{\frac{1}{n}} \le \|T\|. \quad (4.1)$$

The next result says in particular that a C_0-semigroup automatically decays exponentially if it tends to 0 in operator norm as $t \to \infty$.

PROPOSITION 4.2. *Let $T(\cdot)$ be a C_0-semigroup with generator A. Then the following assertions are equivalent.*

a) $T(\cdot)$ is exponentially stable.

b) $\|T(t_0)\| < 1$ for some $t_0 > 0$.

c) $\mathrm{r}(T(t_1)) < 1$ for some $t_1 > 0$.

d) $\omega_0(A) < 0$.

In this case, then statement b) is valid for all sufficiently large $t_0 > 0$, claim c) is true for all $t_1 > 0$, and we have $\mathrm{s}(A) < 0$, cf. (1.12). We further obtain

$$\mathrm{e}^{t\mathrm{s}(A)} \le \mathrm{e}^{t\omega_0(A)} = \mathrm{r}(T(t)) \qquad \textit{and} \qquad \omega_0(A) = \lim_{t\to\infty} \tfrac{1}{t} \ln \|T(t)\| = \inf_{t>0} \tfrac{1}{t} \ln \|T(t)\|$$

for all $t \ge 0$ and with $\ln 0 := -\infty$.

PROOF. Since $\ln\|T(t+s)\| \le \ln\|T(t)\| + \ln\|T(s)\|$ for $t, s \ge 0$, the elementary Lemma IV.2.3 in [7] shows that the limit $\lim_{t\to\infty} \frac{1}{t}\ln\|T(t)\|$ exists and equals $\omega := \inf_{t>0} \frac{1}{t}\ln\|T(t)\|$. This equality yields $\mathrm{e}^{t\omega} \le \|T(t)\|$ for all $t \ge 0$ and thus $\omega \le \omega_0(A)$. Take any $\omega_1 > \omega$. By the description via the limit, there is a time $\tau \ge 0$ such that $\|T(t)\| \le \mathrm{e}^{\omega_1 t}$ for all $t \ge \tau$ so that $\|T(t)\| \le M\mathrm{e}^{\omega_1 t}$ for all $t \ge 0$ and the number $M := \sup\{\mathrm{e}^{-\omega_1 t}\|T(t)\| \,|\, 0 \le t \le \tau\} \in [1,\infty)$. This means that $\omega_1 \ge \omega_0(A)$ and so $\omega = \omega_0(A)$. Using (4.1), we infer the identities

$$\mathrm{r}(T(t)) = \lim_{n\to\infty} \exp\big(t\tfrac{1}{nt}\ln\|T(nt)\|\big) = \exp\big(t\lim_{n\to\infty}\tfrac{1}{nt}\ln(\|T(nt)\|)\big) = \mathrm{e}^{t\omega_0(A)}$$

for $t > 0$. The other claims about $T(\cdot)$ now follow easily. Proposition 1.20 says that $\mathrm{s}(A) \le \omega_0(A)$, which yields the remaining inequality $\mathrm{e}^{t\mathrm{s}(A)} \le \mathrm{e}^{t\omega_0(A)}$. □

For bounded A, Example 5.4 of [**27**] implies the equality $\mathrm{s}(A) = \omega_0(A)$. The next example due to Arendt (1993) shows that $\mathrm{s}(A) < \omega_0(A)$ is possible for unbounded generators. See also Examples IV.2.7 and IV.3.4 as well as Exercises IV.2.13 and IV.3.5 in [7].

EXAMPLE 4.3. Let $X = L^p(1,\infty) \cap L^q(1,\infty)$ for $1 < p \le q < \infty$ which is a reflexive Banach space for the norm $\|f\| = \|f\|_p + \|f\|_q$. We look at the positive operators given by $(T(t)f)(s) = f(s\mathrm{e}^t)$ for $f \in X$, where throughout we let $t \ge 0$, $s > 1$, and $r \in (1,\infty)$. Computing

$$(T(t)T(\tau)f)(s) = (T(\tau)f)(s\mathrm{e}^t) = f(s\mathrm{e}^t\mathrm{e}^\tau) = (T(t+\tau)f)(s)$$

for $\tau \ge 0$, we see that $T(\cdot)$ is a semigroup. Take $f \in L^r(1,\infty)$. We estimate

$$\|T(t)f\|_r^r = \int_1^\infty |f(s\mathrm{e}^t)|^r\,\mathrm{d}s = \int_{\mathrm{e}^t}^\infty |f(\tau)|^r\mathrm{e}^{-t}\,\mathrm{d}\tau \le \mathrm{e}^{-t}\|f\|_r^r,$$

where we substituted $\tau = s\mathrm{e}^t$. For $f \in X$ it follows

$$\|T(t)f\| = \|T(t)f\|_p + \|T(t)f\|_q \le \mathrm{e}^{-t/p}\|f\|_p + \mathrm{e}^{-t/q}\|f\|_q \le \mathrm{e}^{-t/q}\|f\|,$$

so that $T(t)$ belongs to $\mathcal{B}(X)$ with growth bound $\omega_0(T) \le -1/q$.

Let $f \in C_c(1,\infty)$. There is a number $s_0 > 1$ such that $f(s\mathrm{e}^t) = 0$ for all $s\mathrm{e}^t \ge s \ge s_0$. By uniform continuity, the maps $T(t)f$ tend to f uniformly as $t \to 0$, and thus in X due to the bounded support. Lemma 1.7 now yields that $T(\cdot)$ is C_0-semigroup. Let A be its generator. Taking $p = q = r$, we also obtain a C_0-semigroup $T_r(\cdot)$ on $L^r(1,\infty)$ with generator A_r.

Let $f_t = \mathbb{1}_{[\mathrm{e}^t,\mathrm{e}^t+1]}$ for $t \ge 0$. Observe that $\|f_t\|_r = 1$ and so $\|f_t\| = 2$. Since

$$T(t)f_t(s) = \mathbb{1}_{[\mathrm{e}^t,\mathrm{e}^t+1]}(s\mathrm{e}^t) = \mathbb{1}_{[1,1+\mathrm{e}^{-t}]}(s)$$

for $s > 1$, we have $\|T(t)f_t\|_r = \mathrm{e}^{-t/r}$. It follows that

$$\|T(t)f_t\| \ge \|T(t)f_t\|_q = \mathrm{e}^{-t/q} = \tfrac{1}{2}\mathrm{e}^{-t/q}\|f_t\|,$$

and hence $\omega_0(T) = \omega_0(A) = -1/q$.

To determine $\mathrm{s}(A)$, we look at the functions $g_\alpha(s) = s^{-\alpha}$ for $s > 1$ and $\alpha > 1/r$. Then g_α belongs to $L^r(1,\infty)$ and

$$\tfrac{1}{t}(T(t)g_\alpha - g_\alpha) + \alpha g_\alpha = \big(\tfrac{1}{t}(\mathrm{e}^{-\alpha t} - 1) + \alpha\big)g_\alpha.$$

These maps tend to 0 in $L^r(1,\infty)$ as $t \to 0$ so that g_α belongs to $\mathrm{D}(A_r)$ with $A_r g_\alpha = -\alpha g_\alpha$. This means that $-\alpha$ is an eigenvalue of A_r and so $\mathrm{s}(A_r) \geq -1/r$. As $\omega_0(A_r) = -1/r$, Proposition 1.20 shows that $\mathrm{s}(A_r) = \omega_0(A_r) = -1/r$.

We now pass to X. Since $X \hookrightarrow L^p(1,\infty)$ and $T(t) = T_p(t)\restriction_X$, A is the 'part of A_p in X' (i.e., $Af = A_p f$ and $\mathrm{D}(A) = \{f \in \mathrm{D}(A_p) \cap X \mid A_p f \in X\}$) by Proposition II.2.3 in [7]. Proposition 1.20 yields $R(0,A_p)f = \int_0^\infty T_p(t)f\,\mathrm{d}t$. We first take $f \in C_c(1,\infty)$ with $f(s) = 0$ for $s \geq s_0$. Observe that $T_p(t)f = 0$ for all $t > \ln s_0$ and that $t \mapsto T_p(t)f$ is also continuous in supremum norm. The integral thus converges both in $L^p(1,\infty)$ and in $C_0(1,\infty)$. We infer

$$R(0,A_p)f(s) = \Big(\int_0^\infty T_p(t)f\,\mathrm{d}t\Big)(s) = \int_0^\infty f(s\mathrm{e}^t)\,\mathrm{d}t = \int_s^\infty f(\tau)\,\frac{\mathrm{d}\tau}{\tau},$$

substituting $\tau = s\mathrm{e}^t$. Hölder's inequality now implies

$$|R(0,A_p)f(s)| \leq \|f\|_p\Big(\int_s^\infty \tau^{-p'}\,\mathrm{d}\tau\Big)^{\frac{1}{p'}} = \|f\|_p\Big(\frac{s^{1-p'}}{p'-1}\Big)^{\frac{1}{p'}} = \frac{s^{-1/p}}{(p'-1)^{1/p'}}\|f\|_p.$$

We finally take $q > p$. Then $\int_1^\infty s^{-q/p}\,\mathrm{d}s$ is finite, so that $R(0,A_p)$ continuously maps $(C_c(1,\infty), \|\cdot\|_p)$ into X and hence $L^p(1,\infty)$ into X by density. This means that $[\mathrm{D}(A_p)] \hookrightarrow X \hookrightarrow L^p(1,\infty)$. Proposition IV.2.17 of [7] thus shows that $\sigma(A) = \sigma(A_p)$, and so

$$\mathrm{s}(A) = -1/p < -1/q = \omega_0(A)$$

in view of the above results. Rescaling with a number $\omega \in (1/q, 1/p)$, we then obtain a generator $A + \omega I$ of an exponentially growing C_0-semigroup with the negative spectral bound $\omega - 1/p$. ◊

As the best possible identity $\mathrm{s}(A) = \omega_0(A)$ fails in general, one can try to show exponential stability under stronger assumptions. We will first establish it assuming an additional bound of the resolvent. In the next section we actually prove $\mathrm{s}(A) = \omega_0(A)$ (and more) for a class of C_0-semigroups with better regularity properties including analytic ones. We will also comment on results about weaker convergence properties.

If $\dim X = \infty$ it is often more appropriate to complement spectral conditions by resolvent estimates. To establish a corresponding stability theorem, we need some properties of the 'Bochner integral' (where we take $\mathbb{F} \in \{\mathbb{R},\mathbb{C}\}$) and the Fourier transform.

Let $J \subseteq \mathbb{R}$ be an interval. Simple functions $f : J \to X$ and their integral are defined as for $X = \mathbb{R}$. A function $f : J \to X$ is called *strongly measurable* if there are simple functions $f_n : J \to X$ converging to f pointwise almost everywhere. Observe that then the function $t \mapsto \|f(t)\|_X$ is measurable, and that continuous functions are strongly measurable. By Theorem X.1.4 in [2], the map f is strongly measurable if and only if f is Borel measurable and there is a null set $N \subseteq J$ such that $f(J \setminus N)$ is separable. (The latter is true for separable X, of course.) We then define the space

$$L^p(J,X) = \big\{f : J \to X \,\big|\, f \text{ is strongly measurable}, \ \ \|f(\cdot)\|_X \in L^p(J)\big\},$$

$$\|f\|_p = \|\|f(\cdot)\|_X\|_{L^p(J)} = \Big(\int_J \|f(t)\|_X^p\,\mathrm{d}t\Big)^{\frac{1}{p}}$$

for $p \in [1, \infty)$ and analogously for $p = \infty$. Here we identify functions that coincide almost everywhere. One can show that f belongs to $L^1(J, X)$ if and only if there are simple functions converging to f pointwise a.e. such that the sequence $(f_n)_n$ is Cauchy for $\|\cdot\|_1$, see p.90 and Theorem X.3.9 in [**2**]. This fact implies that the integrals $\int_J f_n(t)\,\mathrm{d}t$ converge in X and that their limit is independent of the choice of such a sequence $(f_n)_n$. This limit is denoted by $\int_J f(t)\,\mathrm{d}t$ and called the *(Bochner) integral of* f.

It can be shown that $(L^p(J, X), \|\cdot\|_p)$ is a Banach space and that the Bochner integral satisfies Hölder's inequality and the theorems of Riesz–Fischer, Lebesgue and Fubini, see Chapter X in [**2**] or Chapter 1 in [**13**]. However, the dual of $L^p(J, X)$ for $p \in [1, \infty)$ coincides with $L^{p'}(J, X^*)$ only for a certain class of Banach spaces X, e.g., reflexive ones. (Otherwise the dual is larger.) The duality pairing is given by $\langle f, g\rangle_{L^p(J,X)} = \int_J \langle f(t), g(t)\rangle_X\,\mathrm{d}t$ for $f \in L^p(J, X)$ and $g \in L^{p'}(J, X^*)$. See Theorem 1.3.10 and Corollary 1.3.22 of [**13**].

Let A be closed and $f \in L^1(J, X)$ take values in $\mathrm{D}(A)$ a.e. and Af be integrable. The integral $\int_J f\,\mathrm{d}t$ then belongs to $\mathrm{D}(A)$ and fulfills

$$A \int_J f(t)\,\mathrm{d}t = \int_J Af(t)\,\mathrm{d}t$$

by Theorem C.4 of [**7**].

For $f \in L^1(\mathbb{R}, X)$, we define the *Fourier transform*

$$\widehat{f}(\tau) = \mathcal{F}f(\tau) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}\tau t} f(t)\,\mathrm{d}t, \qquad \tau \in \mathbb{R}.$$

As in the scalar case one shows that $\widehat{f} \in C_0(\mathbb{R}, X)$ and the convolution and inversion theorems, see Theorem 1.8.1 of [**3**]. Let X be a Hilbert space. By Plancherel's Theorem 1.8.2 in [**3**], the Fourier transform then extends from $L^1(\mathbb{R}, X) \cap L^2(\mathbb{R}, X)$ to a unitary operator

$$\mathcal{F} : L^2(\mathbb{R}, X) \to L^2(\mathbb{R}, X)$$

where $L^2(\mathbb{R}, X)$ is a Hilbert space with the inner product

$$(f|g) = \int_{\mathbb{R}} (f(t)|g(t))_X\,\mathrm{d}t, \qquad f, g \in L^2(\mathbb{R}, X).$$

In the theorem below we also need the next auxiliary result by Datko (1970).

Lemma 4.4. *Let $T(\cdot)$ be a C_0-semigroup, $1 \le p < \infty$, and $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. If $T(\cdot)x \in L^p(\mathbb{R}_{\ge 0}, X)$ for all $x \in X$, then $T(\cdot)$ is exponentially stable.*

Proof. Define the bounded operator

$$\Phi_n : X \to L^p(\mathbb{R}_{\ge 0}, X);\ x \mapsto \mathbb{1}_{[0,n]} T(\cdot)x,$$

for each $n \in \mathbb{N}$. The assumption shows that $\sup_{n\in\mathbb{N}} \|\Phi_n x\|$ is finite for each $x \in X$, and hence $C := \sup_{n\in\mathbb{N}} \|\Phi_n\| < \infty$ thanks to the principle of uniform boundedness. It follows $\int_0^t \|T(s)x\|^p\,\mathrm{d}s \le C^p \|x\|^p$ for all $t \ge 0$ and $x \in X$. Fix constants $M \ge 1$ and $\omega > 0$ such that $\|T(t)\| \le M\mathrm{e}^{\omega t}$ for all $t \ge 0$. Let $t \ge 1$ and $x \in X$. We calculate

$$\frac{1 - \mathrm{e}^{-p\omega}}{p\omega} \|T(t)x\|^p \le \frac{1 - \mathrm{e}^{-p\omega t}}{p\omega} \|T(t)x\|^p = \int_0^t \mathrm{e}^{-p\omega s} \|T(s)T(t-s)x\|^p\,\mathrm{d}s$$

$$\leq \int_0^t M^p \mathrm{e}^{\omega s p} \mathrm{e}^{-\omega s p} \|T(t-s)x\|^p \,\mathrm{d}s = M^p \int_0^t \|T(\tau)x\|^p \,\mathrm{d}\tau$$
$$\leq (CM)^p \|x\|^p,$$

so that $\|T(t)\| \leq N$ for all $t \geq 0$, where $N := \max\{M\mathrm{e}^{\omega}, (p\omega)^{1/p} CM(1 - \mathrm{e}^{-p\omega})^{-1/p}\}$. We then derive

$$t\|T(t)x\|^p = \int_0^t \|T(t-s)T(s)x\|^p \,\mathrm{d}s \leq N^p \int_0^t \|T(s)x\|^p \,\mathrm{d}s \leq (CN)^p \|x\|^p,$$

and hence $\|T(t)\| \leq \frac{CN}{t^{1/p}}$. Proposition 4.2 now implies the assertion. □

We first give a heuristic argument for the following stability theorem. Let A generate the C_0-semigroup $T(\cdot)$ on a Hilbert space X. Assume that $\mathrm{s}(A) < 0$. Pick a number $\omega > \omega_0(A)$. We set

$$T_\omega(t) = \begin{cases} \mathrm{e}^{-\omega t} T(t), & t \geq 0, \\ 0, & t < 0. \end{cases}$$

Then there are constants $M \geq 1$ and $\varepsilon > 0$ such that $\|T_\omega(t)\| \leq M\mathrm{e}^{-\varepsilon t}$ for all $t \geq 0$. Take $x \in X$ and $\tau \in \mathbb{R}$. The map $T_\omega(\cdot)x$ belongs to $L^1(\mathbb{R}, X) \cap L^2(\mathbb{R}, X)$ with 2-norm less or equal $M(2\varepsilon)^{-1/2}\|x\|$. Using Proposition 1.20, we compute

$$\mathcal{F}(T_\omega(\cdot)x)(\tau) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \mathrm{e}^{-\mathrm{i}\tau t} \mathrm{e}^{-\omega t} T(t)x \,\mathrm{d}t = \frac{1}{\sqrt{2\pi}} R(\omega + \mathrm{i}\tau, A)x. \tag{4.2}$$

Plancherel's theorem then yields

$$\|R(\omega + \mathrm{i}\cdot, A)x\|_{L^2(\mathbb{R},X)} = \sqrt{2\pi} \|T_\omega(\cdot)x\|_{L^2(\mathbb{R},X)} \leq M\sqrt{\pi/\varepsilon}\|x\|. \tag{4.3}$$

We want to transform this inequality to the imaginary axis by means of the resolvent equation (1.7); i.e.,

$$R(\mathrm{i}\tau, A)x = R(\omega + \mathrm{i}\tau, A)x + \omega R(\mathrm{i}\tau, A)R(\omega + \mathrm{i}\tau, A)x. \tag{4.4}$$

Assuming the boundedness $\|R(\mathrm{i}\cdot, A)\|$ on $\mathbb{R}$, from the above results we infer that $R(\mathrm{i}\cdot, A)x$ is an element of $L^2(\mathbb{R}, X)$. It is now tempting to use Plancherel's theorem once more and to conclude

$$\infty > \|R(\mathrm{i}\cdot, A)x\|_{L^2(\mathbb{R},X)} = \|\mathcal{F}(T_0(\cdot)x)\|_{L^2(\mathbb{R},X)} = \sqrt{2\pi}\|T(\cdot)x\|_{L^2(\mathbb{R}_{\geq 0},X)}.$$

Datko's lemma would then yield $\omega_0(A) < 0$. However, above we need the assertion $\omega_0(A) < 0$ to employ (4.2) for $\omega = 0$ and to apply $\mathcal{F}$ to $T_0(\cdot)x$.

These problems can actually be settled by means of a refined version of (4.2) and an approximation argument, see the proof of Theorem V.1.11 of [**7**]. Below we instead use a shorter argument taken from Theorem 5.2.1 of [**3**]. The resulting *stability theorem of Gearhart* is also special case of Theorem 4.19 below, which has a more involved proof not given in these lectures.

THEOREM 4.5. *Let X be a Hilbert space. A C_0-semigroup $T(\cdot)$ with generator A is exponentially stable if and only if*

$$\mathrm{s}(A) \leq 0 \qquad \textit{and} \qquad C := \sup\nolimits_{\lambda \in \mathbb{C}_+} \|R(\lambda, A)\| < \infty.$$

If this is the case, $\mathrm{s}(A)$ *is negative.*

Proof. The necessity of the conditions and the addendum follow from Proposition 1.20. Let the conditions in display be true. We set $\omega^+ = \max\{0, \omega_0(A)\}$. Let $\omega > \omega^+$, $\tau \in \mathbb{R}$, and $x \in X$. We define $T_\omega(\cdot)$ as above and abbreviate $r_\omega(\tau) = R(\omega + \mathrm{i}\tau, A)x$.

Fix $\overline{\omega} > \omega^+$ and take $\omega \in (\omega^+, \overline{\omega}]$. There exist constants $M \geq 1$ and $\varepsilon > 0$ with $\|T(t)\| \leq M\mathrm{e}^{(\overline{\omega}-\varepsilon)t}$ for all $t \geq 0$, and so $T_{\overline{\omega}}(\cdot)x$ is an element of $L^1(\mathbb{R}, X) \cap L^2(\mathbb{R}, X)$ with 2-norm less or equal $M(2\varepsilon)^{-1/2}\|x\|$. Using $T_\omega(\cdot) \in L^2(\mathbb{R}, X)$ in the last step, as in (4.2)–(4.4) we thus obtain

$$\begin{aligned}
\|r_{\overline{\omega}}\|_{L^2(\mathbb{R},X)} &\leq M\sqrt{\pi/\varepsilon}\|x\|,\\
\|r_\omega\|_{L^2(\mathbb{R},X)} &\leq \|r_{\overline{\omega}}\|_{L^2(\mathbb{R},X)} + |\overline{\omega} - \omega| \sup_{\tau\in\mathbb{R}} \|R(\omega + \mathrm{i}\tau, A)\| \, \|r_{\overline{\omega}}\|_{L^2(\mathbb{R},X)}\\
&\leq M\sqrt{\pi/\varepsilon}(1 + \overline{\omega}C)\|x\| =: (2\pi)^{\frac{1}{2}}\overline{C}\|x\|,\\
\|T_\omega(\cdot)x\|_{L^2(\mathbb{R},X)} &= \tfrac{1}{\sqrt{2\pi}}\|r_\omega\|_{L^2(\mathbb{R},X)} \leq \overline{C}\|x\|.
\end{aligned}$$

Fatou's lemma then yields

$$\|T_{\omega^+}(\cdot)x\|^2_{L^2(\mathbb{R},X)} = \int_0^\infty \lim_{\omega\to\omega^+} \mathrm{e}^{-2\omega t}\|T(t)x\|^2 \mathrm{d}t \leq \varliminf_{\omega\to\omega^+} \|T_\omega(\cdot)x\|^2_{L^2(\mathbb{R},X)} \leq \overline{C}^2\|x\|^2.$$

Datko's Lemma 4.4 now implies that $(T_{\omega^+}(t))_{t\geq 0}$ is exponentially stable. This is impossible in the case $\omega^+ = \omega_0(A)$, so that $\omega_0(A)$ has to be negative. □

In a general (complex) Banach space X the boundedness of the resolvent $R(\cdot, A)$ on $\mathbb{C}_+$ only implies the existence of some constants $M, \varepsilon > 0$ such that

$$\|T(t)x\| \leq M\mathrm{e}^{-\varepsilon t}\|x\|_A \tag{4.5}$$

for all $t \geq 0$ and $x \in \mathrm{D}(A)$ by a theorem from 1996 due to Weis and Wrobel (which also gives improved estimates for certain classes of Banach spaces), see Proposition 5.1.6 and Theorem 5.1.7 in [**3**]. We thus obtain exponential decay of classical solutions only. In Example 4.3, the resolvent of $A + \omega I$ is bounded on $\mathbb{C}_+$ by Theorem 5.3. There are generators A on Hilbert spaces with $\mathrm{s}(A) < 0$ such that (4.5) fails, see Remark 5.5.

Theorem 4.5 is of great importance for applications. We treat damped wave equations in two (relatively simple) examples, checking the resolvent estimate by a contradiction argument. The first one deals with a strictly positive damping.

Example 4.6. We first recall the setting and the results of Example 3.7. Let $G \subseteq \mathbb{R}^3$ be bounded and open with $\partial G \in C^{1-}$, Δ_D be the Dirichlet–Laplacian on $L^2(G)$ from Example 1.54, and $b \in L^\infty(G)$ satisfy $b(x) \geq \beta$ for almost every $x \in G$ and some $\beta > 0$. We set $E = Y \times L^2(G)$, where $Y = W^{1,2}_0(G)$ is endowed with the norm $\|v\|_Y = \||\nabla v|_2\|_2$, and define the operator

$$A = \begin{pmatrix} 0 & I \\ \Delta_D & -b \end{pmatrix} =: A_0 + \begin{pmatrix} 0 & 0 \\ 0 & -b \end{pmatrix} \qquad \text{with } \mathrm{D}(A) = \mathrm{D}(\Delta_D) \times Y$$

on E. It generates a C_0-group $T(\cdot)$ solving the damped wave equation

$$u''(t) = \Delta_D u(t) - bu'(t), \quad t \geq 0, \qquad u(0) = u_0, \quad u'(0) = u_1. \tag{4.6}$$

More precisely, for $(u_0, u_1) \in E$ the orbit $w(t) = T(t)(u_0, u_1)$ has the form $w = (u, u')$ for the unique solution u of (4.6) in $C^2(\mathbb{R}_{\geq 0}, W^{-1,2}(G)) \cap$

$C^1(\mathbb{R}_{\geq 0}, L^2(G)) \cap C(\mathbb{R}_{\geq 0}, Y)$. Here we consider the operator Δ_D also as a map from $Y = W_0^{1,2}(G)$ to $W^{-1,2}(G)$.

Recall from Example 3.7 that the semigroup $(T(t))_{t\geq 0}$ is contractive since $-b \leq 0$. We assert that it is exponentially stable, and thus the energy

$$\tfrac{1}{2}\|T(t)(u_0,u_1)\|_E^2 = \tfrac{1}{2}\||\nabla u|_2\|_2^2 + \tfrac{1}{2}\|\partial_t u(t)\|_2^2$$

of the solution decays as $ce^{-2\varepsilon t}\|(u_0,u_1)\|_E^2$ for some $c, \varepsilon > 0$. This claim is proved by means of Theorem 4.5.

To this end, we first we first note that A is invertible with bounded inverse

$$A^{-1}\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \Delta_D^{-1}(bf+g) \\ f \end{pmatrix}, \qquad (f,g) \in E.$$

Below we show that

$$i\mathbb{R} \subseteq \rho(A) \qquad \text{and} \qquad \sup_{\tau\in\mathbb{R}} \|R(i\tau, A)\| =: \kappa < \infty. \tag{4.7}$$

In view of Remark 1.16, by inequality (4.7) each number $\lambda \in \mathbb{C}$ with $|\operatorname{Re}\lambda| \in \left[0, \frac{1}{2\kappa}\right]$ is an element of $\rho(A)$ and the resolvent is bounded by $\|R(\lambda, A)\| \leq 2\kappa$. Due this bound and the Hille–Yosida estimate (1.15), the assumptions of Theorem 4.5 are fulfilled and the assertion follows.

We establish (4.7). Since $s(A) \leq 0$, any point $i\tau \in \sigma(A)$ would belong to $\partial\sigma(A)$ so that Proposition 1.19 of [**27**] (or (4.17) below) would yield

$$m(\tau) := \inf\left\{\|i\tau w - Aw\|_E \,\middle|\, w \in D(A),\ \|w\|_E = 1\right\} = 0.$$

Note that $\|R(i\tau, A)\| \leq 1/m(\tau)$ if $m(\tau) > 0$. Therefore the lower bound $\inf_{\tau\in\mathbb{R}} m(\tau) =: m_0 > 0$ will imply our claim (4.7) with $\kappa = 1/m_0$.

Since $0 \in \rho(A)$ and $\rho(A)$ is open, there is a number $\tau_0 > 0$ such that $[-i\tau_0, i\tau_0] \subseteq \rho(A)$. For $\tau \in [-\tau_0, \tau_0]$ and $w \in D(A)$ with $\|w\|_E = 1$, we set $i\tau w - Aw = z$ and obtain the first lower bound

$$\|i\tau w - Aw\|_E = \|z\|_E \geq \|R(i\tau, A)\|^{-1}\,\|R(i\tau, A)z\|_E = \|R(i\tau, A)\|^{-1},$$

$$\inf_{|\tau|\leq\tau_0} m(\tau) \geq \Big(\max_{|\tau|\leq\tau_0} \|R(i\tau, A)\|\Big)^{-1} =: \rho > 0. \tag{4.8}$$

Fix $\varepsilon \in (0, \min\{\rho, \frac{\beta}{2}\})$ with $0 < \frac{3\varepsilon\beta}{\beta-2\varepsilon} < \tau_0$. Suppose there are $\tau \in \mathbb{R}$ and $w = (\varphi, \psi) \in D(A)$ such that $\|w\|_E^2 = \||\nabla\varphi|_2\|_2^2 + \|\psi\|_2^2 = 1$ and $\|i\tau w - Aw\|_E \leq \varepsilon$. We infer $|\tau| \geq \tau_0$ from (4.8) and

$$\begin{aligned}
\varepsilon &\geq \left|\left((i\tau I - A)\tbinom{\varphi}{\psi} \,\middle|\, \tbinom{\varphi}{\psi}\right)_E\right| \\
&= \left|\int_G \nabla(i\tau\varphi - \psi)\cdot\nabla\overline{\varphi}\,dx + \int_G \left(-\Delta_D\varphi + (i\tau + b)\psi\right)\overline{\psi}\,dx\right| \\
&= \left|i\tau(\||\nabla\varphi|_2\|_2^2 + \|\psi\|_2^2) - \overline{\int_G \nabla\varphi\cdot\nabla\overline{\psi}\,dx} + \int_G \nabla\varphi\cdot\nabla\overline{\psi}\,dx + \int_G b|\psi|^2\,dx\right| \\
&= \left|i(\tau + 2\operatorname{Im}\int_G \nabla\varphi\cdot\nabla\overline{\psi}\,dx) + \int_G b|\psi|^2\,dx\right|,
\end{aligned}$$

using the definition of Δ_D. The imaginary and real parts thus satisfy

$$\varepsilon \geq \left|\tau + 2\operatorname{Im}\int_G \nabla\varphi\cdot\nabla\overline{\psi}\,dx\right| \quad \text{and} \quad \varepsilon \geq \int_G b|\psi|^2\,dx \geq \beta\|\psi\|_2^2.$$

The second estimate yields $\||\nabla\varphi|_2\|_2^2 = 1 - \|\psi\|_2^2 \geq 1 - \frac{\varepsilon}{\beta}$, and hence

$$1 - 2\||\nabla\varphi|_2\|_2^2 \leq \frac{2\varepsilon}{\beta} - 1 < 0$$

because of $\varepsilon < \frac{\beta}{2}$. We conclude that

$$\begin{aligned}|\tau|\left(1 - \frac{2\varepsilon}{\beta}\right) &\leq |\tau|\,\big|1 - 2\,\||\nabla\varphi|_2\|_2^2\big| = \Big|\tau + 2\,\mathrm{Im}\int_G \nabla\varphi\cdot\overline{\mathrm{i}\tau\nabla\varphi}\,\mathrm{d}x\Big|\\ &\leq \Big|\tau + 2\,\mathrm{Im}\int_G \nabla\varphi\cdot\nabla\overline{\psi}\,\mathrm{d}x\Big| + \Big|2\,\mathrm{Im}\int_G \nabla\varphi\cdot\big(\overline{\mathrm{i}\tau\nabla\varphi - \nabla\psi}\big)\,\mathrm{d}x\Big|\\ &\leq \varepsilon + 2\||\nabla\varphi|_2\|_2\,\||\nabla(\mathrm{i}\tau\varphi - \psi)|_2\|_2 \leq \varepsilon + 2\|(\mathrm{i}\tau I - A)w\|_E \leq 3\varepsilon\end{aligned}$$

by the choice of $w = (\varphi, \psi)$ and the definition of A. It follows $|\tau| \leq \frac{3\varepsilon\beta}{\beta - 2\varepsilon} < \tau_0$. This contradiction yields $m(\tau) \geq \varepsilon > 0$ for $|\tau| \geq \tau_0$, as needed for (4.7). ♢

Above the damping b acts everywhere in space (but only on the second component of the state (u, u')). It is an intriguing question for which $b \geq 0$ the solutions to the wave equation (4.6) tend to 0 exponentially, as the semigroup is unitary if $b = 0$. In general this is a difficult problem beyond the scope of these lectures. But on an interval we can show exponentially stability only assuming that $0 \leq b \in W^{1,\infty}$ is non-zero. The proof is taken from Theorem 3.2.1 in [**17**].

Example 4.7. In Example 4.6 we take $G = (0, 1)$ and a non-zero damping $0 \leq b \in W^{1,\infty}(0, 1)$. We define E, A and $T(\cdot)$ as in Example 4.6. Then $T(\cdot)$ is exponentially stable.

Proof. As in the previous example the claim (4.7) implies $\omega_0(A) < 0$. Now we have $\mathrm{D}(\Delta_D) = W^{2,2}(0, 1) \cap W_0^{1,2}(0, 1)$ by Example 1.48. So $\mathrm{D}(A)$ is compactly embedded into E and thus has only point spectrum, see Theorem 3.34, Remark 2.13 and Theorem 2.15 of [**27**]. Like in Example 4.6 one sees that A is invertible. Note that $W^{1,\infty}(0, 1) = C^{1-}([0, 1])$ by Proposition 3.24 in [**27**].

1) Let $w = (u, v) \in \mathrm{D}(A)$ be an eigenvector for the eigenvalue $\mathrm{i}\tau \in \mathrm{i}\mathbb{R}$. Integrating by parts, we compute

$$\begin{aligned}0 = \big(\mathrm{i}\tau w - Aw\big|w\big)_E &= \mathrm{i}\tau\int_0^1 (|u'|^2 + |v|^2)\,\mathrm{d}s - \int_0^1 (v'\overline{u}' + u''\overline{v})\,\mathrm{d}s + \int_0^1 b|v|^2\,\mathrm{d}s\\ &= \mathrm{i}\tau\int_0^1 (|u'|^2 + |v|^2)\,\mathrm{d}s + 2\mathrm{i}\,\mathrm{Im}\int_0^1 u'\overline{v}'\,\mathrm{d}s + \int_0^1 b|v|^2\,\mathrm{d}s\end{aligned}$$

using $v(0) = 0 = v(1)$. As the real part of the right-hand side, the last integral is 0 which implies $b|v|\,|v| = 0$ and thus $bv = 0$. The eigenvalue equation then yields $\mathrm{i}\tau u = v$ and $u'' = \mathrm{i}\tau v = -\tau^2 u$, so that $|\tau| =: n \in \mathbb{N}$ and $u(s) = c\sin(n\pi s)$ for some $c \neq 0$ by the 0-boundary condition. It follows $0 = b(s)v(s) = \mathrm{i}c\tau b(s)\sin(n\pi s)$ for all $s \in (0, 1)$ which is impossible since $\tau \neq 0$ and b is a non-zero continuous function. Hence, $\mathrm{i}\mathbb{R}$ belongs to the resolvent set of A.

2) Assume that $\|R(\mathrm{i}\tau, A)\|$ was unbounded for $\tau \in \mathbb{R}$. Then there are numbers $\tau_n \in \mathbb{R}$ with $|\tau_n| \to \infty$ and elements $w_n = (u_n, v_n) \in \mathrm{D}(A)$ with $\|u_n'\|_2^2 + \|v_n\|_2^2 = 1$ for all $n \in \mathbb{N}$ and $(f_n, g_n) := \mathrm{i}\tau_n w_n - Aw_n \to 0$ in E as $n \to \infty$. We have

$$f_n = \mathrm{i}\tau_n u_n - v_n \qquad \text{and} \qquad g_n = \mathrm{i}\tau_n v_n - u_n'' + bv_n. \tag{4.9}$$

Hence, the products $\tau_n u_n = -\mathrm{i}(v_n + f_n)$ are uniformly bounded in $L^2(0,1)$ and thus (u_n) tends to 0 in $L^2(0,1)$. Employing that $((f_n, g_n)|w_n)_E$ converges to 0, as in part 1) we infer the limit $\int b|v_n|^2\,\mathrm{d}s \to 0$. It leads to $bv_n \to 0$ in $L^2(0,1)$ as $n \to \infty$ since b is bounded. The equations (4.9) implies

$$-u_n'' - \tau_n^2 u_n = \mathrm{i}\tau_n f_n + g_n - bv_n$$

for $n \in \mathbb{N}$. Pick a map $\varphi \in C^1([0,1],\mathbb{R})$. We multiply the above identity by $\varphi\overline{u}_n'$, integrate over $(0,1)$, and take the real part. Note that $2\,\mathrm{Re}(u_n''\overline{u}_n') = \mathrm{d}/\mathrm{d}s\,|u_n'|^2$ and $2\,\mathrm{Re}(u_n\overline{u}_n') = \mathrm{d}/\mathrm{d}s\,|u_n|^2$. Integration by parts then yields

$$\begin{aligned}\int_0^1 \varphi'(|u_n'|^2 + \tau_n^2|u_n|^2)\,\mathrm{d}s - \varphi|u_n'|^2\big|_0^1 &= 2\,\mathrm{Re}\int_0^1 (g_n - bv_n + \mathrm{i}\tau_n f_n)\varphi\overline{u}_n'\,\mathrm{d}s\\ &= 2\,\mathrm{Re}\int_0^1 \big[(g_n - bv_n)\varphi\overline{u}_n' - \mathrm{i}(\varphi f_n)'\tau_n\overline{u}_n\big]\,\mathrm{d}s\end{aligned}$$

because of $u_n(0) = 0 = u_n(1)$. The right-hand side converges to 0 as $n \to \infty$ by the above observations and Hölder.

We now take special φ to exploit the above limit. Since $\tau_n^2|u_n|^2 = |v_n + f_n|^2$ the integral on the left tends to α if $\varphi' = \alpha\mathbb{1}$. Choosing $\varphi(s) = s$ and $\varphi(s) = 1 - s$, we first deduce $|u_n'(1)|^2 \to 1$ and $|u_n'(0)|^2 \to 1$. Finally, we set $\varphi(s) = \int_0^s b(\sigma)\,\mathrm{d}\sigma$ and $\beta = \varphi(1) > 0$. Using $\varphi(0) = 0$ and $\varphi' = b$, we infer that

$$\beta = \lim_{n\to\infty}\int_0^1 b(|u_n'|^2 + \tau_n^2|u_n|^2)\,\mathrm{d}s = \lim_{n\to\infty}\Big[\int_0^1 b|u_n'|^2\,\mathrm{d}s + \int_0^1 |b^{\frac12}v_n + b^{\frac12}f_n|^2\,\mathrm{d}s\Big]. \tag{4.10}$$

Similar as above the last integral tends to 0 as $n \to \infty$. To treat the penultimate one, we multiply the second equation in (4.9) by $b\overline{u}_n$ and integrate over $(0,1)$. Integration by parts yields

$$\int_0^1 \big[g_n b\overline{u}_n - \mathrm{i}\tau_n\overline{u}_n bv_n - bv_n b\overline{u}_n\big]\,\mathrm{d}s = -\int_0^1 u_n'' b\overline{u}_n\,\mathrm{d}s = \int_0^1 b|u_n'|^2\,\mathrm{d}s + \int_0^1 b'u_n'\overline{u}_n\,\mathrm{d}s.$$

The previous results and Hölder imply that the first and last integral tend 0 as $n \to \infty$. So the same is true for the third one, which yields the contradiction $\beta = 0$ via (4.10). □

We next introduce a more sophisticated concept for the long-time behavior.

Definition 4.8. *A C_0-semigroup $T(\cdot)$ has an* exponential dichotomy *if there are constants $N, \delta > 0$ and a projection $P = P^2 \in \mathcal{B}(X)$ such that $T(t)P = PT(t)$, $T(t) : \mathrm{N}(P) \to \mathrm{N}(P)$ has an inverse denoted by $T_u(-t)$, and we have the estimates $\|T(t)P\| \le N\mathrm{e}^{-\delta t}$ and $\|T_u(-t)(I - P)\| \le N\mathrm{e}^{-\delta t}$ for all $t \ge 0$.*

Clearly, exponential dichotomy coincides with exponential stability if $P = I$. Setting $Q = I - P$, we recall from Lemma 2.16 in [**24**] that $Q = Q^2$ and that $X_s := PX = \mathrm{N}(Q)$ and $X_u := QX = \mathrm{N}(P)$ are closed with $X_s \oplus X_u = X$. Exponential dichotomy then means that $T(t)X_j \subseteq X_j$ for all $t \ge 0$ and $j = \{s, u\}$, that $T_s(\cdot) := T(\cdot)|_{X_s}$ is an exponentially stable C_0-semigroup on X_s, and that $T(\cdot)$ induces a C_0-group $T_u(\cdot)$ on X_u which is exponentially stable in backward time. (Use Lemma 1.28 for the group property.)

We first characterize this notion in terms of the spectrum of $T(t)$.

PROPOSITION 4.9. *A C_0-semigroup $T(\cdot)$ has an exponential dichotomy if and only if $\mathbb{S}^1 := \{\lambda \in \mathbb{C} \,|\, |\lambda| = 1\} \subseteq \rho(T(t))$ for some (and hence all) $t > 0$.*

PROOF. Let $T(\cdot)$ have an exponential dichotomy. Take $t > 0$ and $\lambda \in \mathbb{S}^1$. Then the series

$$R_\lambda = \lambda^{-1} \sum_{n=0}^{\infty} \lambda^{-n} T(nt) P - \lambda^{-1} \sum_{n=1}^{\infty} \lambda^{n} T_u(-nt) Q$$

converges in $\mathcal{B}(X)$. We then compute

$$\begin{aligned}(\lambda I - T(t)) R_\lambda &= \Big(I - \lambda^{-1} T(t)\Big) \left(\sum_{n=0}^{\infty} \Big(\lambda^{-1} T(t)\Big)^n P - \sum_{n=1}^{\infty} \Big(\lambda^{-1} T_u(t)\Big)^{-n} Q \right) \\ &= \sum_{n=0}^{\infty} \Big(\lambda^{-1} T(t)\Big)^n P - \sum_{k=1}^{\infty} \Big(\lambda^{-1} T(t)\Big)^k P \\ &\qquad - \sum_{n=1}^{\infty} \Big(\lambda^{-1} T_u(t)\Big)^{-n} Q + \sum_{k=0}^{\infty} \Big(\lambda^{-1} T_u(t)\Big)^{-k} Q \\ &= P + Q = I.\end{aligned}$$

Similarly one sees that $R_\lambda(\lambda I - T(t)) = I$, and hence $\mathbb{S}^1$ belongs to $\rho(T(t))$.

Conversely, let $\mathbb{S}^1 \subseteq \rho(T(t))$ for some $t > 0$. We define the 'spectral projection'

$$P := \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{S}^1} R(\lambda, T(t)) \,\mathrm{d}\lambda.$$

Theorem 5.5 in [**27**] shows that $P^2 = P \in \mathcal{B}(X)$, $\sigma(T_s(t)) = \sigma(T(t)) \cap B(0,1)$, and $\sigma(T_u(t)) = \sigma(T(t)) \setminus \overline{B}(0,1)$ for all $t > 0$. Moreover, $T(\tau)$ commutes with P for all $\tau \geq 0$ since the same is true for $T(t)$ and thus its resolvent. Because of $\mathrm{r}(T_s(t)) < 1$, Proposition 4.2 yields the exponential stability of $T_s(\cdot)$ on PX. Moreover, $T_u(t)$ is invertible and $\sigma(T_u(t)^{-1}) = \sigma(T_u(t))^{-1} \subseteq B(0,1)$ by Proposition 1.20 in [**27**]. As for $T_s(\cdot)$, we infer that $(T_u(t)^{-1})_{t\geq 0}$ is exponentially stable on QX. Consequently, $T(\cdot)$ has an exponential dichotomy. ☐

In Corollary 4.17 and Theorem 4.19 we characterize exponential dichotomy in terms of A in certain situations. Here we give a typical implication of this property to the long-time behavior of inhomogeneous problems.

PROPOSITION 4.10. *Let A generate the C_0-semigoup $T(\cdot)$ having an exponential dichotomy with projections P and $Q = I - P$. Assume that $u_0 \in X$ and $f \in C_0(\mathbb{R}_{\geq 0}, X)$ satisfy*

$$Qu_0 = -\int_0^{\infty} T_u(-t) Q f(t) \,\mathrm{d}t.$$

Then the mild solution u of the inhomogeneous problem (2.5) on $\mathbb{R}_{\geq 0}$ also belongs to $C_0(\mathbb{R}_{\geq 0}, X)$ and fulfills

$$u(t) = T(t) P u_0 + \int_0^t T(t-s) P f(s) \,\mathrm{d}s - \int_t^{\infty} T_u(t-s) Q f(s) \,\mathrm{d}s, \quad t \geq 0. \tag{4.11}$$

Proof. Let $t \geq 0$. We first note that the integrals in the displayed equations above and those below exist because of the exponential dichotomy. Using Duhamel's formula (2.6), $P + Q = I$ and the assumption, we compute

$$\begin{aligned} u(t) &= T(t)Pu_0 + T_u(t)Qu_0 + \int_0^t T(t-s)Pf(s)\,\mathrm{d}s + \int_0^t T_u(t-s)Qf(s)\,\mathrm{d}s \\ &= T(t)Pu_0 - \int_0^\infty T_u(t-s)Qf(s)\,\mathrm{d}s + \int_0^t T(t-s)Pf(s)\,\mathrm{d}s \\ &\quad + \int_0^t T_u(t-s)Qf(s)\,\mathrm{d}s \end{aligned}$$

so that (4.11) is true.

Let $\varepsilon > 0$. There is a time $s_0 \geq 0$ such that $\|f(s)\| \leq \varepsilon$ for all $s \geq s_0$. Let $t \geq s_0$. Formula (4.11) and the exponential dichotomy lead to the estimate

$$\begin{aligned} \|u(t)\| &\leq N\mathrm{e}^{-\delta t}\|u_0\| + \int_0^{s_0} N\mathrm{e}^{-\delta(t-s)}\|f\|_\infty\,\mathrm{d}s + \int_{s_0}^t N\mathrm{e}^{-\delta(t-s)}\varepsilon\,\mathrm{d}s \\ &\quad + \int_t^\infty N\mathrm{e}^{-\delta(s-t)}\varepsilon\,\mathrm{d}s \\ &\leq N\mathrm{e}^{-\delta t}\big(\|u_0\| + \delta^{-1}(\mathrm{e}^{\delta s_0} - 1)\|f\|_\infty\big) + 2N\delta^{-1}\varepsilon, \end{aligned}$$

which easily implies that $u(t) \to 0$ as $t \to \infty$. □

4.2. Spectral mapping theorems

Let A generate the C_0-semigroup $T(\cdot)$. We say that $T(\cdot)$ or A satisfy the *spectral mapping theorem* if

$$\sigma(T(t)) \setminus \{0\} = \mathrm{e}^{t\sigma(A)} \qquad \text{for all } \; t \geq 0, \tag{4.12}$$

where we put $\mathrm{e}^{t\emptyset} := \emptyset$ for $t > 0$ and $\mathrm{e}^{0\emptyset} := \{1\}$. Observe that we have to exclude 0 on the left-hand side since 0 does not belong to $\mathrm{e}^{t\sigma(A)}$. Theorem 5.3 of [27] shows the identity $\sigma(T(t)) = \mathrm{e}^{t\sigma(A)}$ for $A \in \mathcal{B}(X)$.

Assume for a moment that (4.12) is true for $T(\cdot)$. It follows

$$\begin{aligned} &\mathrm{r}(T(t)) = \max\big\{|\mathrm{e}^{t\mu}| \,\big|\, \mu \in \sigma(A)\big\} = \max\big\{\mathrm{e}^{t\operatorname{Re}\mu} \,\big|\, \mu \in \sigma(A)\big\} = \mathrm{e}^{t\mathrm{s}(A)}, \\ &\omega_0(A) = \mathrm{s}(A), \qquad \omega_0(A) < 0 \iff \mathrm{s}(A) < 0, \end{aligned} \tag{4.13}$$

for all $t \geq 0$, where we employ Proposition 4.2 in the second line. Using also Proposition 4.9, we also deduce from (4.12) the equivalence

$$T(\cdot) \text{ has exp. dichotomy} \iff \mathrm{e}^{\mathrm{i}\mathbb{R}} = \mathbb{S}^1 \subseteq \rho(T(1)) \iff \mathrm{i}\mathbb{R} \subseteq \rho(A). \tag{4.14}$$

Example 4.3 thus tells us that the spectral mapping theorem is not valid for some C_0-semigroups. We first explore which partial results are still true. For this purpose, we recall the following concepts and results from spectral theory for a closed operator A. We define by

$$\begin{aligned} \sigma_{\mathrm{p}}(A) &= \big\{\lambda \in \mathbb{C} \,\big|\, \lambda I - A \;\text{ is not injective}\big\}, \\ \sigma_{\mathrm{ap}}(A) &= \big\{\lambda \in \mathbb{C} \,\big|\, \forall\, n \in \mathbb{N}\ \exists\, x_n \in \mathrm{D}(A) : \|x_n\| = 1,\ \lambda x_n - Ax_n \to 0\ (n \to \infty)\big\}, \\ \sigma_{\mathrm{r}}(A) &= \big\{\lambda \in \mathbb{C} \,\big|\, (\lambda I - A)\mathrm{D}(A) \;\text{ is not dense in } X\big\} \end{aligned}$$

the *point spectrum*, the *approximate point spectrum* and the *residual spectrum* of A, respectively. We call the elements of $\sigma_{\mathrm{ap}}(A)$ *approximate eigenvalues* and the corresponding x_n *approximate eigenvectors.* Proposition 1.19 of [**27**] shows

$$\sigma_{\mathrm{ap}}(A) = \sigma_p(A) \cup \big\{\lambda \in \mathbb{C} \,\big|\, (\lambda I - A)\mathrm{D}(A) \text{ is not closed in } X\big\}, \tag{4.15}$$

$$\sigma(A) = \sigma_{\mathrm{ap}}(A) \cup \sigma_{\mathrm{r}}(A), \tag{4.16}$$

$$\partial\sigma(A) \subseteq \sigma_{\mathrm{ap}}(A). \tag{4.17}$$

Let A be also densely defined. Theorem 1.24 of [**27**] then says that

$$\sigma_{\mathrm{r}}(A) = \sigma_{\mathrm{p}}(A^*), \qquad \sigma(A) = \sigma(A^*), \qquad \text{and} \qquad R(\lambda, A)^* = R(\lambda, A^*) \tag{4.18}$$

for $\lambda \in \rho(A)$. The following *spectral inclusion theorem* provides the easy inclusion in (4.12) and in related formulas for the parts of the spectrum.

Proposition 4.11. *Let A generate the C_0-semigroup $T(\cdot)$, and $t \geq 0$. We then have*

$$\mathrm{e}^{t\sigma(A)} \subseteq \sigma(T(t)) \qquad \textit{and} \qquad \mathrm{e}^{t\sigma_j(A)} \subseteq \sigma_j(T(t)) \quad \textit{for} \;\; j \in \{p, ap, r\}.$$

(Approximate) Eigenvectors of A for the (approximate) eigenvalue λ are (approximate) eigenvectors of $T(t)$ for the (approximate) eigenvalue $\mathrm{e}^{t\lambda}$.

Proof. Let $\lambda \in \mathbb{C}$ and $t \geq 0$. In view of (4.16), we only have to treat the parts σ_j. Recall from Lemma 1.18 that

$$\begin{aligned} \mathrm{e}^{\lambda t}x - T(t)x &= (\lambda I - A)\int_0^t \mathrm{e}^{\lambda(t-s)}T(s)x\,\mathrm{d}s \qquad \text{for } x \in X,\\ &= \int_0^t \mathrm{e}^{\lambda(t-s)}T(s)(\lambda x - Ax)\,\mathrm{d}s \qquad \text{for } x \in \mathrm{D}(A). \end{aligned}$$

Hence, if $\lambda x = Ax$ for some $x \in \mathrm{D}(A) \setminus \{0\}$, then $\mathrm{e}^{\lambda t}x = T(t)x$ and x is an eigenvector of $T(t)$ for the eigenvalue $\mathrm{e}^{\lambda t} \in \sigma_{\mathrm{p}}(T(t))$. If $(\lambda I - A)\mathrm{D}(A)$ is not dense in or not equal to X, then $\mathrm{R}(\mathrm{e}^{\lambda t}I - T(t))$ has the same property. Finally, let x_n be approximate eigenvectors of A for $\lambda \in \sigma_{\mathrm{ap}}(A)$. It follows that

$$\|\mathrm{e}^{\lambda t}x_n - T(t)x_n\| \leq c\|\lambda x_n - Ax_n\| \;\longrightarrow\; 0$$

as $n \to \infty$ so that x_n are approximate eigenvectors for $\mathrm{e}^{\lambda t} \in \sigma_{ap}(T(t))$. □

We have thus shown the inequality $\mathrm{s}(A) \leq \omega_0(A)$ from Proposition 4.2 again. We also obtain the analogous implication for exponential dichotomy.

Corollary 4.12. *Let A generate the C_0-semigroup $T(\cdot)$ having an exponential dichotomy. We then have $\mathrm{i}\mathbb{R} \subseteq \rho(A)$ since $\mathrm{e}^{\mathrm{i}\mathbb{R}} = \mathbb{S}^1 \subseteq \rho(T(1)) \subseteq \mathbb{C} \setminus \mathrm{e}^{\sigma(A)}$ by Propositions 4.9 and 4.11.*

In the following example we use the spectral inclusion to compute the spectra of the translation semigroup on 1-periodic functions. Here the spectral mapping theorem fails for irrational t, but a variant with an additional closure holds. Moreover, the spectrum of $T(t)$ changes drastically under arbitrarily small perturbations of t.

EXAMPLE 4.13. Let $X = \{f \in C(\mathbb{R}) \,|\, \forall\, t \in \mathbb{R} : f(t) = f(t+1)\}$ be endowed with the supremum norm and $T(t)f = f(\cdot + t)$ for $t \in \mathbb{R}$ and $f \in X$. It is easy to see that X is a Banach space and that $T(\cdot)$ is an isometric C_0-group on X (since each $f \in X$ is uniformly continuous). As in Example 1.21 one can verify that the generator A of $T(\cdot)$ is given by $Af = f'$ with $\mathrm{D}(A) = C^1(\mathbb{R}) \cap X$. Let $\Gamma_k = \{\lambda \in \mathbb{C} \,|\, \lambda^k = 1\}$ for $k \in \mathbb{N}$. We claim that

$$\sigma(A) = \sigma_{\mathrm{p}}(A) = 2\pi\mathrm{i}\mathbb{Z},$$

$$\sigma(T(t)) = \begin{cases} \mathbb{S}^1 = \overline{\exp(t\sigma(A))}, & t \in \mathbb{R}_{\geq 0} \setminus \mathbb{Q}, \\ \Gamma_k = \mathrm{e}^{t\sigma(A)}, & t = j/k,\ j,k \in \mathbb{N},\ \text{without common divisors.} \end{cases}$$

PROOF. Clearly, $e_{2\pi\mathrm{i}n}$ belongs to $\mathrm{D}(A)$ and $Ae_{2\pi\mathrm{i}n} = 2\pi\mathrm{i}ne_{2\pi\mathrm{i}n}$ for all $n \in \mathbb{Z}$. Note that $T(n) = I$ for all $n \in \mathbb{N}_0$. Proposition 4.11 thus yields $\mathrm{e}^{\sigma(A)} \subseteq \sigma(T(1)) = \{1\}$ so that $\sigma(A) \subseteq 2\pi\mathrm{i}\mathbb{Z}$. The first assertion is proved.

Since $T(t)$ is isometric and invertible, formula (4.1) implies that

$$\mathrm{r}(T(t)) = 1 = \mathrm{r}(T(t)^{-1}) = \min\big\{|\lambda| \,\big|\, \lambda \in \sigma(T(t))\big\},$$

where we also use Proposition 1.20 of [**27**]. This means that $\sigma(T(t))$ is included in $\mathbb{S}^1$ for $t \geq 0$. If $t \in \mathbb{R}_{\geq 0} \setminus \mathbb{Q}$, it is known that $\mathrm{e}^{t\sigma(A)} = \mathrm{e}^{t2\pi\mathrm{i}\mathbb{Z}}$ is dense in $\mathbb{S}^1$. The second claim then follows from Proposition 4.11 and the closedness of the spectra because of

$$\mathbb{S}^1 = \overline{\mathrm{e}^{t\sigma(A)}} \subseteq \sigma(T(t)) \subseteq \mathbb{S}^1.$$

Let $t = j/k$ for some $j,k \in \mathbb{N}$ without common divisors. The spectral mapping theorem for bounded operators from Theorem 5.3 of [**27**] then yields

$$\sigma(T(t))^k = \sigma\big(T\big(\tfrac{j}{k}\big)^k\big) = \sigma(T(j)) = \{1\};$$

i.e., $\sigma(T(t)) \subseteq \Gamma_k$. On the other hand, the set $\mathrm{e}^{t\sigma(A)} = \exp(2\pi\mathrm{i}\frac{j}{k}\mathbb{Z})$ is equal to Γ_k and contained in $\sigma(T(t))$ by Proposition 4.11, establishing the last assertion. □

In order to use spectral information on A to show exponential stability or dichotomy, we need the converse inclusions in Proposition 4.11. As we have seen they fail in general for the spectrum itself. We next show them for σ_{p} and σ_{r}, starting with the *spectral mapping theorem for the point spectrum.* (In Example 4.13 it implies that $T(t)$ has not only point spectrum for $t \notin \mathbb{Q}$ though $\sigma(A) = \sigma_p(A)$.)

THEOREM 4.14. *Let A generate the C_0-semigroup $T(\cdot)$. We then have*

$$\sigma_{\mathrm{p}}(T(t)) \setminus \{0\} = \mathrm{e}^{t\sigma_{\mathrm{p}}(A)} \qquad \textit{for all} \ \ t \geq 0.$$

PROOF. We have to prove $\sigma_{\mathrm{p}}(T(t)) \setminus \{0\} \subseteq \mathrm{e}^{t\sigma_{\mathrm{p}}(A)}$ since the other inclusion was shown in Proposition 4.11. Let $t > 0$, $\lambda \in \mathbb{C}$ and $x \in X \setminus \{0\}$ such that $\mathrm{e}^{\lambda t}x = T(t)x$. Hence, the function $u(s) = \mathrm{e}^{-\lambda s}T(s)x$ has period $t > 0$. Suppose that all Fourier coefficients

$$\frac{1}{\sqrt{t}} \int_0^t \mathrm{e}^{-\frac{2\pi i n}{t}s} u(s)\,\mathrm{d}s, \qquad n \in \mathbb{Z},$$

would vanish. Therefore all Fourier coefficients of the scalar function $\varphi_{x^*}(s) = \langle u(s), x^* \rangle$ are 0 for any $x^* \in X^*$. Parseval's formula (see Example 3.17 of [**24**]) then yields $\varphi_{x^*} = 0$ for all y $x^* \in X^*$, and so $u = 0$ by the Hahn–Banach theorem. This is wrong and thus there exists an index $m \in \mathbb{Z}$ with

$$y := \int_0^t \mathrm{e}^{-\frac{2\pi \mathrm{i} m s}{t}} \mathrm{e}^{-\lambda s} T(s)x \,\mathrm{d}s \neq 0.$$

Lemma 1.18 shows that $y \in \mathrm{D}(A)$ and

$$\big(A - \big(\lambda + \tfrac{2\pi \mathrm{i} m}{t}\big)I\big)y = \mathrm{e}^{-\lambda t}\mathrm{e}^{-\frac{2\pi \mathrm{i} m}{t} t} T(t)x - x = 0.$$

Therefore the number $\mu := \lambda + \frac{2\pi \mathrm{i} m}{t}$ belongs to $\sigma_{\mathrm{p}}(A)$ and hence $\mathrm{e}^{\lambda t} = \mathrm{e}^{\mu t}$ to $\mathrm{e}^{t\sigma_{\mathrm{p}}(A)}$. We have shown $\sigma_{\mathrm{p}}(T(t)) \subseteq \mathrm{e}^{t\sigma_{\mathrm{p}}(A)}$, as needed. $\square$

Formula (4.18) now suggests to use duality and derive a spectral mapping theorem for σ_{r} from Theorem 4.14. Unfortunately, $T(\cdot)^*$ may fail to be strongly continuous. (For instance the adjoint $T(\cdot)^*$ of the left translations $T(\cdot)$ on $L^1(\mathbb{R})$ are the right translations on $L^\infty(\mathbb{R})$ which are not strongly continuous by Example 1.8.) To deal with this problem, we introduce a new concept.

1) Let A generate the C_0-semigroup $T(\cdot)$ and set $C = \sup_{0\le t\le 1} \|T(t)\|$. We define the *sun dual*

$$X^\odot = \big\{x^* \in X^* \,\big|\, T(t)^*x^* \to x^* \text{ as } t \to 0\big\}.$$

(One has $X^\odot = X^*$ if X is reflexive by Paragraph I.5.14 in [**7**] or an exercise.) We first check that $X^\odot$ is a closed subspace of X^* being invariant under $T(\cdot)^*$.[1]

Let $x_n^* \in X^\odot$ with $x_n^* \to x^*$ in X^* as $n \to \infty$. Take $\varepsilon > 0$. There is an index $k \in \mathbb{N}$ with $\|x_k^* - x^*\| \le \varepsilon$. We fix a time $t_\varepsilon \in (0, 1]$ such that $\|T(t)^*x_k^* - x_k^*\| \le \varepsilon$ for all $t \in [0, t_\varepsilon]$. Since $\|T(t)\| = \|T(t)^*\|$ by Proposition 5.42 of [**24**], it follows

$$\|T(t)^*x^* - x^*\| \le \|T(t)^*\|\,\|x^* - x_k^*\| + \|T(t)^*x_k^* - x_k^*\| + \|x_k^* - x^*\| \le (2 + C)\varepsilon,$$

so that $x^* \in X^\odot$ and $X^\odot$ is closed. Clearly, $T(\cdot)^*$ is a semigroup on X^*. Let $t, \tau \ge 0$ and $x^* \in X^\odot$. We then obtain the invariance of $X^\odot$ by computing

$$T(t)^*T(\tau)^*x^* - T(\tau)^*x^* = T(\tau)^*(T(t)^*x^* - x^*) \longrightarrow 0, \qquad t \to 0,$$

By Lemma 1.7, the operators $T(t)^\odot = T(t)^* \restriction_{X^\odot}$ for $t \ge 0$ thus form a C_0-semigroup on $X^\odot$, endowed with $\|\cdot\|_{X^*}$. Its generator is denoted by $A^\odot$.

2) We have to show that the point spectra of the duals and sun duals are the same. Let $x^* \in \mathrm{D}(A^\odot)$. Take $x \in \mathrm{D}(A)$. We derive $A^\odot \subseteq A^*$ from

$$\langle x, A^\odot x^*\rangle = \lim_{t\to 0} \big\langle x, \tfrac{1}{t}(T(t)^* - I)x^*\big\rangle = \lim_{t\to 0} \big\langle \tfrac{1}{t}(T(t) - I)x, x^*\big\rangle = \langle Ax, x^*\rangle,$$

As restrictions, the operators $A^\odot$ and $T(t)^\odot$ satisfy the inclusions

$$\sigma_{\mathrm{p}}(A^\odot) \subseteq \sigma_{\mathrm{p}}(A^*) \qquad \text{and} \qquad \sigma_{\mathrm{p}}(T(t)^\odot) \subseteq \sigma_{\mathrm{p}}(T(t)^*) \qquad \text{for } \ t \ge 0.$$

3) To show the converse relations, we first prove $\mathrm{D}(A^*) \subseteq X^\odot$. Let $x^* \in \mathrm{D}(A^*)$ and $t \in [0, 1]$. Lemma 1.18 yields

$$\|T(t)^*x^* - x^*\| = \sup_{x\in X, \|x\|\le 1} |\langle x, T(t)^*x^* - x^*\rangle| = \sup_{\|x\|\le 1} |\langle T(t)x - x, x^*\rangle|$$

$$= \sup_{\|x\|\le 1} \Big|\Big\langle A\int_0^t T(s)x \,\mathrm{d}s, x^*\Big\rangle\Big| = \sup_{\|x\|\le 1} \Big|\Big\langle \int_0^t T(s)x \,\mathrm{d}s, A^*x^*\Big\rangle\Big| \le C\|A^*x^*\|t.$$

[1] The next paragraph was omitted in the lectures, but treated in an earlier exercise.

This means that x^* belongs to $X^\odot$ and hence

$$\mathrm{D}(A^*) \subseteq X^\odot. \tag{4.19}$$

4) Let $T(t)^*x^* = \mathrm{e}^{\lambda t}x^*$ for some $x^* \in X^* \setminus \{0\}$, $\lambda \in \mathbb{C}$, and $t \geq 0$. Take $\mu \in \rho(A^*) = \rho(A)$, cf. (4.18). Note that $R(\mu, A)^* = R(\mu, A^*)$ is injective and maps X^* into $\mathrm{D}(A^*) \subseteq X^\odot$ and that it commutes with $T(t)^*$. Hence, $R(\mu, A^*)x^*$ is an eigenvector for $T(t)^\odot$ and the eigenvalue $\mathrm{e}^{\lambda t}$.

Let $x^* \in \mathrm{D}(A^*) \setminus \{0\}$ with $A^*x^* = \lambda x^*$. As above, we obtain the limit

$$\begin{aligned}\Big\|\frac{1}{t}(T(t)^\odot x^* - x^*) - \lambda x^*\Big\| &= \sup_{x\in X, \|x\|\leq 1} \Big|\Big\langle A\frac{1}{t}\int_0^t T(s)x\,\mathrm{d}s, x^*\Big\rangle - \langle x, \lambda x^*\rangle\Big| \\ &= \sup_{\|x\|\leq 1} \Big|\Big\langle x, \frac{1}{t}\int_0^t T(s)^*A^*x^*\,\mathrm{d}s - \lambda x^*\Big\rangle\Big| \\ &\leq \Big\|\frac{1}{t}\int_0^t \lambda T(s)^\odot x^*\,\mathrm{d}s - \lambda x^*\Big\| \;\longrightarrow\; 0\end{aligned}$$

as $t \to 0$, using $A^*x^* = \lambda x^*$ and (4.19). We have thus shown

$$\sigma_{\mathrm{p}}(A^\odot) = \sigma_{\mathrm{p}}(A^*) \qquad \text{and} \qquad \sigma_{\mathrm{p}}(T(t)^\odot) = \sigma_{\mathrm{p}}(T(t)^*) \qquad \text{for all } \ t \geq 0. \tag{4.20}$$

These equalities also hold for the full spectra. For this and further information we refer to Proposition IV.2.18 and §II.2.6 of [**7**].

We now easily obtain the *spectral mapping theorem for the residual spectrum.*

Theorem 4.15. *Let A generate the C_0-semigroup $T(\cdot)$. We then have*

$$\sigma_{\mathrm{r}}(T(t)) \setminus \{0\} = \mathrm{e}^{t\sigma_{\mathrm{r}}(A)} \qquad \textit{for all } \ t \geq 0.$$

Proof. Let $t \geq 0$. Combining (4.18), (4.20) and Theorem 4.14, we obtain

$$\begin{aligned}\sigma_{\mathrm{r}}(T(t)) \setminus \{0\} &= \sigma_{\mathrm{p}}(T(t)^*) \setminus \{0\} = \sigma_{\mathrm{p}}(T(t)^\odot) \setminus \{0\} = \mathrm{e}^{t\sigma_{\mathrm{p}}(A^\odot)} = \mathrm{e}^{t\sigma_{\mathrm{p}}(A^*)} \\ &= \mathrm{e}^{t\sigma_{\mathrm{r}}(A)}.\end{aligned}$$

□

As a result, the spectral mapping theorem can only fail if we are not able to transport approximate eigenvectors from $T(t)$ to A. This can be done if the semigroup has some additional regularity, as stated in the *spectral mapping theorem for eventually norm continuous semigroups* due to Phillips (1951). Besides analytic C_0-semigroups, this class includes various generators arising in retarded problems, see Example 4.18, and also in mathematical biology, see the comments before Theorem 5.8.

Theorem 4.16. *Let A generate the C_0-semigroup $T(\cdot)$ and let the map*

$$(t_0, \infty) \to \mathcal{B}(X); \quad t \mapsto T(t), \tag{4.21}$$

be continuous (in operator norm) for some $t_0 \geq 0$. Then $T(\cdot)$ satisfies the spectral mapping theorem

$$\sigma(T(t)) \setminus \{0\} = \mathrm{e}^{t\sigma(A)} \qquad \textit{for all } \ t \geq 0.$$

Assumption (4.21) is true if $T(\cdot)$ is analytic (then $t_0 = 0$) or if $T(t_0)$ is compact for some $t_0 > 0$.

Proof. Let $T(t_0)$ be compact. Then the closure of $T(t_0)\overline{B}_X(0,1)$ is compact. By an exercise in Functional Analysis, the map

$$[t_0,\infty) \to X;\ t \mapsto T(t)x = T(t-t_0)T(t_0)x,$$

thus is uniformly continuous for $x \in \overline{B}_X(0,1)$ and so (4.21) is true.

In view of Proposition 4.11, Theorem 4.15 and formula (4.16), it remains to show that $\sigma_{\mathrm{ap}}(T(t)) \setminus \{0\} \subseteq \mathrm{e}^{t\sigma_{\mathrm{ap}}(A)}$ for all $t > 0$. To this aim, let $\lambda \in \mathbb{C}$, $\tau > 0$ and $x_n \in X$ satisfy $\|x_n\| = 1$ for all $n \in \mathbb{N}$ and $\lambda x_n - T(\tau)x_n \to 0$ as $n \to \infty$. We look for a number $\mu \in \sigma_{\mathrm{ap}}(A)$ with $\lambda = \mathrm{e}^{\tau\mu}$. Considering the C_0-semigroup $(\mathrm{e}^{-\nu s}T(s\tau))_{s\geq 0}$ with $\lambda = \mathrm{e}^{\nu}$ and its generator $B = \tau A - \nu I$, see Lemma 1.17, we can assume that $\lambda = 1$, $\tau = 1$ and $\mu \in 2\pi\mathrm{i}\mathbb{Z}$.

Fix some $k \in \mathbb{N}$ with $k > t_0$. Let $n \in \mathbb{N}$. By (4.21), the maps $[0,1] \to X;\ s \mapsto T(s)T(k)x_n$, are continuous uniformly for n; i.e., equi-continuous. Moreover,

$$\|T(k)x_n - x_n\| \leq \|T(k-1)(T(1)x_n - x_n)\| + \dots + \|T(1)x_n - x_n\|$$

tends to 0 as $n \to \infty$. This fact implies that also the functions $[0,1] \to X$; $s \mapsto T(s)(T(k)x_n - x_n)$, are equi-continuous. Hence, the same is true for the differences $[0,1] \to X;\ s \mapsto T(s)x_n$.

Choose $x_n^* \in X^*$ such that $\|x_n^*\| \leq 1$ and $\langle x_n, x_n^*\rangle \geq \frac{1}{2}$ for all $n \in \mathbb{N}$, using the Hahn–Banach theorem. Since the functions $\varphi_n : [0,1] \to \mathbb{C};\ s \mapsto \langle T(s)x_n, x_n^*\rangle$, are equi-continuous and uniformly bounded, the Arzelà–Ascoli theorem (see Theorem 1.47 in [**24**]) provides a subsequence $(\varphi_{n_j})_j$ converging in $C([0,1])$ to a function φ. Observe that

$$\|\varphi\|_\infty \geq |\varphi(0)| = \lim_{j\to\infty} |\varphi_{n_j}(0)| = \lim_{j\to\infty} |\langle x_{n_j}, x_{n_j}^*\rangle| \geq \frac{1}{2}$$

showing that $\varphi \neq 0$. Example 3.17 of [**24**] thus implies that φ has a nonzero Fourier coefficient; i.e., there exists an index $m \in \mathbb{Z}$ such that for $\mu := 2\pi\mathrm{i}m$ we have $\int_0^1 \mathrm{e}^{-\mu s}\varphi(s)\,\mathrm{d}s \neq 0$. We now set $z_n = \int_0^1 \mathrm{e}^{-\mu s}T(s)x_n\,\mathrm{d}s$. Lemma 1.18 leads to $z_n \in \mathrm{D}(A)$ and

$$(\mu I - A)z_n = (I - \mathrm{e}^{-\mu}T(1))x_n = x_n - T(1)x_n \ \longrightarrow\ 0$$

as $n \to \infty$. We further compute

$$\begin{aligned}\liminf_{j\to\infty} \|z_{n_j}\| &\geq \liminf_{j\to\infty} |\langle z_{n_j}, x_{n_j}^*\rangle| = \liminf_{j\to\infty} \Big| \int_0^1 \mathrm{e}^{-\mu s}\langle T(s)x_{n_j}, x_{n_j}^*\rangle\,\mathrm{d}s\Big| \\ &= \Big|\int_0^1 \mathrm{e}^{-\mu s}\varphi(s)\,\mathrm{d}s\Big| > 0\end{aligned}$$

so that μ belongs to $\sigma_{\mathrm{ap}}(A)$, as desired. □

The above theorem yields the desired characterizations (4.13) and (4.14).

Corollary 4.17. *Let A generate the C_0-semigroup $T(\cdot)$ satisfying* (4.21). *Then the following equivalences hold.*

a) The semigroup $T(\cdot)$ is exponentially stable if and only if $\mathrm{s}(A) < 0$.

b) The semigroup $T(\cdot)$ has an exponential dichotomy if and only if $\mathrm{i}\mathbb{R} \subseteq \rho(A)$.

In Example 3.16 of [**26**] we apply statement a) to a reaction-diffusion system. Here we discuss a simpler example.

EXAMPLE 4.18. Let $A, B \in \mathbb{C}^{m\times m}$. For a given 'pre-history' $\phi \in X = C([-1,0],\mathbb{C}^m)$ we consider the *delay equation*

$$u'(t) = Au(t) + Bu(t-1), \quad t \geq 0, \qquad u(\theta) = \phi(\theta), \quad \theta \in [-1,0]. \tag{4.22}$$

It can easily be solved as the solution is given iteratively by Duhamel's formula on intervals $[n, n+1]$ for $n \in \mathbb{N}_0$. (For more general retardation terms as in Section VI.6 of [7], one can use fixed-point arguments.) Since we need semigroup theory to study the long-time behavior, we instead look at the operator

$$L\phi = \phi' \quad \text{ with } \quad \mathrm{D}(L) = \big\{\phi \in C^1([-1,0],\mathbb{C}^m) \,\big|\, \phi'(0) = A\phi(0) + B\phi(-1)\big\}.$$

On the space X of 'history functions' $\theta \mapsto v_t(\theta) := v(t+\theta)$ it generates a C_0-semigroup $T(\cdot)$ having the translation property

$$(T(t)\phi)(\theta) = \begin{cases} \phi(t+\theta), & t+\theta \leq 0, \\ (T(t+\theta)\phi)(0), & t+\theta > 0, \end{cases}$$

for $t \geq 0$ and $\theta \in [-1,0]$. Let $\phi \in \mathrm{D}(L)$. Then $u(t) = \phi(t)$ for $t \leq 0$ and $u(t) = (T(t)\phi)(0)$ for $t > 0$ gives the unique solution in $C([-1,\infty),\mathbb{C}^m) \cap C^1(\mathbb{R}_{\geq 0},\mathbb{C}^m)$ of (4.22). These results are special cases of Theorem VI.6.1, Lemma VI.6.2, and Corollary VI.6.3 of [7]; a version of (4.22) was also treated in an exercise.

We first check that $T(\cdot)$ satisfies (4.21). To this aim, note that Duhamel's formula for (4.22) and the above observations yield

$$(T(t)\phi)(0) = \mathrm{e}^{tA}\phi(0) + \int_0^t \mathrm{e}^{(t-s)A}B(T(s)\phi)(-1)\,\mathrm{d}s, \qquad t \geq 0,$$

for $\phi \in \mathrm{D}(L)$ and thus for all $\phi \in X$ by density. For $b \geq t \geq \tau \geq 1$ and $\theta \in [-1,0]$ we deduce

$$\begin{aligned} T(t)\phi(\theta) - T(\tau)\phi(\theta) &= T(t+\theta)\phi(0) - T(\tau+\theta)\phi(0) \\ &= \big[\mathrm{e}^{(t+\theta)A} - \mathrm{e}^{(\tau+\theta)A}\big]\phi(0) + \int_{\tau+\theta}^{t+\theta} \mathrm{e}^{(t+\theta-s)A}B(T(s)\phi)(-1)\mathrm{d}s \\ &\quad + \int_0^{\tau+\theta} \big[\mathrm{e}^{(t+\theta-s)A} - \mathrm{e}^{(\tau+\theta-s)A}\big]B(T(s)\phi)(-1)\,\mathrm{d}s. \end{aligned}$$

Since A is bounded, it follows $\|T(t)\phi - T(\tau)\phi\|_\infty \leq c(b)|t-\tau|\|\phi\|_\infty$ so that (4.21) is valid with $t_0 = 1$.

By Arzela–Ascoli, see Corollary 1.48 in [24], the domain $\mathrm{D}(L)$ is compactly embedded into X. Remark 2.13 and Theorem 2.15 of [27] then show that $\sigma(A)$ consists of eigenvalues only. By the definition of L, we have $\phi \in \mathrm{D}(L) \setminus \{0\}$ and $\lambda\phi = L\phi$ for $\lambda \in \mathbb{C}$ if and only if $\phi = e_\lambda x$ for some non-zero vector $x \in \mathbb{C}^m$ with $\lambda x = Ax + \mathrm{e}^{-\lambda}Bx$ if and only if $\xi(\lambda) := \det(\lambda I - A - \mathrm{e}^{-\lambda}B) = 0$. (The latter identity is called 'characteristic equation.') Theorem 4.16 thus says that the delay semigroup satisfies $T(\cdot)$ satisfies the spectral mapping theorem, and its exponentially stability is equivalent to

$$\mathrm{s}(L) = \sup\big\{\operatorname{Re}\lambda \,\big|\, \lambda \in \mathbb{C},\ \xi(\lambda) = 0\big\} < 0$$

by Corollary 4.17. In Section VI.6.c of [7] this condition is much improved in the positive case $\mathrm{e}^{tA} \geq 0$ and $B \geq 0$. ◊

We add three other important results on the long-time behavior of semigroups without proof, starting with *Gearhart's spectral mapping theorem*. It was shown by Gearhart in 1978 for quasi-contraction semigroups and independently by Herbst (1983), Howland (1984), and Prüss (1984) for general C_0-semigroups. It says that spectral information on A combined with resolvent estimates yield the corresponding spectra for the semigroup, provided that X is a Hilbert space. For a proof we refer to Theorem 2.5.4 in [**20**].

THEOREM 4.19. *Let A generate the C_0-semigroup $T(\cdot)$ on a Hilbert space X. Let $t > 0$ and $\lambda \in \mathbb{C}$. Then*

$$e^{\lambda t} \in \rho(T(t)) \iff \forall\, k \in \mathbb{Z} : \lambda_k := \lambda + \frac{2\pi i k}{t} \in \rho(A), \quad \sup_{k\in\mathbb{Z}} \|R(\lambda_k, A)\| < \infty.$$

We add two results on weaker decay properties, assuming that the semigroup is *bounded*. As in (4.5), the first one deals with classical solutions; i.e., initial values in $D(A)$. Since one looks at estimates of $T(t)$ in $\mathcal{B}(X_1, X)$, one can obtain decay rates which are not exponential in contrast to convergence in $\mathcal{B}(X)$, cf. Proposition 4.2. To obtain polynomial decay, one can allow for a corresponding growth of the resolvent along $i\mathbb{R}$.

THEOREM 4.20. *Let A generate the bounded C_0-semigroup $T(\cdot)$ on a Hilbert space X and let $\alpha > 0$. The follwing two assertions are equivalent.*

a) $\|T(t)x\| \le N t^{-1/\alpha} \|x\|_A$ for some $N > 0$ and all $t \ge 1$ and $x \in D(A)$.

b) $\sigma(A) \subseteq \mathbb{C}_-$ and $\|R(i\tau, A)\| \le C|\tau|^\alpha$ for some $C > 0$ and all $\tau \in \mathbb{R}\setminus[-1, 1]$.

Property b) and Remark 1.16 imply that $|\operatorname{Im}\lambda| \ge c|\operatorname{Re}\lambda|^{-1/\alpha}$ for all $\lambda \in \sigma(A)$ with $\operatorname{Re}\lambda \le -\delta$ for some $c, \delta > 0$. The implication 'b) $\Rightarrow$ a)' is due to Borichev and Tomilov (see [**6**] from 2010), who also constructed an example saying that it fails in an L^1-space. The converse implication was shown by Batty and Duyckaerts in [**5**] from 2008 even for general X and other rates. In this more general framework they also proved a variant of 'b) $\Rightarrow$ a)' with a logarithmic correction. A version of Theorem 4.20 for a large class of decay rates was established in [**23**].

In the setting of the above theorem, by density one obtains *strong stability* of $T(\cdot)$; i.e., $T(t)x$ tends to 0 as $t \to \infty$ for all $x \in X$. But this fact is true in much greater generality, as established already in 1988 by Arendt and Batty as well as, with a different proof, by Lyubich and Vũ.

THEOREM 4.21. *Let A generate the bounded C_0-semigroup $T(\cdot)$ on a Banach space X. Assume that $\sigma(A) \cap i\mathbb{R}$ is countable and that $\sigma_p(A^*) \cap i\mathbb{R} = \emptyset$. (The latter is true if $\sigma_p(A) \cap i\mathbb{R} = \emptyset$ and X is reflexive.) Then $T(\cdot)$ strongly stable.*

The proof by Lyubich and Vũ can be found in Theorem V.2.21 of [**7**], and we refer to Lemma V.2.20 in [**7**] for the addendum. A variety of related results are discussed in [**3**].

CHAPTER 5

Stability of positive semigroups

Evolution[1] equations often describe the behavior of positive quantities, such as the concentration of a species or the distribution of mass or temperature. It is then a crucial property of the system that non-negative initial functions lead to non-negative solutions. This property of *positivity* has to be verified in the applications, of course, and we will see below that it implies many additional useful features of the semigroup solving the equation. To deal with positivity, we consider as state spaces only the following classes of Banach spaces E consisting of scalar-valued functions.

Standing hypothesis. *In this chapter, E denotes a function space of the type $L^p(\mu)$, $C_0(U)$ or $C(K)$, where $p \in [1,\infty)$, $(S,\mathcal{A},\mu)$ is a σ–finite measure space, U is a locally compact metric space (e.g., an open subset of $\mathbb{R}^m$), or K is a compact metric space, respectively.*

We stress that we still take $\mathbb{C}$ as the scalar field in order to use spectral theory. Actually, we could work in the more general class of (complex) *Banach lattices* E, but for simplicity we restrict ourselves to the above indicated setting. It suffices for the typical applications; however for certain deeper investigations one actually needs the more abstract framework. We refer to the monographs [**4**] and [**19**] for a discussion of positive C_0-semigroups in Banach lattices.

In the spaces E given by the standing hypothesis, we have the usual concept of non-negative functions $f \geq 0$, of positive and negative parts $f_\pm$ and domination $f \leq g$ of real-valued functions, and of the absolute value $|f|$. We write $E_+ = \{f \in E \,|\, f \geq 0\}$ for the cone of non-negative functions, which is closed in E. For all $f, g \in E$, it holds $\||f|\| = \|f\|$, and $0 \leq f \leq g$ implies that $\|f\| \leq \|g\|$.

Recall from Example 3.6 that an operator $T \in \mathcal{B}(E)$ is called *positive* if $Tf \geq 0$ for every $f \in E_+$. One then writes $T \geq 0$. A C_0-semigroup $T(\cdot)$ is *positive* if each operator $T(t)$, $t \geq 0$, is positive. We discuss a few basic properties of positive operators $T, S \in \mathcal{B}(E)$ which are used below without further notice. First, products of positive operators are positive. Next,

$$\text{for all } f,g \in E \text{ with } f \geq g \quad \text{we have} \quad T(f-g) \geq 0 \iff Tf \geq Tg.$$

For real-valued f, also the image $Tf = Tf_+ - Tf_-$ has real values. Moreover, $Tf \leq |Tf| \leq Tf_+ + Tf_- = T|f|$. For complex-valued f, we take a point x in $\Omega \in \{S, U, K\}$. Choose a number α such that $|\alpha| = 1$ and $|Tf(x)| = \alpha Tf(x)$, where we fix a representative of Tf if $E = L^p$. It follows that

$$|Tf(x)| = \alpha Tf(x) = T(\mathrm{Re}(\alpha f))(x) + \mathrm{i}T(\mathrm{Im}(\alpha f))(x) = T(\mathrm{Re}(\alpha f))(x)$$

[1]This chapter was not part of the lectures.

$$\leq T(|\operatorname{Re}(\alpha f)|)(x) \leq T(|\alpha f|)(x) = T(|f|)(x).$$

Consequently,

$$|Tf| \leq T|f| \quad \text{holds for all} \quad f \in E.$$

We further write $0 \leq T \leq S$ if $0 \leq Tf \leq Sf$ for all $f \in E_+$. Let $0 \leq T \leq S$. Then $|Tf| \leq T|f| \leq S|f|$ is true for all $f \in E$, and hence

$$\|T\| = \sup_{\|f\|\leq 1} \|Tf\| = \sup_{\|f\|\leq 1} \||Tf|\| \leq \sup_{\|f\|\leq 1} \|S|f|\| \leq \|S\|.$$

We recall from Corollary 3.25 that the semigroup is positive if and only if there exists a number $\omega \geq \omega_0(A)$ such that $R(\lambda, A) \geq 0$ for all $\lambda > \omega$. In Example 3.26 we have seen that the Dirichlet–Laplacian Δ_D with domain $W^{2,p}(G) \cap W_0^{1,p}(G)$ generates a positive C_0-semigroup on $L^p(G)$ for $p \in (1, \infty)$, where $G = \mathbb{R}^m$ or $G \subseteq \mathbb{R}^m$ is bounded and open with $\partial G \in C^2$.

To discuss the Neumann Laplacian we need *Hopf's lemma*. For $w \in C^2(B) \cap C^1(\overline{B})$, it is a special case of the lemma in Section 6.4.2 in [**8**]. Our result can be shown in the same way using Proposition 3.1.10 of [**18**].

Lemma 5.1. *Let $B = B(y, \rho) \subset \mathbb{R}^m$ be an open ball and w belong to $W^{2,p}(B)$ for all $p \in (1, \infty)$ and satisfy $0 \leq \Delta w \in C(\overline{B})$. Assume that there is a apoint $x_0 \in \partial B$ such that $w(x_0) > w(x)$ for all $x \in B$. Then $\partial_\nu w(x_0) > 0$ for the outer normal $\nu(x) = \rho^{-1}(x - y)$ of ∂B.*

Example 5.2. Let $G \subseteq \mathbb{R}^m$ be open and bounded with boundary of class C^2, or let $G = \mathbb{R}^m$. Set $E = L^p(G)$ for $p \in (1, \infty)$. The *Neumann Laplacian* on E is given by $\Delta_N u = \Delta u$ on $\mathrm{D}(\Delta_N) = \{u \in W^{2,p}(G) \,|\, \partial_\nu u = 0\}$. One sees as in Example 2.30 that the operator $\mathrm{e}^{\mathrm{i}\theta}\Delta_N$ is dissipative on $L^p(G)$, if $0 \leq |\theta| \leq \operatorname{arccot}(\frac{|p-2|}{2\sqrt{p-1}}) \in (0, \pi/2]$. Theorem 9.3.5 in [**15**] further implies that that $I - \Delta_N$ is surjective. Consequently, Δ_N generates a contractive analytic C_0-semigroup on E by Corollary 2.27.

To show positivity, let $\lambda > 0$ and $0 \leq f \in C_0(G)$. Set $u = R(\lambda, \Delta_N)f$. Corollary 3.1.24 in [**18**] implies that u belongs to $\mathrm{D}(\Delta_N)$ for all $p \in (1, \infty)$ and Δu to $C(\overline{G})$. As in Example 3.26, we see that u takes real values. Suppose there was a point $x_0 \in G$ such that $u(x_0) < 0$. The function u thus has a minimum $u(x_1) < 0$ for some $x_1 \in \overline{G}$. We then have $\Delta u(x_1) = \lambda u(x_1) - f(x_1) < 0$ and so $\Delta u(x) \leq 0$ for all x in a neighborhood of x_1 in $\overline{G}$. If $x_1 \in G$, Proposition 3.1.10 in [**18**] then yields $\Delta u(x_1) \geq 0$ which is impossible.

So all such minimina occur on ∂G. Since ∂G is C^2, we can find an open ball $B \subseteq G$ with $\overline{B} \cap \partial G = \{x_1\}$ on which $-u$ satisfies the assumptions of Lemma 5.1. Hence, $\partial_\nu v(x_1) < 0$ contradicting $u \in \mathrm{D}(\Delta_N)$. We have shown that $R(\lambda, \Delta_N)f \geq 0$ and by density the resolvent is positive. The positivity of the semigroup then follows from Corollary 3.25. ◊

The next result collects the basic features of the spectral theory of positive semigroups. For a generator A we define two more quantities

$$\mathrm{s}_0(A) = \inf \big\{ r > \mathrm{s}(A) \,\big|\, \sup_{\mu \in \mathbb{C}_r} \|R(\mu, A)\| < \infty \big\},$$

$$\omega_1(A) = \inf \big\{ \omega \in \mathbb{R} \,\big|\, \exists\, M_\omega \geq 1\ \forall\, t \geq 0,\ x \in \mathrm{D}(A) : \|T(t)x\| \leq M_\omega \mathrm{e}^{\omega t} \|x\|_A \big\}.$$

THEOREM 5.3. *Let A generate the positive C_0-semigroup $T(\cdot)$ on E. Then the following assertions hold.*

a) Let $\operatorname{Re}\lambda > \mathrm{s}(A)$ and $f \in E$. Then the improper Riemann integral

$$\int_0^\infty \mathrm{e}^{-\lambda t} T(t) f \,\mathrm{d}t = R(\lambda, A) f \tag{5.1}$$

exists. Moreover, $\|R(\lambda, A)\| \le \|R(\operatorname{Re}\lambda, A)\|$.

b) $\mathrm{s}(A) = \mathrm{s}_0(A)$.

c) Let $\sigma(A) \neq \emptyset$. Then $\mathrm{s}(A)$ belongs to $\sigma(A)$.

d) For $\lambda \in \rho(A)$, the resolvent $R(\lambda, A)$ is positive if and only if $\lambda > \mathrm{s}(A)$.

e) $\mathrm{s}(A) = \omega_1(A)$. In particular, if $\mathrm{s}(A) < 0$, then there are $N, \delta > 0$ such that $\|T(t)x\| \le N\mathrm{e}^{-\delta t}\|x\|_A$ for all $x \in \mathrm{D}(A)$ and $t \ge 0$.

PROOF. a) For $\lambda > \omega_0(A)$, Corollary 3.25 yields that $R(\lambda, A) \ge 0$. If $\mu \in (\mathrm{s}(A), \lambda)$ with $0 < \lambda - \mu < \|R(\lambda, A)\|^{-1}$, the Neumann series gives

$$R(\mu, A) = \sum_{n=0}^\infty (\lambda - \mu)^n R(\lambda, A)^{n+1} \ge 0.$$

Since $\|R(r, A)\|$ is bounded for $r \ge \mathrm{s}(A) + \varepsilon$ and any fixed $\varepsilon > 0$, we deduce the positivity of $R(\mu, A)$ for all $\mu > \mathrm{s}(A)$ (establishing one implication of assertion d)). Let $\mu > \mathrm{s}(A)$, $\operatorname{Re}\alpha > 0$, $f \in E$ and $t \ge 0$. We set

$$V(t)f = \int_0^t \mathrm{e}^{-\mu s} T(s) f \,\mathrm{d}s.$$

From Lemma 1.18 we deduce that

$$0 \le V(t) f = R(\mu, A) f - R(\mu, A)\mathrm{e}^{-\mu t} T(t) f \le R(\mu, A) f$$

for all $f \in E_+$. Hence, $\|V(t)\| \le \|R(\mu, A)\|$ for all $t \ge 0$, and thus the function $\mathbb{R}_{\ge 0} \ni t \mapsto \mathrm{e}^{-\alpha t} V(t) f$ is integrable. Integrating by parts, we deduce

$$\int_0^t \mathrm{e}^{-\alpha s}\mathrm{e}^{-\mu s} T(s) f \,\mathrm{d}s = \int_0^t \alpha \mathrm{e}^{-\alpha s} V(s) f \,\mathrm{d}s + \mathrm{e}^{-\alpha t} V(t) f$$

for all $f \in E$. We can now let $t \to \infty$, obtaining the integral in (5.1) with $\lambda = \mu + \alpha$ on the left-hand side. Proposition 1.20 then yields $\lambda \in \rho(A)$ and (5.1). Since we can vary $\mu > \mathrm{s}(A)$, these results also hold for all $\operatorname{Re}\alpha \ge 0$. It further follows that

$$|R(\mu + \alpha, A) f| \le \int_0^\infty \mathrm{e}^{-(\mu + \operatorname{Re}\alpha)t} |T(t) f| \,\mathrm{d}t \le \int_0^\infty \mathrm{e}^{-\mu t} T(t)|f| \,\mathrm{d}t = R(\mu, A)|f|.$$

This inequality implies that $\|R(\mu + \alpha, A)\| \le \|R(\mu, A)\|$, and thus the second assertion in a) is true.

b) It is clear that $\mathrm{s}(A) \le \mathrm{s}_0(A)$. The converse inequality follows from a) and the fact that $\|R(r, A)\|$ is bounded for $r \ge \mathrm{s}(A) + \varepsilon$ and any fixed $\varepsilon > 0$.

c) Assume that $\sigma(A) \neq \emptyset$. We can find $\lambda_n \in \rho(A)$ tending to $\sigma(A)$ with $\operatorname{Re}\lambda_n > \mathrm{s}(A) > -\infty$. Assertion a) and (1.8) imply that

$$\|R(\operatorname{Re}\lambda_n, A)\| \ge \|R(\lambda_n, A)\| \ge \mathrm{d}(\lambda_n, \sigma(A))^{-1} \longrightarrow \infty$$

as $n \to \infty$. If $\mathrm{s}(A) \in \rho(A)$, then $R(\operatorname{Re}\lambda_n, A)$ would converge to $R(\mathrm{s}(A), A)$ leading to a contradiction. The spectral bound thus belongs to $\sigma(A)$.

d) Let $R(\lambda, A)$ be positive for some $\lambda \in \rho(A)$. Take $0 \neq f \in E_+$. The function $0 \neq u := R(\lambda, A)f$ is also non-negative and $Au = \lim_{t\to 0} \frac{1}{t}(T(t)f - f)$ is real-valued. Hence, $\lambda u = f + Au$ is real, so that $\lambda \in \mathbb{R}$. Let $\mu > \max\{\lambda, \mathrm{s}(A)\}$. Part a) of the proof shows that $R(\mu, A) \geq 0$, and thus

$$R(\lambda, A) = R(\mu, A) + (\mu - \lambda)R(\mu, A)R(\lambda, A) \geq R(\mu, A) \geq 0.$$

Using $\mathrm{s}(A) \in \sigma(A)$ and (1.8), we deduce that

$$\frac{1}{\mu - \mathrm{s}(A)} \leq \frac{1}{\mathrm{d}(\mu, \sigma(A))} \leq \|R(\mu, A)\| \leq \|R(\lambda, A)\|.$$

If $\lambda \leq \mathrm{s}(A)$, the limit $\mu \to \mathrm{s}(A)$ would give a contradiction. Hence, d) holds.

e) Let $\lambda > \mathrm{s}(A)$ and $f \in \mathrm{D}(A)$. Assertion a) then implies that

$$\mathrm{e}^{-\lambda t} T(t) f = f + \int_0^t \mathrm{e}^{-\lambda s} T(s)(A - \lambda I) f \,\mathrm{d}s \;\longrightarrow\; f + R(\lambda, A)(A - \lambda I) f = 0$$

as $t \to \infty$. Hence, $\mathrm{e}^{-\lambda t} T(t)$ is bounded in $\mathcal{B}([\mathrm{D}(A)], X)$ uniformly for $t \geq 0$ by the principle of uniform boundedness. This fact implies that $\omega_1(A) \leq \mathrm{s}(A)$. Conversely, let $\operatorname{Re}\lambda > \omega_1(A)$ and $f \in \mathrm{D}(A)$. Then the integral

$$\int_0^t \mathrm{e}^{-\lambda t} T(t) f \,\mathrm{d}t =: R_\lambda f$$

converges in E. As in the proof of Proposition 1.20, it follows that $R_\lambda f \in \mathrm{D}(A)$ and $(\lambda I - A)R_\lambda f = f$. Moreover, $R_\lambda(\lambda I - A)f = f$ if $f \in \mathrm{D}(A^2)$. We denote by A_1 the restriction of A to $X_1 = [\mathrm{D}(A)]$ with domain $\mathrm{D}(A_1) = \mathrm{D}(A^2)$. We have shown that $\lambda \in \rho(A_1)$. Since A and A_1 are similar via the ismorphism $R(\lambda, A) : \mathrm{D}(A) \to \mathrm{D}(A^2)$, we arrive at $\lambda \in \rho(A)$; i.e., $\mathrm{s}(A) = \omega_1(A)$. □

The next corollary immediately follows from part b) of the above theorem and Gearhart's stability Theorem 4.5.

Corollary 5.4. *Every generator A of a positive semigroup on $E = L^2(\mu)$ satisfies* $\mathrm{s}(A) = \omega_0(A)$.

Remark 5.5. The above corollary actually holds for all our spaces E, see Section 5.3 in [**3**], but it fails already on $L^p \cap L^q$ by Example 4.3. For any generator A, one has $\mathrm{s}(A) \leq \omega_1(A) \leq \mathrm{s}_0(A) \leq \omega_0(A)$. (These inequalities follow from the proof of Theorem 5.3 e), Proposition 5.1.6 and Theorem 5.1.7 in [**3**], and Proposition 1.20.) Hence, in Theorem 5.3 assertion e) is direct consequence of part b) thanks to the (more difficult) general result in [**3**], which is due to Weis and Wrobel. The positive semigroup in Example 4.3 satisfies $\mathrm{s}_0(A) < \omega_0(A)$, see Example 5.1.11 in [**3**]. Moreover, there are (non-positive) semigroups on Hilbert spaces X such that $\mathrm{s}(A) < \omega_1(A) < \mathrm{s}_0(A)$, see Example 5.1.10 in [**3**]. ♢

As an application we look at a *cell division* problem.

Example 5.6. Let $\int_a^b u(t, s)\,\mathrm{d}s$ be the number of cells of a certain species at time $t \geq 0$ of size $s \in [a, b]$. We make the following assumptions on this species.

- Each cell grows linearly with time at (normalized) velocity 1.

- Cells of size $s \geq \alpha > 0$ divide with per capita rate $b(s) \geq 0$ in two daughter cells of equal size, where $b = 0$ on $[1, \infty)$ and on $[\alpha/2, \alpha]$.
- Cells of size s die with per capita rate $\mu(s) \geq 0$.
- The functions $b \neq 0$ and μ are continuous, and $\alpha > 1/2$.
- There are no cells at size $\alpha/2$.

It is just a normalization that the cells divide up size $s = 1$. The assumptions of linear growth and that $\alpha > 1/2$ are made for simplicity, see [**11**] for the general case. The assumptions on b indicate that the interesting cell sizes belong to $J = [\alpha/2, 1]$ (for others one only has growth and death), so that we choose as state space $E = L^1(J)$. Hence, the norm $\|u(t)\|_1$ equals the number of (relevant) cells at time t, if $u \geq 0$. It can be shown that under the above assumptions smooth cell size distributions u satisfy the equations

$$\begin{aligned}
&\partial_t u(t,s) = -\partial_s u(t,s) - \mu(s)u(t,s) - b(s)u(t,s) + 4b(2s)u(t,2s), \quad t \geq 0,\ s \in J,\\
&u(t, \tfrac{\alpha}{2}) = 0, \quad t \geq 0, \qquad\qquad (5.2)\\
&u(0,s) = u_0(s), \quad s \in J.
\end{aligned}$$

Note that $b(2s) = 0$ for $s \geq 1/2$. For such s we put $v(2s) \coloneqq 0$ for any function v on J. We take $0 \leq u_0 \in \mathrm{D}(A) \coloneqq \{v \in W^{1,1}(J) \,|\, v(\alpha/2) = 0\}$ and define

$$Av = -v' - \mu v - bv + Bv, \qquad Bv(s) = 4b(2s)v(2s), \tag{5.3}$$

for $v \in \mathrm{D}(A)$, respectively $v \in E$ and $s \in J$. Observe that B is a bounded (and positive) operator on E because

$$\|Bv\|_1 \leq 4\|b\|_\infty \int_{\alpha/2}^{1/2} |v(2s)| \,\mathrm{d}s \leq 2\|b\|_\infty \|v\|_1.$$

Since $-\frac{\mathrm{d}}{\mathrm{d}s}$ with domain $\mathrm{D}(A)$ generates a positive C_0-semigroup on E (the nilpotent translations), Example 3.6 shows that also A generates a positive C_0-semigroup $T(\cdot)$ on E. It is clear that the non-negative map $u(t,s) = (T(t)u_0)(s)$ with $t \geq 0$ and $s \in J$ belongs to $C^1(\mathbb{R}_+, E) \cap C(\mathbb{R}_+, W^{1,1}(J))$ and satisfies the system (5.2), where the first line holds for a.e. $s \in J$. On the other hand, each solution $u \in C^1(\mathbb{R}_+, E) \cap C(\mathbb{R}_+, W^{1,1}(J))$ of (5.2) is given by $T(\cdot)$. $\Diamond$

In the above example the embedding $\mathrm{D}(A) \hookrightarrow E$ is compact due to Theorem 3.34 in [**27**]. Therefore the resolvent of A is compact and $\sigma(A)$ consists of eigenvalues only, see Remark 2.13 and Theorem 2.15 of [**27**]. We can even determine the eigenvalues by the zeros of a holomorphic function ξ. (The assumption $\alpha > \frac{1}{2}$ is only needed to obtain the simple formula of ξ below.)

Lemma 5.7. *Let A be given by (5.3). Then a number $\lambda \in \mathbb{C}$ belongs to $\sigma(A)$ if and only if*

$$0 = \xi(\lambda) \coloneqq -1 + \int_{\alpha/2}^{1/2} 4b(2\sigma) \exp\Big(-\int_\sigma^{2\sigma} (\lambda + \mu(\tau) + b(\tau)) \,\mathrm{d}\tau\Big) \mathrm{d}\sigma.$$

Proof. As noted above, we have $\sigma(A) = \sigma_p(A)$. Hence, $\lambda \in \mathbb{C}$ belongs to $\sigma(A)$ if and only if there is a map $0 \neq v \in \mathrm{D}(A)$ with $\lambda v = v'$. Equivalently, $0 \neq v \in W^{1,1}(J)$ satisfies

$$v'(s) = -(\lambda + b(s) + \mu(s))v(s), \qquad 1/2 \leq s \leq 1,$$

$$\begin{aligned} v'(s) &= -(\lambda + b(s) + \mu(s))v(s) + 4b(2s)v(2s), \qquad \alpha/2 \le s < 1/2, \\ v(\alpha/2) &= 0. \end{aligned}$$

The differential equations are only fulfilled by the function given by

$$\begin{aligned} v(s) &= c \exp\Big(\int_s^1 (\lambda + b(\sigma) + \mu(\sigma))\,\mathrm{d}\sigma\Big), \qquad \tfrac{1}{2} \le s \le 1, \\ v(s) &= c \exp\Big(\int_s^1 (\lambda + b(\sigma) + \mu(\sigma))\,\mathrm{d}\sigma\Big) \\ &\quad \cdot \Big[1 - \int_s^{1/2} 4b(2\sigma) \exp\Big(-\int_\sigma^{2\sigma} (\lambda + \mu(\tau) + b(\tau))\,\mathrm{d}\tau\Big)\mathrm{d}\sigma\Big], \quad \tfrac{\alpha}{2} \le s < \tfrac{1}{2}, \end{aligned}$$

for any constant $c \neq 0$. Clearly, this map v belongs to $W^{1,1}(J)$, and it satisfies $v(\alpha/2) = 0$ if and only if $\xi(\lambda) = 0$. □

Theorem 5.3 shows that $\omega_1(A) = \mathrm{s}(A)$, and Remark 5.5 even yields $\omega_0(A) = \mathrm{s}(A)$. In Proposition VI.1.4 of [**7**] it is further shown that $t \mapsto T(t)$ is continuous in operator norm for $t > 1 - \frac{\alpha}{2}$. (Here one uses the nilpotency of the semigroup generated by $A_0 := A - B$ and the Dyson–Phillips series (3.7) for $A = A_0 + B$.) Therefore the spectral mapping theorem $\sigma(T(t)) = \mathrm{e}^{t\sigma(A)} \setminus \{0\}$ is true implying again $\omega_0(A) = \mathrm{s}(A)$, see Theorem 4.12 and Corollary 4.17. Positivity even yields a very simple criterion for $\omega_0(A) = \mathrm{s}(A) < 0$.

Theorem 5.8. *The semigroup generated by A from (5.3) is exponentially stable on E if and only if*

$$\xi(0) = -1 + \int_{\alpha/2}^{1/2} 4b(2\sigma) \exp\Big(-\int_\sigma^{2\sigma} (\mu(\tau) + b(\tau))\,\mathrm{d}\tau\Big)\mathrm{d}\sigma < 0.$$

In particular, there are constants $N, \delta > 0$ such that $\|u(t)\|_1 \le N\mathrm{e}^{-\delta t}\|u_0\|_1$ for all $t \ge 0$ and all solutions $u \in C^1(\mathbb{R}_+, E) \cap C(\mathbb{R}_+, W^{1,1}(J))$ of (5.2).

Proof. In view of Lemma 5.7 and the discussion above the statement of the theorem, we have to show that all zeros of ξ have strictly negative real parts. To characterize this property, we use the positivity of the semigroup in a crucial way. Theorem 5.3 says that $\mathrm{s}(A)$ belongs to $\sigma(A)$. Thus $\omega_0(A) < 0$ if and only if all *real* zeros of ξ are strictly negative. On $\mathbb{R}$, the function ξ is continuous and strictly decreasing from ∞ to -1. Consequently, ξ has exactly one real zero, which is strictly negative if and only if $\xi(0) < 0$. □

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