Lecture Notes
Nonlinear Evolution Equations

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These lecture notes are based on my course from summer semester 2024, though there are small corrections and improvements as well as minor changes in the numbering of equations. Typically, the proofs and calculations in the notes are a bit shorter than those given in class. The drawings and many additional oral remarks from the lectures are omitted here. On the other hand, the notes contain a few proofs (mostly of peripheral statements) and additional facts which have not been presented in the lectures. I partly use basic notation, definitions and facts contained in my other lecture notes without further notice.

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CHAPTER 1

Semilinear evolution equations: the basic case

Throughout $X, Y \neq \{0\}$ are Banach spaces over the field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ with norm $\|\cdot\|$. By the same symbol we denote the operator norm on the space of continuous linear operators $\mathcal{B}(X,Y)$, where we put $\mathcal{B}(X) \coloneqq \mathcal{B}(X,X)$. Sometimes we indicate the spaces as a subscript in the norms. For unexplained basic notation we refer to the lectures notes [**30**], [**33**], and [**32**].

This course is devoted to *semilinear evolution equations* of the form

$$u'(t) = Au(t) + F(u(t)), \quad t \in J, \qquad u(0) = u_0, \tag{1.1}$$

where A generates a C_0-semigroup $T(\cdot)$ on X and the nonlinearity F is subordinate to or of 'lower order' than A. In this chapter we start with the simplest case that $F : X \to X$ is Lipschitz on bounded sets. This basic setting serves as a model problem for the area. In particular, in the first section we extend Picard–Lindelöf's local wellposedness Theorem 4.9 in [**31**] to the present situation modulo certain regularity issues. In the second section we apply the developed theory to the cubic semilinear wave equation on subsets of $\mathbb{R}^3$ and discuss global existence and blowup. See also [**7**]. In later chapters we refine and extend these results and methods to treat semilinear parabolic problems like reaction-diffusion systems and the semilinear Schrödinger and wave equation on $\mathbb{R}^m$, where we allow for powers up to 5 in the case $m = 3$.

1.1. Local wellposedness

We study equation (1.1) under the assumptions

$$\begin{aligned}
&A \text{ generates the } C_0\text{-semigroup } T(\cdot) \text{ on } X, \quad M_0 \coloneqq \sup\nolimits_{t\in[0,1]} \|T(t)\|.\\
&J \subseteq \mathbb{R} \text{ is an interval with } \min J = 0,\ \sup J > 0. \quad F : X \to X \text{ satisfies}\\
&\forall\, r > 0\ \ \exists\, L(r) \geq 0\ \ \forall\, x, y \in \overline{B}_X(0,r) : \quad \|F(x) - F(y)\| \leq L(r)\|x - y\|,\\
&\text{where the map} \quad r \mapsto L(r) \ \text{ is non-decreasing.}
\end{aligned} \tag{1.2}$$

If the estimate for F in (1.2) is true for some constants $\tilde{L}(r) \geq 0$, we can replace them by the larger numbers $L(r) = \sup_{0\leq s\leq r} \tilde{L}(s)$ which do not decrease in r.

In essentially the same way one can also treat F which are only defined on an open subset $D \subseteq X$ or explicitely depend on t. Moreover, if $T(\cdot)$ is even a C_0-group one can also consider general time intervals J containing 0, cf. Chapters 4 and 5. We first note that (1.2) is a rather strong condition for substitution operators on L^p spaces with $p \in [1, \infty)$.

Example 1.1. Let $X = L^p(\mu)$ for a measure space $(S, \mathcal{A}, \mu)$, $p \in [1, \infty)$, and $f : \mathbb{F} \to \mathbb{F}$ be Lipschitz with constant L. Moreover, let either $f(0) = 0$

or $\mu(S) < \infty$. Set $(F(v))(s) = f(v(s))$ for $v \in X$ and $s \in S$. Then the map $F : X \to X$ is (globally) Lipschitz.

Indeed, let $v, w \in X$ and $s \in S$. We first note that $F(v)$ belongs to X since

$$|F(v)(s)| \le |f(v(s)) - f(0)| + |f(0)| \le L|v(s)| + |f(0)|$$

and $|f(0)|\mathbb{1} \in X$ by the assumptions. We further estimate

$$|F(v)(s) - F(w)(s)| = |f(v(s)) - f(w(s))| \le L|v(s) - w(s)|,$$

and then the claim follows by taking the p-norm.

Here one cannot allow for locally Lipschitz f, in general. Take $S = (0,1)$, $\mu = \lambda$, and $f(z) = |z|^{\alpha-1}z$ for some $\alpha > 1$ as an example. Fix $\beta = 1/(\alpha p)$. The map $v(s) = s^{-\beta}$ then belongs to X, but $|f(v(s))| = s^{-1/p}$ does not. $\Diamond$

We stress that in (1.1) the existence interval J is part of the problem. Finite-time blowup already occurs for the simple ordinary differential equation

$$u'(t) = u(t)^2, \quad t \ge 0, \qquad u(0) = u_0 > 0, \tag{1.3}$$

whose solution $u(t) = (u_0^{-1} - t)^{-1}$ only exists up to time $1/u_0$.

We first state a natural solution concept. There are several variants in the literature (and we also introduce another notion below), so that we occasionally add the adjective 'classical'. Note that in some areas this word refers to somewhat different solution concepts.

Definition 1.2. *Let* (1.2) *be true and* $u_0 \in X$. *A* (classical) solution u *of* (1.1) *on* J *is a map* $u \in C^1(J, X)$ *satisfying* $u(t) \in \mathrm{D}(A)$ *and* (1.1) *for all* $t \in J$.

We state a few simple properties of solutions. The fixed-point equation (1.4) is the starting point of our approach to semilinear evolution equations.

Remark 1.3. Let (1.2) be true, $u_0 \in X$, and u solve (1.1).

a) The initial value u_0 then must belong to $\mathrm{D}(A)$. The assumptions imply that $F \circ u : J_0 \to X$ is Lipschitz for all compact intervals $J_0 \subseteq J$. Moreover, the solution u is contained in $C(J, [\mathrm{D}(A)])$ since (1.1) yields $Au = u' - F(u)$.

b) From Duhamel's formula in Proposition 2.6 in [**32**] we deduce that

$$u(t) = T(t)u_0 + \int_0^t T(t-s)F(u(s))\,\mathrm{d}s, \qquad t \in J. \tag{1.4}$$

c) Let $v \in C(J, X)$. Then also $F \circ v$ is an element of $C(J, X)$. $\Diamond$

In view of Remark 1.3 c), equation (1.4) makes sense for any continuous function u and can thus serve as a weaker solution concept for (1.1).

Definition 1.4. *Let* (1.2) *be true and* $u_0 \in X$. *A* mild solution *of* (1.1) *on* J *is a function* $u \in C(J, X)$ *satisfying* (1.4).

Observe that a mild solution fulfills $u(0) = u_0$. Notions and results from the lecture Evolution Equations, see Section 2.2 of [**32**], allow us to interpret mild solutions as classical solutions in the (larger) *extrapolation space* X_{-1} of A. We recall that X_{-1} is the completion of X for the norm given by $\|x\|_{-1} = \|R(\omega, A)x\|_X$ for some $\omega \in \rho(A)$, where X is considered as a linear subspace of X_{-1}. The operator A has the extension $A_{-1} \in \mathcal{B}(X, X_{-1})$ which is similar to A and generates the C_0-semigroup $T_{-1}(\cdot)$ on X_{-1} given by extensions of $T(t)$.

REMARK 1.5. Let (1.2) be true, $u_0 \in X$, and $u \in C(J,X)$. Since $F(u) \in C(J,X)$, Propositions 2.6, 2.14, and 2.15 of [**32**] imply that u is a mild solution of (1.1) if and only if u belongs to $C^1(J,X_{-1}) \cap C(J,X)$ and satisfies

$$u'(t) = A_{-1}u(t) + F(u(t)), \quad t \in J, \qquad u(0) = u_0, \tag{1.5}$$

in X_{-1}. Then u is also called classical solution of (1.1) in X_{-1} on J. $\Diamond$

As a first step we solve (1.4) on $[0,b]$ for small times $b > 0$ (only depending $\|u_0\|$, M_0, and F) and uniquely in certain balls of $C([0,b],X)$. Here we proceed as for ordinary differential equations, but now use semigroup theory to treat the linear main part given by $T(\cdot)$. In view of more complicated problems, we stress that one should be careful with the constants here. They must be under control as b tends to 0, and one should specify how they depend on u_0.

LEMMA 1.6. *Let (1.2) be true. Take any $\rho > 0$. Then there is a number $b_0(\rho) > 0$ (given by (1.10) below) such that for each $u_0 \in \overline{B}_X(0,\rho)$ there is a unique mild solution $u \in C([0,b_0(\rho)],X)$ of (1.1) on $[0,b_0(\rho)]$ satisfying $\|u(t)\| \le 1 + M_0\rho =: r$ for all $0 \le t \le b_0(\rho)$. For each $b \in (0,b_0(\rho)]$, the restriction $u{\restriction}_{[0,b]}$ is also the unique mild solution of (1.1) on $[0,b]$ with sup-norm less or equal than r. Finally, the function b_0 is non-increasing.*

PROOF. Let $\rho > 0$ and take $u_0 \in X$ with $\|u_0\| \le \rho$. Fix $r = 1 + M_0\rho$. For $b \in (0,1]$ to be specified below, we define the closed ball

$$E(b) = E(b,r) = \big\{v \in C([0,b],X) \,\big|\, \|v\|_\infty := \max_{t\in[0,b]} \|v(t)\|_X \le r\big\}. \tag{1.6}$$

It is a complete metric space for the metric induced by the sup-norm $\|\cdot\|_\infty$ on $C([0,b],X)$. To solve (1.4), we further introduce the map

$$[\Phi_{u_0}(v)](t) = \Phi(v)(t) := T(t)u_0 + \int_0^t T(t-s)F(v(s))\,\mathrm{d}s \tag{1.7}$$

for $t \in [0,b]$ and $v \in E(b)$. The function $\Phi(v)$ belongs to $C([0,b],X)$ by Remark 1.3 c) and, e.g., dominated convergence as stated in Remark 1.15 of [**32**]. Each mild solution $u \in E(b)$ of (1.1) is a fixed point of Φ in $E(b)$, and vice versa.

Let $v, w \in E(b)$ and $t \in [0,b] \subseteq [0,1]$. Using (1.2) and that $v(s), w(s) \in \overline{B}(0,r)$, we estimate

$$\begin{aligned}\|\Phi(v)(t)\| &\le M_0\|u_0\| + \int_0^t M_0\big(\|F(v(s)) - F(0)\| + \|F(0)\|\big)\,\mathrm{d}s\\ &\le M_0\rho + tM_0(L(r)\max_{s\in[0,t]}\|v(s)\| + \|F(0)\|)\\ &\le M_0\rho + bM_0(L(r)r + \|F(0)\|),\end{aligned} \tag{1.8}$$

$$\|\Phi(v)(t) - \Phi(w)(t)\| \le \int_0^t M_0\|F(v(s)) - F(w(s))\|\,\mathrm{d}s \le bM_0L(r)\|v-w\|_\infty. \tag{1.9}$$

We choose a final time $b \in (0,b_0(\rho)]$, setting

$$b_0(\rho) = \min\big\{1, \big(M_0(L(r)r + \|F(0)\|)\big)^{-1}, \big(2M_0L(r)\big)^{-1}\big\} \in (0,1]. \tag{1.10}$$

It follows that $\|\Phi(v)\|_\infty \le r$ and that $\Phi : E(b) \to E(b)$ is Lipschitz with constant $\frac{1}{2}$. Banach's theorem then gives a unique fixed point $u_b = \Phi(u_b) \in E(b)$ for each $b \in (0, b_0(\rho)]$. Since $u := u_{b_0(\rho)}$ solves (1.4) also on $[0, b]$ and the norm in $E(b)$ does not exceed that in $E(b_0(\rho))$, uniqueness implies that $u\restriction_{[0,b]} = u_b$. $\square$

The above lemma just gives *conditional* uniqueness among functions belonging to a certain ball. To derive unconditional uniqueness (and further properties), we first note that we can glue and shift solutions. Shifted solutions mainly occur since we state and use problems like (1.1) only for the initial time 0.

Remark 1.7. Let (1.2) be true and $u_0 \in X$. Assume that $u \in C([0, b_1], X)$ is a mild solution of (1.1) on $[0, b_1]$. Then the following assertions hold.

a) Let $\beta \in (0, b_1)$. Then the shifted function $u(\,\cdot\, + \beta) \in C([0, b_1 - \beta], X)$ is a mild solution of (1.1) on $[0, b_1 - \beta]$ with the initial value $u(\beta)$.

b) Let $v \in C([0, b_2], X)$ be a mild solution of (1.1) on $[0, b_2]$ with the initial value $u(b_1)$. Then the concatenated function $w \in C([0, b_1 + b_2], X)$ given by

$$w(t) = \begin{cases} u(t), & 0 \le t \le b_1, \\ v(t - b_1), & b_1 < t \le b_1 + b_2, \end{cases}$$

solves (1.1) mildly on $[0, b_1 + b_2]$ with the initial value u_0.

Proof. By Remark 1.5, mild solutions in X coincide with classical ones in X_{-1}. Also note that the left- and right-hand derivatives of w in X_{-1} agree at b_1 because of (1.5) for u and v as well as $v(0) = u(b_1)$. Hence, w belongs to $C^1([0, b_1 + b_2], X_{-1})$ and solves (1.5). The claims then follow easily. $\square$

This lemma can also be shown without passing to extrapolation spaces. We add this more complicated argument since it is of interest in other situations.[1]

In Remark 1.7 b), the function w is continuous and a mild solution of (1.1) for $t \in [0, b_1]$ by its definition. Let $t \in (b_1, b_1 + b_2]$. We use the definition of w, (1.4) for u and v, and the semigroup property of $T(\cdot)$. Also substituting $r = b_1 + s$, we then calculate

$$\begin{aligned} w(t) = v(t - b_1) &= T(t - b_1)u(b_1) + \int_0^{t-b_1} T(t - b_1 - s)F(v(s))\,\mathrm{d}s \\ &= T(t-b_1)\Big[T(b_1)u_0 + \int_0^{b_1} T(b_1-s)F(u(s))\,\mathrm{d}s\Big] + \int_{b_1}^t T(t - r)F(v(r-b_1))\,\mathrm{d}r \\ &= T(t)u_0 + \int_0^t T(t - s)F(w(s))\,\mathrm{d}s. \end{aligned}$$

To prove part a), set $\tilde{u}(t) = u(t + \beta)$ for $t \in [0, b_1 - \beta]$. As above, we obtain

$$\begin{aligned} \tilde{u}(t) = u(t + \beta) &= T(t + \beta)u_0 + \int_0^{t+\beta} T(t + \beta - s)F(u(s))\,\mathrm{d}s \\ &= T(t)\Big[T(\beta)u_0 + \int_0^{\beta} T(\beta - s)F(u(s))\,\mathrm{d}s\Big] + \int_0^t T(t - r)F(u(r + \beta))\,\mathrm{d}r \\ &= T(t)u(\beta) + \int_0^t T(t - s)F(\tilde{u}(s))\,\mathrm{d}s. \end{aligned}$$

[1] This alternative proof was not part of the lectures.

In a second step we now establish *unconditional uniqueness* of all mild solutions to (1.1). It is deduced from the uniqueness statement of Lemma 1.6.

LEMMA 1.8. *Let (1.2) be true and $u_0 \in X$. Assume that u and v are mild solutions of (1.1) on J_u respectively J_v. Then $u = v$ on $J_u \cap J_v$.*

PROOF. Set $J = J_u \cap J_v$. Since $u(0) = v(0)$, the number

$$\tau := \sup \{ b \in J \,\big|\, \forall\, t \in [0, b] : \; u(t) = v(t) \}$$

belongs to $[0, \sup J]$. We assume that $u \neq v$ on J. By continuity, it follows $\tau < \sup J$ and $u(\tau) = v(\tau) =: u_1$. Hence, there are times $t_n \in J$ with $t_n \to \tau^+$ and $u(t_n) \neq v(t_n)$. Fix $\beta_0 > 0$ with $\tau + \beta_0 \in J$. For every $\beta \in (0, \beta_0]$, Remark 1.7 shows that the functions $\tilde{u} = u(\,\cdot\, + \tau)$ and $\tilde{v} = v(\,\cdot\, + \tau)$ are mild solutions of (1.1) on $[0, \beta]$ with initial value u_1.

We now set $\rho = \|u_1\|$ and $r = 1 + M_0 \rho$, and use the number $b_0(\rho)$ from (1.10). For sufficiently small times $0 < \beta \leq \min\{b_0(\rho), \beta_0\}$, the continuous maps $\tilde{u}$ and $\tilde{v}$ have sup-norms less or equal r on $[0, \beta]$ because of $\tilde{u}(0) = \tilde{v}(0) = u_1$. The uniqueness statement of Lemma 1.6 then shows that $\tilde{u}(t) = \tilde{v}(t)$ for $t \in [0, \beta]$, which contradicts the inequality $u(t_n) \neq v(t_n)$ for large n. $\square$

In the above proof, after having fixed τ and β_0 one can also proceed differently. Let r_1 be the maximum of the sup-norms of u and v on $[\tau, \tau + \beta_1]$, where $\beta_1 := \min\{1, \beta_0\}$. Equation (1.4) implies that

$$u(t) - v(t) = \int_0^t T(t-s)[F(u(s)) - F(v(s))]\,\mathrm{d}s = \int_\tau^t T(t-s)[F(u(s)) - F(v(s))]\,\mathrm{d}s$$

for all $t \in [\tau, \tau + \beta_1]$. As in (1.9), assumption (1.2) then leads to

$$\|u(t) - v(t)\| \leq M_0 L(r_1) \int_\tau^t \|u(s) - v(s)\|\,\mathrm{d}s,$$

so that $u = v$ on $[\tau, \tau + \beta_1]$ by Gronwall's inequality, see Lemma 4.5 in [**31**].[2]

In a third step we can now extend solution as much as possible.

DEFINITION 1.9. *Let (1.2) be true. For each initial value $u_0 \in X$ we define its* maximal existence time

$$t^+(u_0) = \sup \{ b > 0 \,\big|\, \exists \text{ mild solution } u_b \text{ of (1.1) on } [0, b] \}.$$

The maximal existence interval *is $J^+(u_0) = [0, t^+(u_0))$. A mild solution u of (1.1) on $J^+(u_0)$ is called* maximal (mild) solution.

The above lemmas easily imply that there is a unique maximal solution.

REMARK 1.10. Let (1.2) be true and $u_0 \in X$.

a) Lemma 1.6 provides a mild solution u of (1.1) on $[0, b_0(\|u_0\|)]$. We can also use this lemma to solve (1.1) with initial value $u(b_0(\|u_0\|))$. Remark 1.7 then yields a concatenated solution of (1.1) on an interval larger than $[0, b_0(\|u_0\|)]$, so that $t^+(u_0)$ belongs to $(b_0(\|u_0\|), \infty]$.

b) Let $b \in (0, t^+(u_0))$. By Definition 1.9, there is a mild solution u_b of (1.1) on $[0, b]$. Lemma 1.8 says that $u_b = u_{b'}$ on $[0, b']$ for $0 < b' < b < t^+(u_0)$.

[2]This variant of the proof was not part of the lectures.

We can thus define a maximal solution of (1.1) by setting $u(t) = u_b(t)$ for $t \in [0,b] \subseteq (0,t^+(u_0))$. It is uniquely determined because of Lemma 1.8.

c) We note that the existence interval of this solution has to be right-open due to Theorem 1.11 b) below. ♢

Local wellposedness means that one has for all (or sufficiently many) initial values unique solutions of (1.1) that continuously depend on the initial values. These properties are necessary to make a prediction of the future behavior of the system that is robust under errors in the initial data and could thus be tested by an experiment. In a fourth step, we now show the continuity of $u_0 \mapsto u(t)$ near u_0 for any compact subinterval of $J^+(u_0)$. This fact is also needed if an argument only works for a dense subset of 'better' initial values, cf. Theorem 1.20. One should also prove continuous dependence on F or A. For F we do this in the exercises and also in Proposition 3.6. The next theorem (and its proof) is the prototype for all local wellposedness results in later chapters.

THEOREM 1.11. *Let (1.2) be true, $u_0 \in X$, and $b_0 = b_0(\|u_0\|) > 0$ be given by (1.10). Then the following assertions hold.*

a) There is a unique maximal mild solution $u = \varphi(\,\cdot\,, u_0) \in C([0,t^+(u_0)), X)$ of (1.1), where $t^+(u_0) \in (b_0(\|u_0\|), \infty]$.

b) If $t^+(u_0) < \infty$, then $\lim_{t\to t^+(u_0)^-} \|u(t)\| = \infty$.

c) Take any $b \in (0,t^+(u_0))$. Then there exists a radius $\delta = \delta(u_0,b) > 0$ such that $t^+(v_0) > b$ for all $v_0 \in \overline{B}_X(u_0,\delta)$. Moreover, the map

$$\overline{B}_X(u_0,\delta) \to C([0,b],X); \quad v_0 \mapsto \varphi(\,\cdot\,, v_0),$$

is Lipschitz continuous.

PROOF. Assertion a) was shown in Remark 1.10. To establish b), let $t^+(u_0) < \infty$. Assume that there were times $t_n \to t^+(u_0)$ for $n \to \infty$ with $t_n \in [0,t^+(u_0))$ and $C := \sup_{n\in\mathbb{N}} \|u(t_n)\| < \infty$. We choose an index $m \in \mathbb{N}$ such that $t_m + b_0(C) > t^+(u_0)$, where $b_0(C) > 0$ is given by (1.10). Lemma 1.6 yields a mild solution of (1.1) on $[0,b_0(C)]$ with initial value $u(t_m)$. By means of Remark 1.7, we thus obtain a mild solution of (1.1) on $[0,t_m + b_0(C)]$ which contradicts the definition of $t^+(u_0)$. So claim b) is shown. We prove part c) by a basic step plus an induction argument in three more steps.

1) Let $b \in (0,t^+(u_0))$ and $u = \varphi(\,\cdot\,,u_0)$. We fix a number $b' \in (b,t^+(u_0))$ and use the radii $\overline{\rho} := 1 + \max_{0\le t\le b'} \|u(t)\|$ and $\overline{r} := 1 + M_0\overline{\rho}$. Let the time $\overline{b} := b_0(\overline{\rho}) \in (0,1]$ be given by (1.10) and the operator Φ_{u_0} by (1.7). Take $v_0, w_0 \in \overline{B}(0,\overline{\rho})$. Part a) and the proof of Lemma 1.6 yield mild solutions $v = \Phi_{v_0}(v) = \varphi(\,\cdot\,,v_0)$ and $w = \Phi_{w_0}(w) = \varphi(\,\cdot\,,w_0)$ of (1.1) on $[0,\overline{b}]$ with the initial values v_0 respectively w_0, where v and w are contained in the space $E(\overline{b},\overline{r})$ from (1.6) endowed with the sup-norm $\|\cdot\|_\infty$ on $[0,\overline{b}]$. Formulas (1.9), (1.10) and (1.7) lead to the estimate

$$\begin{aligned}\|v-w\|_\infty &\le \|\Phi_{v_0}(v) - \Phi_{v_0}(w)\|_\infty + \|\Phi_{v_0}(w) - \Phi_{w_0}(w)\|_\infty \\ &\le \tfrac12\|v-w\|_\infty + \|T(\cdot)(v_0-w_0)\|_\infty \le \tfrac12\|v-w\|_\infty + M_0\|v_0-w_0\|, \\ \|v-w\|_\infty &\le 2M_0\|v_0-w_0\|. \end{aligned} \tag{1.11}$$

2) We next show $t^+(v_0) > b$ inductively. For $j \in \mathbb{N}_0$ we set $b_j = j\overline{b}$. There exists a minimal index $N \in \mathbb{N}$ with $b_N > b$. If $b_N > b'$, we redefine $b_N := b' \in (b, t^+(u_0))$. We choose $\delta = (2M_0)^{-N} \in (0,1)$. We inductively show that for every $v_0 \in \overline{B}(u_0,\delta)$ and $j \in \{0,\dots,N-1\}$ the maximal mild solution $v = \varphi(\,\cdot\,, v_0)$ exists at least on $[0, b_{j+1}]$ and that $v(t)$ is an element of the ball $\overline{B}(u(t), (2M_0)^{j+1-N})$ for $t \in [b_j, b_{j+1}]$, which belongs to $\overline{B}(0,\overline{\rho})$ by the definition of $\overline{\rho}$. This claim then yields $t^+(v_0) > b$.

3) We prove the claim. First let $j = 0$. Since $\delta \le 1$, the vector v_0 is contained in $\overline{B}(0,\overline{\rho})$. From estimate (1.11) with $w = u$ we deduce

$$\|v(t) - u(t)\| \le 2M_0\|v_0 - u_0\| \le 2M_0\delta = (2M_0)^{1-N}$$

for all $t \in [0, b_1]$, as asserted for $j = 0$.

Second, assume that the claim has been established for all $k \in \{0,\dots,j-1\}$ and some $j \in \{1,\dots,N-1\}$. It follows $\|v(b_j)\| \le \overline{\rho}$. Lemma 1.6 and Remark 1.7 thus show that v exists at least on $[0, b_{j+1}]$. Moreover, the inequality (1.11) can be applied to $v(t+b_j) = \varphi(t, v(b_j))$ and $u(t+b_j) = \varphi(t, u(b_j))$ for $t \in [0,\overline{b}]$. Using also the induction hypothesis, we infer the bound

$$\|v(t+b_j) - u(t+b_j)\| \le 2M_0\|v(b_j) - u(b_j)\| \le (2M_0)^{1+j-N}$$

for $t \in [0,\overline{b}]$. So the claim is true.

4) It remains to prove the Lipschitz continuity asserted in c). Let $j \in \{0,\dots,N-1\}$ and $t \in [0,\overline{b}]$. By the claim in 2), the vectors $v(b_j)$ and $w(b_j)$ belong to $\overline{B}(0,\overline{\rho})$. As in step 3), inequality (1.11) implies

$$\begin{aligned}\|v(t+b_j) - w(t+b_j)\| &= \|\varphi(t, v(b_j)) - \varphi(t, w(b_j))\| \le 2M_0\|v(b_j) - w(b_j)\| \\ &\le \dots \le (2M_0)^{j+1}\|v_0 - w_0\| \le (2M_0)^N\|v_0 - w_0\|. \quad \square\end{aligned}$$

We add a simple example for Theorem 1.11, which is considerably improved in Chapter 4. In Section 1.2 we discuss a more sophisticated application.

Example 1.12. Let $X = L^2(\mathbb{R}^m)$ with $\mathbb{F} = \mathbb{C}$ and $A = \mathrm{i}\Delta$ with $\mathrm{D}(A) = W^{2,2}(\mathbb{R}^m)$. This operator generates a unitary C_0-group on X by Example 3.9 in [**32**]. Let $f : \mathbb{C} \to \mathbb{C}$ be (globally) Lipschitz and $f(0) = 0$. Example 1.1 says that $F(v) = f \circ v$ defines a Lipschitz map $F : X \to X$. From Theorem 1.11 we then deduce that for each $u_0 \in L^2(\mathbb{R}^m)$ the nonlinear Schrödinger equation

$$u'(t) = \mathrm{i}\Delta u(t) + \mathrm{i}F(u(t)), \quad t \in J, \qquad u(0) = u_0,$$

has a unique maximal mild solution u, which is locally Lipschitz in u_0. $\Diamond$

A simple general condition for global existence is linear growth of F, as shown next. It applies to Lipschitz $F : X \to X$ as in the above example since then

$$\|F(x)\| \le \|F(x) - F(0)\| + \|F(0)\| \le \max\{L, \|F(0)\|\}(1 + \|x\|), \qquad x \in X.$$

A more refined condition is given in the exercises. We also discuss this issue in some detail for a specific problem in the next section, relying on the regularity theory presented below.

Corollary 1.13. *Let (1.2) be true. Assume that there is a constant $c > 0$ such that $\|F(x)\| \le c(1 + \|x\|)$ for all $x \in X$. We then obtain $t^+(u_0) = \infty$ for every $u_0 \in X$.*

Proof. One usually shows such results via contradiction to reduce the argument to a bounded time interval. So assume that $\tau := t^+(u_0) < \infty$ for some $u_0 \in X$. Then also the number $K := \sup_{t\in[0,\tau]} \|T(t)\|$ is finite. From (1.4) and the assumption, we infer the inequality

$$\|u(t)\| \le \|T(t)u_0\| + \int_0^t \|T(t-s)F(u(s))\| \,\mathrm{d}s \le K\|u_0\| + \int_0^t Kc(1+\|u(s)\|)\,\mathrm{d}s$$
$$\le K(\|u_0\| + c\tau) + cK\int_0^t \|u(s)\|\,\mathrm{d}s$$

for all $t \in [0,\tau)$. Gronwall's Lemma 4.5 in [**31**] now yields the uniform bound $\|u(t)\| \le K(\|u_0\| + c\tau)\mathrm{e}^{cK\tau}$ which contradicts Theorem 1.11 b). □

In a fifth step we will show that the mild solution is actually a classical one on the *same* maximal existence interval provided that $u_0 \in \mathrm{D}(A)$ and F is a bit more regular. To this aim, we need some preparations. For the linear case $F = 0$ the next result is clear since then $u' = Au = T(\cdot)Au_0$ is locally bounded.

Lemma 1.14. *Let (1.2) be true and $u_0 \in \mathrm{D}(A)$. Then the maximal mild solution $u = \varphi(\,\cdot\,, u_0) : [0, t^+(u_0)) \to X$ of (1.1) is locally Lipschitz continuous.*

Proof. Take $b \in [0, t^+(u_0))$ and $0 \le t \le t+h \le b$. Equation (1.4) leads to

$$\begin{aligned} u(t+h)-u(t) &= T(t)(T(h)u_0 - u_0) + \int_0^h T(t+h-s)F(u(s))\,\mathrm{d}s \\ &\qquad + \int_h^{t+h} T(t+h-s)F(u(s))\,\mathrm{d}s - \int_0^t T(t-\tau)F(u(\tau))\,\mathrm{d}\tau \\ &= \int_0^h T(t+s)Au_0\,\mathrm{d}s + \int_0^h T(t+h-s)F(u(s))\,\mathrm{d}s \\ &\qquad + \int_0^t T(t-\tau)(F(u(\tau+h)) - F(u(\tau)))\,\mathrm{d}\tau, \end{aligned} \tag{1.12}$$

where we have used Lemma 1.18 of [**32**] and subtituted $\tau = s-h$. The quantities $r = \sup_{0\le s\le b}\|u(s)\|$, $K = \sup_{0\le s\le b}\|T(s)\|$, and $C = \sup_{0\le s\le b}\|F(u(s))\|$ are finite. Formula (1.12) combined with (1.2) yields

$$\|u(t+h)-u(t)\| \le K\|Au_0\|h + KCh + KL(r)\int_0^t \|u(s+h)-u(s)\|\,\mathrm{d}s.$$

Gronwall's inequality then implies the Lipschitz bound

$$\|u(t+h)-u(t)\| \le K(\|Au_0\| + C)\mathrm{e}^{KL(r)b}h. \qquad \square$$

In our regularity theorem we will require that F is continuously differentiable, where we have to use the more general concept of 'real continuous differentiability' if $\mathbb{F} = \mathbb{C}$. To that purpose, we define

$$\mathcal{B}_{\mathbb{R}}(X,Y) := \Big\{T : X \to Y \;\Big|\; T \text{ is } \mathbb{R}\text{-linear and } \|T\|_{\mathcal{B}_{\mathbb{R}}(X,Y)} := \sup_{\|x\|\le 1}\|Tx\| < \infty\Big\},$$

if $\mathbb{F} = \mathbb{C}$. (For $\mathbb{F} = \mathbb{R}$, one has $\mathcal{B}_{\mathbb{R}}(X,Y) = \mathcal{B}(X,Y)$ of course.) As for $\mathcal{B}(X,Y)$ one shows that $\mathcal{B}_{\mathbb{R}}(X,Y)$ is a Banach space when endowed with $\|\cdot\|_{\mathcal{B}_{\mathbb{R}}(X,Y)}$. Each map T in $\mathcal{B}_{\mathbb{R}}(X,Y)$ is Lipschitz continuous. We clearly have $\mathcal{B}(X,Y) \subseteq$

$\mathcal{B}_{\mathbb{R}}(X,Y)$, but the converse inclusion is false even for $X = Y = \mathbb{C}$. As usual we write $\mathcal{B}_{\mathbb{R}}(X) \coloneqq \mathcal{B}_{\mathbb{R}}(X,X)$.

Let $\emptyset \neq D \subseteq X$ be open. A map $F : D \to Y$ is called *real differentiable* at $x_0 \in D$ if there is an operator $L \in \mathcal{B}_{\mathbb{R}}(X,Y)$ such that the limit

$$\lim_{\substack{h\to 0,\, h\neq 0,\\ x_0+h\in D}} \frac{1}{\|h\|}\|F(x_0+h) - F(x_0) - Lh\| = 0$$

exists. We then set $F'(x_0) \coloneqq L$ and call $F'(x_0)$ the *derivative* of F at x_0. We say that F is *real continuously differentiable* on D if F is real differentiable at each point of D and the function

$$F' : D \to \mathcal{B}_{\mathbb{R}}(X,Y); \quad x \mapsto F'(x),$$

is continuous. In this case we write $F \in C^1_{\mathbb{R}}(D,Y)$. If the derivative is $\mathbb{C}$-linear, the map F is called *differentiable*, and we use the notation $C^1(D,Y) \subseteq C^1_{\mathbb{R}}(D,Y)$ if F' is also continuous. For $\mathbb{F} = \mathbb{R}$, real differentiability and differentiability are of course the same. The usual rules of calculus (up to the implicit function theorem) hold in these settings with analogous proofs and straightforward modifications. See Part 4 in [**18**]. We discuss examples in the next section which also show the necessity to employ real differentiability.

If D is convex and $F \in C^1_{\mathbb{R}}(D,Y)$, the fundamental theorem of calculus (see Remark 1.15 in [**32**]) and the chain rule yield the formula

$$F(z) - F(x) = \int_0^1 \tfrac{\mathrm{d}}{\mathrm{d}t}\, F(x + t(z-x))\,\mathrm{d}t = \int_0^1 F'(x + t(z-x))(z-x)\,\mathrm{d}t \tag{1.13}$$

for all $x, z \in D$. In this situation we thus obtain the inequality

$$\|F(z) - F(x)\| \le \max_{0\le t\le 1} \|F'(x + t(z-x))\|\, \|z - x\| \tag{1.14}$$

for all $z, x \in D$. As a result, a function $F \in C^1_{\mathbb{R}}(X,Y)$ is Lipschitz on bounded sets provided that its derivative is bounded on bounded sets. (Observe that a continuous function on a Banach space does not need to be bounded on a closed ball.) We establish a final prerequisite.

LEMMA 1.15. *Let $u \in C([a,b),X)$ be differentiable from the right and $\frac{\mathrm{d}^+}{\mathrm{d}t}u \eqqcolon v$ be contained in $C([a,b),X)$. Then u belongs to $C^1([a,b),X)$ and $u' = v$.*

PROOF. Fix $h \in (0, b-a)$ and $t \in (a+h, b)$. The Hahn-Banach theorem yields a functional $x_h^\star \in X^\star$ with $\|x_h^\star\| = 1$ and

$$\big|\big\langle \tfrac{1}{h}(u(t) - u(t-h)) - v(t), x_h^\star\big\rangle\big| = \big\|\tfrac{1}{h}(u(t) - u(t-h)) - v(t)\big\| \eqqcolon D_h(t).$$

By Corollary 2.1.2 of [**23**] and the assumption, the map $\varphi_h \coloneqq \langle u, x_h^\star\rangle : [a,b) \to \mathbb{F}$ is continuously differentiable, so that $\varphi_h' = \frac{\mathrm{d}^+}{\mathrm{d}s}\varphi_h = \langle v, x_h^\star\rangle$. We then compute

$$\begin{aligned} D_h(t) &= \big|\tfrac{1}{h}(\varphi_h(t) - \varphi_h(t-h)) - \varphi_h'(t)\big| = \Big|\frac{1}{h}\int_{t-h}^t (\varphi_h'(\tau) - \varphi_h'(t))\,\mathrm{d}\tau\Big| \\ &= \Big|\frac{1}{h}\int_{t-h}^t \langle v(\tau) - v(t), x_h^\star\rangle\,\mathrm{d}\tau\Big| \le \frac{h}{h}\max_{t-h\le\tau\le t}\|v(\tau) - v(t)\|. \end{aligned}$$

Since the right-hand side tends to 0 as $h \to 0$, the map u is differentiable at each $t \in [a,b)$ with the (continuous) derivative v. $\square$

In the next proof we follow a standard strategy to prove additional regularity of a given (e.g., mild) solution. (See also Propositions 4.18 and 4.23 for somewhat different procedures.) *Assume* for a moment that the mild solution u of (1.1) would be contained in $C^1(J,X)$. Moreover, let $u_0 \in \mathrm{D}(A)$ and $F \in C^1_{\mathbb{R}}(X,X)$. One can then differentiate (1.4) in X with respect to t, where we use (1.4) in the form

$$u(t) = T(t)u_0 + \int_0^t T(\tau)F(u(t-\tau))\,\mathrm{d}\tau.$$

This leads to the linear (non-autonomous, integrated) evolution equation

$$\begin{aligned} v(t) := u'(t) &= T(t)Au_0 + T(t)F(u_0) + \int_0^t T(\tau)F(u(t-\tau))u'(t-\tau)\,\mathrm{d}\tau \\ &= T(t)(F(u_0) + Au_0) + \int_0^t T(t-s)B(s)v'(s)\,\mathrm{d}s, \qquad t \in J. \end{aligned} \tag{1.15}$$

(Alternatively one could differentiate (1.1) in X_{-1} and use Duhamel.) Since the perturbations $B(s) := F'(u(s))$ belong to $\mathcal{B}_{\mathbb{R}}(X)$, the above equation is easy to solve. The solution v is a candidate for the time derivative of u. To *verify* the differentiability of u and $u' = v$, we will rewrite the difference quotient of u by means of (1.4).

The theorem is used in the next section where it will be crucial that the classical solutions inherits the existence interval of the mild one.

Theorem 1.16. *Let (1.2) be true, $u_0 \in \mathrm{D}(A)$, and $F \in C^1_{\mathbb{R}}(X,X)$. Then the mild solution u of (1.1) on J in fact solves (1.1) on J classically.*

Proof. Let $u_0 \in \mathrm{D}(A)$ and $b \in (0, \sup J)$. We have to show that u belongs to $C^1([0,b),X)$, because then $F \circ u$ is an element of $C^1([0,b),X)$ and thus the assertion will follow from Theorem 2.9 of [**32**]. Set $K = \sup_{0\le s\le b} \|T(s)\| < \infty$.

1) We first prove a preliminary result. The operators $B(s) := F'(u(s)) \in \mathcal{B}_{\mathbb{R}}(X)$ depend continuously on $s \in [0,b]$ and $L = \sup_{0\le s\le b} \|B(s)\|$ is finite. To solve the ($\mathbb{R}$-linear, non-autonomous) problem (1.15) as in Lemma 1.6, we set $\alpha = 2KL$ and

$$(\Phi v)(t) = T(t)(F(u_0) + Au_0) + \int_0^t T(t-s)B(s)v(s)\,\mathrm{d}s$$

for $t \in [0,b]$ and $v \in E = C([0,b],X)$. We endow E with the equivalent norm

$$\|v\|_\alpha = \max_{t\in[0,b]} \mathrm{e}^{-\alpha t}\|v(t)\|.$$

Let $v, w \in E$. The map Φv clearly belongs to E. We estimate

$$\begin{aligned} \|\Phi v - \Phi w\|_\alpha &\le \max_{t\in[0,b]} \Big\| \int_0^t \mathrm{e}^{-\alpha(t-s)} T(t-s)B(s)\mathrm{e}^{-\alpha s}(v(s)-w(s))\,\mathrm{d}s \Big\| \\ &\le KL \max_{t\in[0,b]} \int_0^t \mathrm{e}^{-\alpha(t-s)}\mathrm{d}s\, \|v-w\|_\alpha \le \frac{KL}{\alpha}\|v-w\|_\alpha = \frac{1}{2}\|v-w\|_\alpha, \end{aligned}$$

employing only real linearity of $B(s)$. The contraction mapping principle hence yields a unique solution $v \in C([0,b],X)$ of (1.15).

2) We now prove that the function v of step 1) is the derivative of u. Let $0 \le t < t+h \le b$. Equations (1.12) and (1.15) imply that

$$\begin{aligned}
w_h(t) &:= \tfrac{1}{h}(u(t+h) - u(t)) - v(t) \\
&= T(t)\tfrac{1}{h}(T(h) - I)u_0 - T(t)Au_0 \\
&\qquad + \frac{1}{h}T(t)\int_0^h T(h-s)F(u(s))\,\mathrm{d}s - T(t)F(u_0) \\
&\qquad + \int_0^t T(t-s)\big[\tfrac{1}{h}(F(u(s+h)) - F(u(s))) - F'(u(s))v(s)\big]\,\mathrm{d}s \\
&=: S_1(h,t) + S_2(h,t) + S_3(h,t).
\end{aligned}$$

We first observe that

$$\begin{aligned}
\|S_1(h,t)\| &\le K\big\|\tfrac{1}{h}(T(h) - I)u_0 - Au_0\big\| =: \alpha_1(h) \ \longrightarrow\ 0, \\
\|S_2(h,t)\| &= \Big\|T(t)\frac{1}{h}\int_0^h (T(h-s)F(u(s)) - F(u_0))\,\mathrm{d}s\Big\| \\
&\le \frac{hK}{h}\sup_{0\le s\le h}\|T(h-s)F(u(s)) - F(u_0)\| =: \alpha_2(h) \ \longrightarrow\ 0
\end{aligned}$$

as $h \to 0^+$, using $u_0 \in \mathrm{D}(A)$ in the first limit and Lemma 1.12 of [**32**] for the second one. We then write

$$\begin{aligned}
S_3(h,t) = \int_0^t T(t-s)\tfrac{1}{h}\big[F(u(s+h)) - F(u(s)) - F'(u(s))(u(s+h) - u(s))\big]\,\mathrm{d}s \\
+ \int_0^t T(t-s)F'(u(s))w_h(s)\,\mathrm{d}s =: S_{3,1}(h,t) + S_{3,2}(h,t).
\end{aligned}$$

Lemma 1.14 shows that u is Lipschitz on $[0,b]$, say with constant ℓ. Using this fact and (1.13), we estimate $\|S_{3,1}(h,t)\|$ by

$$\begin{aligned}
&Kb\sup_{\substack{0\le s\le b\\ 0\le s+h\le b}}\frac{1}{h}\Big\|\int_0^1 [F'(u(s) + \tau(u(s{+}h) - u(s))) - F'(u(s))](u(s{+}h) - u(s))\,\mathrm{d}\tau\Big\| \\
&\quad\le \frac{Kb\ell h}{h}\sup_{\substack{0\le s\le s+h\le b\\ 0\le\tau\le 1}}\|F'(u(s) + \tau(u(s+h) - u(s))) - F'(u(s))\| =: \alpha_3(h).
\end{aligned}$$

Since F' is uniformly continuous on the compact set

$$\big\{u(s) + \tau(u(r) - u(s))\,\big|\,0 \le \tau \le 1,\ 0 \le r, s \le b\big\},$$

the quantity $\alpha_3(h)$ tends to 0 as $h \to 0^+$. Altogether we arrive at the bound

$$\|w_h(t)\| \le \alpha_1(h) + \alpha_2(h) + \alpha_3(h) + KL\int_0^t \|w_h(s)\|\,\mathrm{d}s.$$

Gronwall's inequality (see Lemma 4.5 in [**31**]) then leads to

$$\|w_h(t)\| \le (\alpha_1(h) + \alpha_2(h) + \alpha_3(h))\mathrm{e}^{tKL}$$

for all $t \in [0,b)$. Letting $h \to 0^+$, we derive that u is differentiable from the right and that the right-hand side derivative coincides with v. Since v is continuous on $[0,b]$, Lemma 1.15 finally implies that u belongs to $C^1([0,b),X)$. $\square$

1.2. A semilinear wave equation

In this section we mainly study the cubic nonlinear wave equation on an open subset of $\mathbb{R}^3$. As a preparation we first show the differentiability of substitution operators $F(v) = \varphi \circ v =: \varphi(v)$ on L^p-spaces, which are the prototypical nonlinearities in many situations. Actually, later on we only need the case $\varphi(z) = |z|^{\alpha-1}z$. We often work with complex Banach spaces in order to use spectral theory, complex curve integrals, or the Fourier transform. Moreover, in Chapter 4 we investigate the nonlinear Schrödinger equation which requires complex scalars. On the other hand, our model nonlinearity $F(v) = |v|^{\alpha-1}v$ is only real, but not complex differentiable (for $\alpha > 1$ and $\mathbb{F} = \mathbb{C}$). We thus identify $\mathbb{C}$ with $\mathbb{R}^2$ in the usual way and just require that φ belongs to $C^1(\mathbb{R}^2, \mathbb{R}^2)$ and *not* that it is holomorphic as a function $\varphi : \mathbb{C} \to \mathbb{C}$. To deal with the resulting problems, we introduce a bit of notation.

Let $z = x + \mathrm{i}y \in \mathbb{C}$. For $\varphi = (\varphi_1, \varphi_2) : \mathbb{R}^2 \to \mathbb{R}^2$, we set

$$\varphi(x, y) =: \varphi(z) = \varphi_1(z) + \mathrm{i}\varphi_2(z) \in \mathbb{C}.$$

For $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ and $M = \binom{\xi}{\eta} \in \mathbb{R}^{2\times 2}$, the real scalar product and matrix-vector product on $\mathbb{R}^2$ are written as

$$\xi \cdot z = \xi_1 \operatorname{Re} z + \xi_2 \operatorname{Im} z = \operatorname{Re}((\xi_1 + \mathrm{i}\xi_2)\overline{z}) \in \mathbb{R} \quad \text{and} \quad Mz = \xi \cdot z + \mathrm{i}\eta \cdot z \in \mathbb{C}.$$

Observe that we do not have $\mathbb{C}$-linearity in general, though we use complex notation for convenience. If $\mathbb{F} = \mathbb{R}$, we let $y = \xi_2 = 0$, $\varphi_2 = 0$ and $\eta = 0$ here and in analogous formulas below.

Depending on the growth of φ and φ', we see that $v \mapsto F(v) = \varphi(v)$ maps L^p into L^q with $q < p$. By Example 1.1, this loss of integrability cannot be avoided for $F(v) = |v|^{\alpha-1}v$ if $\alpha > 1$. In Example 3.16 we see that substitution operators are differentiable on C_b without growth restrictions on $\varphi \in C^1(\mathbb{R}^2, \mathbb{R}^2)$. If $\mathbb{F} = \mathbb{R}$, one can drop the subscripts $\mathbb{R}$ below.

LEMMA 1.17. *Let $\varphi \in C^1(\mathbb{R}, \mathbb{R})$ if $\mathbb{F} = \mathbb{R}$ and $\varphi \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ if $\mathbb{F} = \mathbb{C} \cong \mathbb{R}^2$, and assume that $|\varphi(z)| \le c_0|z|^\alpha$ and $|\varphi'(z)| \le c_1|z|^{\alpha-1}$ for all $z \in \mathbb{F}$ and some constants $c_j \ge 0$ and $\alpha > 1$. Let $p \in [\alpha, \infty)$ and $(S, \mathcal{A}, \mu)$ be a measure space. Then the following assertions hold.*

a) The map $F : L^p(\mu) \to L^{p/\alpha}(\mu)$; $F(v) = \varphi(v) = \varphi_1(v) + \mathrm{i}\varphi_2(v)$, belongs to $C^1_{\mathbb{R}}(L^p(\mu), L^{p/\alpha}(\mu))$. Its derivative at $v \in L^p(\mu)$ is given by

$$F'(v)w = \varphi'(v)w = \nabla\varphi_1(v) \cdot w + \mathrm{i}\nabla\varphi_2(v) \cdot w \quad (= \varphi'(v)w \ \text{ if } \ \mathbb{F} = \mathbb{R})$$

and it is bounded by $\|F'(v)\|_{\mathcal{B}_{\mathbb{R}}(L^p, L^{p/\alpha})} \le c_1\|v\|_p^{\alpha-1}$, for all $v, w \in L^p(\mu)$.

b) Let φ be real-valued. The map $\Phi : L^\alpha(\mu) \to \mathbb{R}$; $\Phi(v) = \int_S \varphi(v)\,\mathrm{d}\mu$, is contained in $C^1_{\mathbb{R}}(L^\alpha(\mu), \mathbb{R})$. Its derivative at $v \in L^p(\mu)$ is given by

$$\Phi'(v)w = \int_S \nabla\varphi(v) \cdot w\,\mathrm{d}\mu \quad \Big(= \int_S \varphi'(v)w\,\mathrm{d}\mu \ \text{ if } \ \mathbb{F} = \mathbb{R}\Big)$$

and it is bounded by $\|\Phi'(v)\|_{\mathcal{B}_{\mathbb{R}}(L^\alpha, \mathbb{R})} \le c_1\|v\|_\alpha^{\alpha-1}$, for all $v, w \in L^\alpha(\mu)$.

PROOF. Let $Jg = \int_S g\,\mathrm{d}\mu$ for $g \in L^1(\mu)$. Since $J \in \mathcal{B}_{\mathbb{R}}(L^1(\mu), \mathbb{R})$ and $\Phi = J \circ F$, claim b) follows from a) with $p = \alpha$ by the chain rule and $\|J\| = 1$, To show part a), take $v, w \in L^p(\mu)$ and set $q = p/(\alpha-1) \in [p', \infty)$. Because of $|\varphi(v)| \le c_0|v|^\alpha$ and $|\varphi'(v)| \le c_1|v|^{\alpha-1}$, the functions $F(v) = \varphi(v)$ and $\varphi'(v)$ belong to $L^{p/\alpha}(\mu)$ resp. $L^q(\mu)$. The map $L^p(\mu) \to L^{p/\alpha}(\mu)$; $w \mapsto \varphi'(v)w$, is $\mathbb{R}$-linear and bounded since Hölder's inequality with $\frac{\alpha}{p} = \frac{1}{q} + \frac{1}{p}$ yields

$$\|\varphi'(v)w\|_{\frac{p}{\alpha}} \le c_1 \||v|^{\alpha-1}\|_q \|w\|_p = c_1\|w\|_p \Big[\int_S |v|^p \,\mathrm{d}\mu\Big]^{\frac{\alpha-1}{p}} = c_1\|v\|_p^{\alpha-1}\|w\|_p. \quad (1.16)$$

For the differentiablity we compute

$$\begin{aligned} D(w) &:= F(v+w) - F(v) - \varphi'(v)w = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\tau}\,\varphi(v+\tau w)\,\mathrm{d}\tau - \varphi'(v)w \\ &= \int_0^1 \big(\varphi'(v+\tau w) - \varphi'(v)\big) w \,\mathrm{d}\tau \end{aligned}$$

a.e. on S. Set $f(\tau, w) = \varphi'(v+\tau w) - \varphi'(v)$. Note that $\tau \mapsto f(\tau,w)(s)w(s)$ is continuous on $[0,1]$ for a.e. fixed $s \in S$ and that we have the majorant

$$|f(\tau,w)w| \le c_1\big((|v|+|w|)^{\alpha-1}) + |v|^{\alpha-1}\big)|w| =: g$$

for all $\tau \in [0,1]$ and a.e. $s \in S$. As above, g belongs to $L^{p/\alpha}(\mu)$, and so the map $\tau \mapsto f(\tau,w)w$ is continuous in $L^{p/\alpha}(\mu)$ by dominated convergence. We now use the Riemann-integral (see Remark 1.15 in [**32**]) and Hölder's inequality to infer

$$\Big\|\int_0^1 f(\tau,w)w\,\mathrm{d}\tau\Big\|_{p/\alpha} \le \int_0^1 \|f(\tau,w)w\|_{p/\alpha}\,\mathrm{d}\tau \le \|w\|_p \int_0^1 \|f(\tau,w)\|_q\,\mathrm{d}\tau.$$

Let $I(w)$ denote the last integral. We have to establish $I(w) \to 0$ as $w \to 0$ in $L^p(\mu)$. Due to a standard contradiction argument, for each null sequence (w_n) in $L^p(\mu)$ we have to find a subsequence $(w_{n_j})_j$ such that $I(w_{n_j})$ tends to 0 as $j \to \infty$. So let $w_n \to 0$. The Riesz-Fischer theorem then provides a subsequence $(w_{n_j})_j$ and a function $g \in L^p(\mu)$ such that $w_{n_j} \to 0$ a.e. as $j \to \infty$ and $|w_{n_j}| \le g$ a.e. for all $j \in \mathbb{N}$. Take $\tau \in [0,1]$. Because of the continuity and the growth of φ', the maps $f(\tau, w_{n_j})$ tend to 0 a.e. as $j \to \infty$ and are dominated by $|f(\tau,w_{n_j})| \le c_1\big((|v|+g\big)^{\alpha-1} + |v|^{\alpha-1}\big) =: h \in L^q(\mu)$. By Lebesgue, $(f(\tau,w_{n_j}))_j$ converges to 0 in $L^q(\mu)$ for each fixed τ. Since $\|f(\tau,w_{n_j})\|_q \le \|h\|_q$, dominated convergence also yields the limit $I(w_{n_j}) \to 0$ as $j \to \infty$. We thus have proven that $F : L^p(\mu) \to L^{p/\alpha}(\mu)$ is real differentiable with derivative $F'(v)w = \varphi'(v)w$. The asserted estimate of F' then follows from (1.16).

It remains to show the continuity of the map $L^p(\mu) \to \mathcal{B}_{\mathbb{R}}(L^p(\mu), L^{p/\alpha}(\mu))$; $v \mapsto F'(v)$.[3] As above, for each sequence $v_n \to v$ in $L^p(\mu)$ we need a subsequence such that $F'(v_{n_j})$ tends to $F'(v)$ in $\mathcal{B}_{\mathbb{R}}(L^p(\mu), L^{p/\alpha}(\mu))$ as $j \to \infty$. Take $v_n \to v$ and w in $L^p(\mu)$. Proceeding as in the previous paragraph, we choose an a.e. converging subsequence v_{n_j} with a majorant g in $L^p(\mu)$. The functions $\varphi'(v_{n_j}) - \varphi'(v)$ tend to 0 a.e. and are bounded by $c_1(g^{\alpha-1} + |v|^{\alpha-1}) \in L^q(\mu)$, and hence converge to 0 in $L^q(\mu)$. Hölder's inequality again yields the estimate

$$\|F'(v_{n_j})w - F'(v)w\|_{p/\alpha} \le \|\varphi'(v_{n_j}) - \varphi'(v)\|_q \,\|w\|_p,$$

which implies the continuity in operator norm. $\square$

[3]This part of the proof was omitted in the lectures.

We will mostly use the following special case of the above result.

Corollary 1.18. *Let $\alpha, \beta > 1$, $p \in [\alpha, \infty)$, and $(S, \mathcal{A}, \mu)$ be a measure space. Then the maps*

$$F : L^p(\mu) \to L^{p/\alpha}(\mu); \quad F(v) = |v|^{\alpha-1} v,$$
$$\Phi : L^\beta(\mu) \to \mathbb{R}; \quad \Phi(v) = \int_S |v|^\beta \,\mathrm{d}\mu,$$

are real continuously differentiable, and they are Lipschitz on every bounded set. Their derivatives are given by

$$F'(v)w = |v|^{\alpha-1} w + (\alpha-1)|v|^{\alpha-3} v \operatorname{Re}(v\overline{w}) \qquad \text{for } \ v, w \in L^p(\mu),$$
$$\Phi'(v)w = \beta \int_S |v|^{\beta-2} \operatorname{Re}(v\overline{w}) \,\mathrm{d}\mu \qquad \text{for } \ v, w \in L^\beta(\mu).$$

Proof. We focus on the case $\mathbb{F} = \mathbb{C} \cong \mathbb{R}^2$. For $\mathbb{F} = \mathbb{R}$ one can take $s = 0 = \sigma$ below. For F we look at $\varphi(z) = |z|^{\alpha-1} z$ and for Φ at $\phi(z) = |z|^\beta$, where $z \in \mathbb{C} \cong \mathbb{R}^2$. Writing $x = \operatorname{Re} z$ and $x = \operatorname{Im} z$, we compute

$$\begin{aligned}
\nabla\phi(z) &= \big(\partial_x (x^2+y^2)^{\frac{\beta}{2}}, \partial_y (x^2+y^2)^{\frac{\beta}{2}}\big) = \beta \left((x^2+y^2)^{\frac{\beta}{2}-1} x, (x^2+y^2)^{\frac{\beta}{2}-1} y\right) \\
&= \beta |z|^{\beta-2} z, \\
\varphi'(z) &= \Big(\partial_x \big((x^2+y^2)^{\frac{\alpha-1}{2}} \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\big), \partial_y \big((x^2+y^2)^{\frac{\alpha-1}{2}} \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\big)\Big) \\
&= \begin{bmatrix} (x^2+y^2)^{\frac{\alpha-1}{2}} + (\alpha-1)(x^2+y^2)^{\frac{\alpha-3}{2}} x^2 & (\alpha-1)(x^2+y^2)^{\frac{\alpha-3}{2}} yx \\ (\alpha-1)(x^2+y^2)^{\frac{\alpha-3}{2}} xy & (x^2+y^2)^{\frac{\alpha-1}{2}} + (\alpha-1)(x^2+y^2)^{\frac{\alpha-3}{2}} y^2 \end{bmatrix} \\
&= \begin{pmatrix} |z|^{\alpha-1} & 0 \\ 0 & |z|^{\alpha-1} \end{pmatrix} + (\alpha-1)|z|^{\alpha-3} \begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix} x & y \end{pmatrix}
\end{aligned}$$

for $z \neq 0$. For $z = 0$ the definition of the derivative directly leads to $\phi'(z) = 0 = \varphi'(z)$. For $w \in \mathbb{C}$ with $\xi = \operatorname{Re} w$ and $\eta = \operatorname{Im} w$, it follows

$$\begin{aligned}
\nabla\phi(z) \cdot w &= \beta |z|^{\beta-2} z \cdot w = \beta |z|^{\beta-2} \operatorname{Re}(z\overline{w}), \\
\varphi'(z) w &= |z|^{\alpha-1} w + (\alpha-1)|z|^{\alpha-3} \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) (x\xi + y\eta) \\
&= |z|^{\alpha-1} w + (\alpha-1)|z|^{\alpha-3} z \operatorname{Re}(z\overline{w}).
\end{aligned}$$

It is easy to see that φ' and $\nabla\phi$ are continuous and that the growth assumptions of Lemma 1.17 are satisfied. Combined with (1.14), the assertions follow. □

We now treat the basic semilinear wave equation on an open and bounded set $G \subseteq \mathbb{R}^3$ with Lipschitz boundary. The Dirichlet–Laplacian Δ_D in $L^2(G)$ was introduced in Example 1.54 of [**32**]: A map $v \in W_0^{1,2}(G)$ belongs to $\mathrm{D}(\Delta_D)$ if and only if

$$\exists\, f =: \Delta_D v \in L^2(G) \ \ \forall\, \varphi \in W_0^{1,2}(G) : \quad (f|\varphi)_{L^2} = -\int_G \nabla v \cdot \nabla \overline{\varphi} \,\mathrm{d}x.$$

(The dot denotes the scalar product in $\mathbb{R}^m$.) We often drop the subscript of $(\cdot|\cdot)_{L^2}$. Recall that $W_0^{1,2}(G)$ is the space of functions in $W^{1,2}(G)$ whose trace on ∂G vanishes, see Theorem 3.38 of [**33**]. It is equipped with the (Hilbertian) norm given by $\||\nabla v|_2\|_2$ which is equivalent to $\|\cdot\|_{1,2}$, cf. Theorem 3.36 in [**33**].

The divergence theorem yields $\Delta_D v = \Delta v$ if $v \in W^{2,2}(G) \cap W_0^{1,2}(G)$. As noted after Example 1.53 of [**32**], the domain of Δ_D is equal to $W^{2,2}(G) \cap W_0^{1,2}(G)$ if ∂G is of class C^2. The operator Δ_D is invertible, dissipative and self-adjoint in $L^2(G)$, and $[\mathrm{D}(\Delta_D)]$ is continuously embedded into $W_0^{1,2}(G)$. It has a bounded invertible extension $\Delta_D : W_0^{1,2}(G) \to W^{-1,2}(G) =: W_0^{1,2}(G)^\star$ acting as

$$\forall\, \varphi \in W_0^{1,2}(G): \quad \langle \varphi, \Delta_D v\rangle_{W_0^{1,2}(G)} = -\int_G \nabla v \cdot \nabla \varphi \,\mathrm{d}x.$$

Let $a \in \mathbb{R}$ and J be an interval with $\min J = 0$ and $\sup J > 0$. For given $w_0 \in W_0^{1,2}(G)$ and $w_1 \in L^2(G)$, we will solve the cubic semilinear wave equation

$$\partial_t^2 w(t) = \Delta_D w(t) - a|w(t)|^2 w(t), \quad t \in J, \qquad w(0) = w_0, \quad \partial_t w(0) = w_1. \quad (1.17)$$

We look for a *weak solution*

$$w \in C^2(J, W^{-1,2}(G)) \cap C^1(J, L^2(G)) \cap C(J, W_0^{1,2}(G))$$

of (1.17). If even $w_0 \in \mathrm{D}(\Delta_D)$ and $w_1 \in W_0^{1,2}(G)$, we expect a *classical solution*

$$w \in C^2(J, L^2(G)) \cap C^1(J, W_0^{1,2}(G)) \cap C(J, [\mathrm{D}(\Delta_D)]).$$

Requiring $w(t) \in W_0^{1,2}(G)$, we impose Dirichlet boundary conditions in (1.17).

To treat the nonlinearity in (1.17), we set $f(v) = -a|v|^2 v$. Sobolev's Theorem 3.31 of [**33**] yields the embedding $W_0^{1,2}(G) \hookrightarrow L^6(G)$, since $1 - \frac{3}{2} = -\frac{3}{6}$. Corollar 1.18 thus shows that

$$f \in C^1_{\mathbb{R}}(W_0^{1,2}(G), L^2(G)) \qquad \text{and} \quad f' \text{ is bounded on balls.} \quad (1.18)$$

As in the linear case treated in [**32**], we pass to an equivalent problem which is of first order in time. We introduce

$$Z = W_0^{1,2}(G) \times L^2(G), \quad A = \begin{pmatrix} 0 & I \\ \Delta_D & 0 \end{pmatrix}, \quad \text{and} \quad \mathrm{D}(A) = \mathrm{D}(\Delta_D) \times W_0^{1,2}(G).$$

Example 1.55 of [**32**] says that A is skew-adjoint and thus generates a unitary C_0-group $T(\cdot)$ on Z. The extrapolation space for A is given by $Z_{-1} \cong L^2(G) \times W^{-1,2}(G)$ and the extrapolated generator by

$$A_{-1} = \begin{pmatrix} 0 & I \\ \Delta_D & 0 \end{pmatrix} \qquad \text{with} \quad \Delta_D : W_0^{1,2}(G) \to W^{-1,2}(G),$$

see Example 2.17 of [**32**]. We next set

$$F(\varphi, \psi) = \begin{pmatrix} 0 \\ f(\varphi) \end{pmatrix}.$$

This map belongs to $C^1_{\mathbb{R}}(Z)$ and F' is bounded on balls due to (1.18), and thus F is Lipschitz on bounded sets by (1.14), as required for the results of the previous section. We stress that for $f(z) = |z|^{\alpha-1} z$ with $\alpha > 3$ the nonlinearity F does *not* map Z into itself in three space dimensions.

Arguing as in Examples 2.4, 2.10 and 2.17 of [**32**], one can show that the problem (1.17) is equivalent to

$$u'(t) = Au(t) + F(u(t)), \quad t \in J, \qquad u(0) = u_0 := (w_0, w_1), \quad (1.19)$$

for the above maps A and F. More precisely, $u \in C^1(J,Z) \cap C(J,[\mathrm{D}(\Delta_D)])$ solves (1.19) if and only if w is a classical solution of (1.17). Also, $u \in C(J,Z)$ solves (1.19) mildly (and thus satisfies (1.19) in $Z_{-1} \cong L^2(G) \times W^{-1,2}(G)$ because of Remark 1.5) if and only if w is a weak solution of (1.17). In both cases, we have $u = (w, \partial_t w)$. We can now show local wellposedness of (1.17).

THEOREM 1.19. *Let $G \subseteq \mathbb{R}^3$ be open and bounded with $\partial G \in C^{1-}$ and let $a \in \mathbb{R}$. Then the following assertions are true.*

a) For each $u_0 = (w_0, w_1) \in Z = W_0^{1,2}(G) \times L^2(G)$, there is a maximal existence time $t^+(u_0) > b_0(\|u_0\|_Z) > 0$ and a unique maximal weak solution $w = \varphi(\cdot, u_0)$ of (1.17) on $[0, t^+(u_0)) = J^+(u_0)$.

b) Let $t^+(u_0) < \infty$. Then $\lim_{t\to t^+(u_0)} \||(\nabla w(t), \partial_t w(t))|_2\|_{L^2} = \infty$.

c) Let $b \in (0, t^+(u_0))$. Then there is a radius $\delta = \delta(u_0, b) > 0$ such that for all initial data $\tilde{u}_0 = (\tilde{w}_0, \tilde{w}_1) \in \overline{B}_Z(u_0, \delta)$ we have $t^+(\tilde{u}_0) > b$ and the map $\overline{B}_Z(u_0, \delta) \to C([0,b], Z)$; $\tilde{u}_0 \mapsto \varphi(\,\cdot\,, \tilde{u}_0)$, is Lipschitz.

d) Let $u_0 \in \mathrm{D}(\Delta_D) \times W_0^{1,2}(G)$. Then w solves (1.17) on $J^+(u_0)$ classically.

PROOF. By the remarks before the statement, we can apply Theorems 1.11 and 1.16 to (1.19). The resulting mild and classical solutions u of (1.19) for $u_0 \in Z$, respectively $u_0 \in \mathrm{D}(A)$, yield weak and classical solutions w of the semilinear wave equation (1.17) as observed above. $\square$

Depending on the sign of the coefficient $a \in \mathbb{R}$, we study global existence and blowup of solutions w to (1.17). Our reasoning relies on the *energy* given by

$$\begin{aligned} E : Z \to \mathbb{R}; \quad E(\varphi, \psi) &= \int_G \left(\tfrac{1}{2}|\psi|^2 + \tfrac{1}{2}|\nabla\varphi|_2^2 + \tfrac{a}{4}|\varphi|^4\right) \mathrm{d}x, \\ E_w(t) \coloneqq E(w(t), \partial_t w(t)) &= \int_G \left(\tfrac{1}{2}|\partial_t w(t)|^2 + \tfrac{1}{2}|\nabla w(t)|_2^2 + \tfrac{a}{4}|w(t)|^4\right) \mathrm{d}x \\ &= \tfrac{1}{2}\|(w(t), \partial_t w(t))\|_Z^2 + \tfrac{a}{4}\|w(t)\|_4^4 \end{aligned} \tag{1.20}$$

for $t \in J^+(u_0)$. Since $W_0^{1,2}(G) \hookrightarrow L^4(G)$ by Sobolev's embedding and $1 - \frac{3}{2} \geq -\frac{3}{4}$, Corollary 1.18 shows that E belongs to $C^1_{\mathbb{R}}(Z, \mathbb{R})$. Observe that $E(\varphi, \psi)$ controls the Z–norm of (φ, ψ) *provided* that $a \geq 0$. (See Section 4.1 for the physical meaning of the sign of a.)

We next show that E is constant along weak solutions of (1.17) so that it is a natural quantity for the nonlinear wave equation. (It corresponds to the physical energy where we ignore physical units.) This fact leads to global existence of *all* solutions if $a \geq 0$ and of all solutions with *small* initial values for any $a \in \mathbb{R}$, by means of the blow-up condition in Theorem 1.19.

However, one can only derive the preservation of energy for *classical solutions* of (1.17) by a direct computation. Using the continuous dependence on data and the density of $\mathrm{D}(A)$ in Z, we can then extend the energy equality to weak solutions by approximation. Here it is crucial that a classical solution exists (in $\mathrm{D}(A)$) until its maximal existence time $t^+(w_0, w_1)$ as a weak solution. We thus need the full power of the wellposedness theory of the previous section in the next argument, which is protoypical for many nonlinear systems.

THEOREM 1.20. *Let $G \subseteq \mathbb{R}^3$ be open and bounded with $\partial G \in C^{1-}$, $a \in \mathbb{R}$, and w be maximal weak solution of (1.17) for some $u_0 = (w_0, w_1) \in Z = W_0^{1,2}(G) \times L^2(G)$. Then the following assertions are true.*

a) $E_w(t) = E_w(0) = \frac{1}{2}\|(w_0, w_1)\|_Z^2 + \frac{a}{4}\|w_0\|_4^4$ for $t \in J^+(u_0)$.

b) Let $a \geq 0$. Then $t^+(w_0, w_1) = \infty$ for all initial values $(w_0, w_1) \in Z$.

c) We have $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0]$ there is a radius $\rho = \rho_\varepsilon > 0$ with $t^+(u_0) = \infty$ and $\|w(t)\|_{1,2}^2 + \|\partial_t w(t)\|_2^2 \leq \varepsilon^2$ for all $t \geq 0$ and $u_0 \in \overline{B}_Z(0, \rho)$.

PROOF. a) We first show the equality for $u_0 \in (w_0, w_1) \in \mathrm{D}(A)$ and the corresponding classical solution w of (1.17). Let $t \in [0, t^+(u_0))$. Since the map $t \mapsto \partial_t w$ is differentiable in $L^2(G)$ and $t \mapsto w(t)$ in $W_0^{1,2}(G) \hookrightarrow L^6(G)$, Corollary 1.18 and the chain rule show that E_w has the derivative

$$E_w'(t) = \mathrm{Re} \int_G \Big(\partial_t w(t) \partial_t^2 \overline{w(t)} + \nabla w(t) \nabla \partial_t \overline{w(t)} + a|w(t)|^2 w(t) \partial_t \overline{w(t)} \Big) \mathrm{d}x.$$

In the second summand of the integrand we can use the definition of the Laplacian because of $w(t) \in \mathrm{D}(\Delta_D)$ and $\partial_t \overline{w(t)} \in W_0^{1,2}(G)$. Employing also $\mathrm{Re}\, z = \mathrm{Re}\, \overline{z}$ and the equation (1.17), we infer

$$E_w'(t) = \mathrm{Re} \int_G \big(\partial_t^2 w(t) - \Delta w(t) + a|w(t)|^2 w(t) \big) \partial_t \overline{w(t)} \, \mathrm{d}x = 0,$$

so that $E(w(t), \partial_t w(t)) = E(w_0, w_1)$. (This argument simplifies a bit if $\mathbb{F} = \mathbb{R}$.)

For given data $u_0 = (w_0, w_1)$ in Z we find a sequence $(u_{0,n})_n$ in $\mathrm{D}(A)$ converging to u_0 in Z. Let $b \in (0, t^+(u_0))$. Theorem 1.19 c) says that $t^+(u_{0,n}) > b$ for all sufficiently large n and that the corresponding solution $(w_n, \partial_t w_n)$ tends to $(w, \partial_t w)$ in Z uniformly in $t \in [0, b]$ as $n \to \infty$. We can apply the first part of the proof to w_n on $[0, b]$ in view of Theorem 1.19 d). Part a) then follows from the continuity of $E : Z \to \mathbb{R}$.

b) Statement a) yields $\|(w(t), \partial_t w(t))\|_Z^2 \leq 2E_w(t) = 2E_w(0)$ if $a \geq 0$. The blow-up condition in Theorem 1.19 b) thus implies assertion b).

c) Sobolev's and Poincaré's estimates provide constants $c_{\mathrm{S}}, c_{\mathrm{P}} > 0$ with $\|\varphi\|_4 \leq c_{\mathrm{S}} \|\varphi\|_{1,2}$ and $\|\varphi\|_2 \leq c_{\mathrm{P}} \| |\nabla \varphi|_2 \|_2$ for $\varphi \in W_0^{1,2}(G)$, see Theorems 3.31 and 3.36 in [**33**]. The consequence $\|\varphi\|_{1,2}^2 \leq (c_{\mathrm{P}}^2 + 1) \| |\nabla \varphi|_2 \|^2$ is used several times.

We may let $a \neq 0$. Fix $\varepsilon_0 = \big((c_{\mathrm{P}}^2 + 1) c_{\mathrm{S}}^4 |a| \big)^{-\frac{1}{2}}$. Given $\varepsilon \in (0, \varepsilon_0]$, we define $\rho = \rho_\varepsilon > 0$ by $2(c_{\mathrm{P}}^2 + 1)(2\rho^2 + c_{\mathrm{S}}^4 |a| (c_{\mathrm{P}}^2 + 1)^2 \rho^4) = \varepsilon^2$. Let $u_0 = (w_0, w_1) \in \overline{B}_Z(0, \rho)$ and w be the weak solution of (1.17). The number

$$\beta \coloneqq \sup \big\{ b \in [0, t^+(u_0)) \, \big| \, \forall\, t \in [0, b] : \|w(t)\|_{1,2} \leq \varepsilon \big\}$$

belongs to $(0, t^+(u_0)]$ because of $\|w_0\|_{1,2}^2 \leq (c_{\mathrm{P}}^2 + 1)\rho^2 < \varepsilon^2$.

We have to show $\beta = \infty$. Let $t \in [0, \beta)$. The above estimates, the definition of E_w and part a) lead to

$$\begin{aligned} \|w(t)\|_{1,2}^2 + \|\partial_t w(t)\|_2^2 &\leq 2(c_{\mathrm{P}}^2 + 1)(E_w(t) - \tfrac{a}{4}\|w(t)\|_4^4) \\ &\leq 2(c_{\mathrm{P}}^2 + 1)E_w(0) + \tfrac{1}{2}(c_{\mathrm{P}}^2 + 1)c_{\mathrm{S}}^4 |a| \|w(t)\|_{1,2}^2 \|w(t)\|_{1,2}^2 \\ &\leq 2(c_{\mathrm{P}}^2 + 1)E_w(0) + \tfrac{1}{2}\|w(t)\|_{1,2}^2. \end{aligned}$$

We thus obtain the bound

$$\|w(t)\|_{1,2}^2 + \|\partial_t w(t)\|_2^2 \le 4(c_{\mathrm{P}}^2+1)E_w(0) \le 2(c_{\mathrm{P}}^2+1)\rho^2 + (c_{\mathrm{P}}^2+1)|a|\,\|w_0\|_4^4$$
$$\le (c_{\mathrm{P}}^2+1)\big(2\rho^2 + c_{\mathrm{S}}^4|a|(c_{\mathrm{P}}^2+1)^2\rho^4\big) = \tfrac12\varepsilon^2.$$

Theorem 1.19 b) then yields $t^+(u_0) > \beta$ unless $\infty = \beta = t^+(u_0)$. In the first case, we infer $\|w(t)\|_{1,2} < \varepsilon$ for $t \in [\beta, \beta+\delta]$ and some $\delta > 0$ by continuity. This fact contradicts the definition of β so that $\beta = \infty$ and assertion c) are true. □

In the case $a < 0$ the reasoning in part b) fails since $E_w(t)$ does not control the norm of Z. To show blowup of solutions here, we first look at the case of the 'Neumann boundary condition' $\partial_\nu w = 0$ on ∂G (if $\partial G \in C^2$, say). Spatially constant functions $w(t,x) = \varphi(t)$ satisfy this condition and belong to the kernel of the Laplacian. Such a map w thus solves the Neumann-version of (1.17) if and only if $\varphi'' = |a|\varphi^3$. For given $\varphi(0) = c > 0$ this equation has the solution

$$\varphi_c(t) = \frac{c}{1 - c\sqrt{|a|/2}\,t}$$

with maximal existence time $t_c^+ = \sqrt{2|a|^{-1}c^{-2}}$. Hence, an 'ODE blowup' as in (1.3) is present in (1.17) with Neumann boundary conditions if $a < 0$. Here the initial data $w_0 = c\mathbb{1}$ and $w_1 = c^2\sqrt{|a|/2}\,\mathbb{1}$ can be arbitrarily small in $L^2(G)$ in contrast to the Dirichlet case in Theorem 1.20.

To establish blowup for Dirichlet conditions and a large class of initial values, we will derive a differential inequality for the map $\phi(t) = \|w(t)\|_2^2/4$ which implies the desired explosion. We use the energy preservation and (1.17) to control derivatives of w occuring the argument.

Proposition 1.21. *Let $G \subseteq \mathbb{R}^3$ be open and bounded with $\partial G \in C^{1-}$ and $a < 0$. Assume that the initial data $u_0 = (w_0, w_1) \in Z$ have real values, nonpositive energy $E(w_0, w_1) \le 0$, i.e.,*

$$\tfrac{|a|}{4}\|w_0\|_4^4 \ge \tfrac12\|\,|\nabla w_0|_2\|_2^2 + \tfrac12\|w_1\|_2^2\,, \tag{1.21}$$

and that the inequality

$$\int_G w_0 w_1\,\mathrm{d}x > 0 \tag{1.22}$$

is true. Then $t^+(u_0) < \infty$.

Let $C = 4|a|/\lambda(G)$ and $\tau = 4\sqrt{3}/(\sqrt{C}\,\|w_0\|_2)$. Assume in addition that

$$\int_G w_0 w_1\,\mathrm{d}x \ge \frac{\sqrt{C}}{4\sqrt{3}}\,\|w_0\|_2^3. \tag{1.23}$$

We then have $t^+(u_0) \le \tau$ and $\|w(t)\|_2 \ge \big(\|w_0\|_2^{-1} - \sqrt{C/48}\,t\big)^{-1}$ for $t < t^+(u_0)$.

Conditions (1.21) and (1.22) are satisfied if $w_1 \in W_0^{1,2}(G)$ is real and non-zero and $w_0 = \nu w_1$ for large $\nu > 0$. If w_1 has compact support, we can increase $\lambda(G)$ to fulfill (1.23). Also, if the assumptions are true for u_0 with strict inequalities, then they hold on a ball in Z around u_0. In accordance with Theorem 1.20 c), the above blow-up data have a minimal size $\|w_0\|_6^2 \ge \kappa > 0$. Indeed, Sobolov's estimate, see (3.38) in [**33**], (1.21) and Hölder's inequality imply

$$\|w_0\|_6^2 \le 16\|\,|\nabla w_0|_2\|_2^2 \le 8|a|\,\|w_0\|_4^4 \le 8|a|\lambda(G)^{\frac13}\|w_0\|_6^4 =: \kappa^{-1}\|w_0\|_6^4.$$

Proof. At first, we do not assume (1.23).

1) Theorem 1.19 provides a maximal weak real-valued solution

$$w \in C(J^+, W_0^{1,2}(G)) \cap C^1(J^+, L^2(G)) \cap C^2(J^+, W^{-1,2}(G))$$

of (1.17) for the data u_0 as in the assertion. Let $t \in J^+ = J^+(u_0)$. We set $\phi(t) = \frac{1}{4}\|w(t)\|_2^2 = \frac{1}{4}(w(t)|w(t)) \geq 0$. This function has the derivative

$$\phi'(t) = \frac{1}{2}\int_G w(t)\partial_t w(t)\,\mathrm{d}x = \frac{1}{2}\left\langle w(t), \partial_t w(t)\right\rangle_{W_0^{1,2}}.$$

Assumption (1.22) yields $\phi(0), \phi'(0) > 0$. We can differentiate again and obtain

$$\begin{aligned}
\phi''(t) &= \frac{1}{2}\int_G |\partial_t w(t)|^2\,\mathrm{d}x + \frac{1}{2}\langle w(t), \partial_t^2 w(t)\rangle_{W_0^{1,2}} \\
&= \frac{1}{2}\int_G |\partial_t w(t)|^2\,\mathrm{d}x + \frac{1}{2}\langle w(t), \Delta_D w(t)\rangle_{W_0^{1,2}} - \frac{a}{2}\int_G |w(t)|^4\,\mathrm{d}x \\
&= \frac{1}{2}\int_G \left(|\partial_t w(t)|^2 - |\nabla w(t)|_2^2 - a|w(t)|^4\right)\mathrm{d}x
\end{aligned}$$

where we also insert (1.17) and use the definition of the extended Laplacian. The conservation of energy from Theorem 1.20 a) and assumption (1.21) imply

$$\phi''(t) = \int_G |\partial_t w(t)|^2\,\mathrm{d}x - \frac{a}{4}\int_G |w(t)|^4\,\mathrm{d}x - E(w_0, w_1) \geq \frac{|a|}{4}\int_G |w(t)|^4\,\mathrm{d}x.$$

On the other hand, Hölder's inequality leads to

$$\phi(t)^2 = \frac{1}{16}\Big(\int_G |w(t)|^2\,\mathrm{d}x\Big)^2 \leq \frac{\lambda(G)}{16}\int_G |w(t)|^4\,\mathrm{d}x.$$

Together we derive

$$\phi''(t) \geq \frac{4|a|}{\lambda(G)}\phi(t)^2 =: C\phi(t)^2.$$

Integrating twice, we see that

$$\begin{aligned}
&\phi'(t) \geq \phi'(0) + C\int_0^t \phi(s)^2\,\mathrm{d}s \geq \phi'(0) > 0, \\
&\phi(t) \geq \phi(0) + t\phi'(0) > 0.
\end{aligned} \tag{1.24}$$

Hence, ϕ (strictly) increases. We can now estimate

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\tfrac{1}{2}(\phi'(t))^2 = \phi''(t)\phi'(t) \geq C\phi(t)^2\phi'(t).$$

on J^+. Two more integrations imply the inequality

$$\phi'(t)^2 \geq \phi'(0)^2 + 2C\int_0^t \phi'(s)\phi(s)^2\,\mathrm{d}s = \tfrac{2C}{3}\phi(t)^3 - \tfrac{2C}{3}\phi(0)^3 + \phi'(0)^2. \tag{1.25}$$

2) We suppose that $J^+ = \mathbb{R}_{\geq 0}$. Since $\phi'(0) > 0$, we can fix $t_0 \in \mathbb{R}_{\geq 0}$ with

$$\big(\phi(0) + t_0\phi'(0)\big)^3 \geq 2\phi(0)^3 - 3\phi'(0)^2/C. \tag{1.26}$$

Let $t \geq t_0$. The lower estimate (1.24) leads to

$$\phi(t)^3 \geq \phi(t_0)^3 \geq (\phi(0) + t_0\phi'(0))^3$$

From (1.25) and (1.26) it thus follows

$$\phi'(t)^2 \geq \tfrac{C}{3}\phi(t)^3 + \tfrac{C}{3}(\phi(0) + t_0\phi'(0))^3 - \tfrac{2C}{3}\phi(0)^3 + \phi'(0)^2 \geq \tfrac{C}{3}\phi(t)^3.$$

As a result, ϕ satisfies the differential inequality

$$\phi'(t) \geq \sqrt{C/3}\,\phi(t)^{\frac{3}{2}}, \quad t \geq t_0, \qquad \phi(t_0) = \tfrac{1}{4}\|w(t_0)\|_2^2 > 0.$$

The corresponding equation

$$\psi'(t) = \sqrt{C/3}\,\psi(t)^{\frac{3}{2}}, \quad t \geq 0, \qquad \psi(t_0) = \tfrac{1}{4}\|w(t_0)\|_2^2,$$

has the blow-up solution $\psi(t - t_0) = \big(2\|w(t_0)\|_2^{-1} - \sqrt{C/12}\,(t - t_0)\big)^{-2}$ for $t_0 \leq t < t_0 + \tau$. As in Lemma 5.10 of [**31**] one can show that $\frac{1}{4}\|w(t)\|_2^2 = \phi(t) \geq \psi(t - t_0)$ for $t \in [t_0, t_0 + \tau)$. This fact contradicts the assumption $t^+(u_0) = \infty$.

3) Let (1.23) be true. We can now take $t_0 = 0$ in (1.26) since (1.23) yields

$$\frac{3}{C}\phi'(0)^2 = \frac{3}{4C}\Big(\int_S w_0 w_1 \,\mathrm{d}x\Big)^2 \geq \frac{1}{64}\Big(\int_G w_0^2 \,\mathrm{d}x\Big)^3 = \phi(0)^3.$$

The calculations in step 2) then still work for $t \in J^+$ and lead to $\|w(t)\|_2 \geq 2\psi(t)^{\frac{1}{2}}$, which yields the second assertion. □

The results in this section can be extended to the nonlinearity $-a|v|^{\alpha-1}v$ with $\alpha \in (1, 3]$ by essentially the same arguments. The blow-up example actually works for all $\alpha > 1$ with some modifications. See the exercises, where also further properties of the semilinear wave equation are discussed.

CHAPTER 2

Interpolation theory and regularity

Interpolation theory is an independent branch of functional analysis which has important applications in many fields of mathematics. To explain the basic idea in our context, we look at the spaces $[\mathrm{D}(A)] \hookrightarrow X$ for a generator A of an analytic C_0-semigroup $T(\cdot)$. We want to construct 'interpolation spaces' $[\mathrm{D}(A)] \hookrightarrow Y \hookrightarrow X$, which means that each operator $T \in \mathcal{B}(X)$ having a bounded restriction $T_1 : [\mathrm{D}(A)] \to [\mathrm{D}(A)]$ also leaves invariant Y and the restriction $T_Y : Y \to Y$ is bounded. We use such spaces to establish norm bounds on $T(t) : X \to Y$ for $t > 0$ using the known ones for $T(t) : X \to X$ and $T(t) : X \to [\mathrm{D}(A)]$. This fact will be crucial for the treatment of a large class of reaction-diffusion equations and other semilinear 'parabolic' problems in the next chapter.

Among others, the monographs [**5**], [**21**] and [**37**] are devoted to interpolation theory. The above indicated applications to parabolic evolution equations are stressed in [**20**] and [**21**], for instance. We focus here on these applications and do not develop the general theory explicitely, though it is hidden in some of the proofs. In this sense the next section is similar to Section II.5 of [**8**] (which is concerned with the spaces $D_A(\alpha, \infty)$ and $D_A(\alpha)$ in our notation), but we are closer to interpolation theory omitting certain other aspects investigated in [**8**].

2.1. Real interpolation spaces for semigroups

In this section we always work in the following setting, sometimes adding more restrictions and assumptions.

$$\begin{aligned}&\text{Let } A \text{ generate the } C_0\text{-semigroup } T(\cdot) \text{ on } X, \quad M_0 := \sup\nolimits_{t\in[0,1]} \|T(t)\|,\\ &\alpha \in (0,1), \quad \text{and} \quad q \in [1,\infty].\end{aligned} \tag{2.1}$$

Recall that $T(\cdot)x$ is continuous for $x \in X$ and continuously differentiable for $x \in \mathrm{D}(A)$. One can define the 'real interpolation spaces' between X and $[\mathrm{D}(A)]$ by looking at $x \in X$ such that $T(\cdot)x$ is Hölder continuous (or satisfies an L^q-variant of this property). To that purpose, we define

$$\varphi_{\alpha,x}(s) = s^{-\alpha} \|T(s)x - x\| \qquad \text{for} \quad x \in X, \quad s > 0.$$

If $T(\cdot)$ is a C_0-group, we set $\varphi_{\alpha,x}(s) = |s|^{-\alpha} \|T(s)x - x\|$ for all $s \neq 0$. We further introduce the weighted space $L^q_*(J) = L^q(J, \mathrm{d}s/|s|)$ for any Borel set $J \subseteq \mathbb{R} \setminus \{0\}$, and abbreviate $L^q_* = L^q_*((0,1])$. Observe that $L^\infty_*(J) = L^\infty(J)$ and that $L^q_*(J)$ is endowed with the norm given by

$$\|f\|^q_{L^q_*(J)} = \int_J |f(s)|^q \,\frac{\mathrm{d}s}{|s|}$$

if $q \in [1,\infty)$. Some special features of these spaces are discussed below. We can now formulate the basic notions of this chapter.

Definition 2.1. *Let* (2.1) *be true and* $x \in X$. *We define the quantities*

$$[x]_{\alpha,q} = \|\varphi_{\alpha,x}\|_{L^q_*} \in [0,\infty], \qquad \|x\|_{\alpha,q} = \|x\| + [x]_{\alpha,q}.$$

The real interpolation space *between* X *and* $[\mathrm{D}(A)]$ *of order* $\alpha \in (0,1)$ *and exponent* $q \in [1,\infty]$ *is given by*

$$D_A(\alpha,q) = \big\{x \in X \,\big|\, [x]_{\alpha,q} < \infty\big\},$$

and the continuous interpolation space *by the closure* $D_A(\alpha) = \overline{\mathrm{D}(A)}^{\|\cdot\|_{\alpha,\infty}}$.

It is easy to see that $D_A(\alpha,q)$ is a vector space with norm $\|\cdot\|_{\alpha,q}$ and that $D_A(\alpha)$ is a closed subspace of $D_A(\alpha,\infty)$. Observe that $D_A(\alpha,q)$ is defined just by an estimate (and not by a limit such as the space of continuous functions). In Example 2.3 below we see that $D_A(\alpha) \neq D_A(\alpha,\infty)$, in general.

We first discuss slight variants of the above concepts, where $x \in X$. For $0 < a < b \le n \in \mathbb{N}$, an elementary estimate yields

$$\|x\|+\|\varphi_{\alpha,x}\|_{L^q_*((0,a])} \le \|x\|+\|\varphi_{\alpha,x}\|_{L^q_*((0,b])} \le \|x\|+\|\varphi_{\alpha,x}\|_{L^q_*((0,a])}+c_0\|x\|, \quad (2.2)$$

where for $q \in [1,\infty)$ the constant $c_0 = c_0(a,b,\alpha,M_0)$ is given by

$$c_0 := (M_0^n+1)\Big(\int_a^b s^{-\alpha q-1}\,\mathrm{d}s\Big)^{\frac{1}{q}} \le \frac{M_0^n+1}{(\alpha q)^{1/q}a^\alpha} \le \frac{M_0^n+1}{\alpha a^\alpha}\,,$$

and by $c_0 = a^{-\alpha}(M_0^n+1)$ if $q = \infty$. In Definition 2.1 one can thus replace the interval $J = (0,1]$ by any $J = (0,\tau]$, just obtaining an equivalent norm.

To establish similar results for unbounded intervals, we have to impose more conditions on the semigroup. First, let $T(\cdot)$ be bounded. Setting $M := \sup_{t\ge 0}\|T(t)\| < \infty$, we infer $|\varphi_{\alpha,x}(s)| \le s^{-\alpha}(1+M)\|x\|$ and thus

$$\|\varphi_{\alpha,x}\|_{L^q_*(1,\infty)} \le \alpha^{-1/q}(1+M)\|x\| \le \alpha^{-1}(1+M)\|x\| =: c_1\|x\|.$$

This inequality yields the norm equivalence

$$\|x\|_{\alpha,q} \le \|x\| + \|\varphi_{\alpha,x}\|_{L^q_*(\mathbb{R}_+)} \le \|x\|_{\alpha,q} + (1+c_1)\|x\|. \quad (2.3)$$

Next, let $T(\cdot)$ be a C_0-group bounded by $\tilde{M}$ on $\mathbb{R}$. Using the inequality

$$\varphi_{\alpha,x}(s) = (-s)^{-\alpha}\|T(s)(x - T(-s)x)\| \le \tilde{M}\varphi_{\alpha,x}(-s)$$

for $s < 0$ and the one before (2.3), we estimate

$$\|x\|_{\alpha,q} \le \|x\| + \|\varphi_{\alpha,x}\|_{L^q_*(\mathbb{R}\setminus\{0\})} \le (1+\tilde{M})\|x\|_{\alpha,q} + c_1(1+\tilde{M})\|x\|. \quad (2.4)$$

We further check that also a rescaling of the semigroup leads to an equivalent norm on the interpolation spaces. This fact will be useful in some proofs. Let $\omega \in \mathbb{R}$ and $s \in (0,1]$. Recall that $A - \omega I$ generates the C_0-semigroup $(\mathrm{e}^{-\omega t}T(t))_{t\ge 0}$ by Lemma 1.17 of [**32**]. This 'rescaled' semigroup decays exponentially and $A - \omega I$ is invertible, if ω is larger than the growth bound $\omega_0(A)$ of A, see Definition 1.5 and Proposition 1.20 of [**32**]. (We have $\|(A-\omega I)^{-1}\| \le M/(\omega - \overline{\omega})$ if $\|T(t)\| \le M\mathrm{e}^{\overline{\omega}t}$ for $t \ge 0$ and $\omega > \overline{\omega}$.) By means of the mean value theorem, we compute

$$\varphi_{\alpha,x}(s) \le s^{-\alpha}\|\mathrm{e}^{\omega s}(\mathrm{e}^{-\omega s}T(s)x - x)\| + s^{-1}|\mathrm{e}^{\omega s}-1|\,s^{1-\alpha}\|x\|$$

$$\leq \mathrm{e}^{\omega_+}\big(|\omega|\|x\| + s^{-\alpha}\|\mathrm{e}^{-\omega s}T(s)x - x\|\big), \tag{2.5}$$

$$s^{-\alpha}\|\mathrm{e}^{-\omega s}T(s)x - x\| \leq s^{-\alpha}\mathrm{e}^{-\omega s}\|T(s)x - x\| + s^{-1}|\mathrm{e}^{-\omega s} - 1|\, s^{1-\alpha}\|x\|$$
$$\leq \mathrm{e}^{\omega_-}(|\omega|\|x\| + \varphi_{\alpha,x}(s)).$$

In the next proposition we collect basic properties of the real interpolation spaces, which follow from their definition by elementary arguments and standard semigroup theory. Observe that they become smaller if α increases or if q decreases. Moreover, spaces with larger α are smaller independent of q. In this sense the parameter q provides a 'fine tuning.' As announced above, the space $D_A(\alpha,\infty)$ consists of the vectors x with Hölder continuous orbits $T(\cdot)x$.

Here and below, we write $a \lesssim_K b$ if $a \leq cb$ for all $a, b \in \mathbb{R}$ and a constant $c = c(K) > 0$ depending on K, as well as $a \eqsim_K b$ if $a \lesssim_K b$ and $b \lesssim_K a$.

PROPOSITION 2.2. *Let (2.1) be true, $0 < \alpha < \beta < 1$, $1 \leq p \leq q \leq \infty$, $b > 0$, and $x \in X$. Then the following assertions hold.*

a) $D_A(\alpha, q)$ and $D_A(\alpha)$ are Banach spaces for $\|\cdot\|_{\alpha,q}$ and $\|\cdot\|_{\alpha,\infty}$, respectively.

b) $[\mathrm{D}(A)] \hookrightarrow D_A(\beta,\infty) \hookrightarrow D_A(\alpha, 1) \hookrightarrow X$.

c) Let $q < \infty$. Then $\mathrm{D}(A)$ is dense in $D_A(\alpha, q)$.

d) Let $q<\infty$. Then $D_A(\alpha, 1) \hookrightarrow D_A(\alpha, p) \hookrightarrow D_A(\alpha, q) \hookrightarrow D_A(\alpha) \subseteq D_A(\alpha,\infty)$.

e) $x \in D_A(\alpha,\infty)$ if and only if $T(\cdot)x \in C^\alpha([0, b], X)$.

f) $x \in D_A(\alpha)$ if and only if $t^{-\alpha}(T(t)x - x) \longrightarrow 0$ in X as $t \to 0$.

(In the proof are bounds on the norms of the embeddings. Note $\|x\|_{\alpha,q} \leq \|x\|_{\beta,q}$. The continuous embeddings are given by the respective inclusion maps.)

PROOF. Take $s \in (0, 1]$ and let α, β, p, and q be given as in the statements. We show each part of the proposition separately.

a) In view of the remarks after Definition 2.1, we only have to prove that $(D_A(\alpha, q), \|\cdot\|_{\alpha,q})$ is complete. Let (x_n) be a Cauchy sequence in $D_A(\alpha, q)$. We thus have $c_q := \sup_n \|x_n\|_{\alpha,q} < \infty$, and the vectors x_n converge to some x in X since X is a Banach space. Hence, $\varphi_{\alpha,x_n}(s)$ tends to $\varphi_{\alpha,x}(s)$ as $n \to \infty$. If $q = \infty$, it follows that $\varphi_{\alpha,x}(s) \leq \limsup_n \varphi_{\alpha,x_n}(s) \leq c_\infty$. For $q < \infty$, Fatou's Lemma yields

$$\|\varphi_{\alpha,x}\|^q_{L^q_*} = \int_0^1 \lim_{n\to\infty} \varphi_{\alpha,x_n}(s)^q \frac{\mathrm{d}s}{s} \leq \liminf_{n\to\infty} \int_0^1 \varphi_{\alpha,x_n}(s)^q \frac{\mathrm{d}s}{s} \leq c_q^q.$$

In both cases x belongs to $D_A(\alpha, q)$.

Let $\varepsilon > 0$. There is an index $N_\varepsilon \in \mathbb{N}$ such that $[x_m - x_n]_{\alpha,q} \leq \varepsilon$ for all $n, m \geq N_\varepsilon$. Since φ_{α,x_m-x_n} tends pointwise to $\varphi_{\alpha,x-x_n}$ as $m \to \infty$, we obtain as above the bound

$$[x - x_n]_{\alpha,q} = \|\varphi_{\alpha,x-x_n}\|_{L^q_*} \leq \limsup_{m\to\infty} \|\varphi_{\alpha,x_m-x_n}\|_{L^q_*} \leq \varepsilon$$

for all $n \geq N_\varepsilon$. As a consequence, x_n converges to x in $D_A(\alpha, q)$.

b) The last embedding in part b) is clear. To see the second, for $x \in D_A(\beta,\infty)$ we simply estimate

$$\|\varphi_{\alpha,x}\|_{L^1_*} = \int_0^1 s^{\beta-\alpha-1} s^{-\beta}\|T(s)x - x\|\,\mathrm{d}s \leq \frac{1}{\beta - \alpha}\,\|\varphi_{\beta,x}\|_{L^\infty_*}\,.$$

Let $x \in \mathrm{D}(A)$. Lemma 1.18 of [**32**] then implies the inequality

$$\varphi_{\beta,x}(s) = s^{1-\beta}\Big\|\frac{1}{s}\int_0^s T(\tau)Ax\,\mathrm{d}\tau\Big\| \le M_0\|Ax\|,$$

so that $[\mathrm{D}(A)] \hookrightarrow D_A(\beta,\infty)$.

c) Let $x \in D_A(\alpha,q)$, $q<\infty$, and $\omega_0(A) < n \in \mathbb{N}$. We set $x_n = nR(n,A)x \in \mathrm{D}(A)$ and $c = \sup_{n>\omega_0(A)} \|nR(n,A)\| < \infty$, see Lemma 1.22 of [**32**]. Note that

$$\varphi_{\alpha,x-x_n}(s) = s^{-\alpha}\|(T(s)-I)(x-x_n)\| \ \longrightarrow\ 0, \qquad \text{as } \ n\to\infty,$$
$$0 \le \varphi_{\alpha,x-x_n}(s) \le \varphi_{\alpha,x}(s) + s^{-\alpha}\|nR(n,A)(T(s)-I)x\| \le (1+c)\varphi_{\alpha,x}(s).$$

By dominated convergence, the functions $\varphi_{\alpha,x-x_n}$ thus tend to 0 in L^q_* as $n\to\infty$ which yields assertion c).

d) Let $x \in D_A(\alpha,r)$ for $r\in[1,\infty)$. We compute

$$\begin{aligned}
s^{-\alpha}\|T(s)x-x\| &= \Big(2^{-\alpha r} + \alpha r\int_s^2 \tau^{-\alpha r-1}\,\mathrm{d}\tau\Big)^{\frac1r}\|T(s)x-x\|\\
&\le 2^{-\alpha}(M_0+1)\|x\| + (\alpha r)^{\frac1r}\Big(\int_s^2 \tau^{-\alpha r}\|T(s)x-T(\tau)x+T(\tau)x-x\|^r\,\frac{\mathrm{d}\tau}{\tau}\Big)^{\frac1r}\\
&\le (M_0+1)\|x\| + \mathrm{e}^{1/\mathrm{e}}\Big(\int_s^2 (\tau-s)^{-\alpha r}\|T(s)(x-T(\tau-s)x)\|^r\,\frac{\mathrm{d}\tau}{\tau-s}\Big)^{\frac1r}\\
&\qquad + \mathrm{e}^{1/\mathrm{e}}\Big(\int_s^2 \tau^{-\alpha r}\|T(\tau)x-x\|^r\,\frac{\mathrm{d}\tau}{\tau}\Big)^{\frac1r}\\
&\le \mathrm{e}^{1/\mathrm{e}}(M_0+1)\,\Big(\|x\| + \Big(\int_0^2 \sigma^{-\alpha r}\|T(\sigma)x-x\|^r\,\frac{\mathrm{d}\sigma}{\sigma}\Big)^{\frac1r}\Big)\\
&\lesssim_{M_0,\alpha} \|x\|_{\alpha,r},
\end{aligned}$$

where we substituted $\sigma = \tau - s$ and used (2.2). It follows that $D_A(\alpha,r) \hookrightarrow D_A(\alpha,\infty)$. For $x\in D_A(\alpha,q)$, part b) provides vectors $x_n\in\mathrm{D}(A)$ converging to x in $D_A(\alpha,q)$, and hence in $D_A(\alpha,\infty)$. This means that $D_A(\alpha,q)$ is even embedded into $D_A(\alpha)$. From $D_A(\alpha,p)\hookrightarrow D_A(\alpha,\infty)$ we finally infer

$$\|\varphi_{\alpha,x}\|_{L^q_*} = \Big(\int_0^1 |\varphi_{\alpha,x}|^p\,|\varphi_{\alpha,x}|^{q-p}\,\frac{\mathrm{d}s}{s}\Big)^{\frac1q} \le \|\varphi_{\alpha,x}\|_{L^p_*}^{\frac pq}\,\|\varphi_{\alpha,x}\|_\infty^{1-\frac pq} \lesssim_{\alpha,M_0} \|\varphi_{\alpha,x}\|_{L^p_*},$$

establishing statement d).

e) The implication '$\Leftarrow$' is clear. For the other implication, let $x\in D_A(\alpha,\infty)$, $0\le s<t\le b$, and $K := \sup_{t\in[0,b]}\|T(t)\|$, where we may assume that $b\ge 1$. If $t-s\ge1$, one trivially has $(t-s)^{-\alpha}\|T(t)x-T(s)x\|\le 2K\|x\|$. For $t-s\le 1$, the semigroup property yields

$$(t-s)^{-\alpha}\|T(t)x-T(s)x\| \le K(t-s)^{-\alpha}\|T(t-s)x-x\| \le K[x]_{\alpha,\infty}.$$

f) For $x\in D_A(\alpha)$ and $\varepsilon>0$, there is a vector $y\in\mathrm{D}(A)$ such that $\|x-y\|_{\alpha,\infty}\le\varepsilon$. We can thus estimate

$$\varlimsup_{t\to0} t^{-\alpha}\|T(t)x-x\| \le [x-y]_{\alpha,\infty} + \varlimsup_{t\to0} t^{1-\alpha}\,t^{-1}\|T(t)y-y\| \le \varepsilon + 0\,\|Ay\| = \varepsilon,$$

proving the 'only if' part. Conversely, assume that $\varphi_{\alpha,x}(s)$ tends to 0 as $s \to 0$. Let $\varepsilon > 0$ and $s, t \in (0,1]$. First, there thus exists $\delta_\varepsilon \in (0,1)$ with

$$s^{-\alpha}\|(T(s)-I)(T(t)x-x)\| \le (1+M_0)s^{-\alpha}\|T(s)x-x\| \le \varepsilon$$

for all $s \in (0,\delta_\varepsilon]$ and $t \in (0,1]$. Second, we find a number $\eta \in (0,1)$ such that

$$s^{-\alpha}\|(T(s)-I)(T(t)x-x)\| \le (1+M_0)\delta_\varepsilon^{-\alpha}\|T(t)x-x\| \le \varepsilon$$

for all $s \in [\delta_\varepsilon, 1]$ and $t \in (0,\eta]$. This means that $T(t)x$ tends to x in $D_A(\alpha,\infty)$ as $t \to 0$ so that the vectors $y_n = n\int_0^{1/n} T(t)x\,\mathrm{d}t$ converge to x in $D_A(\alpha,\infty)$ as $n \to \infty$. Hence, x belongs to $D_A(\alpha)$ as $y_n \in \mathrm{D}(A)$ by Lemma 1.18 in [**32**]. $\square$

We now describe the interpolation spaces for the translation group.

Example 2.3. Let $X = L^p(\mathbb{R})$ for some $p \in [1,\infty)$ or $X = C_0(\mathbb{R})$ for $p = \infty$. We consider the (isometric) translation group on X given by $T(t)f = f(\cdot+t)$ for $f \in X$ and $t \in \mathbb{R}$. It has the generator $A = \mathrm{d}/\mathrm{d}s$ with domain $\mathrm{D}(A) = W^{1,p}(\mathbb{R})$ respectively $\mathrm{D}(A) = C_0^1(\mathbb{R})$, cf. Examples 1.42 and 1.21 in [**32**]. Due to (2.4), the interpolation norms are given by

$$\begin{aligned}
\|f\|_{\alpha,q} &\eqsim_\alpha \|f\|_p + \Big(\int_{\mathbb{R}} |t|^{-\alpha q-1}\,\|f(\cdot+t)-f\|_{L^p(\mathbb{R})}^q\,\mathrm{d}t\Big)^{\frac{1}{q}}\\
&= \|f\|_p + \Big(\int_{\mathbb{R}}\Big(\int_{\mathbb{R}} \frac{|f(s+t)-f(s)|^p}{|t|^{\alpha p+\frac{p}{q}}}\,\mathrm{d}s\Big)^{\frac{q}{p}}\,\mathrm{d}t\Big)^{\frac{1}{q}},\\
\|f\|_{\alpha,\infty} &= \|f\|_\infty + \sup_{t\in\mathbb{R}\setminus\{0\}}\Big(\int_{\mathbb{R}} \frac{|f(s+t)-f(s)|^p}{|t|^{\alpha p}}\,\mathrm{d}s\Big)^{\frac{1}{p}}
\end{aligned}$$

for $p, q \in [1,\infty)$, and for $p = q = \infty$ by the Hölder norm

$$\|f\|_{\alpha,\infty} = \|f\|_\infty + \sup_{t\in\mathbb{R}\setminus\{0\},\, s\in\mathbb{R}} \frac{|f(s+t)-f(s)|}{|t|^\alpha}.$$

For $p < \infty$, the space $D_A(\alpha,q)$ coincides with the *Besov space* $B^\alpha_{pq}(\mathbb{R})$, see [**37**] and [**38**].[1] In the special case $p = q \in [1,\infty)$, the space $B^\alpha_{pp}(\mathbb{R}) =: W^{\alpha,p}(\mathbb{R})$ is called *Slobodetskii space* (or *fractional Sobolev space*) and has the simpler norm

$$\|f\|_{\alpha,p} \eqsim \|f\|_p + \Big(\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|f(\tau)-f(s)|^p}{|\tau-s|^{\alpha p+1}}\,\mathrm{d}\tau\,\mathrm{d}s\Big)^{\frac{1}{p}}.$$

(Here one uses Fubini's theorem and the substitution $\tau = s+t$.)

There are functions f in $C_0(\mathbb{R})$ with a finite Hölder norm $\|f\|_{\alpha,\infty}$ such that $f(s) = s^\alpha$ for $s \in [0,1]$, and thus $t^{-\alpha}\|T(t)f-f\|_\infty \ge t^{-\alpha}|f(t)-f(0)| = 1$ for all $t \in (0,1]$. Proposition 2.2 then yields $f \in D_A(\alpha,\infty)\setminus D_A(\alpha)$. $\Diamond$

We next see that $T(\cdot)$ behaves nicely on its interpolation spaces. But, in general, it is not strongly continuous on $D_A(\alpha,\infty)$. E.g., consider the translation group on $C_0(\mathbb{R})$ in Example 2.3, and take a map $f \in C_0(\mathbb{R}) \cap C_b^\alpha(\mathbb{R})$ with $f(t) = |t-n|^\alpha$ if $|t-n| \le \frac{1}{n}$ and $2 \le n \in \mathbb{N}$. For $g_n = T(\frac{1}{n})f - f$, it then holds

$$\big\|T(\tfrac{1}{n})f - f\big\|_{C^\alpha} \ge n^\alpha\big|g_n(n) - g_n(n-\tfrac{1}{n})\big| = n^\alpha|f(n+\tfrac{1}{n}) + f(n-\tfrac{1}{n})| = 2$$

[1] To obtain the Besov spaces $B^\alpha_{\infty q}(\mathbb{R})$ or Hölder spaces, one has to start with $L^\infty(\mathbb{R})$ or $C_b(\mathbb{R})$ instead of $C_0(\mathbb{R})$, which is not possible in our setting, but see p.13 in [**21**] or [**37**], [**38**].

for all $2 \leq n \in \mathbb{N}$. Nevertheless one could also work on $D_A(\alpha, \infty)$, cf. Section II.5 b) in [**8**] as well as Section 2.2 in [**20**] for analytic semigroups.

PROPOSITION 2.4. *Let* (2.1) *be true. Then the following assertions hold.*

a) We have $T(t)D_A(\alpha,q) \subseteq D_A(\alpha,q)$ and $T(t)D_A(\alpha) \subseteq D_A(\alpha)$ for all $t \geq 0$. The norms of the restrictions $T_{\alpha,q}(t) := T(t)\restriction_{D_A(\alpha,q)}$ and $T_\alpha(t) := T(t)\restriction_{D_A(\alpha)}$ are smaller or equal $\|T(t)\|_{\mathcal{B}(X)}$ for each $t \geq 0$. The operator families $T_{\alpha,q}(\cdot)$ if $q < \infty$ and $T_\alpha(\cdot)$ are C_0-semigroups on $D_A(\alpha,q)$ and $D_A(\alpha)$, respectively.

b) The generators of $T_{\alpha,q}(\cdot)$ and $T_\alpha(\cdot)$ are the restrictions (‘parts’*) $A_{\alpha,q}$ (with $q < \infty$) and A_α of A in the respective spaces endowed with the domains*

$$\mathrm{D}(A_{\alpha,q}) = \big\{x \in \mathrm{D}(A) \,\big|\, Ax \in D_A(\alpha,q)\big\} =: D_A(1+\alpha,q),$$
$$\mathrm{D}(A_\alpha) = \big\{x \in \mathrm{D}(A) \,\big|\, Ax \in D_A(\alpha)\big\} =: D_A(1+\alpha).$$

c) Let $\lambda \in \rho(A)$. Then λ belongs to $\rho(A_{\alpha,q})$ and $\rho(A_\alpha)$, with $R(\lambda, A_{\alpha,q}) = R(\lambda, A)\restriction_{D_A(\alpha,q)}$ if $q < \infty$ and $R(\lambda, A_\alpha) = R(\lambda, A)\restriction_{D_A(\alpha)}$. These resolvents have norm smaller or equal $\|R(\lambda, A)\|_{\mathcal{B}(X)}$.

d) We have $\sigma(A) = \sigma(A_{\alpha,q}) = \sigma(A_\alpha)$, where $q < \infty$.

PROOF. Let $x \in X$, $t \geq 0$, and $s \in (0, 1]$. Observe that

$$\varphi_{\alpha,T(t)x}(s) = s^{-\alpha}\|T(t)(T(s)x - x)\| \leq \|T(t)\|\,\varphi_{\alpha,x}(s).$$

Hence, the semigroups leave invariant the interpolation spaces and their restrictions have norms smaller or equal $\|T(t)\|$. Of course, they are still semigroups on these spaces. Let $x \in \mathrm{D}(A)$. Proposition 2.2 yields that

$$\|T(t)x - x\|_{\alpha,q} \leq c\|T(t)x - x\|_A \;\longrightarrow\; 0$$

as $t \to 0$. Since the restrictions are locally bounded, $T(\cdot)$ is strongly continuous on $D_A(\alpha)$ and, due to the density proved in Proposition 2.2, also on $D_A(\alpha,q)$ if $q < \infty$. We have shown assertion a).

From now on we take $q < \infty$. Let B be the generator of $T_{\alpha,q}(\cdot)$ and $A_{\alpha,q}$ be defined as in the statement. Let $x \in \mathrm{D}(B) \subseteq D_A(\alpha,q)$. Then $\frac{1}{t}(T(t)x - x)$ converges to Bx in $D_A(\alpha,q)$, as $t \to 0$. Since $D_A(\alpha,q) \hookrightarrow X$, these vectors also tend to Bx in X. This means that x belongs to $\mathrm{D}(A)$ and $Ax = Bx$ to $D_A(\alpha,q)$; i.e., $B \subseteq A_{\alpha,q}$.

Let $\lambda \in \rho(A)$. We show that λ is contained in $\rho(A_{\alpha,q})$, implying that $\rho(B)$ and $\rho(A_{\alpha,q})$ both contain a right halfplane, and hence $B = A_{\alpha,q}$ by Lemma 1.23 of [**32**]. Let $x \in D_A(\alpha,q)$. Then $AR(\lambda, A)x = \lambda R(\lambda, A)x - x$ is also an element of $D_A(\alpha,q)$, so that $R(\lambda, A)x \in D_A(\alpha + 1, q)$ and

$$(\lambda I - A_{\alpha,q})R(\lambda, A)x = (\lambda I - A)R(\lambda, A)x = x$$

because of the definition of $A_{\alpha,q}$. For $x \in D_A(\alpha + 1, q)$, we further have

$$R(\lambda, A)(\lambda I - A_{\alpha,q})x = R(\lambda, A)(\lambda I - A)x = x.$$

It follows that λ belongs to $\rho(A_{\alpha,q})$ and $R(\lambda, A_{\alpha,q}) = R(\lambda, A)|_{D_A(\alpha,q)}$. The estimate for $R(\lambda, A_{\alpha,q})$ is then shown as for $T_{\alpha,q}(t)$. The results in b) and c) for $D_A(\alpha)$ are proved in the same way.

Statement d) now is a direct consequence of Proposition IV.2.17 in [**8**]. □

We extend Example 2.3 to second derivatives and several dimensions.

Example 2.5. Let $p = q \in [1,\infty)$ for simplicity. We first look at $X = L^p(\mathbb{R})$, $A = \mathrm{d}/\mathrm{d}s$ and $\mathrm{D}(A) = W^{1,p}(\mathbb{R})$ as in Example 2.3. One defines the space

$$W^{1+\alpha,p}(\mathbb{R}) := \big\{u \in W^{1,p}(\mathbb{R}) \,\big|\, u' \in W^{\alpha,p}(\mathbb{R})\big\} = D_A(1+\alpha,p).$$

Based on somewhat deeper interpolation theory, Example 5.14 in [21] yields the identity $W^{1+\alpha,p}(\mathbb{R}) = D_{A^2}(\frac{1}{2}+\frac{\alpha}{2},p)$ and also $W^{\alpha,p}(\mathbb{R}) = D_{A^2}(\frac{\alpha}{2},p)$. Here $A^2 = \mathrm{d}^2/\mathrm{d}s^2$ has the domain $\mathrm{D}(A^2) = \{u \in W^{1,p}(\mathbb{R}) \,|\, u' \in W^{1,p}(\mathbb{R})\} = W^{2,p}(\mathbb{R})$. It generates a C_0-semigroup, cf. Example 1.48 in [32].

One introduces the Slobodetskii spaces on $\mathbb{R}^m$ as for $m = 1$ by

$$W^{\alpha,p}(\mathbb{R}^m) = \left\{u \in L^p(\mathbb{R}^m) \,\middle|\, \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \frac{|u(y)-u(x)|^p}{|y-x|^{\alpha p+m}}\,\mathrm{d}x\,\mathrm{d}y < \infty\right\},$$
$$W^{1+\alpha,p}(\mathbb{R}^m) = \big\{u \in W^{1,p}(\mathbb{R}^m) \,\big|\, \nabla u \in W^{\alpha,p}(\mathbb{R}^m)^m\big\}.$$

Let $p \in (1,\infty)$. Example 2.30 in [32] says that Δ with $\mathrm{D}(\Delta) = W^{2,p}(\mathbb{R}^m)$ generates an (analytic) C_0-semigroup. Again employing more interpolation theory, it can be shown that

$$W^{\alpha,p}(\mathbb{R}^m) = D_\Delta(\tfrac{\alpha}{2},p) \qquad \text{and} \qquad W^{1+\alpha,p}(\mathbb{R}^m) = D_\Delta(\tfrac{1}{2}+\tfrac{\alpha}{2},p).$$

See Examples 5.15 and 5.16 (combined with Proposition 5.7) in [21], where also the cases $p \in \{1,\infty\}$ are treated. ◊

Two equivalent definitions. We next characterize the interpolation spaces in terms of the resolvent of A and, for analytic semigroups, by the time derivative $\frac{\mathrm{d}}{\mathrm{d}t}T(\cdot) = AT(\cdot)$.

In the proof of the first equivalence, we need the following facts which highlight the role of the weight $1/t$ of the spaces $L^q_*(J)$. The multiplicative group $\mathbb{R}_+$ posseses the invariant measure $\mathrm{d}t/t$; i.e., the equality

$$\int_0^\infty f(\lambda s)\,\frac{\mathrm{d}s}{s} = \int_0^\infty f(\tau)\,\frac{\mathrm{d}\tau}{\tau} \tag{2.6}$$

holds for every measurable function $f : \mathbb{R}_+ \to \mathbb{R}_{\ge 0}$ and each $\lambda > 0$, due to the substitution $\tau = \lambda s$. Similarly, one obtains

$$\int_0^\infty f(s^{-1})\,\frac{\mathrm{d}s}{s} = \int_0^\infty f(\tau)\,\frac{\mathrm{d}\tau}{\tau}. \tag{2.7}$$

By means of these identities, as for the additive group $\mathbb{R}$ and the Lebesgue measure one can prove Young's inequality for the convolution integral

$$(f * g)(t) := \int_0^\infty f(ts^{-1})g(s)\,\frac{\mathrm{d}s}{s} = (g * f)(t), \qquad t > 0,$$

for $f \in L^p_*(\mathbb{R}_+)$, $g \in L^1_*(\mathbb{R}_+)$ and $p \in [1,\infty]$, namely

$$\|f * g\|_{L^p_*(\mathbb{R}_+)} \le \|f\|_{L^p_*(\mathbb{R}_+)}\,\|g\|_{L^1_*(\mathbb{R}_+)}. \tag{2.8}$$

To use the above results directly, we take the interval $J = \mathbb{R}_+$ in the next result and thus assume that the semigroup is bounded by $M := \sup_{t\ge 0}\|T(t)\| < \infty$ and hence $\mathbb{R}_+ \subseteq \rho(A)$ by Proposition 1.20 of [32]. This is always true if we replace the given A in (2.1) by $A - \omega I$ for some $\omega > \omega_0(A)$. The constants below would then depend on ω, too, see (2.5).

For $x \in X$, $\alpha \in (0,1)$ and $\lambda > 0$, we introduce the function

$$\chi_{\alpha,x}(\lambda) = \|\lambda^\alpha AR(\lambda, A)x\| = \|\lambda^{\alpha+1}R(\lambda, A)x - \lambda^\alpha x\|. \tag{2.9}$$

Observe that $\chi_{\alpha,x}$ is bounded for $\alpha = 0$ and $x \in X$ as well as for $\alpha = 1$ and $x \in \mathrm{D}(A)$ by the Hille–Yosida estimate $\|\lambda R(\lambda, A)\| \le M$ from Proposition 2.20 in [**32**]. The result belows says that the interpolation spaces fill the gap between these two extreme cases. Note that the limit $s \to 0$ is replaced by $\lambda \to \infty$ compared to Proposition 2.2 f). Besides (2.6)–(2.8), the proofs rely on basic formulas from [**32**] relating generators with their resolvent and semigroup.

Proposition 2.6. *Let* (2.1) *hold with* $M = \sup_{t\ge 0} \|T(t)\| < \infty$. *We then have*

$$D_A(\alpha, q) = \{x \in X \mid \chi_{\alpha,x} \in L^q_*(\mathbb{R}_+)\},$$

and the norm $\|\cdot\|_{\alpha,q}$ *is equivalent to* $x \mapsto \|x\| + \|\chi_{\alpha,x}\|_{L^q_*(\mathbb{R}_+)}$. *(In the proof one finds estimates on the corresponding constants.) Moreover, a vector* $x \in X$ *belongs to* $D_A(\alpha)$ *if and only if* $\chi_{\alpha,x}(\lambda) \to 0$ *as* $\lambda \to \infty$.

Proof. Let $x \in X$, $\alpha \in (0,1)$, $q \in [1,\infty]$, and $\lambda, s > 0$.

1) Let $x \in D_A(\alpha, q)$. The formula for $R(\lambda, A)$ in Proposition 1.20 of [**32**] and (2.6) imply

$$AR(\lambda, A)x = \lambda R(\lambda, A)x - x = \int_0^\infty \lambda \mathrm{e}^{-\lambda t} t^\alpha t^{-\alpha}(T(t)x - x)\,\mathrm{d}t,$$

$$\chi_{\alpha,x}(\lambda) \le \int_0^\infty (\lambda t)^{1+\alpha}\mathrm{e}^{-\lambda t}\varphi_{\alpha,x}(t)\,\frac{\mathrm{d}t}{t} = \int_0^\infty s^{1+\alpha}\mathrm{e}^{-s}\varphi_{\alpha,x}(\lambda^{-1}s)\,\frac{\mathrm{d}s}{s}. \tag{2.10}$$

Setting $\tilde\varphi_{\alpha,x}(\tau) = \varphi_{\alpha,x}(\tau^{-1})$, we infer from Young's inequality (2.8) the bound

$$\|\chi_{\alpha,x}\|_{L^q_*(\mathbb{R}_+)} \le \|\tilde\varphi_{\alpha,x}\|_{L^q_*(\mathbb{R}_+)} \int_0^\infty s^{1+\alpha}\mathrm{e}^{-s}\,\frac{\mathrm{d}s}{s} = \Gamma(1+\alpha)\|\varphi_{\alpha,x}\|_{L^q_*(\mathbb{R}_+)}$$

with the classical Gamma function and also using (2.7).

Let $x \in D_A(\alpha)$. We then have $\varphi_{\alpha,x}(s/\lambda) \le c(\alpha, M)\|x\|_{\alpha,\infty}$ and $\varphi_{\alpha,x}(s/\lambda) \to 0$ as $\lambda \to \infty$ because of (2.3) respectively Proposition 2.2 f). By dominated convergence, estimate (2.10) leads to $\chi_{\alpha,x}(\lambda) \to 0$ as $\lambda \to \infty$.

2) For the converse, let $\chi_{\alpha,x}$ belong to $L^q_*(\mathbb{R}_+)$. We decompose x into a vector in $\mathrm{D}(A)$ with a weight and one in X, writing

$$x = s^{-1}R(s^{-1}, A)x - AR(s^{-1}, A)x =: x_1 - x_2.$$

Lemma 1.18 of [**32**] and (2.9) now yield

$$\|T(s)x_1 - x_1\| \le \int_0^s \|T(\tau)Ax_1\|\,\mathrm{d}\tau \le sM\|Ax_1\| = s^\alpha M\chi_{\alpha,x}(s^{-1}),$$

$$\|T(s)x_2 - x_2\| \le (M+1)\|x_2\| = (1+M)s^\alpha\chi_{\alpha,x}(s^{-1}).$$

It follows that $\varphi_{\alpha,x}(s) \le (1+2M)\chi_{\alpha,x}(s^{-1})$ and hence

$$\|\varphi_{\alpha,x}\|_{L^q_*(\mathbb{R}_+)} \le (1+2M)\,\|\chi_{\alpha,x}\|_{L^q_*(\mathbb{R}_+)}$$

by (2.7). In view of estimate (2.3), the first assertion is shown.

Let $\chi_{\alpha,x}(\lambda) \to 0$ as $\lambda \to \infty$. Then $\varphi_{\alpha,x}(s)$ tends to 0 as $s \to 0$ by the above pointwise inequality, and the second claim results from Proposition 2.2 f). □

As a preparation for the next characterization, we prove an important estimate for power-weighted L^p–norms called *Hardy's inquality.*

LEMMA 2.7. *Let $a \in (0,\infty]$, $\alpha > 0$, $p \in [1,\infty)$ and $\varphi : (0,a) \to \mathbb{R}_{\geq 0}$ be measurable. We then have*

$$\int_0^a t^{-\alpha p}\Big(\int_0^t \varphi(s)\,\frac{\mathrm{d}s}{s}\Big)^p\,\frac{\mathrm{d}t}{t} \leq \frac{1}{\alpha^p}\int_0^a s^{-\alpha p}\varphi(s)^p\,\frac{\mathrm{d}s}{s}.$$

PROOF. We can assume that the right-hand side in the above inequality is finite. We set $f(\tau,t) = (f(\tau))(t) = t^{-\alpha}\varphi(\tau t)$ for $t \in (0,a)$ and $\tau \in (0,1]$. The substitution $\tau = s/t$ yields

$$N := \Big(\int_0^a t^{-\alpha p}\Big(\int_0^t \varphi(s)\,\frac{\mathrm{d}s}{s}\Big)^p\,\frac{\mathrm{d}t}{t}\Big)^{\frac{1}{p}} = \Big(\int_0^a \Big(\int_0^1 (f(\tau))(t)\,\frac{\mathrm{d}\tau}{\tau}\Big)^p\,\frac{\mathrm{d}t}{t}\Big)^{\frac{1}{p}}.$$

Below we show that $f : (0,1] \to L^q_*(0,a)$ is strongly measurable, compare the discussion before Lemma 4.4 in [**32**]. We can thus rewrite the above formula invoking a Bochner integral. Substituting also $s = \tau t$, it then follows

$$\begin{aligned} N &= \Big\|\int_0^1 f(\tau)\,\frac{\mathrm{d}\tau}{\tau}\Big\|_{L^p_*(0,a)} \leq \int_0^1 \|f(\tau)\|_{L^p_*(0,a)}\,\frac{\mathrm{d}\tau}{\tau} = \int_0^1 \Big[\int_0^a t^{-\alpha p}\varphi(\tau t)^p\,\frac{\mathrm{d}t}{t}\Big]^{\frac{1}{p}}\,\frac{\mathrm{d}\tau}{\tau}\\ &= \int_0^1 \Big(\int_0^{a\tau} \tau^{\alpha p}s^{-\alpha p}\varphi(s)^p\,\frac{\mathrm{d}s}{s}\Big)^{\frac{1}{p}}\,\frac{\mathrm{d}\tau}{\tau} \leq \int_0^1 \tau^{\alpha}\Big(\int_0^a s^{-\alpha p}\varphi(s)^p\,\frac{\mathrm{d}s}{s}\Big)^{\frac{1}{p}}\,\frac{\mathrm{d}\tau}{\tau}\\ &= \frac{1}{\alpha}\Big(\int_0^a s^{-\alpha p}\varphi(s)^p\,\frac{\mathrm{d}s}{s}\Big)^{\frac{1}{p}}. \end{aligned}$$

We finally indicate a proof of the claimed strong measurability.[2] Let $0 \leq g \in L^{p'}_*(0,a)$. Since the function $(\tau,t) \mapsto f(\tau,t)g(t)$ is measurable and non-negative, Fubini's theorem shows the measurability of the map

$$(0,1] \to \mathbb{R};\ \tau \mapsto \langle f(\tau), g\rangle_{L^p_*(0,a)} = \int_0^a f(\tau,t)g(t)\,\frac{\mathrm{d}t}{t}.$$

This fact is then true for all $g \in L^{p'}_*(0,a)$. Pettis' measurability theorem now yields that f is strongly measurable, see Theorem 1.1.6 in [**14**]. □

The next proposition describes the interpolation spaces of an analytic semigroup in terms of its time derivative $\frac{\mathrm{d}}{\mathrm{d}t}T(t) = AT(t)$. This result will be crucial for our appplications to parabolic problems. To this aim, we define the quantity

$$\psi_{\alpha,x}(s) = s^{1-\alpha}\|AT(s)x\|$$

for $x \in X$, $\alpha \in (0,1)$, and $s > 0$. Observe that it becomes bounded (for $s \in (0,1]$, say) if $x \in \mathrm{D}(A)$ and $\alpha = 1$, as well as for $x \in X$ and $\alpha = 0$ by Theorem 2.23 of [**32**]. (To use this result we take $\mathbb{F} = \mathbb{C}$.) Again we want to interpolate between these two starting points. As in the previous result we employ basic semigroup theory, but now also Hardy's inequality.

[2] This part was omitted in the lectures. The above calculation can also be justified using *Minkowski's inequality for integrals*, see Theorem 2.4 in [**19**].

Proposition 2.8. *Let (2.1) hold, $\mathbb{F}=\mathbb{C}$, and $T(\cdot)$ be analytic. We then have*

$$D_A(\alpha,q)=\{x\in X\mid \psi_{\alpha,x}\in L^q_*\},$$

and the norm $\|\cdot\|_{\alpha,q}$ is equivalent to $x\mapsto \|x\|+\|\psi_{\alpha,x}\|_{L^q_*}$. (In the proof one finds estimates on the corresponding constants.) For $q\in(1,\infty)$, it follows

$$x\in D_A(1-\tfrac{1}{q},q)\iff AT(\cdot)x\in L^q((0,1],X).$$

Moreover, x belongs to $D_A(\alpha)$ if and only if $\psi_{\alpha,x}(s)\to 0$ as $s\to 0$.

Proof. Let $x\in X$, $\alpha\in(0,1)$, $q\in[1,\infty]$, and $s\in(0,1]$.

1) Using Lemma 1.18 of [**32**], we estimate

$$\begin{aligned}\varphi_{\alpha,x}(s)&=\lim_{\varepsilon\to0}s^{-\alpha}\|T(s)x-T(\varepsilon)x\|=\lim_{\varepsilon\to0}s^{-\alpha}\left\|\int_\varepsilon^s\tau^{\alpha-1}\tau^{1-\alpha}AT(\tau)x\,\mathrm{d}\tau\right\|\\&\le\limsup_{\varepsilon\to0}s^{-\alpha}\int_\varepsilon^s\tau^{\alpha-1}\psi_{\alpha,x}(\tau)\,\mathrm{d}\tau\le\frac{1}{\alpha}\sup_{0<\tau\le s}\psi_{\alpha,x}(\tau).\end{aligned}$$

This inequality yields the first half of the first assertion for $q=\infty$ and of the last assertion because of Propsition 2.2 f). Let $q\in[1,\infty)$ and $\psi_{\alpha,x}\in L^q_*$. Proceeding as above, from Hardy's Lemma 2.7 we infer the bound

$$\begin{aligned}\|\varphi_{\alpha,x}\|^q_{L^q_*}&\le\int_0^1 s^{-\alpha q}\Big(\int_0^s\tau\|AT(\tau)x\|\,\frac{\mathrm{d}\tau}{\tau}\Big)^q\,\frac{\mathrm{d}s}{s}\\&\le\frac{1}{\alpha^q}\int_0^1\tau^{-\alpha q}\tau^q\|AT(\tau)x\|^q\,\frac{\mathrm{d}\tau}{\tau}=\alpha^{-q}\|\psi_{\alpha,x}\|^q_{L^q_*}.\end{aligned}$$

2) For the converse, we put $M_1:=\sup_{0<s\le1}\|sAT(s)\|$. (See Theorem 2.25 and Remark 2.26 of [**32**].) Let $x\in D_A(\alpha,q)$. Lemma 1.18 of [**32**] implies

$$\begin{aligned}s^{1-\alpha}AT(s)x&=s^{-\alpha}T(s)(T(s)x-x)-s^{-\alpha}AT(s)\int_0^s\tau^\alpha\tau^{-\alpha}(T(\tau)x-x)\,\mathrm{d}\tau,\\\psi_{\alpha,x}(s)&\le M_0\varphi_{\alpha,x}(s)+M_1s^{-1-\alpha}\int_0^s\tau^\alpha\,\varphi_{\alpha,x}(\tau)\,\mathrm{d}\tau.\end{aligned}$$

In the case $q=\infty$, we derive

$$\psi_{\alpha,x}(s)\le M_0\varphi_{\alpha,x}(s)+\frac{M_1}{1+\alpha}\sup_{\tau\in(0,s]}\varphi_{\alpha,x}(\tau),$$

and deduce the asserted equivalence for $q=\infty$ and the final assertion. Let $q\in[1,\infty)$. The previous estimate and Hardy's inequality lead to

$$\begin{aligned}\psi_{\alpha,x}(s)&\le M_0\varphi_{\alpha,x}(s)+M_1s^{-\alpha}\int_0^s\tau^\alpha\,\varphi_{\alpha,x}(\tau)\,\frac{\mathrm{d}\tau}{\tau},\\\|\psi_{\alpha,x}\|_{L^q_*}&\le M_0\|\varphi_{\alpha,x}\|_{L^q_*}+\frac{M_1}{\alpha}\Big[\int_0^1\tau^{-\alpha q}\tau^{\alpha q}\varphi_{\alpha,x}(\tau)^q\,\frac{\mathrm{d}\tau}{\tau}\Big]^{\frac1q}=(M_0+\tfrac{M_1}{\alpha})\|\varphi_{\alpha,x}\|_{L^q_*},\end{aligned}$$

concluding the proof of the first assertion.

3) Let $q\in(1,\infty)$. The second claim then follows from the formula

$$\|\psi_{1-1/q,x}\|^q_{L^q_*}=\int_0^1 s^{q(1-(1-1/q))}\,\|AT(s)x\|^q\,\frac{\mathrm{d}s}{s}=\int_0^1\|AT(s)x\|^q\,\mathrm{d}s.\qquad\square$$

Let $b > 0$, where allow for $b = \infty$ if $T(\cdot)$ and $s \mapsto sAT(s)$ are bounded on $\mathbb{R}_+$, cf. Theorem 2.25 of [**32**]. We set $M_0' = \sup_{0<s<b} \|T(s)\|$ and $M_1' = \sup_{0<s<b} \|sAT(s)\|$. As in the above proof one shows that

$$\|\psi_{\alpha,x}\|_{L_*^q(0,b)} \le \big(M_0' + \tfrac{M_1'}{\alpha}\big) \|\varphi_{\alpha,x}\|_{L_*^q(0,b)}. \tag{2.11}$$

The next theorem shows that our spaces $D_A(\alpha, q)$ and $D_A(\alpha)$ are indeed *interpolation spaces* in the usual sense of this concept. For an operator $T \in \mathcal{B}(X, Y)$ mapping a subspace $X_0 \subseteq X$ into a subspace $Y_0 \subseteq Y$, we denote the restriction of T acting from X_0 to Y_0 by the same symbol. In the proof below we implicitly use the standard definition of real interpolation spaces via the 'k-functional,' see Remark 2.11c).

THEOREM 2.9. *Assume that A and B generate C_0-semigroups $T(\cdot)$ and $S(\cdot)$ on X and Y, respectively, and that the operator $V \in \mathcal{B}(X, Y)$ satisfies $V\mathrm{D}(A) \subseteq \mathrm{D}(B)$ and $V \in \mathcal{B}(X_1, Y_1)$, where $X_1 := [\mathrm{D}(A)]$ and $Y_1 := [\mathrm{D}(B)]$. Let $0 < \alpha < 1$ and $1 \le q \le \infty$. Then V maps $D_A(\alpha, q)$ into $D_B(\alpha, q)$ and $D_A(\alpha)$ into $D_B(\alpha)$, we have $V \in \mathcal{B}(D_A(\alpha, q), D_B(\alpha, q))$ and $V \in \mathcal{B}(D_A(\alpha), D_B(\alpha))$, and it holds*

$$\|V\|_{\mathcal{B}(D_A(\alpha,q),D_B(\alpha,q))},\ \|V\|_{\mathcal{B}(D_A(\alpha),D_B(\alpha))} \le c\|V\|_{\mathcal{B}(X,Y)}^{1-\alpha}\, \|V\|_{\mathcal{B}(X_1,Y_1)}^{\alpha}$$

for a constant only depending on α and the exponential growth bounds of $T(\cdot)$ and $S(\cdot)$, as indicated in the proof.

PROOF. In view of (2.5), after rescaling if necessary, we may assume that the semigroups are exponentially stable. So let $\mathbb{R}_{\ge 0} \subseteq \rho(A) \cap \rho(B)$ and the semigroups be bounded.

Take $x \in X$, $s \in (0, 1]$, and $t > 0$. Since the result is trivially true for $V = 0$, we may assume that $V \neq 0$. We set $\|V\|_0 = \|V\|_{\mathcal{B}(X,Y)}$, $\|V\|_1 = \|V\|_{\mathcal{B}(X_1,Y_1)}$, and $N_0 = \sup_{0\le s\le 1} \|S(s)\|$. Let $x = x_0 + x_1$ for some $x_0 \in X$ and $x_1 \in X_1$. As $Vx_1 \in Y_1$, Lemma 1.18 of [**32**] yields

$$\begin{aligned}
\|S(s)Vx - Vx\|_Y &\le \|S(s)Vx_0 - Vx_0\|_Y + \|S(s)Vx_1 - Vx_1\|_Y \\
&\le (N_0 + 1)\|Vx_0\|_Y + \int_0^s \|S(\tau)BVx_1\|_Y \,\mathrm{d}\tau \\
&\le (N_0 + 1)\|V\|_0\|x_0\|_X + sN_0\|Vx_1\|_{Y_1} \\
&\le (N_0 + 1)\|V\|_0\big(\|x_0\|_X + s\|V\|_1\, \|V\|_0^{-1}\|x_1\|_{X_1}\big).
\end{aligned}$$

We now define the *k-functional* by

$$k(t, x) = \inf\big\{\|x_0\|_X + t\|x_1\|_{X_1} \,\big|\, x = x_0 + x_1,\ x_0 \in X,\ x_1 \in X_1\big\}. \tag{2.12}$$

Below we use the decomposition of x given in (2.13), which already appeared in the proof of Proposition 2.8. However, our proof also requires the infimum in (2.12) over all decompositions. Taking this infimum, we deduce

$$\varphi^B_{\alpha,Vx}(s) \le (N_0 + 1)\|V\|_0 s^{-\alpha} k(s\|V\|_1\|V\|_0^{-1}, x).$$

Here and below we use the superscript B in an obvious way. We have the decomposition $x = x_0 + x_1$ with

$$x_0 = -AR(t^{-1}, A)x \in X \quad \text{ and } \quad x_1 = t^{-1}R(t^{-1}, A)x \in X_1. \tag{2.13}$$

The k-functional can thus be dominated by

$$k(t,x) \le \|AR(t^{-1},A)x\|_X + t\|t^{-1}R(t^{-1},A)x\|_{X_1} \le (2+\|A^{-1}\|)\|AR(t^{-1},A)x\|_X.$$

Let $q<\infty$. The substitution $t = s\|V\|_1\,\|V\|_0^{-1}$, (2.6), and (2.7) then imply

$$\begin{aligned}
[Vx]^B_{\alpha,q} &\le (N_0+1)\|V\|_0 \left[\int_0^1 s^{-\alpha q} k(s\|V\|_1\,\|V\|_0^{-1},x)^q\,\frac{\mathrm{d}s}{s}\right]^{1/q}\\
&\le (N_0+1)\|V\|_0 \left[\int_0^\infty t^{-\alpha q}\|V\|_1^{\alpha q}\|V\|_0^{-\alpha q} k(t,x)^q\,\frac{\mathrm{d}t}{t}\right]^{1/q}\\
&\le (N_0+1)(2+\|A^{-1}\|)\|V\|_0^{1-\alpha}\|V\|_1^{\alpha}\left[\int_0^\infty t^{-\alpha q}\|AR(t^{-1},A)x\|_X^q\,\frac{\mathrm{d}t}{t}\right]^{1/q}\\
&= (N_0+1)(2+\|A^{-1}\|)\|V\|_0^{1-\alpha}\|V\|_1^{\alpha}\,\|\chi_{\alpha,q}\|_{L^q_*(\mathbb{R}_+)}\,.
\end{aligned}$$

The norm of $D_B(\alpha,q)$ also contains the term $\|y\|_Y$.[3] To deal with it, let $x = x_0+x_1$ for some $x_0\in X$ and $x_1\in X_1$. We estimate

$$\begin{aligned}
\|Vx\|_Y &\le \|V\|_0\big(\|x_0\|_X + \|V\|_1\,\|V\|_0^{-1}\|x_1\|_{X_1}\big) = \|V\|_0 k(\|V\|_1\,\|V\|_0^{-1},x)\\
&\le \|V\|_0 \sup_{t>0} t^{-\alpha} k(t\|V\|_1\,\|V\|_0^{-1},x) = \|V\|_0 \sup_{s>0}(s\|V\|_1^{-1}\|V\|_0)^{-\alpha}k(s,x)\\
&= \|V\|_0^{1-\alpha}\|V\|_1^{\alpha}\sup_{s>0}(\alpha q)^{\frac1q}\Big(\int_s^\infty \tau^{-\alpha q-1}k(s,x)^q\,\mathrm{d}\tau\Big)^{1/q}\\
&\le \alpha \mathrm{e}^{\frac{1}{\mathrm{e}}}\|V\|_0^{1-\alpha}\|V\|_1^{\alpha}\Big(\int_0^\infty \tau^{-\alpha q}k(\tau,x)^q\,\frac{\mathrm{d}\tau}{\tau}\Big)^{1/q}\\
&\le \mathrm{e}^{\frac{1}{\mathrm{e}}}(1+\|A^{-1}\|)\|V\|_0^{1-\alpha}\|V\|_1^{\alpha}\,\|\chi_{\alpha,q}\|_{L^q_*(\mathbb{R}_+)}\,,
\end{aligned}$$

using that $k(\cdot,x)$ is non-decreasing and the above computation at the end.

Proposition 2.6 then yields the assertion for $q<\infty$. The case $q=\infty$ can be handled in a similar, but simpler way. The remaining result then follows from $VD_A(\alpha) = V\overline{\mathrm{D}(A)} \subseteq \overline{V\mathrm{D}(A)} \subseteq \overline{\mathrm{D}(B)} = D_B(\alpha)$ with closures in the (α,∞) norms, employing the continuity of V. □

The above result implies the 'interpolation estimate' for the norms, which can also be proved by elementary methods in many cases.

Corollary 2.10. *Let (2.1) hold and $x\in \mathrm{D}(A)$. We then have the inequality*

$$\|x\|_{\alpha,q} \le c\|x\|_X^{1-\alpha}\,\|x\|_A^{\alpha}$$

for a constant $c = c(A,\alpha)>0$, which is given by the above proof.

Proof. For $x\in\mathrm{D}(A)$, we consider the map $V_x:\mathbb{F}\to X_1$ given by $V_x\mu = \mu x$. On $\mathbb{F}$ we choose the semigroup $R(\cdot)=I$ generated by $B=0$ with domain $\mathbb{F}$. Note that V_x has the norms $\|x\|_X$ in $\mathcal{B}(\mathbb{F},X)$, $\|x\|_{\alpha,q}$ in $\mathcal{B}(\mathbb{F},D_A(\alpha,q))$, and $\|x\|_A$ in $\mathcal{B}(\mathbb{F},X_1)$. The claim now follows from Theorem 2.9. □

Remark 2.11. a) We point out that the interpolation estimate in Corollary 2.10 does not imply the interpolation property expressed by Theorem 2.9.

[3]The following argument was not given in the lectures.

b) Let $B \in \mathcal{B}(D_A(\alpha,q),X)$ for some $\alpha \in (0,1)$ and $q \in [1,\infty]$. Let $a > 0$ and $x \in \mathrm{D}(A)$. Corollary 2.10 and Young's inequality with $p = 1/\alpha$ yield

$$\begin{aligned}\|Bx\| &\le \|B\|\,\|x\|_{\alpha,q} \le c\|B\|\,(\alpha/a)^{\alpha}\|x\|_X^{1-\alpha}\,(a/\alpha)^{\alpha}\|x\|_A^{\alpha}\\ &\le a\|x\|_A + (1-\alpha)\big(c\|B\|\,(\alpha/a)^{\alpha}\big)^{1/(1-\alpha)}\|x\|_X.\end{aligned}$$

This estimate allows us to apply the perturbation theorems for analytic or dissipative semigroups for suitable A and B, see Section 3.1 in [**32**].

c) Let X and X_1 be Banach spaces which are linear subspaces of a vector space Z whose addition and scalar multiplication are continuous for some metric on Z and $X, X_1 \hookrightarrow Z$. One can then define the k–functional as in (2.12). Setting $\kappa_{\alpha,x}(s) = s^{-\alpha}k(s,x)$ for $s > 0$, one now introduces the real interpolation space

$$(X,X_1)_{\alpha,q} = \big\{x \in X + X_1 \,\big|\, \kappa_{\alpha,x} \in L^q_*(\mathbb{R}_+)\big\}$$

endowed with the norm $\|\kappa_{\alpha,x}\|_{L^q_*(\mathbb{R}_+)}$. (See §2.2 E) in [**30**] for the sum space.) Arguing as in the proof of Theorem 2.9, one sees that this space coincides with our real interpolation space with equivalent norms if X_1 is the domain of a generator, see Proposition 5.7 in [**21**].

This fact tells us that $D_A(\alpha,q)$ and $D_A(\alpha)$ do *not* depend on the generator itself, but only on the Banach spaces X and $[\mathrm{D}(A)]$.

Any spaces E and F satisfying the conclusion of Theorem 2.9 are called interpolation spaces (of order α) between X and $[\mathrm{D}(A)]$ and between Y and $[\mathrm{D}(B)]$, respectively. Another important class of such spaces are the 'complex interpolation spaces' $[X,X_1]_\alpha$ of order $\alpha \in (0,1)$. It can be shown that $[L^p(\mu),L^q(\mu)]_\alpha = L^r(\mu)$ for $\frac{1}{r} = (1-\alpha)\frac{1}{p} + \alpha\frac{1}{q}$ and $1 \le p,q \le \infty$, see e.g. Example 2.11 in [**21**]. In this case the assertion of Corollary 2.10 is a standard consequence of Hölder's inequality. The real interpolation spaces between $L^p(\mu)$ and $L^q(\mu)$ are the 'Lorentz spaces', see Example 1.27 in [**21**]. $\Diamond$

We next give a typical application of the interpolation property to the theory of function spaces, which is the basis of our investigations in Chapter 3.

Example 2.12. Let $G \subseteq \mathbb{R}^m$ be bounded and open with $\partial G \in C^2$, $\alpha \in (0,1)$, and $p \in (1,\infty)$. On $L^p(G)$ we consider $A = \Delta$ with $\mathrm{D}(A) = W^{2,p}(G) \cap W^{1,p}_0(G)$, and on $L^p(\mathbb{R}^m)$ the operator $B = \Delta$ with $\mathrm{D}(B) = W^{2,p}(\mathbb{R}^m)$. These operators generate (analytic) C_0-semigroups and their graph norms are equivalent to the norms of $W^{2,p}(G)$ respectively $W^{2,p}(\mathbb{R}^m)$, cf. Example 2.30 in [**32**].

There is an (extension) operator $E \in \mathcal{B}(L^p(G),L^p(\mathbb{R}^m))$ whose restriction belongs to $\mathcal{B}(W^{2,p}(G),W^{2,p}(\mathbb{R}^m))$ such that $Ef = f$ on G for all $f \in L^p(G)$, see Theorem 3.28 in [**33**]. Theorem 2.9 then implies that E induces a map in $\mathcal{B}(D_A(\alpha,q),D_B(\alpha,q))$. Example 5.15 in [**21**] yields $D_B(\alpha,p) = W^{2\alpha,p}(\mathbb{R}^m)$ if $\alpha \ne \frac{1}{2}$, see also Example 2.5 above. Using the restriction operator $Rg = g{\restriction_G}$ on $L^p(\mathbb{R}^m)$, we thus obtain the embedding

$$RE : D_A(\alpha,p) \hookrightarrow W^{2\alpha,p}(G) := \big\{u \in L^p(G) \,\big|\, \exists\, v \in W^{2\alpha,p}(\mathbb{R}^m) : v{\restriction_G} = u\big\}.$$

By the same reasoning, we have $D_C(\alpha,q) \hookrightarrow W^{2\alpha,p}(G)$ for any generator C on $L^p(G)$ such that $\mathrm{D}(C) \subseteq W^{2,p}(G)$ and $\|\cdot\|_C \eqsim \|\cdot\|_{2,p}$.

In the above definition, the norm of u in $W^{2\alpha,p}(G)$ is given by the infimum of all norms $\|v\|_{W^{2\alpha,p}(\mathbb{R}^m)}$ with $v\restriction_G= u$. However, $W^{2\alpha,p}(G)$ also possesses an equivalent 'intrinsic' norm of the same type as those in Example 2.5, cf. Theorem 4.4.2.2 in [**37**].

In the exceptional case $\alpha = 1/2$, one again needs more interpolation theory. In this case the above results are true with $W^{2\alpha,p}(G)$ replaced by the integer Besov space $B^1_{pp}(G)$, see Example 5.14 of [**21**] or Theorem 4.3.1.2 and 4.4.2.2 of [**37**]. This space differs from $W^{1,p}(G)$ if $p \neq 2$.

The embeddings $D_A(\alpha, p) \hookrightarrow W^{2\alpha,p}(G)$ and $D_A(\frac{1}{2}, p) \hookrightarrow B^1_{pp}(G)$ are sufficient for later applications. But actually much more is known. We first note that the trace operator tr maps $W^{2\alpha,p}(G)$ and $B^1_{pp}(G)$ continuously into $L^p(\partial G)$ if $\alpha > 1/(2p)$ due to the fractional trace theorem, cf. Theorem 4.7.1 in [**37**]. Since $\mathrm{D}(A)$ is dense in $D_A(\alpha, q)$ by Proposition 2.2, we infer that

$$D_A(\alpha,p) \hookrightarrow \begin{cases} \{u \in W^{2\alpha,p}(G) \,\big|\, \operatorname{tr} u = 0\}, & \frac{1}{2p} < \alpha < 1,\ \alpha \neq \frac{1}{2}, \\ \{u \in B^1_{pp}(G) \,\big|\, \operatorname{tr} u = 0\}, & \alpha = \frac{1}{2}. \end{cases}$$

With considerably more effort here one can establish equalities instead of embeddings, namely

$$D_A(\alpha,p) = \begin{cases} W^{2\alpha,p}(G), & 0 < \alpha < \frac{1}{2p}, \\ \{u \in W^{2\alpha,p}(G) \,\big|\, \operatorname{tr} u = 0\}, & \frac{1}{2p} < \alpha < 1,\ \alpha \neq \frac{1}{2}, \\ \{u \in B^1_{pp}(G) \,\big|\, \operatorname{tr} u = 0\}, & \alpha = \frac{1}{2}, \end{cases}$$

see Theorem 4.3.3 in [**37**] and the references therein. $\Diamond$

Finally, we state a result on compact embeddings needed in the next chapter. The proof requires more facts from interpolation theory not presented here. See Corollary 3.8.2 of [**5**].

PROPOSITION 2.13. *Let* (2.1) *be true. Set* $X_0 = X$ *and* $X_1 = [\mathrm{D}(A)]$. *For* $0 < \alpha < \beta < 1$ *we consider* $X_\alpha \in \{D_A(\alpha, p), D_A(\alpha) \,|\, p \in [1,\infty]\}$ *and* $X_\beta \in \{D_A(\beta, q), D_A(\beta) \,|\, q \in [1,\infty]\}$. *Assume that* $I : X_1 \to X_0$ *is compact. Then also the embeddings* $X_\beta \hookrightarrow X_\alpha$ *are compact for* $0 \le \alpha < \beta \le 1$.

2.2. Regularity of analytic semigroups

In this section we treat basic regularity properties of linear parabolic evolution equations complementing the results established in Theorems 2.23 and 2.31 of [**32**]. We will first look at semigroup orbits and then at the inhomogeneous problem. The term 'parabolic' means that we assume that A generates the analytic C_0-semigroup $T(\cdot)$ on X, and it is motivated by the applications to diffusion-type equations. These examples will be discussed in the next chapter.

Recall that a C_0-semigroup $T(\cdot)$ is analytic if and only if it maps X into $[\mathrm{D}(A)]$ with norm less or equal c/t for $t \in (0, 1]$. This property is also equivalent to a resolvent estimate for the generator A (i.e., the sectoriality of angle $\varphi > \pi/2$ of $A - \omega I$ for some $\omega \ge 0$). See Theorem 2.25 and Remark 2.26 of [**32**]. If $\omega = 0$, we have a *bounded analytic* semigroup with $\sup_{t>0}(\|T(t)\| + \|tAT(t)\|) < \infty$.

For convenience, we write X_α for any of the spaces $D_A(\alpha, q)$ or $D_A(\alpha)$ with $\alpha \in (0, 1)$ and $q \in [1,\infty)$. We further let $\mathbb{F} = \mathbb{C}$ and set $X_0 = X$, $X_1 = [\mathrm{D}(A)]$, and $\|x\|_\alpha = \|x\|_{X_\alpha}$ for $\alpha \in [0,1]$. In Proposition 2.4 we have seen that the 'parts' $A_{\alpha,q}$ and A_α of A in $D_A(\alpha,q)$ and $D_A(\alpha)$ generate C_0-semigroups in these spaces, respectively, which are the restrictions of $T(\cdot)$ to the respective space. For simplicity, we now use the symbols A and $T(t)$ for all these objects. We sometimes write $C^0(J,X)$ instead of $C(J,X)$.

We first describe the regularizing effect of an analytic semigroup in the scale of interpolation spaces. The main point is that the norm of $T(t) : X \to X_\beta$ is bounded by $ct^{-\beta}$ and thus integrable on $(0,1]$ for $\beta < 1$. Integrability fails if $\beta = 1$, which was the only case studied in [**32**]. We also show 'full' regularity of the orbits away from $t = 0$ even for initial values $x \in X$.

THEOREM 2.14. *Let A generate the analytic C_0-semigroup $T(\cdot)$ on X with $\mathbb{F} = \mathbb{C}$, $\alpha, \beta \in [0,1]$, and $b \geq 1$. Then the following assertions hold.*

a) The restrictions of $T(\cdot)$ to X_α are also analytic C_0-semigroups.

b) Let $k \in \{0,1\}$, $\alpha \in [0,\beta]$ if $k = 0$, and $x \in X_\alpha$. For $t \in (0,b]$ we then obtain $\|A^k T(t)x\|_\beta \leq c(b,\alpha,\beta,A)t^{\alpha-\beta-k}\|x\|_\alpha$. If $T(\cdot)$ is bounded analytic and A invertible, the constant does not depend on $b \geq 1$.

c) Let $x \in X_\alpha$ and $\gamma \in [0,\alpha]$. Then $T(\cdot)x$ belongs to $C^{\alpha-\gamma}([0,b],X_\gamma)$. Let $x \in X$ and $\varepsilon \in (0,b)$. Then $T(\cdot)x$ is an element of $C^{1-\gamma}([\varepsilon,b],X_\gamma)$ for all $\gamma \in [0,1]$.

PROOF. a) Let $\alpha \in (0,1)$. As a generator of an analytic C_0-semigroup, $A - \omega I$ is sectorial of angle $\varphi > \pi/2$ on X for some $\omega \geq 0$. This resolvent estimate is transfered to X_α via Proposition 2.4 c). For $\alpha = 1$ this fact simply follows from the formula $AR(\lambda,A) = R(\lambda,A)A$ on $\mathrm{D}(A)$ for $\lambda \in \rho(A)$. The first assertion is then a consequence of Theorem 2.25 and Remark 2.26 of [**32**].

b) Let $x \in X_\alpha$ and $t \in (0,b]$. We omit the dependence on A. The constants do not depend on b if $T(\cdot)$ is bounded analytic and A invertible (use $\|x\|_1 \leq c\|Ax\|$).

1) Let $\alpha = 0$. In the first main step we interpolate between the estimates for $T(t) : X \to X$ and $T(t) : X \to X_1$ using Corollary 2.10. Indeed, Theorem 2.25 and Remark 2.26 of [**32**] and Corollary 2.10 (if $\beta \in (0,1)$) yield

$$\begin{aligned}\|A^k T(t)x\|_\beta &\lesssim_\beta \|T(t/2)A^k T(t/2)x\|_1^\beta \; \|T(t/2)A^k T(t/2)x\|_0^{1-\beta}\\ &\lesssim_b (t/2)^{-\beta}\|A^k T(t/2)x\|_0 \lesssim t^{-\beta-k}\|x\|_0 \,.\end{aligned}$$

2) Let $\alpha \in (0,1)$; for $\alpha = 1$ the proof is a bit simpler. From step 1), (2.11), (2.2) (or (2.3) for bounded $T(\cdot)$), and Proposition 2.2 we deduce

$$\begin{aligned}\|tAT(t)x\|_\beta &\leq 2(t/2)^\alpha \|T(t/2)\|_{\mathcal{B}(X,X_\beta)} \,\|(t/2)^{1-\alpha}AT(t/2)x\|_0\\ &\lesssim_{\alpha,\beta,b} t^{\alpha-\beta}\|x\|_{\alpha,\infty} \lesssim_\alpha t^{\alpha-\beta}\,\|x\|_\alpha.\end{aligned}$$

3) Take $\alpha > 0$ and $k = 0$. The case $\alpha = \beta$ follows from part a). Let $0 < \alpha < \beta$. Recall that $\frac{\mathrm{d}}{\mathrm{d}t}T(t) = AT(t)$. Steps 1) and 2) as well as Proposition 2.2 imply

$$\begin{aligned}\|T(t)x\|_\beta &= \Big\|T(b)x - \int_t^b AT(s)x\,\mathrm{d}s\Big\|_\beta \leq \|T(b)x\|_\beta + \int_t^b \|AT(s)x\|_\beta\,\mathrm{d}s\\ &\lesssim_{\alpha,\beta,b} b^{-\beta}\|x\|_0 + \int_t^b s^{\alpha-\beta-1}\|x\|_\alpha\,\mathrm{d}s \lesssim_{\beta-\alpha} (b^{-\beta} - b^{\alpha-\beta} + t^{\alpha-\beta})\|x\|_\alpha\\ &\leq t^{\alpha-\beta}\|x\|_\alpha.\end{aligned}$$

(Here we use $b \geq 1$. If $b < 1$, one could replace b by 1 in the above computation.)

c) The first part of assertion c) follows from part a) if $\alpha = \gamma$, and from Proposition 2.2 if $\gamma = 0$. So let $\gamma \in (0, \alpha)$, $0 \leq s < t \leq b$, and $x \in X_\alpha$. Statement b) then yields

$$\begin{aligned} \|T(t)x - T(s)x\|_\gamma &= \|(T(t-s) - I)T(s)x\|_\gamma \leq \int_0^{t-s} \|AT(\tau)T(s)x\|_\gamma \,\mathrm{d}\tau \\ &\lesssim_{b,\alpha,\gamma} \int_0^{t-s} \tau^{\alpha-\gamma-1} \,\mathrm{d}\tau \;\|T(s)x\|_\alpha \lesssim_{b,\alpha-\gamma} (t-s)^{\alpha-\gamma} \|x\|_\alpha. \end{aligned}$$

The last claim then follows from $T(t)x - T(s)x = (T(t-\varepsilon) - T(s-\varepsilon))T(\varepsilon)x$ and $T(\varepsilon)x \in \mathrm{D}(A)$. ☐

In Example 2.2.11 of [**20**] one can find an analytic semigroup which is unbounded in $\mathcal{B}(D_A(\alpha,\infty), D_A(\alpha,q))$ if $q \in [1,\infty)$ for $t \in (0,1]$. So one also has to pay a price if one only decreases the 'fine tuning parameter' q. By induction, one can define the scale X_α to all $\alpha \geq 0$ and extend the above theorem to this setting, see e.g. Proposition 2.2.9 in [**20**].

We turn our attention to the inhomogeneous problem

$$u'(t) = Au(t) + f(t), \quad t \in J, \qquad u(0) = x, \tag{2.14}$$

for $J = (0,b]$, $J' = [0,b]$, and given $x \in X$ and $f \in C(J', X)$. A *(classical) solution of* (2.14) *on* J is a function $u \in C(J', X) \cap C^1(J, X)$ such that $u(t) \in \mathrm{D}(A)$ for all $t \in J$ and (2.14) holds. It is a *solution on* J' if we can replace here J by J' throughout. If a solution of (2.14) on J exists, it is uniquely given by the *mild solution*

$$u(t) = T(t)x + \int_0^t T(t-s)f(s)\,\mathrm{d}s =: T(t)x + v(t), \qquad t \in J', \tag{2.15}$$

see Proposition 2.6 in [**32**]. The summand $T(\cdot)x$ has been studied above.

In the proof of Theorem 2.31 of [**32**] we have seen that $v : J' \to X$ is Hölder continuous of any exponent less than 1 and that it is continuously differentiable if $f \in C^\alpha(J', X)$ for some $\alpha > 0$. Example 4.1.7 of [**20**] shows that one cannot take $\alpha = 0$, in general. We now improve the results from [**32**] by using interpolation spaces instead of X. By the next result, for $f \in C(J', X)$ the orders of space and time regularity of v sum up to 1, provided that none is zero. We denote by $B(M, Y)$ the space of bounded functions from a set M to Y, endowed with the supnorm $\|f\|_{\infty,Y}$.

Theorem 2.15. *Let A generate the analytic C_0-semigroup $T(\cdot)$ on X, $\mathbb{F} = \mathbb{C}$, $f \in C(J', X)$, and v be given by* (2.15). *Then the following assertions hold.*

a) Let either $\alpha \in (0,1)$ and $\beta \in [0,\alpha]$ or $\alpha = 1$ and $\beta \in [0,1)$. Then v belongs to $C^{1-\alpha}(J', X_\beta)$ with norm bounded by $c(\alpha,b,A)\|f\|_{\infty,X}$.

b) Let $f \in C^\alpha(J', X)$ or $f \in B(J', X_\alpha)$ for some $\alpha \in (0,1)$. Then v solves (2.14) *on J' with $x = 0$, and the quantity $\|v'\|_{\infty,X} + \|Av\|_{\infty,X}$ is bounded by $c(\alpha,b,A)\|f\|_{C^\alpha}$ repectively $c(\alpha,b,A)\|f\|_{\infty,X_\alpha}$.*

The constants do not depend on b if $T(\cdot)$ is bounded analytic and A is invertible.

PROOF. a) Let $\alpha \in (0,1)$. It is enough to show v belongs to $C^{1-\alpha}(J', X_\alpha)$ and the corresponding estimate since $\|x\|_{\beta,q} \le \|x\|_{\alpha,q}$ and $\|f\|_\infty \le \|f\|_{C^{1-\beta}}$.

Let $0 < s < t \le b$. The dependence of the constants on A is not indicated, and they are independent of b if $T(\cdot)$ is bounded analytic and A is invertible. Theorem 2.14 b) yields

$$\|v(t)\|_\alpha \lesssim_{b,\alpha} \int_0^t (t-\tau)^{-\alpha} \|f(\tau)\|_0 \, \mathrm{d}\tau \le \frac{1}{1-\alpha} t^{1-\alpha} \|f\|_{\infty,X}\,.$$

Using the differentiability of $T(\cdot)$ and this theorem, we further compute

$$\begin{aligned} v(t) - v(s) &= \lim_{\varepsilon\to 0} \int_0^{s-\varepsilon} (T(t-\tau) - T(s-\tau)) f(\tau)\,\mathrm{d}\tau + \int_s^t T(t-\tau) f(\tau)\,\mathrm{d}\tau \\ &= \lim_{\varepsilon\to 0} \int_0^{s-\varepsilon} \int_{s-\tau}^{t-\tau} AT(\sigma) f(\tau)\,\mathrm{d}\sigma\,\mathrm{d}\tau + \int_s^t T(t-\tau) f(\tau)\,\mathrm{d}\tau, \\ \|v(t)-v(s)\|_\alpha &\lesssim_{b,\alpha} \limsup_{\varepsilon\to 0} \int_0^{s-\varepsilon} \int_{s-\tau}^{t-\tau} \sigma^{-1-\alpha} \|f\|_\infty \,\mathrm{d}\sigma\,\mathrm{d}\tau + \int_s^t (t-\tau)^{-\alpha} \|f\|_\infty \,\mathrm{d}\tau \\ &\le \Big(\frac{1}{\alpha} \int_0^s ((s-\tau)^{-\alpha} - (t-\tau)^{-\alpha})\,\mathrm{d}\tau + \frac{1}{1-\alpha}(t-s)^{1-\alpha}\Big) \|f\|_{\infty,X} \\ &\le \tfrac{2}{\alpha(1-\alpha)} (t-s)^{1-\alpha} \|f\|_{\infty,X}\,. \end{aligned}$$

b) The case $f \in C^\alpha$ was shown in Theorem 2.31 of [**32**]. Let $f \in B(J', X_\alpha)$ for $\alpha \in (0,1)$. Take $0 < s < t \le b$. Due to Theorem 2.14, the function given by $\varphi(s) = \|AT(t-s)f(s)\| \lesssim_{b,\alpha} (t-s)^{\alpha-1} \|f\|_{\infty,X_\alpha}$ is integrable on $[0,t]$. Since A is closed, we deduce that $v(t) \in \mathrm{D}(A)$. As in step a), we then obtain

$$\begin{aligned} \|Av(t)\| &\le \int_0^t \|T(t-s)f(s)\|_1 \,\mathrm{d}s \lesssim_{b,\alpha} \int_0^t (t-s)^{\alpha-1} \|f\|_{\infty,X_\alpha} \,\mathrm{d}s \\ &= \tfrac{1}{\alpha} t^\alpha \|f\|_{\infty,X_\alpha}, \\ \|Av(t) - Av(s)\| &\le \overline{\lim_{\varepsilon\to 0}} \int_0^{s-\varepsilon} \int_{s-\tau}^{t-\tau} \|AT(\sigma) f(\tau)\|_1 \,\mathrm{d}\sigma\,\mathrm{d}\tau + \int_s^t \|T(t-\tau) f(\tau)\|_1 \,\mathrm{d}\tau \\ &\lesssim_{b,\alpha} \int_0^s \int_{s-\tau}^{t-\tau} \sigma^{\alpha-2} \|f(\tau)\|_\alpha \,\mathrm{d}\sigma\,\mathrm{d}\tau + \int_s^t (t-\tau)^{\alpha-1} \|f(\tau)\|_\alpha \,\mathrm{d}\tau \\ &\le \Big(\frac{1}{1-\alpha} \int_0^s ((s-\tau)^{\alpha-1} - (t-\tau)^{\alpha-1})\,\mathrm{d}\tau + \frac{(t-s)^\alpha}{\alpha}\Big) \|f\|_{\infty,X_\alpha} \\ &\le \tfrac{2}{\alpha(1-\alpha)} (t-s)^\alpha \|f\|_{\infty,X_\alpha}\,. \end{aligned}$$

So Av belongs to $C([0,b],X)$ as $v(0)=0$. The claim follows from $v' = Av+f$. $\square$

In part b) of the above proof we have even shown that $Av \in C^\alpha([0,b],X)$. Actually, it can proved that the terms u' and Au have the same regularity as the given function f, if we work in $D_A(\alpha,\infty)$ and assume that $x \in \mathrm{D}(A)$ and $Ax+f(0) \in D_A(\alpha,\infty)$ in the case $f \in C^\alpha(J',X)$, respectively $x \in D_A(\alpha+1,\infty)$ if $f \in C(J',X) \cap B(J', D_A(\alpha,\infty))$. This property of 'maximal regularity of type C^α' is shown in Theorem 4.3.1 and Corollary 4.3.9 of [**20**]. Chapter 4 of [**20**] provides many variants and refinements of these results. See also the exercises.

CHAPTER 3

Semilinear parabolic problems

In this chapter we again study semilinear evolution equations of the form

$$u'(t) = Au(t) + F(u(t)), \quad t \in J, \qquad u(0) = u_0, \tag{3.1}$$

but now requiring that A generates an analytic C_0-semigroup $T(\cdot)$. Based on the previous chapter, we allow for nonlinearities F mapping an interpolation space of A into X. So they are still of 'lower order', but one is far less restricted than in the first chapter where we could only treat the case $F : X \to X$. Typical examples are reaction-diffusion systems which we study below in some detail.

The wellposedness theory of semilinear parablic equations can be developed similar as for ordinary differential equations since $T(t) : X \to X_\alpha$ is bounded by the integrable function $t \mapsto ct^{-\alpha}$ on $(0, 1]$ due to Theorem 2.14. In the study of the long-time behavior one has partly to assume that A has compact resolvent. On the other hand, concrete applications may depend on detailed knowledge of the properties of a differential operator A with boundary conditions which is much harder to study than a matrix.

We state the setting of this chapter. Let $\mathbb{F} = \mathbb{C}$, $X_0 = X$, $X_1 = [\mathrm{D}(A)]$, and

$$X_\alpha \in \{D_A(\alpha, q), D_A(\alpha) \mid q \in [1, \infty)\} \quad \text{ for } \ \alpha \in (0, 1),$$

as well as $\|x\|_\alpha = \|x\|_{X_\alpha}$ and $\overline{B}_\alpha(x, r) = \overline{B}_{X_\alpha}(x, r)$ for $\alpha \in [0, 1]$. We assume that

$$\begin{aligned}
&A \text{ generates the analytic } C_0\text{-semigroup } T(\cdot) \text{ on } X, \quad \alpha \in [0, 1), \quad u_0 \in X_\alpha,\\
&M_0 \coloneqq \sup\nolimits_{t\in[0,1]} \|T(t)\|_{\mathcal{B}(X_\alpha)}, \quad M_1 \coloneqq \sup\nolimits_{t\in[0,1]} \|t^\alpha T(t)\|_{\mathcal{B}(X,X_\alpha)};\\
&\emptyset \neq J \subseteq \mathbb{R} \text{ is an interval with } \inf J = 0 \text{ and } 0 \notin J, \quad J' \coloneqq J \cup \{0\};\\
&F : X_\alpha \to X \text{ satisfies } \forall\, r > 0 \ \exists\, L(r) > 0 \ \forall\, x, y \in \overline{B}_\alpha(0, r) :\\
&\|F(x) - F(y)\|_0 \le L(r)\|x - y\|_\alpha, \quad \text{where } r \mapsto L(r) \text{ is non-decreasing.}
\end{aligned} \tag{3.2}$$

The numbers $M_0 \ge 1$ and $M_1 > 0$ are finite because of Theorem 2.14. As after (1.2) one sees that the last restriction in (3.2) is made without loss of generality. In contrast to the first chapter, now the time interval J does not include 0.

3.1. Local wellposedness and global existence

A function u is called a *(classical) solution of* (3.1) *on* J if it belongs to $C(J', X_\alpha) \cap C^1(J, X) \cap C(J, X_1)$ and satisfies (3.1). It is a *(classical) solution on* J' if even $u \in C^1(J', X) \cap C(J', X_1)$ and (3.1) is true for $t \in J'$. In both cases, $f = F(u) : J' \to X$ is continuous by (3.2) so that Proposition 2.6 of [**32**] says that u is a *mild solution* of (3.1); i.e., a function $u \in C(J', X_\alpha)$ fulfilling the fixed-point problem

$$u(t) = T(t)u_0 + \int_0^t T(t-s)F(u(s))\,\mathrm{d}s =: T(t)u_0 + v(t), \qquad t \in J'. \tag{3.3}$$

If the semigroup $T(\cdot)$ is not analytic, there is a significant difference between mild and classical solutions, cf. Theorem 1.16. In the parabolic case, however, from the regularity results of the previous section we can deduce that a mild solution u immediately becomes regular; i.e., $u(t)$ belongs to X_1 for all $t > 0$ and mild solutions coincide with classical ones on J. In Chapter 7 of the monograph [**20**] one finds much more refined versions of the next fundamental lemma.

LEMMA 3.1. *Let* (3.2) *be true and* $u \in C(J', X_\alpha)$ *be a mild solution of* (3.1). *Then* u *is a classical solution on* J, *and on* J' *if* $u_0 \in X_1$.

PROOF. Let v be given by (3.3), $b \in J$, and $\varepsilon \in (0, b)$. Since $T(\cdot)$ is analytic, the orbit $T(\cdot)u_0 \in C(\mathbb{R}_+, X_1) \cap C^1(\mathbb{R}_+, X)$ satisfies $\mathrm{d}/\mathrm{d}t\, T(t)u_0 = AT(t)u_0$ for $t > 0$. Theorems 2.14 and 2.15 a) yield that $T(\cdot)u_0$ is an element of $C^{1-\alpha}([\varepsilon, b], X_\alpha) \cap C(\mathbb{R}_{\geq 0}, X_\alpha)$ and v of $C^{1-\alpha}([0, b], X_\alpha)$, respectively. Hence, u belongs to $C^{1-\alpha}([\varepsilon, b], X_\alpha)$, and thus $F(u)$ to $C^{1-\alpha}([\varepsilon, b], X)$ by (3.2). (If $\alpha = 0$, one has to replace $C^{1-\alpha}$ by C^β for any $\beta \in (0, 1)$.) Using

$$v(t) = \int_\varepsilon^t T(t-s)F(u(s))\,\mathrm{d}s + T(t-\varepsilon)\int_0^\varepsilon T(\varepsilon - s)F(u(s))\,\mathrm{d}s \quad \text{for } t \geq \varepsilon,$$

from Theorems 2.14 and 2.15 b) plus a time shift we infer that v belongs to $C^1((\varepsilon, b], X) \cap C((\varepsilon, b], X_1)$. As $0 < \varepsilon < b$ are arbitrary, Lemma 2.8 in [**32**] implies that v solves (3.1) with $u_0 = 0$ on J. So the first claim follows from the properties of $T(\cdot)u_0$.

Let $u_0 \in X_1$. Then $T(\cdot)u_0$ is even contained in $C^{1-\alpha}([0, b], X_\alpha)$ and we can take $\varepsilon = 0$ in the above reasoning. (See Theorem 2.14.) □

The above result easily implies that one can shift and glue mild (or classical) solutions as in Remark 1.7, which we will use below often without further notice.

We next solve the fixed-point problem (3.3) on a possibly small time interval by means of the contraction mapping principle. We proceed as in Section 1.1, but now exploit the (linear) regularity results from Theorems 2.14 and 2.15.

LEMMA 3.2. *Let* (3.2) *be true. Take any* $\rho > 0$. *Then there is a time* $b_0(\rho) > 0$ *such that for every* $u_0 \in \overline{B}_\alpha(0, \rho)$ *there is a unique solution* u *of* (3.1) *on* $(0, b_0(\rho)]$ *satisfying* $\|u(t)\|_\alpha \leq r := 1 + M_0\rho$ *for all* $t \in [0, b_0(\rho)]$. *The map* b_0 *is defined in* (3.7), *and it is non-increasing.*

PROOF. Let $b \in (0, 1]$, $\rho > 0$, and $r = 1 + M_0\rho > \rho$. We introduce the space

$$E(b) = E(b, r) = \big\{ v \in C([0, b], X_\alpha) \,\big|\, \|v\|_{\infty,\alpha} := \sup_{0 \leq t \leq b} \|v(t)\|_\alpha \leq r \big\}. \tag{3.4}$$

Note that $E(b)$ is complete for the metric $\mathrm{d}(v, w) = \|v - w\|_{\infty,\alpha}$. Let $u_0 \in \overline{B}_\alpha(0, \rho)$ be the initial value, $v, w \in E(b)$, and $t \in [0, b]$. We define the map

$$[\Phi_{u_0}(v)](t) = \Phi(v)(t) = T(t)u_0 + \int_0^t T(t-s)F(v(s))\,\mathrm{d}s. \tag{3.5}$$

Clearly, a fixed point $u = \Phi(u)$ in $E(b)$ is a solution of (3.3), and u then solves (3.1) on $(0, b]$ by Lemma 3.1. We will obtain such a fixed point for small $b > 0$.

Since $F(v) \in C([0, b], X)$ and $u_0 \in X_\alpha$, Theorems 2.14 and 2.15 imply that $\Phi(v) \in C([0, b], X_\alpha)$. Using also (3.2), we estimate

$$\begin{aligned} \|\Phi(v)(t)\|_\alpha &\le M_0 \|u_0\|_\alpha + M_1 \int_0^t (t-s)^{-\alpha} \|F(v(s)) - F(0) + F(0)\|_0 \, \mathrm{d}s \\ &\le M_0 \rho + M_1 \int_0^t (t-s)^{-\alpha} \left(L(r) \|v(s) - 0\|_\alpha + \|F(0)\|_0 \right) \mathrm{d}s \\ &\le M_0 \rho + \frac{M_1}{1-\alpha} (r L(r) + \|F(0)\|_0)\, b^{1-\alpha} \le r, \end{aligned}$$

where we choose times $b \in (0, 1]$ with

$$b \le b_1(\rho) := \Big(\frac{1-\alpha}{M_1 (r L(r) + \|F(0)\|_0)} \Big)^{\frac{1}{1-\alpha}}.$$

In the same way, we compute

$$\begin{aligned} \|\Phi(v)(t) - \Phi(w)(t)\|_\alpha &\le M_1 \int_0^t (t-s)^{-\alpha} \|F(v(s)) - F(w(s))\|_0 \, \mathrm{d}s \\ &\le M_1 \int_0^t (t-s)^{-\alpha} L(r) \|v(s) - w(s)\|_\alpha \, \mathrm{d}s \\ &\le \frac{M_1 L(r)}{1-\alpha}\, b^{1-\alpha} \, \|v - w\|_{\infty,\alpha} \le \frac{1}{2} \|v - w\|_{\infty,\alpha} \end{aligned} \tag{3.6}$$

for every final time

$$0 < b \le b_0(\rho) := \min \Big\{ 1, b_1(\rho), \Big(\frac{1-\alpha}{2 M_1 L(r)} \Big)^{\frac{1}{1-\alpha}} \Big\}. \tag{3.7}$$

As a result, the map $\Phi : E(b) \to E(b)$ is a strict contraction and we obtain a unique solution $u = \Phi(u)$ of (3.1) on $(0, b]$ which belongs to $E(b)$. $\square$

Exactly as in Lemma 1.8 we next derive a uniqueness result without the condition that the functions are bounded by r. One can it also prove as discussed after that lemma, now using the singular Gronwall inequality (3.9) below.[1]

Lemma 3.3. *Let (3.2) be true and u and v be solutions of (3.1) on J_u respectively J_v. Then u and v coincide on $J_u \cap J_v$.*

Proof. Set $J = J_u \cap J_v$. Since $u(0) = v(0)$, the number

$$\tau := \sup \big\{ b \in J \,\big|\, \forall\, t \in [0, b] : \ u(t) = v(t) \big\}$$

belongs to $[0, \sup J]$. We assume that $u \neq v$ on J. By continuity, it follows $\tau < \sup J$ and $u(\tau) = v(\tau) =: u_1 \in X_\alpha$. There are times $t_n \in J$ with $t_n \to \tau^+$ and $u(t_n) \neq v(t_n)$. Fix $\beta_0 > 0$ with $\tau + \beta_0 \in J$. For $\beta \in (0, \beta_0]$, the functions $\tilde{u} = u(\,\cdot + \tau)$ and $\tilde{v} = v(\,\cdot + \tau)$ solve (3.1) on $(0, \beta]$ with initial value u_1.

We now set $\rho = \|u_1\|_\alpha$ and $r = 1 + M_0 \rho > \rho$, and use the number $b_0(\rho)$ from (3.7). For sufficiently small times $0 < \beta \le \min\{b_0(\rho), \beta_0\}$, the maps $\tilde{u}$ and $\tilde{v}$ have norms smaller or equal r in $C([0, \beta], X_\alpha)$ because of $\tilde{u}(0) = \tilde{v}(0) = u_1$. The

[1] The next proof was omitted in the lectures.

uniqueness statement (in the proof) of Lemma 3.2 then shows that $\tilde{u}(t) = \tilde{v}(t)$ for $t \in [0, \beta]$, which contradicts the inequality $u(t_n) \neq v(t_n)$ for large n. □

Following the first chapter, we next introduce the *maximal existence time*

$$t_+(u_0) = \sup \{ b > 0 \,\big|\, \exists \text{ solution } u_b \text{ of (3.1) on } (0, b] \}$$

of (3.1) still assuming (3.2). The *maximal existence interval* is

$$J_+(u_0) = (0, t_+(u_0)) \quad \text{or} \quad J'_+(u_0) = [0, t_+(u_0)).$$

Lemma 3.2 yields $t_+(u_0) \geq b_0(\|u_0\|_\alpha)$. We actually have $t_+(u_0) > b_0(\|u_0\|_\alpha)$ since we can restart (3.1) with initial value $u(b_0(\|u_0\|_\alpha))$. If $b < \beta < t_+(u_0)$, then $u_b = u_\beta$ on $[0, b]$ by Lemma 3.3. This fact allows us to define a *maximal solution* of (3.1) by setting $u(t) = u_b(t)$ for $t \in [0, b] \subseteq [0, t_+(u_0))$. It is uniquely determined because of Lemma 3.3.

We can now show the local wellposedness of (3.1).[2] Continuous dependence on F is established in Proposition 3.6.

THEOREM 3.4. *Let (3.2) be true and $b_0 = b_0(\|u_0\|_\alpha) > 0$ be given by (3.7). Then the following assertions hold.*

a) There is a unique maximal solution $u = \varphi(\,\cdot\,, u_0)$ in $C(J'_+(u_0), X_\alpha) \cap C^1(J_+(u_0), X) \cap C(J_+(u_0), X_1)$ of (3.1), where $t_+(u_0) \in (b_0, \infty]$. If $u_0 \in X_1$, we have $u \in C^1(J'_+(u_0), X) \cap C(J'_+(u_0), X_1)$.

b) Let $t_+(u_0) < \infty$. Then we have $\lim_{t \to t_+(u_0)^-} \|u(t)\|_\alpha = \infty$.

c) Take any $b \in J_+(u_0)$. Then there exists a radius $\delta = \delta(u_0, b) > 0$ such that $t_+(v_0) > b$ for all $v_0 \in \overline{B}(u_0, \delta)$. Moreover, the map

$$\overline{B}_\alpha(u_0, \delta) \to C([0, b], X_\alpha); \quad v_0 \mapsto \varphi(\,\cdot\,, v_0),$$

is Lipschitz continuous.

PROOF. Assertion a) was shown above and in Lemma 3.1. To establish b), let $t_+(u_0) < \infty$. Assume that there were times $t_n \to t_+(u_0)$ for $n \to \infty$ with $t_n \in J'_+(u_0)$ and $C := \sup_{n \in \mathbb{N}} \|u(t_n)\|_\alpha < \infty$. We choose an index $m \in \mathbb{N}$ such that $t_m + b_0(C) > t_+(u_0)$, where $b_0(C) > 0$ is given by (3.7). Lemma 3.2 yields a solution $\tilde{u}$ of (3.1) on $(0, b_0(C)]$ with initial value $u(t_m)$. Glueing u and the shifted $\tilde{u}$, we then obtain a solution of (3.1) on $(0, t_m + b_0(C)]$ which contradicts the definition of $t_+(u_0)$. So assertion b) is shown. We next prove part c) by a basic step plus an induction argument in three more steps.

1) Let $b \in J_+(u_0)$ and $u = \varphi(\,\cdot\,, u_0)$. We fix a number $b' \in (b, t_+(u_0))$ and use the radii $\overline{\rho} := 1 + \max_{0 \leq t \leq b'} \|u(t)\|_\alpha$ and $\overline{r} := 1 + M_0 \overline{\rho}$. The uniform bound by $\overline{\rho}$ will crucially be used below. Let the time $\overline{b} := b_0(\overline{\rho}) \in (0, 1]$ be given by (3.7) and the operator Φ_{u_0} by (3.5). Take $v_0, w_0 \in \overline{B}_\alpha(0, \overline{\rho})$. Lemma 3.2 and its proof provide solutions $v = \Phi_{v_0}(v) = \varphi(\,\cdot\,, v_0)$ and $w = \Phi_{w_0}(w) = \varphi(\,\cdot\,, w_0)$ of (3.1) on $(0, \overline{b}]$ with the initial values v_0 respectively w_0, where v and w belong to the space $E(\overline{b}, \overline{r})$ from (3.4) endowed with the norm $\|\cdot\|_{\infty,\alpha}$. Formulas (3.6) and (3.5) lead to the contraction estimate

$$\|v - w\|_{\infty,\alpha} \leq \|\Phi_{v_0}(v) - \Phi_{v_0}(w)\|_{\infty,\alpha} + \|\Phi_{v_0}(w) - \Phi_{w_0}(w)\|_{\infty,\alpha}$$

[2] The next proof was omitted in the lectures as it is very close to that of Theorem 1.11.

$$
\begin{aligned}
&\le \tfrac{1}{2}\|v-w\|_{\infty,\alpha} + \|T(\cdot)(v_0-w_0)\|_{\infty,\alpha}\\
&\le \tfrac{1}{2}\|v-w\|_{\infty,\alpha} + M_0\|v_0-w_0\|_\alpha,\\
\|v-w\|_{\infty,\alpha} &\le 2M_0\|v_0-w_0\|_\alpha. \qquad (3.8)
\end{aligned}
$$

2) We next show $t_+(v_0) > b$ by iteration. For $j \in \mathbb{N}_0$ we set $b_j = j\bar{b}$. There exists a minimal index $N \in \mathbb{N}$ with $b_N > b$. If $b_N > b'$ we redefine $b_N := b' \in (b, t_+(u_0))$. We fix the radius $\delta = (2M_0)^{-N} \in (0,1)$ for the initial values. We inductively show that for every $v_0 \in \overline{B}_\alpha(u_0,\delta)$ and $j \in \{0,\dots,N-1\}$ the maximal solution $v = \varphi(\,\cdot\,, v_0)$ exists at least on $[0, b_{j+1}]$ and that $v(t)$ is an element of the ball $\overline{B}_\alpha(u(t), (2M_0)^{j+1-N})$ for $t \in [b_j, b_{j+1}]$, which belongs to $\overline{B}_\alpha(0,\overline{\rho})$ because of the basic bound $\overline{\rho} \ge 1 + \|u(t)\|_\alpha$ for $t \in [0, b_N]$. This claim then yields $t_+(v_0) > b$.

3) We prove the claim. First let $j = 0$. Since $\delta < 1$, the vector v_0 is contained in $\overline{B}_\alpha(0,\overline{\rho})$. From estimate (3.8) with $w = u$ we deduce

$$
\|v(t)-u(t)\|_\alpha \le 2M_0\|v_0-u_0\|_\alpha \le 2M_0\delta = (2M_0)^{1-N}
$$

for all $t \in [0, b_1]$, as asserted for $j = 0$.

Second, assume that the claim has been established for all $k \in \{0,\dots,j-1\}$ and some $j \in \{1,\dots,N-1\}$. It follows $\|v(b_j)\|_\alpha \le \overline{\rho}$. Lemma 3.2 thus shows that v exists at least on $[0, b_{j+1}]$. Moreover, the inequality (3.8) can be applied to $v(t+b_j) = \varphi(t, v(b_j))$ and $u(t+b_j) = \varphi(t, u(b_j))$ for $t \in [0,\bar{b}]$. Using also the induction hypothesis, we infer the bound

$$
\|v(t+b_j) - u(t+b_j)\|_\alpha \le 2M_0\|v(b_j)-u(b_j)\|_\alpha \le (2M_0)^{j+1-N}
$$

for $t \in [0,\bar{b}]$. So the claim is true.

4) It remains to prove the Lipschitz continuity asserted in c). Let $j \in \{0,\dots,N-1\}$ and $t \in [0,\bar{b}]$. By the claim in 2), the vectors $v(b_j)$ and $w(b_j)$ belong to $\overline{B}_\alpha(0,\overline{\rho})$. As in step 3), inequality (3.8) implies

$$
\begin{aligned}
\|v(t+b_j) - w(t+b_j)\|_\alpha &= \|\varphi(t,v(b_j)) - \varphi(t,w(b_j))\|_\alpha \le 2M_0\|v(b_j)-w(b_j)\|_\alpha\\
&\le \dots \le (2M_0)^{j+1}\|v_0-w_0\|_\alpha \le (2M_0)^N\|v_0-w_0\|_\alpha. \qquad \square
\end{aligned}
$$

We stress that X_α is the adequate norm to decribe the behavior of the solutions to (3.1): They are continuous in the X_α–norm up to 0 and this norm gives the Lipschitz continuity and blow-up condition in assertions b) and c). We state an extra regularity´property provided by our setting

Remark 3.5. Let (3.2) be true, $\beta \in [\alpha, 1)$, and $u_0 \in X_\beta$. Proposition 2.2 yields the embedding $X_\beta \hookrightarrow X_\alpha$, so that the assumptions of Theorem 3.4 also hold for β. We thus obtain a solution u_β in X_β on a maximal existence interval $J'_{+,\beta}(u_0)$. By uniqueness and $X_\alpha \hookrightarrow X_\beta$, this solution coincides with u from the theorem on $J'_{+,\beta}(u_0) \subseteq J'_+(u_0)$. Since $u \in C(J_+(u_0), X_1)$, the blow-up condition in X_β finally implies that $J'_{+,\beta}(u_0) = J'_+(u_0)$. Using Proposition 2.2 once more, we have shown that u belongs to $C(J'_+(u_0), X_\kappa)$ for all $\kappa \in [0,\beta]$. $\Diamond$

The regularity of mild and classical solutions to (3.1) is studied in great detail in Chapter 7 of [20], also for $u_0 \in X$ under additional restrictions on F. The above proofs completely break down if $\alpha = 1$; i.e., when the nonlinearity has the same order as the linear part. Under certain additional assumptions, one can also develop a theory on wellposedness and asymptotic behavior for such problems, which is similar to the semilinear case discussed here. This is done in Chapters 8 and 9 of [**20**] based on the results on maximal regularity mentioned at the end of the previous chapter; see also the last chapter of this notes.

Since the data of equation (3.1) are not known exactly in applications, it is very important to know that the solution depends continuously on the system operators, where we restrict ourselves to F for simplicity. When discussing the positivity of reaction diffusion systems we will actually use a very special case of this continuous dependence (whose proof would not be much simpler). For this and other purposes, we need the *singular Gronwall inequality.* Let $0 \le \varphi \in C(J')$, $\beta \in [0, 1)$ and $a, \kappa \ge 0$. Assume that

$$\varphi(t) \le a + \kappa \int_0^t (t-s)^{-\beta}\, \varphi(s)\, \mathrm{d}s$$

holds for all $t \in J'$. Then there is a constant $c_0 > 0$ such that

$$\varphi(t) \le a + a\kappa c_0 t^{1-\beta}\, \mathrm{e}^{c(\beta)\kappa^{1/(1-\beta)} t} \tag{3.9}$$

holds for all $t \in J$, where $c(\beta) := 2\Gamma(1-\beta)^{\frac{1}{1-\beta}}$, see Theorem II.3.3.1 in [**2**].

Proposition 3.6. *Let* (3.2) *be true,* $v_0 \in X_\alpha$*,* $G : X_\alpha \to X$ *be Lipschitz on bounded sets. Let* u *solve* (3.1) *and* v *solve* (3.1) *with nonlinearity* G *and initial value* v_0*. Take any* $b \in (0, t_+(u_0, F))$*. Then there are constants* $\delta_0, \rho, c > 0$ *(depending on* b *and* u_0*) with the following property: Let* $\|u_0 - v_0\|_\alpha \le \rho$ *and* $\|F(u(t)) - G(u(t))\|_0 \le \delta \le \delta_0$ *for* $t \in [0, b]$*. We then have* $t_+(v_0, G) > b$ *and*

$$\|u(t) - v(t)\|_\alpha \le c(\delta + \|u_0 - v_0\|_\alpha) \qquad \text{for} \quad t \in [0, b].$$

Proof. Fix $r > 0$ and $b \in (0, t_+(u_0, F))$, and let L be the Lipschitz constant of G on the bounded set $\bigcup \big\{\overline{B}_\alpha(u(t), r) \,\big|\, t \in [0, b]\big\}$. Take $\rho \in (0, r)$ and $v_0 \in \overline{B}_\alpha(u_0, \rho)$. The numbers

$$N_0 = \sup_{t\in[0,b]} \|T(t)\|_{\mathcal{B}(X_\alpha)} \qquad \text{and} \qquad N_1 = \sup_{t\in[0,b]} \|t^\alpha\, T(t)\|_{\mathcal{B}(X, X_\alpha)}$$

are finite by Theorem 2.14. Let b^* be the supremum of all times $\beta \in (0, b]$ with $\beta < t_+(v_0, G)$ and $\|v(t) - u(t)\|_\alpha \le r$ for all $t \in [0, \beta]$. The continuity of $u - v$ yields $b^* > 0$ and $\|v(b^*) - u(b^*)\|_\alpha \le r$. Using the mild formulation of both evolution equations, we obtain

$$\begin{aligned} u(t) - v(t) &= T(t)(u_0 - v_0) + \int_0^t T(t-s)(F(u(s)) - G(u(s)))\, \mathrm{d}s \\ &\qquad + \int_0^t T(t-s)(G(u(s)) - G(v(s)))\, \mathrm{d}s, \\ \|u(t) - v(t)\|_\alpha &\le N_0\|u_0 - v_0\|_\alpha + N_1\delta \int_0^t \frac{\mathrm{d}s}{(t-s)^\alpha} + N_1 L \int_0^t \frac{\|u(s) - v(s)\|_\alpha}{(t-s)^\alpha}\, \mathrm{d}s \\ &\le N_0\|u_0 - v_0\|_\alpha + \frac{\delta N_1 b^{1-\alpha}}{1-\alpha} + N_1 L \int_0^t (t-s)^{-\alpha} \|u(s) - v(s)\|_\alpha\, \mathrm{d}s \end{aligned}$$

for all $t \in [0, b^*)$. The inequality (3.9) thus yields

$$\|u(t) - v(t)\|_\alpha \le c(\delta + \|u_0 - v_0\|_\alpha) \le c(\delta + \rho), \tag{3.10}$$

for all $t \in [0, b^*)$, where $c > 0$ depends on b, L, N_j and α, but not on t, δ or ρ. The blow-up condition then yields $b^* < t_+(v_0, G)$. Hence, by continuity (3.10) with constant $2c$ is true for $t \le b^* + \eta$ and some $\eta > 0$. Fixing sufficiently small $\delta_0, \rho > 0$, we infer $\|u(t) - v(t)\|_\alpha \le r/2$ for all $t \in [0, b^* + \eta]$ and $\delta \le \delta_0$. Hence, b^* equals b because of its definition, and so (3.10) holds for all $t \in [0, b]$. □

In the next example we give an introduction to the L^p-approach to *reaction-diffusion systems*, whereas in Section 7.3 of [**20**] sup-norm setting is discussed.

Example 3.7. 1) We first recall reaction systems without diffusion. As a simple example, we consider the chemical reaction $A + 2B \rightleftharpoons C$, where one mol of the substance A reacts with 2 mols of B to one mol of the product C, which in turn can decompose into one mol of A and two mols of B.

Let $a(t)$, $b(t)$ and $c(t)$ be the concentrations at time $t \ge 0$ of the species A, B and C, respectively. Roughly speaking, the two reactions take place with a 'probability' proportional to the products of $a(t)\, b(t)\, b(t)$ and $c(t)$ of the concentrations, where we denote the proportionality constants by k_+ and k_-, respectively. Each concentration then increases and decreases according to the two reactions, where the rate is given by the 'probability' times the number of mols needed of the respective substance. We arrive at the system

$$\begin{aligned}
a'(t) &= -k_+ a(t) b(t)^2 + k_- c(t), \qquad t \ge 0,\\
b'(t) &= -2k_+ a(t) b(t)^2 + 2k_- c(t), \qquad t \ge 0,\\
c'(t) &= k_+ a(t) b(t)^2 - k_- c(t), \qquad t \ge 0,\\
a(0) &= a_0, \quad b(0) = b_0, \quad c(0) = c_0,
\end{aligned}$$

with initial concentrations $a_0, b_0, c_0 \ge 0$.

This problem has a unique local non-negative solution due to the Picard–Lindelöf Theorem 4.9 and the positivity criterion Satz 4.12 of [**31**]. Since $a' + c' = 0$ and $b' + 2c' = 0$, we have $a(t) + c(t) = a_0 + c_0$ and $b(t) + 2c(t) = b_0 + 2c_0$ as long as the solutions exist. Thanks to the positivity, the solutions thus stay bounded on their existence interval, so that they exist for all $t \ge 0$. These facts hold in much greater generality, see Section 8.7 of [**29**].

2) In a reaction–diffusion system one takes into account that the concentrations of the species may differ at different points of the container G which is an open and bounded subset of $\mathbb{R}^m$ with $\partial G \in C^2$ and outer unit normal ν. For given ℓ species we thus consider concentration densities $u(t, x) = (u_1(t, x), \ldots, u_\ell(t, x))$ at every time $t \ge 0$ and spatial point $x \in G$.

We assume that at each x a reaction-convection term $f(u(t, x), \nabla u(t, x))$ acts. Later we will focus on pure reaction terms $f(u(t, x))$ depending only on the concentrations $u(t, x)$ as in the ordinary differential equation above. If spatial gradients of $u(t, x)$ are involved, we also have (possibly nonlinear) convective effects. The function $f : \mathbb{C}^{\ell + m\ell} \to \mathbb{C}^\ell$ (or later $f : \mathbb{C}^\ell \to \mathbb{C}^\ell$) is given and assumed to be locally Lipschitz.

Moreover, the species shall move in the container driven by 'homogeneous' and 'isotropic' diffusion with constants $a_1, \dots, a_\ell > 0$, resulting in diffusion terms $a_j \Delta u_j(t,x)$. We require that the species do not move through the boundary ∂G. It can be seen that this behaviour is decribed by the Neumann boundary condition $\partial_\nu u_j(t,x) = 0$ saying that in normal direction at the boundary the concentration does not change. Summing up, we arrive at the system

$$\begin{aligned}
\partial_t u_j(t,x) &= a_j \Delta u_j(t,x) + f_j(u(t,x), \nabla u(t,x)), \quad t>0,\ x\in G,\ j\in\{1,\dots,\ell\},\\
\partial_\nu u_j(t,x) &= 0, \quad t>0,\ x\in\partial G,\ j\in\{1,\dots,\ell\}, \\
u_j(0,x) &= u_{j,0}(x), \quad x\in G,\ j\in\{1,\dots,\ell\},
\end{aligned} \tag{3.11}$$

for given initial distributions $u_{j,0} \geq 0$. One could also treat more complicated diffusion phenomena. In the linear case, heterogeneous and anisotropic diffusion is described by the term $\operatorname{div}(a_j \nabla u_j)$ for coefficient functions a_j on G taking values in the symmetric positive definite matrices, which could even depend on time. If the diffusion coefficients $a_j = a_j(x,u)$ depend on the solution u itself, one has 'quasilinear diffusion' which can be treated by more sophisticated methods, see [**20**] or the last chapter. Moreover, interactions between the species can lead to nondiagonal diffusion terms.

3) By Example 5.2 of [**32**], the Neumann Laplacian Δ_N with domain $\mathrm{D}(\Delta_N) = \{v \in W^{2,p}(G) \,|\, \partial_\nu v = 0\}$ generates a contractive, positive, analytic C_0-semigroup $S(\cdot)$ on $L^p(G)$ for $p\in(1,\infty)$. Here we let $\partial_\nu v = \sum_{k=1}^m \nu_k \operatorname{tr} \partial_k v$, and positivity of an operator T on $L^p(G)$ means that $Tg \geq 0$ a.e. for every $0 \leq g \in L^p(G)$. We now introduce $E = L^p(G)^\ell$, $0 \leq u_0 = (u_{1,0}, \dots, u_{\ell,0}) \in E$,

$$A = \begin{pmatrix} a_1\Delta_N & 0 & 0 & \cdots & 0 \\ 0 & a_2\Delta_N & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & a_\ell \Delta_N \end{pmatrix}, \quad \mathrm{D}(A) = \mathrm{D}(\Delta_N)^\ell =: E_1, \tag{3.12}$$

and $[F(v)](x) = f(v(x), \nabla v(x))$ for $v\in W^{1,p}(G)^\ell$ and $x \in G$. We say that $v \geq 0$ in E if all components of v_k are non-negative. It is easy to see that A generates the contractive, positive, analytic C_0-semigroup

$$T(t) = \begin{pmatrix} S(a_1 t) & 0 & 0 & \cdots & 0 \\ 0 & S(a_2 t) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & S(a_m t) \end{pmatrix}, \quad t \geq 0,$$

on E. Set $E_\alpha = D_A(\alpha, p)$ for $\alpha \in (0,1)$. By Proposition 2.4 and Theorem 2.14 the numbers $N_0 = \sup_{t\geq 0} \|T(t)\|_{\mathcal{B}(E_\alpha)}$ and $N_1 = \sup_{t\geq 0} \|\min\{1, t^\alpha\} T(t)\|_{\mathcal{B}(E,E_\alpha)}$ are finite. (Use $T(1) = T(1)T(t-1)$ for N_1 if $t\geq 1$.)

4) We look for a framework in which F becomes Lipschitz on bounded sets. We first let $v,w\in C^1(\overline{G})^\ell$ with C^1-norm less or equal r, where $C^1(\overline{G})$ contains the C^1–functions v on G such that v and ∇v have continuous extensions to ∂G. Denoting by $L_0(r)$ the Lipschitz constant of f on $\overline{B}_{|\cdot|_\infty}(0,r)$, we estimate

$$\begin{aligned}
|F(v)(x) - F(w)(x)|_\infty &\leq L_0(r) \max\{|v(x)-w(x)|_\infty, |\nabla v(x) - \nabla w(x)|_\infty\} \\
&\leq L_0(r) \|v-w\|_{C^1} \qquad \text{for } \ x\in G.
\end{aligned}$$

This means that $F : C^1(\overline{G})^\ell \to C(\overline{G})^\ell$, and thus $F : C^1(\overline{G})^\ell \to L^p(G)^\ell$, are Lipschitz on bounded sets. Let $p > m$ and $\alpha \in (\frac{1}{2} + \frac{m}{2p}, 1)$, so that $2\alpha - \frac{m}{p} > 1$. Example 2.12 and the fractional Sobolev embedding theorem (see Theorem 4.6.1 in [**37**]) then imply that

$$E_\alpha \hookrightarrow W^{2\alpha,p}(G)^\ell \hookrightarrow C^1(\overline{G})^\ell. \tag{3.13}$$

As a result, $F : E_\alpha \to E$ is Lipschitz on bounded sets. If we have a pure reaction term $f(u)$, then the same arguments show that $F : E_\alpha \to E$ is Lipschitz on bounded sets already if $p > m/2$ and $\alpha \in (\frac{m}{2p}, 1)$.

5) We can thus write (3.11) as the semilinear parabolic problem (3.1) on E, and apply Theorem 3.4. For $u_0 \in E_\alpha$ it yields a unique solution u in $C(J'_+(u_0), E_\alpha) \cap C^1(J_+(u_0), E) \cap C(J_+(u_0), E_1)$ of (3.1). By (3.13), u belongs to $C(J'_+(u_0), C^1(\overline{G})^\ell)$. Such a solution satisfies the first line of (3.11) for a.e. $x \in G$ and the second line for all $x \in \partial G$. If $u_0 \in \mathrm{D}(A)$, we can include $t = 0$ in these statements. By Remark 3.5 the regularity properties hold for all $\alpha \in (\frac{1}{2} + \frac{m}{2p}, 1)$. Let $q \in (p, \infty)$ and note $\alpha \in (\frac{1}{2} + \frac{m}{2q}, 1)$. As in this remark, one can show analogous properties for the restriction of u to L^q-based spaces. $\Diamond$

In the following we restrict ourselves to pure reaction equations with $F(u)(x) = f(u(x))$ for a function $f : \mathbb{C}^\ell \to \mathbb{C}^\ell$ being locally Lipschitz. To study positivity, we further need *Hopf's lemma.* For functions $w \in C^2(B) \cap C^1(\overline{B})$, it is a special case of the lemma in Section 6.4.2 in [**9**]. Our result is shown in the same way using Proposition 3.1.10 of [**20**]. Sobolev's embedding implies that $w \in C^1(\overline{G})$ under the assumptions of the lemma.

Lemma 3.8. *Let $B = B(y, \rho) \subseteq \mathbb{R}^m$ be an open ball and w belong to $W^{2,p}(B)$ for all $p \in (1, \infty)$ and satisfy $0 \le \Delta w \in C(\overline{B})$. Assume that there is a point $x_0 \in \partial B$ such that $w(x_0) > w(x)$ for all $x \in B$. Then $\partial_\nu w(x_0) > 0$ for the outer normal $\nu(x) = \rho^{-1}(x - y)$ of ∂B.*

We show local wellposedness for the reaction-diffusion system (3.11) with $f(u)$. Besides basic properties already discussed in Example 3.7, the theorem contains a much improved blow-up condition, a compactness result, and a positivity criterion analogous to the ODE case. These facts are the basis for later studies of the long-term behavior of (3.11). Note that only non-negative solutions are relevant here. A blow-up example for $\ell = 1$ is given in the exercises.

Theorem 3.9. *Consider the situation of Example 3.7 with a locally Lipschitz map $f : \mathbb{C}^\ell \to \mathbb{C}^\ell$. Let $m < p < \infty$, $\alpha \in (\frac{m}{2p}, 1 - \frac{m}{2p})$, $F(v) = f \circ v$, A be given by (3.12), and $u_0 \in E_\alpha = D_A(\alpha, q)$. Then the following assertions hold.*

a) Problem (3.11) has a unique maximal solution u in $C([0, t_+(u_0)), E_\alpha) \cap C^1((0, t_+(u_0)), E) \cap C((0, t_+(u_0)), E_1)$. If $u_0 \in \mathrm{D}(A)$, then we can replace the interval $(0, t_+(u_0))$ by $[0, t_+(u_0))$. Assertion c) of Theorem 3.4 holds analogously. Moreover, the maps $\partial_t u$ and $\Delta_N u$ belong to $C((0, t_+(u_0)) \times \overline{G})^\ell$ and u to $C((0, t_+(u_0)), W^{2,q}(G)^\ell)$ for all $q \in (1, \infty)$.

b) Let $t_+(u_0) < \infty$. Then we have $\limsup_{t \to t_+(u_0)^-} \|u(t)\|_\infty = \infty$.

c) Let $t_+(u_0) = \infty$ and u be bounded on $\mathbb{R}_{\ge 0} \times \overline{G}$. Then the orbit $\{u(t) \mid t \ge 0\}$ is relatively compact in E_α.

d) Let $f(\mathbb{R}^\ell) \subseteq \mathbb{R}^\ell$ and $u_0 \geq 0$. Then u also takes real values. Let f also satiesfy the positivity condition

$$f_k(r_1, \dots, r_{k-1}, 0, r_{k+1}, \dots, r_\ell) \geq 0 \quad \textit{for all} \quad r_j \geq 0,\ j,k \in \{1, \cdots, \ell\}. \tag{3.14}$$

Then $u(t)$ is non-negative for all $t \in [0, t_+(u_0))$.

Proof. a) The first part follows from Theorem 3.4 and Example 3.7, except for the additional regularity results. To prove them, take $0 < \varepsilon < b < t_+(u_0)$. As in Lemma 3.1, Theorems 2.14 and 2.15 show that $F(u) \in C^{1-\alpha}([\varepsilon, b], E)$. Corollary 4.3.6 of [**20**] then implies that $u' = \partial_t u : [\varepsilon, b] \to D_A(1-\alpha, \infty)$ is bounded. Let $0 < \varepsilon \leq s \leq t \leq b$ and $\beta \in (\frac{m}{2p}, 1-\alpha)$. Corollary 1.24 in [**21**] and Corollary 2.10 yield

$$\|u'(t)-u'(s)\|_\beta \lesssim \|u'(t)-u'(s)\|_{1-\alpha}^{\frac{\beta}{1-\alpha}} \|u'(t)-u'(s)\|_0^{1-\frac{\beta}{1-\alpha}} \lesssim \|u'(t)-u'(s)\|_0^{1-\frac{\beta}{1-\alpha}},$$

where the right-hand side tends to 0 as $t-s \to 0$. We deduce that $\partial_t u$ belongs to $C([\varepsilon, b]), E_\beta) \hookrightarrow C([\varepsilon, b], C(\overline{G})^\ell)$, using also the fractional Sobolev embedding Theorem 4.6.1 in [**37**], Example 2.12 and $2\beta - \frac{m}{p} > 0$, cf. (3.13). Hence, $\partial_t u$ is contained in $C((0, t_+(u_0)) \times \overline{G})^\ell$ and the same holds for $\Delta_N u$ because of (3.11) and $F(u) \in C([0, t_0(u_0)) \times \overline{G})^\ell$.

In particular, u_j and $\Delta_N u_j$ are continuous with values in $L^q(G)$ for all $q < \infty$. Since $\mathrm{D}(\Delta_N, L^q(G)) \subseteq W^{2,q}(G)$ and $I - \Delta_N$ is invertible in $L^q(G)$ by Example 5.2 in [**32**], we obtain that u is an element of $C((0, t_+(u_0)), W^{2,q}(G)^\ell)$.

b) Let $t_+ = t_+(u_0) < \infty$ and $|u(t,x)|_\infty \leq R$ on $[0, t_+) \times \overline{G}$. It follows that

$$\begin{aligned} |f(u(t,x))|_\infty &\leq |f(u(t,x)) - f(0)|_\infty + |f(0)|_\infty \leq L_0(R)|u(t,x)|_\infty + |f(0)|_\infty \\ &\leq RL_0(R) + |f(0)|_\infty \end{aligned}$$

for all $(t,x) \in [0, t_+(u_0)) \times \overline{G}$ and the Lipschitz constant $L_0(R)$ of f, and thus

$$\|F(u(t))\|_E \leq \mathrm{vol}(G)^{\frac{1}{p}} (RL_0(R) + |f(0)|_\infty) =: C.$$

Employing Example 3.7, we can then control u via the mild formula (3.3) by

$$\begin{aligned} \|u(t)\|_\alpha &\leq \|T(t)u_0\|_\alpha + \int_0^t \|T(t-s)\|_{\mathcal{B}(E,E_\alpha)} \|F(u(s))\|_0 \,\mathrm{d}s \\ &\leq N_0 \|u_0\|_\alpha + CN_1 \Big(\int_{(t-1)_+}^t \frac{\mathrm{d}s}{(t-s)^\alpha} + (t-1)_+ \Big) \\ &\leq N_0 \|u_0\|_\alpha + \frac{CN_1}{1-\alpha} \max\{1, t_+\} \end{aligned}$$

for all $t \in [0, t_+)$. This bound contradicts the blow-up condition in Theorem 3.4, and thus either $t_+ = \infty$ or u is unbounded as $t \to t_+$.

c) Let $t_+(u_0) = \infty$ and u be bounded by $\tilde{R}$ on $\mathbb{R}_{\geq 0} \times \overline{G}$. First note that $\{u(t) \,|\, t \in [0,1]\}$ is compact in E_α by continuity. Let $\beta \in (\alpha, 1)$. Then E_β is compactly embedded into E_α by Proposition 2.13 since $E_1 \hookrightarrow E$ is compact by Theorem 3.34 in [**33**]. Using that u solves (3.1) on $[t-1, \infty)$ with initial value $u(t-1)$, we conclude as in part b) that

$$\|u(t)\|_\beta \leq \|T(1)u(t-1)\|_\beta + \int_{t-1}^t \|T(t-s)\|_{\mathcal{B}(E,E_\beta)} \|F(u(s))\|_0 \,\mathrm{d}s$$

$$\leq \tilde{N}_1 \|u(t-1)\|_0 + \tfrac{\tilde{C}\tilde{N}_1}{1-\beta} \leq \tilde{N}_1 \tilde{R} \operatorname{vol}(G)^{\frac{1}{p}} + \tfrac{\tilde{C}\tilde{N}_1}{1-\beta}$$

for some constants $\tilde{C}, \tilde{N}_1 > 0$ and for all $t \geq 1$. Hence, the set $\{u(t) \,|\, t \geq 1\}$ is compact in E_α and part c) is true.

d) Let f be real and $u_0 \geq 0$. When discussing (3.11), one can then replace throughout the scalar field $\mathbb{C}$ by $\mathbb{R}$, and thus obtain a real-valued solution. (We use $\mathbb{C}$ only to construct the analytic semigroup $S(\cdot)$ generated by Δ_N. But since it is positive, it leaves invariant real-valued functions. See also Remark 3.15.)

Let $\varepsilon > 0$ and $b \in (0, t_+(u_0))$. We set $u_{0,\varepsilon} = u_0 + \varepsilon \mathbb{1} > 0$ and $F_\varepsilon(v) = F(v) + \varepsilon \mathbb{1}$. Let u_ε solve (3.11) for $u_{0,\varepsilon}$ and F_ε. Proposition 3.6 yields a number $\varepsilon_0(b) > 0$ such that $u_\varepsilon(t)$ exists for all $t \in [0, b]$ and $\varepsilon \in (0, \varepsilon_0(b)]$ and such that $u_\varepsilon(t)$ tends to $u(t)$ in $E_\alpha \hookrightarrow C(\overline{G})^\ell$ as $\varepsilon \to 0$ uniformly for $t \in [0, b]$. It thus suffices to show that $u_\varepsilon(t) > 0$ for all $t \in [0, b]$ and $\varepsilon \in (0, \varepsilon_0(b)]$.

Suppose that this was not the case. Since $u_\varepsilon(0) > 0$, there would exist $\varepsilon \in (0, \varepsilon_0(b)]$, $t_0 > 0$, $x_0 \in \overline{G}$, and $k \in \{1, \ldots, \ell\}$ such that $v(t_0, x_0) = 0$, $u_\varepsilon(t, x) > 0$ and $u_\varepsilon(t_0, x) \geq 0$ for all $t \in [0, t_0)$ and $x \in \overline{G}$, where we put $v = (u_\varepsilon)_k$. Hence, $\partial_t v(t_0, x_0) \leq 0$ and $v(t_0, x_0)$ is a minimum of the function $x \mapsto v(t_0, x)$ on $\overline{G}$. Moreover, the condition (3.14) shows that $f_k(u_\varepsilon(t_0, x_0)) \geq 0$. Thus the differential equation in (3.11) implies that

$$a_k \Delta v(t_0, x_0) = \partial_t v(t_0, x_0) - f_k(u_\varepsilon(t_0, x_0)) - \varepsilon \leq -\varepsilon < 0.$$

Note that then $\Delta v(t_0, x) \leq 0$ for $x \in \overline{G}$ which are close to x_0, as $\Delta v(t_0)$ is continuous on $\overline{G}$ by part a).

If $x_0 \in G$, Proposition 3.1.10 of [**20**] yields that $\Delta v(t_0, x_0) \geq 0$ which is impossible. We can thus assume that $x_0 \in \partial G$ and $v(t_0, x) > 0$ for all $x \in G$. Since $\partial G \in C^2$, there a ball $B \subseteq G$ with $x_0 \in \partial B$, $\nu(x_0) = \nu_B(x_0)$, and $\Delta v(t_0) \leq 0$ on $\overline{B}$. We apply Hopf's Lemma 3.8 for $w = -v(t_0)$ obtaining $\partial_\nu v(t_0, x_0) < 0$, which contradicts the boundary condition in (3.11). It follows $u_\varepsilon(t) > 0$ for all $t \in [0, b]$ and $\varepsilon \in (0, \varepsilon_0(b)]$, as needed. (The extra regularity from step a) is needed to apply Proposition 3.1.10 of [**20**] and Lemma 3.8.) $\square$

We next show the weak maximum parabolic principle in our regularity framework. This result and slight variants are used in examples below to obtain sup-norm estimates on solutions.

Proposition 3.10. *Let $G \subseteq \mathbb{R}^m$ be bounded and open with $\partial G \in C^2$, $a > 0$, and let the real-valued function v belong to $C([0, T] \times \overline{G}) \cap C^1((0, T], C(\overline{G})) \cap C((0, T], W^{2,p}(G))$ for all $p \in (1, \infty)$ and satisfy $\Delta v \in C((0, T] \times \overline{G})$. Assume that $\partial_t v - a\Delta v \leq 0$ on $(0, T] \times \overline{G}$ and $\partial_\nu v \leq 0$ on $(0, T] \times \partial G$. We then have*

$$\max_{(t,x) \in [0,T] \times \overline{G}} v(t, x) \leq \max_{x \in \overline{G}} v(0, x).$$

Proof. We first observe that the maxima in the assertion exist. Let $v_\varepsilon(t, x) = v(t, x) - \varepsilon t$ for $\varepsilon > 0$ and $(t, x) \in [0, T] \times \overline{G}$. The maps v_ε satisfy the same assumptions with $\partial_t v_\varepsilon - a\Delta v_\varepsilon \leq -\varepsilon < 0$ on $(0, T] \times \overline{G}$.

Suppose there were $\varepsilon > 0$, $t_0 \in (0, T]$ and $x_0 \in \overline{G}$ such that $M := v_\varepsilon(t_0, x_0) > \max_{\overline{G}} v(0)$. We define

$$t_1 = \sup \left\{ t \in (0, T] \,\middle|\, \max_{x \in \overline{G}} v_\varepsilon(s, x) < M \ \text{ for all } s \in [0, t] \right\} \in (0, t_0].$$

Note that there is a point $x_1 \in \overline{G}$ with $v_\varepsilon(t_1, x_1) = M$. Hence, $\partial_t v_\varepsilon(t_1, x_1) \geq 0$ and $v_\varepsilon(t_1, x_1)$ is a maximum of $v_\varepsilon(t_1, \cdot)$ on $\overline{G}$. As in the proof of Theorem 3.9 d), these facts lead to a contradiction implying the assertion. □

In Example 3.7 we have indicated that many (ordinary) reaction equations are globally solvable under reasonable assumptions. If combined with diffusion, the situation is much more complicated as discussed in the survey article [**24**]. To give a flavor of this topic, we investigate a simple example.

Example 3.11. In the framework of Example 3.7, we let $p > m$, $p > 3/2$ if $m = 1$, $\ell = 2$ with the (different) species u and v, and $f(u, v) = (u^j v^k, -u^j v^k)$ for some $j, k \in \mathbb{N}_0$. We show global existence for all data $0 \leq (u_0, v_0) \in E_\alpha$.

Theorem 3.9 yields a unique nonnegative maximal solution (u, v) on $(0, t_+)$ since (3.14) holds. We suppose that $t_+ < \infty$. We first proceed as in the ODE system in Example 3.7 and deduce that

$$\partial_t(u + v) = a_1 \Delta_N(u + v) + \frac{a_2 - a_1}{a_1} a_1 \Delta_N v + u^j v^k - u^j v^k,$$

$$u(t) + v(t) = S(a_1 t)(u_0 + v_0) + \frac{a_2 - a_1}{a_1} a_1 \Delta_N \int_0^t S(a_1(t - s))v(s)\,\mathrm{d}s$$

for $t \in [0, t_+)$. Observe that $a_1 \neq a_2$ as we have two different species. Therefore the sum $u + v$ satisfies an equation with the inhomogeneity $g = (a_2 - a_1)\Delta_N v$ which has probably not a fixed sign. In the second equation above we see that the first summand is bounded on $\mathbb{R}_+ \times \overline{G}$ since the semigroup $S(\cdot)$ generated by Δ_N is bounded on $L^p(G)$, and thus on $D_{\Delta_N}(\alpha, p) \hookrightarrow C(\overline{G})$, see Example 3.7. The second term is defined by Theorem 2.15 since v takes values in an interpolation space of Δ_N. However, the norm of v in these spaces could blow up as $t \to t_+$. Which uniform bounds do we know for v?

Since $\partial_t v - a_2 \Delta_N v = -u^j v^k \leq 0$ by (3.11), Proposition 3.10 and Theorem 3.9 show that $\|v\|_\infty \leq \|v_0\|_\infty$ and hence

$$\|v\|_{L^q([0,t_+),L^q(G))} \leq t_+^{1/q} \operatorname{vol}(G)^{\frac{1}{q}} \|v_0\|_\infty.$$

for all $q \in (1, \infty)$. Unfortunately, the theory presented so far does not allow to use this global bound due to the presence of $a_1 \Delta_N$ in front of the integral.

Here the deeper theory of *maximal regularity* of type L^q helps. It says that for certain classes of Banach spaces X and generators B of analytic semigroups $R(\cdot)$ on X (including Δ_N on $L^r(G)$ for all $r \in (1, \infty)$) the function $w(t) = \int_0^t R(t - s)g(s)\,\mathrm{d}s$ takes values in $\mathrm{D}(B)$ for a.e. $t \geq 0$ and that

$$\|Bw\|_{L^q([0,b],X)} \lesssim_{b,q} \|g\|_{L^q([0,b],X)}$$

for every $g \in L^q([0, b], X)$. (See Theorem 6.3.2 in [**28**] and Section 7 in [**17**] for this result. We refer to these two works, Chapter 17 in [**15**], and our last chapter for a treatment of this theory.) We further recall that $\|BR(\cdot)x\| \lesssim_{b,q} \|x\|_{1-1/q,q}$ by Proposition 2.8. To use this result, we restrict to $t > \delta$ and thus replace (u_0, v_0) by $(u(\delta), v(\delta))$ for some $\delta > 0$ which is contained in $W^{2,q}(G)^2$ for all $q \in (1, \infty)$ by Theorem 3.9,a).

With $B = a_1 \Delta_N$ and $X = L^q(G)$ these facts imply that $u + v$, and thus u due to positivity, belong to $L^q([0, t_+), L^q(G))$ for every $q \in (1, \infty)$. Taking

$q = jp$ and using that v is uniformly bounded, we deduce that $u^j v^k$ belongs to $L^p([0,t_+), L^p(G))$. We next employ the equation $\partial_t u = a_1 \Delta_N u + u^j v^k$ and again maximal regularity to conclude that $\Delta_N u$ is an element of $L^p([0,t_+), L^p(G))$. The equation then implies that $\partial_t u \in L^p([0,t_+), L^p(G))$. Finally, we employ the interpolative embedding

$$\begin{aligned} &L^p\big((0,t_+), [\mathrm{D}(\Delta_N)]\big) \cap W^{1,p}\big((0,t_+), L^p(G)\big) \\ &\hookrightarrow C_b\big([0,t_+), D_{\Delta_N}(1-\tfrac{1}{p}, p)\big) \hookrightarrow C_b\big([0,t_+), W^{2-\frac{2}{p},p}(G)\big) \hookrightarrow C_b\big([0,t_+), C(\overline{G})\big), \end{aligned}$$

see Theorem III.4.10.2 in [2] or the exercises, Example 2.12, and the fractional Sobolev embedding Theorem 4.6.1 in [37] with $2 - \frac{2}{p} > \frac{m}{p}$.[3] Here we use that $p > 3/2$ if $m = 1$. Summing up, the solution (u,v) is bounded on $[0,t_+) \times \overline{G}$ so that Theorem 3.9 b) yields that $t_+ = \infty$.

One can show global existence of (3.11) for reactions $A + B \rightleftharpoons C$ with a refined version of the above L^p-approach, see [27]. The more general case $A + B \rightleftharpoons C + D$ was settled recently by different methods in [10]. $\Diamond$

3.2. Convergence to equilibria

Quite often one is particularly interested in certain classes of special solutions u_*. Here we only look at the simplest case of equilibria; a different one would be time-periodic solutions. We first characterize the stationary solutions of (3.1).

Lemma 3.12. *Let (3.2) be true. Then $u_* \in X$ is a time-independent solution* (equilibrium) *of (3.1) with $u_0 = u_*$ if and only if $u_* \in \mathrm{D}(A)$ and $Au_* = -F(u_*)$.*

Proof. If $u_* \in \mathrm{D}(A)$ and $Au_* = -F(u_*)$, then the function $u(t) = u_*$ clearly solves (3.1) for all $t \geq 0$. Conversely, if (3.1) has a stationary solution $u(t) = u_*$ for $t \geq 0$, then the mild formula (3.3) yields

$$\frac{1}{t}\,(T(t)u_* - u_*) = -\frac{1}{t}\int_0^t T(t-s)F(u_*)\,\mathrm{d}s \;\longrightarrow\; -F(u_*)$$

as $t \to 0$ by strong continuity, and hence $u_* \in \mathrm{D}(A)$ with $Au_* = -F(u_*)$. $\square$

At least some equilibria of (3.11) are easy to obtain.

Remark 3.13. The reaction diffusion system (3.11) possesses the spatially constant equilibrium $u_* = r_* \mathbb{1}$ if there is a vector $r_* \in \mathbb{R}^\ell$ with $f(r_*) = 0$; i.e., (3.11) inherits the equilibria of the correspondig ordinary differential equation $y' = f(y)$. (This works since we have chosen Neumann boundary conditions.) The construction of other, spatially heterogeneous equilibria is part of the theory of semilinear elliptic equation, not treated here.

More generally, for every constant initial value $u_0 = r_0 \mathbb{1}$ the system (3.11) has the solution $u(t) = r(t)\mathbb{1}$ where $r' = f(r)$ and $r(0) = r_0 \in \mathbb{R}^\ell$, so that the reaction ODE is contained in the reaction-diffusion system (3.11). $\Diamond$

[3] $W^{1,q}(I,X)$ is the space of functions $g \in L^q(I,X)$ with $\partial_t g \in L^q(I,X)$ for $q \in [1,\infty]$ and an open interval $I \subseteq \mathbb{R}$. Here the weak derivative is defined as for $X = \mathbb{F}$ and has the same properties as in Theorem 3.22 in [33].

In the applications one cannot exactly prescribe an equilibrium u_* as an initial value. Thus it is important whether small deviations of the initial value lead to small deviations of the solution for all $t \geq 0$. This property is called stability. In this section we treat the slightly different, but closely related, topic of convergence to u_*. We consider two basic results, both due to Lyapunov in the ODE case: a local one using the spectrum of the linearization and a global one using Lyapunov functions, cf. [**31**] for ordinary differential equation.

The first result is called *principle of linearized stability*, and it is based on the idea that near an equilibrium u_* the problem $u'(t) = Au(t) + F(u(t))$ is very close to the linearized equation $w'(t) = Aw(t) + F'(u_*)w(t)$. We show that the exponential stability of the latter problem implies the 'local exponential stability' of u_* for (3.1). Recall that the spectral bound $\mathrm{s}(B)$ of a closed operator is the supremum of the real parts of all $\lambda \in \sigma(A)$.

THEOREM 3.14. *Let* (3.2) *be true. Assume that u_* is an equilibrium of* (3.1), *F is differentiable at u_*, and $\mathrm{s}(A+F'(u_*)) < 0$. Take $\kappa \in (0, -\mathrm{s}(A+F'(u_*)))$. Then there are constants $c, \rho > 0$ such that for each $u_0 \in \overline{B}_\alpha(u_*, \rho)$ we have $t_+(u_0) = \infty$ and*

$$\|u(t) - u_*\|_\alpha \leq c\mathrm{e}^{-\kappa t}\|u_0 - u_*\|_\alpha$$

for all $t \geq 0$, where u solves (3.1).

PROOF. Since $F'(u_*) \in \mathcal{B}(X_\alpha, X)$, the sum $B := A + F'(u_*)$ on $\mathrm{D}(B) = \mathrm{D}(A)$ generates an analytic C_0-semigroup $S(\cdot)$ on X by Theorem 3.10 of [**32**] and Remark 2.11. Corollary 4.17 in [**32**] and the assumption thus yield $\omega_0(B) = \mathrm{s}(B) < -\kappa < 0$, and hence $\|S(t)\| \leq M\mathrm{e}^{-\delta' t}$ for some $M \geq 1$ and $\delta' \in (\kappa, -\mathrm{s}(B))$ and all $t \geq 0$. We write X_α^B for the interpolation spaces of B and take $\delta \in (\kappa, \delta')$. Proposition 2.4 implies $\|S(t)\|_{\mathcal{B}(X_\alpha^B)} \leq M\mathrm{e}^{-\delta' t} \leq M\mathrm{e}^{-\delta t}$ for $t \geq 0$. From Theorem 2.14 we infer $\|S(t)\|_{\mathcal{B}(X, X_\alpha^B)} \leq ct^{-\alpha} \leq c\mathrm{e}^{\delta} t^{-\alpha}\mathrm{e}^{-\delta t}$ for $t \in (0,1]$, as well as

$$\|S(t)\|_{\mathcal{B}(X,X_\alpha^B)} \leq \|S(1)\|_{\mathcal{B}(X,X_\alpha^B)}\,\|S(t-1)\| \leq c\mathrm{e}^{-\delta' t} \leq ct^{-\alpha}\mathrm{e}^{-\delta t}$$

for $t \geq 1$. To transfer these estimates to X_α, we note that $I : [\mathrm{D}(A)] \to [\mathrm{D}(B)]$ is an isomorphism by Theorem 3.10 in [**32**]. Interpolation (see Theorem 2.9) then shows that $I : X_\alpha \to X_\alpha^B$ is also isomorphic, resulting in

$$\|S(t)\|_{\mathcal{B}(X_\alpha)} \leq M_0\,\mathrm{e}^{-\delta t} \qquad \text{and} \qquad \|S(t)\|_{\mathcal{B}(X,X_\alpha)} \leq M_1 t^{-\alpha}\mathrm{e}^{-\delta t} \tag{3.15}$$

for all $t > 0$ and some constants $M_0, M_1 \geq 1$.

Since $Au_* = -F(u_*)$, the function $v := u - u_*$ with initial value $v(0) = u_0 - u_* =: v_0$ satisfies the equation

$$\begin{aligned} v'(t) &= u'(t) = Au(t) + F(u(t)) - Au_* - F(u_*) \\ &= Bv(t) + F(u_* + v(t)) - F(u_*) - F'(u_*)v(t) =: Bv(t) + G(v(t)) \end{aligned} \tag{3.16}$$

for all $t \in [0, t_+(u_0))$. We can fix a number $\varepsilon > 0$ with

$$c_0 M_1 \varepsilon t^{1-\alpha} \exp(c(\alpha) M_1^{\frac{1}{1-\alpha}}\,\varepsilon^{\frac{1}{1-\alpha}}\,t)\,\mathrm{e}^{(\kappa-\delta)t} \leq \frac{1}{2}$$

for all $t \geq 0$, where the constants c_0 and $c(\alpha)$ are taken from (3.9). Because F is differentiable at u_*, there is a radius $r > 0$ such that

$$\forall\, x \in \overline{B}_\alpha(r) : \qquad \|G(x)\|_\alpha \leq \varepsilon\|x\|_\alpha. \tag{3.17}$$

To use this estimate, we have to restrict ourselves to times $t \ge 0$ such that $\|v(t)\|_\alpha \le r$. Set $\rho = (2M_0)^{-1}r \in (0,r)$ and take $u_0 \in \overline{B}_\alpha(u_*,\rho)$ so that $\|v_0\|_\alpha \le \rho < r$. We introduce the number

$$\tau = \sup\big\{t \in (0,t_+(u_0)) \,\big|\, \|v(s)\|_\alpha \le r \ \text{ for all } s \in [0,t]\big\} \ \in \ (0,t_+(u_0)].$$

Equation (3.16) and estimates (3.15) and (3.17) now yield

$$\begin{aligned}
\|v(t)\|_\alpha &\le \|S(t)v_0\|_\alpha + \int_0^t \|S(t-s)G(v(s))\|_\alpha \,\mathrm{d}s\\
&\le M_0 \mathrm{e}^{-\delta t}\|v_0\|_\alpha + \varepsilon M_1 \int_0^t (t-s)^{-\alpha}\mathrm{e}^{-\delta(t-s)}\|v(s)\|_\alpha \,\mathrm{d}s,\\
\mathrm{e}^{\delta t}\|v(t)\|_\alpha &\le M_0\|v_0\|_\alpha + \varepsilon M_1 \int_0^t (t-s)^{-\alpha}\,\mathrm{e}^{\delta s}\|v(s)\|_\alpha \,\mathrm{d}s
\end{aligned}$$

for all $t \in [0,\tau)$. The singular Gronwall inequality (3.9) then leads to

$$\begin{aligned}
\mathrm{e}^{\delta t}\|v(t)\|_\alpha &\le M_0\|v_0\|_\alpha\big[1 + c_0 M_1 \varepsilon t^{1-\alpha}\exp\big(c(\alpha)M_1^{\frac{1}{1-\alpha}}\,\varepsilon^{\frac{1}{1-\alpha}}\,t\big)\big],\\
\|v(t)\|_\alpha &\le M_0\mathrm{e}^{-\kappa t}\|v_0\|_\alpha\big[1 + c_0 M_1 \varepsilon t^{1-\alpha}\exp\big(c(\alpha)M_1^{\frac{1}{1-\alpha}}\,\varepsilon^{\frac{1}{1-\alpha}}\,t\big)\mathrm{e}^{(\kappa-\delta)t}\big]\\
&\le \tfrac{3}{2}M_0\mathrm{e}^{-\kappa t}\|v_0\|_\alpha \le \tfrac{3}{4}r,
\end{aligned}$$

for all $t \in [0,\tau)$, due to our choice of ε and ρ. If $\tau < t_+(u_0)$, we would obtain $\|v(\tau)\|_\alpha \le 3r/4 < r$ by continuity, contradicting the definition of τ. Hence, $\tau = t_+(u_0)$ so that $\|v(t)\|_\alpha$ is bounded on $[0,t_+(u_0))$. Theorem 3.4 thus yields $t_+(u_0) = \infty$. The claim with $c := \frac{3M_0}{2}$ now follows from the above estimate. $\square$

There are refinements of Theorem 3.14 that describe the neighborhood of an equilibrium depending on $\sigma(A + F'(u_*))$, see e.g. Chapter 9 of [**20**].

In Theorem 3.14 we have employed $\mathbb{C}$–linear derivatives. As we have seen in Corollary 1.18, there are many important nonlinearities which are only real differentiable. On the other hand, we have used complex scalars for spectral theory or to construct and study analytic semigroups. In the next remark we indicate how to pass from real to complex scalars in our setting.

Remark 3.15. Let X be a real Banach space. We define its complexification

$$X_{\mathbb{C}} = X \oplus \mathrm{i}X = \big\{z = x + \mathrm{i}y \,\big|\, x,y \in X\big\},$$

and write $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$. It is straightforward to check that $X_{\mathbb{C}}$ is a complex vector space for the scalar multiplication

$$(\alpha + \mathrm{i}\beta)(x + \mathrm{i}y) := (\alpha x - \beta y) + \mathrm{i}(\beta x + \alpha y)$$

for $\alpha,\beta \in \mathbb{R}$ and $x,y \in X$. Moreover, $X_{\mathbb{C}}$ is a Banach space when endowed with

$$\|z\|_{X_{\mathbb{C}}} = \sup_{\theta\in[0,2\pi]} \|\operatorname{Re}(\mathrm{e}^{\mathrm{i}\theta}z)\|_X.$$

Note that $\|z\|_{X_{\mathbb{C}}}$ is equivalent to $\|\operatorname{Re} z\|_X + \|\operatorname{Im} z\|_X$. Typical examples are the real-valued function spaces $L^p(\mu)$ or $C(K)$ whose complexifications are the corresponding $\mathbb{C}$–valued spaces. (Compare Appendix B.4 of [**14**].)

For a real-linear operator B on X, one sets $\mathrm{D}(B_{\mathbb{C}}) = \mathrm{D}(B) \oplus \mathrm{i}\mathrm{D}(B)$ and $B_{\mathbb{C}}z = Bx + \mathrm{i}By$. Routine calculations show that $B_{\mathbb{C}}$ is $\mathbb{C}$-linear and closed if B is closed. In addition, if B is bounded, then $B_{\mathbb{C}}$ has the same norm. We define analytic C_0-semigroups as in Theorem 2.25 d) of [**32**] if $\mathbb{F}=\mathbb{R}$.

We assume that A and F satisfy (3.2) on a real Banach space X, that $u_* \in \mathrm{D}(A)$ is a stationary solution for (3.1) with A and F, and that F is real differentiable at u_*. One then verifies that $A_{\mathbb{C}}$ generates the analytic C_0-semigroup $(T(t)_{\mathbb{C}})_{t\geq 0}$ and that $D_{A_{\mathbb{C}}}(\alpha,p)$ and $D_{A_{\mathbb{C}}}(\alpha)$ are isomorphic to the complexifications of $D_A(\alpha,p)$ and $D_A(\alpha)$, respectively. Setting $F_{\mathbb{C}}(z) = F(\operatorname{Re} z)+\mathrm{i}F(\operatorname{Im} z)$, we obtain that $F_{\mathbb{C}}$ fulfills (3.2) on $X_{\mathbb{C}}$. Moreover, $u_*+\mathrm{i}u_*$ is an equilibrium for $A_{\mathbb{C}}$ and $F_{\mathbb{C}}$, and $F_{\mathbb{C}}$ possesses the $\mathbb{C}$-derivative $F'(u_*)_{\mathbb{C}}$ at $u_*+\mathrm{i}u_*$.

We also assume that $\mathrm{s}(A_{\mathbb{C}}+F'(u_*)_{\mathbb{C}})<0$. We can now apply Theorem 3.14 to the solution u of (3.1). Let $u_0\in X_\alpha$ be real. Since $\operatorname{Re} A_{\mathbb{C}}(z) = A(\operatorname{Re} z)$ and $\operatorname{Re} F_{\mathbb{C}}(z) = F(\operatorname{Re} z)$, the map $\operatorname{Re} u$ solves (3.1) with A, F, and initial value u_0, and it fulfills the conclusions of Theorem 3.14 with the same exponent.

Let $F(0)=0$ and $w=\operatorname{Im} z$. Take $b>0$ and set $r=\max_{0\leq t\leq b}\|w(t)\|_\alpha$. Since w solves (3.1) with $w(0)=0$, by means of the properties of $T(\cdot)$ we estimate $\|w(t)\|_\alpha \leq c(b)L(r)\int_0^t (t-s)^{-\alpha}\|w(s)\|_\alpha\,\mathrm{d}s$ for $t\in[0,b]$. Gronwall's inequality (3.9) then yields $w=0$. Consequently, $u=\operatorname{Re} u$ is real if $F(0)=0$. $\Diamond$

As a simple application of the principle of linearized stability, we discuss spatially constant equilibria of reaction-diffusion systems with two species. The treatment of general equilibria requires deeper investigations of spectral properties of elliptic systems with space-depending (heterogeneous) coefficients.

Example 3.16. We continue to work in the framework of Example 3.7 with $\ell=2$, $m=3$, $p\geq 2$, $f\in C^1(\mathbb{R}^2,\mathbb{R}^2)$, and $f(0,0)=0$. We assume that $f(r_*,s_*)=0$ for some $(r_*,s_*)\in\mathbb{R}^2_{\geq 0}$, and consider the equilibrium $(u_*,v_*) = (r_*,s_*)\mathbb{1}$. We write

$$f'(r_*,s_*) = \begin{pmatrix} c_{11} & c_{12}\\ c_{21} & c_{22}\end{pmatrix} =: C.$$

To check that F is real differentiable, let $(u,v),(\hat u,\hat v)\in C(\overline{G})^2$ be real and $x\in\overline{G}$. We compute

$$\begin{aligned} D(u,v)(x) &:= F(\hat u+u,\hat v+v)(x) - F(\hat u,\hat v)(x) - f'(\hat u(x),\hat v(x))(u(x),v(x))\\ &= \int_0^1 \big[f'(\hat u(x)+\tau u(x),\hat v(x)+\tau v(x)) - f'(\hat u(x),\hat v(x))\big](u(x),v(x))\,\mathrm{d}\tau,\end{aligned}$$
$$\|D(u,v)\|_\infty \leq \|(u,v)\|_\infty \max_{|r|,|s|\leq\|(u,v)\|_\infty}\ \max_{x\in\overline{G}} |f'(\hat u(x)+r,\hat v(x)+s) - f'(\hat u(x),\hat v(x))|.$$

Since f', $\hat u$ and $\hat v$ are uniformly continuous on $\overline{G}$, the map $F: C(\overline{G},\mathbb{R}^2)\to C(\overline{G},\mathbb{R}^2)$ has the (real) derivative given by

$$[F'(\hat u,\hat v)(u,v)](x) = f'(\hat u(x),\hat v(x))(u(x),v(x)).$$

One can show in a similar way that $F': C(\overline{G},\mathbb{R}^2)\to\mathcal{B}(C(\overline{G},\mathbb{R}^2))$ is continuous. Because $E_\alpha\hookrightarrow C(\overline{G})^2$ by (3.13) and $C(\overline{G})^2\hookrightarrow E = L^p(G)^2$, we conclude that F also belongs to $C^1_{\mathbb{R}}(E_\alpha,E)$ with the same formula for $F'(\hat u,\hat v)$, and hence

$$B(u,v) := [A+F'((u_*,v_*))](u,v) = \begin{pmatrix} a_1\Delta_N + c_{11}I & c_{12}I\\ c_{21}I & a_2\Delta_N + c_{22}I\end{pmatrix}\begin{pmatrix} u\\ v\end{pmatrix}.$$

Here we pass to $\mathbb{F} = \mathbb{C}$ as in Remark 3.15. If $\mathrm{s}(B) < 0$, then this remark and Theorem 3.14 show that all real solutions starting close to (u_*, v_*) converge to (u_*, v_*) in E_α exponentially as $t \to \infty$.

It remains to study $\mathrm{s}(B)$. The domain $\mathrm{D}(B) = \{(u, v) \in W^{2,p}(G)^2 \,|\, \partial_\nu u = \partial_\nu v = 0 \text{ on } \partial G\}$ is compactly embedded into E by the Rellich–Kondrachov Theorem 3.34 in [**33**]. Thus, B has pure point spectrum without accumulation points by Remark 2.13 and Theorem 2.15 of [**33**]. To use the powerful L^2–theory, we consider the restriction B_2 of B to $L^2(G)^2$ with the analogous domain. As for B, one obtains $\sigma(B_2) = \sigma_{\mathrm{p}}(B_2)$. We check that $\sigma_{\mathrm{p}}(B_2) = \sigma(B)$.

From $p \geq 2$ it follows $\mathrm{D}(B) \subseteq \mathrm{D}(B_2)$ and hence each eigenvector of B is one of B_2 for the same eigenvalue; i.e., $\sigma(B) \subseteq \sigma_{\mathrm{p}}(B_2)$. Conversely, let $w = (u_1, u_2) \in \mathrm{D}(B_2)$ satisfy $\lambda w = B_2 w$. By Sobolev's embedding Theorem 3.31 of [**33**], the eigenfunction w belongs to $C(\overline{G})^2 \hookrightarrow L^p(G)^2$ because of $2 - 3/2 > 0$. We further have $\partial_\nu u_j = 0$ on ∂G and

$$\Delta u_j - u_j = a_j^{-1}(\lambda u_j - c_{j1} u_1 - c_{j2} u_2) - u_j =: \varphi_j \ \in \ L^p(G)$$

for $j \in \{1, 2\}$. Since $\Delta_N - I$ is bijective on $L^p(G)$ by Example 5.2 of [**32**], there is a function $v_j \in W^{2,p}(G) \subseteq W^{2,2}(G)$ such that $\partial_\nu v_j = 0$ and $\Delta v_j - v_j = \varphi_j$. The injectivity of $\Delta_N - I$ on $L^2(G)$ implies that $u_j = v_j$ for $j \in \{1, 2\}$, and so w is an eigenfunction of B for the eigenvalue λ. We have shown $\sigma(B) = \sigma_p(B_2)$.[4]

Arguing as in Example 4.8 of [**33**] for the Dirichlet–Laplacian, we see that Δ_N is self-adjoint on $L^2(G)$. The spectral theorem in the compact case (see Theorem 4.15 in [**33**]) thus yields an orthonormal basis of eigenfunctions e_n for the eigenvalues $\mu_n \leq 0$ of Δ_N, where $\mu_n \to -\infty$ as $n \to \infty$. Let $(u, v) \in \mathrm{D}(B_2)$. We use the orthonormal series $u = \sum_{n \geq 0} \alpha_n \mathrm{e}_n$ and $v = \sum_{n \geq 0} \beta_n \mathrm{e}_n$ with the coefficients $\alpha_n = (u|\mathrm{e}_n)$ and $\beta_n = (v|\mathrm{e}_n)$. It further holds $\Delta_N u = \sum_{n \geq 0} \alpha_n \mu_n \mathrm{e}_n$ and $\Delta_N v = \sum_{n \geq 0} \beta_n \mu_n \mathrm{e}_n$. Hence, the equation $B_2(u, v) = \lambda(u, v)$ for some $\lambda \in \mathbb{C}$ is equivalent to

$$\sum_{n=0}^{\infty} \left[\begin{pmatrix} a_1 \mu_n \alpha_n \mathrm{e}_n \\ a_2 \mu_n \beta_n \mathrm{e}_n \end{pmatrix} + \begin{pmatrix} c_{11} \alpha_n \mathrm{e}_n + c_{12} \beta_n \mathrm{e}_n \\ c_{21} \alpha_n \mathrm{e}_n + c_{22} \beta_n \mathrm{e}_n \end{pmatrix} \right] = \sum_{n=0}^{\infty} \begin{pmatrix} \lambda \alpha_n \mathrm{e}_n \\ \lambda \beta_n \mathrm{e}_n \end{pmatrix}.$$

Since the functions e_n are orthonormal, this identity is true if and only if

$$M_n \begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix} := \begin{pmatrix} a_1 \mu_n + c_{11} & c_{12} \\ c_{21} & a_2 \mu_n + c_{22} \end{pmatrix} \begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix} = \lambda \begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix} \qquad \text{for every } n \in \mathbb{N}_0.$$

This means that an eigenvalue λ of some M_k with eigenvector (α_k, β_k) is an eigenvalue of B_2 with eigenfunction $(\alpha_k \mathrm{e}_k, \beta_k \mathrm{e}_k) \in \mathrm{D}(B_2)$. Conversely, every eigenfunction $w = (u, v)$ for an eigenvalue of B_2 gives an eigenvalue of M_n for those n such that $(\alpha_n, \beta_n) \neq 0$. The spectrum of B is thus given by

$$\sigma(B) = \sigma_{\mathrm{p}}(B_2) = \bigcup\nolimits_{n \in \mathbb{N}_0} \sigma(M_n).$$

Since B is sectorial and has compact resolvent, the set $\sigma(B) \cap \{\lambda \in \mathbb{C} \,|\, \mathrm{Re}\, \lambda \geq \mathrm{s}(B) - 1\}$ contains only finitely many points so that

$$\mathrm{s}(B) < 0 \iff \forall\, n \in \mathbb{N}_0 : \ \mathrm{s}(M_n) < 0 \iff \forall\, n \in \mathbb{N}_0 : \ \mathrm{tr}\, M_n < 0, \ \det M_n > 0.$$

[4] The spectrum of restrictions of generators to subspaces is studied in §IV.2.b of [**8**].

With these equivalences the stability problem is reduced to a finite family of inequalities involving the coefficients a_j and c_{jk} and the Neumann eigenvalues μ_n. We discuss this result a bit. As $\mu_0 = 0$, we have $M_0 = C = f'(r_*, s_*)$ so that $\mathrm{s}(C) < 0$ is a necessary condition for the local exponential stability of the reaction-diffusion equation; i.e., (r_*, s_*) must be locally exponential stable for the ODE describing the pure reaction. If this is the case, we have $\operatorname{tr} C = c_{11} + c_{22} < 0$ and hence $\operatorname{tr} M_n = (a_1 + a_2)\mu_n + c_{11} + c_{12} < 0$ since $\mu_n \le 0$ for all $n \in \mathbb{N}_0$. The limit $\mu_n \to -\infty$ yields $\det M_n > 0$ for all large n. However, for small $n \ge 1$ it can happen that

$$0 > \det M_n = a_1 a_2 \mu_n^2 + \mu_n(a_1 c_{22} + a_2 c_{11}) + \det C$$

despite $\det C > 0$ and $a_1 a_2 \mu_n^2 > 0$, as $\mu_n < 0$. For instance, this behaviour can occur if $\det C$ is close to 0, a_1 is small, and $c_{11} \gg a_1 |\mu_1|$. (Typically, diffusion coefficients are small.) Here one has 'diffusion induced instability' $\mathrm{s}(B) > 0$. $\Diamond$

It would be nice to know that the solutions converge to u_* for a 'larger' set of initial values than in Theorem 3.14 (e.g., all positive ones). To obtain sucxh results, we need two more tools: Lyapunov functions and omega limit sets.

Definition 3.17. *Let (3.2) be true and $\emptyset \neq D \subseteq X_\alpha$. A map $\Phi \in C(D, \mathbb{R})$ is called* Lyapunov function *for (3.1) on D if the function $\psi_u(t) = \Phi(u(t))$ does not increase for each solution u of (3.1) with $u(J') \subseteq D$. It is* strict *if $u(t)$ equals an equilibrium for $t \ge t_0$ if $\psi_u(t_1) = \psi_u(t_0)$ for some $t_1 > t_0 \ge 0$.*

(Note that we do not have backward uniqueness in general.) As we will see below, the existence of a Lyapunov function has a significant impact on the qualitative behavior of the system (3.1). So it is no surprise that there is not general recipe to find them. Often, Lyapunov functions are related to physical quantities such as energy or entropy for systems arising in sciences.

We first show that a Lyapunov function Φ allows to detect invariant sets for (3.1) and global solutions, if Φ blows up at ∂D and for large $\|x\|_\alpha$, respectively.

Proposition 3.18. *Let (3.2) be true, $D \subseteq X_\alpha$ be open, and $\Phi \in C(D, \mathbb{R})$ be a Lyapunov function for (3.1). Then the following assertions hold.*

a) Let $D \neq X_\alpha$ and $\Phi(x) \to \infty$ as $x \to \partial D$ in X_α. Then D is invariant*; i.e., for every $u_0 \in D$ we have $\varphi(t, u_0) \in D$ for all $t \in [0, t_+(u_0))$.*

b) Let D be invariant and $\Phi(x) \to \infty$ as $\|x\|_\alpha \to \infty$ for $x \in D$. Then $t_+(u_0) = \infty$ for all $u_0 \in D$.

Proof. Let $u_0 \in D$. The states $u(t) \coloneqq \varphi(t, u_0)$ then belong to the set $\{x \in D \mid \Phi(x) \le \Phi(u_0)\}$ as long as they stay in D and $t \in [0, t_+(u_0))$.

a) Let $\Phi(x) \to \infty$ as $x \to \partial D$ in X_α. Suppose there was a time $t_1 \in (0, t_+(u_0))$ with $u(t_1) \notin D$. Then the number

$$\tau \coloneqq \sup \big\{ t \in [0, t_1) \,\big|\, \forall\, s \in [0, t] : \; u(s) \in D \big\}$$

is contained in $(0, t_1]$. Since D is open and $u \in C(J'_+(u_0), X_\alpha)$, we have $u(\tau) \notin D$. For $t \in [0, \tau)$ the vectors $u(t)$ are elements of D and converge to $u(\tau)$ in X_α as $t \to \tau$. So $\Phi(u(t))$ tends to infinity, contradicting the initial observation.

b) Similarly, the conditions in b) imply that $\|u(t)\|_\alpha$ is bounded on $[0, t_+(u_0))$ so that $t_+(u_0) = \infty$ due to Theorem 3.4. $\square$

One can use Lyapunov functions to detect (asymptotic) stability, see Theorem 4.1.4 in [**13**]. We omit such results and focus on global convergence properties. To this end, we establish the basic properties of omega limit sets; i.e., the sets of accumulation points of solutions as $t \to \infty$.

We need the concept of connectedness in a metric space M. A set $C \subseteq M$ is *disconnected* if it decomposes into $C = C_1 \cup C_2$ for relatively open, disjoint, and non-empty subsets C_j. A subset is called *connected* if it is not disconnected.

We note some basic properties of connected sets, see Section III.4 of [**3**]. The space M is connected if and only if only $\emptyset$ and M are the only open and closed subsets of M. For open subsets of a normed vector space, connectedness and pathwise connectedness coincide. Let N be a metric space, $f : M \to N$ be continuous, and $C \subseteq M$ be connected. Then the range $f(C)$ is connected.

Proposition 3.19. *Let* (3.2) *be true. Let $u = \varphi(\,\cdot\,, u_0)$ be a solution of* (3.1) *on $\mathbb{R}_+$ such that the orbit $\gamma(u_0) := \{u(t) \,|\, t \geq 0\}$ is relatively compact in X_α. Then the following assertions hold.*

a) The omega limit set

$$\omega(u_0) := \big\{x \in X_\alpha \,\big|\, \exists\, \mathbb{R}_{\geq 0} \ni t_n \to \infty : u(t_n) \to x \text{ in } X_\alpha \text{ as } n \to \infty\big\}$$

is non-empty, compact and connected in X_α, and it is invariant for (3.1)*.*

b) We have $\mathrm{d}_\alpha(u(t), \omega(u_0)) := \mathrm{dist}_{X_\alpha}(u(t), \omega(u_0)) \to 0$ *as $t \to \infty$.*

c) Let $v_0 \in \omega(u_0)$. Then $t_+(v_0) = \infty$ and there is a solution v of (3.1) *on $\mathbb{R}$ with $v(0) = v_0$ and $v(t) \in \omega(u_0)$ for all $t \in \mathbb{R}$. In particular, $v_0 \in \mathrm{D}(A)$.*

Proof. 1) The set $\omega(u_0)$ is not empty, since $\gamma(u_0)$ has a compact closure in X_α. Let $x_n \in \omega(u_0)$ converge to x in X_α as $n \to \infty$. For each $n \in \mathbb{N}$ we choose $t_n \geq n$ such that $\|u(t_n) - x_n\|_\alpha \leq 1/n$. Hence, $\|x - u(t_n)\|_\alpha \leq \|x - x_n\|_\alpha + 1/n$ tends to 0 as $n \to \infty$. The vector x thus belongs to $\omega(u_0)$, and $\omega(u_0)$ is closed in X_α. This fact also yields the compactness of $\omega(u_0) \subseteq \overline{\gamma(u_0)}^{\alpha}$.

2) Suppose $\mathrm{d}_\alpha(u(t_n), \omega(u_0)) \geq \eta > 0$ for a sequence $t_n \to \infty$. Then there is a subsequence $(u(t_{n_j}))_j$ with a limit $x \in \omega(u_0)$ in X_α, which is a contradiction.

3) To show the connectedness of $\omega(u_0)$, we assume that there were non-empty disjoint subsets ω_j of $\omega(u_0)$ such that $\omega_j = O_j \cap \omega(u_0)$ for open sets $O_j \subseteq X_\alpha$ for $j \in \{1, 2\}$ and $\omega_1 \cup \omega_2 = \omega(u_0)$. Writing $\omega_j = \omega(u_0) \cap (X_\alpha \setminus O_i)$ for $i \neq j$, we see that ω_j is closed, hence compact, in X_α. It follows that $\|x_1 - x_2\|_\alpha \geq \delta > 0$ for some $\delta > 0$ and all $x_j \in \omega_j$.

Take some $x \in \omega_1$ and $y \in \omega_2$. There are times $s_n, t_n \to \infty$ such that $s_n < t_n < s_{n+1}$, $\|u(s_n) - x\|_\alpha < \delta/3$ and $\|v(t_n) - y\|_\alpha < \delta/3$ for all $n \in \mathbb{N}$. By continuity, we can then find a time $r_n \in (s_n, t_n)$ such that $\mathrm{d}_\alpha(u(r_n), \omega_1) = \delta/3$ and thus $\mathrm{d}_\alpha(u(r_n), \omega_2) \geq 2\delta/3$. But for a subsequence the states $u(r_{n_j})$ tend to some $z \in \omega(u_0)$ in X_α as $j \to \infty$, which is impossible. Thus, $\omega(u_0)$ is connected.

4) It remains to show the last part of a) and assertion c). Let $v_0 \in \omega(u_0)$. Then there are $t_n \to \infty$ such that $u(t_n) \to v_0$ in X_α as $n \to \infty$. By Theorem 3.4, the continuous dependence on initial data and the uniqueness of (3.1) imply

$$v(t) := \varphi(t, v_0) = \lim_{n \to \infty} \varphi(t, \varphi(t_n, u_0)) = \lim_{n \to \infty} \varphi(t + t_n, u_0)$$

in X_α for $t \in J_+(v_0)$. Hence, $v(t)$ belongs to $\omega(u_0)$ and $\omega(u_0)$ is invariant. As a compact set, $\omega(u_0)$ is bounded in X_α. Theorem 3.4 c) now yields $t_+(v_0) = \infty$.

Inductively, for all $j \in \{1, \dots, m\}$ and $m \in \mathbb{N}$ we obtain vectors $v_j \in \omega(u_0)$ and subsequences $(t_{\nu_m(k)})_k$ of $(t_{\nu_{m-1}(k)})_k$ such that $\varphi(t_{\nu_m(k)} - j, u_0)$ tends to v_j in X_α as $k \to \infty$ (with v_0 and $t_{\nu_0}(n) = t_n$ from above). Set $v^m = \varphi(\,\cdot\,, v_m)$ on $\mathbb{R}_{\geq 0}$ for $m \in \mathbb{N}_0$. For $t \geq -j \geq -m$ we compute

$$\begin{aligned} v^m(t+m) &= \lim_{k\to\infty} \varphi(t+m, \varphi(t_{\nu_m(k)} - m, u_0)) = \lim_{k\to\infty} \varphi(t + t_{\nu_m(k)}, u_0) \\ &= \lim_{k\to\infty} \varphi(t+j, \varphi(t_{\nu_m(k)} - j, u_0)) = v^j(t+j). \end{aligned}$$

For $j = 0$ this means $v^m(t+m) = v(t)$ if $t \geq 0$. So we can extend v to a solution of (3.1) on $\mathbb{R}$ by setting $v(t) = v^m(t+m)$ for $t \geq -m$ and $m \in \mathbb{N}$. In particular, $v_0 = v(0)$ is an element of $\mathrm{D}(A)$. Since $v(-m) = v_m \in \omega(u_0)$ for all $m \in \mathbb{N}_0$, the invariance of $\omega(u_0)$ implies that $v(t)$ belongs to $\omega(u_0)$ for each $t \in \mathbb{R}$. □

These results (except for the very last assertion) hold in a much more general setting, see Chapter 4 of [**13**]. The proof of the above proposition indicates again that one should describe the behavior of (3.1) in the norm of X_α. The compactness assumption in the above proposition is crucial, of course, as seen by the solution e^t of the ODE $u'(t) = u(t)$.

The above result only says that the solution tends to its omega limit set. In our next convergence theorem we can describe $\omega(u_0)$ in better way and then deduce that the solution has a limit. It is based on the above proposition and the observation that $\omega(u_0)$ only contains equlibria if the problem possesses a strict Lyapunov function.

Theorem 3.20. *Let (3.2) be true, $D \subseteq X_\alpha$, and $\Phi \in C(D, \mathbb{R})$ be a strict Lyapunov function for (3.1). Assume that $u_0 \in D$ satisfies $t_+(u_0) = \infty$ and that $\overline{\gamma(u_0)}^{\alpha}$ is compact in X_α and contained in D.*

Then the omega limit set $\omega(u_0)$ belongs to $\mathcal{E}_D$ and $\mathrm{d}_\alpha(u(t), \mathcal{E}_D)$ tends to 0 as $t \to \infty$, where $\mathcal{E}_D = \{u_* \in D \cap \mathrm{D}(A) \,|\, Au_* = -F(u_*)\}$ is the set of equilibria in D. If $\mathcal{E}_D$ is discrete, then u even converges to a vector $u_* \in \mathcal{E}_D$.

Proof. Since $\psi_u = \Phi \circ u$ does not increase and Φ is bounded on the compact set $\overline{\gamma(u_0)}^{\alpha}$, the function $\psi_u(t)$ converges to some $\ell \in \mathbb{R}$. Take any $x \in \omega(u_0)$ and $t_n \to \infty$ with $u(t_n) \to x$ in X_α. The vector x then belongs to D and $\Phi(x) = \lim_{n\to\infty} \psi_u(t_n) = \ell$ which means that Φ is constant on $\omega(u_0)$. Since $\varphi(t, x)$ stays in $\omega(u_0) \subseteq D$ by Proposition 3.19 for all $t \in \mathbb{R}$, the vector x belongs to $\mathcal{E}_D$ because Φ is strict. The assertions now follow from Proposition 3.19. (Note that a discrete, connected, non-empty set has to be a singleton.) □

There is a variant of this theorem without the strictness assumption, called *LaSalle's invariance principle*, see Theorem 4.3.4 of [**13**]. If $\mathcal{E}_D$ is not discrete, the solution does not tend to an equilibrium, in general; though in some situations one can still show convergence using more tools. See Sections 8.8 and 10.4 in [**29**] for such results in an ODE setting.

Despite its surprisingly elementary proof, the above theorem is very powerful in applications. For the reaction-diffusion system (3.11), Theorem 3.9 shows that uniform boundedness already implies compactness of the orbit. For (strict)

Lyapunov functions there are at least some candidates as the one used in the next example treating a predator-prey model. (Compare Beispiel 6.17 of [**31**].)

Example 3.21. Again we work in the framework of Example 3.7 with $\ell = 2$, now assuming that G is connected. We consider the 'reaction term'

$$f(u,v) = \begin{pmatrix} (1-\lambda_1 u - v)u \\ (\mu - \lambda_2 v + u)v \end{pmatrix}$$

with $\lambda_1, \lambda_2 > 0$ and $\mu \in \mathbb{R}$, describing the (normalized) interaction between the prey species u and the predators v. Let $0 \le u_0, v_0 \in E_\alpha \hookrightarrow C(\overline{G})^2$. Since the positivity condition (3.14) holds, there is a unique maximal solution $(u(t), v(t)) \ge 0$ of (3.1) with the above f on $[0, t_+)$ by Theorem 3.9.

1) We first show that $u \le \kappa := \max\{\|u_0\|_\infty, 1/\lambda_1\}$. Suppose that there were $(t_0, x_0) \in (0, t_+) \times \overline{G}$ and $\delta > 0$ such that $u(t_0, x_0) \ge \delta + \kappa$. Set

$$t_1 := \sup\big\{t \in (0, t_0] \,\big|\, \forall\, s \in [0,t],\ x \in \overline{G}:\ u(s,x) < \delta + \kappa\big\} \ \in\ (0, t_0].$$

There thus exists a point $x_1 \in \overline{G}$ with $u(t_1, x_1) = \delta + \kappa$. Then $u(t_1, x_1)$ is a maximum of $u(t_1, \cdot)$ on $\overline{G}$ and $\partial_t u(t_1, x_1) \ge 0$. We further have

$$\partial_t u(t_1, x_1) - a_1 \Delta u(t_1, x_1) \le u(t_1, x_1) - \lambda_1 u(t_1, x_1)^2 < 0$$

since $u(t_1, x_1) > 1/\lambda_1$ and $v \ge 0$. As in the proof of the parabolic maximum principle Proposition 3.10, we obtain a contradiction so that $u \le \kappa$.

2) Step 1) implies the inequality $(\mu - \lambda_2 v + u)v \le (\mu + \kappa)v - \lambda_2 v^2$. We can now proceed as in 1) and show that $v \le \max\{(\mu+\kappa)/\lambda_2, \|v_0\|_\infty\}$. Theorem 3.9 thus yields that $t_+ = \infty$ and that the orbit is relatively compact in E_α.

3) From now, let $u_0 \ne 0$ and $v_0 \ne 0$. Below we need the positivity of the solution. To this aim, set $\omega = \lambda_1 \|u\|_\infty + \|v\|_\infty - 1$ for the given solution. By (3.11), the rescaled function $\varphi(t) = \mathrm{e}^{\omega t} u(t)$ then satisfies the inequality

$$\partial_t \varphi - a_1 \Delta \varphi = \varphi(\omega + 1 - \lambda_1 u - v) \ge 0$$

on $\mathbb{R}_{\ge 0} \times \overline{G}$. If $u(t_0, x_0) = 0$ for some $(t_0, x_0) \in \mathbb{R}_+ \times \overline{G}$, then $\varphi = 0$ on $[0, t_0] \times \overline{G}$ due to the above inequality and the strong parabolic maximum principle (see Theorems 3.5 and 3.6 in [**26**][5]), contradicting $u_0 \ne 0$. In the same way one treats v. We have shown that $u, v > 0$ on $\mathbb{R}_+ \times \overline{G}$.

4) We only consider strictly positive equilibria. It is easy to see that f has a strictly positive zero (r_*, s_*) if and only if $\lambda_2 > \mu > -1/\lambda_1$, and then

$$(r_*, s_*) = \frac{1}{1 + \lambda_1 \lambda_2} (\lambda_2 - \mu, 1 + \mu \lambda_1).$$

This condition means that the internal reproduction coefficient μ of the predators is neither too negative nor bigger than the internal 'damping' coefficient λ_2 reflecting the competion between the predators. We thus study the equilibrium $(u_*, v_*) = (r_*, s_*)\mathbb{1}$ of (3.11), assuming $\lambda_2 > \mu > -1/\lambda_1$.

5) We introduce the set $D = \{w \in E_\alpha \,|\, w > 0 \ \text{ on } \overline{G}\}$ and the functional

$$\Phi(w) = \int_G \Big((w_1 - u_* \ln w_1) + (w_2 - v_* \ln w_2)\Big) \mathrm{d}x = \int_G \Psi(w)\,\mathrm{d}x,$$

[5] Here one needs the connectness of G. The proofs in [**26**] can be extended to our situation.

recalling that w_k is strictly positive on $\overline{G}$. (See Beispiel 6.13 of [**31**] for a derivation in the ODE setting.) As in Example 3.16, one sees that $\Psi \in C^1_{\mathbb{R}}(D, C(\overline{G}))$ with derivative $\Psi'(w)[\tilde{w}] = (1 - u_*/w_1)\tilde{w}_1 + (1 - v_*/w_2)\tilde{w}_2$ for all $\tilde{w} \in E_\alpha$. The integral is just a linear functional on $C(\overline{G})$ so that $\Phi \in C^1_{\mathbb{R}}(D, \mathbb{R})$. Since $(u(t), v(t)) \in D$ for all $t > 0$, the chain rule and (3.11) yield

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Phi(u,v) &= \int_G \Big(\big(1 - \frac{u_*}{u}\big)\partial_t u + \big(1 - \frac{v_*}{v}\big)\partial_t v\Big)\mathrm{d}x\\ &= \int_G \Big(\big(1 - \frac{u_*}{u}\big)a_1\Delta u + \big(1 - \frac{v_*}{v}\big)a_2\Delta v\Big)\mathrm{d}x\\ &\quad + \int_G \Big((u - u_*)(1 - \lambda_1 u - v) + (v - v_*)(\mu - \lambda_2 v + u)\Big)\mathrm{d}x,\end{aligned}$$

omitting the variables $(t, x) \in \mathbb{R}_+ \times G$. We denote the integrals in the last two lines by J_1 and J_2, respectively. An integration by parts shows that

$$J_1 = -\int_G \Big(\frac{a_1 u_* |\nabla u|^2}{u^2} + \frac{a_2 u_* |\nabla v|^2}{v^2}\Big)\mathrm{d}x \le 0.$$

Using $1 = \lambda_1 r_* + s_*$ and $\mu = \lambda_2 s_* - r_*$, we compute

$$\begin{aligned}J_2 &= \int_G \Big((u - u_*)(\lambda_1 u_* + v_* - \lambda_1 u - v) + (v - v_*)(\lambda_2 v_* - u_* - \lambda_2 v + u)\Big)\mathrm{d}x\\ &= -\int_G \Big(\lambda_1 (u - u_*)^2 + \lambda_2 (v - v_*)^2\Big)\mathrm{d}x \le 0.\end{aligned}$$

Summing up, we arrive at

$$\Phi(u(t), v(t)) \le \Phi(u(s), v(s)) - \int_s^t \int_G \Big(\lambda_1 (u(\tau) - u_*)^2 + \lambda_2 (v(\tau) - v_*)^2\Big)\mathrm{d}x\,\mathrm{d}\tau.$$

for all $t \ge s > 0$. As a consequence, $\Phi(u, v)$ does not increase along each orbit with nonzero, non-negative initial data. If it is constant on $[s, t]$ for some $t > s > 0$, we infer $u = u_*$ and $v = v_*$ on $[s, t]$, and thus on $[s, \infty)$. In particular, (u_*, v_*) is the only equilibrium in D and Φ is a strict Lyapunov function on D.

6) To apply Theorem 3.20, we have to check that the closure of $\gamma((u_0, v_0))$ is contained in D. Let $\widehat{w}_0 = (\widehat{u}_0, \widehat{v}_0) \in \omega((u_0, v_0))$. We have $\widehat{w}_0 \ge 0$ as a uniform limit of $(u(t_n), v(t_n)) \ge 0$. Observe that $\Psi(u, v) \ge c_*$ for a number $c_* \in \mathbb{R}$ and all $u, v \ge 0$. A variant of Fatou's lemma thus implies that

$$\Phi(\widehat{w}_0) \le \liminf_{n\to\infty} \Phi(u(t_n), v(t_n)) \le \Phi(u(1), v(1)) < \infty$$

so that $\widehat{u}_0(x) > 0$ and $\widehat{v}_0(x) > 0$ must hold for a.e. $x \in G$. Recall from Proposition 3.19, that $\widehat{w}_0 = \widehat{w}(0)$ for a solution $\widehat{w}$ of (3.1) on $\mathbb{R}$ belonging to $\omega((u_0, v_0))$. In particular, $\widehat{u}(-1)(x) > 0$ and $\widehat{v}(-1)(x) > 0$ for a.e. $x \in G$, and hence $\widehat{w}_0 > 0$ on $\overline{G}$ by step 3); i.e, $\overline{\gamma((u_0, v_0))}^{\alpha} \subseteq D$.

7) Theorem 3.20 now implies that the solution $(u(t), v(t))$ converges to (u_*, v_*) in E_α as $t \to \infty$ if $\lambda_2 > \mu > -1/\lambda_1$ and $0 \le (u_0, v_0) \in E_\alpha$ with $u_0 \ne 0$ and $v_0 \ne 0$. (Observe that the decay of the L^1-type quantity $\Phi(u(t), v(t))$ already implies the convergence of $(u(t), v(t))$ to the equilibrium in all norms $\|\cdot\|_\alpha$.) $\Diamond$

CHAPTER 4

The nonlinear Schrödinger equation

In this chapter we investigate the nonlinear Schrödinger equation

$$\begin{aligned} \mathrm{i}\partial_t u(t,x) &= -\Delta u(t,x) + \mu |u(t,x)|^{\alpha-1} u(t,x), \qquad t\in J,\ x\in\mathbb{R}^m, \\ u(0,x) &= u_0(x), \qquad x\in\mathbb{R}^m. \end{aligned} \tag{4.1}$$

Equivalently, one can write

$$\partial_t u(t,x) = \mathrm{i}\Delta u(t,x) - \mathrm{i}\mu |u(t,x)|^{\alpha-1} u(t,x).$$

Most of the time, it is assumed that

$$\mu\in\{-1,1\}, \quad 1<\alpha<\frac{m+2}{(m-2)_+} =: \alpha_c, \quad 0\in J \text{ is an interval}, \ J^\circ \neq \emptyset. \tag{4.2}$$

(We also allow for negative times.) Note that $\alpha_c=\infty$ for $m\in\{1,2\}$, $\alpha_c=5$ for $m=3$, $\alpha_c=3$ for $m=4$, and $\alpha_c \searrow 1$ as $m\to\infty$. The results below can be extended to more general nonlinearities, see [**6**], but the model equation (4.1) already gives a very good insight in the field. In some cases we also treat the *critical exponent* $\alpha=\frac{m+2}{m-2}$ for $m\ge 3$, which is harder to study. An extended survey is given in [**35**]. If $\mu=1$ one has the (in some respects easier) *defocusing case* and for $\mu=-1$ the *focusing case.*

The nonlinear Schrödinger equation is one of the prototypical systems exhibiting 'dispersive behavior', and much recent research has concentrated on it. Variants of it appear in quantum field theory; e.g., in the study of so called Bose–Einstein condensates. It is also used to describe (approximately) the amplitudes of wave packages in nonlinear optics, see [**22**]. Natural numbers α play a significant role when nonlinear material laws are given by power series. Due to symmetry constraints one then often considers odd α.

In this chapter we write $W^{k,q}=W^{k,p}(\mathbb{R}^m)$ for $k\in\mathbb{N}_0$ and $q\in[1,\infty]$, $H^k=W^{k,2}$, and similarly for other function spaces on $\mathbb{R}^m$, where $W^{0,q}=L^q$. We usually drop the domain $\mathbb{R}^m$ in integrals over $\mathbb{R}^m$. The norm on $W^{k,q}$ is denoted by $\|v\|_{k,q}$ and we set $\|v\|_{0,q}=\|v\|_q$.

4.1. Preparations

We start with a few (more or less explicit) special solutions of the differential equation in (4.1), which illustrate some phenomena occuring in the nonlinear Schrödinger equation. In the exercises we discuss symmetries and scaling properties of (4.1) which allow to construct new solutions out of a given one.

EXAMPLE 4.1. We want to construct *plane waves.* For a given vector $\xi\in\mathbb{R}^m\backslash\{0\}$, we look for a function $\phi:\mathbb{R}\to\mathbb{C}$ such that the map $w_\xi(t,x):=\phi(t)\mathrm{e}^{\mathrm{i}\xi\cdot x}$ solves (4.1). We compute $\partial_t w_\xi(t,x)=\phi'(t)\mathrm{e}^{\mathrm{i}\xi\cdot x}$, $\partial_k u(t,x)=\mathrm{i}\xi_k w_\xi(t,x)$, and

$\Delta w_\xi(t,x) = -|\xi|_2^2 w_\xi(t,x)$ for $(t,x) \in \mathbb{R}^{1+m}$. Since $|w_\xi| = |\phi|$, the map w_ξ satisfies (4.1) if and only if

$$\phi'(t) = -\mathrm{i}(|\xi|_2^2 + \mu|\phi(t)|^{\alpha-1})\phi(t).$$

This ordinary differential equation can be solved leading to the plane wave

$$w_\xi(t,x) = a\mathrm{e}^{\mathrm{i}\xi\cdot x}\mathrm{e}^{-\mathrm{i}|\xi|_2^2 t}\mathrm{e}^{-\mathrm{i}\mu|a|^{\alpha-1}t}, \qquad (t,x) \in \mathbb{R}^{1+m}.$$

Here $a := \phi(0)$, $|a| = |w_\xi|$ is the amplitude, ξ is the wave vector, and $\omega = |\xi|_2^2 + \mu|a|^{\alpha-1}$ is proportional to the (temporal) frequency. Observe that the summand $|\xi|_2^2$ in ω comes from $-\Delta$, whereas $\mu|a|^{\alpha-1}$ is the contribution of the nonlinear part which depends on $|a|$. For $\mu = 1$ these two terms add up and so the nonlinearity increases the frequency and thus the time oscillation, whereas for $\mu = -1$ the oscillations partly cancel. We further have

$$w_\xi(t,x) = a \iff \frac{1}{|\xi|_2}\xi \cdot x = \Big(|\xi|_2 + \mu\frac{|a|^{\mu-1}}{|\xi|_2}\Big)t =: v(\xi)t.$$

This plane moves along its unit normal vector $\frac{1}{|\xi|_2}\xi$ with the *phase velocity* $v(\xi)$ which depends on the length of the wave vector.

This behavior is called *dispersion*. Dispersion causes plane waves with different wave vectors ξ_j (even if they have the same direction $\frac{1}{|\xi_j|_2}\xi_j$) to spread out in space as time evolves. This effect will be stronger in the defocusing case $\mu = 1$, since again the nonlinear effect adds to the linear one. In the case $\mu = -1$ the waves exhibit less dispersion, they longer stay *focused*. This explains the terminology of the two cases. $\Diamond$

In focusing case one can construct standing waves, which is not possible in the defocusing case, see Theorem 7.3.1 in [**6**]. In the latter situation dispersion destroys such persistent patterns.

Example 4.2. We look for a *standing wave* for (4.1); i.e., a solution given by

$$u_\omega(t,x) = \mathrm{e}^{\mathrm{i}\omega t}\varphi_\omega(x), \qquad (t,x) \in \mathbb{R}^{1+m},$$

for a frequency $\omega \in \mathbb{R}$ and a wave profile $\varphi_\omega \in H^2 \setminus \{0\}$. Such a function u_ω solves (4.1) if and only if

$$\begin{aligned} \mathrm{i}\partial_t u_\omega(t,x) &= -\omega\mathrm{e}^{\mathrm{i}\omega t}\varphi_\omega(x) = -\mathrm{e}^{\mathrm{i}\omega t}\Delta\varphi_\omega(x) + \mu|\varphi_\omega(x)|^{\alpha-1}\mathrm{e}^{\mathrm{i}\omega t}\varphi_\omega(x), \\ -\Delta\varphi_\omega + \omega\varphi_\omega &= -\mu|\varphi_\omega|^{\alpha-1}\varphi_\omega \end{aligned} \tag{4.3}$$

on $\mathbb{R}^m$. The resulting semilinear elliptic problem for φ_ω can be solved in H^2 in the focusing case $\mu = -1$ if $\omega > 0$ and $1 < \alpha < \alpha_c$. One can even prove that φ_ω and $\nabla\varphi_\omega$ decay exponentially and find radial, decreasing solutions $\varphi_\omega > 0$. (See §8.1 of [**6**] and the references given there.)

For $m = 1$, $\omega = 1$ and $\mu = -1$ we have the explicit solution

$$\varphi_1(x) = \left(\frac{\sqrt{\beta+1}}{\cosh(\beta x)}\right)^{\frac{1}{\beta}}, \qquad x \in \mathbb{R},$$

where we set $\beta = \frac{\alpha-1}{2}$. For $\alpha = 3$, one has $\beta = 1$ and $\varphi_1 = \frac{\sqrt{2}}{\cosh}$. $\Diamond$

In the next example, these standing waves lead to *blow-up* solutions in the focusing case $\mu = -1$ with $\alpha = 1 + \frac{4}{m}$. Blow up actually occurs for all $\alpha \in [1 + \frac{4}{m}, \alpha_c)$ as shown in Theorem 6.5.10 of [**6**]. (The basic idea of its proof is similar to that of Proposition 1.21.) On the other hand, Theorem 4.19 below yields global existence if $\mu = 1$ and $\alpha < \alpha_c$ and if $\mu = -1$ and $\alpha < 1 + \frac{4}{m}$. Hence, stronger dispersion or weaker nonlinear effects prevent blow up.

EXAMPLE 4.3. Let $\mu = -1$, $\alpha = 1 + \frac{4}{m} < \alpha_c$, $\omega > 0$, and take $0 < \varphi_\omega \in H^2$ as in Example 4.2. For $t \in [0, 1)$ and $x \in \mathbb{R}^m$ we define

$$u(t,x) = (\mathrm{i}(t-1))^{-\frac{m}{2}} \mathrm{e}^{\mathrm{i}\frac{|x|^2}{4(t-1)}} \mathrm{e}^{-\mathrm{i}\frac{\omega}{(t-1)}} \varphi_\omega\big(\tfrac{1}{t-1}x\big).$$

One can directly (and tediously) check that u solves (4.1), cf. p.116 in [**35**]. Moreover, the substition $y = \frac{1}{t-1}x$ yields

$$\begin{aligned}
\|u(t)\|_2^2 &= \int_{\mathbb{R}^m} |t-1|^{-m} |\varphi_\omega\big(\tfrac{1}{t-1}x\big)|^2 \,\mathrm{d}x = \|\varphi_\omega\|_2^2,\\
\||\nabla u(t)|_2\|_2^2 &= \int_{\mathbb{R}^m} |t-1|^{-m} \Big|\tfrac{\mathrm{i}}{2(t-1)}\varphi_\omega\big(\tfrac{1}{t-1}x\big)x + \tfrac{1}{t-1}\nabla\varphi_\omega\big(\tfrac{1}{t-1}x\big)\Big|_2^2 \,\mathrm{d}x\\
&\geq \int_{\mathbb{R}^m} |t-1|^{-m-2} \big|\nabla\varphi_\omega\big(\tfrac{1}{t-1}x\big)\big|_2^2 \,\mathrm{d}x = |t-1|^{-2}\||\nabla\varphi_\omega|_2\|_2^2.
\end{aligned}$$

As a result, this solution explodes in H^1 as $t \to 1^-$ though it stays bounded in L^2 and the initial value $u(0)$ belongs to H^2 by the properties of φ_ω mentioned in Example 4.2. $\Diamond$

As we will see below, solutions of (4.1) preserve the L^2-norm (interpreted as 'mass') and the 'energy'. These conservation laws are the basic tools to study the longterm properties of solutions and in particular to understand the blow-up behavior. To define the energy, we need some preparations.

Let $\alpha \in (1, \alpha_c]$ if $m \neq 2$ and $\alpha \in (1, \alpha_c)$ if $m = 2$. Sobolev's embedding shows

$$\begin{aligned}
H^1 &\hookrightarrow L^{1+\alpha}, \quad &\|v\|_{1+\alpha} &\leq C_{\mathrm{So}}\|v\|_{1,2},\\
H^2 &\hookrightarrow L^{2\alpha}, \quad &\|w\|_{2\alpha} &\leq C_{\mathrm{So}}\|w\|_{2,2},
\end{aligned} \tag{4.4}$$

for all $v \in H^1$ and $w \in H^2$. See Theorem 3.31 in [**33**] and observe that, e.g.,

$$\begin{aligned}
&m \geq 3: \quad 1 + \alpha_c = 1 + \frac{m+2}{m-2} = \frac{2m}{m-2} \quad \Longrightarrow \quad 1 - \frac{m}{2} = \frac{2-m}{2} \geq -\frac{m}{1+\alpha},\\
&m \geq 4: \quad 2\alpha_c < \frac{2m}{m-4} \quad \Longrightarrow \quad 2 - \frac{m}{2} = \frac{4-m}{2} > -\frac{m}{2\alpha}.
\end{aligned}$$

For the above α, we can define the 'energy' $\mathcal{E}: H^1 \to \mathbb{R}$ by

$$\mathcal{E}(v) = \frac{1}{2}\int_{\mathbb{R}^m} |\nabla v|_2^2 \,\mathrm{d}x + \frac{\mu}{\alpha+1}\int_{\mathbb{R}^m} |v|^{\alpha+1} \,\mathrm{d}x \tag{4.5}$$

for $v \in H^1 \hookrightarrow L^{1+\alpha}$. We stress that $2\mathcal{E}(v) + \|v\|_2^2 \geq \|v\|_{1,2}^2$ in the defocusing case $\mu = 1$, but that the energy may become negative if $\mu = -1$. These properties lead to global existence in the first case and the occurence of blow up in the second one (if $\alpha \geq 1 + \frac{4}{m}$), as noted above.

The embedding (4.4), the chain rule and Corollary 1.18 yield $\mathcal{E} \in C^1_{\mathbb{R}}(H^1, \mathbb{R})$ with derivative

$$\mathcal{E}'(v)w = \operatorname{Re} \int_{\mathbb{R}^m} \big(\nabla v \cdot \nabla \overline{w} + \mu |v|^{\alpha-1} v \overline{w}\big)\, \mathrm{d}x \qquad \text{for} \quad v, w \in H^1. \tag{4.6}$$

We next show that regular solutions preserve the L^2-norm and the energy.

Remark 4.4. Let $u \in C(J, H^2) \cap C^1(J, L^2)$ solve (4.1) on J. Let $t \in J$.

a) From equation (4.1) and an integration by parts we infer

$$\begin{aligned} \partial_t \|u(t)\|_2^2 &= 2 \operatorname{Re} \int \partial_t u(t)\, \overline{u(t)}\, \mathrm{d}x = 2 \operatorname{Re} \mathrm{i} \int (\Delta u(t) - \mu |u(t)|^{\alpha-1} u(t)) \overline{u(t)}\, \mathrm{d}x \\ &= 2 \operatorname{Im} \int (|\nabla u(t)|_2^2 + \mu |u(t)|^{\alpha+1})\, \mathrm{d}x = 0, \\ \|u(t)\|_2 &= \|u_0\|_2. \end{aligned}$$

b) We cannot directly treat $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(u(t))$ by the chain rule and (4.6) as u may not belong to $C^1(J,H^1)$. To regularize, we use the Yosida approximations $R_n = nR(n, \Delta)$ which tend to I strongly in L^2 and $H^2 \hookrightarrow L^{2\alpha}$ (and hence uniformly on compact sets) as $n \to \infty$, see Lemma 1.22 of [**32**], Lemma 4.5, and (4.4). Since $R_n u \in C^1(J, H^2)$, formula (4.6) and an integration by parts imply

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(R_n u(t)) &= \operatorname{Re} \int \big(\nabla R_n u(t) \cdot \nabla \partial_t R_n \overline{u}(t) + \mu |R_n u(t)|^{\alpha-1} R_n u(t) \partial_t R_n \overline{u}(t)\big) \mathrm{d}x \\ &= \operatorname{Re} \int \big(-\Delta R_n u(t) + \mu |R_n u(t)|^{\alpha-1} R_n u(t)\big) R_n \partial_t \overline{u}(t)\, \mathrm{d}x. \end{aligned}$$

Using the above mentioned properties of R_n and Hölder's inequality, one can let $n \to \infty$ locally uniformly in t on the right-hand side. Hence, $\mathcal{E}(u)$ is continuously differentiable with

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(u(t)) &= \operatorname{Re} \int \big(\mu |u(t)|^{\alpha-1} u(t) - \Delta u(t)\big) \partial_t \overline{u}(t) \mathrm{d}x = \operatorname{Re} \int \mathrm{i} \partial_t u(t) \partial_t \overline{u}(t) \mathrm{d}x = 0, \\ \mathcal{E}(u(t)) &= \mathcal{E}(u_0), \end{aligned}$$

where we also employed (4.1). $\Diamond$

The above results indicate that one can control the H^1-norm of solutions at least in the defocusing case. We also note that there are no conservation laws involving second space derivatives, in general. Aiming at global existence, we are thus looking for a local wellposedness theory for the nonlinear Schrödinger equation (4.1) in which we can derive a blow-up condition involving only the H^1-norm. For such a theory it is reasonable to consider solutions which are continuous with values in H^1 (for which one must then also show the above conservation laws). Hence, one has to extend the Laplacian from H^2 to H^1. We first develop the necessary functional analytic framework.

To that purpose, we define the negative Sobolev spaces on $\mathbb{R}^m$ by

$$W^{-k,r} = W^{-k,r}(\mathbb{R}^m) := W^{k,r'}(\mathbb{R}^m)^\star, \qquad H^{-k} := (H^k)^\star$$

for $k \in \mathbb{N}$ and $r \in (1, \infty]$. (The norms in $W^{1,r}$ depend on the norm $|\cdot|_s$ chosen on $\mathbb{R}^m$, but they are equivalent with constants $c(m)$. This fact is mostly ignored below and we usually omit the subscript s.) The norm of the Banach space $W^{-k,r}$ is given by $\|\varphi\|_{-k,r} = \sup_{\|v\|_{k,r'}=1} |\varphi(v)|$. If $r \in (1,\infty)$, then $W^{-k,r}$ is separable and reflexive with a dual space isomorphic to $W^{k,r'}$. (See Propositions 4.19 and 5.28 of [30].)

Let $\alpha \in (1,\alpha_c]$ if $m \geq 3$ and $\alpha \in (1,\alpha_c)$ if $m \in \{1,2\}$. We set $q = \alpha+1$. By (4.4) and Proposition 4.13 of [30], the inclusion $I : H^1 \to L^q$ is continuous and injective with dense range. Proposition 5.46 of [30] thus yields that

$$I^\star : L^{q'} \to H^{-1} \quad \text{is continuous and injective with dense range.} \tag{4.7}$$

(Here we can also allow for $\alpha = 1$.) In this context we note that

$$q = \alpha + 1, \qquad q' = \frac{q}{q-1} = \frac{q}{\alpha} = 1 + \frac{1}{\alpha} \geq 1 + \frac{1}{\alpha_c} \qquad (> \quad \text{if } m \leq 2). \tag{4.8}$$

Here and below, for $r \in [1,\infty)$ we identify $L^{r'}$ with $(L^r)^\star$ via

$$\forall\, f \in L^r,\, g \in L^{r'} : \quad \langle f, g\rangle_{L^r \times (L^r)^\star} = \int fg\,\mathrm{d}x.$$

(We do *not* identify H^1 with $(H^1)^\star = H^{-1}$.) In this way a map $g \in L^{r'}$ with $r' \in [1+\frac{1}{\alpha_c}, 2]$ ($r' \in (1+\frac{1}{\alpha_c}, 2]$ if $m \leq 2$) induces a functional $\varphi_g = I^\star g$ acting on H^1 via

$$\forall\, v \in H^1 : \quad \langle v, \varphi_g\rangle_{H^1 \times H^{-1}} = \langle Iv, g\rangle_{L^r \times (L^r)^\star} = \int vg\,\mathrm{d}x.$$

We set $F(v) = -\mathrm{i}\mu|v|^{\alpha-1}v$ for $v \in H^1$. Lemma 1.17, Corollary 1.18, (4.4), (4.7) and (4.8) yield that

$$F \in C^1_{\mathbb{R}}(L^{\alpha+1}, L^{\frac{\alpha+1}{\alpha}}) \cap C^1_{\mathbb{R}}(H^1, H^{-1}), \quad \|F'(v)\|_{\mathcal{B}(L^{\alpha+1}, L^{\frac{\alpha+1}{\alpha}})} \leq \alpha \|v\|_{\alpha+1}^{\alpha-1}, \tag{4.9}$$

$$F \in C^1_{\mathbb{R}}(H^2, L^2), \quad \|F'(w)\|_{\mathcal{B}(H^2,L^2)} \leq \alpha C_{\mathrm{So}}^{\alpha-1} \|w\|_{2,2}^{\alpha-1}, \tag{4.10}$$

for all $v \in L^{\alpha+1}$ and $w \in H^2$. In particular, the derivatives of F are bounded on bounded sets of H^1, respectively H^2.

We now turn our attention to the Laplacian. We first extend the partial derivative $\partial_j : H^1 \to L^2$ to a bounded map $\partial_j : L^2 \to H^{-1}$ via

$$\forall\, v \in H^1 : \quad \langle v, \partial_j u\rangle_{H^1 \times H^{-1}} := -\int \partial_j v\, u\,\mathrm{d}x$$

for $j \in \{1,\ldots,m\}$ and $u \in L^2$. In this way we obtain bounded extensions $\partial_{jk}, \Delta : H^1 \to H^{-1}$. As in Example 1.54 of [32] one shows the invertibility of $I - \Delta : H^1 \to H^{-1}$ using the sesquilinear form

$$a(u,v) = \int (u\overline{v} + \nabla u \cdot \nabla \overline{v})\,\mathrm{d}x \qquad \text{for} \quad u, v \in H^1.$$

For $u \in H^1$ we then compute

$$\begin{aligned}\|(I-\Delta)u\|_{-1,2} &= \sup_{\|v\|_{1,2}=1} \left|\langle \overline{v}, u - \Delta u\rangle_{H^1}\right| = \sup_{\|v\|_{1,2}=1} \left|\int (u\overline{v} + \nabla u \cdot \nabla\overline{v})\,\mathrm{d}x\right| \\ &= \sup_{\|v\|_{1,2}=1} \left|(u|v)_{H^1}\right| = \|u\|_{1,2},\end{aligned} \tag{4.11}$$

so that $I-\Delta : H^1 \to H^{-1}$ is also isometric and thus unitary by Proposition 5.52 in [**30**]. Hence, H^{-1} is a Hilbert space with the scalar product $(\varphi|\psi)_{H^{-1}} = \big((I-\Delta)^{-1}\varphi \big| (I-\Delta)^{-1}\psi\big)_{H^1}$.

By Example 1.45 of [**32**] the Laplace operator Δ with domain H^2 is self-adjoint in L^2, and hence $\mathrm{i}\Delta$ is skew-adjoint in L^2. Stone's Theorem 1.44 in [**32**] thus shows that $\mathrm{i}\Delta$ generates a unitary C_0-group $T(\cdot)$ on L^2, which is called the *free Schrödinger group*. By the next result, this group looks like the diffusion semigroup with 'imaginary time' $\mathrm{i}t$. The resulting representation formula implies that $T(t)v \in C^\infty(\mathbb{R}^m)$ if $v \in L^2$ has compact support. However, there is no smoothing effect in the full space L^2 since $T(t)$ is bijective. We further extend $T(\cdot)$ to H^{-1} and show further regularity properties. Extensions and restrictions of $T(\cdot)$ and Δ are denoted by the same symbols.[1]

Lemma 4.5. *a) For $k \in \mathbb{Z}$ with $k \geq -1$, the operator Δ with $\mathrm{D}(\Delta) = H^{k+2}$ is self-adjoint and dissipative in H^k. The unitary group generated by $\mathrm{i}\Delta$ on H^k is an extension, respectively restriction, of $T(\cdot)$ on L^2. Moreover, $\partial_j T(t)u = T(t)\partial_j u$ in H^{-1} for all $u \in L^2$ and $j \in \{1, \dots, m\}$.*

b) For $v \in L^1 \cap L^2$, $t \in \mathbb{R} \setminus \{0\}$, and $x \in \mathbb{R}^m$ we have

$$T(t)v(x) = \frac{1}{(4\pi \mathrm{i} t)^{m/2}} \int_{\mathbb{R}^m} \mathrm{e}^{\mathrm{i}\frac{|x-y|^2}{4t}} v(y)\,\mathrm{d}y. \tag{4.12}$$

Proof. 1) Let $k \in \mathbb{N}_0$ and $\mathcal{F} : L^2 \to L^2$; $\mathcal{F}u = \hat{u}$, be the (unitary) Fourier transform. Theorem 3.25 in [**33**] yields the characterization $H^k = \{u \in L^2 \mid |\xi|^k \hat{u} \in L^2\}$. As in Example 1.45 of [**32**] one can then check that Δ with $\mathrm{D}(\Delta) = H^{k+2}$ is self-adjoint and dissipative in H^k, so that $\mathrm{i}\Delta$ generates a unitary C_0-group on H^k by Theorem 1.44 in [**32**]. The uniqueness of the Cauchy problem implies that the groups on H^k extend each other.

2) For $\lambda > 0$ the operators $(I-\Delta)^{-1}$ and $R(\lambda, \mathrm{i}\Delta)$ commute on H^k. The resolvent approximation in Corollary 3.24 of [**32**] then shows that also $(I-\Delta)^{-1}$ and $T(t)$ commute on H^k for $t \in \mathbb{R}$. Using the isomorphism $I - \Delta : H^1 \to H^{-1}$, see (4.11), we can thus extend $T(\cdot)$ to a unitary C_0-group on H^{-1} which is generated by $\mathrm{i}\Delta$ with domain H^1. Theorem 1.44 of [**32**] now yields that Δ is self-adjoint in H^{-1}. For self-adjoint operators A, dissipativity means that $\sigma(A) \subseteq \mathbb{R}_{\leq 0}$, cf. Corollary 2.28 of [**32**]. Hence the dissipativity of Δ on H^1 implies that on H^{-1}.

3) Using $\partial_j \Delta = \Delta \partial_j$ on H^3, we compute $\partial_j R(\lambda, \mathrm{i}\Delta) = R(\lambda, \mathrm{i}\Delta)\partial_j$ on H^1 for $j \in \{1, \dots, m\}$ and $\lambda > 0$. As in step 2) one concludes that $\partial_j T(t) = T(t)\partial_j$ on H^1 for $t \in \mathbb{R}$. The last part of assertion a) then follows by approximation.

4) The right-hand side of (4.12) defines a bounded map from L^1 to L^∞ for $t \neq 0$. Moreover, C_c^∞ is dense in $L^1 \cap L^2$ with respect to the sum norm $\|\cdot\|_1 + \|\cdot\|_2$ by Proposition 4.13 of [**30**]. It thus suffices to show (4.12) for $v \in C_c^\infty$.

By Theorem 3.25 in [**33**], we have $\mathcal{F}(\Delta\varphi) = -|\xi|^2 \mathcal{F}\varphi$ for $\varphi \in H^2$. Let $t \in \mathbb{R}$ and $x \in \mathbb{R}^m$. For $v \in C_c^\infty$, the map $u = T(\cdot)v$ belongs to $C(\mathbb{R}, H^2) \cap C^1(\mathbb{R}, L^2)$ and satisfies $u'(t) = \mathrm{i}\Delta u(t)$. It is then easy to check that $\widehat{u} \in C^1(\mathbb{R}, L^2)$ and

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\widehat{u}(t) = \mathcal{F}(\mathrm{i}\Delta u(t)) = -\mathrm{i}|\xi|^2 \widehat{u}(t), \qquad \widehat{u}(0) = \widehat{v}.$$

[1] The following proof was only sketched in the lectures.

Solving this ordinary differential equation for fixed $\xi \in \mathbb{R}^m$, we arrive at

$$\widehat{u}(t,\xi) = \mathrm{e}^{-\mathrm{i}t|\xi|^2}\widehat{v}(\xi) = \gamma_{\mathrm{i}t}(\xi)\widehat{v}(\xi),$$

where $\gamma_z(\xi) := \mathrm{e}^{-z|\xi|^2}$ for $\xi \in \mathbb{R}^m$ and $z \in \mathbb{C}$. Since $\gamma_{\mathrm{i}t}$ is bounded, we deduce $u(t) = \mathcal{F}^{-1}(\gamma_{\mathrm{i}t}\widehat{v})$.

5) As $\gamma_{\mathrm{i}t}$ is not the Fourier transform of an L^1-function, we cannot directly apply the convolution formulas in Theorem 3.11 of [33]. Instead we employ the regularization $m_\varepsilon(t) = \gamma_{\mathrm{i}t+\varepsilon} \in L^1 \cap L^2$ for $\varepsilon > 0$. Using the inversion formula for $\mathcal{F}$ from this theorem, we first compute

$$[\mathcal{F}^{-1}m_\varepsilon(t)](x) = (2\pi)^{-\frac{m}{2}}\int \mathrm{e}^{\mathrm{i}x\cdot\xi}\mathrm{e}^{-\mathrm{i}t|\xi|^2}\mathrm{e}^{-\varepsilon|\xi|^2}\,\mathrm{d}\xi = \prod_{k=1}^m \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}x_k\xi_k-(\mathrm{i}t+\varepsilon)\xi_k^2}\,\mathrm{d}\xi_k.$$

By means of complex contour integrals, we establish in step 6) the identity

$$\int_{\mathbb{R}} \mathrm{e}^{-(\mathrm{i}t+\varepsilon)s^2}\mathrm{e}^{\mathrm{i}x_k s}\,\mathrm{d}s = \sqrt{\frac{\pi}{\mathrm{i}t+\varepsilon}}\,\mathrm{e}^{\frac{-x_k^2}{4(\mathrm{i}t+\varepsilon)}} \tag{4.13}$$

for $t \neq 0$. Hence, $\mathcal{F}^{-1}m_\varepsilon$ belongs to L^1.

We can thus apply Theorem 3.11 of [33] to $m_\varepsilon\widehat{v}$. Since $|m_\varepsilon| \le \mathbb{1}$ and $m_\varepsilon(t)$ tends pointwise to $\gamma_{\mathrm{i}t}$, Lebesgue's theorem and the continuity of $\mathcal{F}^{-1}$ now yield

$$u(t) = \mathcal{F}^{-1}(\gamma_{\mathrm{i}t}\widehat{v}) = \lim_{\varepsilon\to 0}\mathcal{F}^{-1}(m_\varepsilon(t)\widehat{v}) = \lim_{\varepsilon\to 0}(2\pi)^{-\frac{m}{2}}(\mathcal{F}^{-1}m_\varepsilon(t)) * v,$$

$$u(t,x) = \frac{1}{(4\pi)^{\frac{m}{2}}}\lim_{\varepsilon\to 0}\int \frac{1}{(\mathrm{i}t+\varepsilon)^{\frac{m}{2}}}\mathrm{e}^{-\frac{|x-y|^2}{4(\mathrm{i}t+\varepsilon)}}v(y)\,\mathrm{d}y.$$

For fixed $t \neq 0$ and $x \in \mathbb{R}^m$, Lebesgue's theorem allows to let $\varepsilon \to 0$ in the integral since $v \in C_c^\infty$, and hence (4.12) holds.

6) It remains to check (4.13), where $x := x_k \in \mathbb{R}$, $t \neq 0$ and $\varepsilon > 0$ are fixed. We set $\zeta = \frac{\mathrm{i}x}{2(\mathrm{i}t+\varepsilon)}$ and compute

$$\int_{\mathbb{R}} \mathrm{e}^{-(\mathrm{i}t+\varepsilon)s^2}\mathrm{e}^{\mathrm{i}xs}\,\mathrm{d}s = \mathrm{e}^{-\frac{x^2}{4(\mathrm{i}t+\varepsilon)}}\int_{\mathbb{R}} \mathrm{e}^{-(\mathrm{i}t+\varepsilon)(s-\zeta)^2}\,\mathrm{d}s = \mathrm{e}^{-\frac{x^2}{4(\mathrm{i}t+\varepsilon)}}\int_{\mathbb{R}-\zeta} \mathrm{e}^{-(\mathrm{i}t+\varepsilon)z^2}\,\mathrm{d}z.$$

Writing $f(z) = \mathrm{e}^{-(\mathrm{i}t+\varepsilon)z^2}$ and $I = \int_{\mathbb{R}-\zeta} f\,\mathrm{d}z$, we have to prove $I = \big(\frac{\pi}{\mathrm{i}t+\varepsilon}\big)^{1/2}$.

To this purpose, we consider the counterclockwise oriented curve $\Gamma_n = \Gamma_n^{\mathrm{b}} \cup \Gamma_n^{\mathrm{r}} \cup (-\Gamma_n^{\mathrm{t}}) \cup (-\Gamma_n^{\mathrm{l}})$, where

$$\Gamma_n^{\mathrm{b}} = \{z = \tau n - \zeta \mid -1 \le \tau \le 1\}, \qquad \Gamma_n^{\mathrm{r}} = \{z = n + \tau\zeta \mid -1 \le \tau \le 0\},$$
$$\Gamma_n^{\mathrm{t}} = [-n, n], \qquad \Gamma_n^{\mathrm{l}} = \{z = -n + \tau\zeta \mid -1 \le \tau \le 0\},$$

and $n \in \mathbb{N}$. Cauchy's theorem shows $\int_{\Gamma_n} f\,\mathrm{d}z = 0$. There is a constant $c = c(\varepsilon, t, x) > 0$ such that

$$\sup_{z\in\Gamma_n^{\mathrm{r}}\cup\Gamma_n^{\mathrm{l}}} |\mathrm{e}^{-(\mathrm{i}t+\varepsilon)z^2}| \le \mathrm{e}^{-\varepsilon n^2}\mathrm{e}^{cn},$$

and hence $\int_{\Gamma_n^j} f\,\mathrm{d}z \to 0$ as $n \to \infty$ for $j \in \{\mathrm{l}, \mathrm{r}\}$. By a similar estimate one sees that $\int_{\Gamma_n^{\mathrm{b}}} f\,\mathrm{d}z$ tends to I. Letting $n \to \infty$, we then deduce that $I = 2\int_0^\infty f(s)\,\mathrm{d}s$.

Let $t > 0$ and set $\beta = \frac{1}{2}\arg(\mathrm{i}t+\varepsilon) \in (0, \frac{\pi}{4})$. Since $|\sqrt{\mathrm{i}t+\varepsilon}|\,\mathrm{e}^{\mathrm{i}\beta} = \sqrt{\mathrm{i}t+\varepsilon}$, the substitution $\tau = \sqrt{\mathrm{i}t+\varepsilon}\, s$ yields

$$I = \frac{2}{\sqrt{\mathrm{i}t+\varepsilon}} \int_{\mathrm{e}^{\mathrm{i}\beta}\mathbb{R}_{\geq 0}} \mathrm{e}^{-\tau^2}\,\mathrm{d}\tau.$$

To evaluate this integral, we use the curve

$$\Gamma_n' = [0,n] \cup \{n\mathrm{e}^{\mathrm{i}\sigma} \mid 0 \leq \sigma \leq \beta\} \cup (-\mathrm{e}^{\mathrm{i}\beta}[0,n])$$

with positive orientation. Since $|\mathrm{e}^{-n^2\mathrm{e}^{2\mathrm{i}\sigma}}| \leq \mathrm{e}^{-n^2\cos 2\beta}$ for $\sigma \in [0,\beta]$, Cauchy's theorem now implies

$$I = \frac{2}{\sqrt{\mathrm{i}t+\varepsilon}} \int_0^\infty \mathrm{e}^{-s^2}\,\mathrm{d}s = \frac{\sqrt{\pi}}{\sqrt{\mathrm{i}t+\varepsilon}},$$

as asserted. The case $t < 0$ is treated in the same way. $\square$

This representation formula allows to describe the *dispersive* behavior of $T(t)$ in quantitative way. The next corollary says that $T(t)$ flattens initial data in $L^1 \cap L^2$ which become bounded immediately and then tend to 0 in all L^p–norms for $p > 2$ as $t \to \infty$. Since the L^2–norm is preserved, local concentrations of $T(t)v$ must be pushed towards infinity in $\mathbb{R}^m$.

Corollary 4.6. *Let $q \in [2,\infty]$. Then $T(t)$ extends from $L^1 \cap L^2$ to an operator in $\mathcal{B}(L^{q'}, L^q)$ for all $t \in \mathbb{R}\setminus\{0\}$, with norm less or equal $(4\pi|t|)^{m(\frac{1}{q}-\frac{1}{2})}$.*

Proof. By (4.12), $T(t)$ maps $(L^1 \cap L^2, \|\cdot\|_1)$ into L^∞ with norm less or equal $(4\pi|t|)^{-m/2}$. Moreover, it has norm 1 as an operator on L^2. Let $q \in (2,\infty)$. The Riesz–Thorin interpolation theorem then shows that we can extend $T(t)$ to an operator from $L^{q'}$ to L^q with norm less or equal $(4\pi|t|)^{-m/2(1-2/q)} = (4\pi|t|)^{m/q-m/2}$. See Theorem 2.26 in [**30**] with $\theta := 2/q \in (0,1)$ and

$$\frac{1}{q'} = \frac{1-\theta}{1} + \frac{\theta}{2}, \qquad \frac{1}{q} = \frac{1-\theta}{\infty} + \frac{\theta}{2}. \qquad \square$$

4.2. Strichartz estimates

We first introduce our solution concepts. Recall that $F(v) = -\mathrm{i}\mu|v|^{\alpha-1}v$ induces a map $F \in C^1_{\mathbb{R}}(H^1, H^{-1}) \cap C^1_{\mathbb{R}}(H^2, L^2)$ by (4.9) and (4.10), and that $\mathrm{i}\Delta$ generates the unitary C_0-semigroup $T(\cdot)$ on H^k with $k \geq -1$ by Lemma 4.5.

Definition 4.7. *Let $k \in \{1,2\}$ and (4.2) be true, where we also allow that $\alpha = \alpha_c$ if $m \geq 3$. A function $u \in C(J,H^k) \cap C^1(J,H^{k-2})$ satisfying (4.1) in H^{k-2} is called* H^k-solution *(on J).*

Since F is only defined on subspaces of H^{-1} or L^2, we cannot use the results of the first chapter to solve the nonlinear Schrödinger equation (4.1). We still want to follow the approach based on mild solutions and Duhamel's formula

$$u(t) = T(t)u_0 - \mathrm{i}\mu \int_0^t T(t-s)(|u(s)|^{\alpha-1}u(s))\,\mathrm{d}s, \qquad t \in J. \tag{4.14}$$

In Chapter 3, this was possible since an analytic semigroup maps into interpolation spaces of its generator on which F was defined. As the free Schrödinger group is bijective, this does not work here. On the other hand, in (4.14) we do not need regularization of $T(t-s)$ to treat the integral, it only has to improve integrability to counteract the power nonlinearity.

Corollary 4.6 already says that $T(t)$ improves integrability, though the corresponding norms blow up as $t \to 0$. Using also L^p-spaces in time, one can describe this dispersive behavior in a more convenient way, and it is possible to deal also with inhomogeneities as needed in (4.14). The resulting *Strichartz estimates* in Theorem 4.10 are crucial for the following sections.

Before proving the Strichartz estimates, we collect some tools needed to state and prove them. We refer to pp.94–95 of [**32**] for a few remarks about Banach space valued integration and Bochner–Lebesgue spaces. The basic results (up to Fubini's theorem) are analogous to the scalar-valued case. Using these facts and Lemma 4.5, for $f \in L^1_{\mathrm{loc}}(J, H^{-1})$ we can define the one-sided convolution

$$(T *_+ f)(t) = \int_0^t T(t-s)f(s)\,\mathrm{d}s, \qquad t \in J,$$

in H^{-1}. Let $J \subseteq \mathbb{R}$ be any interval for a moment. We have the duality

$$L^p(J, W^{k,q})^\star = L^{p'}(J, W^{-k,q'}) \quad \text{via} \quad \langle f, g\rangle_{L^p_J W^{k,q}} = \int_J \langle f(t), g(t)\rangle_{W^{k,q}}\,\mathrm{d}t \quad (4.15)$$

for $k \in \mathbb{Z}$, $1 \le p < \infty$, $1 < q < \infty$, $f \in L^p(J, W^{k,q}) = L^p_J W^{k,q}$ and $g \in L^{p'}(J, W^{-k,q'})$, see Corollary 1.3.22 in [**14**]. Moreover, the space $L^p(J, W^{k,q})$ is reflexive for $p, q \in (1, \infty)$. By means of the density of simple $u : J \to W^{k,q}$, one can show that $L^p(J, W^{k,q})$ is separable if $p, q \in [1, \infty)$, cf. Proposition 1.2.29 in [**14**]. We need a density result for Bochner spaces.[2] (Recall that $W^{l,r} \cap W^{k,q}$ is a Banach space when endowed with the norm given by $\|v\|_{l,r} + \|v\|_{k,q}$.)

Lemma 4.8. *Let $J \subseteq \mathbb{R}$ be open, $k \in \mathbb{N}_0$, and $1 \le p, q < \infty$. Then all spaces $C_c^\infty(J, W^{l,r} \cap W^{k,q})$ with $l \in \mathbb{N}_0$ and $r \in [1, \infty]$ are dense in $L^p(J, W^{k,q})$.*

Proof. Fix $f \in L^p(J, W^{k,q})$, $l \in \mathbb{N}_0$ and $r \in [1, \infty]$. Take $\varepsilon > 0$.

1) The standard mollifiers $G_{1/n}$ with $n \in \mathbb{N}$ are uniformly bounded in $W^{k,q}$ and tend strongly to I as $n \to \infty$. The same is true for the cut-off map $v \mapsto \phi_j v$ where $\phi_j(x) = \phi(|x|/j)$ for $j \in \mathbb{N}$ and $\phi \in C^\infty(\mathbb{R})$ with $0 \le \phi \le 1$, $\phi = 1$ on $[-1, 1]$ and $\operatorname{supp}\phi \subseteq (-2, 2)$. Using dominated convergence, we can thus fix indices $n, j \in \mathbb{N}$ and a function $g = G_{1/n}(\phi_j f)$ in $L^p(J, W^{l,r} \cap W^{k,q})$ such that $\|f - g\|_{L^p(J,W^{k,q})} \le \varepsilon$, cf. Theorem 4.21 in [**30**]. (Actually, $g(t) \in C_c^\infty$.)

2) Let $J_n \subseteq \overline{J_n} \subseteq J_{n+1} \subseteq J$ be open bounded intervals whose union is J. Pick maps $\psi_n \in C_c(J)$ with $0 \le \psi_n \le 1$, $\psi_n = 1$ on J_n, and $\operatorname{supp}\psi_n \subseteq J_{n+1}$. Lebesgue's theorem gives an index $N \in \mathbb{N}$ with $\|g - \psi_N g\|_{L^p(J,W^{l,r})} \le \varepsilon$. Using that $\psi_N g$ has compact support in J and mollifiers on $\mathbb{R}$, we then find a function h in $C_c^\infty(J, W^{l,r} \cap W^{k,q})$ satisfying $\|g - h\|_{L^p(J,W^{l,r})} \le 2\varepsilon$. (The usual properties of mollifiers also work in the Banach space valued case.) $\square$

We state the Hardy–Littlewood–Sobolev inequality, see Theorem 4.3 in [**19**]. The conditions in the lemma are sharp, see Section 1.2 in [**12**].

[2] The next proof was omitted in the lectures.

LEMMA 4.9. *Let $1 < r < s < \infty$ and $0 < \lambda < n$ satisfy $1 + \frac{1}{s} = \frac{\lambda}{n} + \frac{1}{r}$. Then there is a constant $c > 0$ such that*

$$\Big(\int_{\mathbb{R}^n} \Big[\int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^\lambda}\,\mathrm{d}y\Big]^s \mathrm{d}x\Big)^{\frac{1}{s}} \le c\|f\|_r \qquad \textit{for all} \;\; f \in L^r(\mathbb{R}^n).$$

This result resembles Young's convolution estimate from Theorem 2.14 in [**30**] applied to $f * \varphi_\lambda$ with $\varphi_\lambda(x) = |x|^{-\lambda}$ for $x \in \mathbb{R}^n \setminus \{0\}$ and $\varphi_\lambda(0) = 0$. However, to apply Young one would need that φ_λ belonged to $L^{n/\lambda}(\mathbb{R}^n)$; i.e., $\int_{\mathbb{R}^n} |x|^{-n}\,\mathrm{d}x < \infty$, which is not quite true. The Hardy–Littlewood–Sobolev inequality still yields $\|f * \varphi_\lambda\|_s \le c\|f\|_r$ for $1 + \frac{1}{s} = \frac{\lambda}{n} + \frac{1}{r}$ and non-negative f.

Strichartz estimates bound the two summands $T(\cdot)\varphi$ and $T *_* f$ of (4.14) in the space $E = L^p(\mathbb{R}, L^q)$ for *(Schrödinger)-admissible* exponents (p, q); i.e.,

$$2 \le p, q \le \infty, \quad \frac{2}{p} + \frac{m}{q} = \frac{m}{2} \qquad \text{and} \;\; (p,q) \ne (2,\infty) \;\; \text{if } m = 2. \tag{4.16}$$

The 'critical endpoint' $(2, \frac{2m}{m-2}) = (2, 1+\alpha_c)$ for $m \ge 3$ and the 'trivial endpoint' $(\infty, 2)$ are admissible. The inverses $(\frac{1}{p}, \frac{1}{q})$ of admissible exponents (p, q) belong to the line from $(0, \frac{1}{2})$ to $(\frac{1}{2}, \frac{(m-2)_+}{2m})$ if $m \ge 2$, excluding the latter point if $m = 2$. So we have the embedding $L^{q'} \hookrightarrow H^{-1}$ for admissible (p, q) due to (4.7) and (4.8). This means that the convolution $T *_+ f$ is well-defined in H^{-1} for $f \in L^{p'}(J, L^{q'})$ as $T(\cdot)$ is a C_0-group on H^{-1}.

We explain admissibility by a scaling argument. Let the map $\varphi \mapsto u = T(\cdot)\varphi$ be bounded from L^2 to E. For $\lambda > 0$ we set $u_\lambda(t, x) = u(\lambda^2 t, \lambda x)$ and $\varphi_\lambda(x) = \varphi(\lambda x)$ for $t \in \mathbb{R}$ and $x \in \mathbb{R}^m$. Then also u_λ solves $\partial_t u_\lambda = \mathrm{i}\Delta u_\lambda$ and thus $u_\lambda = T(\cdot)\varphi_\lambda$. Let $\varphi \in L^2 \setminus \{0\}$. Substituting $y = \lambda x$ and $t = \lambda^2 s$, we compute

$$\|\varphi_\lambda\|_2 = \Big(\int |\varphi(\lambda x)|^2\,\mathrm{d}x\Big)^{\frac{1}{2}} = \lambda^{-\frac{m}{2}}\|\varphi\|_2,$$
$$\|u_\lambda\|_E^p = \Big(\int_{\mathbb{R}} \Big(\int |u(\lambda^2 s, \lambda x)|^q\,\mathrm{d}x\Big)^{\frac{p}{q}}\mathrm{d}s\Big)^{\frac{1}{p}} = \lambda^{-\frac{m}{q}}\lambda^{-\frac{2}{p}}\|u\|_E.$$

As a result, the claimed boundedness implies that $\lambda^{-\frac{m}{q}}\lambda^{-\frac{2}{p}}\|u\|_E \le C\lambda^{-\frac{m}{2}}\|\varphi\|_2$ for all $\lambda > 0$, and hence (p, q) must satisfy the equality in (4.16).

We now come to the main theorem of this section, containing the *homogeneous* Strichartz' estimate a) and the *inhomogeneous* estimate b).

THEOREM 4.10. *Let (p, q) and $(\overline{p}, \overline{q})$ be admissible as in (4.16), $k \in \mathbb{N}_0$, $\varphi \in H^k$, $J \subseteq \mathbb{R}$ be an interval containing 0, and $f \in L^{\overline{p}'}(J, W^{k,\overline{q}'}) =: \overline{E}_k'(J)$. Then the maps $T(\cdot)\varphi$ and $T *_+ f$ belong to $L^p(J, W^{k,q}) =: E_k(J)$, and there is a constant $C_{\mathrm{St}} \ge 1$ (independent of φ, f and J) such that*

a) $\|T(\cdot)\varphi\|_{E_k(J)} \le C_{\mathrm{St}}\|\varphi\|_{k,2}$,

b) $\|T *_+ f\|_{E_k(J)} \le C_{\mathrm{St}}\|f\|_{\overline{E}_k'(J)}$.

If $p = \infty$ and $q = 2$, we can replace $L^\infty(J, W^{k,q})$ by $C_b(J, W^{k,q})$. If $p = \overline{p} > 2$, the integral $T *_+ f(t)$ exists in $W^{k,q}$ for a.e. $t \in J$,

(We will use the constant C_{St} also for finitely many admissible pairs at the same time.) Compared to the L^2–setting, one gains space integrability from

$q = 2$ to $q > 2$ and one looses time integrability from $p = \infty$ to $p < \infty$ (but gains some decay as $t \to \infty$). Moreover, in b) the exponents on the right-hand side are smaller than 2, whereas they are larger than 2 on the left-hand side. We point out that $(\overline{p}, \overline{q})$ can be chosen independently of (p, q) in assertion b).

Part a) is wrong for non-admissible (p, q) as seen above and in Section 2.4 of [**6**], whereas part b) is true for some non-admissible exponents, cf. Section 2.4 of [**6**] and the exercises. The theorem and variants for the wave equation were proved by several authors in the case $p > 2$ and $\overline{p} > 2$ starting with Strichartz in 1977, see Section 2.3 of [**6**]. The much more difficult endpoint case $p = 2$ or $\overline{p} = 2$ was established by Keel and Tao in [**16**].

We will prove Theorem 4.10 only for $p, \overline{p} > 2$ and either for $(p, q) = (\overline{p}, \overline{q})$ or for $(p, q) = (\infty, 2)$ and any admissible $(\overline{p}, \overline{q})$, since we mostly work with these cases later on. In this situation the results follow from Corollary 4.6 and Lemma 4.9 in a nice way without deeper difficulties.

Exponents $(p, q) \neq (\overline{p}, \overline{q})$ can be used to treat (4.1) with more general nonlinearities, for instance. This case can be handled in an additional step using also the 'Christ–Kiselev' lemma, see Section 2.3 in [**35**]. A somehow different approach is employed in Theorem 2.3.3 in [**6**]. The endpoint case $(2, \frac{2m}{m-2})$ for $m \geq 3$ is needed to study (4.1) in the critical case $\alpha = \alpha_c$, see Theorem 4.17.

Proof of Theorem 4.10. As noted above we restrict ourselves to the cases $p, \overline{p} > 2$ and either $(p, q) = (\overline{p}, \overline{q})$ or $(p, q) = (\infty, 2)$. Lemma 4.5 says that $\partial_j T(t) = T(t)\partial_j$ in H^{-1}, and ∂_j is closed in H^{-1} on its maximal domain (which can be used to take it out of integrals). Hence it is enough to show the result for $k = 0$. We will prove the assertions for $J = \mathbb{R}$. The case of general J can be reduced to $J = \mathbb{R}$ by extending f by 0 to $\mathbb{R}$ and by restricting $T(\cdot)\varphi$ and $T *_+ f$ from $\mathbb{R}$ to J. We write $E = E_0(\mathbb{R})$, $\overline{E}' = \overline{E}_0'(\mathbb{R})$, and note that $E^\star = L^{p'}(\mathbb{R}, L^{q'})$ by (4.15).

1) Let $(p, q) = (\overline{p}, \overline{q})$ be admissible, $2 < q < 1 + \alpha_c$, $\varphi \in L^2$, and $f \in E^\star$. We first prove estimate b). Corollary 4.6 and Lemma 4.9 (with $\lambda = \frac{m}{2} - \frac{m}{q}$, $n = 1$, $r = p'$, $s = p$) imply the crucial estimate

$$\begin{aligned} I_1 &:= \Big[\int_{\mathbb{R}} \Big[\int_0^t \|T(t-s)f(s)\|_q \,\mathrm{d}s\Big]^p \mathrm{d}t\Big]^{\frac{1}{p}} \leq \Big[\int_{\mathbb{R}} \Big[\int_{\mathbb{R}} \frac{\|f(s)\|_{q'}}{(4\pi|t-s|)^{\frac{m}{2}-\frac{m}{q}}} \,\mathrm{d}s\Big]^p \mathrm{d}t\Big]^{\frac{1}{p}} \\ &\leq C_0 \Big[\int_{\mathbb{R}} \|f(s)\|_{q'}^{p'} \,\mathrm{d}s\Big]^{\frac{1}{p'}} = C_0 \|f\|_{E^\star}, \end{aligned} \tag{4.17}$$

where C_0 only depends on m, p and q. The conditions of Lemma 4.9 hold since (p, q) is admissible and $2 < q < 1 + \alpha_c$. (The measurability of the integrand of I_1 is verified below.)

From this estimate assertion b) will follow by means of Fubini's theorem, but the details concerning integrability are a bit tricky. To this aim, take $l \in \mathbb{N}$ with $l \geq \frac{m}{2} - \frac{m}{q}$ so that $H^l \hookrightarrow L^q$ by Sobolev's embedding Theorem 3.31 in [**33**]. Lemma 4.8 yields functions $g_n \in C_c(\mathbb{R}, H^l \cap L^{q'})$ that converge to f in $E^\star$ as $n \to \infty$. The map $\psi_n : \mathbb{R}^2 \to L^q$; $(t, s) \mapsto T(t-s)g_n(s)$, is continuous for each $n \in \mathbb{N}$, since it is continuous in H^l by Lemma 4.5. There is a subsequence such that the functions $g_{n_j}(s)$ converge in $L^{q'}$ to $f(s)$ as $j \to \infty$

for a.e. $s \in \mathbb{R}$. Corollary 4.6 says that $T(t-s)$ maps $L^{q'}$ continuously into L^q for $t \neq s$. Therefore $(t,s) \mapsto T(t-s)f(s)$ is strongly measurable in L^q outside a set of measure. Hence, the integral I_1 is defined. Similarly, one sees that $T(\cdot)\varphi : \mathbb{R} \to L^q$ is strongly measurable if $\varphi \in L^2 \cap L^{q'}$.

It now follows from Fubini's theorem and (4.17) that the integral $(T *_+ f)(t)$ exists in L^q for a.e. $t \in \mathbb{R}$ and that $T *_+ f : \mathbb{R} \to L^q$ is strongly measurable. Since $\|T *_+ f\|_E \le I_1$, assertion b) is shown in our case.

In the same way one derives $\|T * f\|_E \le C_0 \|f\|_{E^\star}$ for the usual convolution.

2) Keeping the assumptions of 1), we show part a) by a duality argument. Let $g \in C_c(\mathbb{R}, L^2 \cap L^{q'})$. Using step 1), we compute

$$\begin{aligned} I_2 &:= \int_{\mathbb{R}} \int \Big(\int_{\mathbb{R}} T(t-s) g(s)\, \mathrm{d}s \Big) \overline{g}(t)\, \mathrm{d}x\, \mathrm{d}t = \langle T * g, \overline{g} \rangle_E, \\ |I_2| &\le \|T * g\|_E \|g\|_{E^\star} \le C_0 \|g\|_{E^\star}^2. \end{aligned}$$

The continuity of the scalar product and the unitarity of $T(\cdot)$ on L^2 yield

$$\begin{aligned} I_2 &= \int_{\mathbb{R}} \int_{\mathbb{R}} (T(t-s)g(s)|g(t))_{L^2}\, \mathrm{d}s\, \mathrm{d}t = \int_{\mathbb{R}} \int_{\mathbb{R}} (T(-s)g(s)|T(-t)g(t))_{L^2}\, \mathrm{d}s\, \mathrm{d}t \\ &= \Big(\int_{\mathbb{R}} T(-s)g(s)\, \mathrm{d}s \,\Big|\, \int_{\mathbb{R}} T(-t)g(t)\, \mathrm{d}t \Big)_{L^2} = \Big\| \int_{\mathbb{R}} T(-t) g(t)\, \mathrm{d}t \Big\|_2^2, \end{aligned}$$

where all integrals are $\mathbb{C}$- or L^2-valued Riemann integrals. We have thus shown

$$\Big\| \int_{\mathbb{R}} T(-t) g(t)\, \mathrm{d}t \Big\|_2 \le \sqrt{C_0} \|g\|_{E^\star}. \tag{4.18}$$

(If $(p,q) = (2,\infty)$, this fact can directly be proven with $C_0 = 1$.) Let $\varphi \in L^2 \cap L^{q'}$. Observe that the scalar function $t \mapsto \langle T(t)\varphi, \overline{g}(t)\rangle_{L^q} = (T(t)\varphi|g(t))_{L^2}$ is continuous. Estimate (4.18) leads to

$$\begin{aligned} \Big| \int_{\mathbb{R}} \langle T(t)\varphi, \overline{g}(t) \rangle_{L^q}\, \mathrm{d}t \Big| &= \Big| \int_{\mathbb{R}} (T(t)\varphi|g(t))_{L^2}\, \mathrm{d}t \Big| = \Big| \int_{\mathbb{R}} (\varphi|T(-t)g(t))_{L^2}\, \mathrm{d}t \Big| \\ &= \Big| \Big(\varphi \,\Big|\, \int_{\mathbb{R}} T(-t) g(t)\, \mathrm{d}t \Big)_{L^2} \Big| \le \sqrt{C_0}\, \|\varphi\|_2 \|g\|_{E^\star}. \end{aligned}$$

Since $C_c(\mathbb{R}, L^2 \cap L^{q'})$ is dense in $E^\star = L^{p'}(\mathbb{R}, L^{q'})$ by Lemma 4.8, we infer

$$\|T(\cdot)\varphi\|_E = \sup_{\|g\|_{E^\star} \le 1} |\langle T(\cdot)\varphi, \overline{g}\rangle_E| \le \sqrt{C_0}\, \|\varphi\|_2.$$

In particular, $T(\cdot)\varphi$ belongs to $(E^\star)^\star = E = L^p(\mathbb{R}, L^q)$, see (4.15). The assertions are shown for $(p,q) = (\overline{p}, \overline{q})$ and $2 < q < 1 + \alpha_c$.

3) Let $(p,q) = (\infty, 2)$ and $(\overline{p}, \overline{q})$ be admissible with $\overline{p} > 2$. Then part a) is true with C_b instead of L^∞ since $T(\cdot)$ is a unitary C_0-group on L^2. To prove b), we set $f_t = \mathbb{1}_{[0,t]} f$ for $t \ge 0$ and $f_t = \mathbb{1}_{[t,0]} f$ for $t < 0$. We write (p,q) instead of $(\overline{p}, \overline{q})$. First, let $f \in C_c(\mathbb{R}, L^2 \cap L^{p'})$. Using also (4.18), we obtain $T *_+ f \in C_b(\mathbb{R}, L^2)$ and

$$\begin{aligned} \|T *_+ f\|_{C_b(\mathbb{R}, L^2)} &= \sup_{t \in \mathbb{R}} \Big\| \int_0^t T(t) T(-s) f_t(s)\, \mathrm{d}s \Big\|_2 = \sup_{t \in \mathbb{R}} \Big\| \int_{\mathbb{R}} T(-s) f_t(s)\, \mathrm{d}s \Big\|_2 \\ &\le \sup_{t \in \mathbb{R}} \sqrt{C_0}\, \|f_t\|_{E^\star} = \sqrt{C_0}\, \|f\|_{E^\star}. \end{aligned}$$

We approximate the given inhomogeneity f in $E^\star$ by $f_n \in C_c(\mathbb{R}, L^2 \cap L^{q'})$, The above estimate then shows that $(T *_+ f_n)_n$ converges to a function u in $C_b(\mathbb{R}, L^2)$. On the other hand, by step 1) for a subsequence the functions $(T *_+ f_{n_j})(t)$ tend to $(T *_+ f)(t)$ in L^q for a.e. $t \in \mathbb{R}$. Hence, $T *_+ f = u$ belongs to $C_b(\mathbb{R}, L^2)$ and assertion b) is true in the present case. □

Before we solve (4.1) in the next section, we briefly explain why one calls the case $\alpha = \alpha_c$ *(energy)-critical*, where $m \geq 3$. Let u be an H^1-solution of (4.1) and $\lambda > 0$. As in the exercises, the rescaled function $u_\lambda(t,x) := \lambda^{2/(\alpha-1)} u(\lambda^2 t, \lambda x)$ also solves (4.1) with initial value $\lambda^{2/(\alpha-1)} u_0(\lambda\,\cdot)$. Observe that

$$\||\nabla u_\lambda(t)|\|_2 = \lambda^{\frac{2}{\alpha-1}} \lambda^{1-\frac{m}{2}} \||\nabla u(\lambda^2 t)|\|_2 \quad \text{and} \quad \alpha \gtreqless \alpha_c \iff \frac{2}{\alpha-1} + 1 - \frac{m}{2} \lesseqgtr 0.$$

In the *energy-supercritical* case $\alpha > \alpha_c$ the possibly 'bad' behavior of a solution u at a time $t_0 \approx 1$ with $\||\nabla u(0)|\|_2 \approx 1$ is transfered to u_λ at small times $t = \lambda^{-2} t_0$ for large λ, and one even has small $\nabla u_\lambda(0)$. This makes it hard to prove wellposedness. In the energy-critical case $\alpha = \alpha_c$ we obtain $\||\nabla u_\lambda(t)|\|_2 = \||\nabla u(\lambda^2 t)|\|_2$ but $\|u_\lambda(t)\|_2 = \lambda^{-1} \|u(\lambda^2 t)\|_2$, so that the above effect is weaker, and there is hope for some wellposedness. In the *energy-subcritical* case $\alpha < \alpha_c$, the behavior is reversed: Good properties for small $\nabla u(0)$ should lead to good properties for large data at small times.

One can discuss in a similar way mass-criticality dropping the derivatives above. This leads to the alternative

$$\alpha \gtreqless \alpha_c^0 \iff \frac{2}{\alpha-1} - \frac{m}{2} \lesseqgtr 0 \qquad \text{with} \quad \alpha_c^0 := 1 + \frac{4}{m}.$$

(See pp.118–120 of [**35**] for more details.)

4.3. Local wellposedness

In this section we establish the local wellposedness theory of the semilinear problem (4.1). The strategy of the proofs goes back to T. Kato. It is similar to the approach in Section 3.1 in the parabolic case. However, the smoothing effect of analytic semigroups is now replaced by Strichartz estimates, and many of the arguments are more sophisticated. In the critical case we restrict ourselves to the case $m = 3$ (where $\alpha = 5$) in Theorem 4.17. Here one obtains a much less convenient blow-up condition, and one needs the endpoint case of Strichartz estimates. We start with some preparations.

We again reformulate (4.1) as fixed-point problem for the operator

$$\Phi(u)(t) = [\Phi_{u_0}(u)](t) := T(t)u_0 + \int_0^t T(t-s)F(u(s))\,\mathrm{d}s, \qquad t \in J, \tag{4.19}$$

for a given initial value $u_0 \in H^1$ and $F(v) = -\mathrm{i}\mu|v|^{\alpha-1}v$. Still assuming (4.2), we recall from (4.8) and (4.16) our assumptions and definitions

$$q = \alpha{+}1 \in (2, \alpha_c{+}1), \quad \alpha_c = \frac{m+2}{(m-2)_+}, \quad q' = \frac{\alpha+1}{\alpha} = \frac{q}{\alpha}, \quad \frac{2}{p} = \frac{m}{2} - \frac{m}{q}, \tag{4.20}$$

in the subcritical case. Moreover, F belongs to $C^1_{\mathbb{R}}(H^1, H^{-1}) \cap C^1_{\mathbb{R}}(L^q, L^{q'})$ by (4.9). The Strichartz estimates from Theorem 4.10 will compensate the loss of integrability caused by F in (4.19). For $k \in \mathbb{N}_0$ we introduce the spaces

$$E_k(J) = L^p(J, W^{k,q}), \quad E_k'(J) = L^{p'}(J, W^{k,q'}), \quad G_k(J) = L^\infty(J, H^k),$$
$$\mathcal{F}_k(J) = E_k(J) \cap G_k(J) \quad \text{endowed with} \quad |||v|||_{k,J} = \max\big\{\|v\|_{E_k(J)},\ \|v\|_{G_k(J)}\big\}.$$

(For $J = [-b, b]$ we replace J by b.) We first collect the mapping properties of F needed below.

Lemma 4.11. *Let (4.20) be true (with $\alpha \le \alpha_c$ if $m \ge 3$) and J be an interval of length $a > 0$. Take $u, v \in G_1(J) \hookrightarrow E_0(J)$, $w \in \mathcal{F}_1(J)$, $\varphi, \psi \in L^q$, and $\chi \in W^{1,q}$. Set $r = \operatorname{ess\,sup}\big\{\|u(t)\|_{1,2}, \|v(t)\|_{1,2}, \|w(t)\|_{1,2} \,\big|\, t \in J\big\}$. Then we have $F(\varphi) \in L^{q'}$, $F(\chi) \in W^{1,q'}$, $F(u) \in E_0'(J)$, $F(w) \in E_1'(J)$, and the inequalities*

a) $\|F(\varphi) - F(\psi)\|_{q'} \le C_F\big(\|\varphi\|_q^{\alpha-1} + \|\psi\|_q^{\alpha-1}\big)\|\varphi - \psi\|_q$,

b) $\|F(u) - F(v)\|_{E_0'(J)} \le C_F r^{\alpha-1} a^{\frac{1}{p'}-\frac{1}{p}}\|u - v\|_{E_0(J)}$,

c) $\||\nabla F(\chi)|\|_{q'} \le C_F\|\chi\|_q^{\alpha-1}\||\nabla\chi|\|_q$,

d) $\||\nabla F(w)|\|_{E_0'(J)} \le C_F r^{\alpha-1} a^{\frac{1}{p'}-\frac{1}{p}}\||\nabla w|\|_{E_0(J)}$,

e) $\|F(w)\|_{E_1'(J)} \le C_F r^{\alpha-1} a^{\frac{1}{p'}-\frac{1}{p}}\|w\|_{E_1(J)}$

for a constant C_F only depending on α and m.

Proof. From (4.9) and (4.20) we deduce

$$\begin{aligned}
\|F(\varphi) - F(\psi)\|_{q'} &= \Big\|\int_0^1 F'(\psi + \tau(\varphi - \psi))(\varphi - \psi)\,\mathrm{d}\tau\Big\|_{q'} \\
&\le \int_0^1 \|F'(\psi + \tau(\varphi - \psi))\|_{\mathcal{B}(L^q, L^{q'})}\|\varphi - \psi\|_q\,\mathrm{d}\tau \\
&\le \alpha \sup_{\tau\in[0,1]} \|(1-\tau)\psi + \tau\varphi\|_q^{\alpha-1}\|\varphi - \psi\|_q \\
&\le c_\alpha\big(\|\varphi\|_q^{\alpha-1} + \|\psi\|_q^{\alpha-1}\big)\|\varphi - \psi\|_q.
\end{aligned}$$

In this estimate we insert $u(t)$ and $v(t)$, and take the p-norm in time. Using Sobolev's embedding (4.4) and $\|u(t)\|_{1,2}, \|v(t)\|_{1,2} \le r$, we arrive at

$$\begin{aligned}
\|F(u) - F(v)\|_{L^p(J, L^{q'})} &\le c_\alpha\Big(\int_J \big(\|u(t)\|_q^{\alpha-1} + \|v(t)\|_q^{\alpha-1}\big)^p\|u(t) - v(t)\|_q^p\,\mathrm{d}t\Big)^{\frac{1}{p}} \\
&\le 2c_\alpha C_{\mathrm{So}}^{\alpha-1} r^{\alpha-1}\|u - v\|_{E_0(J)}.
\end{aligned}$$

With $\frac{1}{p'} = \frac{1}{p} + \frac{1}{p'} - \frac{1}{p}$ and $p \ge 2 \ge p'$, Hölder's inequality then yields

$$\|F(u) - F(v)\|_{E'(J)} \le a^{\frac{1}{p'}-\frac{1}{p}}\|F(u) - F(v)\|_{L^p(J, L^{q'})} \le c r^{\alpha-1} a^{\frac{1}{p'}-\frac{1}{p}}\|u - v\|_{E_0(J)}.$$

For c), we let $\phi(z) = -\mathrm{i}\mu|z|^{\alpha-1}z$ and $j \in \{1, \dots, m\}$. There are functions $\chi_n \in C_c^\infty$ converging to χ in $W^{1,q}$ as $n \to \infty$ by Theorem 3.27 in [**33**]. Hölder's inequality with exponents $\frac{1}{q'} = \frac{\alpha-1}{q} + \frac{1}{q}$ yields

$$\|\partial_j F(\chi_n)\|_{q'} = \|\phi'(\chi_n)\partial_j\chi_n\|_{q'} \le \alpha\||\chi_n|^{\alpha-1}\partial_j\chi_n\|_{q'} \le \alpha\|\chi_n\|_q^{\alpha-1}\|\partial_j\chi_n\|_q.$$

Lemma 1.17 and Hölder's inequality show that the maps $\partial_j F(\chi_n) = \phi'(\chi_n)\partial_j \chi_n$ tend to $\phi'(\chi)\partial_j\chi$ and $F(\chi_n)$ to $F(\chi)$ in $L^{q'}$ as $n \to \infty$. From Lemma 3.16 in [**33**] we then deduce that $F(\chi) \in W^{1,q'}$, and so inequality c) is true.

The estimates d) and e) follow as above (using also b) with $v = 0$ for e)). $\square$

Observe that the above estimates are uniform on a ball in H^1, which is often used below. Lemma 4.11 and the Strichartz estimates from Theorem 4.10 show that the operator Φ from (4.19) maps $\mathcal{F}_1(b)$ into itself and that it is Lipschitz on bounded sets of $\mathcal{F}_1(b)$, but only with respect to the metric of $\mathcal{F}_0(b)$. Fortunately, these properties still allow one to apply Banach's fixed point theorem, as seen in the next lemma. It relies on the Banach–Alaoglu theorem.

Lemma 4.12. *Let (4.20) be true, $r > 0$, and $J \subseteq \mathbb{R}$ be an interval. Then the ball $\Sigma(J, r) = \{v \in \mathcal{F}_1(J) \mid |||v|||_{1,J} \le r\}$ is a complete metric space when endowed with the metric induced by $|||\cdot|||_{0,J}$.*

Proof. Let $(u_n)_n$ be a Cauchy sequence in $\Sigma(J, r)$ for $|||\cdot|||_{0,J}$. Since $\mathcal{F}_0(J)$ is a Banach space, $(u_n)_n$ converges in $\mathcal{F}_0(J)$ to a function $u \in \mathcal{F}_0(J)$ as $n \to \infty$. We have to show that $u \in \Sigma(J, r)$. As indicated after (4.15), the space $G_1(J)$ is the dual of $L^1(J, H^{-1})$ and $E_1(J)$ is reflexive with dual $L^{p'}(J, W^{-1,q'})$. Moreover, $L^1(J, H^{-1})$ is separable. The Banach–Alaoglu theorem thus provides a subsequence $(u_{n_j})_j$ which converges weakly in $E_1(J)$ to some v with $\|v\|_{E_1(J)} \le r$ and weakly* in $G_1(J)$ to some w with $\|w\|_{G_1(J)} \le r$ as $j \to \infty$. Since $E_0(J)^\star \hookrightarrow E_1(J)^\star$ and $L^1(J, L^2) \hookrightarrow L^1(J, H^{-1})$, the functions u_{n_j} also tend weakly in $E_0(J)$ to v and weak* in $G_0(J)$ to w. On the other hand, $(u_{n_j})_j$ has the limit u in both $E_0(J)$ and $G_0(J)$ so that $u = v$ and $u = w$ by the uniqueness of weak and weak* limits; i.e., u is an element $\Sigma(J, r)$. $\square$

As Remark 1.7 b) one can concatenate H^k-solutions.

Remark 4.13. Let (4.20) be true (with $\alpha \le \alpha_c$ if $m \ge 3$), $k \in \{1, 2\}$, and u and v be H^k-solutions of (4.1) on $[a, b]$ and $[b, c]$, respectively. Assume that $u(b) = v(b)$. Then the function w given by $w(t) = u(t)$ for $t \in [a, b]$ and $w(t) = v(t)$ for $t \in (b, c]$ is an H^k-solution of (4.1) with $w(a) = u(a)$. $\lozenge$

As a first part of local wellposedness, we show uniqueness of H^1-solutions which easily follows from Strichartz estimates and Lemma 4.11 if $\alpha < \alpha_c$.

Lemma 4.14. *Let (4.20) be true and $u_0 \in H^1$. Let u and v be H^1-solutions of (4.1) on intervals J_u and J_v containing 0, respectively. Then $u = v$ on $J_u \cap J_v$.*

Proof. We can assume that $J_u \cap J_v$ has positive length. If the assumption was not true, there would exist $\tilde\tau \le 0 \le \tau$ in $J_u \cap J_v$ such that $u = v$ on $[\tilde\tau, \tau]$ and $u(t_n) \ne v(t_n)$ for certain $t_n \in J_u \cap J_v$ with, say, $t_n \to \tau^+$ as $n \to \infty$. (The case that $t_n \to \tilde\tau^-$ is treated similarly.) Take $\delta_0 > 0$ with $J_0 := [\tau, \tau + \delta_0] \subseteq J_u \cap J_v$. Let $r = \max\big\{\|u(t)\|_{1,2}, \|v(t)\|_{1,2} \,\big|\, t \in J_0\big\}$. Because u and v are H^1-solutions of (4.1), the maps $F(u)$ and $F(v)$ belong to $C(J_0, H^{-1})$ by (4.9). Proposition 2.6 of [**32**] and Lemma 4.5 thus imply the mild formulas

$$u(t + \tau) = T(t)u(\tau) + \int_0^t T(t - s)F(u(s + \tau))\,\mathrm{d}s,$$

$$v(t+\tau) = T(t)u(\tau) + \int_0^t T(t-s)F(v(s+\tau))\,\mathrm{d}s$$

for all $t \in [0, \delta_0]$. Take any interval $J = [\tau, \tau+\delta] \subseteq J_0$. After a time shift, the Strichartz inequality from Theorem 4.10 and Lemma 4.11 then yield

$$\|u-v\|_{E_0(J)} \le C_{\mathrm{St}} \|F(u) - F(v)\|_{E_0'(J)} \le C_{\mathrm{St}} C_F r^{\alpha-1} \delta^{\frac{1}{p'}-\frac{1}{p}} \|u-v\|_{E_0(J)}.$$

Since $p' < p$, for small $\delta \in (0, \delta_0]$ the right-hand side is less than $\frac{1}{2}\|u-v\|_{E_0(J)}$. As $u, v \in E_0(J)$ by (4.4), we infer $u = v$ on $[\tau, \tau+\delta]$ contradicting $t_n \to \tau^+$. $\square$

We can now establish the basic existence result for (4.1) employing the same tools as in the previous lemma and the fixed point space of Lemma 4.12.

Lemma 4.15. *Let (4.20) be true and $\rho > 0$. Then there is a number $b_0(\rho) > 0$ (see (4.22) below) such that for each initial value $u_0 \in \overline{B}_{H^1}(0,\rho)$ there is a unique H^1-solution u in $\mathcal{F}_1(b_0(\rho))$ of (4.1) on the time interval $[-b_0(\rho), b_0(\rho)] =: J_0$. Moreover, $|||u|||_{1,J_0} \le r := 1 + C_{\mathrm{St}}\rho$, where $C_{\mathrm{St}} \ge 1$ is taken from Theorem 4.10. We further have $u = \Phi_{u_0}(u)$ on J_0, cf. (4.19).*

Proof. Let $\rho > 0$ and $u_0 \in H^1$ with $\|u_0\|_{1,2} \le \rho$. Take $b > 0$ to be specified below. Fix $r = 1 + C_{\mathrm{St}}\rho$. Lemma 4.12 provides the complete metric space $\Sigma(b, r)$ with the metric $|||u-v|||_{0,b}$. Let $\Phi(u) = \Phi_{u_0}(u)$ be defined by (4.19) for $u \in \Sigma(b,r)$. Combining the Strichartz inequalities from Theorem 4.10 and the mapping properties of F in Lemma 4.11, we estimate

$$|||\Phi(u)|||_{1,b} \le C_{\mathrm{St}}(\|u_0\|_{1,2} + \|F(u)\|_{E_1'(b)}) \le C_{\mathrm{St}}\rho + C_{\mathrm{St}} C_F r^{\alpha} (2b)^{\frac{1}{p'}-\frac{1}{p}}, \tag{4.21}$$

$$|||\Phi(u) - \Phi(v)|||_{0,b} \le C_{\mathrm{St}} \|F(u) - F(v)\|_{E_0'(b)} \le C_{\mathrm{St}} C_F r^{\alpha-1} (2b)^{\frac{1}{p'}-\frac{1}{p}} \|u-v\|_{E_0(b)}$$

for $u, v \in \Sigma(b,r)$, using that $p > 2 > p'$ by (4.20) and $\alpha < \alpha_c$. We now define

$$b_0 = b_0(\rho) = \tfrac{1}{2} \min\Big\{ \big(C_{\mathrm{St}} C_F r^{\alpha}\big)^{\frac{p'p}{p'-p}}, \big(2C_{\mathrm{St}} C_F r^{\alpha-1}\big)^{\frac{p'p}{p'-p}} \Big\} > 0. \tag{4.22}$$

Let $b_0(\rho)$ and $J_0 = [-b_0(\rho), b_0(\rho)]$. It follows that $\Phi(u) \in \Sigma(b_0, r)$ and $|||\Phi(u) - \Phi(v)|||_{0,J_0} \le \frac{1}{2}|||u-v|||_{0,J_0}$. The contraction mapping principle then yields a unique fixed point $u = \Phi(u)$ in $\Sigma(b_0, r)$.

Theorem 4.10 further shows that u belongs to $C(J_0, H^1)$, and hence $f := F(u)$ to $C(J_0, H^{-1})$ by (4.9). Since $u \in C(J_0, H^1)$ is a mild solution of $u' = \mathrm{i}\Delta u + f$ in H^{-1} with $u(0) = u_0$, Lemma 2.8 of [**32**] and Lemma 4.5 imply that u is an H^1-solution of (4.1) on J_0. Uniqueness follows from Lemma 4.14. $\square$

The above proof acually yields solutions in $\Sigma(b,r)$ for each $b \in (0, b_0(\rho)]$. For the local wellposedness theorem, we define the *maximal existence times*

$$t^+(u_0) = \sup\big\{ b > 0 \,\big|\, \exists\, H^1\text{-solution } u_b \in C([0,b], H^1) \text{ of (4.1)} \big\},$$
$$t^-(u_0) = \inf\big\{ b < 0 \,\big|\, \exists\, H^1\text{-solution } u_b \in C([b,0], H^1) \text{ of (4.1)} \big\}.$$

Lemma 4.15 implies that $-t^-(u_0)$ and $t^+(u_0)$ belong to $[b_0(\|u_0\|_{1,2}), \infty]$. Using also Remark 4.13 we can restart the system with initial value $u(t_\pm(u_0))$ and thus obtain $-t^-(u_0), t^+(u_0) > b_0(\|u_0\|_{1,2})$. As in Remark 1.10, Lemma 4.14 allows us to define H^1-solutions of (4.1) on $(t^-(u_0), 0]$ and $[0, t^+(u_0))$, and thus on $J(u_0) := (t^-(u_0), t^+(u_0))$ by Remark 4.13. The H^1-solution u of (4.1) on

$J(u_0)$ is called *maximal*. (In view of Theorem 4.16 c), we have to take an open time interval here.) It is unique by Lemma 4.14.

Our local wellposedness theorem follows the pattern of Theorem 1.11 though the underlying function spaces are adapted to the Strichartz estimates. Moreover, we only show the continuity of the solution map $u_0 \mapsto \varphi(\cdot, u_0)$ and not its Lipschitz continuity on balls as in Theorems 1.11 and 3.4. One obtains such a Lipschitz property if $\alpha \in (2, \alpha_c)$, see the exercises, or in weaker norms, cf. (4.24). We stress that we obtain a blow-up condition in H^1 which is the space of the initial values (as in Theorems 1.11 and 3.4).

THEOREM 4.16. *Let* (4.20) *be true,* $\rho > 0$, $u_0 \in H^1$ *with* $\|u_0\|_{1,2} \le \rho$, *and* $b_0(\rho)$ *be defined by* (4.22). *Then the following assertions hold.*

a) There are numbers $0 < b_0(\rho) < \pm t^{\pm}(u_0) \le \infty$ *and a unique maximal* H^1*-solution* $u = \varphi(\cdot, u_0)$ *of* (4.1) *on* $J(u_0) = (t^-(u_0), t^+(u_0))$.

b) Let $[a, b] \subseteq J(u_0)$. *Then* u *belongs to* $L^p([a, b], W^{1,q})$.

c) Let $\pm t^{\pm}(u_0) < \infty$. *We then have* $\lim_{t \to t^{\pm}(u_0)} \|u(t)\|_{1,2} = \infty$.

d) Let $J \subseteq J(u_0)$ *be a compact interval with* $0 \in J$. *Then there is a radius* $\delta = \delta(J, u_0) > 0$ *such that for* $v_0 \in \overline{B}_{H^1}(u_0, \delta)$ *we have* $J \subseteq J(v_0)$ *and the map*

$$\overline{B}_{H^1}(u_0, \delta) \to C(J, H^1) \cap L^p(J, W^{1,q}), \quad v_0 \mapsto \varphi(\cdot, v_0),$$

is continuous.

PROOF. 1) Part a) was proved before the theorem. Let $\tau \in J(u_0)$. Lemma 4.15 yields a time $\beta(\tau) > 0$ and an H^1-solution v of (4.1) with $v(0) = u(\tau)$ belonging to $\mathcal{F}_1(\beta(\tau))$. By the uniqueness Lemma 4.14, v is a restriction of $u(\tau + \cdot)$ and thus u belongs to $L^p([\tau - \beta(\tau), \tau + \beta(\tau)], W^{1,q})$. A compactness argument then yields assertion b).

Suppose that $t^+(u_0) < \infty$ and there were times $t_n \to t^+(u_0)^-$ with $\sup_n \|u(t_n)\|_{1,2} =: C < \infty$. Take an index with $t_N + b_0(C) > t^+(u_0)$. Using Lemma 4.15 and Remark 4.13, we can extend the given H^1-solution to $[0, t_N + b_0(C)]$ by considering (4.1) with initial value $u(t_N)$. This fact contradicts the definition of $t^+(u_0)$. One treats $t^-(u_0)$ in the same way. Hence, claim c) holds.

2) Fix $J = [T_0, T_1] \subseteq J(u_0)$ with $0 \in J$. We show that every sequence $(\psi_n)_n$ with limit u_0 in H^1 as $n \to \infty$ has a subsequence $(\psi_{n_j})_j$ such that $J \subseteq J(\psi_{n_j})$ for all j and the solutions $u_{n_j} = \varphi(\cdot, \psi_{n_j})$ tend to u in $\mathcal{F}_1(J)$ as $j \to \infty$. By a straightforward contradiction argument, this fact implies the first part of assertion d) and the continuity of $v_0 \mapsto \varphi(\cdot, v_0)$ at u_0.

So let $(\psi_n)_n$ converge to u_0 in H^1 and set $u_n = u(\cdot, \psi_n)$. The uniform bound $\overline{\rho} := 1 + \max_{t \in J} \|u(t)\|_{1,2} < \infty$ will be crucial for our reasoning. There is an index $n_0 \in \mathbb{N}$ such that $\|\psi_n\|_{1,2} \le \overline{\rho}$ for all $n \ge n_0$. Lemma 4.15 then implies that $-t^-(\psi_n), t^+(\psi_n) > b_0(\overline{\rho}) =: b_0$ for all $n \ge n_0$. Let $n \ge n_0$. By Lemma 4.14, the restrictions of u and u_n to $J_0 := [-b_0, b_0]$ coincide with the solutions from Lemma 4.15 for the initial value u_0 and ψ_n, respectively. We thus have the equations $u = \Phi_{u_0}(u)$ and $u_n = \Phi_{\psi_n}(u_n)$ on J_0, and the core bound

$$|||u|||_{1,b_0}, \ |||u_n|||_{1,b_0} \le \overline{r} := 1 + C_{\mathrm{St}}\overline{\rho}. \tag{4.23}$$

Due to (4.21) and the choice of b_0 in (4.22), the operator Φ_{u_0} is Lipschitz on $\Sigma(b_0,\overline{r}) = \overline{B}_{\mathcal{F}_1(b_0)}(0,\overline{r})$ for the metric induced by $|||\cdot|||_{0,b_0}$ with a constant bounded by $\frac{1}{2}$. Combining these facts with the Strichartz estimates of Theorem 4.10, we derive the basic 0-order Lipschitz inequality

$$\begin{aligned}
|||u-u_n|||_{0,b_0} &\le |||\Phi_{u_0}(u)-\Phi_{u_0}(u_n)\,|||_{0,b_0} + |||\Phi_{u_0}(u_n)-\Phi_{\psi_n}(u_n)|||_{0,b_0}\\
&\le \tfrac{1}{2}|||u-u_n\,|||_{0,b_0} + |||T(\cdot)(u_0-\psi_n)|||_{0,b_0}\\
&\le \tfrac{1}{2}|||u-u_n\,|||_{0,b_0} + C_{\mathrm{St}}\|u_0-\psi_n\|_2,\\
|||u-u_n|||_{0,b_0} &\le 2C_{\mathrm{St}}\|u_0-\psi_n\|_2 \qquad (4.24)
\end{aligned}$$

for all $n \ge n_0$. Unfortunately, this argument does not give the desired continuity of $v_0 \mapsto u(\,\cdot\,,v_0)$ from H^1 to $\mathcal{F}_1(b_0)$, so that we need a more sophisticated analysis than in Theorems 1.11 and 3.4.

3) The inequality (4.24) shows that, after passing to a subsequence $(u_k)_k$, the functions $u_k(t)$ tend to $u(t)$ in L^q as $k \to \infty$, for a.e. $t \in J$. Recall that

$$u_k - u = T(\cdot)(\psi_k-u_0) + T *_+ (F(u_k)-F(u)) \qquad \text{on } J_0.$$

Let $b \in (0,b_0]$ and $k \ge n_0$. Theorem 4.10 and Lemma 4.11 yield

$$|||u_k-u|||_{1,b} \le C_{\mathrm{St}}\big(\|\psi_k-u_0\|_{1,2} + \|F(u_k)-F(u)\|_{E_1'(b)}\big). \qquad (4.25)$$

Let $\phi(z) = -\mathrm{i}\mu|z|^{\alpha-1}z$ for $z \in \mathbb{R}^2$ and $j \in \{1,\dots,m\}$. To bound the F-term, as in the proof of Lemma 4.11 we compute

$$\begin{aligned}
&\|\partial_j F(u_k(t)) - \partial_j F(u(t))\|_{q'}\\
&\quad\le \|\phi'(u_k(t))[\partial_j u_k(t)-\partial_j u(t)]\|_{q'} + \|[\phi'(u_k(t))-\phi'(u(t))]\partial_j u(t)\|_{q'}\\
&\quad\le \alpha\|u_k(t)\|_q^{\alpha-1}\|\partial_j u_k(t)-\partial_j u(t)\|_q + \|[\phi'(u_k(t))-\phi'(u(t))]\partial_j u(t)\|_{q'}\\
&\quad\le \alpha C_{\mathrm{So}}^{\alpha-1}\overline{r}^{\alpha-1}\|u_k(t)-u(t)\|_{1,q} + \|[\phi'(u_k(t))-\phi'(u(t))]\partial_j u(t)\|_{q'},
\end{aligned}$$

for $t \in J_0$ by Hölder, Sobolev's embedding (4.4) and estimate (4.23). Taking the norm in $L^{p'}([-b,b])$, Lemma 4.11 b) and Hölder's inequality then lead to

$$\|F(u_k)-F(u)\|_{E_1'(b)} \le c_1\overline{r}^{\alpha-1}b^{\frac{1}{p'}-\frac{1}{p}}\|u_k-u\|_{E_1(b)} + c_2\|(\phi'(u_k)-\phi'(u))|\nabla u|\|_{E_0'(b)}.$$

Here and below, $c_j > 0$ are constants only depending on m and α. We fix

$$\overline{b} = \overline{b}(\overline{\rho}) := \min\big\{b_0, \big(2c_1C_{\mathrm{St}}\overline{r}^{\alpha-1}\big)^{\frac{p'p}{p'-p}}\big\} \qquad (4.26)$$

and insert the above inequality into (4.25), arriving at

$$\begin{aligned}
|||u_k-u|||_{1,\overline{b}} &\le C_{\mathrm{St}}\|\psi_k-u_0\|_{1,2} + \tfrac{1}{2}\|u_k-u\|_{E_1(\overline{b})}\\
&\qquad + c_2C_{\mathrm{St}}\|(\phi'(u_k)-\phi'(u))|\nabla u|\|_{E_0'(\overline{b})},\\
|||u_k-u|||_{1,\overline{b}} &\le 2C_{\mathrm{St}}\|\psi_k-u_0\|_{1,2} + 2c_2C_{\mathrm{St}}\|(\phi'(u_k)-\phi'(u))|\nabla u|\|_{E_0'(\overline{b})}. \qquad (4.27)
\end{aligned}$$

It remains to control

$$\|(\phi'(u_k)-\phi'(u))|\nabla u|\|_{E_0'(\overline{b})}^{p'}v = \int_{-\overline{b}}^{\overline{b}} \|(\phi'(u_k(t))-\phi'(u(t)))|\nabla u(t)|\|_{q'}^{p'}\,\mathrm{d}t.$$

Lemma 1.17 and Hölder with $\frac{1}{q'} = \frac{\alpha-1}{q} + \frac{1}{q}$ imply that the functions $\phi'(u_k(t))|\nabla u(t)|$ tend to $\phi'(u(t))|\nabla u(t)|$ in $L^{q'}$ as $k \to \infty$ for a.e. $t \in [-\overline{b},\overline{b}]$, since $u_k(t) \to u(t)$ in L^q. Combining Lemma 1.17 with (4.20), (4.4) and (4.23), we further estimate

$$\begin{aligned}\|(\phi'(u_k(t)) - \phi'(u(t)))|\nabla u(t)|\|_{q'} &\le c_3(\|u_k(t)\|_q^{\alpha-1} + \|u(t)\|_q^{\alpha-1})\||\nabla u(t)|\|_q \\ &\le 2c_3 C_{\mathrm{So}}^{\alpha-1}\overline{r}^{\alpha-1}\|u(t)\|_{1,q}.\end{aligned}$$

Since $p' \le p$, the function u belongs to $L^{p'}([-\overline{b}, \overline{b}], W^{1,q})$. Due to dominated convergence, the last term in (4.27) thus tends to 0 as $k \to \infty$. As a result, (u_k) converges to u in $\mathcal{F}_1(\overline{b})$, and we can fix an index k_1 such that $\|u_k(\pm\overline{b})\|_{1,2} \le \overline{\rho}$ for all $k \ge k_1$. (Here we use that $u, u_k \in C([-\overline{b}, \overline{b}], H^1)$.)

4) Let $T_0 < -\overline{b}$ or $T_1 > \overline{b}$. As $\|u_k(\pm\overline{b})\|_{1,2} \le \overline{\rho}$ for $k \ge k_1$, we can repeat the above argument with initial time $-\overline{b}$ or $\overline{b}$, passing to further subsequences. This can be done with the step size $\overline{b}$ from (4.26) which only depends on $\overline{\rho}$, m and α. In finitely many steps, we thus construct a subsequence $(\psi_{n_j})_j$ with $J \subseteq J(\psi_{n_j})$ for $j \in \mathbb{N}$ and $u_{n_j} \to u$ in $\mathcal{F}_1(J)$ as $j \to \infty$.

As noted in step 2, the above fact yields a radius $\delta > 0$ with $J \subseteq J(v_0)$ if $v_0 \in \overline{B}_{H^1}(u_0, \delta)$. Replacing u_0 by v_0 in steps 2)-4), we derive the continuity of the map $w_0 \mapsto \varphi(\,\cdot\,, w_0)$ from $\overline{B}_{H^1}(u_0, \delta)$ to $\mathcal{F}_1(J)$, and thus claim d) holds. □

The above arguments fail in the *critical case* $\alpha = \alpha_c$ and $m \ge 3$, where $p = p' = 2$ and the factor $b^{\frac{1}{p'} - \frac{1}{p}} = 1$ does not vanish as $b \to 0^+$. To treat this case, one uses the structure of the power nonlinearity in a more clever way. By Hölder and Sobelev inqualities one can bound a part of $F(v)$ in the $L^s((-b, b), L^s)$-norm for suitable $s > 1$. This space-time norm can be made small without restricting the size of v in $G_1(b)$ and hence of u_0 in H^1. This approach requires a more sophisticated fixed-point space (involving a smallness condition). For $m = 3$, it leads to a lower bound on the existence time depending on the behavior of $T(\cdot)u_0$ in $L^{10}(\mathbb{R}, L^{10})$ besides $\|u_0\|_{1,2}$, and to a blow-up condition in terms of the norm of u in $L^{10}([0, t^+), L^{10})$, which is much harder to control than the H^1-norm of $u(t)$. Moreover, 'unconditional uniqueness' of H^1-solutions is far more difficult to prove than for $\alpha < \alpha_c$.

We establish local wellposedness for $m = 3$ and $\alpha = 5$. Dimensions $m \in \{4, 5\}$ can be handled similarly. See [**36**] and Theorem 4.5.1 in [**6**] for the general case.

THEOREM 4.17. *Let $\mu \in \{-1, 1\}$, $m = 3$, $\alpha = 5 = \alpha_c$, and $u_0 \in H^1$. Then the following assertions hold.*

a) There is a unique maximal H^1-solution u of (4.1) on $(t^-(u_0), t^+(u_0)) = J(u_0)$. It belongs to $L^p_{\mathrm{loc}}(J(u_0), W^{1,q})$. The numbers $\pm t^\pm(u_0)$ are larger than $b(u_0, \varepsilon)$ given by (4.30) and (4.33).

b) If $t^+(u_0) < \infty$, then $\|u\|_{L^{10}([0,t^+(u_0))\times\mathbb{R}^3)} = \infty$; and analogously for $t^-(u_0)$.

c)[3] *Let $J \subseteq J(u_0)$ be a compact interval with $0 \in J$. Then there is $\delta(J, u_0) = \delta > 0$ such that for $v_0 \in \overline{B}_{H^1}(u_0,\delta)$ we have $J \subseteq J(v_0)$ and the Lipschitz continuity of*

$$\overline{B}_{H^1}(u_0, \delta) \to C(J, H^1) \cap L^p(J, W^{1,q}), \quad v_0 \mapsto \varphi(\,\cdot\,, v_0).$$

[3]The continuous dependence on data was not treated in the lectures.

PROOF. 1) We first discuss the basic spaces and estimates needed in the proof. As before we use the Strichartz pairs $(\infty, 2)$ and $(2, 6)$, where $6 = \alpha_c + 1 = q$. In addition we involve the pair $(10, r)$ with $r = \frac{30}{13} > 2$ noting that $\frac{2}{10} + \frac{39}{30} = \frac{3}{2}$. Since $1 - \frac{3}{r} = -\frac{3}{10}$ we have the Sobolev embedding

$$W^{1,r} \hookrightarrow L^{10}, \qquad \text{and hence} \quad L^{10}(J, W^{1,r}) \hookrightarrow L^{10}(J, L^{10}) \tag{4.28}$$

with a constant C'_{So} independent of the interval J. We can thus control the norm of $L^{10}(J, L^{10})$ by the (Strichartz) norm of $L^{10}(J, W^{1,r})$. This fact shall be used to bound the nonlinearity in the (dual Strichartz) norm of $L^2(J, W^{k,\frac{6}{5}}) = E'_k(J)$, where $\frac{6}{5} = 6'$. To this aim, let $v \in L^{10}(J, L^{10})$ and $w \in L^{10}(J, L^r)$. Using Hölder's inequality with $\frac{5}{6} = \frac{2}{5} + \frac{1}{r}$ and with $\frac{1}{2} = \frac{2}{5} + \frac{1}{10}$, we estimate

$$\begin{aligned} \||v|^4 w\|_{E'_0(J)} &\le \bigg(\int_J \Big[\Big(\int_{\mathbb{R}^3} |v(t)|^{\frac{4\cdot 5}{2}} \, \mathrm{d}x \Big)^{\frac{2\cdot 2}{5\cdot 2}} \Big(\int_{\mathbb{R}^3} |w(t)|^r \, \mathrm{d}x \Big)^{\frac{1}{r}} \Big]^2 \, \mathrm{d}t \bigg)^{\frac{1}{2}} \qquad (4.29) \\ &= \Big(\int_J \big(\|v(t)\|_{10}^4 \|w(t)\|_r \big)^2 \, \mathrm{d}t \Big)^{\frac{1}{2}} \\ &\le \Big[\int_J \|v(t)\|_{10}^{10} \, \mathrm{d}t \Big]^{\frac{2}{5}} \Big[\int_J \|w(t)\|_r^{10} \, \mathrm{d}t \Big]^{\frac{1}{10}} = \|v\|_{L^{10}_J L^{10}}^4 \|w\|_{L^{10}_J L^r}, \end{aligned}$$

writing $L^{10}_J L^r$ instead $L^{10}(J, L^r)$ and b instead of $J = [0, b]$ etc. We now define

$$\mathcal{V}_k(J) = C_b(J, H^k) \cap L^2(J, W^{k,6}) \cap L^{10}(J, W^{k,r})$$

for $k \in \{0, 1\}$, and put $\mathcal{V}_k(J) = \mathcal{V}_k(b)$ if $J = [0, b]$. They are Banach spaces with the norm $[\![v]\!]_{k,b}$ where we let $[\![v]\!]_{0,b}$ be the sum of the norms in $C_b(J, L^2)$, $L^2(J, L^6)$ and $L^{10}(J, L^r)$, and we set $[\![v]\!]_{1,b} = [\![v]\!]_{0,b} + [\![|\nabla v|_1]\!]_{0,b}$. (Accordingly, we use the equivalent norm $\|\varphi\|_{W^{1,r}} = \|\varphi\|_r + \||\nabla\varphi|_1\|_r$ on $W^{1,r}$.)

Let $u_0 \in H^1$ and $\rho \ge \|u_0\|_{1,2}$. Fix $R := 2C_{\mathrm{St}}\rho + 1$ with C_{St} from Theorem 4.10. For a time $b > 0$ and a radius $\varepsilon > 0$ we introduce the fixed-point space

$$\Sigma(b, R, \varepsilon) = \big\{ v \in \mathcal{V}_1(b) \,\big|\, [\![v]\!]_{1,b} \le R, \ \|v\|_{L^{10}_b L^{10})} \le \varepsilon \big\}.$$

and endow it with the complete metric $[\![v - w]\!]_{1,b}$. convergence a.e. of a subsequence.) For $v \in \mathcal{V}_1(b)$ we further define the fixed-point operator

$$\Phi(v)(t) = T(t)u_0 - \mathrm{i}\mu \int_0^t T(t-s)(|v(s)|^4 v(s)) \, \mathrm{d}s, \qquad t \in [0, b].$$

2) We show that Φ is a strict contraction on $\Sigma(b, R, \varepsilon)$ for small $b, \varepsilon > 0$. The Strichartz estimate in Theorem 4.10 a) and inequality (4.28) imply that $T(\cdot)u_0$ belongs to $L^{10}_b W^{1,r} \hookrightarrow L^{10}_b L^{10}$. We can thus fix a time $b = b(u_0, \varepsilon) > 0$ with

$$\|T(\cdot)u_0\|_{L^{10}_b L^{10}} \le \tfrac{\varepsilon}{2}. \tag{4.30}$$

Let $v \in \mathcal{V}_1(b)$. From Theorem 4.10 and (4.29) we deduce the inequalities

$$\begin{aligned} [\![\Phi(v)]\!]_{0,b} &\le C_{\mathrm{St}}(\|u_0\|_2 + \||v|^4 v\|_{E'_0(b)}) \le C_{\mathrm{St}}\rho + C_{\mathrm{St}}\varepsilon^4 R, \\ [\![|\nabla\Phi(v)|]\!]_{0,b} &\le C_{\mathrm{St}}(\|u_0\|_{1,2} + \||\nabla(|v|^4 v)|\|_{E'_0(b)}) \le C_{\mathrm{St}}\rho + 5C_{\mathrm{St}}\||v|^4|\nabla v|\|_{E'_0(b)} \\ &\le C_{\mathrm{St}}\rho + 5C_{\mathrm{St}}\varepsilon^4 R. \end{aligned} \tag{4.31}$$

For $0 < \varepsilon \le \varepsilon_0 := (6C_{\mathrm{St}}R)^{-1/4}$, the definition of R thus leads to

$$[\![\Phi v]\!]_{1,b} \le 2C_{\mathrm{St}}\rho + 6C_{\mathrm{St}}\varepsilon^4 R \le R.$$

Using also (4.30) and (4.28), we further estimate

$$\begin{aligned}\|\Phi(v)\|_{L_b^{10}L^{10}} &\le \tfrac{\varepsilon}{2} + C'_{\mathrm{So}}\|T *_+ (|v|^4 v)\|_{L_b^{10}W^{1,r}} \\ &\le \tfrac{\varepsilon}{2} + C'_{\mathrm{So}}C_{\mathrm{St}}\big(\||v|^4 v\|_{E_0(b)'} + 5\||v|^4|\nabla v|\|_{E_0(b)'}\big) \\ &\le \tfrac{\varepsilon}{2} + C'_{\mathrm{So}}C_{\mathrm{St}}(\varepsilon^3 R + 5\varepsilon^3 R)\varepsilon \le \varepsilon\end{aligned}$$

provided that $0 < \varepsilon \le \varepsilon_1 := \min\{\varepsilon_0, (12RC'_{\mathrm{So}}C_{\mathrm{St}})^{-1/3}\}$. For such ε and fixed $R = R(\rho)$ and $b = b(u_0, \varepsilon) > 0$, the operator Φ thus maps $\Sigma(b, R, \varepsilon)$ into itself.

Let $w \in \Sigma(b, R, \varepsilon)$. By means of Young's inequality, we can write

$$\begin{aligned}|v|^4 v - |w|^4 w &= (v-w)|v|^4 + w(\overline{v} - \overline{w})|v|^2 v + |w|^2(v-w)|v|^2 \\ &\quad + |w|^2 w(\overline{v} - \overline{w})v + |w|^4(v-w), \\ \big||v|^4 v - |w|^4 w\big| &\le \tfrac{5}{2}(|v|^4 + |w|^4)|v-w|, \\ \big|\nabla(|v|^4 v - |w|^4 w)\big| &\le |\nabla(v-w)|\,\tfrac{5}{2}(|v|^4 + |w|^4) + |v-w|\,\big[|\nabla v|\big(4|v|^3 + 3|v|^2|w| \\ &\quad + 2|v||w|^2 + |w|^3\big) + |\nabla w|\big(|v|^3 + 2|v|^2|w| + 3|v||w|^2 + 4|w|^3\big)\big] \\ &\le \tfrac{5}{2}|\nabla[v-w]|(|v|^4 + |w|^4) + 10|v-w|(|\nabla v| + |\nabla w|)[|v|^3 + |w|^3].\end{aligned}$$

The Strichartz estimate in Theorem 4.10 b) and (4.29) now yield

$$\begin{aligned}[\![\Phi(v) - \Phi(w)]\!]_{0,b} &\le C_{\mathrm{St}}\||v|^4 v - |w|^4 w\|_{E'_0(b)} \le \tfrac{5}{2}C_{\mathrm{St}}\|(|v|^4 + |w|^4)|v-w|\|_{E'_0(b)} \\ &\le 5C_{\mathrm{St}}\varepsilon^4 [\![v-w]\!]_{0,b} \le \tfrac{1}{2}[\![v-w]\!]_{0,b}, \\ [\![\nabla(\Phi(v) - \Phi(w))]\!]_{0,b} &\le C_{\mathrm{St}}\|\nabla(|v|^4 v - |w|^4 w)\|_{E'_0(b)} \\ &\le 5C_{\mathrm{St}}\varepsilon^4\||\nabla(v-w)|\|_{L_b^{10}L^r} + 10C_{\mathrm{St}}\big(\|v-w\|_{L_b^{10}L^{10}} \\ &\qquad \cdot \||\nabla v| + |\nabla w|\|_{L_b^{10}L^r}\,\||v|^3 + |w|^3\|_{L_b^{\frac{10}{3}}L^{\frac{10}{3}}}\big) \\ &\le \tfrac{1}{2}[\![|\nabla(v-w)|]\!]_{0,b} + 40C_{\mathrm{St}}R\varepsilon^3[\![v-w]\!]_{0,b} \\ &\le \tfrac{1}{2}[\![|\nabla(v-w)|]\!]_{0,b} + \tfrac{1}{4}[\![v-w]\!]_{0,b},\end{aligned} \tag{4.32}$$

where we let $\varepsilon \in (0, \varepsilon_2)$ and the number

$$\varepsilon_2 = \varepsilon_2(\rho) := \min\{\varepsilon_1, (10C_{\mathrm{St}})^{-1/4}, (160C_{\mathrm{St}}R)^{-1/3}\} \tag{4.33}$$

is independent of b. For the gradient term we have also employed Hölder's inequality with $\frac{5}{6} = \frac{1}{10} + \frac{1}{r} + \frac{3}{10}$ in space and with $\frac{1}{2} = \frac{1}{10} + \frac{1}{10} + \frac{3}{10}$ in time.

So there is a (unique) fixed point u of Φ in $\Sigma(b, R, \varepsilon)$ with $\varepsilon = \varepsilon_2$, which an H^1-solution of (4.1) on $[0, b]$ due to Lemma 2.8 of [**32**] and Lemma 4.5.

3) Let u and v be H^1-solutions of (4.1) on J_u and J_v, respectively, such that $J := J_u \cap J_v$ contains more points than 0. If $0 \ne \max J$, we set

$$\tau = \sup\{t \in J \cap \mathbb{R}_{\ge 0} \,|\, \forall\, s \in [0, t] : u(s) = v(s)\}.$$

We suppose $\tau < \sup J$. Then there are times $t_n > \tau$ in J with $t_n \to \tau$ as $n \to \infty$ and $u(t_n) \ne v(t_n)$ for all $n \in \mathbb{N}$. The continuity of u and v implies the equality $u(\tau) = v(\tau) =: \varphi \in H^1$.

Fix a time $\delta' > 0$ with $\tau + \delta' \in J$ and $\delta' \le b(\varphi, \varepsilon')$, where $\varepsilon' = \varepsilon_2(\rho')$ and $b(\varphi, \varepsilon') =: b'$ are chosen as in step 2) for $\rho' := \|\varphi\|_{1,2}$. Set $R' = 2C_{\mathrm{St}}\|\varphi\|_{1,2} + 1$. This step yields a solution $\tilde z \in \Sigma(\delta', R', \varepsilon')$ of (4.1) with $\tilde z(0) = \varphi$. Set $z = \tilde z(\cdot + \tau)$. It suffices to show that $u = z$ and $v = z$ on $[\tau, \tau + \delta]$ for some $\delta \in (0, \delta']$ to obtain a contradiction. We can thus assume that $v = z$ on $[\tau, \tau + \delta']$; i.e., $v_\tau := v(\cdot - \tau)$ is contained in $\Sigma(\delta', R', \varepsilon')$.

We set $w = u(\cdot - \tau) - v(\cdot - \tau)$ on $J' = [0, \delta']$. This function satisfies

$$\partial_t w = \mathrm{i}\Delta w + F(w + v_\tau) - F(v_\tau) =: \mathrm{i}\Delta w + f, \quad t \in J', \qquad w(0) = 0.$$

Moreover, w belongs to $C(J', H^1)$ and thus to $C(J', L^6)$. These properties are not enough to use directly the estimates of step 2). But we can exploit the better behavior of $v_\tau \in \mathcal{V}_1(\delta')$. In particular, we need smallness of w and v_τ to absorb the error term f. Here we have for any given $\eta > 0$ (fixed below) a time $\delta = \delta(\eta) \in (0, \delta']$ such that $\|w(t)\|_{1,2} \le \eta$ for all $t \in [0, \delta]$ and $\|v_\tau\|_{L^{10}_\delta L^{10}} \le \eta$.

Writing $|w + v_\tau|^4 = (w + v_\tau)^2(\overline{w} + \overline{v_\tau})^2$, we can expand $f = f_1 + \cdots + f_5$, where $f_5 = -\mathrm{i}\mu|w|^4 w$ and the other summands f_j are linear combinations of product containing j factors from $\{w, \overline{w}\}$ and $5 - j$ factors from $\{v_\tau, \overline{v_\tau}\}$. The Strichartz estimate in Theorem 4.10 b) thus implies the bound

$$\|w\|_{L^2([0,\delta], L^6)} \le C_{\mathrm{St}} \sum_{j=1}^{5} \|f_j\|_{L^{p_j'}([0,\delta], L^{q_j'})} \le c_0 \sum_{j=1}^{5} \||w|^j |v_\tau|^{5-j}\|_{L^{p_j'}([0,\delta], L^{q_j'})}$$

for a constant $c_0 > 0$ and the Strichartz pairs

$$(p_1, q_1) = (10, r) = \big(10, \tfrac{30}{13}\big), \quad (p_2, q_2) = \big(5, \tfrac{30}{11}\big), \quad (p_3, q_3) = \big(\tfrac{10}{3}, \tfrac{10}{3}\big),$$
$$(p_4, q_4) = \big(\tfrac{5}{2}, \tfrac{30}{7}\big), \quad (p_5, q_5) = (2, 6).$$

Similar as in (4.29), on each product $|w|^j |v_\tau|^{5-j}$ we apply Hölder's inequality first in space and then in time with the exponent

$$\begin{aligned}
j = 1: &\quad \frac{1}{q_1'} = \frac{17}{30} = \frac{1}{6} + \frac{4}{10}, \quad \frac{1}{p_1'} = \frac{9}{10} = \frac{1}{2} + \frac{4}{10},\\
j = 2: &\quad \frac{1}{q_2'} = \frac{19}{30} = \frac{1}{6} + \frac{1}{6} + \frac{3}{10}, \quad \frac{1}{p_2'} = \frac{4}{5} = \frac{1}{2} + \frac{1}{\infty} + \frac{3}{10},\\
j = 3: &\quad \frac{1}{q_3'} = \frac{7}{10} = \frac{1}{6} + \frac{2}{6} + \frac{2}{10}, \quad \frac{1}{p_3'} = \frac{7}{10} = \frac{1}{2} + \frac{2}{\infty} + \frac{2}{10},\\
j = 4: &\quad \frac{1}{q_4'} = \frac{23}{30} = \frac{1}{6} + \frac{3}{6} + \frac{1}{10}, \quad \frac{1}{p_4'} = \frac{3}{5} = \frac{1}{2} + \frac{3}{\infty} + \frac{1}{10},\\
j = 5: &\quad \frac{1}{q_5'} = \frac{5}{6} = \frac{1}{6} + \frac{4}{6}, \quad \frac{1}{p_5'} = \frac{1}{2} = \frac{1}{2} + \frac{4}{\infty}.
\end{aligned}$$

We thus obtain

$$\begin{aligned}
\|w\|_{L^2_\delta L^6} \le c_0 \big(&\|w\|_{L^2_\delta L^6} \|v_\tau\|^4_{L^{10}_\delta L^{10}} + \|w\|_{L^2_\delta L^6} \|w\|_{L^\infty_\delta L^6} \|v_\tau\|^3_{L^{10}_\delta L^{10}}\\
&+ \|w\|_{L^2_\delta L^6} \|w\|^2_{L^\infty_\delta L^6} \|v_\tau\|^2_{L^{10}_\delta L^{10}} + \|w\|_{L^2_\delta L^6} \|w\|^3_{L^\infty_\delta L^6} \|v_\tau\|_{L^{10}_\delta L^{10}}\\
&+ \|w\|_{L^2_\delta L^6} \|w\|^4_{L^\infty_\delta L^6}\big)\\
\le\ & 5c_0 \eta^4 \|w\|_{L^2_\delta L^6}.
\end{aligned}$$

We recall that w is an element of $C(J', L^6)$ and thus the norm $\|w\|_{L^2_\delta L^6}$ is finite. Fixing $\eta = (10c_0)^{-1/4}$ and consequently $\delta = \delta(\eta)$, we conclude that $w(t) = 0$ for every $t \in [0, \delta]$ and hence $u(t) = v(t)$ for $t \in [\tau, \tau + \delta]$. This is the desired contradiction. Here and in step 2), negative times are treated analogously.[4]

4) Again, we define $t^+(u_0) > 0$ as the supremum of those $b > 0$ for which we have an H^1-solution u_b of (4.1) on $[0, b]$. Step 3) then allows us to construct a unique H^1-solution u on $[0, t^+(u_0))$. Analogously one argues for $t < 0$ and, using also Remark 4.13, we obtain a unique maximal H^1-solution u on $J(u_0) = (t^-(u_0), t^+(u_0))$. As in Theorem 4.16 one can show that u belongs to $L^p_{\mathrm{loc}}(J(u_0), W^{1,q})$. So part a) is shown.

We suppose that $t^+(u_0) =: t^+$ and $\|u\|_{L^{10}_{t^+} L^{10}} =: M$ are finite. Take $\kappa > 0$ to be fixed below. Since M is finite and $[0, t^+]$ is compact, there are times $t_0 = 0 < t_1 < \cdots < t_N = t^+$ with $\|u\|_{L^{10}(J_k, L^{10})} \le \kappa$ for all intervals $J_k = [t_k, t_{k+1}]$.

Using this bound, as in (4.31) we find a constant $C > 0$ such that

$$[\![u]\!]_{1,J_k} \le C(\|u(t_k)\|_{1,2} + \kappa^4 \|u\|_{E_1(b)}) \le C(\|u(t_k)\|_{1,2} + \kappa^4 [\![u]\!]_{1,J_k})$$

for all $k \in \{0, 1, \dots, N-1\}$. With $\kappa = (2C)^{-1/4}$ it follows

$$[\![u]\!]_{1,J_k} \le 2C\|u(t_k)\|_{1,2} \le 2C[\![u]\!]_{1,J_{k-1}} \le \cdots \le (2C)^{N-1}[\![u]\!]_{1,J_0} \le (2C)^N \|u_0\|_{1,2}$$

for all $k \le N-1$. Recall that $\|u_0\|_{1,2} \le \rho$. As a result, $\|u(t)\|_{1,2}$ is bounded by $\overline{\rho} := (2C)^N \rho$ for all $t \in [0, t^+)$ and thus $[\![u]\!]_{1,t^+}$ by $2CN\overline{\rho}$. Set $\overline{R} = 2C_{\mathrm{St}}\overline{\rho} + 1$. We define the number $\overline{\varepsilon} > 0$ corresponding to $\overline{R}$ as ε_2 in step 2).

Let $J_\tau := [\tau, t^+) \subseteq \mathbb{R}_{\ge 0}$. Theorem 4.10 and estimates (4.28) and (4.29) yield

$$\begin{aligned}
\|u - T(\cdot - \tau)u(\tau)\|_{L^{10}_{J_\tau} L^{10}} &= \|T *_+ F(u)\|_{L^{10}_{J_\tau} L^{10}} \le C'_{\mathrm{So}} \|T *_+ F(u)\|_{L^{10}_{J_\tau} W^{1,r}} \\
&\le C'_{\mathrm{So}} C_{\mathrm{St}} \big(\| |u|^5 \|_{E'_0(J_\tau)} + 5 \| |u|^4 |\nabla u| \|_{E'_0(J_\tau)} \big) \\
&\le 6 C'_{\mathrm{So}} C_{\mathrm{St}} \|u\|^4_{L^{10}_{J_\tau} L^{10}} \|u\|_{L^{10}_{J_\tau} W^{1,r}} \\
&\le 12 C'_{\mathrm{So}} C_{\mathrm{St}} C N \overline{\rho}\, \|u\|^4_{L^{10}_{J_\tau} L^{10}} \;\longrightarrow\; 0
\end{aligned}$$

as $\tau \to (t^+)^-$, because $\|u\|_{L^{10}_{J_\tau} L^{10}}$ tends to 0. So we can fix an initial time $\tau < t^+$ with $\|T(\cdot - \tau)u(\tau)\|_{L^{10}_{J_\tau} L^{10}} \le \overline{\varepsilon}/4$. Since $T(\cdot - \tau)u(\tau)$ belongs to $L^{10}(\mathbb{R}, W^{1,r}) \hookrightarrow L^{10}(\mathbb{R}, L^{10})$ by Theorem 4.10 and (4.28), there is a time step $\beta > 0$ with

$$\|T(\cdot - \tau)u(\tau)\|_{L^{10}([\tau, t^+ + \beta], L^{10})} \le \overline{\varepsilon}/2.$$

Step 2) now allows us to solve (4.1) on $[\tau, t^+ + \beta]$ with intial value $u(\tau)$, giving a solution of (4.1) on $[0, t^+ + \beta]$. This fact contradicts the definition of $t^+(u_0) = t^+$, and the blow-up criterion is proved. Negative times are treated analogously.

5) To show c), we take a compact interval $J \subseteq J(u_0)$ with $0 = \min J$. Let $v_0 \in H^1$ with $\|u_0 - v_0\|_{1,2} \le \delta' \le 1$ so that $\|v_0\|_{1,2} \le \rho + 1 =: \tilde{\rho}$. Set $\tilde{R} = 2C_{\mathrm{St}}\tilde{\rho} + 1$ and $\tilde{\varepsilon} = \varepsilon_2(\tilde{\rho})$ as in step 2). For the iteration of the initial estimate, we need a variant of (4.30) along the compact set $u(J)$ in H^1. By Theorem 4.10 the map

[4] Some parts of this step were only sketched in the lectures.

$\varphi \mapsto \mathbb{1}_{[0,b]}T(\cdot)\varphi$ tends 0 in $\mathcal{V}_1(\mathbb{R})$ strongly on H^1, hence uniformly on $u(J)$. We can thus fix a time $\tilde{b} > 0$ with

$$\|T(\cdot)u(t)\|_{L^{10}_{\tilde{b}}L^{10}} \leq \tfrac{\tilde{\varepsilon}}{8} \leq \tfrac{\tilde{\varepsilon}}{4}, \tag{4.34}$$

for all $t \in J$. Set $\tilde{t}_k = k\tilde{b}$ for $k \in \{0, 1, \ldots, K\}$, where $\tilde{t}_K$ is redefined as $\max J$ if $K\tilde{b} > \max J$ and $(K-1)\tilde{b} < \max J$. Let $\tilde{J}_k = [\tilde{t}_k, \tilde{t}_{k+1}]$. For $j = 0$, Theorem 4.10 and inequalities (4.28) and (4.34) yield

$$\|T(\cdot)v_0\|_{L^{10}_{\tilde{b}}L^{10}} \leq \|T(\cdot)(v_0 - u_0)\|_{L^{10}_{\tilde{b}}L^{10}} + \|T(\cdot)u_0\|_{L^{10}_{\tilde{b}}L^{10}} \leq C_{\mathrm{St}}C'_{\mathrm{So}}\delta' + \tfrac{\tilde{\varepsilon}}{4} \leq \tfrac{\tilde{\varepsilon}}{2}$$

for $0 \leq \delta' \leq \delta_0 := \min\{1, \tilde{\varepsilon}/(8C_{\mathrm{St}}C'_{\mathrm{So}})\}$. (The extra factor $\frac{1}{2}$ is needed below.)

Step 2) now provides a solution $v = \Phi_{v_0}(v) \in \Sigma(\tilde{b}, \tilde{R}, \tilde{\varepsilon})$ of (4.1) on $[0, \tilde{b}]$. To iterate the argument we let $\|u_0 - v_0\|_{1,2} \leq (4C_{\mathrm{St}})^{-K}\delta_0 =: 2\delta$. Using (4.32) and the definition of Φ, we compute

$$\begin{aligned}
[\![u - v]\!]_{1,\tilde{b}} &\leq [\![\Phi_{u_0}(u) - \Phi_{u_0}(v)]\!]_{1,\tilde{b}} + [\![\Phi_{u_0}(v) - \Phi_{v_0}(v)]\!]_{1,\tilde{b}} \\
&\leq \tfrac{3}{4}[\![u - v]\!]_{1,\tilde{b}} + [\![T(\cdot)(u_0 - v_0)]\!]_{1,\tilde{b}} \leq \tfrac{3}{4}[\![u - v]\!]_{1,\tilde{b}} + C_{\mathrm{St}}\|u_0 - v_0\|_{1,2}, \\
[\![u - v]\!]_{1,\tilde{b}} &\leq 4C_{\mathrm{St}}\|u_0 - v_0\|_{1,2} \leq (4C_{\mathrm{St}})^{1-K}\delta_0,
\end{aligned}$$

and hence $\|u(\tilde{t}_1) - v(\tilde{t}_1)\|_{1,2} \leq (4C_{\mathrm{St}})^{1-K}\delta_0 \leq \delta_0$. We deduce

$$\|T(\cdot)v(\tilde{t}_1)\|_{L^{10}_{\tilde{b}}L^{10}} \leq C'_{\mathrm{So}}C_{\mathrm{St}}\|u(\tilde{t}_1) - v(\tilde{t}_1)\|_{1,2} + \tfrac{\tilde{\varepsilon}}{4} \leq \tfrac{\tilde{\varepsilon}}{2}.$$

As in Theorem 1.11, based on (4.34) we can repeat the above estimate $K-1$ times so that $J \subseteq J(v_0)$ and

$$[\![u - v]\!]_{1,\tilde{J}_k} \leq 4C_{\mathrm{St}}\|u(\tilde{t}_k) - v(\tilde{t}_k)\|_{1,2} \leq \cdots \leq (4C_{\mathrm{St}})^K\|u_0 - v_0\|_{1,2} \leq \delta_0 \tag{4.35}$$

for all $k \in \{0, 1, \ldots, K-1\}$. It follows $[\![u - v]\!]_{1,J} \leq K(4C_{\mathrm{St}})^K\|u_0 - v_0\|_{1,2}$ by the triangle inequality.

Finally, we replace in this argument u_0 by any initial map $w_0 \in \overline{B}_{H^1}(u_0, \delta)$, noting that $J \subseteq J(w_0)$. By (4.35) with w instead of v we obtain $\|u(\tilde{t}_k) - w(\tilde{t}_k)\|_{1,2} \leq \delta_0$. As above, Sobolev's and Strichartz' inequalies and (4.34) imply

$$\begin{aligned}
\|T(\cdot)w(\tilde{t}_k)\|_{L^{10}_{\tilde{b}}L^{10}} &\leq \|T(\cdot)(w(\tilde{t}_k) - u(\tilde{t}_k))\|_{L^{10}_{\tilde{b}}L^{10}} + \|T(\cdot)u(\tilde{t}_k)\|_{L^{10}_{\tilde{b}}L^{10}} \\
&\leq C_{\mathrm{St}}C'_{\mathrm{So}}\delta_0 + \tfrac{\tilde{\varepsilon}}{8} \leq \tfrac{\tilde{\varepsilon}}{4}.
\end{aligned}$$

for all $k \in \{0, 1, \cdots, K-1\}$. Since $\|w_0 - v_0\|_{1,2} \leq 2\delta$, the same reasoning as for u and v yields $[\![w - v]\!]_{1,J} \leq K(4C_{\mathrm{St}})^K\|w_0 - v_0\|_{1,2}$ and thus claim c). □

There are illposedness results for the supercritical case $\alpha > \alpha_c$, but for $\mu = 1$ and e.g. $\alpha \in (5, 6)$ and $m = 3$ one has existence (not uniqueness) of solutions in $L^\infty(J, H^1)$, see Section 3.8 respectively Exercise 3.56 in [**35**].

4.4. Asymptotic behavior

In this section we show global existence of solutions for $1 < \alpha < \alpha_c = \frac{m+2}{(m-2)_+}$ in the defocusing case, and in the focusing case if either α or u_0 are small. Moreover, we state a result on asymptotic stability in the defocusing case. A main ingredient of the proofs is the preservation of the L^2-norm and of the energy $\mathcal{E}$ for solutions to (4.1).

So far these conservation laws have been shown only for H^2-solutions in Remark 4.4. For such solutions one can expect a blow-up condition in H^2 which does not fit to the energy bound. We thus have to extend these laws to H^1-solutions. This seems to be feasible since $\mathcal{E} \in C^1_{\mathbb{R}}(H^1, \mathbb{R})$, H^2 is dense in H^1, and the solutions depend continuously on data in H^1 by Theorem 4.16. However, if we approximate a given initial value u_0 in H^1 by functions $\psi_n \in H^2$, we do not yet know whether the solutions $u_n = \varphi(\,\cdot\,, \psi_n)$ belong to $C(J, H^2)$ for an n-independent interval. (Theorem 4.16 only shows that $u_n \in C(J, H^1)$, for any compact $J \subseteq J(u_0)$ and large n.)

Fortunately, one can show that for $v_0 \in H^2$ the corresponding maximal H^1-solution v of (4.1) actually belongs to $C(J(v_0), H^2) \cap C^1(J(v_0), L^2)$, where $J(v_0)$ is the maximal existence interval as an H^1-solution from Theorem 4.16. We show this fact here only for $\alpha > 2$. By means of a different approach, this restriction is removed in Proposition 4.23 treated in an appendix. We have established such a regularity result already in Theorem 1.16 for general semilinear evolution equations based on difference quotients in time. Since we now study a partial differential equation on $\mathbb{R}^m$, we can instead use differences in space exploiting the characterization

$$v \in L^r \text{ belongs to } W^{1,r} \iff C := \sup_{h \in \mathbb{R}^m \setminus \{0\}} |h|^{-1} \|v(\cdot + h) - v\|_r < \infty \quad (4.36)$$

for $r \in (1, \infty]$, where $C = \||\nabla v|\|_r$. See Theorems 5.8.3 and 5.8.4 in [9]. For $r = 1$ the implication '$\Leftarrow$' is wrong, and for $r = \infty$ one cannot get $v \in C^1$ in this way (which was our aim in Theorem 1.16 for the X-valued solution).

Proposition 4.18. *Let (4.20) be true, $\alpha > 2$, and $u_0 \in H^2$. Then the maximal solution u of (4.1) from Theorem 4.16 is an H^2-solution on $J(u_0)$.*

Proof. We fix $b \in (0, t^+(u_0))$ and set $r = \|u\|_{\mathcal{F}_1(0,b)}$ and $\rho = \|u_0\|_{2,2}$. Let $t \in [0, b]$ and $h \in \mathbb{R}^m \setminus \{0\}$. We define $u_h(t) = u(t, \cdot + h)$ and $v_h = |h|^{-1}(u_h - u)$. Observe that $\|u_h(t)\|_q = \|u(t)\|_q$ and $\||\nabla u_h(t)|\|_q = \||\nabla u(t)|\|_q$. Since u_h fulfills (4.1) with initial value $u_0(\cdot + h)$, the difference quotient satisfies the new equation

$$\begin{aligned} \partial_t v_h(t) &= \mathrm{i}\Delta v_h(t) + |h|^{-1}\big(F(u_h(t)) - F(u(t))\big) \\ &= \mathrm{i}\Delta v_h(t) + \int_0^1 \phi'(u(t) + \tau(u_h(t) - u(t)))\,\mathrm{d}\tau\, v_h(t) \end{aligned} \quad (4.37)$$

and $v_h(0) = |h|^{-1}(u_0(\cdot + h) - u_0)$, where $\phi(z) = -\mathrm{i}\mu|z|^{\alpha-1}z$. We abbreviate $u_{\tau,h} = u + \tau(u_h - u)$ and denote the integral by $g_h(u(t))$. Because of $\alpha > 2$, we can differentiate

$$\nabla(g_h(u)v_h) = g_h(u)\nabla v_h + \int_0^1 \phi''(u_{\tau,h})[\nabla u_{\tau,h}, v_h]\,\mathrm{d}\tau$$

omitting the variable t. Hölder's inequality with $\frac{1}{q'} = \frac{\alpha-k}{q} + \frac{k}{q}$ for $k \in \{1,2\}$ and Sobolev's embedding $H^1 \hookrightarrow L^q$ from (4.4) yield

$$\begin{aligned} \|g_h(u)v_h\|_{q'} &\le c\|u_{\tau,h}\|_q^{\alpha-1}\|v_h\|_q \le cr^{\alpha-1}\|v_h\|_q, \\ \||\nabla(g_h(u)v_h)|\|_{q'} &\le c\big(\|u_{\tau,h}\|_q^{\alpha-1}\||\nabla v_h|\|_q + \|u_{\tau,h}\|_q^{\alpha-2}\||\nabla u_{\tau,h}|\|_q\|v_h\|_q\big) \end{aligned}$$

$$\leq c\big(r^{\alpha-1}\|v_h\|_{1,q} + r^{\alpha-2}\|v_h\|_{1,2}\|u\|_{1,q}\big).$$

for some constants $c > 0$. Let $\beta \in (0, b]$. Applying the $L^{p'}$-norm and Hölder in time, we infer

$$\begin{aligned}\|g_h(u)v_h\|_{E_1'(0,\beta)} &\leq c\big(r^{\alpha-1}\|v_h\|_{L^{p'}_\beta W^{1,q}} + r^{\alpha-2}\|v_h\|_{L^\infty_\beta H^1}\|u\|_{L^{p'}_\beta W^{1,q}}\big)\\ &\leq \bar{c} r^{\alpha-1}\beta^{\frac{1}{p'}-\frac{1}{p}}\, |||v_h|||_{1,\beta}.\end{aligned}$$

We now set $\beta = \min\{b, (2C_{\mathrm{St}}\bar{c}r^{\alpha-1})^{-\gamma}\}$ with $\gamma = \frac{p'p}{p-p'}$. Equation (4.37), the Strichartz estimates from Theorem 4.10, and (4.36) then imply

$$|||v_h|||_{1,\beta} \leq C_{\mathrm{St}}\big(\|v_h(0)\|_{1,2} + \|g_h(u)v_h\|_{E_1'(0,\beta)}\big) \leq cC_{\mathrm{St}}\rho + \tfrac{1}{2}|||v_h|||_{1,\beta},$$

and hence $\||\nabla v_h|\|_{C([0,\beta],L^2)} \leq |||v_h|||_{1,\beta} \leq 2cC_{\mathrm{St}}\rho$. From the characterization (4.36) we deduce $\|u(t)\|_{2,2} \leq \hat{c}\rho$ for $t \in [0,\beta]$. Since $\hat{c}$ and β only depend on α, m and r, we can iterate this argument to obtain boundedness of $u : [0,b] \to H^2$.

Moreover, $u : [0,b] \to H^2$ is strongly measurable by (Pettis') Theorem 1.1.6 of [**14**], since H^2 is separable and $t \mapsto (u(t)|\varphi)_{L^2}$ is continuous for $\varphi \in L^2$. To obtain continuity in H^2, we note that $\partial_{jk}F(v)$ is a linear combination of terms of the form like $F_1(v) = |v|^{\alpha-1}\partial_{jk}v$ and $F_2(v) = |v|^{\alpha-2}\partial_j v\partial_k v$. As above we can estimate $F_2(v)$ in $L^{q'}$ by $\|v\|_q^{\alpha-2}\||\nabla v|\|_q^2 \leq c\|v\|_{2,2}^\alpha$. For the other term, we remark that $m \leq 5$ because of $2 \leq \alpha < \alpha_c$. Sobolev's Theorem 3.31 in [**33**] yields $H^2 \hookrightarrow L^r$ for all $r \in [2,\infty]$ if $m \leq 3$, all $r \in [2,\infty)$ if $m = 4$ and for $r \in [2,10]$ if $m = 5$. In the first two cases, we easily obtain $\|F_1(v)\|_{q'} \leq c\|v\|_{2,2}^\alpha$. For $m = 5$ we use Hölder with $\frac{1}{q'} = \frac{\alpha}{1+\alpha} = \frac{\alpha-1}{2(\alpha+1)} + \frac{1}{2}$. Since $\alpha < \frac{7}{3}$, the exponent $r = 2(\alpha+1) < \frac{20}{3}$ is in the admissible range $[2,10]$, and we infer again $\|F_1(v)\|_{q'} \leq c\|v\|_{2,2}^\alpha$. Hence, $\partial_{jk}F(u)$ belongs to $L^\infty([0,b], L^{q'})$ for $j,k \in \{1,\cdots,m\}$. Note that $w := \partial_{jk}u$ satisfies

$$w(t) = T(t)\partial_{jk}u_0 + \int_0^t T(t-s)\partial_{jk}F(u(s))\,\mathrm{d}s, \qquad t \in [0,b],$$

in H^{-1}. Theorem 4.10 then yields $w \in C([0,b], L^2)$ and thus the desired continuity of $u : [0,b] \to H^2$. Negative times are treated in the same way. □

Based on the above result we can show global existence in the defocusing and in the focusing case if $\alpha < 1 + \frac{4}{m}$ or u_0 is small in H^1. We stress that there is blow up if $\alpha \in [1+\frac{4}{m}, \alpha_c)$ and $\mu = -1$ by Theorem 6.5.10 in [**6**]. As noted after Proposition 1.21, the focusing semilinear wave equation admits blow up for all $\alpha > 1$. This striking difference relies on the possible growth of the L^2-norm of solutions in the wave case.

Theorem 4.19. *Let (4.20) be true, $u_0 \in H^1$, and u be the maximal H^1-solution of (4.1) on $J(u_0)$. Then the following assertions are true.*

a) We have $\|u(t)\|_2 = \|u_0\|_2$ and $\mathcal{E}(u(t)) = \mathcal{E}(u_0)$ for all $t \in J(u_0)$.

b) Let $\mu = 1$. Then $J(u_0) = \mathbb{R}$ for all $u_0 \in H^1$.

c) Let $\mu = -1$ and $\alpha < 1 + \frac{4}{m}$. Then $J(u_0) = \mathbb{R}$ for all $u_0 \in H^1$.

d) Let $\mu \in \{-1,1\}$ and $\alpha \in (1,\alpha_c)$. Then there is a constant $\varepsilon_0 > 0$ with the following property: For all $\varepsilon \in (0,\varepsilon_0]$ there is a radius $\rho = \rho(\varepsilon) > 0$ such that for all $u_0 \in \overline{B}_{H^1}(0,\rho)$ we have $J(u_0) = \mathbb{R}$ and $\|u(t)\|_{1,2} \le \varepsilon$ for all $t \in \mathbb{R}$.

In all three cases the nonlinearity is relatively tame so that it does not destroy the global existence that we have in the linear case: In b) we have the good sign $\mu = 1$. In c) the nonlinearity does not grow too much. In d) the solution u is small initially which leads to an even smaller nonlinearity $|u|^{\alpha-1}u$. Moreover, the trivial fixed point $u_* = 0$ is stable in all cases. As we will see in the proof, in these situations the mass and energy of a solution control its H^1-norm, so that the blow-up condition implies global existence.

These results and their proofs are typical for evolution equations possessing

- a 'full' local wellposedness theory as in Theorem 4.16,
- a conserved quantity which dominates the norm of the blow-up condition (directly or under a smallness condition) at least for a class of 'good solutions',
- and a regularity result which shows that these good solutions exist as long as those from the local wellposedness theorem.

Proof of Theorem 4.19. a) The assertion is true for H^2-solutions by Remark 4.4. There are functions $\psi_n \in H^2$ tending to u_0 in H^1 as $n \to \infty$. Set $u_n = \varphi(\,\cdot\,,\psi_n)$. Proposition 4.23 says that u_n is an H^2-solution on the maximal interval $J(\psi_n)$ from Theorem 4.16. Take any compact interval $J \subseteq J(u_0)$. Due to Theorem 4.16, there is an index $N_J \in \mathbb{N}$ such that $J \subseteq J(\psi_n)$ for $n \ge N_J$ and $u_n(t)$ converges to $u(t)$ in H^1 as $n \to \infty$, uniformly for $t \in J$. Assertion a) then follows by approximation since $\mathcal{E} \in C(H^1,\mathbb{R})$, cf. (4.6).

b) Let $\mu = 1$. In this case, from step a) and (4.5) we derive

$$\|u(t)\|_{1,2}^2 \le 2\mathcal{E}(u(t)) + \|u(t)\|_2^2 = \mathcal{E}(u_0) + \|u_0\|_2^2$$

for all $t \in J(u_0)$. The blow-up criterion in Theorem 4.16 now yields $J(u_0) = \mathbb{R}$.

c) Let $1 < \alpha < 1 + \frac{4}{m}$ and $\mu = -1$. We consider $m \ge 3$, the proof for $m \in \{1,2\}$ is similar. Observe that

$$\frac{1}{\alpha+1} = \frac{1-\theta}{2} + \theta\frac{m-2}{2m} \qquad \text{for} \ \ \theta = \frac{m}{2} - \frac{m}{\alpha+1} \in (0,1).$$

The interpolation and Sobolev inequalities (see (3.38) in [**33**]) thus imply the 'Gagliardo–Nirenberg' inequality

$$\|v\|_{\alpha+1}^{\alpha+1} \le \Big(\|v\|_2^{1-\theta}\|v\|_{\frac{2m}{m-2}}^{\theta}\Big)^{\alpha+1} \le c\|v\|_2^{\alpha+1-m(\alpha-1)/2}\,\||\nabla v|\|_2^{m(\alpha-1)/2}$$

for all $v \in H^1$. We have $\beta := \frac{4}{m(\alpha-1)} > 1$ by our assumption. Young's inequality with β and β' leads to

$$\frac{1}{\alpha+1}\|v\|_{\alpha+1}^{\alpha+1} \le \frac{1}{4}\||\nabla v|\|_2^2 + c\|v\|_2^{\beta'(\alpha+1-m(\alpha-1)/2)}$$

for a constant only depending on α and m. Denoting the last summand by $k(\|v\|_2)$, we infer from step a) that

$$\mathcal{E}(u_0) = \mathcal{E}(u(t)) = \frac{1}{2}\||\nabla u(t)|\|_2^2 - \frac{1}{\alpha+1}\|u(t)\|_{\alpha+1}^{\alpha+1}$$

$$\geq \frac{1}{4}\||\nabla u(t)|\|_2^2 - k(\|u(t)\|_2) = \frac{1}{4}\||\nabla u(t)|\|_2^2 - k(\|u_0\|_2)$$

for all $t \in J(u_0)$. Hence, $\|u(t)\|_{1,2}^2 \leq 4\mathcal{E}(u_0)+4k(\|u_0\|_2)+\|u_0\|_2^2$ for all $t \in J(u_0)$, and again it follows that $J(u_0) = \mathbb{R}$.

d) Definition (4.5), part a), and Sobolev's embedding (4.4) yield

$$\begin{aligned}\frac{1}{2}\|u(t)\|_{1,2}^2 &= \frac{1}{2}\|u(t)\|_2^2 + \mathcal{E}(u(t)) - \frac{\mu}{\alpha+1}\|u(t)\|_{\alpha+1}^{\alpha+1} \\ &\leq \frac{1}{2}\|u_0\|_2^2 + \mathcal{E}(u_0) + c_0\|u(t)\|_{1,2}^{\alpha-1}\|u(t)\|_{1,2}^2 \end{aligned} \tag{4.38}$$

for all $t \in J(u_0)$ and with $c_0 = C_{\mathrm{So}}^{\alpha+1}/(\alpha+1)$. We set $\varepsilon_0 = (4c_0)^{1/(1-\alpha)}$ and take any $0 < \rho < \varepsilon \leq \varepsilon_0$. Let $\|u_0\|_{1,2} \leq \rho$. We now define

$$\tau = \sup\big\{t \in (0, t^+(u_0)) \,\big|\, \|u(s)\|_{1,2} \leq \varepsilon \text{ for all } s \in [0,t]\big\}$$

and observe that $\tau \in (0, t^+(u_0)]$. Estimates (4.38) and (4.4) then lead to

$$\begin{aligned}\frac{1}{2}\|u(t)\|_{1,2}^2 &\leq \frac{1}{2}\|u_0\|_2^2 + \mathcal{E}(u_0) + \frac{1}{4}\|u(t)\|_{1,2}^2, \\ \|u(t)\|_{1,2}^2 &\leq 2\|u_0\|_{1,2}^2 + \frac{4}{\alpha+1}\|u_0\|_{\alpha+1}^{\alpha+1} \leq c_1(\rho^2 + \rho^{\alpha+1}) \end{aligned} \tag{4.39}$$

for all $t \in [0,\tau)$ and a constant $c_1 \geq 1$ depending only on m. We finally choose $\rho = \rho(\varepsilon) \in (0,\varepsilon)$ such that $c_1(\rho^2 + \rho^{\alpha+1}) \leq \varepsilon^2/4$. Estimate (4.39) then implies $\|u(t)\|_{1,2}^2 \leq \varepsilon^2/4$ for $t < \tau$. If $\tau < t^+(u_0)$, we would obtain the contradiction $\|u(\tau)\|_{1,2} = \varepsilon/2$ by continuity. Hence, $\tau = t^+(u_0)$ and so Theorem 4.16 d) and gives $t^+(u_0) = \infty$ and the asserted bound for $t \geq 0$ follows. Similarly one treats negative times, possibly decreasing ρ. □

Global existence holds in the defocusing case also if $\alpha = \alpha_c$ and $m \geq 3$. This deep result is far beyond the scope of these lectures, see Chapter 5 of [**35**] for an extended survey.

The next result is a direct consequence of the (hard to prove) 'pseudo-confomal' conservation law for (4.1), Theorem 7.2.1 in [**6**]. The proposition says that solutions decay to 0 as $|t| \to \infty$ in the defocusing case.

PROPOSITION 4.20. *Let $1 + 4/m \leq \alpha < \alpha_c$, $\mu = 1$ and $u_0 \in H^1$ with $|x|u_0 \in L^2$, and u be the corresponding H^1-solution of (4.1) on $\mathbb{R}$. Then there is a constant $c > 0$ such that*

$$\|u(t)\|_{\alpha+1} \leq c|t|^{-\frac{2}{\alpha+1}}\||x|u_0\|_2^{\frac{2}{\alpha+1}} \qquad \textit{for} \ \ t \in \mathbb{R}.$$

PROOF. Let $t \in J(u_0) = \mathbb{R}$. Theorem 7.2.1 in [**6**] yields

$$\|(x+2\mathrm{i}t\nabla)u(t)\|_2^2 + \frac{8t^2}{\alpha+1}\|u(t)\|_{\alpha+1}^{\alpha+1} = \|xu_0\|_2^2 + 4\frac{m+4-\alpha m}{\alpha+1}\int_0^t s\|u(s)\|_{\alpha+1}^{\alpha+1}\,\mathrm{d}s.$$

Since the last summand is non-positive, the assertion follows. □

A more precise version of this result is given in Theorem 7.3.1 of [**6**]. It covers the full range $\alpha \in (1,\alpha_c)$ and estimates $\|u(t)\|_r$ for $r \in (2, 1+\alpha_c]$ ($r \in (2, 1+\alpha_c)$ if $m = 2$). In many cases one obtains the decay $c|t|^{\frac{m}{r}-\frac{m}{2}}$ as in Corollary 4.6.

In the setting of the proposition *scattering* can be shown; i.e., there are maps $u_\pm \in H^1$ with $|x|u_\pm \in L^2$ such $T(-t)u(t) \to u_\pm$ as $t \to \pm\infty$ in the norm $\|v\|_{1,2} + \||x|v\|_2$. See Theorem 7.4.1 of [**6**]. In Sections 7.7 and 7.8 in [**6**] one can find variants of these results in H^1 without weights.

In the focusing case, the qualitative behavior is completey different. In Example 4.2 we have seen that (4.1) admits standing waves if $\mu = -1$ and $\alpha \in (1, \alpha_c)$, see Section 8.1 of [**6**]. Because of symmetries, these standing waves form a finite dimensional manifold in H^1. It is unstable in H^1 (due to blowup) if $\alpha \geq 1 + \frac{4}{m}$, and stable if $\alpha < 1 + \frac{4}{m}$ by Sections 8.2 and 8.3 of [**6**].

4.5. An improved version of Proposition 4.18

In[5] this section we establish Proposition 4.18 without the restriction $\alpha > 2$ (and thus $m \leq 5$) by means of a different approach. To this aim, we modify the proof of Lemma 4.15 involving extra time regularity. In this argument we use the *vector-valued Sobolev space* $W^{1,r}(J, X)$ for $r \in [1, \infty]$, an open interval $J \subset \mathbb{R}$ and a Banach space X. A function u belongs to $W^{1,r}(J, X)$ if $u \in L^r(J, X)$ and there is a map $v \in L^r(J, X)$ with

$$u(t) = u(a) + \int_a^t v(s)\,\mathrm{d}s \tag{4.40}$$

for a.e. $t, a \in J$. (Section 2.5 in [**14**] provides a more thorough treatment of this topic.) One can then fix a number $a \in J$ (and thus a representative of u) such that (4.40) is valid for a.e. $t \in J$, see Proposition 2.5.9 in [**14**]. Hence, u is continuous on J and has a continuous extension to $\overline{J}$ by dominated convergence, and so (4.40) holds for all $t, a \in \overline{J}$. We set $u' = v$.

We discuss a few properties of these spaces needed later on. The space $W^{1,r}(J, X)$ is a Banach space when equipped with the norm given by

$$\|u\|_{1,r} = \begin{cases} \left(\|u\|_r^r + \|u'\|_r^r\right)^{\frac{1}{r}}, & \text{if } 1 \leq r < \infty, \\ \max\{\|u\|_\infty, \|u'\|_\infty\}, & \text{if } r = \infty, \end{cases}$$

where $\|\cdot\|_r$ is the norm on $L^r(J, X)$. Moreover $W^{1,r}(J, X)$ is isometrically isomorphic to a closed subspace of $L^r(J, X)^2$ via the map $u \mapsto (u, u')$. The remarks after (4.15) and standard results from functional analysis then yield parts a) and b) of the next result.

Remark 4.21. a) Let $1 \leq r < \infty$ and X is separable. Then $W^{1,r}(J, X)$ is separable.

b) Let $1 < r < \infty$ and X is reflexive. Then $W^{1,r}(J, X)$ is reflexive.

c) Let X be reflexive. Then $W^{1,\infty}(J, X)$ is isometrically isomorphic to the space of bounded Lipschitz functions $u : \overline{J} \to X$, cf. Theorem 2.5.12 in [**14**].

d) Let $a < b < c$ and $r \in [1, \infty)$. Let $u \in W^{1,r}((a, b), X)$ and $v \in W^{1,r}((b, c), X)$ satisfy $u(b) = v(b)$. Define $w(t) = u(t)$ for $t \in (a, b)$, $w(b) = u(b)$ and $w(t) = v(t)$ for $t \in (b, c)$. Set $g(t) = u'(t)$ for $t \in (a, b)$ and $g(t) = v'(t)$ for $t \in (b, c)$. It is then straightforward to check that $g \in L^r((a, c), X)$ is the derivative of w. The concatenation w thus belongs to $W^{1,r}((a, c), X)$. $\Diamond$

[5] This section was not part of the lectures.

We further need a simple density and embedding result for $W^{1,r}(J,X)$.

Lemma 4.22. *Let $J \subseteq \mathbb{R}$ be an open and bounded interval.*

a) Let $1 \le r < \infty$. Then $C^1(\overline{J},X)$ is dense in $W^{1,r}(J,X)$.

b) Let $1 \le r \le \infty$. Then $W^{1,r}(J,X)$ is continuously embedded into $C(\overline{J},X)$.

Proof. Let $u \in W^{1,r}(J,X)$. As noted after (4.40), we can fix a representative in $C(\overline{J},X)$. Part b) is shown as in Remark 1.41 of [**32**].

Let $1 \le r < \infty$ and $\overline{J} = [a,b]$. We can approximate u' in $L^r(J,X)$ by $v_n \in C_c(J,X)$, $n \in \mathbb{N}$, as in the proof of Lemma 4.8. Setting

$$u_n(t) = u(a) + \int_a^t v_n(s)\,\mathrm{d}s, \qquad t \in \overline{J},\ n \in \mathbb{N},$$

we obtain functions $u_n \in C^1(\overline{J},X)$ with $u_n' = v_n \to u'$ in $L^r(J,X)$. Hölder's inequality further yields

$$\|u_n(t) - u(t)\|_X^r \le \Big(\int_a^t \|v_n(s) - u'(s)\|_X\,\mathrm{d}s\Big)^r \le |J|^{\frac{r}{r'}} \int_J \|v_n(s) - u'(s)\|_X^r\,\mathrm{d}s,$$
$$\|u_n - u\|_r \le |J|\,\|v_n - u'\|_r \;\longrightarrow\; 0$$

as $n \to \infty$, where $|J|$ is the length of J. Assertion a) has been shown. $\square$

We can now show the improved version of Proposition 4.18.

Proposition 4.23. *Let (4.20) be true and $u_0 \in H^2$. Then the maximal solution u obtained in Theorem 4.16 is an H^2-solution on $J(u_0)$. Moreover, u belongs to $W^{1,p}((a,b),L^q)$ for all intervals $[a,b] \subseteq J(u_0)$.*

Proof. Let $u_0 \in H^2$ and let u be the maximal H^1-solution of (4.1) obtained in Theorem 4.16. Take any compact interval $J_0 \subseteq J(u_0)$ containing 0. We have to show that u is an H^2-solution on J_0 with $u \in W^{1,p}(J_0^\circ, L^q)$. By a refinement of the fixed-point argument in the proof of Lemma 4.15, we first prove this claim on an interval $J_1 = [-b_1, b_1]$. It turns out that this time $b_1 > 0$ only depends on α, m, and the size

$$\overline{\rho} \coloneqq \max_{t \in J_0} \|u(t)\|_{1,2}$$

of u in H^1. We can thus repeat the argument for the initial values $u(\pm b_1)$ with the same time step b_1 and deduce the assertion in finitely many iterations. Throughout we use the setting and the notation of the proof of Lemma 4.15 and Theorem 4.16, e.g., the spaces $\mathcal{F}_k(b)$ and their norm $|||v|||_{k,b}$.

1) We fix $\overline{r} \coloneqq 1 + C_{\mathrm{St}}\overline{\rho}$. Lemma 4.15 and its proof say that the H^1-solution u is a fixed point of the operator Φ (see (4.19)) in the set $\Sigma(b,\overline{r}) = \overline{B}_{\mathcal{F}_1(b)}(0,\overline{r})$, where $0 < b \le b_0(\overline{\rho})$, see (4.22). Set $J = (-b,b)$. We will show that u is also a fixed point in a subset of more regular functions, using that $u_0 \in H^2$. To this aim, for $R \ge C_{\mathrm{St}}\|\Delta u_0\|_2 =: R_0$ (to be fixed below) we define the spaces

$$\mathcal{Z}(b) = \mathcal{F}_1(b) \cap W^{1,p}(J,L^q) \cap W^{1,\infty}(J,L^2),$$
$$\Theta = \Theta(b,R) = \big\{ v \in \mathcal{Z}(b) \,\big|\, v(0) = u_0,\ |||v|||_{1,b} \le \overline{r},\ |||v'|||_{0,b} \le R \big\}.$$

The set Θ is non-empty since the Strichartz estimate in Theorem 4.10 a) yields $|||T(\cdot)u_0|||_{1,b} \le C_{\mathrm{St}}\overline{\rho} \le \overline{r}$ and $|||\frac{\mathrm{d}}{\mathrm{d}t}T(\cdot)u_0|||_{0,b} = |||T(\cdot)\Delta u_0|||_{0,b} \le C_{\mathrm{St}}\|\Delta u_0\|_2 \le R$. We endow $\Theta(b,R)$ with the metric given by $|||v-w|||_{0,b}$.

We recall from Remark 4.21 that $W^{1,\infty}(J,L^2)$ is isometrically isomorphic to the space of Lipschitz functions $f:\overline{J}\to L^2$. We first claim that $\Theta(b,R)$ is complete. In fact, take a Cauchy sequence $(v_n)_n$ in $\Theta(b,R)$. Lemma 4.12 yields that $(v_n)_n$ tends in $\mathcal{F}_0(b)$ to a function $v\in\mathcal{F}_1(b)$ with $|||v|||_{1,b}\le\overline{r}$ as $n\to\infty$. Since the maps $v_n:J\to L^2$ converge in $L^\infty(J,L^2)$ to v and have the uniform Lipschitz bound R, we conclude that $v_n\to v$ in $C(\overline{J},L^2)$ as $n\to\infty$, $v(0)=u_0$, and $v:\overline{J}\to L^2$ is Lipschitz with bound R. By Remark 4.21, the limit v belongs to $W^{1,\infty}(J,L^2)$ with $\|v'\|_{L^\infty(J,L^2)}\le R$. Further, after passing to a subsequence, $(v_{n_j})_j$ has a weak limit w in $W^{1,p}(J,L^q)$ with $\|w'\|_{E_0(J)}\le R$. Since $E_0(J)^\star = L^p(J,L^q)^\star \hookrightarrow W^{1,p}(J,L^q)^\star$ and $\frac{\mathrm{d}}{\mathrm{d}t}: W^{1,p}(J,L^q)\to E_0(J)$ is linear and bounded, v_{n_j} and v'_{n_j} converge weakly in $E_0(J)$ to w and w', respectively. It follows $v=w$, $v\in\mathcal{Z}(b)$, and $|||v'|||_{0,b}\le R$; i.e., $\Theta(b,R)$ is complete.

2) Let $t\in J$ and $v,w\in\Theta(b,R)$ for $R\ge R_0$ and $b\in(0,b_0(\overline{\rho})]$. We rewrite Φ as

$$\Phi(v)(t) = T(t)u_0 + \int_0^t T(t-s)F(v(s))\,\mathrm{d}s = T(t)u_0 + \int_0^t T(s)F(v(t-s))\,\mathrm{d}s.$$

We want to choose R and b so that $\Phi:\Theta(b,R)\to\Theta(b,R)$ becomes a strict contraction. In Lemma 4.15, by (4.21) and (4.22) we have already shown that $\Phi(v)$ belongs to $\mathcal{F}_1(b)\cap C(\overline{J},H^1)$ and satisfies

$$|||\Phi(v)-\Phi(w)|||_{0,b} \le \tfrac{1}{2}|||v-w|||_{0,b} \qquad \text{and} \qquad |||\Phi(v)|||_{1,b}\le\overline{r}. \tag{4.41}$$

3) To treat $\frac{\mathrm{d}}{\mathrm{d}t}\Phi(v)$ in the next step, we first differentiate the convolution term with respect to t. This is done via an approximation argument. Corollary 1.18 and (4.20) show that $F:L^q\to L^{q'}$ is real continuously differentiable with derivative given by $F'(\varphi)\psi = \phi'(\varphi)\psi$ for $\varphi,\psi\in L^q$ and $\phi(z) = -\mathrm{i}\mu|z|^{\alpha-1}z$ for $z\in\mathbb{R}^2$. Moreover,

$$\|F'(\varphi)\psi\|_{q'} \le c_1\|\varphi\|_q^{\alpha-1}\|\psi\|_q. \tag{4.42}$$

Here and below $c_j>0$ is a constant only depending on α and m.

Lemma 4.22 allows us to approximate v in $W^{1,p}(J,L^q)$ by $w_n\in C^1(\overline{J},L^q)$. Passing to a subsequence if necessary, the maps $w_n'(t)$ converge in L^q as $n\to\infty$ and $\|w_n'(t)\|_p\le h(t)$ for all $n\in\mathbb{N}$, a.e. $t\in J$, and a function $h\in L^p(J)\hookrightarrow L^{p'}(J)$, where we note that $p'<2<p$. Lemma 4.22 b) and its proof yield that $w_n(0)=v(0)=u_0$ and the sequence $(w_n)_n$ converges to v in $C(\overline{J},L^q)$. It is thus bounded by a constant $\overline{c}$ in this space. By the properties of F, the functions $F'(w_n(t))w_n'(t)$ tend to $F'(v(t))v'(t)$ in $L^{q'}$ as $n\to\infty$ and satisfy

$$\sup_{n\in\mathbb{N}}\|F'(w_n(t))w_n'(t)\|_{q'} \le \sup_{s\in\overline{J},n\in\mathbb{N}} c_1\|w_n(s)\|_q^{\alpha-1}\|w_n'(t)\|_q \le c_1\overline{c}^{\alpha-1}h(t).$$

From dominated convergence we deduce that the maps $F'(w_n)w_n'$ have the limit to $F'(v)v'$ in $L^{p'}(J,L^{q'})$ as $n\to\infty$.

Because $L^{q'} \hookrightarrow H^{-1}$ by (4.7), the function $F(w_n) : \overline{J} \to H^{-1}$ is continuously differentiable and so the derivative

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t T(s)F(w_n(t-s))\,\mathrm{d}s &= T(t)F(w_n(0)) + \int_0^t T(s)F'(w_n(t-s))w_n'(t-s)\,\mathrm{d}s\\ &= T(t)F(u_0) + \int_0^t T(t-s)F'(w_n(s))w_n'(s)\,\mathrm{d}s\end{aligned}$$

exists in H^{-1}. (In this calculation we identify $\mathbb{C}$ with $\mathbb{R}^2$.) Observe that $F(u_0)$ belongs to L^2 since $u_0 \in H^2$ and $F \in C^1(H^2, L^2)$, see (4.10). The Strichartz estimate in Theorem 4.10 b) thus implies that the right-hand side of the above identity is continuous in L^2 and converges to

$$T(t)F(u_0) + \int_0^t T(t-s)F'(v(s))v'(s)\,\mathrm{d}s$$

in L^2 uniformly in t as $n \to \infty$. Similarly, the integral on the left-hand side tends to $T *_+ F(v)(t)$ in L^2 uniformly in t. We can thus differentiate $T *_+ F(v)$ in L^2 and obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t T(s)F(v(t-s))\,\mathrm{d}s &= T(t)F(u_0) + \int_0^t T(t-s)F'(v(s))v'(s)\,\mathrm{d}s,\\ \frac{\mathrm{d}}{\mathrm{d}t}\Phi(v)(t) &= T(t)(\mathrm{i}\Delta u_0 + F(u_0)) + \int_0^t T(t-s)F'(v(s))v'(s)\,\mathrm{d}s.\end{aligned} \tag{4.43}$$

4) In this step we prove that $\frac{\mathrm{d}}{\mathrm{d}t}\Phi(v)$ is an element of $\mathcal{F}_0(b)$ with $|||\frac{\mathrm{d}}{\mathrm{d}t}\Phi(v)|||_{0,b} \le R$. It is crucial that R will enter only linearly. Using inequality (4.42), Sobolev's embedding (4.4) and $\|v(s)\|_{1,2} \le \overline{r}$, we derive

$$\|F'(v(s))v'(s)\|_{q'} \le c_1 C_{\mathrm{So}}^{\alpha-1}\overline{r}^{\alpha-1}\|v'(s)\|_q$$

for all $s \in J$. The inhomogeneous Strichartz estimate and Hölder's inequality now allow us to bound the $\mathcal{F}_0(b)$-norm of the integral term in (4.43) by

$$\begin{aligned}C_{\mathrm{St}}\|F'(v)v'\|_{E_0'(b)} &\le C_{\mathrm{St}}c_1 C_{\mathrm{So}}^{\alpha-1}\overline{r}^{\alpha-1}\|v'\|_{L^{p'}(J,L^q)} \le c_2\overline{r}^{\alpha-1}b^{\frac{1}{p'}-\frac{1}{p}}\|v'\|_{E_0(b)}\\ &\le c_2\overline{r}^{\alpha-1}b^{\frac{1}{p'}-\frac{1}{p}}R,\end{aligned} \tag{4.44}$$

using $v \in \Theta(R,b)$. By Sobolev's embedding (4.4), we have $H^2 \hookrightarrow L^{2\alpha}$ and hence $\|F(u_0)\|_2 = \|u_0\|_{2\alpha}^\alpha \le c_3\|u_0\|_{2,2}^\alpha$. Equation (4.43), the Strichartz estimates and estimate (4.44) thus yield that

$$|||\tfrac{\mathrm{d}}{\mathrm{d}t}\Phi(v)|||_{0,b} \le C_{\mathrm{St}}(\|\Delta u_0\|_2 + c_3\|u_0\|_{2,2}^\alpha) + c_2\overline{r}^{\alpha-1}b^{\frac{1}{p'}-\frac{1}{p}}R \tag{4.45}$$

and that $\frac{\mathrm{d}}{\mathrm{d}t}\Phi(v)$ belongs to $C(\overline{J}, L^2)$. We fix

$$R_1 = 2C_{\mathrm{St}}(\|\Delta u_0\|_2 + c_3\|u_0\|_{2,2}^\alpha) \ge R_0 \quad \text{and} \quad b_1 = \min\bigl\{b_0(\overline{\rho}), (2c_2\overline{r}^{\alpha-1})^{\frac{p'-p}{p'p}}\bigr\}.$$

Since $\overline{r} = 1 + C_{\mathrm{St}}\overline{\rho}$, the number b_1 only depends on $\overline{\rho}$, α and m. The inequalities (4.41) and (4.45) show that $\Phi : \Theta(b_1, R_1) \to \Theta(b_1, R_1)$ is a strict contraction.

We thus obtain a fixed point $v_* = \Phi(v_*)$ in $\Theta(b_1, R_1)$ contained in $C(J_1, H^1) \cap C^1(J_1, L^2) \cap W^{1,q}(J_1^\circ, L^q)$, where $J_1 = [-b_1, b_1]$. The function v_* is an H^1-solution of (4.1) on J_1 by Lemma 2.8 of [**32**]. Hence, $u = v_*$ by the uniqueness of (4.1).

5) We still have to show that $u \in C(J_1, H^2)$. To prove this fact, we use a 'boot-strapping' argument based on the invertibility of $I - \Delta : W^{2,r} \to L^r$ for all $r \in (1, \infty)$, see Example 2.30 in [**32**]. If $\varphi \in C_c^\infty$, then there is a unique solution $v \in \bigcap_r W^{2,r}$ of $v - \Delta v = \varphi$, see (1.25) in [**32**]. By density, we see tha the inverses $(I - \Delta)^{-1}$ coincide on $L^r \cap L^s$ for $r, s \in (1, \infty)$.

As a starting point, we rewrite (4.1) as

$$u - \Delta u = u + \mathrm{i}u' - \mathrm{i}F(u) = f + g,$$

where $f \coloneqq u + \mathrm{i}u'$ belongs to $C(J_1, L^2)$ and $g \coloneqq -\mathrm{i}F(u)$ to $C(J_1, L^{q/\alpha})$ since $u \in C^1(J_1, L^2(\mathbb{R}^d)) \cap C(J_1, L^q)$. We have $\frac{q}{\alpha} = q' \in (1,2)$ by (4.20). It follows that $(I - \Delta)^{-1} f \in C(J_1, H^2)$ and $(I - \Delta)^{-1} g \in C(J_1, W^{2,q/\alpha})$. Because of $2 - \frac{m}{2} > 2 - \frac{m\alpha}{q}$, Sobolev's embedding Theorem 3.31 of [**33**] yields

$$u = (I - \Delta)^{-1}(f + g) \in C(J_1, L^{r_1})$$

for $r_1 = \frac{m}{m\alpha - 2q} q =: \gamma q > q$ if $m\alpha > 2q$ and for any $r_1 \in (2, \infty)$ otherwise (e.g. if $m \in \{1, 2\}$). Observe that $\gamma > 1$ if $m\alpha > 2q$ since $q = \alpha + 1$ and $\alpha < \frac{m+2}{m-2}$.

This extra integrability of u implies that g is an element of $C(J_1, L^{r_1/\alpha})$, by (4.9). If $r_1 \geq 2\alpha$, the function g belongs to $C(J_1, L^2)$ since then $L^{q/\alpha} \cap L^{r_1/\alpha} \hookrightarrow L^2$ by Hölder's inequality. As a result, $u = (I - \Delta)^{-1}(f + g)$ is contained in $C(J_1, H^2)$ in this case.

If $r_1 < 2\alpha$, as above we infer that $u \in C(J_1, L^{r_2})$ for $r_2 = \frac{m r_1}{m\alpha - 2r_1} \geq \gamma r_1 = \gamma^2 q$ if $m\alpha > 2r_1$ and for any $r_2 \in (2, \infty)$ if $m\alpha \leq 2r_1$. Since $\gamma > 1$, in finitely many steps we arrive at $r_N \geq \gamma^N q \geq 2\alpha$, and hence $u \in C(J_1, H^2)$.

6) We can now finish the proof. If $J_0 \subseteq J_1$ we are done. If not, assume that $\max J_0 > b_1$. Since $u(b_1) \in H^2$, we can repeat steps 2) – 5) with initial value $u(b_1)$ and the same time step b_1. We then obtain an H^2-solution u_1 of (4.1) on $[b_1, 2b_1]$ with $u_1(b_1) = u(b_1)$. Remark 4.13 allows us to glue together these functions to an H^2-solution v on $[-b_1, 2b_1]$ with $v \in W^{1,p}((-b_1, 2b_1), L^q)$. The uniqueness of H^1-solutions yields that $v = u$ on $[-b_1, 2b_1]$. The assertion now follows in finitely many iterations. $\square$

CHAPTER 5

The semilinear wave equation

In this chapter we treat the semilinear wave equation

$$\partial_t^2 u = \Delta u - \mu|u|^{\alpha-1}u, \qquad u(0) = u_0, \quad \partial_t u(0) = u_1, \qquad x \in \mathbb{R}^m, \;\; t \in J, \tag{5.1}$$

now on $\mathbb{R}^m$, for $\mu \in \{-1, 1\}$, an interval J of positive length containing 0, and initial functions $u_0, u_1 : \mathbb{R}^m \to \mathbb{C}$. We focus on the case $m = 3$ and $\alpha \in [3, 5]$ for conciseness, though several preliminary facts are stated for all $m \in \mathbb{N}$ or for $m \geq 2$. In Section 1.2 the case $\alpha = 3 = m$ was already studied on a bounded domain G with Dirichlet boundary conditions, using only L^2-based estimates and Sobolev embeddings. This approach can be extended to $\alpha \in (1, 3)$ and $\mathbb{R}^3$. However, for $\alpha \in (3, 5]$ and $m = 3$ we again need Strichartz estimates of the inhomogeneous linear problem

$$\partial_t^2 u = \Delta u + f, \qquad u(0) = u_0, \quad \partial_t u(0) = u_1, \qquad x \in \mathbb{R}^m, \quad t \in J, \tag{5.2}$$

for a given forcing $f : J \times \mathbb{R}^m \to \mathbb{C}$. To avoid a case distinction, below we do not consider the much easier range $\alpha \in (1, 3)$.

Let $w = (u, \partial_t u)$. As in Theorem 1.20, one can show the energy equality

$$\mathcal{E}_w(w(t)) := \int_{\mathbb{R}^m} \Big(\frac{1}{2}|\partial_t u(t)|^2 + \frac{1}{2}|\nabla u(t)|^2 + \frac{\mu}{1+\alpha}|u|^{\alpha+1}\Big) \mathrm{d}x = \mathcal{E}_w(w(0)) \tag{5.3}$$

at least for regular solutions. In contrast to $W_0^{1,2}(G)$ in Section 1.2, now the energy does not control the norm of $L^2(\mathbb{R}^m)$ which is an obstacle for the investigation of the long-term behavior. To deal with this difficulty, we introduce a new class of Sobolev spaces below. Semigroup methods do not work well in the resulting setting, but we can replace them by the Fourier transform. Anyway, the latter is the most natural tool to solve a linear PDE on $\mathbb{R}^m$ with constant coefficients like (5.2). The needed concepts and results are discussed in the next preparatory section, where we take $\mathbb{F} = \mathbb{C}$. We still write L^q for $L^q(\mathbb{R}^m)$ etc.

5.1. Fourier transform and fractional Sobolev spaces

The approach of this section is based on *Schwartz space*

$$\mathcal{S} = \mathcal{S}_m = \big\{ v \in C^\infty(\mathbb{R}^m) \,\big|\, \forall\, k \in \mathbb{N}_0,\, \alpha \in \mathbb{N}_0^m : p_{k,\alpha}(v) := \||x|^k \partial^\alpha v\|_\infty < \infty \big\}$$

($|x|$ stands for the map $x \mapsto |x|$ etc.). A sequence (v_n) *converges* to v in $\mathcal{S}$ if $p_{k,\alpha}(v_n - v) \to 0$ as $n \to \infty$ for all $k \in \mathbb{N}_0$ and $\alpha \in \mathbb{N}_0^m$. This limit concept can be expressed by a complete metric. (See Section 3.1 in [**33**] for proofs omitted here and more information.) For further definitions, let $a > 0$, $x, y, \xi \in \mathbb{R}^m$,

$e_{\mathrm{i}y}(x) = \mathrm{e}^{\mathrm{i}y\cdot x}$, and $v, w \in \mathcal{S}$. We define translations, dilations, reflection, Fourier transform, and convolution on $\mathcal{S}$ by

$$\tau_y v(x) = v(x+y), \quad \sigma_a v(x) = v(ax), \quad Rv(x) = v(-x),$$
$$\mathcal{F}v(\xi) = \hat{v}(\xi) = (2\pi)^{-\frac{m}{2}} \int_{\mathbb{R}^m} \mathrm{e}^{-\mathrm{i}\xi\cdot x} v(x)\,\mathrm{d}x, \quad (v * w)(x) = \int_{\mathbb{R}^m} v(x-y)w(y)\,\mathrm{d}y.$$

Extensions of these operators are denoted by the same symbols. We collect their basic properties.

Remark 5.1. Under the avove conditions, the maps $\tau_y, \sigma_a, R : \mathcal{S} \to \mathcal{S}$ and $* : \mathcal{S} \times \mathcal{S} \to \mathcal{S}$ are continuous. The same is true for the derivative $\partial^\alpha : \mathcal{S} \to \mathcal{S}$ and the multiplication $M_g : \mathcal{S} \to \mathcal{S}$; $v \mapsto gv$, if g belongs to

$$\mathcal{E} = \mathcal{E}_m := \big\{ h \in C^\infty(\mathbb{R}^m) \,\big|\, \forall\, \alpha \in \mathbb{N}_0^m \;\exists\, n_\alpha \in \mathbb{N}_0 : \sup_{|x|\ge 1} \big||x|^{-n_\alpha} h\big| < \infty \big\}$$

Moreover, the Fourier transform $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ is a homeomorphism satisfying

$$\mathcal{F}^{-1} = R\mathcal{F} = \mathcal{F}^3, \quad R = \mathcal{F}^2, \qquad \int v\hat{w}\,\mathrm{d}x = \int \hat{v}w\,\mathrm{d}x,$$
$$\mathcal{F}(\tau_y v) = e_{\mathrm{i}y}\hat{v}, \quad \mathcal{F}(e_{\mathrm{i}y}v) = \tau_{-y}\hat{v}, \qquad \mathcal{F}(\sigma_a v) = a^{-m}\sigma_{1/a}\hat{v},$$
$$\partial^\alpha \hat{v} = (-\mathrm{i})^{|\alpha|}\mathcal{F}(x^\alpha v), \qquad \mathcal{F}(\partial^\alpha v) = \mathrm{i}^{|\alpha|}\xi^\alpha \hat{v},$$
$$\mathcal{F}(v * w) = (2\pi)^{\frac{m}{2}}\,\hat{v}\hat{w}, \quad \mathcal{F}(vw) = (2\pi)^{-\frac{m}{2}}\,\hat{v} * \hat{w}.$$

Finally, $\mathcal{F}$ extends to a unitary operator on L^2 (Plancherel's theorem). $\Diamond$

As in previous chapters we also need Sobolev spaces of negative order. So far they were defined via duality. On $\mathbb{R}^m$ one can introduce such spaces also by means of the Fourier transform and (if $p = 2$) Plancherel in a convenient way. This is described now, even with a regularity parameter $s \in \mathbb{R}$. We only treat L^2-based spaces; see [**4**], [**12**] or [**38**] for the case $p \neq 2$.

These Sobolev spaces are contained in the space of *tempered distributions*

$$\mathcal{S}^\star = \mathcal{S}_m^\star := \big\{ u : \mathcal{S}_m \to \mathbb{C} \,\big|\, u \text{ linear, continuous} \big\}.$$

We write $\langle v, \varphi\rangle_{\mathcal{S}} = \varphi(v)$ for $\varphi \in \mathcal{S}^\star$ and $v \in \mathcal{S}$. It is clear that $\mathcal{S}^\star$ is a vector space. It is equipped with the weak* convergence $\varphi_n \to \varphi$, meaning that $\varphi_n(v) \to \varphi(v)$ as $n \to \infty$ for each $v \in \mathcal{S}$. (For missing proofs and more information concerning $\mathcal{S}^\star$, we refer to Section 3.6 in [**33**].)

We mention two (simple) types of elements of $\mathcal{S}$. First, *regular* tempered distributions are given by $\varphi_g : \mathcal{S} \to \mathbb{C}$; $\varphi_g(v) = \int gv\,\mathrm{d}x$, for a function g satisfying $\int_{j\le|x|\le j+1} |g|\,\mathrm{d}x \le cj^\kappa$ for some constants $c, \kappa \ge 0$ and all $j \in \mathbb{N}_0$. For instance, g could be polynomially bounded. One usually writes g instead of φ_g. Second, the *Dirac* distributions $\delta_y^\alpha : v \mapsto \partial^\alpha v(y)$ also belong to $\mathcal{S}^\star$ for every $\alpha \in \mathbb{N}_0^m$ and $y \in \mathbb{R}^m$, where we set $\delta_y = \delta_y^{(0)}$.

To extend the operators from Remark 5.1 to $\mathcal{S}^\star$, let $a > 0$, $\alpha \in \mathbb{N}_0^m$ $x \in \mathbb{R}^m$, $g \in \mathcal{E}$, $\varphi \in \mathcal{S}^\star$, and $v \in \mathcal{S}$. We set

$$M_g\varphi(v) = (g\varphi)(v) = \varphi(gv), \quad \partial^\alpha\varphi(v) = (-1)^{|\alpha|}\varphi(\partial^\alpha v), \quad \sigma_a\varphi(v) = a^{-m}\varphi(\sigma_{\frac{1}{a}} v),$$
$$(\mathcal{F}\varphi)(v) = \hat{\varphi}(v) = \varphi(\hat{v}), \quad R\varphi(v) = \varphi(Rv), \quad (v * \varphi)(x) = \varphi(\tau_{-x}Rv).$$

Note that $\tau_{-x}Rv(y) = v(x-y)$. These maps behave on $\mathcal{S}^\star$ similar as in $\mathcal{S}$.

Remark 5.2. In the above setting, the operators $M_g, \partial^\alpha, \sigma_a, R : \mathcal{S}^\star \to \mathcal{S}^\star$ and $* : \mathcal{S} \times \mathcal{S}^\star \to \mathcal{S}^\star$ are continuous with $v * \varphi \in \mathcal{E}$. Also, the Fourier transform $\mathcal{F} : \mathcal{S}^\star \to \mathcal{S}^\star$ is a homeomorphism satisfying

$$\mathcal{F}^{-1} = R\mathcal{F} = \mathcal{F}^3, \quad R = \mathcal{F}^2, \quad \mathcal{F}\delta_y = (2\pi)^{-\frac{m}{2}} e_{-\mathrm{i}y}, \qquad \mathcal{F}e_{\mathrm{i}y} = (2\pi)^{\frac{m}{2}}\delta_y,$$
$$\mathcal{F}(\sigma_a\varphi) = a^{-m}\sigma_{1/a}\hat{\varphi}, \quad \partial^\alpha\hat{\varphi} = (-\mathrm{i})^{|\alpha|}\mathcal{F}(x^\alpha v), \quad \mathcal{F}(\partial^\alpha\varphi) = \mathrm{i}^{|\alpha|}\xi^\alpha\hat{\varphi},$$
$$\partial^\alpha(v*\varphi) = \partial^\alpha v*\varphi = v*\partial^\alpha\varphi, \quad \mathcal{F}(v*\varphi) = (2\pi)^{\frac{m}{2}}\hat{v}\hat{\varphi}, \quad \mathcal{F}(v\varphi) = (2\pi)^{-\frac{m}{2}}\hat{v}*\hat{\varphi}. \ \Diamond$$

As weight function on $\mathbb{R}^m$ we will use powers of $\langle\xi\rangle = (1+|\xi|_2^2)^{\frac{1}{2}}$. For $s \in \mathbb{R}$ we define the *fractional Sobolev spaces*

$$H^s = H^s(\mathbb{R}^m) = \big\{v \in \mathcal{S}_m^\star \,\big|\, \langle\xi\rangle^s\hat{v} \in L^2(\mathbb{R}^m)\big\}, \quad \|v\|_{H^s} = \|v\|_{s,2} = \|\langle\xi\rangle^s\hat{v}\|_{L^2}.$$

Note that $\hat{v}$ belongs to $L^2_{\mathrm{loc}}(\mathbb{R}^m)$ if $v \in H^s$ and that $\mathcal{F} : H^s \to L^2(\langle\xi\rangle^s\lambda) =: L^2_s$ is an isometric isomorphism. Hence, H^s is a Hilbert space for the scalar product

$$(v|w)_{H^s} = (v|w)_s = \int_{\mathbb{R}^m} \langle\xi\rangle^{2s}\hat{v}\overline{\hat{w}}\,\mathrm{d}\xi.$$

It is clear that $H^t \hookrightarrow H^s$ for $t \geq s$ and that $H^0 = L^2$ by Plancherel. Moreover, we have $H^k = W^{k,2}$ for $k \in \mathbb{N}$ by Theorem 3.25 in [**33**]. We show important properties of these spaces.

Proposition 5.3. *Let $s \in \mathbb{R}$. Then $\mathcal{S}$ is dense in H^s. Moreover, $(H^s)^\star$ is isomorphic to H^{-s} via the map $\ell \mapsto \varphi$ given by $\ell(v) = \int \varphi v\,\mathrm{d}x$ for $v \in H^s$.*

Proof. 1) Take $\varphi \in H^s$ with $0 = (v|\varphi)_s = \int \langle\xi\rangle^{2s}\hat{v}\overline{\hat{\varphi}}\,\mathrm{d}\xi$ for all $v \in \mathcal{S}$. Let $\chi \in C_c^\infty$. Choosing $v := \mathcal{F}^{-1}(\langle\xi\rangle^{-2s}\chi) \in \mathcal{S}$, we see that $\int \chi\overline{\hat{\varphi}}\,\mathrm{d}\xi = 0$, and hence $\hat{\varphi} = 0$ by Lemma 4.15 in [**30**]. Since then $\varphi = 0$, the density of $\mathcal{S}$ in H^s follows from the Projection Theorem 3.8 of [**30**].

2) For $v, w \in \mathcal{S}$, Remark 5.1 and Hölder' inequality yield

$$\Big|\int vw\,\mathrm{d}x\Big| = \Big|\int v\mathcal{F}\mathcal{F}^{-1}w\,\mathrm{d}\xi\Big| = \Big|\int \hat{v}\,R\hat{w}\,\mathrm{d}\xi\Big| = \Big|\int \langle\xi\rangle^s\hat{v}(\xi)\,\langle-\xi\rangle^{-s}\hat{w}(-\xi)\,\mathrm{d}\xi\Big|$$
$$\leq \|v\|_{H^s}\|w\|_{H^{-s}}.$$

By step 1) we can extend $(v,w) \to \int vw\,\mathrm{d}x$ to a bilinear and contractive map from $H^s \times H^{-s}$ to $\mathbb{C}$ so that $H^{-s} \hookrightarrow (H^s)^\star$.

Conversely, let $\ell \in (H^s)^\star$. To reduce to L^2, we define $\tilde{\ell} : L^2 \to \mathbb{C}$ via $\tilde{\ell}(v) = \ell\big(\mathcal{F}^{-1}(\langle\xi\rangle^{-s}v)\big)$. Since $\mathcal{F}^{-1}\langle\xi\rangle^{-s} : L^2 \to H^s$ is an isometric isomorphism, the norm of $\tilde{\ell}$ in $(L^2)^\star$ is equal to $\|\ell\|_{(H^s)^\star}$. Riesz' Theorem 3.10 in [**30**] then provides a function $g \in L^2$ with $\|g\|_2 = \|\tilde{\ell}\|_{(L^2)^\star}$ and $\tilde{\ell}(v) = \int v\overline{g}\,\mathrm{d}x$ for all $v \in L^2$. Observe that $\varphi := \mathcal{F}(\langle\xi\rangle^s\overline{g}) = \mathcal{F}^{-1}R(\langle\xi\rangle^s\overline{g})$ belongs to H^{-s} with norm $\|g\|_2 = \|\ell\|_{(H^s)^\star}$. Computing

$$\langle w, \varphi\rangle_{\mathcal{S}} = \langle\hat{w}, \langle\xi\rangle^s\overline{g}\rangle_{\mathcal{S}} = \int \langle\xi\rangle^s\hat{w}\overline{g}\,\mathrm{d}\xi = \tilde{\ell}(\langle\xi\rangle^s\hat{w}) = \ell(w)$$

for every $w \in \mathcal{S}$, we derive $\ell = \varphi$ and the second assertion. $\square$

As indicated above, we need a variant of H^s that does not contain L^2 for $s > 0$. For $s \in \mathbb{R}$, we thus define the *homogeneous fractional Sobolev space*

$$\dot H^s = \dot H^s(\mathbb{R}^m) = \big\{ v \in \mathcal{S}_m^\star \,\big|\, \hat v \in L^1_{\mathrm{loc}}(\mathbb{R}^m),\ |\xi|^s \hat v \in L^2(\mathbb{R}^m)\big\}, \quad \|v\|_{\dot H^s} = \||\xi|^s \hat v\|_{L^2}.$$

Clearly, $\dot H^s$ is a normed vector space. Observe that we have $\hat v \in L^2_{\mathrm{loc}}$ for $s \le 0$ if the above norm is finite, but not for $s > 0$. In the latter case $\hat v$ may develop singularities at $\xi = 0$ if $v \in \dot H^s$; which are restricted by the extra condition in the definition, though. For v this leads to reduced decay at infinity, as quantified in Remark 5.4 f). (Also compare $\mathcal{F}\mathbb{1} = (2\pi)^{\frac{m}{2}} \delta_0$.) To the contrary, for $s < 0$ the Fourier transform $\hat v$ has to be small near $\xi = 0$ providing spatial decay.

We have $\dot H^0 = L^2$ by Plancherel, as well as $L^2 \cap \dot H^s = H^s$ (hence $H^s \hookrightarrow \dot H^s$) if $s \ge 0$ since here $\langle \xi \rangle^s \eqsim \max\{1, |\xi|^s\}$. It can be seen that there is no inclusion between $\dot H^s$ and $\dot H^t$ if $s \neq t$. (There are radial functions $u(x) = \phi(|x|)$ such that $\psi_s(r) := r^{2s+m-1}|\hat\phi(r)|^2$ is integrable on $\mathbb{R}_+$, but not ψ_t near ∞ (if $t > s$) or near 0 (if $t < s$).) We again collect basic properties, see Section 1.3 of [4] for missing details and more information. We set $\dot L^2_s = L^2(|\xi|^s \lambda)$.

Remark 5.4. a) For $s_0 < s < s_1$, Hölder's inequality yields $\dot H^{s_0} \cap \dot H^{s_1} \hookrightarrow \dot H^s$.

b) Let $s < \frac{m}{2}$. Then $\mathcal{F} : \dot H^s \to \dot L^2_s$ is isometric and surjective. Indeed, take $\psi \in \dot L^2_s$. Since $|\xi|^{-s} \in L^2(B(0,1))$ by assumption, $\psi = |\xi|^{-s}|\xi|^s \psi$ belongs to L^1_{loc} and thus to $\mathcal{S}^\star$. So we can set $\varphi = \mathcal{F}^{-1}\psi \in \dot H^s$ which shows surjectivity. Isometry is clear by definition of $\dot H^s$.

c) Part b) implies that $\dot H^s$ is a Hilbert space if $s < \frac{m}{2}$. (Conversely, if $\dot H^s$ is complete, one can deduce $s < \frac{m}{2}$.)

d) Using claim c), like in Proposition 5.3 one can show that $\mathcal{S}_0 = \{v \in \mathcal{S} \,|\, \hat v = 0 \text{ near } 0\}$ is dense in $\dot H^s$ if $s < \frac{m}{2}$. For $s > -\frac{m}{2}$, also $\mathcal{S}$ belongs to $\dot H^s$ since then $|\xi|^s$ is contained in L^2_{loc}. Hence, $\mathcal{S}$ is dense in $\dot H^s$ if $|s| < \frac{m}{2}$.

e) Let $|s| < \frac{m}{2}$. Then $\dot H^{-s}$ is isomorphic to $(\dot H^s)^\star$ as in Proposition 5.3.

f) Let $s \in [0, \frac{m}{2})$. Then $\dot H^s$ is embedded into L^p for $p = \frac{2m}{m-2s}$; i.e., $s - \frac{m}{2} = -\frac{m}{p}$. An example for this sharp Sobolev embedding is $\dot H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$. (By Remark 3.30 in[**33**], $\dot H^1$ does not embed into L^q for $q \in [2, \frac{2m}{m-2})$ and $m \ge 3$.)

Proof. Let $v \in \mathcal{S}_0$. Set $w = \mathcal{F}^{-1}(|\xi|^s \hat v)$. Proposition 1.29 in [4] implies $v = \mathcal{F}^{-1}(|\xi|^{-s}\hat w) = c(s,m)|\xi|^{s-m} * w$. Using Lemma 4.9 with $1 + \frac{1}{p} = \frac{m-s}{m} + \frac{1}{2}$, we deduce $\|v\|_p \le c\|w\|_2 = c\|v\|_{\dot H^s}$. The claim now follows by property d). □

Below we (mostly) restrict to the Hilbert space range $s < \frac{m}{2}$ when using $\dot H^s$. There are L^p-variants $H^{s,p}$ and $\dot H^{s,p}$ of these spaces, often called Bessel potential spaces, see [4], [**12**], [**38**] or [**39**]. The definitions of the homogeneous spaces differ in the literature, at least for large s.

In this chapter we heavily use *Fourier multipliers* $a(D)$ with symbol a. Let $a : \mathbb{R}^m \to \mathbb{C}$ be measurable and bounded by $|a| \le c\langle \xi \rangle^\alpha$ for some $\alpha \in \mathbb{R}$. Then $a(D)v := \mathcal{F}^{-1}(a\hat v)$ maps $\mathcal{S}$ into $\mathcal{S}^\star$ and satisfies

$$\|a(D)v\|^2_{H^{s-\alpha}} = \int \langle \xi \rangle^{2s-2\alpha}|a|^2|\hat v|^2 \,\mathrm{d}\xi \le c\int \langle \xi \rangle^{2s}|\hat v|^2 \,\mathrm{d}\xi = c\|v\|^2_{H^s}$$

for $v \in H^s$ and $s \in \mathbb{R}$. Hence, $a(D) : H^s \to H^{s-\alpha}$ is bounded. Observe the special case of bounded a where $\alpha = 0$. A core example is $a(\xi) = \xi^\alpha$ yielding $a(D) = (-\mathrm{i})^{|\alpha|}\partial^\alpha$ by Remark 5.1. In homogeneous spaces, for $\alpha > 0$ the product $|\xi|^{-2\alpha}\langle\xi\rangle^{2\alpha}$ is not bounded anymore. But for $\alpha \le 0$ and $s - \alpha < \frac{m}{2}$, one obtains as above the boundedness of $a(D) : \dot H^s \to \dot H^{s-\alpha}$. These operators commute.

For $a(\xi) = \langle\xi\rangle^\alpha$ and $\alpha \in \mathbb{R}$, we set $a(D) = \langle\nabla\rangle^\alpha$. This map is an isometric isomorphism from H^s to $H^{s-\alpha}$ for each $s \in \mathbb{R}$ with inverse $\langle\nabla\rangle^{-\alpha}$. The symbol $a(\xi) = |\xi|^\alpha$ is better suited to $\dot H^s$. We write here $a(D) = |\nabla|^\alpha$. Let $\alpha \ge 0$. Then $|\nabla|^\alpha$ belongs to $\mathcal{B}(H^s, H^{s-\alpha})$ by the above observations. Moreover, with $\hat v$ also $|\xi|^\alpha \hat v$ is locally integrable, and we can then check that $|\nabla|^\alpha : \dot H^s \to \dot H^{s-\alpha}$ is isometric. On the other hand, $|\xi|^{-\alpha}\hat v = |\xi|^{-\alpha-s}|\xi|^s\hat v$ is locally integrable if $v \in \dot H^s$ and $\alpha + s < \frac{m}{2}$. In this case $|\nabla|^{-\alpha} : \dot H^s \to \dot H^{s+\alpha}$ is the bounded inverse of $|\nabla|^\alpha : \dot H^{s+\alpha} \to \dot H^s$. We will use $|\nabla|^{-1} : L^2(\mathbb{R}^3) \to \dot H^1(\mathbb{R}^3)$ with inverse $|\nabla|$.[1]

In contrast to $\langle\nabla\rangle^\alpha$, the operator $|\nabla|^\alpha$ has good scaling properties. Let $a > 0$, $\alpha < \frac{m}{2}$ and $v \in \dot H^\alpha$. As expected for $\alpha \in \mathbb{N}$, Remark 5.2 yields

$$|\nabla|^\alpha(\sigma_a v) = a^{-m}\mathcal{F}^{-1}(|\xi|^\alpha \sigma_{\frac{1}{a}}\hat v) = a^{-m}\mathcal{F}^{-1}\big(\sigma_{\frac{1}{a}}(a^\alpha|\xi|^\alpha \hat v)\big) = a^\alpha \sigma_a|\nabla|^\alpha v. \tag{5.4}$$

We can now solve the linear problem (5.2), using the following concepts and the bounded operators

$$C(t) = \mathcal{F}^{-1}\cos(t|\xi|)\mathcal{F}, \quad S(t) = \mathcal{F}^{-1}\sin(t|\xi|)\mathcal{F}, \quad Sc(t) = \mathcal{F}^{-1}\operatorname{sinc}(t|\xi|)\mathcal{F}$$

on H^s or $\dot H^s$. (The map $\operatorname{sinc}(r) = \frac{1}{r}\sin r$ is bounded by 1.)

Definition 5.5. *Let $s \in \mathbb{R}$, $u_0 \in H^s$, $u_1 \in H^{s-1}$, and $f \in L^1(J, H^{s-1})$. An* H^s-solution *of (5.2) on J is a map u in $C(J,H^s) \cap C^1(J,H^{s-1}) \cap C^2(J,H^{s-2})$ satisfying (5.2).*

Let $s < \frac{m}{2}$, $u_0 \in \dot H^s$, $u_1 \in \dot H^{s-1}$, and $f \in L^1(J, \dot H^{s-1})$. We say that $u \in C(J, \dot H^s)$ is a $\dot H^s$-solution *of (5.2) on J if it fulfills (5.2) in $\dot H^{s-1} + \dot H^{s-2} \subseteq \mathcal{S}^\star$ and has derivatives $\partial_t u \in C(J, \dot H^{s-1})$ and $\partial_t^2 u \in C(J, \dot H^{s-1} + \dot H^{s-2})$.*

We refer to Section 2.2 E) in [**30**] for the sum space. For $\dot H^s$-solutions, the time derivatives are defined in $\mathcal{S}^\star$, see also the next proof.

Proposition 5.6. *Let $s \in \mathbb{R}$, $u_0 \in H^s$, $u_1 \in H^{s-1}$, and $f \in L^1(J, H^{s-1})$. Then the unique H^s-solution u of (5.2) is given by*

$$u(t) = C(t)u_0 + tSc(t)u_1 + \int_0^t (t-\tau)Sc(t-\tau)f(\tau)\,\mathrm{d}\tau, \qquad t \in J. \tag{5.5}$$

We further have

$$\begin{aligned} |\nabla|u(t) &= C(t)|\nabla|u_0 + S(t)u_1 + \int_0^t S(t-\tau)f(\tau)\,\mathrm{d}\tau, \qquad t \in J,\\ \partial_t u(t) &= -S(t)|\nabla|u_0 + C(t)u_1 + \int_0^t C(t-\tau)f(\tau)\,\mathrm{d}\tau, \qquad t \in J. \end{aligned} \tag{5.6}$$

If $s < \frac{m}{2}$ one can replace H^r by $\dot H^r$ throughout.

[1] The L^p-theory of Fourier multipliers is more difficult and treated in harmonic analysis.

Proof. 1) Let u be an H^s-solution and $t \in J$. Since $\mathcal{F} \in \mathcal{B}(H^r, L^2_r)$, we obtain $\mathcal{F}\partial_t v = \partial_t \mathcal{F} v$ for $v \in C^1(J, H^r)$. Setting $\hat{u}(t,\xi) = (\mathcal{F}u(t))(\xi)$, we infer from (5.2) the ordinary differential equation

$$\partial_t^2 \hat{u}(t,\xi) + |\xi|^2 \hat{u}(t,\xi) = \hat{f}(t,\xi), \qquad \hat{u}(0,\xi) = \widehat{u_0}(\xi), \quad \partial_t \hat{u}(0,\xi) = \widehat{u_1}(\xi),$$

for (a.e.) fixed $\xi \in \mathbb{R}^m$. (The equation actually holds in L^2_{s-2}.) We thus obtain

$$\hat{u}(t,\xi) = \cos(t|\xi|)\widehat{u_0}(\xi) + \sin(t|\xi|)\tfrac{1}{|\xi|}\widehat{u_1}(\xi) + \int_0^t \sin((t-\tau)|\xi|)\tfrac{1}{|\xi|}\hat{f}(\tau,\xi)\,\mathrm{d}\tau. \quad (5.7)$$

Applying $\mathcal{F}^{-1} \in \mathcal{B}(L^2_{s-1}, H^{s-1})$, we derive (5.5).

Conversely, we can differentiate (5.7) in t and the derivative has the majorant

$$\langle\xi\rangle^{s-1}\Big(|\xi||\widehat{u_0}(\xi)| + |\widehat{u_1}(\xi)| + \int_J |\hat{f}(\tau,\xi)|\,\mathrm{d}\tau\Big)$$

in L^2. (The weighted L^2-norm can be taken in the time integral.) By Lebesgue the t-derivative exists in L^2_{s-1} (and $\partial_t^2\hat{u}$ in L^2_{s-2}). Similarly one sees that the derivatives are continuous in t. Applying again $\mathcal{F}^{-1}$, we infer that u from (5.5) satisfies (5.6) and that $\partial_t u$ and $\partial_t^2 u$ are continuous H^{s-1}, repectively H^{s-2}. Since $|\nabla|^2 = \mathcal{F}^{-1}|\xi|^2\mathcal{F} = -\Delta$, we see that u satisfies (5.2). Finally, the first line of (5.6) shows that besides u also $|\nabla|u$ is continuous in H^{s-1}, i.e., $u \in C(J, H^s)$.

2)[2] One proceeds similarly for $\dot{H}^s$ if $s < \frac{m}{2}$, so that we only highlight the differences. To deduce (5.7), we now take the Fourier transform in $\mathcal{S}^\star$. If one differentiates this formula in t, it is enough to look at the symbol $\mathrm{e}^{\mathrm{i}t|\xi|}$ instead of sin and cos, see (5.12). For $g_h(\xi) = h^{-1}(\mathrm{e}^{\mathrm{i}(t+h)|\xi|}\hat{\varphi}(\xi) - \mathrm{e}^{\mathrm{i}t|\xi|})\hat{\varphi}(\xi) - \mathrm{i}|\xi|\mathrm{e}^{\mathrm{i}t|\xi|}\hat{\varphi}(\xi)$ with $h \neq 0$ in $\dot{L}^2_{s-1}$, we use the majorant

$$|\xi|^{s-1}\big(|h|^{-1}(\mathrm{e}^{\mathrm{i}h|\xi|}-1) - \mathrm{i}|\xi|\big)|\hat{\varphi}(\xi)| \le \frac{1}{|h|}\int_0^{|h|} |\mathrm{e}^{\mathrm{i}\tau|\xi|}-1|\,\mathrm{d}\tau\,|\xi|^s|\hat{\varphi}(\xi)| \le 2|\xi|^s|\hat{\varphi}(\xi)|.$$

Since $g_h \to 0$ pointwise, dominated convergence yields $g_h \to 0$ in $\dot{L}^2_{s-1}$ as $h \to 0$. Using Plancherel, we can differentiate (5.5) in $\dot{H}^{s-1}$ to obtain the second line in (5.6). The first two terms of this equation can be treated analogously in $\dot{H}^{s-2}$. The difference quotient of integral is decomposed into

$$\frac{1}{h}\int_t^{t+h} C(t+h-\tau)f(\tau)\,\mathrm{d}\tau + \int_0^t \frac{1}{h}(C(t+h-\tau) - C(t,\tau))f(\tau)\,\mathrm{d}\tau$$

(where $h > 0$, say). The first summand tends to $f(t)$ in $\dot{H}^{s-1}$, and the second one to $-\int_0^t S(t-\tau)|\nabla|f(\tau)\,\mathrm{d}\tau$ in $\dot{H}^{s-2}$ (arguing as above). $\square$

Since $|\xi_k| \le |\xi|$, we obtain $\|\partial_k v\|_{H^s} \le \||\nabla|v\|_{H^s}$ and also $\|\partial_k v\|_{\dot{H}^s} \le \||\nabla|v\|_{\dot{H}^s}$. Hence Proposition 5.6 implies the basic linear energy estimate.

Corollary 5.7. *Let $u_0 \in H^1$, $u_1 \in L^2$, and $f \in L^1(J, L^2)$. Then the H^1-solution u of (5.2) from (5.5) satisfies*

$$\|(u(t), \partial_t u(t))\|_{H^1\times L^2} \le c\big(\|u_0\|_{1,2} + (1+|t|)\|u_1\|_2 + (1+|t|)\|f\|_{L^1_t L^2}\big),$$
$$\||\nabla_{t,x} u(t)|\|_2 \le c\big(\|(|\nabla|u_0, u_1)\|_2 + \|f\|_{L^1_t L^2}\big)$$

[2]This part was not treated in the lectures

for $t \in J$. The second estimate is also true if $u_0 \in \dot H^1$ (and $m \ge 3$).

(Here and below c denotes a m-depending constant.) We stress that only the inequality in $\dot H^1$ is uniform in t. Moreover, the map $(u_0, u_1) \mapsto u$ just keeps the regularity, but in $f \mapsto u$ we gain a derivative and boundedness in time.

5.2. Strichartz estimates

We again use Strichartz estimates to deal with a stronger nonlinearity in the semilinear wave equation (5.1), namely, $\alpha \in (3, 5]$ if $m = 3$. Compared to Section 4.2, in the wave case they contain new features and their proof requires more tools and is more demanding (though the basic strategy stays the same).

Besides the time and space exponents $p, q \in [2, \infty]$ we now also need a regularity parameter $\gamma \in \mathbb{R}$. Such numbers form an *admissible triple* (for $m = 3$ and the wave equation) if

$$\frac{1}{p} + \frac{1}{q} \le \frac{1}{2}, \qquad \frac{1}{p} + \frac{3}{q} = \frac{3}{2} - \gamma, \qquad (p, q, \gamma) \ne (2, \infty, 1). \tag{5.8}$$

We call a triple *sharp* if the first relation in (5.8) is an equality. In this case we obtain $\frac{1}{p} = \frac{\gamma}{2}$, $\frac{1}{q} = \frac{1}{2} - \frac{\gamma}{2}$, and $\gamma \in [0, 1)$. In general one has larger $\gamma \in [\frac{2}{p}, \frac{3}{2}]$. The 'trivial endpoint' $(\infty, 2, 0)$ is the only triple with $\gamma = 0$. In view of (5.10) it corresponds to the energy estimate. Below we use the non-sharp triples $(p_\alpha, 3(\alpha - 1), 1)$ with $\alpha \in [3, 5]$ and $p_\alpha = 2\frac{\alpha-1}{\alpha-3}$, which gives $(4, 12, 1)$ for $\alpha = 5$.

For other[3] $m \ge 2$ (and the wave equation), a triple is called *admissible* if

$$\frac{2}{p} + \frac{m-1}{q} \le \frac{m-1}{2}, \qquad \frac{1}{p} + \frac{m}{q} = \frac{m}{2} - \gamma. \tag{5.9}$$

Note that $\frac{m+1}{p(m-1)} \le \gamma \le \frac{m}{2}$. For $m > 3$, one has the 'critical endpoint' $p = 2$, $q = 2\frac{m-1}{m-3}$ and $\gamma = \frac{m+1}{2(m-1)} < 1$. Compared to Schrödinger pairs in (4.16), the dimension m is replaced by $m-1$ (which are the numbers of non-zero curvatures of the 'light cones' $-\tau = |\xi|^2$, respectively $\pm\tau = |\xi|$, in $\mathbb{R}^{1+m} \setminus \{0\}$. Moreover the somewhat different form of the dispersive estimate for the wave equation (see (III.1.18') in [**34**] and also [**16**]), allows one to use non-sharp triples.

The parameter γ coresponds to a loss in regularity for the map $(u_0, u_1) \mapsto u$ in (5.10) and to a reduced gain of $1-\gamma$ derivatives in the map $f \mapsto u$ compared to the energy estimate from Corollary 5.7. Thus the Strichartz estimates below trade regularity *and* time integrability to improve spatial integrability (and to obtain some decay as $|t| \to \infty$).

Theorem 5.8. *Let (p, q, γ) satisfy (5.9), $m \ge 2$, $u_0 \in \dot H^1$, $u_1 \in L^2$, and $f \in L^1(\mathbb{R}, L^2)$. Then the $\dot H^1$-solution u of (5.2) satisfies*

$$\| |\nabla|^{-\gamma} \nabla_{t,x} u \|_{L^p_{\mathbb{R}} L^q} \le C_{\mathrm{St}} \big(\| (|\nabla| u_0, u_1) \|_{L^2} + \|f\|_{L^1_{\mathbb{R}} L^2} \big) \tag{5.10}$$

for a constant $C_{\mathrm{St}} \ge 1$. If $|\nabla| u_0, u_1 \in \dot H^\gamma$ and $f \in L^1(\mathbb{R}, \dot H^\gamma)$, we obtain

$$\| |\nabla_{t,x} u| \|_{L^p_{\mathbb{R}} L^q} \le C_{\mathrm{St}} \big(\| (|\nabla| u_0, u_1) \|_{\dot H^\gamma} + \|f\|_{L^1_{\mathbb{R}} \dot H^\gamma} \big). \tag{5.11}$$

[3]For $m = 1$ the wave equation has no dispersive behavior since, e.g., $u(t, x) = \frac{1}{2}(u_0(x + t) + u_0(x - t))$ is a solution of (5.2) with $u_1 = 0$ and $f = 0$.

As in the Schrödinger case, such estimates were proven by several authors, starting with Strichartz in 1977 and culminating in the Keel–Tao paper [**16**] which settled the critical endpoint case. In the theorem one can replace the (dual) admissible triple $(1, 2, 0)$ for f by any other one. Moreover, we prove this result only for $m = 3$ for simplification. The general case is shown in [**16**] or in Corollary IV.1.2 of the monograph [**34**]. The estimate (5.10) is used in the local wellposedness theory below, and it has the advantage to require less derivatives. However, its direct proof would involve L^q-based homogeneous Sobolev spaces, which we by-pass via (5.11). We discuss the theorem.

REMARK 5.9. a) For $q = 2$ we obtain $p = \infty$ and $\gamma = 0$ so that (5.10) becomes the energy inequality (in $\dot H^1$) from Corollary 5.7. As we see already there, we need homogeneous spaces in a global estimate for (5.2). We let $q > 2$ below.

b) For $q = \infty$ one has to replace L^∞ by the Besov space $\dot B^0_{\infty,2}$. For simplicity we assume $q < \infty$ below, which entails $\gamma < \frac{m}{2}$ fitting to our approach to $\dot H^s$.

c) Since $\psi_k(\xi) \coloneqq \xi_k/|\xi|$ is 0-homogeneous and smooth on the unit sphere, ψ_k satisfies the 'Mikhlin condition' $\sup_{\xi\neq 0} |\xi|^{|\alpha|}|\partial^\alpha\psi_k| < \infty$ for each $\alpha \in \mathbb{N}_0^m$. Hence, the 'Riesz transform' $\partial_k|\nabla|^{-1}$ is bounded on L^q by Theorem 6.2.7 in [**11**]. It is thus enough to show (5.10) and (5.11) with $|\nabla|$ instead of ∇_x.

d) The equalities in (5.8) and (5.9) are needed for the homogeneous Strichartz estimate. As in the Schrödinger case, this can be seen by a scaling argument. Let u solve (5.2) with $f = 0$. Then also $u_\lambda(t, x) = u(\lambda t, \lambda x)$ is a solution with non-zero initial values $\sigma_\lambda u_0$ and $\lambda\sigma_\lambda u_1$, for $\lambda > 0$. Let (5.11) hold for (p, q, γ) and set $E = L^q(\mathbb{R}, L^q)$. By the transformation rule and (5.4), we conclude

$$\begin{aligned}\lambda^{1-\frac{1}{p}-\frac{m}{q}}\|\nabla_{t,x}u\|_E &= \lambda\|(\nabla_{t,x}u)(\lambda\cdot,\lambda\cdot)\|_E = \|\nabla_{t,x}u_\lambda\|_E\\ &\le C_{\mathrm{St}}\|(|\nabla|^{1+\gamma}\sigma_\lambda u_0, \lambda|\nabla|^\gamma\sigma_\lambda u_1)\|_2 = C_{\mathrm{St}}\lambda^{1+\gamma}\|\sigma_\lambda(|\nabla|^{1+\gamma}u_0, |\nabla|^\gamma u_1)\|_2\\ &= C_{\mathrm{St}}\lambda^{1+\gamma-\frac{m}{2}}\|(|\nabla|^{1+\gamma}u_0, |\nabla|^\gamma u_1)\|_2.\end{aligned}$$

Letting $\lambda \to 0$ and $\lambda \to \infty$, we infer $\frac{1}{p} + \frac{m}{q} = \frac{m}{2} - \gamma$.

e) The inequalities in (5.8) and (5.9) are necesssary for Theorem 5.8 because of Knapp's example, which we present for $m = 3$: Let $\varepsilon \in (0, 1]$, $R_\varepsilon = [1, 2] \times [-\varepsilon, \varepsilon]^2$, and $\varphi = \mathcal{F}^{-1}\mathbb{1}_{R_\varepsilon}$ (which belongs to H^k for all $k \in \mathbb{N}$). The H^2-solution of (5.2) with $u_0 = \varphi$, $u_1 = -\mathrm{i}|\nabla|\varphi$ and $f = 0$ is given by

$$u(t, x) = (2\pi)^{-\frac{3}{2}}\int_{R_\varepsilon} \mathrm{e}^{\mathrm{i}x\cdot\xi}\mathrm{e}^{-\mathrm{i}|\xi|t}\,\mathrm{d}\xi = \mathcal{F}^{-1}(\mathrm{e}^{-\mathrm{i}|\xi|t}\mathbb{1}_{R_\varepsilon}),$$

cf. (5.12). By means of Plancherel, we first estimate (with $|\xi| = |\xi|_2$)

$$\||\nabla|\varphi\|_2^2 = \int_{R_\varepsilon} |\xi|^2\,\mathrm{d}\xi \le 6\lambda(R_\varepsilon) = 24\varepsilon^2.$$

To obtain a lower bound for u, we fix $\kappa = \frac{1}{4}\arccos\frac{1}{2} > 0$ and define

$$S_\varepsilon = \big\{(t, x) \in \mathbb{R}^4 \,\big|\, 2|t| \le \kappa\varepsilon^{-2},\ |x_1 - t| \le \kappa,\ |x_2| + |x_3| \le \kappa\varepsilon^{-1}\big\}.$$

Let $(t, x) \in S_\varepsilon$ and $\xi \in R_\varepsilon$. We aim at the inequality

$$\tfrac{1}{2} \le \operatorname{Re}\mathrm{e}^{\mathrm{i}(x\cdot\xi-|\xi|t)} = \cos\big[(x_1 - t)\xi_1 + x_2\xi_2 + x_3\xi_3 + t\xi_1(1 - |\xi|/\xi_1)\big].$$

This lower bound is true since the definitions of S_ε and R_ε imply

$$|[\dots]| \le 2\kappa + \frac{\kappa}{\varepsilon}\varepsilon + \frac{\kappa}{2\varepsilon^2}2\Big(\sqrt{1+(\xi_2^2+\xi_3^2)\xi_1^{-2}}-1\Big) \le 4\kappa$$

by a standard estimate for the square root. We infer

$$\begin{aligned}\||\nabla|^{1-\gamma}u\|_E &= \|\mathcal{F}^{-1}(|\xi|^{1-\gamma}\mathrm{e}^{-\mathrm{i}|\xi|t}\mathbb{1}_{R_\varepsilon})\|_E = c\Big\|\Big|\int_{R_\varepsilon}|\xi|^{1-\gamma}\mathrm{e}^{\mathrm{i}(x\cdot\xi-|\xi|t)}\,\mathrm{d}\xi\Big|\Big\|_{L_t^pL_x^q}\\ &\ge c\Big\|\int_{R_\varepsilon}\operatorname{Re}\mathrm{e}^{\mathrm{i}(x\cdot\xi-|\xi|t)}\,\mathrm{d}\xi\Big\|_{L_t^pL_x^q(S_\varepsilon)}\\ &\ge \tfrac{c}{2}\lambda(R_\varepsilon)\|\mathbb{1}_{S_\varepsilon}\|_{L_\mathbb{R}^pL^q} = c\varepsilon^2\varepsilon^{-\frac{2}{q}}\varepsilon^{-\frac{2}{p}}.\end{aligned}$$

If (5.10) holds, it implies $\||\nabla|^{1-\gamma}u\|_E \le c\varepsilon$ for all $\varepsilon\in(0,1]$ so that $1-\frac{2}{p}-\frac{2}{q}\ge 0$.

f) The last condition in (5.8) is needed due to an example by Stein, see Exercise 2.44 in [**35**]. The inequality in (5.8) already implies that $p,q\ge 2$, and similarly for q in (5.9). For $m>3$ the condition $p\ge 2$ can be justified by a more complicated argument, see [**16**]. ◊

We prove Theorem 5.8 in a series of reduction steps formulated as lemmas, which partly need additional facts from harmonic analysis. The proof of the theorem is then given at the end of the section.

The analysis will be reduced to the *half-wave group* given by $G(t)=\mathrm{e}^{\mathrm{i}t|\nabla|}$ for $t\in\mathbb{R}$. These are unitary operators on H^s and $\dot H^s$ forming a group. The map $t\mapsto|\xi|^s\mathrm{e}^{\mathrm{i}t|\xi|}\hat v(\xi)$ is bounded by $|\xi|^s|\hat v|$ and C^1 in t for fixed $\xi\ne 0$ with derivative $\mathrm{i}|\xi|^{s+1}\mathrm{e}^{\mathrm{i}t|\xi|}\hat v(\xi)$, which is bounded by $|\xi|^{s+1}|\hat v|$. Dominated convergence and Plancherel then imply that $t\mapsto G(t)$ is strongly continuous in $\dot H^s$ and that $t\mapsto G(t)v$ has the derivative $\mathrm{i}|\nabla|G(t)v$ in $\dot H^s$ if $v\in\dot H^{s+1}\cap\dot H^s$. Here one can omit the dots. Similarly, $G(\cdot)$ is a unitary C_0-group on H^s for $s\in\mathbb{R}$ with generator $\mathrm{i}|\nabla|$ defiend on H^{s-1}. (Note that this operator has the resolvent $\mathcal{F}^{-1}(\lambda-\mathrm{i}|\xi|)\mathcal{F}\in\mathcal{B}(H^s)$ for $\lambda\notin\mathrm{i}\mathbb{R}$.)

The half-wave group is closely tied to the wave equation since

$$C(t)=\tfrac12(G(t)+G(-t)),\qquad S(t)=\tfrac{1}{2\mathrm{i}}(G(t)-G(-t)),\qquad t\in\mathbb{R}. \tag{5.12}$$

Formulas (5.6), Remark 5.9 c) and the transformation $t\mapsto -t$ then yield the first simplification, where we get rid of the extra derivatives in (5.10) and (5.11).

Lemma 5.10. *In the setting of Theorem 5.8, estimate (5.10) follows from the inequalities*

$$\||\nabla|^{-\gamma}G(\cdot)\varphi\|_{L_\mathbb{R}^pL^q}\le c\|\varphi\|_{L^2},\qquad \||\nabla|^{-\gamma}G*_+f\|_{L_\mathbb{R}^pL^q}\le c\|f\|_{L_\mathbb{R}^1L^2} \tag{5.13}$$

for $\varphi\in L^2$ and $f\in L^1(\mathbb{R},L^2)$. Moreover, (5.11) is a consequence of

$$\|G(\cdot)\varphi\|_{L_\mathbb{R}^pL^q}\le c\|\varphi\|_{\dot H^\gamma},\qquad \|G*_+f\|_{L_\mathbb{R}^pL^q}\le c\|f\|_{L_\mathbb{R}^1\dot H^\gamma} \tag{5.14}$$

for $\varphi\in\dot H^\gamma$ and $f\in L^1(\mathbb{R},\dot H^\gamma)$.

As we have already noted, in our approach it is more convenient to prove (5.14). We thus show that these inequalities imply (5.13).

Lemma 5.11. *In the setting of Lemma 5.10, formula (5.14) implies (5.13).*

Proof. Let (5.14) be valid. Take $\varphi \in L^2$ and $f \in L^1_{\mathbb{R}} L^2$. Set $\psi = |\nabla|^{-\gamma}\varphi \in \dot{H}^\gamma$ and $g = |\nabla|^{-\gamma} f \in L^1(\mathbb{R}, \dot{H}^\gamma)$. Since the Fourier multipliers commute and $|\nabla|^{-\gamma}$ belongs to $\mathcal{B}(L^2, \dot{H}^\gamma)$, we obtain $|\nabla|^{-\gamma} G(t)\varphi = G(t)\psi$ and $|\nabla|^{-\gamma} G *_+ f(t) = G *_+ g(t)$ in $\dot{H}^\gamma \subseteq \mathcal{S}^\star$ for $t \in \mathbb{R}$. The estimates (5.14) then imply

$$\begin{aligned} \| |\nabla|^{-\gamma} G(\cdot)\varphi\|_{L^p_{\mathbb{R}} L^q} &\le c\|\psi\|_{\dot{H}^\gamma} = c\|\varphi\|_2, \\ \| |\nabla|^{-\gamma} G *_+ f\|_{L^p_{\mathbb{R}} L^q} &\le c\|g\|_{L^1_{\mathbb{R}} \dot{H}^\gamma} = c\|f\|_{L^1 L^2}. \end{aligned} \qquad \square$$

Our main argument only works for sharp triples. Fortunately, the general case then follows by Sobolev's inequality.

Lemma 5.12. *Let (5.14) hold for sharp admissible triples $(\tilde{p}, \tilde{q}, \tilde{\gamma})$. Then it is true for all admissible tripes (p, q, γ).*

Proof. Let (p, q, γ) be non-sharp admissible. The numbers

$$\frac{1}{\tilde{q}} := \frac{1}{2} - \frac{2}{p(m-1)} > \frac{1}{q}, \qquad \tilde{\gamma} := \frac{m}{2} - \frac{1}{p} - \frac{m}{\tilde{q}} < \gamma,$$

yield a sharp admissible triple $(p, \tilde{q}, \tilde{\gamma})$. (Note that $p > 2$ if $m = 3$.) By admissibility we have $s := \gamma - \tilde{\gamma} = \frac{m}{\tilde{q}} - \frac{m}{q} > 0$ and $s + \tilde{\gamma} = \gamma < \frac{m}{2}$. Theorem 1.2.3 in [**12**] and (5.14) with $(p, \tilde{q}, \tilde{\gamma})$ then imply

$$\begin{aligned} \|G *_+ f\|_{L^p_{\mathbb{R}} L^q} &= \| |\nabla|^{-s} |\nabla|^s G *_+ f\|_{L^p_{\mathbb{R}} L^q} \le c_{\mathrm{So}} \| |\nabla|^s G *_+ f\|_{L^p_{\mathbb{R}} L^{\tilde{q}}} \\ &\le c_{\mathrm{So}} C_{\mathrm{St}} \| |\nabla|^{\tilde{\gamma}} |\nabla|^{\gamma - \tilde{\gamma}} f\|_{L^1_{\mathbb{R}} L^2} = c_{\mathrm{So}} C_{\mathrm{St}} \|f\|_{L^1_{\mathbb{R}} \dot{H}^\gamma}. \end{aligned}$$

Here we commute the operators as above. The homogeneous estimate is treated in the same way. $\square$

We next reduce the inhomogeneous to the homogeneous case, by reversing the order of the duality argument in the proof of Theorem 4.10.

Lemma 5.13. *Let (p, q, γ) be sharp admissible. Then the first part of (5.14) implies the second one.*

Proof. Set $E = L^p(\mathbb{R}, L^q)$. The first part of (5.14) yields the boundedness of the orbit map $T : \dot{H}^\gamma \to E;\ \varphi \mapsto G(\cdot)\varphi$, and hence of its adjoint $T^\star : E^\star \to \dot{H}^{-\gamma}$. Let $\varphi \in H^\gamma \hookrightarrow \dot{H}^\gamma$ and $g \in C_c(\mathbb{R}, \mathcal{S}) =: F$, where $F \subseteq C_c(\mathbb{R}, \dot{H}^{-\gamma})$ by Remark 5.4. Using this regularity and the continuity of the scalar product, we compute $T^\star$ via

$$\begin{aligned} \langle \varphi, T^\star g\rangle_{\dot{H}^\gamma} &= \langle T\varphi, g\rangle_E = \int_{\mathbb{R}} \big(G(t)\varphi \,\big|\, \overline{g}(t)\big)_{L^2} \,\mathrm{d}t = \Big(\varphi \,\Big|\, \int_{\mathbb{R}} G(-t)\overline{g}(t)\,\mathrm{d}t\Big)_{L^2} \\ &= \Big\langle \varphi, \int_{\mathbb{R}} G(-t) g(t)\,\mathrm{d}t \Big\rangle_{\dot{H}^\gamma}. \end{aligned}$$

By density (see Remark 5.4 and the proof of Lemma 4.8), we obtain $T^\star g = \int_{\mathbb{R}} G(-t) g(t)\,\mathrm{d}t$ on $E^\star = L^{p'}(\mathbb{R}, L^{q'})$.

We next take $f \in L^1(\mathbb{R}, H^{\frac{3}{p}})$ which belongs to $L^1(\mathbb{R}, \dot{H}^\gamma)$ as $\frac{3}{p} > \frac{2}{p} = \gamma$. Then $G *_+ f(t)$ is defined in $H^{\frac{3}{p}}$ and contained in $C_b(\mathbb{R}, H^{\frac{3}{p}})$ and thus in $C_b(\mathbb{R}, L^q)$

due to Sobolev's theorem and admissibility $\frac{3}{p} - \frac{m}{2} = \gamma + \frac{1}{p} - \frac{m}{2} = -\frac{m}{q}$. Let $g \in F$. By means of Fubini, as above we compute

$$\Gamma := \langle G *_+ f, g\rangle_E = \int_{\mathbb{R}} (G *_+ f(t)|\overline{g}(t))_{L^2} \mathrm{d}t = \int_{\mathbb{R}} \int_{-\infty}^{t} (G(t-\tau)f(\tau)|\overline{g}(t))_{L^2} \mathrm{d}\tau\, \mathrm{d}t$$
$$= \int_{\mathbb{R}} \int_{\tau}^{\infty} (f(\tau)|G(\tau - t)\overline{g}(t))_{L^2}\, \mathrm{d}t\, \mathrm{d}\tau = \int_{\mathbb{R}} \Big\langle f(\tau) \,\Big|\, \int_{\tau}^{\infty} G(\tau)G(-t)g(t)\, \mathrm{d}t \Big\rangle_{\dot{H}^\gamma} \mathrm{d}\tau.$$

We can now estimate

$$|\Gamma| \le \|f\|_{L^1_{\mathbb{R}}\dot{H}^\gamma} \sup_{\tau\in\mathbb{R}} \|G(\tau)T^*(\mathbb{1}_{[\tau,\infty)}g)\|_{\dot{H}^{-\gamma}} \le \|f\|_{L^1_{\mathbb{R}}\dot{H}^\gamma} \|T^*\| \sup_{\tau\in\mathbb{R}} \|\mathbb{1}_{[\tau,\infty)}g\|_{E^\star}$$
$$= \|T^*\|\|g\|_{E^\star}\|f\|_{L^1_{\mathbb{R}}\dot{H}^\gamma}.$$

Because of density, this estimate holds for all $g \in E^\star$. Hence, the Duhamel term $G *_+ f$ belongs to $E = E^{\star\star}$ with norm bounded by $\|T^*\|\|f\|_{L^1_{\mathbb{R}}\dot{H}^\gamma}$. Moreover, $L^1(\mathbb{R}, H^{\frac{3}{p}})$ is dense in $L^1(\mathbb{R}, \dot{H}^\gamma)$ as can be see by approximation of simple functions using Remark 5.4. So $f \mapsto G *_+ f$ has an extension $\mathcal{G} : L^1(\mathbb{R}, \dot{H}^\gamma) \to E$ with the same norm bound, where $\mathcal{G}$ is given by the Duhamel integral in $\dot{H}^\gamma$. □

To show the first inequality in (5.14) we need the *Littlewood–Paley* decomposition of L^q and of $\dot{H}^\gamma$. (This core result from harmonic analysis holds in greater generality, see Sections 6.1 and 6.2 of [**11**] and Section 1.3 of [**12**].) To this aim, we work on $\mathbb{R}^m$ and fix a radial function $\chi \in \mathcal{S}$ such that $\chi \ge 0$, $\operatorname{supp}\chi \subseteq \{\frac{6}{7} \le |\xi| \le 2\}$, $\chi = 1$ on $\{1 \le |\xi| \le \frac{12}{7}\}$, and $\chi(\xi) + \chi(\frac{1}{2}\xi) = 1$ for $1 \le |\xi| \le \frac{24}{7}$, see §1.3.2 in [**12**]. Set $\chi_j = \sigma_{2^{-j}}\chi$. One obtains $\sum_{j\in\mathbb{Z}} \chi_j(\xi) = 1$ for all $\xi \ne 0$. We define the *Littlewood–Paley operators* $P_j = \mathcal{F}^{-1}\chi_j\mathcal{F}$. We stress that $\chi_j\hat{v}$ has support in the annulus with radii 2^{j-1} and 2^{j+1}.

Having a bounded symbol, each P_j maps each of the spaces $\mathcal{S}$, $\mathcal{S}^*$, H^s and $\dot{H}^s$ in itself, and it is bounded by 1 on the latter two. For $q \in (1,\infty)$ they are also bounded on L^q uniformly in $j \in \mathbb{Z}$ by the Mikhlin–Hörmander Theorem 6.2.7 in [**11**]. (For each $\alpha \in \mathbb{N}_0^m$, the functions $\xi^\alpha\partial^\alpha\chi_j(\xi) = \xi^\alpha 2^{-j|\alpha|}(\partial^\alpha\chi)(2^{-j}\xi)$ are bounded on $\mathbb{R}^m$ uniformly in $j \in \mathbb{Z}$ by the support of χ.)

We also have $P_j = (P_{j-1}+P_j+P_{j+1})P_j =: \tilde{P}_jP_j$, where $\tilde{\chi}_j := \chi_{j-1}+\chi_j+\chi_{j+1}$. In this sense these operators are almost projections. Remark 5.2 implies the representation $P_jv = (2\pi)^{-\frac{m}{2}}2^{mj}\psi(2^j\cdot) * v$ as a convolution, with $\psi = \mathcal{F}^{-1}\chi$.

These operators allow us to reduce our analysis to frequency-localized functions. They yield an 'almost orthogonal' decomposition of L^q, as expressed by the Littlewood–Paley Theorem 1.3.8 in [**12**]: Let $q \in (1,\infty)$. If $v \in L^q$, we have

$$\|v\|_q \le c\Big\|\Big(\sum_{j\in\mathbb{Z}} |P_jv|^2\Big)^{\frac{1}{2}}\Big\|_q =: c\|v\|_q^*.$$

Conversely, if $v \in \mathcal{S}^*$ satisfies $\|v\|_q^* < \infty$, then v belongs to L^q and $\|v\|_q^* \le c\|v\|_q$. These results are also true for $\tilde{P}_j$ with different constants.

For L^2 or $\dot{H}^s$ the Littlewood–Paley characterization can be shown in a rather elementary way by means of Plancherel. We present this argument assuming $s < \frac{m}{2}$. Let $v \in \mathcal{S}_0$. Then $\chi_j\hat{v}$ vanishes for all $j \le j_v$ and an index $j_v \in \mathbb{Z}$, and for $j > j_v$ Plancherel yields $\|P_j v\|_s = \|\chi_j \hat{v}\|_2 \le c(v,s)2^{-2sj}$ since $\hat{v} \in \mathcal{S}$. Using the properties of χ_j and Plancherel, we compute

$$\begin{aligned}\|v\|_{\dot{H}^s}^2 &= \int \Big|\sum_{j\in\mathbb{Z}} \chi_j |\xi|^s \hat{v}\Big|^2 \mathrm{d}\xi \eqsim_s \int \Big|\sum_{j\in\mathbb{Z}} 2^{js}\chi_j \hat{v}\Big|^2 \mathrm{d}\xi = \sum_{j,k\in\mathbb{Z}} \int 2^{js}2^{ks}\chi_j\chi_k|\hat{v}|^2\,\mathrm{d}\xi \\ &\lesssim_s \sum_{j\in\mathbb{Z}} 2^{2js} \int \chi_j(\chi_{j-1}+\chi_j+\chi_{j+1})|\hat{v}|^2\,\mathrm{d}\xi \le \sum_{j\in\mathbb{Z}} 2^{2js}\int |\tilde{\chi}_j\hat{v}|^2\,\mathrm{d}\xi \\ &= \sum_{j\in\mathbb{Z}} 2^{2js}\int |\tilde{P}_j v|^2\,\mathrm{d}x \lesssim_s \int \sum_{k\in\mathbb{Z}} 2^{2ks}|P_k v|^2\,\mathrm{d}x.\end{aligned}$$

Here we inserted $\tilde{P}_j = P_{j-1} + P_j + P_{j+1}$, and the final $L^2\ell^2$-norm is finite due the observations above. The converse inequality is shown similarly. We obtain the equivalence

$$\|v\|_{\dot{H}^s} \eqsim \Big\|\Big(\sum\nolimits_{j\in\mathbb{Z}} 2^{2js}|P_j v|^2\Big)^{\frac12}\Big\|_2 \tag{5.15}$$

for $v \in \mathcal{S}_0$. By density, it can be extended to $v \in \dot{H}^s$, see Remark 5.4 and also Theorem 1.3.8 in [**12**]. One can replace P_k by $\tilde{P}_k$ modifying the constants.

In the next step, the Littlewood–Paley decomposition leads to the crucial reduction to frequency localized estimates. The argument also works for other $m \ge 2$ if $q < \infty$, but from now on we let $m = 3$.

LEMMA 5.14. *Let $m = 3$, (p, q, γ) be sharp admissible, and $q > 2$. Assume there exists a constant $C > 0$ with*

$$\|P_j G(\cdot)\varphi\|_{L^p_{\mathbb{R}}L^q} \le C2^{j\gamma}\,\|\tilde{P}_j\varphi\|_{L^2} \tag{5.16}$$

for all $j \in \mathbb{Z}$ and $\varphi \in \dot{H}^\gamma$. Then the first part of (5.14) *and thus Theorem 5.8 are true.*

PROOF. We have $p, q < \infty$ by (5.8) and the assumptions. Let $\varphi \in H^2$ and J be a compact interval. Then $G(t)\varphi$ belongs to $H^2 \hookrightarrow L^q$ by Sobolev's embedding. The Littlewood–Paley decomposition yields

$$\|G(\cdot)\varphi\|^2_{L^p_J L^q} \lesssim \Big\|\Big\|\Big(\sum\nolimits_{j\in\mathbb{Z}} |P_j G(\cdot)v|^2\Big)^{\frac12}\Big\|_q\Big\|^2_{L^p_J} = \Big\|\Big\|\|(|P_jG(\cdot)v|^2)_j\|_{\ell^1}\Big\|_{\frac{q}{2}}\Big\|_{L^{\frac{p}{2}}_J}.$$

For fixed t, we interpret the inner terms as the norm in $L^{\frac{q}{2}}(\mathbb{R}^3)$ of the $L^{\frac{q}{2}}$-valued sum $\sum_j |P_j G(t)\varphi|^2$. We can take this norm in the sum since $q \ge 2$, obtaining

$$\|G(\cdot)\varphi\|^2_{L^p_J L^q} \lesssim \Big\|\sum\nolimits_{j\in\mathbb{Z}} \|P_j G(\cdot)v\|_q^2\Big\|_{L^{\frac{p}{2}}_J}.$$

This procedure also works for the t-integral so that

$$\|G(\cdot)\varphi\|^2_{L^p_J L^q} \lesssim \sum_{j\in\mathbb{Z}} \|P_j G(\cdot)\varphi\|^2_{L^p_J L^q}$$

Estimates (5.16) and (5.15) now yield

$$\|G(\cdot)\varphi\|^2_{L^p_J L^q} \le cC^2 \sum_{j\in\mathbb{Z}} 2^{2j\gamma}\|\tilde{P}_j\varphi\|^2_{L^2} = cC^2\Big\|\sum\nolimits_{j\in\mathbb{Z}} 2^{2j\gamma}|\tilde{P}_j\varphi|^2\Big\|_2^2 \lesssim \|\varphi\|^2_{\dot{H}^\gamma}.$$

Fatou's lemma allows us to replace J by $\mathbb{R}$. The first claim then follows from the density of H^2 in $\dot{H}^\gamma$. The second one is consequence of Remark 5.9 and Lemmas 5.10–5.13. $\square$

It is even enough to show (5.16) at unit frequencies $j = 0$ thanks to a scaling argument and admissibility. Observe that $\|\sigma_a v\|_r = a^{-\frac{m}{r}}\|v\|_r$ by the transformation rule. The next proof directly extends to $m \geq 2$.

LEMMA 5.15. *Under the assumptions of Lemma 5.14, estimate* (5.16) *holds for all $j \in \mathbb{Z}$ if it is true for $j = 0$.*

PROOF. Again it is enough to show the result for $\varphi \in H^2$ and compact J. Remark 5.2 and the assumption imply

$$\begin{aligned}
\|P_j G(\cdot)\varphi\|_{L^p_J L^q} &= 2^{-\frac{j}{p}} 2^{-\frac{3j}{q}} \big\|\sigma_{2^{-j}} \mathcal{F}^{-1}\big(\sigma_{2^{-j}}(\chi)\mathcal{F}(G(2^{-j}\cdot)\varphi)\big)\big\|_{L^p_J L^q} \\
&= 2^{-\frac{j}{p}-\frac{3j}{q}} 2^{3j} \big\|\mathcal{F}^{-1}\big(\chi\sigma_{2^j}\mathcal{F}(G(2^{-j}\cdot)\varphi)\big)\big\|_{L^p_J L^q} \\
&= 2^{-\frac{j}{p}-\frac{3j}{q}} \big\|\mathcal{F}^{-1}\big(\chi\mathcal{F}(\sigma_{2^{-j}}\mathcal{F}^{-1}[\mathrm{e}^{\mathrm{i}2^{-j}t|\xi|}\hat{\varphi}])\big)\big\|_{L^p_J L^q} \\
&= 2^{-\frac{j}{p}-\frac{3j}{q}} \big\|P_0\mathcal{F}^{-1}(\mathrm{e}^{\mathrm{i}2^{-j}t|2^j\xi|}\mathcal{F}(\sigma_{2^{-j}}\varphi)\big\|_{L^p_J L^q} = 2^{-\frac{j}{p}-\frac{3j}{q}} \big\|P_0 G(\cdot)\sigma_{2^{-j}}\varphi\big\|_{L^p_J L^q} \\
&\leq C 2^{-\frac{j}{p}-\frac{3j}{q}} \big\|\mathcal{F}^{-1}(\tilde{\chi}\mathcal{F}\sigma_{2^{-j}}\varphi)\big\|_2 = C 2^{-\frac{j}{p}-\frac{3j}{q}} 2^{3j} \big\|\mathcal{F}^{-1}(\tilde{\chi}\sigma_{2^j}\hat{\varphi})\big\|_2 \\
&= C 2^{-\frac{j}{p}-\frac{3j}{q}} \big\|\sigma_{2^{-j}}\mathcal{F}^{-1}(\sigma_{2^{-j}}(\tilde{\chi})\hat{\varphi})\big\|_2 = C 2^{-\frac{j}{p}-\frac{3j}{q}} 2^{\frac{3j}{2}} \|\tilde{P}_j\varphi\|_2 = C 2^{\gamma j}\|\tilde{P}_j\varphi\|_2,
\end{aligned}$$

where we used admissibility at the end. $\square$

We prove (5.16) for $j = 0$ in a similar way as the homogeneous Strichartz estimate for the Schrödinger equation in Theorem 4.10 a). Again we start with a dispersive estimate for fixed t. Observe that

$$\begin{aligned}
P_0 G(t)\varphi(x) &= (2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} \mathrm{e}^{\mathrm{i}(x\cdot\xi + t|\xi|)}\chi(\xi)\hat{\varphi}(\xi)\,\mathrm{d}\xi \\
&= (2\pi)^{-\frac{3}{2}}\mathcal{F}^{-1}(\mathrm{e}^{\mathrm{i}t|\xi|}\chi) * \varphi\,(x) \qquad (5.17)
\end{aligned}$$

for $x \in \mathbb{R}^3$. If $|t|$ and $|x|$ differ much, we can rewrite this integral as, e.g.,

$$P_0 G(t)\varphi(x) = (2\pi)^{-\frac{3}{2}} \int_{|\xi|\approx 1} \frac{|\xi|}{\mathrm{i}(x_1|\xi| + t\xi_1)} \partial_{\xi_1} \mathrm{e}^{\mathrm{i}(x\cdot\xi + t|\xi|)}\chi(\xi)\hat{\varphi}(\xi)\,\mathrm{d}\xi.$$

After integrating by parts, one can estimate $|P_0 G(t)\varphi(x)| \leq c(|t| + |x|)^{-1}$. By iteration one obtains strong decay in space and time. Near the 'light cone' $\{t = \pm|x|\}$ this reasoning fails. In general one needs deeper tools from harmonic analysis at this point, but for $m = 3$ we can argue in a quite elementary way. We first record an auxiliary result (which requires the cut-off χ), see (III.1.18') in [**34**] for an improved version.

LEMMA 5.16. *Let $m = 3$. We then have $\|\mathcal{F}^{-1}(\mathrm{e}^{\mathrm{i}t|\xi|}\chi)\|_{L^\infty} \leq c|t|^{-1}$ for alle $t \neq 0$ and a constant $c > 0$.*

Taking this inequality for granted, we can prove the Strichartz estimate for the wave equation.

Proof of Theorem 5.8. Let $m = 3$ and $q > 2$. By Lemmas 5.14 and 5.15 we have to show the boundedness of $P_0G(\cdot) : \dot{H}^\gamma \to E = L^p(\mathbb{R}, L^q)$. Let $\varphi \in L^1 \cap L^2$. Lemma 5.16 and (5.17) yield the basic (frequency-localized) dispersive estimate

$$\|P_0G(t)\varphi\|_\infty \le c|t|^{-1}\|\varphi\|_1.$$

Interpolating with $\|P_0G(t)\|_{\mathcal{B}(L^2)} \le 1$ as in Corollary 4.6 with $m = 2$, we derive

$$\|P_0G(t)\varphi\|_q \le c|t|^{\frac{2}{q}-1}\|\varphi\|_{q'}\,.$$

Since $1 + \frac{1}{p} = 1 - \frac{2}{q} + \frac{1}{p'}$ by strict admissibility, Lemma 4.9 implies

$$\|P_0G * f\|_E \le c\Big\||t|^{\frac{2}{q}-1} * \|f(\cdot)\|_{q'}\Big\|_{L^p_{\mathbb{R}}} \le c\|f\|_{E^\star} \tag{5.18}$$

for $f \in E^\star = L^{p'}(\mathbb{R}, L^{q'})$. Here we also need $\frac{2}{q} \in (0,1)$ which follows from the assumption and (5.8).

We now conclude the proof of (5.16) for $j = 0$ by a duality argument as in Theorem 4.10 a). Set $S : E^\star \to L^2$; $Sf = \int_{\mathbb{R}} P_0G(-t)f(t)\,\mathrm{d}t$. Let $f \in C_c(L^2 \cap L^{q'})$. Using (5.18), we compute

$$\begin{aligned}\|Sf\|_2^2 &= \int_{\mathbb{R}}\int_{\mathbb{R}} \big(G(-t)P_0f(t)\big|P_0G(-\tau)f(\tau)\big)_{L^2}\,\mathrm{d}\tau\,\mathrm{d}t\\ &= \int_{\mathbb{R}} \Big(P_0f(t)\Big|\int_{\mathbb{R}} P_0G(t-\tau)f(\tau)\,\mathrm{d}\tau\Big)_{L^2}\,\mathrm{d}t\\ &= \langle P_0f, P_0G * \overline{f}\rangle_{E^\star} \le c\|f\|_{E^\star}^2\end{aligned}$$

By density, S and thus the adjoint $S^* : L^2 \to E$ are bounded. As in the proof of Lemma 5.13, we see that $S^*\varphi = P_0G(\cdot)\varphi = P_0G(\cdot)\tilde{P}_0\varphi$. $\square$

It only remains to show the above lemma.[4]

Proof of Lemma 5.16. By rotation, it is enough to look at $x = x_3e_3 \neq 0$. Using polar coordinates and the radiality of χ, we calculate

$$J := \int_{\mathbb{R}^3} \mathrm{e}^{\mathrm{i}(x_3\xi_3+t|\xi|)}\chi(\xi)\,\mathrm{d}\xi = \int_0^\infty\int_{-\pi}^{\pi}\int_{-\pi/2}^{\pi/2} \mathrm{e}^{\mathrm{i}x_3r\sin\theta}\cos\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi\,r^2\chi(r)\mathrm{e}^{\mathrm{i}rt}\,\mathrm{d}r.$$

In the θ-integral we substitute $\tau = \sin\theta$ and obtain

$$\int_{-\pi/2}^{\pi/2} \mathrm{e}^{\mathrm{i}x_3r\sin\theta}\cos\theta\,\mathrm{d}\theta = \int_{-1}^{1} \mathrm{e}^{\mathrm{i}x_3r\tau}\,\mathrm{d}\tau = \frac{2\sin(rx_3)}{rx_3}.$$

Integration by parts then yields

$$\begin{aligned}J &= 4\pi\int_{6/7}^{2} r^2\chi(r)\,\mathrm{sinc}(rx_3)\,(\mathrm{i}t)^{-1}\partial_r\mathrm{e}^{\mathrm{i}rt}\,\mathrm{d}r\\ &= \frac{4\pi\mathrm{i}}{t}\int_{6/7}^{2}\mathrm{e}^{\mathrm{i}rt}\Big[\big(2r\chi(r) + r^2\chi'(r)\big)\,\mathrm{sinc}(rx_3) + r\chi(r)\big(\cos(rx_3) - \mathrm{sinc}(rx_3)\big)\Big]\mathrm{d}r.\end{aligned}$$

The result follows from the boundedness of the integrand. $\square$

[4]The next proof was omitted in the lectures.

5.3. Local wellposedness and global existence

We now study the semilinear wave equation

$$\partial_t^2 u = \Delta u - \mu |u|^{\alpha-1} u, \quad u(0) = u_0, \quad \partial_t u(0) = u_1, \quad x \in \mathbb{R}^3, \ t \in J, \qquad (5.19)$$

for $\mu \in \{-1, 1\}$, $\alpha \in [3, 5]$, an interval J of positive length containing 0, and initial maps $u_0, u_1 : \mathbb{R}^3 \to \mathbb{C}$. We focus on the 'physical' case $m = 3$ for conciseness, and use the Strichartz estimates from Theorem 5.8 for the admissible triple $(p_\alpha, 3(\alpha-1), 1)$ with $p_\alpha = 2\frac{\alpha-1}{\alpha-3}$, besides the energy triple $(\infty, 2, 0)$.

We introduce H^1- and $\dot{H}^1$-solutions u of (5.19) on J as in Definition 5.5, but require in addition $u \in E_{\rm loc}(J)$; i.e., $u \in E(J_0) := L^{p_\alpha}(J_0, L^{3(\alpha-1)})$ for every compact $J_0 \subseteq J$. Below we show that then $\phi(u) = -\mu |u|^{\alpha-1} u$ belongs to $L^1_{\rm loc}(J, L^2)$ as needed for Proposition 5.6.

Let $X = H^1 \times L^2$, $\dot{X} = \dot{H}^1 \times L^2$, $G(J) = C^1(J, L^2) \cap C(J, H^1)$, $\dot{G}(J) = \big\{ v \in C(J, \dot{H}^1) \,\big|\, \exists\, \partial_t u \in C(J, L^2) \big\}$, $\dot{\mathcal{F}}(J) = \dot{G}(J) \cap E(J)$, $\dot{\mathcal{F}}_{\rm loc}(J) = \dot{G}(J) \cap E_{\rm loc}(J)$, $\mathcal{F}(J) = G(J) \cap E(J)$, and $\mathcal{F}_{\rm loc}(J) = G(J) \cap E_{\rm loc}(J)$. In the definition of $\dot{G}$ the time derivative is at first understood in $Z := L^2 + L^6$ using that $\dot{H}^1 \hookrightarrow L^6 \hookrightarrow Z$ and $L^2 \hookrightarrow Z$, see Section 2.2 E) in [**30**] and Remark 5.4. These embeddings can be also used to show that $\dot{G}(J)$ is complete for its natural norm if J is compact. We write $w = (u, \partial_t u)$, $w_0 = (u_0, u_1)$, and $E(b)$ if $J = [-b, b]$ etc.

We first state the local wellposedness result in the energy-subcritical case $\alpha < 5$. The $\dot{X}$-norm will give the blow-up condition so that it is natural to look for $\dot{H}^1$-solutions. The $\dot{H}^s$-norms are also needed for the long-time behavior, cf. Theorem 5.18 and Remark 5.19. The proofs follow the pattern of those in Sections 4.3 and 4.4, but the function spaces are partly different since now the energy estimate gains a derivative, which is lost in the Strichartz estimate with $\gamma = 1$. The proof of continuous dependence is easier since the fixed-point space is a closed ball in $\dot{\mathcal{F}}(b)$ endowed with the metric of $\dot{\mathcal{F}}(b)$ itself. We only show *conditional* uniqueness of $\dot{H}^1$-solutions belonging also to $E(J)$. In [**25**] uniqueness is shown without such extra conditions for $\alpha \in [3, 5)$ and also in critical cases in higher dimensions. We will omit or sketch parts of the proofs below which are straightforward modifications of previous arguments.[5]

Theorem 5.17. *Let $m = 3$, $\alpha \in [3, 5)$, $w_0 = (u_0, u_1) \in \dot{X}$ with $\|u_0\|_{\dot{H}^1} + \|u_1\|_2 \le \rho$, and $(p_\alpha, 3(\alpha-1), 1)$ be admissible. The following assertions hold.*

a) There is a unique maximal *$\dot{H}^1$-solution $u = \varphi(\cdot, w_0)$ in $\dot{\mathcal{F}}_{\rm loc}(J(w_0)$ of (5.19) on $J(w_0) = (t^-(w_0), t^+(w_0))$, where $[-b_0(\rho), b_0(\rho)] \subseteq J(w_0)$, see (5.23).*

b) If $t^+(w_0) < \infty$, then $\|w(t)\|_{\dot{X}} \to \infty$ as $t \to t^+(w_0)$; analogously for $t^-(w_0)$.

c) Let $J \subseteq J(w_0)$ be compact. Then there is a radius $\delta = \delta(w_0, J) > 0$ such that for all $\tilde{w}_0 \in \overline{B}_{\dot{X}}(w_0, \delta)$ we have $J \subseteq J(\tilde{w}_0)$ and the map $\overline{B}_{\dot{X}}(w_0, \delta) \to \dot{\mathcal{F}}(J)$; $\tilde{w}_0 \to \varphi(\cdot, \tilde{w}_0)$, is Lipschitz continuous.

d) Let u_0 and u_1 be real-valued. Then the same is true for u.

e) Let $u_0 \in H^1$. Then $u = \varphi(\cdot, w_0)$ is an H^1-solution. Claim c) is true in X.

[5] In the lectures further parts of the proofs in this section were omitted.

PROOF. 1) Let $b>0$, set $r=C_{\mathrm{St}}\rho+1$, and equip $\dot{\mathcal{F}}(b)$ with the norm $|||v|||_b=\max\{\|v\|_{L_b^\infty \dot H^1},\|\partial_t v\|_{L_b^\infty L^2},\|v\|_{E(b)}\}$. The fixed-point space is $\Sigma(b,r)=\overline{B}_{\dot{\mathcal{F}}(b)}(0,r)$ with the metric $|||v-\tilde v|||_b$. Take $v,\tilde v\in\Sigma(b,r)$ and $t\in[-b,b]$. We use the fixed-point operator

$$\Phi(v)(t)=[\Phi_{w_0}(v)](t)=C(t)u_0+S(t)|\nabla|^{-1}u_1+\int_0^t S(t-\tau)|\nabla|^{-1}\phi(v(\tau))\,\mathrm{d}\tau. \tag{5.20}$$

As in (5.6), it follows

$$|\nabla|\Phi(v)(t)=C(t)|\nabla|u_0+S(t)u_1+\int_0^t S(t-\tau)\phi(v(\tau))\,\mathrm{d}\tau,$$
$$\partial_t\Phi(v)(t)=-S(t)|\nabla|u_0+C(t)u_1+\int_0^t C(t-\tau)\phi(v(\tau))\,\mathrm{d}\tau.$$

The Strichartz estimate in Theorem 5.8 bounds $\|\Phi(v)\|_{\dot G(b)}$ via the (energy) triple $(p,q,\gamma)=(\infty,2,0)$ and $\|\Phi(v)\|_{E(b)}=\||\nabla|^{-1}|\nabla|\Phi(v)\|_{E(b)}$ via $(p_\alpha,3(\alpha-1),1)$. Using also Hölder's inequality with $\frac12=\frac13+\frac16$ and Sobolev's embedding $\dot H^1\hookrightarrow L^6$, we derive

$$\begin{aligned}|||\Phi(v)|||_b&\le C_{\mathrm{St}}\rho+C_{\mathrm{St}}\int_{-b}^b\||v(\tau)|^{\alpha-1}v(\tau)\|_2\,\mathrm{d}\tau\\&\le C_{\mathrm{St}}\rho+C_{\mathrm{St}}\int_{-b}^b\|v(\tau)\|_{3(\alpha-1)}^{\alpha-1}\|v(\tau)\|_6\,\mathrm{d}\tau\\&\le C_{\mathrm{St}}\rho+rC_{\mathrm{St}}C_{\mathrm{So}}\int_{-b}^b\|v(\tau)\|_{3(\alpha-1)}^{\alpha-1}\,\mathrm{d}\tau.\end{aligned}$$

We then employ Hölder in time with $\frac{1}{\alpha-1}=\frac{1}{p_\alpha}+\frac{5-\alpha}{2(\alpha-1)}$, obtaining

$$|||\Phi(v)|||_b\le C_{\mathrm{St}}\rho+rC_{\mathrm{St}}C_{\mathrm{So}}(2b)^{\frac{5-\alpha}{2}}\|v\|_{E(b)}^{\alpha-1}\le C_{\mathrm{St}}\rho+r^\alpha C_{\mathrm{St}}C_{\mathrm{So}}(2b)^{\frac{5-\alpha}{2}}\le r \tag{5.21}$$

for $0<b\le b_1(\rho):=\frac12\big(C_{\mathrm{St}}C_{\mathrm{So}}r^\alpha\big)^{\frac{2}{\alpha-5}}$. (This argument fails if $\alpha\ge5$.) In particular, $\phi(v)$ belongs to $L_b^1L^2$ as noted above (also for $\alpha=5$).

We again use $\phi(v)-\phi(\tilde v)=\int_0^1\phi'\big(\sigma v+(1-\sigma)\tilde v\big)(v-\tilde v)\,\mathrm{d}\sigma$. In a similar way as in (5.21) we then estimate

$$\begin{aligned}|||\Phi(v)-\Phi(\tilde v)|||_b&\le C_{\mathrm{St}}\int_{-b}^b\|\phi(v(\tau))-\phi(\tilde v(\tau))\|_2\,\mathrm{d}\tau\\&\le\alpha C_{\mathrm{St}}\int_{-b}^b\big\|(|v(\tau)|+|\tilde v(\tau)|)^{\alpha-1}(v(\tau)-\tilde v(\tau))\big\|_2\,\mathrm{d}\tau\\&\le\alpha C_{\mathrm{St}}C_{\mathrm{So}}\int_{-b}^b\||v(\tau)|+|\tilde v(\tau)|\|_{3(\alpha-1)}^{\alpha-1}\|v(\tau)-\tilde v(\tau)\|_{\dot H^1}\,\mathrm{d}\tau\\&\le\alpha C_{\mathrm{St}}C_{\mathrm{So}}(2b)^{\frac{5-\alpha}{2}}\big(\|v\|_{E(b)}+\|\tilde v\|_{E(b)}\big)^{\alpha-1}\|v-\tilde v\|_{L_b^\infty\dot H^1}\\&\le\alpha2^{\frac{\alpha-3}{2}}C_{\mathrm{St}}C_{\mathrm{So}}r^{\alpha-1}b^{\frac{5-\alpha}{2}}|||v-\tilde v|||_b\le\tfrac12|||v-\tilde v|||_b\end{aligned} \tag{5.22}$$

if $0 < b \le b_0(\rho)$ with

$$b_0 = b_0(\rho) := \min\big\{b_1(\rho), \big(\alpha 2^{1+\frac{\alpha-3}{2}} C_{\mathrm{St}} C_{\mathrm{So}} r^{\alpha-1}\big)^{\frac{2}{\alpha-5}}\big\}. \tag{5.23}$$

We thus obtain a unique fixed point $u = \Phi(u)$ in $\Sigma(b,r)$, which is a $\dot H^1$-solution of (5.19) by Proposition 5.6.

2) For the Lipschitz continuity, we take $\tilde w_0 \in \dot X$ with $\|\tilde w_0\|_{\dot X} \le \rho$. Step 1) provides a $\dot H^1$-solution $\tilde u \in \Sigma(b_0, r)$ of (5.19) with initial value $\tilde w_0$ given by $\tilde u = \Phi_{\tilde w_0}(\tilde u)$. Using (5.22), (5.20) and Theorem 5.8, we estimate

$$\begin{aligned}
|||u-\tilde u|||_{b_0} &\le |||\Phi_{w_0}(u) - \Phi_{w_0}(\tilde u)|||_{b_0} + |||\Phi_{w_0}(\tilde u) - \Phi_{\tilde w_0}(\tilde u)|||_{b_0}\\
&\le \tfrac12 |||u-\tilde u|||_{b_0} + |||C(\cdot)(w_0-\tilde w_0) + S(\cdot)|\nabla|^{-1}(w_0-\tilde w_0)|||_{b_0},\\
|||u-\tilde u|||_{b_0} &\le 2C_{\mathrm{St}}\|w_0-\tilde w_0\|_{\dot X}\,.
\end{aligned} \tag{5.24}$$

3) Based on step 1), uniqueness in $\dot{\mathcal F}(J)$ can be shown following the reasoning in Lemma 1.8. A maximal solution $u = \varphi(\,\cdot\,, w_0)$ on $J(w_0)$ is constructed as before Theorem 1.11, but now using solutions in $\dot{\mathcal F}([0,b])$ and $\dot{\mathcal F}([-b,0])$, so that u belongs to $\dot{\mathcal F}_{\mathrm{loc}}(J(w_0))$. Claims b) and c) are shown as in Theorem 1.11 starting from step 1) and (5.24), respectively. (See also the proof of Theorem 5.18 c).)

Let u_0 and u_1 be real-valued. Fix a compact interval $J \subseteq J(w_0)$ with $0 \in J^\circ$ and put $R = |||u|||_J$. Set $v = \operatorname{Im} u$. It follows $v(0) = 0 = \partial_t v(0)$ and $\partial_t^2 v = \Delta v - \mu|u|^{\alpha-1}v$. Using Strichartz' Theorem 5.8 as in (5.21), we estimate

$$|||v|||_b \le C_{\mathrm{St}}\int_{-b}^{b} \||u(\tau)|^{\alpha-1}v(\tau)\|_2 \,\mathrm{d}\tau \le C_{\mathrm{St}} C_{\mathrm{So}} R^{\alpha-1}(2b)^{\frac{5-\alpha}{2}}\|v\|_{L_b^\infty \dot H^1}$$

for $b > 0$ with $[-b,b] \subseteq J$. For small b, we infer $|||v|||_b = 0$ and thus $v = 0$ on $[-b,b]$. As $v(\pm b) = 0 = \partial_t v(\pm b)$, an iteration yields $v = 0$.

Finally, let $u_0 \in H^1$. Since $u(t) = u_0 + \int_0^t \partial_t u(\tau)\,\mathrm{d}\tau$, the solution then belongs to $C(J(w_0), H^1)$. This equation also implies

$$\|u-\tilde u\|_{L_J^\infty L^2} \le \|u_0-\tilde u_0\|_2 + \lambda(J)\|\partial_t u - \partial_t \tilde u\|_{L_J^\infty L^2} \le (1+\lambda(J)L)\|w_0-\tilde w_0\|_X,$$

where L is the Lipschitz constant from part c). □

Various authors contributed to the above and the next theorems, references can be found in [**34**] and [**35**]. In Theorem 5.18 we treat the critical case $\alpha = 5$. As in Theorem 4.17, the smallness condition for the contraction argument is incorporated in the fixed-point space via a resctriction on a certain Bochner norm, and not by choosing small time intervals. Again this approach leads to more complicated conditions on blowup and the minimal existence time. In contrast to Theorem 4.17 we directly use a Strichartz norm, and not a norm controlled via the Strichartz and Sobolev inequalities. One has ill-posedness results if $\alpha > 5$ and $\mu = -1$, cf. Exercise 3.64 in [**35**]. In part f) we show global existence for small data by the basic fixed-point argument on $\mathbb R$ (and not just on a small interval). This convenient argument requires a global estimate for the linear part as our Strichartz inequality.

Theorem 5.18. *Let $m = 3$, $\alpha = 5$, and $w_0 = (u_0, u_1) \in \dot X$ with $\|u_0\|_{\dot H^1} + \|u_1\|_2 \le \rho$. Then the following assertions are true.*

a) There is a unique maximal $\dot H^1$*-solution* $u = \varphi(\,\cdot\,, w_0)$ *in* $\dot{\mathcal F}_{\mathrm{loc}}(J(w_0))$ *of* (5.19) *on* $J(w_0) = (t^-(w_0), t^+(w_0))$. *There is a number* $b_0(w_0) > 0$ *with* $[-b_0(w_0), b_0(w_0)] \subseteq J(w_0)$*, see* (5.25) *and* (5.29).

b) If $t^+(w_0) < \infty$*, then* $\|u\|_{E([0,t^+(w_0)))} = \infty$*; and analogously for* $t^-(w_0)$.

c) Let $J \subseteq J(w_0)$ *be compact. Then there is a radius* $\delta = \delta(w_0, J) > 0$ *such that for all* $\tilde w_0 \in \overline{B}_{\dot X}(w_0, \delta)$ *we have* $J \subseteq J(\tilde w_0)$ *and the map* $\overline{B}_{\dot X}(w_0,\delta) \to \dot{\mathcal F}(J)$*;* $\tilde w_0 \to \varphi(\,\cdot\,, \tilde w_0)$*, is Lipschitz continuous.*

d) Let u_0 *and* u_1 *be real-valued. Then the same is true for* u.

e) Let $u_0 \in H^1$. *Then* $u = \varphi(\,\cdot\,, w_0)$ *is an* H^1*-solution. Claim c) is true in* X.

f) There is a radius $\overline{\rho} > 0$ *such that* $J(w_0) = \mathbb{R}$ *if* $\|u_0\|_{\dot H^1} + \|u_1\|_2 \le \overline{\rho}$.

Proof. 1) We use the admissible triple $(4, 12, 1)$ and write $E(b) = L^4([-b, b], L^{12})$. To obtain smallness in e.g. (5.21), we will require small Strichartz norms in the fixed-point space. Set $r = C_{\mathrm{St}}\rho + 1$. Take $\varepsilon \in (0, r]$ to be fixed below. Set $u^0 = C(\cdot)u_0 + S(\cdot)|\nabla|^{-1}u_1$ which solves (5.2) with $f = 0$. Because of Theorem 5.8 with $\gamma = 1$, the map u^0 belongs to $E(\mathbb{R})$ with norm bounded by $C_{\mathrm{St}}\rho$. Hence, there is a time $b' = b_0(w_0, \varepsilon) \in (0, \infty]$ with

$$\|u^0\|_{E(b')} \le \tfrac{\varepsilon}{2}. \tag{5.25}$$

The fixed-point space

$$\Sigma_c(b', r, \varepsilon) = \big\{ v \in \Sigma(b', r) \,\big|\, \|v\|_{E(b')} \le \varepsilon \big\}$$

is endowed with the metric of $\dot{\mathcal F}(b')$. Let $v, \tilde v \in \Sigma_c(b', r, \varepsilon)$. As in (5.21) and (5.22), using also (5.25) we obtain

$$\begin{aligned} |||\Phi(v)|||_{b'} &\le C_{\mathrm{St}}\rho + C_{\mathrm{St}}\int_{-b'}^{b'} \||v|^4 v\|_2 \,\mathrm{d}\tau \le C_{\mathrm{St}}\rho + rC_{\mathrm{St}}C_{\mathrm{So}}\int_{-b'}^{b'} \|v\|_{12}^4 \,\mathrm{d}\tau \\ &\le C_{\mathrm{St}}\rho + rC_{\mathrm{St}}C_{\mathrm{So}}\varepsilon^4 \le C_{\mathrm{St}}\rho + 1, \end{aligned} \tag{5.26}$$

$$\|\Phi(v)\|_{E(b')} \le \|u^0\|_{E(b')} + C_{\mathrm{St}}\int_{-b'}^{b'} \||v|^4 v\|_2 \,\mathrm{d}\tau \le \frac{\varepsilon}{2} + rC_{\mathrm{St}}C_{\mathrm{So}}\varepsilon^4 \le \varepsilon, \tag{5.27}$$

$$\begin{aligned} |||\Phi(v) - \Phi(\tilde v)|||_{b'} &\le C_{\mathrm{St}}\||v|^4 v - |\tilde v|^4\tilde v\|_{L^1_{b'}L^2} \le 5C_{\mathrm{St}}\int_{-b'}^{b'} \||v| + |\tilde v|\|_{12}^4 \,\|v - \tilde v\|_6 \,\mathrm{d}\tau \\ &\le 5C_{\mathrm{St}}C_{\mathrm{So}}\big(\|v\|_{E(b')} + \|\tilde v\|_{E(b')}\big)^4 \,\|v - \tilde v\|_{L^\infty_{b'}\dot H^1} \\ &\le 80C_{\mathrm{St}}C_{\mathrm{So}}\varepsilon^4 |||v - \tilde v|||_{b'} \le \tfrac12 |||v - \tilde v|||_{b'}, \end{aligned} \tag{5.28}$$

where we fix $\varepsilon = \varepsilon' = \varepsilon_0(\rho)$ given by

$$\varepsilon_0(\rho) = \min\big\{ \big(rC_{\mathrm{St}}C_{\mathrm{So}}\big)^{-\frac14}, \big(2rC_{\mathrm{St}}C_{\mathrm{So}}\big)^{-\frac13}, \big(160C_{\mathrm{St}}C_{\mathrm{So}}\big)^{-\frac14} \big\}. \tag{5.29}$$

Hence, Φ is a strict contraction on $\Sigma_c(b', r, \varepsilon')$ and there thus exists a unique fixed point $u = \Phi(u)$ in $\Sigma_c(b', r, \varepsilon')$. It is a $\dot H^1$ solution of (5.19) on $[-b', b']$ by Proposition 5.6.

2) For claim f), take $\overline{\rho} \in (0, 1]$ and set $\overline{r} = C_{\mathrm{St}} + 1$ as well as $\overline{\varepsilon} = \varepsilon_0(1)$. Let $\|w_0\|_{\dot X} \le \overline{\rho}$. From Theorem 5.8 we obtain

$$\|u^0\|_{E(\mathbb{R})} = \||\nabla|^{-1}|\nabla| u^0\|_{E(\mathbb{R})} \le C_{\mathrm{St}}\overline{\rho} \le \tfrac{\overline{\varepsilon}}{2}$$

fixing $\overline{\rho} = \min\{1, \overline{\varepsilon}/(2C_{\mathrm{St}})\}$. Step 1) then yields a solution u of (5.19) on $\mathbb{R}$.

To show uniqueness, let $u \in \dot{\mathcal{F}}_{\mathrm{loc}}(J)$ and $\tilde{u} \in \dot{\mathcal{F}}_{\mathrm{loc}}(\tilde{J})$ solve (5.19) on J and $\tilde{J}$, respectively, where $J^* := J \cap \tilde{J}$ contains more points than 0. Suppose that $u \neq \tilde{u}$ on J^*. Then there is a number $\tau \in J^*$, say $\tau \geq 0$ such that $u(\tau) = \tilde{u}(\tau) =: u_0^*$ and $\partial_t u(\tau) = \partial_t \tilde{u}(\tau) =: u_1^*$, as well as times $\tau < t_n$ in J^* with $t_n \to \tau$ as $n \to \infty$ and $u(t_n) \neq \tilde{u}(t_n)$ for all n. The numbers ρ^*, r^*, b_0^* and ε_0^* are defined for u_0^* and u_1^* as in step 1). There exists a time $\beta \in (0, b^*]$ with $J_\beta := [\tau, \tau + \beta] \subseteq J^*$ and $|||u|||_{J_\beta}, |||\tilde{u}|||_{J_\beta} \leq \varepsilon^*$. Hence, $u{\restriction}_{|J_\beta}$ and $\tilde{u}{\restriction}_{|J_\beta}$ have to coincide with the unique solution produced in step 1) which contradicts $u(t_n) \neq \tilde{u}(t_n)$ for large n.

One can now define a unique maximal solution $= \varphi(\,\cdot\,, w_0)$ as asserted in statement a), proceeding as in Theorem 5.17.

Let u_0 and u_1 be real. Take a compact subinterval $J \subseteq J(w_0)$ with $0 \in J^\circ$. Set $v = \operatorname{Im} u$ and fix $\kappa = (2C_{\mathrm{St}}C_{\mathrm{So}})^{-\frac{1}{4}}$. We can decompose J into intervals $J_1, \cdots, J_N$ with $\|u\|_{E(J_k)} \leq \kappa$ for all k, where $0 \in J_1$. Combing step 3) of the previous proof and (5.26), we estimate

$$|||v|||_{J_1} \leq C_{\mathrm{St}} \int_{J_1} \||u(\tau)|^4 v(\tau)\|_2 \,\mathrm{d}\tau \leq C_{\mathrm{St}}C_{\mathrm{So}}\kappa^4 \|v\|_{L^\infty_{J_1}\dot{H}^1} \leq \frac{1}{2}|||v|||_{J_1},$$

which yields $v = 0$ on J_1. This procedure can be iterated, implying claim d).

3) To show part b), we suppose $t^+ = t^+(w_0) < \infty$ and $u \in L^4(J^+, L^{12})$ with $J^+ = [0, t^+)$. We use $\kappa > 0$ from the previous step and decompose J^+ into intervals $J_k = [t_k, t_{k+1}]$ with $0 = t_0 < t_1 < \cdots < t_K = \max J$ and $\|u\|_{E(J_k)} \leq \kappa$ for all k. Using $w = (u, \partial_t u)$, we argue as above to deduce

$$|||u|||_{J_k} \leq C_{\mathrm{St}}\|w(t_k)\|_{\dot{X}} + C_{\mathrm{St}}C_{\mathrm{So}}\kappa^4\|u\|_{L^\infty_{J_k}\dot{H}^1} \leq C_{\mathrm{St}}\|w(t_k)\|_{\dot{X}} + \tfrac{1}{2}|||v|||_{J_k}$$

and hence $|||u|||_{J_k} \leq 2C_{\mathrm{St}}\|w(t_k)\|_{\dot{X}}$. This inequality can iterated to

$$|||u|||_{J_k} \leq 2C_{\mathrm{St}}|||u|||_{J_{k-1}} \leq (2C_{\mathrm{St}})^2\|w(t_{k-1})\|_{\dot{X}} \leq \cdots \leq (2C_{\mathrm{St}})^K\|w_0\|_{\dot{X}} =: \hat{\rho}$$

for all $k \in \{0, \cdots, K-1\}$, so that $\|w(t)\|_{\dot{X}} \leq \hat{\rho}$ for $t \in J$. Define $\hat{r} = C_{\mathrm{St}}\hat{\rho} + 1$ and $\hat{\varepsilon} = \varepsilon_0(\hat{\rho})$ as in step 1). Set $J_\tau = [\tau, t^+) \subseteq \mathbb{R}_{\geq 0}$. As in (5.26) we estimate

$$\|u - u^0\|_{E(J_\tau)} = \||\nabla|^{-1}S *_+ \phi(u)\|_{E(J_\tau)} \leq C_{\mathrm{St}}\||u|^4 u\|_{L^1_{J_\tau}L^2} \leq C_{\mathrm{St}}C_{\mathrm{So}}\hat{\rho}\|u\|^4_{L^4_{J_\tau}L^{12}}$$

which tends to 0 as $\tau \to t^+$. We can thus fix $\tau \in [0, t^+)$ with $\|u^0\|_{E(J_\tau)} \leq \hat{\varepsilon}/4$. Since u^0 belongs to $E(\mathbb{R})$, there is a time $\beta > 0$ such that $\|u^0\|_{E(J_\beta)} \leq \hat{\varepsilon}/2$ for $J_\beta = [\tau, t^+ + \beta]$. Step 1 now yields a solution $v \in \dot{\mathcal{F}}(J_\beta)$ of (5.19) with $v(\tau) = u(\tau)$, which contradicts the assumption. Negative times are treated analoguously, so that statement b) is shown.

4) For c), fix a compact interval $J \subset J(w_0)$ with $0 = \min J$. Take $\tilde{w}_0 = (\tilde{u}_0, \tilde{u}_1) \in \overline{B}_{\dot{X}}(w_0, \delta')$ for some $\delta' \in (0, 1]$. We then have $\|\tilde{w}_0\|_{\dot{X}} \leq \rho + 1 =: \tilde{\rho}$. Set $\tilde{r} = C_{\mathrm{St}}\tilde{\rho} + 1$, $\tilde{\varepsilon} = \varepsilon_0(\tilde{\rho})$, see (5.29), $\tilde{u} = \varphi(\,\cdot\,, \tilde{w}_0)$, $\mathcal{T}(\varphi, \psi) = C(\cdot)\varphi + S(\cdot)|\nabla|^{-1}\psi$ for $(\varphi, \psi) \in \dot{X}$, and $\tilde{u}^0 = \mathcal{T}\tilde{w}_0$. We need the smallness condition (5.25) not only at time 0, but also along the given solution w to iterate the estimates. To this aim, we note that maps $\mathcal{T}_b(\varphi, \psi) = \mathbb{1}_{[0,b]}\mathcal{T}(\varphi, \psi)$ tend to 0 in $E(\mathbb{R})$ as $b \to 0$ for each $(\varphi, \psi) \in \dot{X}$ so that $\|\mathcal{T}w(t)\|_{E(0,b)}$ converges to 0 for each $t \in J$ by the

compactness of the orbit $\{w(t)\,|\,t\in J\}$ in $\dot X$. We can thus fix a time step $\tilde b>0$ such that
$$\|\mathcal{T}w(t)\|_{E(\tilde b)}\le \tfrac{\tilde\varepsilon}{4},\qquad j\in\{0,\dots,M-1\}. \tag{5.30}$$
form all $t\in J$. There is a number $M\in\mathbb{N}$ with $(M-1)\tilde b<\max J\le M\tilde b$ and we set $\tilde t_j=j\tilde b$ for $j\in\{0,\dots,M-1\}$, $\tilde t_M=\max J$, and $\tilde J_j=[\tilde t_j,\tilde t_{j+1}]$.

We start with $\tilde J_0$. Using again Strichartz' Theorem 5.8 with $\gamma=1$, we deduce
$$\|\tilde u^0\|_{E(\tilde J_0)}\le\|\tilde u^0-u^0\|_{E(\tilde J_0)}+\|u^0\|_{E(\tilde J_0)}\le C_{\rm St}\|w_0-\tilde w_0\|_{\dot X}+\tfrac{\tilde\varepsilon}{4}\le\tfrac{\tilde\varepsilon}{2}$$
if $\|w_0-\tilde w_0\|_{\dot X}\le\delta_0:=\min\{1,\tilde\varepsilon/(4C_{\rm St})\}$. So step 1) yields a solution in $\Sigma_c(\tilde J_0,\tilde r,\tilde\varepsilon)$ of (5.19) with $\tilde w(0)=\tilde w_0$ on $\tilde J_0$ given as a fixed point. By uniqueness, it follows $\tilde u=\Phi_{\tilde w_0}(\tilde u)$ on $\tilde J_0$.

Let $\|w^0-\tilde w^0\|_{\dot X}\le(2C_{\rm St})^{-M}\delta_0=:\delta$. By means of an obvious variant of (5.28) and Theorem 5.8, we estimate
$$\begin{aligned}|||u-\tilde u|||_{\tilde J_0}&\le|||\Phi_{w_0}(u)-\Phi_{w_0}(\tilde u)|||_{\tilde J_0}+|||\Phi_{w_0}(\tilde u)-\Phi_{\tilde w_0}(\tilde u)|||_{\tilde J_0}\\&\le\tfrac12|||u-\tilde u|||_{\tilde J_0}+|||u^0-\tilde u^0|||_{\tilde J_0}\le\tfrac12|||u-\tilde u|||_{\tilde J_0}+C_{\rm St}\|w_0-\tilde w_0\|_{\dot X},\\|||u-\tilde u|||_{\tilde J_0}&\le 2C_{\rm St}\|w^0-\tilde w^0\|_{\dot X}\le(2C_{\rm St})^{1-M}\delta_0.\end{aligned}$$
As in Theorem 4.17, based on (5.30) the above procedure can now be iterated $M-1$ times, obtaining $J\subseteq J(\tilde w_0)$ and $|||u-\tilde u|||_J\le M(2C_{\rm St})^M\|w^0-\tilde w^0\|_{\dot X}$. Negative times can be treated in the same way. Here one can replace w_0 by $v_0\in\overline{B}_{\dot X}(w_0,\delta)$ to obtain c), slightly modifying some constants (cf. Theorem 4.17).

The remaining claim e) can now be shown as in the previous theorem. $\square$

REMARK 5.19. Let $\alpha\in[3,5)$ and $m=3$. Then there is a number $\overline\rho_\alpha>0$ such that $J(w_0)=\mathbb{R}$ whenever $w_0\in\dot X$ satisfies $\|u_0\|_{\dot H^\theta}+\|u_1\|_{\dot H^{\theta-1}}\le\overline\rho_\alpha$ with $\theta=\theta_\alpha:=\frac32-\frac{2}{\alpha-1}\in[\frac12,1)$. See Theorem IV.3.1 in [**34**].

Using the preservation of the energy, we finally show global existence in the defocusing case if $\alpha\in[3,5)$. The arguments are similar to those in Section 4.4, but a bit simpler. We need the continuity of the energy, and thus work in H^1 exploiting $H^1\hookrightarrow L^{\alpha+1}$. (One can reduce the extra assumption $u_0\in L^2$, see Theorem 8.41 in [**4**].) One has blowup in the focusing case $\mu=-1$, see Exercise 3.9 in [**35**]. In the critical case $\alpha=5$, global existence for $\mu=1$ was shown by Shatah–Struwe and Kapitanski in 1994, see Section 5.1 in [**35**].

THEOREM 5.20. *Let $m=3$, $\alpha\in[3,5)$, $w_0=(u_0,u_1)\in X$, and $w=\varphi(\,\cdot\,,w_0)$ be the maximal H^1-solution of (5.19) from Theorem 5.17. We then have $\mathcal{E}_w(w(t))=\mathcal{E}_w(w_0)$ for all $t\in J(w_0)$. If $\mu=1$, we obtain $J(w_0)=\mathbb{R}$.*

PROOF. 1) For H^2-solutions u one can show $\mathcal{E}_w(w(t))=\mathcal{E}_w(w_0)$ as in Theorem 1.20. Let $u_0\in H^2$ and $u_1\in H^1$ with $\|u_0\|_{2,2}+\|u_1\|_{1,2}\le\rho$. We have to show that the H^1-solution u from Theorem 5.17 is an H^2-solution on $J(w_0)$. To this aim, as in Proposition 4.18 we set $u_h(t)=u(t,\cdot+h)$ and $v_h=(u_h-u)/|h|$ for $h\in\mathbb{R}^3\setminus\{0\}$. Fix a compact interval $J\subseteq J(w_0)$, set $r=|||u|||_J$ and $J_b=J\cap[-b,b]$ for $b>0$. Because of (5.19), the difference quotient satisfies
$$\partial_t^2v_h=\Delta v_h+|h|^{-1}(\phi(u_h)-\phi(u))=\Delta v_h+\int_0^1\phi'(u+\tau(u_h-u))v_h\,\mathrm{d}\tau.$$

Using Theorem 5.8 and (4.36), we then estimate as in (5.21)

$$\begin{aligned}\|v_h\|_{\dot G(J_b)} &\le C_{\mathrm{St}}\big(\|v_h(0)\|_{1,2}+\|\partial_t v_h(0)\|_2+\alpha\big\|(|u_h|+|u|)^{\alpha-1}v_h\big\|_{L^1_{J_b}L^2}\big)\\ &\le C_{\mathrm{St}}\big(\|u_0\|_{2,2}+\|u_1\|_{1,2}+\alpha C_{\mathrm{So}}(2b)^{\frac{5-\alpha}{2}}(2\|u\|_{E(J_b)})^{\alpha-1}\|v_h\|_{L^\infty_{J_b}\dot H^1}\big)\\ &\le C_{\mathrm{St}}\rho+\alpha C_{\mathrm{St}}C_{\mathrm{So}}(2b)^{\frac{5-\alpha}{2}}(2r)^{\alpha-1}\|v_h\|_{\dot G(J_b)}.\end{aligned}$$

Choosing a small $b=b(r)$, we can absorb the last term obtaining

$$\|v_h\|_{\dot G(J_b)}\le 2C_{\mathrm{St}}\rho.$$

In finitely many iterations we infer that v_h is uniformly bounded in $\dot G(J)$. The characterization (4.36) then yields the boundedness of $w=(u,\partial_t u):J\to H^2\times H^1$. As in Proposition 4.18 we see that w belongs to $L^\infty(J,H^2\times H^1)$, and hence ∇u to $L^\infty(J,L^6)$ by Sobolev's embedding. Note that $|\partial_k\phi(u)|=\alpha|u|^{\alpha-1}|\partial_k u|$. Since $u\in E(J)$ by Theorem 5.17, Hölder's inequality with $\frac12=\frac13+\frac16$ implies that $\partial_k\phi(u)$ is an element of $L^1(J,L^2)$. Moreover, we have

$$\partial_{jk}u(t)=C(t)\partial_{jk}u_0+S(t)\partial_j|\nabla|^{-1}\partial_k u_1+\int_0^t S(t-\tau)\partial_j|\nabla|^{-1}\partial_k\phi(u(\tau))\,\mathrm{d}\tau.$$

for $t\in J$ so that $u:J\to H^2$ is continuous as $u_0\in H^2$ and $u_1\in H^1$. Similarly one obtains the continuity of $\partial_t u:J\to H^1$. Equation (5.19) yields $u\in C^2(J,L^2)$ since $\phi(u)\in C(J,L^2)$ as $H^2\hookrightarrow L^\infty$ by Sobolev's embedding.

2) Let $w_0=(u_0,u_1)\in X$. There are functions $w_{0,n}=(u_{0,n},u_{1,n})\in H^2\times H^1$ converging to w_0 in X as $n\to\infty$. Let $J\subseteq J(w_0)$ be compact. Theorem 5.17 shows that $J\subseteq J(w_{0,n})$ for all large n and that the solutions $w_n(t)$ tend to $w(t)$ in X as $n\to\infty$ for $t\in J$. By step 1), w_n satisfies $\mathcal{E}_w(w_n(t))=\mathcal{E}_w(w_{0,n})$ for $t\in J$ which then also holds for w due to the continuity of $\mathcal{E}_w:X\to\mathbb{R}$. Hence, the first asertion is shown.

Let $\mu=1$. Then the energy preservation yields $\|w(t)\|_{\dot X}\le 2\mathcal{E}_w(w(t))=2\mathcal{E}_w(w_0)$ for $t\in J(w_0)$ so that $J(w_0)=\mathbb{R}$ by the blow-up condition in Theorem 5.17. □

CHAPTER 6

Quasilinear parabolic problems

So[1] far we have investigated semilinear problems in which a generator was perturbed by a nonlinear term of 'lower order'. In a quasilinear problem the linear operator itself depends nonlinearly on the solution though this dependence is again of 'lower order'.

In this short chapter we prove the local wellposedness of a basic class of such systems in the parabolic case. We study the equation

$$u'(t) = A(u(t))u(t) + F(u(t)), \quad t \in J, \qquad u(0) = u_0. \tag{6.1}$$

Here $A(v)$ with *fixed* domain X_1 is a sectorial operator in X of angle $\varphi > \pi/2$ depending on vectors $v \in X_\gamma := (X, X_1)_{1-\frac{1}{p},p}$ for a fixed $p \in (1,\infty)$, cf. Section 2.1. We assume that the mappings $A : X_\gamma \to \mathcal{B}(X_1, X)$ and $F : X_\gamma \to X$ are Lipschitz on balls, that $u_0 \in X_\gamma$, and that $J \subseteq \mathbb{R}$ is a non-empty open interval with $0 = \inf J$. We start with a prototypical example.

EXAMPLE 6.1. Let $G \subseteq \mathbb{R}^m$ be an open and bounded set with a C^2-boundary and let $a \in C^2(\mathbb{R}, \mathbb{R}^{m\times m})$ satisfy $a = a^\top \geq \eta I$ for a number $\eta > 0$. Fixing an exponent $p \in (m+2,\infty)$, we set $X = L^p(G)$ and $X_1 = W^{2,p}(G) \cap W^{1,p}(G)$. As explained in Example 2.12, one has the embedding $X_\gamma \hookrightarrow W^{2-\frac{2}{p},p}(G)$. The fractional Sobolev embedding theorem then implies that $X_\gamma \hookrightarrow C^1(\overline{G})$ since $2 - \frac{2}{p} - \frac{m}{p} > 0$, cf. Theorem 4.6.1 in [**37**]. We define the operator $A(v)$ by $A(v)u = \operatorname{div}(a(v)\nabla u)$ for $u \in X_1$ and $v \in X_\gamma$, and let $F(v) = f(v, \nabla v)$ be a reaction-convection term as in Example 3.7.

In this case the quasilinear problem (6.1) becomes the reaction-diffusion equation with state-depending anisotropic diffusion coefficients

$$\begin{aligned}
\partial_t u(t,x) &= \sum_{j,k=1}^m \partial_j\big(a_{jk}(u(t,x)\partial_k u(t,x)\big) + f(u(t,x), \nabla u(t,x)), \quad t > 0,\ x \in G,\\
u(t,x) &= 0, \quad t > 0,\ x \in \partial G, \qquad\qquad (6.2)\\
u(0,x) &= u_0(x), \quad x \in G.
\end{aligned}$$

We come back to this equation in Example 6.8. One can also treat analogous systems. Neumann-type boundary conditions are not covered by our setting since then the domain of $A(v)$ contains the condition $\operatorname{tr}(\nu \cdot a(v)\nabla u) = 0$ and thus depends on v.[2] See also the comments after Theorem 6.7. $\Diamond$

[1] This chapter was not part of the lectures.

[2] The domain becomes v-independent, however, if one replaces X_1 by $W^{1,q}(G)$ and passes to a weak formulation.

We want to solve (6.1) by a fixed-point procedure again. For a solution $u \in C(\overline{J}, X_1)$ we write

$$u'(t) = A(u_0)u(t) + (A(u(t)) - A(u_0))u(t) + F(u(t)). \tag{6.3}$$

One now replaces u in the last two summands by a given function v in our fixed-point space, say $v \in C(\overline{J}, X_1)$. Using the analytic C_0-semigroup $T(\cdot)$ generated by $A(u_0)$, Duhamel's formula then yields

$$u(t) = T(t)u_0 + \int_0^t T(t-s)\big(((A(v(s)) - A(u_0))v(s) + F(v(s))\big)\,\mathrm{d}s =: \Phi(v).$$

In a fixed-point procedure, the image $\Phi(v)$ must also belong to $C(\overline{J}, X_1)$. However, the inhomogeneity $f = (A(v) - A(u_0))v + F(v)$ is just an element of $C(\overline{J}, X)$ so that

$$T *_+ f(t) = \int_0^t T(t-s)f(s)\,\mathrm{d}s$$

is not contained in X_1, in general, since $T(t-s)$ has norm $c/(t-s)$ in $\mathcal{B}(X, X_1)$. See Example 4.1.7 in [**20**].

To overcome this problem, one can pass to classes of more regular functions, e.g., v being Hölder continuous in time or taking values in interpolation spaces smaller than X_1. This is done in [**1**], [**20**] or [**40**] in various ways.

Here we follow a different route by reducing the regularity level a bit. For this we have to introduce a new concept and discuss its basic properties. Let A be a closed, densely defined operator in X, where $X_1 = [\mathrm{D}(A)]$. We say that A has *maximal regularity of type L^p* on J if for all $f \in L^p(J, X) =: E_0(J)$ there exists a unique function $u \in W^{1,p}(J, X) \cap L^p(J, X_1) =: E_1(J)$ solving[3]

$$u'(t) = Au(t) + f(t), \quad \text{a.e. } t \in J, \qquad u(0) = 0. \tag{6.4}$$

We then write $A \in \mathrm{MR}_p(J)$.

Let $S : E_0(J) \to E_1(J);\ f \mapsto u$, be the solution operator of (6.4). Let $f_n \to 0$ in $E_0(J)$ and $u_n = Sf_n \to u$ in $E_1(J)$ as $n \to \infty$. Then u solves (6.4) with $f = 0$ and it thus must be equal to 0 by uniqueness. The closed graph theorem now shows that $\|u\|_{E_1(J)} \le C_p \|f\|_{E_0(J)}$ for a constant $C_p > 0$ and all $f \in E_0(J)$.

We next collect other basic facts about these spaces and this concept; many of them are employed in the proof of the local wellposedness result below.

Remark 6.2. We use the notation and definions introduced above.

a) We have the embeddings $E_1(J) \hookrightarrow C(\overline{J}, X_\gamma)$ and $E_1(\mathbb{R}_+) \hookrightarrow C_0(\mathbb{R}_{\ge 0}, X_\gamma)$. See Theorem III.4.10.2 in [**2**] or the exercises. The initial condition of (6.4) is understood in this sense. Moreover, the constant $c(J)$ of the embeddings can be bounded uniformly for intervals J whose length b is larger than a fixed number $b_0 > 0$. As $b \to 0$ the constant $c(b)$ will blow up e.g. for functions $t \mapsto v(t) = v_0$ with $v_0 \in X_1 \setminus \{0\}$, which is a severe obstacle in a fixed point argument. On a subspace of $E_1(J)$ this does not happen:

Let $b > 0$ and $v \in E_1(b)$ with $v(0) = 0$. We reflect v at b and extend the resulting function by 0 for $t > 2b$. This yields an extension $\tilde{v} \in E_1(\mathbb{R}_+)$ with

[3] We write $E_k(b)$ if $J = (0, b)$.

norm less or equal $2\|v\|_{E_1(b)}$. We then obtain the uniform bound

$$\|v\|_{C([0,b],X_\gamma)} = \|\tilde{v}\|_{C_0(\mathbb{R}_{\geq 0},X_\gamma)} \leq c(\mathbb{R}_+)\|\tilde{v}\|_{E_1(\mathbb{R}_+)} \leq 2c(\mathbb{R}_+)\|v\|_{E_1(b)}.$$

b) Let $A \in \mathrm{MR}_p(J)$ for an interval J and some $p \in (1,\infty)$. Then there is a number $\omega \geq 0$ such that A is sectorial of angle $\varphi > \pi/2$. In the case $J = \mathbb{R}_+$ one can choose $\omega = 0$, and A is invertible. See Theorem 17.2.25 in [**15**] or Proposition 3.5.2 in [**28**]. The proof implies that ω and the sectoriality constants (K,ϕ) of A are bounded by constants only depending on J, p, and C_p above. In turn, Lemma 2.22 of [**32**] gives an exponential bound of the semigroup in terms of ω, K and ϕ.

The solution of (6.4) is thus given by $u = T *_+ f$ for the analytic C_0-semigroup $T(\cdot)$ generated by A due to an L^p-variant of Proposition 2.6 of [**32**]. If one includes an initial value $u_0 \neq 0$ in (6.4), we have the solution $u = T(\cdot)u_0 + T *_+ f$. In the next item we treat the orbit $T(\cdot)u_0$.

c) Let A generate an analytic C_0-semigroup $T(\cdot)$ and $p \in (1,\infty)$. By Theorem 2.14, $T(\cdot)$ induces an analytic C_0-semigroup on X_γ by restriction. Proposition 2.8 says that a vector u_0 belongs to X_γ if and only if the orbit $T(\cdot)u_0$ is an element of $E_1(1)$. In this case one has $\|T(\cdot)u_0\|_{E_1(1)} \leq c\|u_0\|_{X_\gamma}$ for some constant $c > 0$. It follows that $\|T(\cdot)u_0\|_{E_1(b)} \leq c\|u_0\|_{X_\gamma}$ for $b \in (0,1)$. We further estimate

$$\|T(\cdot)u_0\|_{E_1(1,2)} = \|T(\cdot)T(1)u_0\|_{E_1(1)} \leq c\|T(1)\|_{\mathcal{B}(X_\gamma)}\|u_0\|_{X_\gamma}.$$

Iteratively we obtain that $\|T(\cdot)\|_{E_1(b)} \leq c(b_0)\|u_0\|_{X_\gamma}$ for $b \in (0,b_0]$ and any given $b_0 > 0$. If $\omega_0(A) < 0$, one finds a uniform constant for all $b > 0$ by this argument.

d) Let $A \in \mathrm{MR}_p(J)$. Let $J_0 \subseteq J = (0,b)$ be an open subinterval with $\inf J_0 = 0$. We extend $f \in E_0(J_0)$ by 0 to a function $\tilde{f} \in E_0(J)$. Then one has $T *_+ f = T *_+ \tilde{f}$ on J_0 and hence

$$\|u\|_{E_1(J_0)} \leq \|u\|_{E_1(J)} \leq c(J)\|\tilde{f}\|_{E_0(J)} = c(J)\|f\|_{E_0(J_0)}$$

for the solution u of (6.4). Let $f \in E_0(2b)$. For $t \in (b,2b)$ we compute

$$\begin{aligned}\int_0^t T(t-s)f(s)\,\mathrm{d}s &= \int_b^t T(t-s)f(s)\,\mathrm{d}s + T(t-b)\int_0^b T(b-s)f(s)\,\mathrm{d}s \\ &= \int_0^{t-b} T(t-b-r)f(r+b)\,\mathrm{d}r + T(t-b)\int_0^b T(b-s)f(s)\,\mathrm{d}s.\end{aligned}$$

The first term can be estimated in $E_1((b,2b))$ by the maximal regularity on $(0,b)$. Part a) implies that the norm in X_γ of the last integral is bounded by $\|f\|_{E_0(b)}$. Hence, the second summand is controlled in $E_1(b,2b)$ using part c). Summing up, we have maximal regularity on $(0,2b)$. This procedure can be iterated, so that $A \in \mathrm{MR}_p(b')$ for all $b' > 0$. This argument also yields that A has maximal regularity on $\mathbb{R}_+$ if $T(\cdot)$ is exponentially stable, in addition.

e) It is easy to check that A has maximal regularity on a bounded interval J if and only $A - \omega I \in \mathrm{MR}_p(J)$, since the later operator generates $\mathrm{e}^{-\omega t}T(t)$. For $\omega > \omega_0(A)$, the operator $A - \omega I$ then has maximal regularity on $\mathbb{R}_+$ by part d).

f) If an operator has maximal regularity of type L^p for some $p \in (1,\infty)$, it has this property for all $p \in (1,\infty)$. See Theorem 17.2.31 in [**15**]. (This result requires some harmonic analysis.)

g) There exist quite explicit examples of sectorial operators of angle $\varphi > \pi/2$ on spaces $L^1(\mu)$ or $C(K)$ without maximal regularity of type L^p. On L^q spaces with $q \neq 2$ such examples also exist, but they are not very explicit. See Section 17.4.a and b in [**15**]. $\Diamond$

We describe two classes of operators having maximal regularity of type L^p. The first one says that this property comes for free on Hilbert spaces, and the second one suffices for our Example 6.1.

Example 6.3. Let X be a Hilbert space and A be sectorial of angle $\varphi > \pi/2$ on X. Then A has maximal regularity of type L^p.

Proof. By Remark 6.2 it is enough to consider the case $p = 2$, and one can assume that $\omega_0(A) < 0$ and $J = \mathbb{R}_+$. We extend $f \in E_0(\mathbb{R}_+)$ and $T(\cdot)$ by 0 to $t < 0$ so that $T *_+ f(t) = T * f(t)$ for $t \geq 0$.

First take $f \in L^p(\mathbb{R}_+, X_1)$. Since X is a Hilbert space, Plancherel's theorem is valid for the Fourier transform $\mathcal{F}$ on $L^2(\mathbb{R}, X)$, see Theorem 2.4.9 in [**14**]. On further has the usual convolution theorem for $\mathcal{F}$ by Lemma 2.4.8 there. Using also the resolvent formula (4.2) from [**32**], we compute

$$\begin{aligned}
\|T *_+ f\|_{L^2(\mathbb{R}_+,X_1)} &\leq \|T * f\|_{L^2(\mathbb{R},X)} + \|AT * f\|_{L^2(\mathbb{R},X)} \\
&= \|\mathcal{F}(T * f)\|_{L^2(\mathbb{R},X)} + \|\mathcal{F}(T * Af)\|_{L^2(\mathbb{R},X)} \\
&= \sqrt{2\pi}\big(\|\widehat{T}\widehat{f}\|_{L^2(\mathbb{R},X)} + \|\widehat{T}A\widehat{f}\|_{L^2(\mathbb{R},X)}\big) \\
&= \|R(\mathrm{i}\,\cdot, A)\widehat{f}\|_{L^2(\mathbb{R},X)} + \|AR(\mathrm{i}\,\cdot, A)\widehat{f}\|_{L^2(\mathbb{R},X)} \\
&\leq c\|\widehat{f}\|_{L^2(\mathbb{R},X)} = c\|f\|_{L^2(\mathbb{R}_+,X)}.
\end{aligned}$$

Note that the operators $R(\mathrm{i}\tau, A)$ and $AR(\mathrm{i}\tau, A) = \mathrm{i}\tau R(\mathrm{i}\tau, A) - I$ are uniformly bounded for $\tau \in \mathbb{R}$ as A generates an exponentially stable analytic C_0-semigroup. By approximation the result then follows. $\square$

Example 6.4. Let A generate an analytic C_0-semigroup $T(\cdot)$ on $L^2(\mu)$ for a measure space $(S, \mathcal{A}, \mu)$. Assume that there is a constant $\omega \geq 0$ such that $\|\mathrm{e}^{-\omega t}T(t)f\|_q \leq \|f\|_q$ for all $f \in L^2(\mu) \cap L^q(\mu)$, $q \in [1,\infty]$, and $t \geq 0$. Then A has maximal regularity of type L^p, see Note 1.13 in [**17**].

Moreover, one has maximal regularity for each generator A of a positive and contractive analytic C_0-semigroup on $L^q(\mu)$ for some $q \in (1,\infty)$ by Corollary 17.3.9 in [**15**]. These results rely on deeper tools from operator theory and harmonic analysis. $\Diamond$

In view of the above example, it should be noted that semigroups generated by elliptic systems often fail to be contractive. Fortunately there is a quite convenient characterization of maximal regularity in 'good' Banach spaces due to Lutz Weis (2001). We present it now without giving many details.

We first describe the relevant class of Banach spaces. Let $f \in C_c^1(\mathbb{R}, X)$. By p. 374 of [**14**] the limit

$$Hf(t) = \lim_{\varepsilon \to 0^+, r \to \infty} \int_{\varepsilon < |s| < r} \frac{f(s)}{\pi(t-s)} \, \mathrm{d}s$$

exist for all $t \in \mathbb{R}$. One calls Hf the *Hilbert transform* of f. We call X a *UMD space*[4] if H has a continuous extension to $L^p(\mathbb{R}, X)$ for some (and then all) $p \in (1, \infty)$, cf. Theorem 5.1.1 in [**14**]. The spaces needed in our examples belong to this class.

Example 6.5. The spaces $X = L^q(\mu)$ for a measure space $(S, \mathcal{A}, \mu)$ and $q \in (1, \infty)$ are of class UMD by Proposition 4.2.15 of [**14**]. Proposition 4.2.17 of this monograph further shows that Cartesian products, closed subspaces, quotient spaces, duals, isomorphic images, and real (and complex) interpolation spaces with exponent $r \in (1, \infty)$ of UMD spaces have the same property. Hence, (closed subspaces of) Sobolev–Slobodetski $W^{\alpha,q}(G)$ and Besov spaces $B^\alpha_{q,r}(G)$ are UMD if $\alpha \geq 0$ and $1 < q, r < \infty$, cf. Example 2.3 and 2.5. $\Diamond$

We also need a stronger sectoriality concept. Let $\varepsilon_n : \Omega \to \{-1, 1\}$ be measurable functions on a probability space which are (stochastically) independent and have expectation 0 for $n \in \mathbb{N}$. An example are the Rademacher functions $r_n(t) = \operatorname{sign} \sin(2^n \pi t)$ on $\Omega = (0, 1)$ with the Lebesgue measure. A set $\mathcal{T} \subseteq \mathcal{B}(X)$ is called $\mathcal{R}$-*bounded* if there is a constant $C > 0$ such that

$$\forall\, N \in \mathbb{N},\ x_n \in X,\ T_n \in \mathcal{T}: \quad \Big\| \sum_{n=1}^N \varepsilon_n T_n x_n \Big\|_{L^2(\Omega;X)} \leq C \Big\| \sum_{n=1}^N \varepsilon_n x_n \Big\|_{L^2(\Omega;X)}.$$

See Paragraph 1.9 and Remark 2.6 of [**17**]. Roughly speaking, this means that we can estimate the operators T_n in sums with random signs if we take averages. Uniform boundedness follows from $\mathcal{R}$-boundedness (take $N = 1$), and these notions are equivalent in a Hilbert space X, see Paragraph 1.9 of [**17**].

A closed and densely defined operator A is called $\mathcal{R}$-*sectorial* of angle $\phi \in (0, \pi]$ if the set $\mathcal{T} = \{AR(\lambda, A) \,|\, \lambda \in \Sigma_\phi\}$ is $\mathcal{R}$-bounded.

We state Weis' result, see Theorem 17.3.1 of [**15**] or Theorem 4.4.4 of [**28**].

Theorem 6.6. *Let X be an UMD space. A closed and densely defined operator A on X has maximal regularity of type L^p if and only if it is $\mathcal{R}$-sectorial of angle $\phi > \pi/2$.*

In Hilbert spaces X one already knows this characterization in view of the above remarks and Example 6.3. Remark 6.2 g) shows that one needs a stronger condition than sectoriality even on L^q with $q \in (1, \infty) \setminus \{2\}$. Theorem 17.4.1 of [**17**] also indicates that the UMD property is crucial here.

Since $\mathcal{R}$-boundedness is a complicated property, one wonders whether $\mathcal{R}$-sectoriality can be checked in examples. Fortunately, a powerful theory is available to deal with this concept. First, there are more accessible sufficient conditions for it (like the properties mentioned in Example 6.4). Moreover, one has

[4] UMD stands for 'uniform martingal difference' which refers to the standard definition of UMD spaces in the literature, see Definition 4.2.1 in [**14**].

developed a machinery that allows to show $\mathcal{R}$-sectoriality based on standard techniques for the proof of sectoriality of elliptic operators with boundary conditions on $L^q(G)$. The monographs [**15**], [**17**] and [**28**] give an introduction to this theory and its applications.

We now come to a local wellposedness result for the quasilinear parabolic problem (6.1). In contrast to earlier sections we allow for nonlinearities defined only on an open subset $V_\gamma \subseteq X_\gamma$. By a *solution* of (6.1) on J we mean a function $u \in E_1(J) \hookrightarrow C(\overline{J}, X_\gamma)$ with values in V_γ and $u(0) = u_0$ which solves the differential equation (6.1) pointwise a.e. in X. Then all terms in (6.1) belong to $L^p(J, X)$. In quasilinear problems local boundedness is not enough to ensure global existence even if $X_\gamma = V_\gamma$. Below we require uniform continuity in X_γ, compare Theorem 4.17 for a different condition. The subscript γ refers to the norm of X_γ.

Theorem 6.7. *Let X_1 be densely embdded into X, $X_\gamma = (X, X_1)_{1-\frac{1}{p},p}$ for some $p \in (1, \infty)$, $V_\gamma \subseteq X_\gamma$ be open and nonempty. Assume that the maps $A : V_\gamma \to \mathcal{B}(X_1, X)$ and $F : V_\gamma \to X$ are Lipschitz on bounded subsets of V_γ and that $A(v)$ has maximal regularity of type L^q for each $v \in V_\gamma$. Let $\overline{u}_0 \in V_\gamma$. Then the following assertions are true.*

a) There is a radius $\rho = \rho(\overline{u}_0) > 0$ and a time $b = b(\overline{u}_0) > 0$ such that for all $u_0 \in \overline{B}_\gamma(\overline{u}_0, \rho) \subseteq V_\gamma$ we have a unique solution $u = \varphi(\,\cdot\,, u_0) \in E_1(b)$ of (6.1). These solutions satisfy

$$\|\varphi(\,\cdot\,, u_0) - \varphi(\,\cdot\,, v_0)\|_{E_1(b)} \le c\|u_0 - v_0\|_\gamma \tag{6.5}$$

for a constant $c = c(\overline{u}_0) > 0$ and all $u_0, v_0 \in \overline{B}_\gamma(\overline{u}_0, \rho)$.

b) We can extend u from a) to a unique solution $u = \varphi(\,\cdot\,, u_0) \in E_1(t^+(u_0))$ of (6.1) on the maximal existence interval $J(u_0) = [0, t^+(u_0))$. Here $t^+(u_0) < \infty$ implies that $u : J(u_0) \to X_\gamma$ is not uniformly continuous or that the distance $\operatorname{dist}_\gamma(u(t), \partial V_\gamma)$ tends to 0 as $t \to t^+(u_0)^-$.

c) Assume that the constants of maximal regularity of $A(v)$ are uniformly bounded for $v \in V_\gamma$. Let $\overline{b} \in J(\overline{u}_0)$. Then there is a radius $\delta = \delta(\overline{u}_0, \overline{b}) > 0$ such that for all $u_0, v_0 \in \overline{B}_\gamma(\overline{u}_0, \delta) \subseteq V_\gamma$ we have $t^+(u_0), t^+(v_0) > \overline{b}$ and the estimate (6.5) is true for $t \in [0, \overline{b}]$.

In contrast to Theorem 3.4 the above result does not provide a pointwise regularization of the solution. This can be achieved by an extension of our approach using weights in time, see Theorem 5.1.1 of [**28**]. In this way one can also obtain compactness in X_γ of bounded orbits if X_1 is compactly embedded into X as shown in Theorem 5.7.1 of [**28**] combined with Proposition 2.13.

In Theorem 6.7 all operators $A(v)$ have the same domain. One can show variants of it without this assumption using 'maximal regularity of type C^α,' roughly speaking. We refer to [**1**], [**20**] or [**40**] for such results. In the context of quasilinear parabolic partial differential equations results of this type have been achieved since the sixties, also employing maximal regularity proved directly for a specific class of PDEs. The methods discussed here can also be extended to such PDE with nonlinear boundary conditions or with moving interfaces as presented in [**28**]. Actually, such applications were a main motivation for this line of research.

Proof of Theorem 6.7. 1) We first collect auxiliary facts and prove the basic estimates. Let $b \in (0,1]$ and denote by $T(\cdot)$ the semigroup generated by $A(\overline{u}_0)$. Remark 6.2 yields the following inequalities with constants ≥ 1 not depending on b, where M_γ, C_{MR} and C^0_{MR} are functions of $\overline{u}_0$. We write $L^\infty_b X_\gamma$ for $L^\infty((0,b), X_\gamma)$ etc.

$$\forall\, v \in E_1(b) \text{ with } v(0) = 0: \quad v \in C([0,b], X_\gamma), \quad \|v\|_{L^\infty_b X_\gamma} \leq C_\gamma \|v\|_{E_1(b)}, \tag{6.6}$$

$$\forall\, v_0 \in X_\gamma: \quad T(\cdot)v_0 \in C([0,b], X_\gamma) \cap E_1(b), \quad \|T(\cdot)v_0\|_{L^\infty_b X_\gamma} \leq M_\gamma \|v_0\|_\gamma, \tag{6.7}$$

$$\|T(\cdot)v_0\|_{E_1(b)} \leq C^0_{\mathrm{MR}} \|v_0\|_\gamma, \tag{6.8}$$

$$\forall\, f \in E_0(b): \quad T *_+ f \in E_1(b), \quad \|T *_+ f\|_{E_1(b)} \leq C_{\mathrm{MR}} \|f\|_{E_0(b)}. \tag{6.9}$$

Fix a radius $\rho_0 > 0$ with $B_0 := \overline{B}_\gamma(\overline{u}_0, \rho_0) \subseteq V_\gamma$. Take $\rho \in (0, \rho_0]$ and $u_0 \in B_\gamma(\overline{u}_0, \rho)$. We set $u_* = T(\cdot)u_0$ and $\overline{u}_* = T(\cdot)\overline{u}_0$. To obtain smallness below, we need the limit

$$\kappa_0(b) := \max_{t\in[0,b]} \|\overline{u}_*(t) - \overline{u}_0\|_\gamma \;\longrightarrow\; 0, \qquad b \to 0.$$

The estimate (6.6) is only uniform as $b \to 0$ for functions vanishing at 0. For this reason, we incorporate the initial condition $v(0) = v_0$ in our fixed point space and substract u_* from v. The other constants are tied to $\overline{u}_0$ and we will thus linearize the equation at $\overline{u}_0$ (and not at the initial value u_0 as in (6.3)). So the difference $v - \overline{u}_0$ appears naturally. Let $v \in E_1(b)$ with $v(0) = u_0$ and $\|v - \overline{u}_*\|_{E_1(b)} \leq r$ for some $r > 0$. Using (6.6), (6.7) and (6.8), we estimate

$$\begin{aligned}
\|v - \overline{u}_0\|_{L^\infty_b X_\gamma} &\leq \|v - u_*\|_{L^\infty_b X_\gamma} + \|T(\cdot)(u_0 - \overline{u}_0)\|_{L^\infty_b X_\gamma} + \|\overline{u}_* - \overline{u}_0\|_{L^\infty_b X_\gamma} \\
&\leq C_\gamma \|v - u_*\|_{E_1(b)} + M_\gamma \|u_0 - \overline{u}_0\|_\gamma + \kappa_0(b) \\
&\leq C_\gamma \|v - \overline{u}_*\|_{E_1(b)} + C_\gamma \|T(\cdot)(u_0 - \overline{u}_0)\|_{E_1(b)} + M_\gamma \rho + \kappa_0(b) \\
&\leq C_\gamma r + C_\gamma C^0_{\mathrm{MR}} \rho + M_\gamma \rho + \kappa_0(b) \\
&\leq C_\gamma r + \frac{r}{3} + \frac{r}{3} \leq \rho_0,
\end{aligned} \tag{6.10}$$

where we take $r \in (0, r_0]$, $\rho \in (0, \rho_1]$ and $b \in (0, b_0]$ with $b_0 \leq 1$,

$$r_0 = \frac{\rho_0}{3C_\gamma} \leq \rho_0, \qquad \rho_1 = \frac{r}{3(C_\gamma C^0_{\mathrm{MR}} + M_\gamma)} \leq \rho_0, \qquad \kappa_0(b_0) \leq \frac{r}{3}. \tag{6.11}$$

Observe that the numbers r_0, $\rho_1 = \rho_1(r)$ and $b_0 = b_0(r)$ only depend on $\overline{u}_0$, ρ_0, and r. As in the above computation, we infer

$$\|u_* - \overline{u}_*\|_{E_1(b)} \leq C^0_{\mathrm{MR}} \|u_0 - \overline{u}_0\|_\gamma \leq C^0_{\mathrm{MR}} \rho \leq r/3. \tag{6.12}$$

2) For ρ, b and r as above, we take $u_0 \in \overline{B}_\gamma(\overline{u}_0, \rho)$ and define

$$\Sigma(b, r) := \{v \in E_1(b) \,|\, v(0) = u_0,\; \|v - \overline{u}_*\|_{E_1(b)} \leq r\}.$$

This set contains u_* due to (6.12). It is complete for the metric $\|v - w\|_{E_1(b)}$ because of (6.6). Let $v, w \in \Sigma(b, r)$. Inequality (6.10) shows that $v(t) \subseteq B_0 \subseteq$

V_γ for all $t \in [0,b]$. Let L_0 be the maximum of the Lipschitz constants of A and F on B_0. We set

$$G(v) = (A(v) - A(\overline{u}_0))v + F(v) \ \in E_0(b).$$

The assumption and Remark 6.2 provide a solution $u =: \Phi(v) = \Phi_{u_0}(v) \in E_1(b)$ of the linear problem

$$u'(t) = Au(t) + G(v(t)), \quad t \in (0,b), \qquad u(0) = u_0. \tag{6.13}$$

It is given by $u = T(\cdot)u_0 + T *_+ G(v)$.

We set $\kappa_1(b) = \|\overline{u}_*\|_{E_1(b)}$ and note that this number tends to 0 as $b \to 0$. The inequalities (6.8), (6.9), (6.10), and (6.12) imply

$$\begin{aligned}
&\|\Phi(v) - \overline{u}_*\|_{E_1(b)} \le \|u_* - \overline{u}_*\|_{E_1(b)} + \|T *_+ G(v)\|_{E_1(b)} \\
&\quad \le C^0_{\mathrm{MR}}\|u_0 - \overline{u}_0\|_\gamma + C_{\mathrm{MR}}\|(A(v) - A(\overline{u}_0))v\|_{E_0(b)} + C_{\mathrm{MR}}\|F(v) \mp F(\overline{u}_0)\|_{E_0(b)} \\
&\quad \le C^0_{\mathrm{MR}}\rho + C_{\mathrm{MR}}L_0\|v - \overline{u}_0\|_{L^\infty_b X_\gamma}\|v - \overline{u}_* + \overline{u}_*\|_{E_1(b)} \\
&\qquad\quad + C_{\mathrm{MR}}b^{\frac{1}{p}}(L_0\|v - \overline{u}_0\|_{L^\infty_b X_\gamma} + \|F(\overline{u}_0)\|_\gamma) \\
&\quad \le \tfrac{r}{3} + +C_{\mathrm{MR}}L_0 r(C_\gamma + \tfrac{2}{3})(r + \kappa_1(b)) + C_{\mathrm{MR}}b^{\frac{1}{p}}(L_0\rho_0 + \|F(\overline{u}_0)\|_\gamma) \\
&\quad \le \tfrac{r}{3} + \tfrac{r}{3} + \tfrac{r}{3} = r.
\end{aligned} \tag{6.14}$$

Here we fix $r = r_1$ and $\rho = \rho_1(r_1)$ and take $b \in (0, b_2]$ with

$$\begin{aligned}
&r_1 = \min\{r_0, (6\hat{c})^{-1}\}, \qquad \hat{c} = C_{\mathrm{MR}}L_0(C_\gamma + \tfrac{2}{3}), \qquad \kappa_1(b_1) \le r, \\
&b_2 = \min\big\{b_0, b_1, r^p\big(3C_{\mathrm{MR}}(L_0\rho_0 + \|F(\overline{u}_0)\|_\gamma)\big)^{-p}\big\}.
\end{aligned} \tag{6.15}$$

Similarly, using estimates (6.9), (6.10), (6.6), (6.14), (6.15) and $(v-w)(0) = 0$, we compute

$$\begin{aligned}
&\|\Phi(v) - \Phi(w)\|_{E_1(b)} \le C_{\mathrm{MR}}\|G(v) - G(w)\|_{E_0(b)} \\
&\quad \le C_{\mathrm{MR}}\big(\|(A(v) - A(\overline{u}_0))(v - w)\|_{E_0(b)} + \|(A(v) - A(w))w\|_{E_0(b)} \\
&\qquad\quad + \|F(v) - F(w)\|_{E_0(b)}\big) \\
&\quad \le C_{\mathrm{MR}}L_0\big(\|v - \overline{u}_0\|_{L^\infty_b X_\gamma}\|v - w\|_{L^p_b X_1} + \|v - w\|_{L^\infty_b X_\gamma}\|w - \overline{u}_* + \overline{u}_*\|_{L^p_b X_1} \\
&\qquad\quad + \|v - w\|_{L^p_b X_\gamma}\big) \\
&\quad \le \big(\hat{c}r + C_{\mathrm{MR}}L_0C_\gamma(r + \kappa_1(b)) + C_{\mathrm{MR}}L_0 b^{\frac{1}{p}}c_\gamma\big)\|v - w\|_{E_1(b)} \\
&\quad \le (\tfrac{1}{6} + \tfrac{1}{3} + \tfrac{1}{6})\|v - w\|_{E_1(b)} = \tfrac{2}{3}\|v - w\|_{E_1(b)},
\end{aligned} \tag{6.16}$$

Here c_γ is the norm of the embedding $X_\gamma \hookrightarrow X_1$ from Proposition 2.2, and we have taken

$$0 < b \le b_3 := \min\{b_2, (6C_{\mathrm{MR}}L_0c_\gamma)^{-p}\}. \tag{6.17}$$

As a consequence, $\Phi = \Phi_{u_0} : \Sigma(r_1, b) \to \Sigma(r_1, b)$ is a strict contraction for each initial value $u_0 \in \overline{B}_\gamma(\overline{u}_0, \rho_1(r_1))$. The fixed point $u \in \Sigma(r_1, b)$ solves (6.1) on $[0,b]$ uniquely in $\Sigma(r_1, b)$.

3) Let $u \in E_1(J_u)$ and $v \in E_1(J_v)$ solve (6.1) on open intervals J_u and J_v, respectively. We suppose that $u \ne v$ on $J = J_u \cap J_v$. Then there are times t_n in J with limit $\tau \in J \cup \{0\}$ such that $u = v$ on $[0,\tau]$, $t_n > \tau$, and $u(t_n) \ne v(t_n)$ for all $n \in \mathbb{N}$. We replace in steps 1) and 2) the vector $\overline{u}_0 \in V_\gamma$ by

$\overline{u}_0' = u(\tau) = v(\tau) \in V_\gamma$, where we fix a radius $\rho_0' > 0$ such that $\overline{B}_\gamma(\overline{u}_0', \rho_0') \subseteq V_\gamma$. This leads to numbers r_1' and b_3' as in (6.11), (6.15), and (6.17). We thus have a unique fixed point $w_b \in \Sigma(r_1', b)$ of (6.1) with initial value $\overline{u}_0'$ on $(0, b)$ for each $b \in (0, b_3']$.

On the other hand, there is a number $\beta \in (0, b_3']$ such that $\tau + \beta \in J$, $\|u(\cdot - \tau) - T(\cdot)\overline{u}_0'\|_{E_1(\beta)} \le r_1'$, and $\|v(\cdot - \tau) - T(\cdot)\overline{u}_0'\|_{E_1(\beta)} \le r_1'$. This means that $u(\cdot - \tau)$ and $v(\cdot - \tau)$ both belong to $\Sigma(r_1', \beta)$ and thus have to coincide. This contradiction shows that $u = v$ on $J_u \cap J_v$.

4) Let u and v solve (6.1) on $[0, b_3] = [0, b]$ for initial data $u_0, v_0 \in \overline{B}(\overline{u}_0, \rho_1(r_1))$ as found in step 2). We thus have $u = \Phi_{u_0}(u)$ and $v = \Phi_{v_0}(v)$. Observing that $\Phi_{u_0}(u) - \Phi_{v_0}(u) = T(\cdot)(u_0 - v_0)$, we derive from (6.8) and (6.16) the bound

$$\begin{aligned} \|u - v\|_{E_1(b)} &\le \|T(\cdot)(u_0 - v_0)\|_{E_1(b)} + \|\Phi_{v_0}(u) - \Phi_{v_0}(v)\|_{E_1(b)} \\ &\le C_{\mathrm{MR}}^0 \|u_0 - v_0\|_\gamma + \tfrac{2}{3}\|u - v\|_{E_1(b)}, \\ \|u - v\|_{E_1(b)} &\le 3C_{\mathrm{MR}}^0 \|u_0 - v_0\|_\gamma. \end{aligned} \tag{6.18}$$

We thus have shown assertion a).

5) Let $u_0 \in V_\gamma$. Based on step 3), we obtain as usual a unique solution $\varphi(\cdot, u_0)$ of (6.1) on $J(u_0) = [0, t^+(u_0))$ with

$$t^+(u_0) = \sup\{b > 0 \,|\, \exists\, u_b \in E_1(b) \text{ solving (6.1) on } (0, b)\}.$$

Note that $t^+(u_0) \ge b_3(u_0)$ for the number b_3 from (6.17) if one replaces $\overline{u}_0$ by u_0 as in step 3). One can also restart the problem at $b_3(u_0)$ and obtains a solution on a larger interval so that $t^+(u_0) > b_3(u_0)$.

Suppose that $t^+(u_0) < \infty$ and that $u : J(u_0) \to X_\gamma$ is uniformly continuous and $\operatorname{dist}_\gamma(u(t), \partial V_\gamma) \ge \delta > 0$ for all $t \in [0, t^+(u_0))$. Then the limit $u(t) \to u_1$ as $t \to t^+(u_0)$ exists in X_γ and u_1 still belongs to V_γ. One can thus extend the solution as above and obtains a contradiction, so that b) holds.

6) Take $\overline{b} \in (0, t^+(\overline{u}_0))$ and fix $b' \in (\overline{b}, t^+(\overline{u}_0))$. We write $\overline{u}(t) = \varphi(t, \overline{u}_0)$ and note that the orbit $\Gamma = \{\overline{u}(t) \,|\, t \in [0, b']\}$ is compact in V_γ. It thus has a positive distance to ∂V_γ, and we can redefine $\rho_0 > 0$ from step 1) such that $\operatorname{dist}_\gamma(\Gamma, \partial V_\gamma) > \rho_0$. In parts 1) and 2) we replace $\overline{u}_0$ by $v \in \Gamma$ in the definition of all constants and of κ_j keeping the notation. By the assumption in c), the constants C_{MR}^0 and C_{MR} are then uniformly bounded $v \in \Gamma$. The same is true for M_γ by Remark 6.2 and Proposition 2.4. Moreover, the functions $\kappa_j(b)$ tend to 0 as $b \to 0$ uniformly for $v \in \Gamma$ by the compactness of Γ and (6.8). As a result, the numbers $r = r_1$, $\rho = \rho_1(r_1) \le \rho_0$, and $b = b_3$ in (6.15) and (6.17) can be chosen uniformly for the vectors $\overline{u}(t)$ with $t \in [0, b']$ instead of $\overline{u}_0$.

Let $t_k = kb$ and $J_k = [t_{k-1}, t_k]$ for $k \in \mathbb{N}$ and N be the first integer with $t_N > b$. If $Nb > b'$, we redefine $T_N = b'$. We set $C = 3C_\gamma C_{\mathrm{MR}}^0 \ge 1$ and $\delta = C^{-N}\rho$. Let $u_0 \in \overline{B}_\gamma(\overline{u}_0, \delta)$. Estimates (6.18) and (6.6) imply that

$$\|u(b) - \overline{u}(b)\|_\gamma \le C_\gamma \|u - \overline{u}\|_{E_1(b)} \le 3C_\gamma C_{\mathrm{MR}}^0 \|u_0 - \overline{u}_0\|_\gamma \le C\delta \le \rho.$$

We can thus extend u to $[0, t_2]$ by step 1) and deduce

$$\|u(t_2) - \overline{u}(t_2)\|_\gamma \le C\|u(t_1) - \overline{u}(t_1)\|_\gamma \le C^2\delta \le \rho.$$

Iteratively we see that u exists on $[0, b_N]$; i.e., $t^+(u_0) > \overline{b}$.

Let also $v_0 \in \overline{B}_\gamma(\overline{u}_0, \delta)$. We can now replace in the above argument $\overline{u}$ by v, obtaining

$$\|u - v\|_{E_1(J_k)} \leq 3C^0_{\mathrm{MR}} \|u(t_{k-1}) - v(t_{k-1})\|_\gamma \leq 3C^0_{\mathrm{MR}} C^{N-1} \|u_0 - v_0\|_\gamma$$

for all $k \in \{1, \dots, N\}$. So assertion c) follows by the triangle inequality. $\square$

One can apply Theorem 6.7 to the quasilinear reaction-diffusion problem (6.1) in Example 6.1 as we briefly sketch.

Example 6.8. In the setting of Example 6.1, let $V_\gamma = X_\gamma$ and write $\partial_j(a_{jk}(u)\partial_k u) = a_{jk}(u)\partial_{jk} u + \partial_k a_{jk}(u)\partial_k u$. (One would employ sets of the form $V_\gamma = \{v \in X_\gamma \,|\, a(v) > \alpha I > 0\}$ if one only assumes that $a = a^\top > 0$.) Using $X_\gamma \hookrightarrow C^1(\overline{G})$, one can the check that $A : V_\gamma \to \mathcal{B}(X_1, X)$ and $F : V_\gamma \to X$ are Lipschitz on bounded subsets of X_γ similar as in Example 3.7. The maximal regularity of $A(v)$ can de deduced from Example 6.3 above and Theorem 7.3.6 of [**23**], where positivity is shown as in Example 5.2 of [**32**].

As in Theorem 3.9 global existence here follows again from uniform boundedness if $F(v) = f(v)$. See Theorem 5.2 of [**1**], where even certain elliptic systems were studied in a somewhat different setting. $\Diamond$

The asymptotic behavior of problems such as (6.1) was investigated e.g. in [**20**], [**27**] or [**28**].

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