Lecture Notes
Nonlinear Maxwell Equations

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These lecture notes are based on my course from winter semester 2024/25, though there are minor corrections and improvements as well as small changes in the numbering of equations. Typically, the proofs and calculations in the notes are a bit shorter than those given in the course. Many additional oral remarks from the lectures are omitted here. It is assumed that the reader has a solid background in functional analysis and Sobolev spaces. Occasionally I use notation and definitions of my lecture notes Functional Analysis and Spectral Theory without further notice.

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CHAPTER 1

The Maxwell system on $\mathbb{R}^3$

The Maxwell equations are the fundamental laws of electro-magnetism and play an important role as building blocks of many coupled systems. They relate the *electric field* $E(t,x) \in \mathbb{R}^3$, the (electric) *displacement field* $D(t,x) \in \mathbb{R}^3$, the *magnetic field* $B(t,x) \in \mathbb{R}^3$ and the *magnetizing field* $H(t,x) \in \mathbb{R}^3$ via the Maxwell–Ampère and Maxwell–Faraday laws

$$\partial_t D = \operatorname{curl} H - J_e, \qquad \partial_t B = -\operatorname{curl} E, \qquad t \geq 0,\ x \in \mathbb{R}^3, \tag{1.1}$$

where $J_e(t,x) \in \mathbb{R}^3$ is the *current density.* Here E and B can be measured in experiments via the Lorentz force they exert on charges. (See e.g. [**29**] for the background in physics.) On spatial domains $G \neq \mathbb{R}^3$ one has to add boundary conditions to (1.1) as discussed in Chapter 2 of [**44**]. We use the standard differential expressions

$$\operatorname{curl} u = \nabla \times u = \begin{pmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \quad \operatorname{div} u = \nabla \cdot u = \partial_1 u_1 + \partial_2 u_2 + \partial_3 u_3,$$

where the derivatives are interpreted in a weak sense if needed (cf. Section 2.1). Roughly speaking, the Maxwell equations say that the electric field is changed by a current or by magnetic vortices, and that the magnetic field is changed by electric vortices in the opposite way.

We note that these fields have the SI units $NC^{-1} = Vm^{-1}$ for E, Cm^{-2} for D, tesla $T = NA^{-1}m^{-1}$ for B, Am^{-1} for H, and Am^{-2} for J_e, using the more basic units newton N, volt V, coulomb C, and ampère A.

To complete the Maxwell system (1.1), we have to connect the fields via *material laws.* They involve the *polarization* $P = D - \varepsilon_0 E$ and the *magnetization* $M = \mu_0^{-1} B - H$ which describe the material response to the fields E and B, namely the density of induced electric, respectively magnetic, dipole moments. Here $\varepsilon_0 \approx 8.854 \cdot 10^{-12}\, Fm^{-1}$ is the vacuum permittivity and $\mu_0 \approx 1.257 \cdot 10^{-6}\, Hm^{-1}$ the vacuum permeability, with the units farad $F = CV^{-1}$ and henry $H = Tm^2A^{-1}$. In the following we ignore units and set $\varepsilon_0 = 1 = \mu_0$ so that the speed of light in vacuum becomes $c = 1$. Otherwise one has $c = (\varepsilon_0\mu_0)^{-1/2}$.

We collect **some formulas**. First let $u \in W^{2,1}_{\mathrm{loc}}(U, \mathbb{R}^3)$ and $\varphi \in W^{2,1}_{\mathrm{loc}}(U, \mathbb{R})$, where $U \subseteq \mathbb{R}^3$ is open. (We often omit the range spaces.) First, one obtains

$$\operatorname{div} \operatorname{curl} u = \partial_1(\partial_2 u_3 - \partial_3 u_2) + \partial_2(\partial_3 u_1 - \partial_1 u_3) + \partial_3(\partial_1 u_2 - \partial_2 u_1) = 0, \tag{1.2}$$

$$\operatorname{curl} \nabla\varphi = \begin{pmatrix} \partial_2\partial_3\varphi - \partial_3\partial_2\varphi \\ \partial_3\partial_1\varphi - \partial_1\partial_3\varphi \\ \partial_1\partial_2\varphi - \partial_2\partial_1\varphi \end{pmatrix} = 0. \tag{1.3}$$

Because of (1.2), solutions to (1.1) fulfill *Gauß' laws*

$$\rho_e(t) := \operatorname{div} D(t) = \operatorname{div} D(0) - \int_0^t \operatorname{div} J_e(s)\,\mathrm{d}s, \qquad \operatorname{div} B(t) = \operatorname{div} B(0) \tag{1.4}$$

for $t \geq 0$. The *electric charge density* ρ_e (with unit Cm^{-3}) is thus determined by the initial charge and the current density. As there are no magnetic charges and currents in physics, one typically requires $\operatorname{div} B(0) = 0$. However, in the analytic treatment such quantities often appear and will be included later on. A control on the charges is crucial to counteract the large kernel of curl, cf. (1.3).

For $u \in W^{1,p}_{\mathrm{loc}}(U, \mathbb{R}^3)$ and $\varphi \in W^{1,p'}_{\mathrm{loc}}(U, \mathbb{R})$ with $p \in [1, \infty]$ we further have the product formulas

$$\operatorname{div}(\varphi u) = \partial_1(\varphi u_1) + \partial_2(\varphi u_2) + \partial_3(\varphi u_3) = \nabla\varphi \cdot u + \varphi \operatorname{div} u, \tag{1.5}$$

$$\begin{aligned} \operatorname{curl}(\varphi u) &= \begin{pmatrix} \partial_2(\varphi u_3) - \partial_3(\varphi u_2) \\ \partial_3(\varphi u_1) - \partial_1(\varphi u_3) \\ \partial_1(\varphi u_2) - \partial_2(\varphi u_1) \end{pmatrix} = \begin{pmatrix} \partial_2\varphi u_3 - \partial_3\varphi u_2 \\ \partial_3\varphi u_1 - \partial_1\varphi u_3 \\ \partial_1\varphi u_2 - \partial_2\varphi u_1 \end{pmatrix} + \varphi \operatorname{curl} u \\ &= \nabla\varphi \times u + \varphi \operatorname{curl} u. \end{aligned} \tag{1.6}$$

The dot denotes the scalar product in $\mathbb{R}^m$, and the cross product in $\mathbb{R}^3$ is given by

$$a \times b = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix} = \begin{pmatrix} a \cdot \hat{b}^1 \\ a \cdot \hat{b}^2 \\ a \cdot \hat{b}^3 \end{pmatrix}$$

with $\hat{b}^1 = (0, b_3, -b_2)$, $\hat{b}^2 = (-b_3, 0, b_1)$, and $\hat{b}^3 = (b_2, -b_1, 0)$. For $u \in W^{1,p}(U)$, $v \in W^{1,p'}(U)$ and a (say, compact) Lipschitz boundary ∂U with outer unit normal ν, the divergence theorem yields

$$\begin{aligned} \int_U \operatorname{curl} u \cdot v \,\mathrm{d}x &= \sum_{j=1}^3 \int_U v_j \operatorname{div} \hat{u}^j \,\mathrm{d}x = \sum_{j=1}^3 \Big(-\int_U \hat{u}^j \cdot \nabla v_j \,\mathrm{d}x + \int_{\partial U} \nu \cdot \hat{u}^j\, v_j \,\mathrm{d}\sigma \Big) \\ &= \int_U [u_2\partial_3 v_1 - u_3\partial_2 v_1 + u_3\partial_1 v_2 - u_1\partial_3 v_2 - u_2\partial_1 v_3 + u_1\partial_2 v_3]\mathrm{d}x + \int_{\partial U} \nu \times u \cdot v \,\mathrm{d}\sigma \\ &= \int_U u \cdot \operatorname{curl} v \,\mathrm{d}x + \int_{\partial U} u \cdot (v \times \nu)\,\mathrm{d}\sigma. \end{aligned} \tag{1.7}$$

The boundary term disappears if $U = \mathbb{R}^3$, or if u or v have compact support. Here we also used the first of the formulas (with $a, b, c \in \mathbb{R}^3$)

$$a \cdot (b \times c) = b \cdot (c \times a) = c \cdot (a \times b), \quad a \times (b \times c) = b(a \cdot c) - c(a \cdot b). \tag{1.8}$$

We briefly discuss **material laws**, see [1], [7], or [12] for a systematic treatment in the context of nonlinear optics.

1) In these notes we focus on *instantaneous constitutive relations*. At first we look at the general case

$$(D, B) = \theta(x, E, H) = (\theta_e(x, u), \theta_m(x, u)) \quad \text{for} \quad \theta\colon U \times \mathbb{R}^6 \to \mathbb{R}^6. \tag{1.9}$$

Here we choose $u = (E, H)$ as state which suits best to energy estimates. The choice $v = (D, B)$ is also possible since $\theta(x, \cdot)$ is invertible under our assumptions, at least locally. This state fits better to (1.4) and is also used later on.

Our main hypothesis will be that $a_0(x,u) := \partial_u\theta(x,u)$ belongs to the space $\mathbb{R}^{6\times 6}_{\geq\eta}$ of symmetric real matrices with $a_0 \geq \eta I$ for some number $\eta > 0$. In an exercise, the typical autonomous linear case $\theta(x,E,H) = (\varepsilon(x)E, \mu(x)H)$ with $\varepsilon, \mu \in L^\infty(U, \mathbb{R}^{3\times 3}_{\geq\eta})$ is studied. We discuss some basic nonlinear material laws.

Example 1.1. A standard example in nonlinear optics is the Kerr law

$$D = \varepsilon_{\mathrm{lin}}(x)E + \kappa(x)|E|^2\, E, \qquad H = B,$$

for bounded functions $\varepsilon_{\mathrm{lin}}, \kappa\colon U \to \mathbb{R}$ with $\varepsilon_{\mathrm{lin}}(x) \geq 2\eta > 0$ for all x, see [1], [7], or [20]. It is isotropic; i.e., $D(t,x)$ and $E(t,x)$ have the same direction. The Kerr law satisfies our assumption $a_0 = a_0^\top \geq \eta I$ for small E (and for all E if $\kappa \geq 0$) since $\partial_E\theta_E(E) = (\varepsilon_{\mathrm{lin}} + \kappa|E|^2)I + 2\kappa EE^\top$. The assumption also holds for the more general laws $D = \varepsilon_{\mathrm{lin}}(x)E + \beta_e(x,|E|^2)E$ and $B = \mu_{\mathrm{lin}}(x)H + \beta_m(x,|H|^2)H$ for coefficients $\varepsilon_{\mathrm{lin}}, \mu_{\mathrm{lin}} \in C_b(U, \mathbb{R}^{3\times 3}_{\geq 2\eta})$ and $\beta_j, \partial_2\beta_j \in C(U\times\mathbb{R}, \mathbb{R})$ which are bounded in x and satisfy $\beta_j(x,0) = 0$.[1] $\Diamond$

A typical anisotropic relation is the following polynomial one.

Example 1.2. Let $\theta(x,E,H) = (\varepsilon_{\mathrm{lin}}(x)E + \varepsilon_{\mathrm{nl}}(x,E)E, \mu_{\mathrm{lin}}(x)H)$. As above we assume that $\varepsilon_{\mathrm{lin}}, \mu_{\mathrm{lin}} \in C_b(U, \mathbb{R}^{3\times 3}_{\geq 2\eta})$, and we set

$$\varepsilon_{\mathrm{nl}}(x,E) = \Big(\sum\nolimits_{j,k,l=1}^{3} \kappa_i^{jkl}(x)E_jE_k\Big)_{il}$$

for scalar coefficients $\kappa_i^{jkl} \in C_b(U)$, cf. [12]. Because of the triple sum, the tensor $(\kappa_i^{jkl})_{il}$ has to be symmetric in $\{j,k,l\}$. Using this symmetry, we compute

$$\partial_E\theta_e(E) = \varepsilon_{\mathrm{lin}}I + 3\Big(\sum\nolimits_{j,k=1}^{3} \kappa_i^{jkl}E_jE_k\Big)_{il}$$

which is symmetric if also $\kappa_i^{jkl} = \kappa_l^{jki}$, i.e., we can only prescribe κ_i^{jkl} for, say, $1 \leq i \leq j \leq k \leq l \leq 3$. For $|E| \leq r$ with a suitable $r \in (0,\infty]$ and all $x, H \in \mathbb{R}^3$ we then obtain $\partial_u\theta(x,u) \geq \eta I$. $\Diamond$

2) In nonlinear optics, the material response is often descibed by a retardation in time, see [1] or [12]. A rather general *retarded material law* is given by

$$\begin{aligned} D(t,x) = \varepsilon_{\mathrm{lin}}(x)E(t,x) &+ \int_{-\infty}^{t} k_1(t-\tau,x)E(\tau,x)\,\mathrm{d}\tau \\ &+ \int_{-\infty}^{t}\int_{-\infty}^{t} k_2(t-\tau_1,t-\tau_2,x)[E(\tau_1,x),E(\tau_2,x)]\,\mathrm{d}\tau_1\mathrm{d}\tau_2 + \dots \end{aligned} \tag{1.10}$$

for tensor-valued kernels $k_n \in L^1(\mathbb{R}^n_{\geq 0}, L^\infty(U, L_n(\mathbb{R}^3,\mathbb{R}^3)))$. Instantaneous laws as in Example 1.2 result as (formal) singular limits of such retarded ones. The components of k_j could be decaying exponentials times trigonometric polynomials in basic cases, cf. Section 4.2.1 of [20].

3) In *dynamical material laws* the polarization or magnetization are given by evolution equations coupled with the Maxwell systems, e.g., in the Maxwell–Bloch, Maxwell–Lorentz or the Maxwell–Landau–Lifschitz systems, see Section 4.2 of [20], Section 4a of [40] and also [18].

[1] The subscript b means that the functions and all occuring derivatives are bounded.

4) In many basic models, the current is described as the sum

$$J_e = \sigma(x, E, H)E + J_0 \tag{1.11}$$

of a given external current density $J_0 \colon \mathbb{R}_{\geq 0} \times U \to \mathbb{R}^3$ and a current induced via Ohm's law for a (possibly state-depending) *conductivity* $\sigma \colon U \times \mathbb{R}^6 \to \mathbb{R}^{3\times 3}$. In several models J_e is coupled to another evolution equation, e.g., in magnetohydrodynamics or the Maxwell–Schrödinger system. Such systems and the dynamical laws from item 3) are not treated in these lectures.

Sometimes it is useful to pass to second-order versions of (1.1), i.e., to **formulations as wave systems**. We first treat time-depending anisotropic linear relations $D = \varepsilon(t, x)E$ and $B = \mu(t, x)H$, before we discuss special cases. Nonlinear laws are treated similarly. We focus on the equation for the electric field E, one can handle D, B, H analogously. Here, Ampère's equation in (1.1) yields

$$\partial_t \varepsilon E + \varepsilon \partial_t E = \partial_t(\varepsilon E) = \operatorname{curl}(\mu^{-1}B) - J_e.$$

Differentiating in t, we deduce

$$\varepsilon \partial_t^2 E = \operatorname{curl}\big(\mu^{-1}\partial_t B\big) - \operatorname{curl}\big(\mu^{-1}\partial_t\mu\mu^{-1}B\big) - 2\partial_t\varepsilon\partial_t E - \partial_t^2\varepsilon E - \partial_t J_e.$$

Faraday's equation in (1.1) and $B = \mu H$ then lead to

$$\partial_t^2 E + \varepsilon^{-1}\operatorname{curl}(\mu^{-1}\operatorname{curl} E) = -\varepsilon^{-1}\Big(2\partial_t\varepsilon\partial_t E + \partial_t^2\varepsilon E + \operatorname{curl}\big(\mu^{-1}\partial_t\mu H\big) + \partial_t J_e\Big). \tag{1.12}$$

Observe that the equation is still coupled to H in first order. Besides $E(0) = E_0$ one has the initial condition

$$\partial_t E(0) = \varepsilon^{-1}\big(\operatorname{curl} H_0 - \partial_t\varepsilon(0)E_0 - J_e(0)\big).$$

invoking $\operatorname{curl} H_0$. The second-order term in (1.12) is symmetric in the weighted space $L^2(\varepsilon\,\mathrm{d}x)$ if we impose the boundary condition $E \times \nu = 0$ and enough regularity, see (1.7). In the equations for H or B inhomogeneities as $\operatorname{curl}(\varepsilon^{-1}J_e)$ appear, so that E is present via $J_e = \sigma E$ if one has nonzero conductivity.

If ε and μ do not depend on time, (1.12) simplifies considerably and the H term diappears. But still the components of E are coupled in the term of highest order. For scalar ε and μ, the product rule (1.6) further yields

$$\varepsilon\mu\partial_t^2 E + \operatorname{curl}(\operatorname{curl} E) = \tfrac{1}{\mu}\nabla\mu \times \operatorname{curl} E - \mu\partial_t J_e.$$

In addition let $\operatorname{div}(\varepsilon E_0) = 0 = \operatorname{div} J_e$ and thus the charges $\operatorname{div}(\varepsilon E)$ vanish by (1.4). As $\operatorname{curl}\operatorname{curl} = \nabla\operatorname{div} - \Delta I$, cf. (1.8), formula (1.5) implies

$$\varepsilon\mu\partial_t^2 E = \Delta E + \tfrac{1}{\mu}\nabla\mu\times\operatorname{curl} E + \tfrac{1}{\varepsilon}\nabla\varepsilon\cdot\nabla E + \tfrac{1}{\varepsilon}D^2\varepsilon E - \tfrac{1}{\varepsilon}\nabla\varepsilon(\tfrac{1}{\varepsilon}\nabla\varepsilon\cdot E) - \mu\partial_t J_e. \tag{1.13}$$

The dot term reads as $\sum_k \partial_k\varepsilon\,\partial_j E_k$. So in the time-independent, isotropic and charge-free case, the terms of highest order form a decoupled wave equation with coefficients, ingnoring boundary conditions. The coupling occurs in lower order only. The system completely decouples into the standard wave equation if ε and μ are also constant.

It is often convenient to rewrite (1.1) with (1.9) and (1.11) as a **quasilinear symmetric hyperbolic system**. To this end, we first introduce the matrices

$$S_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{satisfying}$$

$$\operatorname{curl} = S_1\partial_1 + S_2\partial_2 + S_3\partial_3 \quad \text{and} \quad a \times b = (a_1 S_1 + a_2 S_2 + a_3 S_3) b$$

for vectors $a, b \in \mathbb{R}^3$. We then define $\partial_0 = \partial_t$,

$$A_j^{\mathrm{co}} = \begin{pmatrix} 0 & -S_j \\ S_j & 0 \end{pmatrix}, \quad a_0(u) = \partial_u \theta(\cdot, u), \quad d = \begin{pmatrix} \sigma \\ 0 \end{pmatrix}, \quad f = \begin{pmatrix} -J_0 \\ 0 \end{pmatrix} \tag{1.14}$$

for $j \in \{1, 2, 3\}$. Note that the matrices A_j^{co} are symmetric.

Then the Maxwell system (1.1) with material laws (1.9) and (1.11) becomes

$$L(u)u := a_0(u)\partial_t u + \sum_{j=1}^{3} A_j^{\mathrm{co}} \partial_j u + d(u)u = f. \tag{1.15}$$

Our strategy to solve this problem originates from Kato [**31**]. One freezes a function v from a suitable space $\mathcal{E}$ in the nonlinearities, setting $A_0 = a_0(v)$, $A_j = A_j^{\mathrm{co}}$ and $D = d(v)$. One next solves the resulting non-autonomous linear problem $L(v)u = \sum_{j=0}^{3} A_j \partial_j u + Du = f$ in the space $\mathcal{E}$. On small time intervals $(0, T)$ one finds a fixed point of the map $v \mapsto u$ which then solves (1.15) and (1.1). The first linear step is more difficult; here it is crucial to control very well how the constants in the estimates depend on the coefficients. We carry out this program on $\mathbb{R}^3$ in the following chapter.

CHAPTER 2

Local wellposedness in $\mathcal{H}^3$

In this chapter we develop a local wellposedness theory for the quasilinear Maxwell equations on $\mathbb{R}^3$. Our approach is based on energy methods and a fixed-point argument, which make use of the linear system with time-depending coefficients. One has to work in Sobolev spaces $\mathcal{H}^s$ with $s > \frac{5}{2}$ in this context, where we take $s = 3$ for simplicity. Actually we treat general symmetric hyperbolic systems on $\mathbb{R}^3$. In the first subsection we introduce Maxwell equations and discuss some facts used throughout these notes. We then investigate the linear case, first in L^2 and then in $\mathcal{H}^3$, also establishing the finite speed of propagation. Our main tools are energy estimates, duality arguments for existence in L^2, approximation by mollifiers for regularity and uniqueness, and finally a transformation from L^2 to $\mathcal{H}^3$. The non-linear problem is solved by means of fixed-point arguments going back to Kato [**31**] at least, where the derivation of blow-up conditions in $W^{1,\infty}$ and the continuous dependence of data in $\mathcal{H}^3$ require significant additional efforts. Finally, for the isotropic Maxwell system, we show the preservation of energy and construct a blow-up example in $\mathcal{H}^1$.

The wellposedness results on $\mathbb{R}^3$ are due to Kato [**32**], but our proof differs from Kato's and instead uses (well known) energy methods from the theory of symmetric hyperbolic PDE, see [**5**], [**6**], [**13**] or [**38**]. The corresponding problem on domains can also be treated by means of energy methods, but this is much harder. See [**56**], [**57**] and [**58**] for the core theory, [**50**] and [**51**] for different boundary conditions, as well as my lecture notes [**44**] or the shorter version [**45**] for the easier accessible halfspace case.

2.1. The linear problem in L^2

We often omit range spaces as $\mathbb{R}^6$ in the notation, and write $L^p_J X = L^p(J, X)$ for function spaces from an interval J to a Banach space X (also with subscript T if $J = (0, T)$), as well as L^p instead of $L^p(\mathbb{R}^m)$, etc. Chapter 1 of [**26**] discusses the theory of X-valued L^p-spaces. It is quite similar to the Lebesgue case $X = \mathbb{R}$, and we will highlight differences if they play a role below. Let $J = (0, T)$. We solve the linear problem in the space $C(\overline{J}, L^2(\mathbb{R}^3, \mathbb{R}^6)) = C_{\overline{J}} L^2$ for coefficients and data subject to the assumptions

$$\begin{aligned} &A_j = A_j^\top \in \mathcal{W}^{1,\infty}_J := W^{1,\infty}(J \times \mathbb{R}^3, \mathbb{R}^{6\times 6}),\ j \in \{0,1,2,3\}, \quad A_0 = A_0^\top \geq \eta I > 0, \\ &D \in \mathcal{L}^\infty_J := L^\infty(J \times \mathbb{R}^3, \mathbb{R}^{6\times 6}), \quad u_0 \in L^2, \quad f \in L^2_J L^2 = L^2(J \times \mathbb{R}^3, \mathbb{R}^6). \end{aligned} \tag{2.1}$$

(See Proposition 1.2.4 of [**26**] for the last equality, which is an isomorphism actually.) Then A_0^{-1} belongs to $\mathcal{W}^{1,\infty}_J$ and $A_0^{-1} = (A_0^{-1})^\top \geq \|A_0\|_\infty^{-1} I$.

Compared to (1.15) we allow for D and f with non-zero 'magnetic' components, as needed in our analysis. We also deal with general symmetric (t,x)-depending coefficients A_1, A_2 and A_3, and thus with linear *symmetric hyperbolic systems.* Those occur in many applications, see [**6**], [**31**], [**38**] or the exercises; and our reasoning would not differ much if we restricted to $A_j = A_j^{\mathrm{co}}$. Moreover, when treating the Maxwell system on domains by localization arguments, one obtains x-depending coefficients. It is useful to see them first in an easier case.

Assuming (2.1), we look for a solution $u \in C(\overline{J}, L^2)$ of the system

$$Lu \coloneqq \sum_{j=0}^{3} A_j \partial_j u + Du = f, \quad t \geq 0, \qquad u(0) = u_0, \tag{2.2}$$

with $\partial_0 = \partial_t$. Here the derivatives are understood in a weak sense.

To explain this, we assume that the reader is familiar with Sobolev spaces $W^{k,p}(U) = W^{k,p}$ for an open subset U of $\mathbb{R}^m$, $k \in \mathbb{N}_0$, and $p \in [1,\infty]$. (See [**9**] or [**47**], for instance.) We work with real scalars in this chapter almost entirely, endow $W^{k,p}$ with the (complete) norm $\|v\|_{k,p}^p = \sum_{0\leq|\alpha|\leq k} \|\partial^\alpha v\|_p^p$ (obvious modification for $p = \infty$), and write $\mathcal{H}^k \coloneqq W^{k,2}$ (which is a Hilbert space), $L^p = W^{0,p}$ and $\|v\|_p \coloneqq \|v\|_{0,p}$. By $W_0^{k,p}(U)$ we denote the closure of test functions $C_c^\infty(U)$ in $W^{k,p}(U)$. If ∂U is compact and C^k (or Lipschitz if $k = 1$), say, then $W_0^{k,p}$ is the closed subspace in $W^{k,p}$ of functions whose (weak) derivatives of order up to $k-1$ have trace 0. One can check that $W_0^{k,p}(\mathbb{R}^m) = W^{k,p}(\mathbb{R}^m)$.

Let $\mathcal{H}^{-k}(U)$ be the dual space $\mathcal{H}_0^k(U)^\star$, where we restrict ourselves to $p = 2$ for simplicity. Since $\mathcal{H}_0^k(U) \hookrightarrow L^2(U)$ with dense range, the space $L^2(U) \cong L^2(U)^\star$ (and thus $C_c^\infty(U)$) is densely embedded into $\mathcal{H}^{-k}(U)$, where $\varphi \in L^2(U)$ acts as $\varphi(v) = \int \varphi v \,\mathrm{d}x$ on $v \in \mathcal{H}_0^k(U)$. One also has $\mathcal{H}^{-k}(U) \hookrightarrow \mathcal{H}^{-l}(U)$ for $k \leq l \in \mathbb{N}$.

For $\varphi \in L^2(U)$, $j \in \{1,\dots,m\}$ and $v \in \mathcal{H}_0^1(U)$, we define the weak derivative $\partial_j \varphi \in \mathcal{H}^{-1}(U)$ by setting

$$(\partial_j \varphi)(v) = \langle v, \partial_j \varphi \rangle_{\mathcal{H}_0^1} \coloneqq -\langle \partial_j v, \varphi \rangle_{L^2}.$$

(The brackets $\langle\cdot,\cdot\rangle_X$ designate the duality pairing between a Banach space X and its dual $X^\star$.) Since $|\langle \partial_j v, \varphi\rangle| \leq \|v\|_{1,2}\,\|\varphi\|_2$, the linear map $\partial_j \colon L^2(U) \to \mathcal{H}^{-1}(U)$ is bounded. Iteratively, one obtains bounded maps $\partial_j \colon \mathcal{H}^{-k}(U) \to \mathcal{H}^{-k-1}(U)$, and analogously $\partial^\alpha \colon \mathcal{H}^{-k}(U) \to \mathcal{H}^{-k-|\alpha|}(U)$ for multi-indices $\alpha \in \mathbb{N}_0^m$ and $k \in \mathbb{N}_0$. The definitions imply that these derivatives commute.

For $a \in W^{1,\infty}(U)$ and $\varphi \in \mathcal{H}^{-1}(U)$, we next define the map $a\varphi \in \mathcal{H}^{-1}(U)$ by

$$(a\varphi)(v) = \langle v, a\varphi\rangle_{\mathcal{H}_0^1} \coloneqq \langle av, \varphi\rangle_{\mathcal{H}_0^1}, \qquad v \in \mathcal{H}_0^1(U).$$

Because of $\|av\|_{1,2} \lesssim \|a\|_{1,\infty}\,\|v\|_{1,2}$, we see as above that the multiplication operator $M_a \colon \varphi \mapsto a\varphi$ is bounded on $\mathcal{H}^{-1}(U)$. (Here and below $A \lesssim_\alpha B$ stands for $A \leq cB$ for a generic constant $c = c(\alpha)$ which is non-decreasing in each component of $\alpha \in \mathbb{R}^n_{\geq 0}$.) These facts easily extend to $\mathbb{R}^l$–valued functions.

We infer that $Lu \in \mathcal{H}^{-1}(J \times \mathbb{R}^3)$ if $u \in L_J^2 L^2$. Let $Lu = f$ be contained in $L_J^2 L^2$. We stress that a summand $A_j \partial_j u$ may only belong to $L^2(\mathbb{R}^3, \mathcal{H}^{-1}(J))$ if $j = 0$ and to $L^2(J \times \mathbb{R}^2, \mathcal{H}^{-1}(\mathbb{R}))$ otherwise. More precisely, for the time derivative we obtain

$$\partial_t u = A_0^{-1} f - \sum\nolimits_{j=1}^{3} A_0^{-1} A_j \partial_j u - A_0^{-1} D u \ \in\ L^2_J \mathcal{H}^{-1}, \tag{2.3}$$

and so u is an element of $\mathcal{H}^1_J \mathcal{H}^{-1} \hookrightarrow C(\overline{J}, \mathcal{H}^{-1})$. (See the beginning of Section 4.5 of [**46**] for Banach space valued Sobolev spaces.) Accordingly, the initial condition in (2.2) is understood in $\mathcal{H}^{-1}$.

We will first show the basic *energy* (or *apriori*) *estimate*. Here we use the temporal weights $e_{-\gamma}(t) := \mathrm{e}^{-\gamma t}$ for $\gamma \geq 0$ and $t \in J$ (or $t \in \mathbb{R}$) and the weighted spaces $L^2_{\gamma,J} X = L^2_\gamma(J, X)$ ($= L^2_{\gamma,T} X$ if $J = (0,T)$) of functions with finite norm

$$\|v\|_{L^2_{\gamma,J} X} := \|e_{-\gamma} v\|_{L^2_J X} = \Big(\int_J \mathrm{e}^{-2\gamma t} \|v(t)\|_X^2 \,\mathrm{d}t \Big)^{\frac{1}{2}}.$$

We have the equivalence $\mathrm{e}^{\gamma a} \|v\|_{L^2_{\gamma,J} X} \leq \|v\|_{L^2_J X} \leq \mathrm{e}^{\gamma b} \|v\|_{L^2_{\gamma,J} X}$ if $J = (a,b)$ is bounded. Taking large γ in these norms, we can produce small constants in front of the contribution of f in the inequality below. This fact will be used to absorb error terms by the left-hand side, for instance. The estimate and the precise form of the constants is also crucial for the nonlinear problem. We write $\operatorname{div} A = \sum_{j=0}^3 \partial_j A_j$ and use $\|\cdot\|_\infty$ for the sup-norm in (t,x).

Lemma 2.1. *Assume that* (2.1) *is true and* $u \in \mathcal{H}^1(J \times \mathbb{R}^3)$ *solves* (2.2). *Let* $C := \frac{1}{2} \operatorname{div} A - D$, $\gamma \geq \gamma_0'(L) := \max\{1, 4\eta^{-1} \|C\|_\infty\}$, *and* $t \in \overline{J}$. *We then obtain*

$$\tfrac{\gamma\eta}{4} \|u\|^2_{L^2_{\gamma,t} L^2} + \tfrac{\eta}{2} \mathrm{e}^{-2\gamma t} \|u(t)\|^2_{L^2} \leq \tfrac{1}{2} \|A_0(0)\|_{L^\infty} \|u_0\|^2_{L^2} + \tfrac{1}{2\gamma\eta} \|f\|^2_{L^2_{\gamma,t} L^2}.$$

Proof. Set $v = e_{-\gamma} u$ and $g = e_{-\gamma} f$ for $\gamma \geq 0$. We have $\gamma A_0 v + L v = g$. Using the symmetry of A_j, we derive

$$\begin{aligned}
\langle g, v \rangle &= \gamma \langle A_0 v, v \rangle + \sum_{j=0}^{3} \langle A_j \partial_j v, v \rangle + \langle D v, v \rangle \\
&= \gamma \langle A_0 v, v \rangle + \frac{1}{2} \sum_{j=0}^{3} \Big(\int_0^t \int_{\mathbb{R}^3} \partial_j (A_j v \cdot v) \,\mathrm{d}x \,\mathrm{d}s - \langle \partial_j A_j v, v \rangle \Big) + \langle D v, v \rangle,
\end{aligned}$$

where we drop the subscript $L^2_t L^2$ of the brackets and denote the scalar product in $\mathbb{R}^6$ by a dot. Integration yields

$$\gamma \langle A_0 v, v \rangle + \tfrac{1}{2} \langle A_0(t) v(t), v(t) \rangle_{L^2} = \tfrac{1}{2} \langle A_0(0) v(0), v(0) \rangle_{L^2} + \langle C v, v \rangle + \langle g, v \rangle.$$

We now replace $v = e_{-\gamma} u$, $g = e_{-\gamma} f$ as well as $u(0) = u_0$, and use (2.1) and $\gamma \geq \gamma_0'(L)$. It follows

$$\begin{aligned}
&\gamma\eta \|u\|_{L^2_{\gamma,t} L^2} + \tfrac{\eta}{2} \mathrm{e}^{-2\gamma t} \|u(t)\|^2_{L^2} \\
&\qquad \leq \tfrac{1}{2} \|A_0(0)\|_{L^\infty} \|u_0\|^2_{L^2} + \|C\|_\infty \|u\|^2_{L^2_{\gamma,t} L^2} + \tfrac{\sqrt{\gamma\eta}}{\sqrt{\gamma\eta}} \|u\|_{L^2_{\gamma,t} L^2} \|f\|_{L^2_{\gamma,t} L^2} \\
&\qquad \leq \tfrac{1}{2} \|A_0(0)\|_{L^\infty} \|u_0\|^2_{L^2} + \big(\tfrac{\gamma\eta}{4} + \tfrac{\gamma\eta}{2} \big) \|u\|^2_{L^2_{\gamma,t} L^2} + \tfrac{1}{2\gamma\eta} \|f\|^2_{L^2_{\gamma,t} L^2},
\end{aligned}$$

which implies the assertion. $\square$

Below we use the above estimate for

$$\gamma \geq \gamma_0(r,\eta) \coloneqq \max\{1, 12r/\eta\} \geq \gamma_0'(L) \tag{2.4}$$

where $\|\partial_j A_j\|_\infty, \|D\|_\infty \leq r$. For $\gamma = 0$ its proof yields the *energy equality*

$$\int_{\mathbb{R}^3} A_0(t)u(t)\cdot u(t)\,\mathrm{d}x = \int_{\mathbb{R}^3} A_0(0)u_0\cdot u_0\,\mathrm{d}x + 2\int_0^t\int_{\mathbb{R}^3} \big(C(s)u(s)+f(s)\big)\cdot u(s)\,\mathrm{d}x\,\mathrm{d}s. \tag{2.5}$$

In the term with $C = \frac{1}{2}\operatorname{div} A - D$ we have damping effects (if $D = D^\top \gneqq 0$) and extra errors terms coming from the t- or x-dependence of A_j.

Lemma 2.1 yields uniqueness of $\mathcal{H}^1$-solutions to (2.2). However, we need uniqueness (and the energy estimate) for solutions in $C(\overline{J}, L^2)$. This fundamental gap can be closed by a crucial regularization argument based on mollifiers. We recall the definition and basic properties of this core tools, see e.g. [9].

We set $g_\varepsilon(x) = \varepsilon^{-m} g(\varepsilon^{-1}x)$ for any function g on $\mathbb{R}^m$, $\varepsilon > 0$, and $x \in \mathbb{R}^m$. Take $0 \leq \rho \in C_c^\infty(\mathbb{R}^m)$ with $\int \rho\,\mathrm{d}x = 1$, support $\operatorname{supp}\rho$ in the closed unit ball $\overline{B}(0,1)$, and $\rho(x) = \rho(-x)$ for $x \in \mathbb{R}^m$. Note that $\|\rho_\varepsilon\|_1 = 1$. For $\varepsilon > 0$ and $v \in L^1_{\mathrm{loc}}(\mathbb{R}^m)$, we define the *mollifiers* R_ε by

$$R_\varepsilon v(x) = \rho_\varepsilon * v(x) = \int_{\mathbb{R}^m} \rho_\varepsilon(x-y)v(y)\,\mathrm{d}y, \qquad x \in \mathbb{R}^m.$$

One can check that $R_\varepsilon v \in C^\infty(\mathbb{R}^m)$, $\operatorname{supp} R_\varepsilon v \subseteq \operatorname{supp} v + \overline{B}(0,\varepsilon)$, and $\partial^\alpha R_\varepsilon v = R_\varepsilon \partial^\alpha v$ for $v \in W^{|\alpha|,p}(\mathbb{R}^m)$. Young's inequality for convolutions yields $\|R_\varepsilon v\|_{k,p} \leq \|v\|_{k,p}$ for $p \in [1,\infty]$ and $k \in \mathbb{N}_0$. Using this estimate, one derives that $R_\varepsilon v \to v$ in $W^{k,p}(\mathbb{R}^m)$ for $v \in W^{k,p}(\mathbb{R}^m)$ as $\varepsilon \to 0$ if $p < \infty$, since this limit is true for test functions v. Differentiating $\rho_\varepsilon(x-y)$ in x, one also obtains the smoothing estimate $\|R_\varepsilon v\|_{k,p} \lesssim_{\varepsilon,k} \|v\|_p$.

Finally, for $\varphi \in \mathcal{H}^{-k}(\mathbb{R}^m)$, $v \in \mathcal{H}^k(\mathbb{R}^m)$ and $k \in \mathbb{N}$, we set

$$(R_\varepsilon\varphi)(v) = \langle v, R_\varepsilon\varphi\rangle_{\mathcal{H}^k} \coloneqq \langle R_\varepsilon v, \varphi\rangle_{\mathcal{H}^k}.$$

This definition is consistent with the symmetry $R_\varepsilon^\star = R_\varepsilon$ on $L^2(\mathbb{R}^m)$ which follows from the symmetry of ρ and Fubini's theorem. By means of its properties in $\mathcal{H}^k(\mathbb{R}^m)$, one can show that R_ε is contractive on $\mathcal{H}^{-l}(\mathbb{R}^m)$ and that it maps this space into $\mathcal{H}^k(\mathbb{R}^m)$ for all $l \in \mathbb{N}$. Moreover, it commutes with ∂^α.

Hence, the commutator $[R_\varepsilon, M_a] \coloneqq R_\varepsilon M_a - M_a R_\varepsilon$ tends to 0 strongly in L^2 if $a \in L^\infty$ (and is bounded uniformly in $\varepsilon > 0$). It even gains a derivative if $a \in W^{1,\infty}$, which is crucial for our analysis.

Proposition 2.2. *Let $a \in W^{1,\infty}(\mathbb{R}^m)$, $\varphi \in L^2(\mathbb{R}^m)$, $j \in \{1,\dots,m\}$, and $\varepsilon > 0$. Set $C_\varepsilon\varphi \coloneqq R_\varepsilon(a\partial_j\varphi) - a\partial_j(R_\varepsilon\varphi)$. Then there is a constant $c = c(\rho)$ such that*

$$\|C_\varepsilon\varphi\|_2 \leq c\|a\|_{1,\infty}\|\varphi\|_2 \qquad \text{and} \qquad C_\varepsilon\varphi \to 0 \quad \text{in } L^2 \quad \text{as } \varepsilon \to 0.$$

Proof. Let $v \in \mathcal{H}^1(\mathbb{R}^m)$. Using the above indicated facts, we compute

$$\langle v, C_\varepsilon\varphi\rangle_{\mathcal{H}^1} = \langle aR_\varepsilon v, \partial_j\varphi\rangle_{\mathcal{H}^1} - \langle av, R_\varepsilon\partial_j\varphi\rangle_{\mathcal{H}^1} = \langle \partial_j(R_\varepsilon(av) - aR_\varepsilon v), \varphi\rangle_{L^2}.$$

We set $C'_\varepsilon v = \partial_j(R_\varepsilon(av) - aR_\varepsilon v)$ and R^j_ε for the convolution with $(|\partial_j\rho|)_\varepsilon$. For a.e. $x \in \mathbb{R}^m$, differentiation and $|x-y| \le \varepsilon$ yield

$$C'_\varepsilon v(x) = \int_{B(x,\varepsilon)} \varepsilon^{-m}(\partial_j\rho)(\varepsilon^{-1}(x-y))\,\varepsilon^{-1}(a(y)-a(x))v(y)\,\mathrm{d}y - \partial_j a(x)R_\varepsilon v(x),$$

$$|C'_\varepsilon v(x)| \le \|a\|_{1,\infty}\,(R^j_\varepsilon|v|(x) + |R_\varepsilon v(x)|).$$

(Recall that $W^{1,\infty}(\mathbb{R}^m)$ is isomorphically isomorpic to the space of bounded Lipschitz functions by Proposition 9.3 in [9].) Young's inequality now implies the first assertion. The second one is true for u in the dense subspace $\mathcal{H}^1(\mathbb{R}^m)$ and thus on $L^2(\mathbb{R}^m)$ by the uniform estimate. $\square$

With this tool we can extend Lemma 2.1 to all solutions of (2.2) in $C(\overline{J}, L^2)$.

Proposition 2.3. *Let (2.1) hold and $u \in C(\overline{J}, L^2)$ solve (2.2). Then the statement of Lemma 2.1 and (2.5) are also valid for u. Hence, (2.2) has at most one solution in $C(\overline{J}, L^2)$.*

Proof. We note that $R_\varepsilon u$ belongs to $C(\overline{J}, \mathcal{H}^k)$ for all $\varepsilon > 0$ and $k \in \mathbb{N}$. Moreover, $R_\varepsilon u$ tends to u in $C(\overline{J}, L^2)$ as $\varepsilon \to 0$ since $u(\overline{J})$ is compact and $R_\varepsilon \to I$ strongly in L^2. As $\|R_\varepsilon f(t)\|_2 \le \|f(t)\|_2$, dominated convergence also yields $R_\varepsilon f \to f$ in $L^2_J L^2$. Using $Lu = f$ and (2.3), we compute

$$LR_\varepsilon u = R_\varepsilon f + [D, R_\varepsilon]u + \sum_{j=1}^{3}[A_j, R_\varepsilon]\partial_j u + [A_0, R_\varepsilon]\partial_t u \tag{2.6}$$

$$= R_\varepsilon f + [D, R_\varepsilon]u + [A_0, R_\varepsilon]A_0^{-1}(f - Du) + \sum_{j=1}^{3}\big([A_j, R_\varepsilon] - [A_0, R_\varepsilon]A_0^{-1}A_j\big)\partial_j u.$$

Proposition 2.2 shows that the right-hand side belongs to $L^2_J L^2$ with uniform bounds. Hence, $R_\varepsilon u$ is also contained $\mathcal{H}^1_J L^2$ by (2.3). Arguing as above, we further see that the commutator terms tend to 0 in $L^2_J L^2$ and thus in $L^2_{\gamma,J} L^2$. Lemma 2.1 and (2.5) for $R_\varepsilon u$ now lead to the first assertion letting $\varepsilon \to 0$. The second one follows from linearity. $\square$

Combining the energy estimate with a clever duality argument, one can also deduce the existence of a solution. As a starting point, we note that a closed operator C from X to Y with dense domain is surjective if its adjoint $C^\star$ is bounded from below, i.e., $\|C^\star y^\star\| \ge c\|y^\star\|$ for some $c > 0$ and all $y^\star \in \mathrm{D}(C^\star)$. See Theorem 2.20 in [9]. Below we avoid to invoke the adjoint explicitly.

Theorem 2.4. *Let (2.1) be true. Then there is a unique map u in $C(\overline{J}, L^2)$ solving (2.2). It satisfies the estimate in Lemma 2.1 and (2.5).*

Proof. 1) We need the (formal) adjoint $L^\circ = -\sum_{j=0}^{3} A_j\partial_j + D^\circ$ of L with $D^\circ = D^\top - \operatorname{div} A$. Let $V = \{v \in \mathcal{H}^1(J \times \mathbb{R}^3, \mathbb{R}^6) \,|\, v(T) = 0\}$, $v \in V$, and $L^\circ v = h$. We introduce $\tilde{v}(t) = v(T-t)$ and $f(t) = h(T-t)$ for $t \in \overline{J}$ and the operator $\tilde{L}$ with coefficients $\tilde{A}_0(t) = A_0(T-t)$, $\tilde{A}_j(t) = -A_j(T-t)$ for

$j \in \{1,2,3\}$ and $\tilde{D}(t) = D^\circ(T-t)$. Note that $\tilde{L}\tilde{v} = f$ and $\tilde{v}(0) = 0$. Applied at time $T-s$ to $\tilde{L}$, $\tilde{v}$ and $\gamma = \gamma_0(r,\eta)$ from (2.4), Lemma 2.1 yields the estimate

$$\begin{aligned}
\|v(s)\|_2^2 = \|\tilde{v}(T-s)\|_2^2 &\le \frac{2\mathrm{e}^{2\gamma(T-s)}}{\eta\cdot 2\eta\gamma}\int_0^{T-t} \mathrm{e}^{-2\gamma\tau}\|h(T-\tau)\|_2^2 \,\mathrm{d}\tau \\
&\le \frac{\mathrm{e}^{2\gamma T}}{\gamma\eta^2}\int_s^T \|h(\tau')\|_2^2 \,\mathrm{d}\tau', \qquad s \in (0,T),
\end{aligned}$$

$$\|v\|_{L^2_J L^2} \le \kappa\sqrt{T}\,\|L^\circ v\|_{L^2_J L^2}, \qquad \kappa := \tfrac{1}{\eta\sqrt{\gamma}}\mathrm{e}^{\gamma T}. \tag{2.7}$$

Hence, $L^\circ\colon V \to L^2(J\times\mathbb{R}^3)^6$ is injective. We can thus define the functional

$$\ell_0\colon L^\circ V \to \mathbb{R}; \quad \ell_0(L^\circ v) = \langle v, f\rangle_{L^2_J L^2} + \langle v(0), A_0(0)u_0\rangle_{L^2}.$$

The Cauchy–Schwarz inequality and estimate (2.7) imply

$$|\ell_0(L^\circ v)| \le \big(\|f\|_{L^2_J L^2} + \|A_0(0)u_0\|_{L^2}\big)\,\kappa\big(\sqrt{T}+1\big)\|L^\circ v\|_{L^2_J L^2}.$$

By the Hahn–Banach theorem, ℓ_0 has an extension ℓ in $(L^2_J L^2)^\star$ which can be represented by a function $u \in L^2_J L^2 \cong L^2(J\times\mathbb{R}^3)$ via

$$\begin{aligned}
\langle v, f\rangle_{L^2_J L^2} + \langle v(0), A_0(0)u_0\rangle_{L^2} &= \ell(L^\circ v) = \langle L^\circ v, u\rangle_{L^2_J L^2} \qquad (2.8)\\
&= \langle v, Du\rangle - \sum_{j=0}^{3}\int_0^T\!\!\int_{\mathbb{R}^3} \partial_j(A_j v)\cdot u \,\mathrm{d}x\,\mathrm{d}t \quad (\forall\, v \in V).
\end{aligned}$$

2) To evaluate (2.8), we first take $v \in \mathcal{H}^1_0(J\times\mathbb{R}^3)$. The definition of weak derivatives then leads to $\langle v, f\rangle_{L^2_J L^2} = \langle v, Lu\rangle_{\mathcal{H}^1_0}$; i.e., $Lu = f$ in $\mathcal{H}^{-1}(J\times\mathbb{R}^3)$. Hence, u belongs to $\mathcal{H}^1_J\mathcal{H}^{-1}$ because of (2.3) and $f \in L^2_J L^2$. For $v \in V$, we can now integrate by parts the summand in (2.8) with $j=0$ in $\mathcal{H}^{-1}$; the others are treated as before. As $v(T) = 0$, it follows

$$\langle v, f\rangle_{L^2_J L^2} + \langle v(0), A_0(0)u_0\rangle_{L^2} = \langle v, Lu\rangle_{\mathcal{H}^1_0} + \langle A_0(0)v(0), u(0)\rangle_{L^2}\,.$$

Since $A_0(0)$ is symmetric and $Lu = f$, we have also shown that $u(0) = u_0$.

3) We next use (2.6) for $w_{n,m} = R_{1/n}u - R_{1/m}u$. As in the proof of Proposition 2.3, Proposition 2.2 implies that $w_{n,m}$ is contained in $\mathcal{H}^1(J\times\mathbb{R}^3)$ and satisfes $Lw_{n,m} \to 0$ in $L^2_J L^2$ and $w_{n,m}(0) \to 0$ in L^2 as $n,m\to\infty$. So $(R_{1/n}u)$ is a Cauchy sequence in $C_{\overline{J}}L^2$ by Lemma 2.1, and it converges to u in $L^2_J L^2$. Thus, u belongs to $C_{\overline{J}}L^2$. The other assertions were proven in Proposition 2.3. □

There are blow-up solutions even for the wave equation on $\mathbb{R}$ with Hölder continuous and x-independent coefficients, as shown in [**15**].

As indicated in Chapter 1 and described in the next example, the above result can easily be applied to the linear Maxwell system

$$\partial_t(\varepsilon E) = \operatorname{curl} H - \sigma E - J_0, \qquad \partial_t(\mu H) = -\operatorname{curl} E, \qquad t \ge 0,\ x \in \mathbb{R}^3, \tag{2.9}$$

which is (1.1) with the material laws $D = \varepsilon(t,x)E$ and $B = \mu(t,x)H$.

Example 2.5. Let $\varepsilon,\mu \in W^{1,\infty}(J\times\mathbb{R}^3, \mathbb{R}^{3\times 3}_{\ge\eta})$ for some $\eta > 0$, $\sigma \in L^\infty(J\times\mathbb{R}^3,\mathbb{R}^{3\times 3})$, $E_0, H_0 \in L^2(\mathbb{R}^3,\mathbb{R}^3)$ and $J_0 \in L^2(J\times\mathbb{R}^3,\mathbb{R}^3)$. As in (1.14), we set $A_0 = \operatorname{diag}(\varepsilon,\mu)$, $A_j = A_j^{\mathrm{co}}$ for $j = \{1,2,3\}$, $D = \operatorname{diag}(\sigma + \partial_t\varepsilon, \partial_t\mu)$, $f =$

$(-J_0, 0)$, and $u_0 = (E_0, H_0)$. Theorem 2.4 then yields a unique solution $(E, H) \in C(\overline{J}, L^2)$ of (2.9) with $E(0) = E_0$ and $H(0) = H_0$. It satisfies the energy equality

$$\|\varepsilon(t)^{\frac{1}{2}} E(t)\|_2^2 + \|\mu(t)^{\frac{1}{2}} H(t)\|_2^2 = \|\varepsilon(0)^{\frac{1}{2}} E_0\|_2^2 + \|\mu(0)^{\frac{1}{2}} H_0\|_2^2 - \int_0^t \int_{\mathbb{R}^3} \big((2\sigma E + \partial_t \varepsilon\, E + 2J_0) \cdot E + \partial_t \mu H \cdot H\big) \,\mathrm{d}x \,\mathrm{d}s.$$

In the autonomous case it suffices that $\varepsilon, \mu \in L^\infty(\mathbb{R}^3, \mathbb{R}^{3\times 3}_{\geq \eta})$, see the exercises and also Theorem 5.2.5 in [2] or §7.8 in [20]. $\Diamond$.

One of the key features of hyperbolic systems is the finite propagation speed of their solutions. As a simple example, we first look at the standard wave equation $\partial_t^2 u = c^2 \partial_x^2 u$ on $\mathbb{R}$ for the wave speed $c > 0$ equipped with the initial conditions $u(0) = u_0 \in C^2(\mathbb{R})$ and $\partial_t u(0) = v_0 \in C^1(\mathbb{R})$. (To pass to the above first-order framework, use the state $(\partial_t u, \sqrt{c}\partial_x u)$ and $A_1 v = -\sqrt{c}\,(v_2, v_1)$.) The pointwise solution of this wave problem is given by d'Alembert's formula

$$u(t, x) = \tfrac{1}{2}(u_0(x + ct) + u_0(x - ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} v_0(s) \,\mathrm{d}s, \qquad t \geq 0,\ x \in \mathbb{R}.$$

Hence, the solution at (t, x) only depends on the initial data on $[x - ct, x + ct]$; for instance, $u(t, x) = 0$ if u_0 and v_0 vanish on $[x - ct, x + ct]$. Conversely, the value of u_0 and v_0 at y influences u at most for (t, x) with $|x - y| \leq ct$; i.e., on a triangle with vertex $(y, 0)$ and lateral sides of slope $\pm\frac{1}{c}$. In this sense, c is the speed of propagation.

We extend these observations to the system (2.2), assuming (2.1). In the statement we use the backward 'light' cone

$$\Gamma(x_0, R, K) = \big\{(t, x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^3 \,\big|\, |x - x_0| < R - Kt\big\}$$

with base $B(x_0, R)$ at $t = 0$ and apex $(\frac{R}{K}, x_0)$, where $x_0 \in \mathbb{R}$ and $R, K > 0$. Set

$$k_0^2 = \|A_1\|_\infty^2 + \|A_2\|_\infty^2 + \|A_3\|_\infty^2$$

with the operator norm for $|\cdot|_2$ on $\mathbb{R}^{6\times 6}$. Note that $k_0 = \sqrt{3}$ in Example 2.5.

Below we see (for $f = 0$) that u vanishes on $\Gamma(x_0, R, k_0/\eta)$ if $u_0 = 0$ on $B(x_0, R)$. Hence, if two initial functions u_0 and $\tilde{u}_0$ coincide on $B(x_0, R)$ then the corresponding solutions u and $\tilde{u}$ are equal on $\Gamma(x_0, R, k_0/\eta)$ by linearity. In other words, the values of u_0 outside $B(x_0, R)$ influence $u(t)$ only off $\Gamma(x_0, R, k_0/\eta)$, that is, with maximal speed k_0/η. Our proof is based on energy estimates with an exponential weight, and the arguments are taken from §4.2.2 of [5].

THEOREM 2.6. *Let (2.1) be true. Assume that $u_0 = 0$ on $B(x_0, R)$ and $f = 0$ on $\Gamma(x_0, R, k_0/\eta)$ for some $R > 0$ and $x_0 \in \mathbb{R}^3$. Then the solution $u \in C(\overline{J}, L^2)$ of (2.2) also vanishes on $\Gamma(x_0, R, k_0/\eta)$.*

PROOF. 1) Let $\delta, R > 0$ and $x_0 \in \mathbb{R}^3$ be given. There is a function $\psi \in C^\infty(\mathbb{R}^3)$ with $|\nabla\psi| \leq \eta/k_0$ (for the euclidean norm) and

$$-2\delta + \eta k_0^{-1}(R - |x - x_0|) \leq \psi(x) \leq -\delta + \eta k_0^{-1}(R - |x - x_0|), \qquad x \in \mathbb{R}^3. \quad (2.10)$$

We construct ψ as in Theorem 6.1 of [56]. Take $\chi(s) = -\frac{3}{2}\delta + \eta k_0^{-1}(R - |s|)$ for $s \in \mathbb{R}$. This function is Lipschitz with constant η/k_0. The same is true for the mollified map $\chi_\varepsilon = R_\varepsilon \chi$ as $\nabla \chi_\varepsilon = R_\varepsilon \nabla \chi$. Also, χ_ε tends uniformly to χ as $\varepsilon \to 0$ since

$$|\chi_\varepsilon(s) - \chi(s)| \le \int_{\mathbb{R}} \varepsilon^{-1} \rho(\varepsilon^{-1}\tau) |\chi(s-\tau) - \chi(s)| \,\mathrm{d}\tau \le \eta k_0^{-1} \varepsilon \int_{\mathbb{R}} \rho(\sigma)|\sigma| \,\mathrm{d}\sigma.$$

We fix a small $\varepsilon > 0$ such that χ_ε satisfies (2.10) with s instead of $|x - x_0|$ and $5/3$ instead of 2. Then $\psi(x) = \chi_\varepsilon((\delta_0^2 + |x - x_0|^2)^{1/2})$ does the job, where $\delta_0 = k_0 \delta (3\eta)^{-1}$.

Set $\phi(t,x) = \psi(x) - t$ and $u_\tau = \mathrm{e}^{\tau\phi} u$ for $\tau > 0$. Inequality (2.10) yields $\psi(x) \le -\delta + t$ if $|x - x_0| \ge R - k_0 t/\eta$ (i.e., $(t,x) \notin \Gamma(x_0, R, k_0/\eta)$), so that $\mathrm{e}^{\tau\phi} \le \mathrm{e}^{-\tau\delta} \le 1$ off $\Gamma(x_0, R, k_0/\eta)$ and $\mathrm{e}^{\tau\phi}$ is bounded on $J \times \mathbb{R}^3$. We further have $\nabla \mathrm{e}^{\tau\phi} = \tau \mathrm{e}^{\tau\phi} \nabla\psi$ and $\partial_t \mathrm{e}^{\tau\phi} = -\tau \mathrm{e}^{\tau\phi}$. As a result, u_τ is an element of $C(\overline{J}, L^2)$ and the right-hand side of

$$L u_\tau = \mathrm{e}^{\tau\phi} f - \tau \Big(A_0 - \sum_{j=1}^3 A_j \partial_j \psi \Big) u_\tau$$

belongs to $L_J^2 L^2$. The matrix in parentheses is denoted by M.

2) For $\xi \in \mathbb{R}^6$ we have $M\xi \cdot \xi \ge (\eta - k_0 |\nabla\psi|)|\xi|^2 \ge 0$. Set $C = \frac{1}{2} \operatorname{div} A - D$ and $\kappa = \|C\|_\infty$. By Theorem 2.4, the function u_τ satisfies the energy equality

$$\|A_0(t)^{\frac{1}{2}} u_\tau(t)\|_{L^2}^2 = \|A_0(0)^{\frac{1}{2}} u_\tau(0)\|_{L^2}^2 + 2\langle (C - \tau M) u_\tau + \mathrm{e}^{\tau\phi} f, u_\tau \rangle_{L_J^2 L^2}.$$

Using Cauchy–Schwarz, the above inequalities and Gronwall, we estimate

$$\eta \|u_\tau(t)\|_{L^2}^2 \le \|A_0(0)\|_{L^\infty} \|\mathrm{e}^{\tau\phi} u_0\|_{L^2}^2 + \|\mathrm{e}^{\tau\phi} f\|_{L_J^2 L^2}^2 + (2\kappa + 1) \int_0^t \|u_\tau(s)\|_{L^2}^2 \,\mathrm{d}s,$$

$$\|\mathrm{e}^{\tau\phi} u(t)\|_{L^2}^2 \lesssim_T \|\mathrm{e}^{\tau\phi} u_0\|_{L^2}^2 + \|\mathrm{e}^{\tau\phi} f\|_{L_J^2 L^2}^2.$$

The right-hand side tends to 0 as $\tau \to \infty$ since u_0 and f vanish on $\Gamma(x_0, R, k_0/\eta)$ and $\mathrm{e}^{\tau\phi} \to 0$ uniformly off $\Gamma(x_0, R, k_0/\eta)$. Hence, $u(t)$ has to be 0 on $\{\phi > \delta\} = \{\psi > t + \delta\}$. By (2.10), this set includes points (t,x) with $|x - x_0| < R - k_0 \eta^{-1}(t + 3\delta)$. Since $\delta > 0$ is arbitrary here, u equals 0 on $\Gamma(x_0, R, k_0/\eta)$. □

2.2. The linear problem in $\mathcal{H}^3$

As noted in Chapter 1, to solve the nonlinear problem (1.15) we will set $A_0 = a_0(v)$ for functions v having the same regularity as the desired solution u. Since A_0 has to be Lipschitz in Theorem 2.4, the same must be true for v. Working in $\mathcal{H}^k$ spaces, we thus need solutions in $L_J^\infty \mathcal{H}^3 \cap W_J^{1,\infty} \mathcal{H}^2$ at least. We want to reduce the problem in $\mathcal{H}^3$ to that in L^2 by means of a transformation. (One could also perform the proof of Theorem 2.4 in $\mathcal{H}^3$ instead of L^2, see e.g. [**6**] or [**13**], which would require more work in our context.)

To this end, we define the operator $\Lambda = \mathcal{F}^{-1}(1 + |\xi|^2)^{1/2} \mathcal{F}$ via the Fourier transform $\mathcal{F}$ on tempered distributions at first. Using standard properties of $\mathcal{F}$, one sees that $\Lambda = (I - \Delta)^{1/2}$ can be restricted to isomorphisms $\mathcal{H}^k \to \mathcal{H}^{k-1}$ for $k \in \mathbb{Z}$ with inverse given by $\Lambda^{-1} = (I - \Delta)^{-1/2} = \mathcal{F}^{-1}(1 + |\xi|^2)^{-1/2} \mathcal{F}$ and that it commutes with derivatives. Powers of Λ behave analogously. See also Section 3.1. Moreover, Λ^{-1} is a convolution operator with positive kernel by Proposition 1.2.5 in [**24**], so that $\Lambda = (I - \Delta)\Lambda^{-1}$ leaves invariant real-valued functions.

Our analysis relies on a commutator estimate for Λ^3 and $M_a : \varphi \mapsto a\varphi$ which gains a derivative. We use the proof of Lemma A2 in [**31**], which works for real $k > \frac{m}{2} + 1$ on $\mathbb{R}^m$. More results can be found in e.g. [**65**] or in Proposition 3.9.

Proposition 2.7. *Let $k \in \mathbb{N}$ and $a \in W^{1,\infty}(\mathbb{R}^3)$ with $\Delta a \in L^3(\mathbb{R}^3)$ if $k = 2$ and $\nabla a \in \mathcal{H}^{k-1}(\mathbb{R}^3)$ if $k \geq 3$. Then the commutator $[\Lambda^k, M_a] = \Lambda^k M_a - M_a\Lambda^k : \mathcal{H}^{k-1}(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ is bounded.*

Proof. The result for $k = 1$ requires more work and is a special case of Proposition 4.1.A in [**65**] or of Proposition 3.9. For $k = 2$, we have $-[\Lambda^2, M_a]v = \Delta a\, v + 2\nabla a \cdot \nabla v$ so that the result easily follows from Hölder and Sobolev. To simplify a bit, we now restrict ourselves to $k = 3$. By Plancherel, we have to show that $T = \mathcal{F}[\Lambda^3, M_a]\Lambda^{-2}\mathcal{F}^{-1}$ is bounded on $L^2(\mathbb{R}^3)$. Observe that T is the integral operator with kernel

$$\kappa(\xi, \zeta) = (2\pi)^{-\frac{3}{2}}\big[(1 + |\xi|^2)^{\frac{3}{2}} - (1 + |\zeta|^2)^{\frac{3}{2}}\big]\hat{a}(\xi - \zeta)(1 + |\zeta|^2)^{-1}$$

with $\xi, \zeta \in \mathbb{R}^3$. We can bound

$$\big|[\dots]\big| \leq \int_0^1 \big|\partial_\tau(1 + |\xi + \tau(\zeta - \xi)|^2)^{\frac{3}{2}}\big|\,\mathrm{d}\tau \leq \tfrac{15}{2}|\xi - \zeta|\big|(1 + |\xi|^2) + (1 + |\zeta|^2)\big|.$$

Hence, κ is dominated by $|\kappa| \leq c_0\kappa_1 + c_0\kappa_2$ with $c_0 = \frac{15}{2}(2\pi)^{-\frac{3}{2}}$ and

$$\kappa_1(\xi, \zeta) = (1+|\xi|^2)\tilde{b}(\xi-\zeta)(1+|\zeta|^2)^{-1}, \ \ \kappa_2(\xi, \zeta) = \tilde{b}(\xi-\zeta), \quad \tilde{b} = |\xi\hat{a}| = |\mathcal{F}(\nabla a)|,$$

so that we need the L^2-boundedness of the corresponding integral operators. Again by Plancherel, it thus suffices to show that $T_1 = \Lambda^2 M_b \Lambda^{-2}$ and $T_2 = M_b$ are bounded on L^2, where $b = \mathcal{F}^{-1}\tilde{b}$. Since we have

$$\|b\|_{2,2}^2 \leq c \int_{\mathbb{R}^3}(1 + |\xi|^2)^2|\tilde{b}(\xi)|^2\,\mathrm{d}\xi \leq c\|\nabla a\|_{2,2}^2$$

for some constants, Lemma 2.8 below and Sobolev's embedding indeed yield

$$\begin{aligned}
&\|T_1\varphi\|_2 \leq c\|b\Lambda^{-2}\varphi\|_{2,2} \leq c\|b\|_{2,2}\|\Lambda^{-2}\varphi\|_{2,2} \leq c\|\nabla a\|_{2,2}\|\varphi\|_2,\\
&\|T_2\varphi\|_2 \leq \|b\|_\infty\|\varphi\|_2 \leq c\|\nabla a\|_{2,2}\|\varphi\|_2, \qquad \varphi \in L^2. \qquad \square
\end{aligned}$$

Guided by Proposition 2.7 and (2.1), we introduce the space

$$\mathcal{F}^k(J) = \mathcal{F}^k(T) = \big\{A \in W^{1,\infty}(J\times\mathbb{R}^3, \mathbb{R}^{6\times6}) \,\big|\, (\nabla, \partial_t)A \in L_J^\infty\mathcal{H}^{k-1}\big\}, \qquad k \in \mathbb{N},$$

for the coefficients, endowed with its natural norm. We will usually take $k = 3$. We employ the same notation for vector- or scalar-valued functions of the same regularity. The subscript sym will refer to symmetric matrices and $\geq \eta$ to those with $A = A^\top \geq \eta I$. We state the hypotheses of the present section:

$$\begin{aligned}
&A_0 \in \mathcal{F}^3_{\geq\eta}(J), \quad \eta > 0, \quad A_1, A_2, A_3 \in \mathcal{F}^3_{\mathrm{sym}}(J), \quad D \in \mathcal{F}^3(J), \qquad (2.11)\\
&u_0 \in \mathcal{H}^3 = \mathcal{H}^3(\mathbb{R}^3, \mathbb{R}^6), \quad f \in \mathcal{Z}^3(J) = \mathcal{Z}^3(T) := L^2(J, \mathcal{H}^3) \cap \mathcal{H}^1(J, \mathcal{H}^2).
\end{aligned}$$

Set $\|f\|^2_{\mathcal{Z}^k_\gamma(J)} = \|e_{-\gamma}f\|^2_{L^2_J\mathcal{H}^k} + \|e_{-\gamma}\partial_t f\|^2_{L^2_J\mathcal{H}^{k-1}}$ for $\gamma \geq 0$ and $k \in \mathbb{N}$. We also use

$$\hat{\mathcal{H}}^k = \big\{v \in L^\infty(\mathbb{R}^3) \,\big|\, \nabla_x v \in \mathcal{H}^{k-1}\big\}, \quad \mathcal{G}^k(\overline{J}) = \mathcal{G}^k(T) = C(\overline{J}, \mathcal{H}^k) \cap C^1(\overline{J}, \mathcal{H}^{k-1})$$

with their natural norms, as well as $\|v\|^2_{\mathcal{G}^k_\gamma(J)} = \|e_{-\gamma}v\|^2_{L^\infty_J\mathcal{H}^k} + \|e_{-\gamma}\partial_t v\|^2_{L^\infty_J\mathcal{H}^{k-1}}$. (Such spaces will also be considered on time intervals different from $J = (0, T)$.)

We state product and inversion rules which are often used in this chapter, cf. [58]. Here one can replace $\mathbb{R}^3$ by all Lipschitz domains. In the proof and also later on, we employ Sobolev embeddings such as $\mathcal{H}^2 \hookrightarrow L^p$ for $p \in [2,\infty]$ and $\mathcal{H}^1 \hookrightarrow L^q$ for $q \in [2,6]$ on (Lipschitz domains in) $\mathbb{R}^3$.

Lemma 2.8. *Let $k, j \in \mathbb{N}_0$ with $k \geq \max\{j,2\}$ and $l \in \mathbb{N}$.*

a) For $v \in \mathcal{H}^k$ and $w \in \mathcal{H}^j$ we have $\|vw\|_{\mathcal{H}^j} \lesssim \|v\|_{\mathcal{H}^k}\|w\|_{\mathcal{H}^j}$. Here one can replace $\mathcal{H}^k$ by $\hat{\mathcal{H}}^k$, as well as $\mathcal{H}^j$ and $\mathcal{H}^k$ by $\mathcal{G}^j(\overline{J})$ and $\mathcal{G}^k(\overline{J})$ (or $\mathcal{F}^k(J)$), or by $\mathcal{F}^j(J)$ and $\mathcal{F}^k(J)$, taking $j \geq 1$ if $\mathcal{G}^j(\overline{J})$ or $\mathcal{F}^j(J)$ is involved.

b) Let $A \in \hat{\mathcal{H}}^l_{\geq\eta}$. Then A^{-1} belongs to $\hat{\mathcal{H}}^l_{\geq\mu}$ with norm bounded by $c(\mu,k)(1+\|A\|_{\hat{\mathcal{H}}^l})^{l-1}\|A\|_{\hat{\mathcal{H}}^l}$ and $\mu = \|A\|_\infty^{-1}$.

Proof. a) For the first claim, by the product rule (and interpolative inequalities) we have to control $\partial^\beta v \partial^{\alpha-\beta} w$ for multi-indices $0 \leq \beta \leq \alpha$ with $|\alpha| = j$. Observe that $\partial^\beta v \in \mathcal{H}^{k-|\beta|}$ and $\partial^{\alpha-\beta} w \in \mathcal{H}^{|\beta|}$. This product can be estimated in L^2, as needed, if $k - |\beta| \geq 2$ or $|\beta| \geq 2$ since then $\partial^\beta v$ or $\partial^{\alpha-\beta} w$ are bounded, respectively. As $k \geq 2$, only the case $|\beta| = 1$ remains. Here $\partial^\beta v$ and $\partial^{\alpha-\beta} w$ belong to $\mathcal{H}^1 \hookrightarrow L^4$ and thus the product to L^2. The other variants are proved analogously.

b) We take $l = 3$, the other cases are similar. Observe that $\partial_x^3 A^{-1}$ is a linear conbination of terms like

$$A^{-1}\partial^3 A A^{-1}, \quad A^{-1}\partial^2 A A^{-1}\partial A A^{-1}, \quad A^{-1}\partial A A^{-1}\partial A A^{-1}\partial A A^{-1}.$$

(Here and below we occasionally use somewhat informal notation in such expressions.) These terms satisfy the asserted estimate as in part a), since $\|A^{-1}\|_\infty \leq 1/\eta$. The lower-order ones are treated in the same way. $\square$

We look for a solution $u \in \mathcal{G}^3(\overline{J})$ of (2.2) assuming (2.11). The basic idea is to solve a modified problem for $w = \Lambda^3 u$ in $C(\overline{J}, L^2)$. Since the commutator result Proposition 2.7 only improves space regularity, we first replace the equation $Lu = f$ by $\hat{L}u = \hat{f} := A_0^{-1} f$ where $\hat{L}$ has the coefficients $\hat{A}_0 = I$, $\hat{A}_j = A_0^{-1}A_j$ and $\hat{D} = A_0^{-1}D$. We then obtain

$$\hat{L}w = \Lambda^3 \hat{f} + \sum_{j=1}^{3} [\hat{A}_j, \Lambda^3]\partial_j u + [\hat{D}, \Lambda^3]u,$$

$$Lw = A_0\Lambda^3 \hat{f} + \sum_{j=1}^{3} A_0[\hat{A}_j, \Lambda^3]\partial_j u + A_0[\hat{D}, \Lambda^3]u =: g(f,u). \tag{2.12}$$

We now replace in g the unknown u by a given map $v \in C(\overline{J}, \mathcal{H}^3)$. Theorem 2.4 provides a solution $w \in C(\overline{J}, L^2)$ of $Lw = g(f,v)$ with $w(0) = \Lambda^3 u_0$. The energy estimate from Lemma 2.1 (with a large γ) then implies that $\Phi\colon v \mapsto \Lambda^{-3}w$ is a strict contraction on $L^\infty_{\gamma,J}\mathcal{H}^3$. This fact will lead to the desired regularity result. Let λ be the maximum of $\|\Lambda^k\|_{\mathcal{B}(\mathcal{H}^k,L^2)}$ and $\|\Lambda^{-k}\|_{\mathcal{B}(L^2,\mathcal{H}^k)}$ for $k \in \{2,3\}$. It will be important in the fixed-point argument for the nonlinear problem that the constant c_0 in (2.13) only depends on r_0 (and η), but not on r.

THEOREM 2.9. *Let* (2.11) *be true with* $\|A_j(0)\|_{\hat{\mathcal{H}}^2}, \|D(0)\|_{\hat{\mathcal{H}}^2} \le r_0$ *and* $\|A_j\|_{\mathcal{F}^3(J)}, \|D\|_{\mathcal{F}^3(J)} \le r$ *for* $j \in \{0,1,2,3\}$. *Then there exists a unique solution* u *of* (2.2) *in* $C(\overline{J}, \mathcal{H}^3) \cap C^1(\overline{J}, \mathcal{H}^2)$. *For* $t \in \overline{J}$ *and* $\gamma \ge \gamma_1(r,\eta) := \max\{\gamma_0(r,\eta), \sqrt{c_1}\}$, *see* (2.4), *it satisfies*

$$\begin{aligned} \gamma\|u\|^2_{\mathcal{Z}^3_\gamma(0,t)} &+ \mathrm{e}^{-2\gamma t}(\|u(t)\|^2_{\mathcal{H}^3} + \|\partial_t u(t)\|^2_{\mathcal{H}^2}) \\ &\le c_0(\|u_0\|^2_{\mathcal{H}^3} + \|f(0)\|^2_{\mathcal{H}^2}) + \tfrac{c_1}{\gamma}\, \|f\|^2_{\mathcal{Z}^3_\gamma(0,t)} \end{aligned} \tag{2.13}$$

for constants $c_0 = c_0(r_0, \eta)$ *and* $c_1 = c_1(r, \eta)$ *described in the proof.*

PROOF. 1) Take $v \in C(\overline{J}, \mathcal{H}^3)$ and $\gamma \ge \gamma_0(r,\eta)$ from (2.4). Let $t \in J$. Using Proposition 2.7 and Lemma 2.8, we see that the square of the norm in $L^2_{\gamma,t}L^2$ of $g(f,v)$ from (2.12) is bounded by $c_1'(\|f\|^2_{L^2_{\gamma,t}\mathcal{H}^3} + \|v\|^2_{L^2_{\gamma,t}\mathcal{H}^3})$ for a constant $c_1' = c_1'(r,\eta)$. Theorem 2.4 yields a solution $w \in C(\overline{J}, L^2)$ of $Lw = g(f,v)$ and $w(0) = \Lambda^3 u_0 =: w_0$ which satisfies

$$\tfrac{\gamma\eta}{4}\|w\|^2_{L^2_{\gamma,t}L^2} + \tfrac{\eta}{2}\|w\|^2_{L^\infty_{\gamma,t}L^2} \le c_0'\|u_0\|^2_{\mathcal{H}^3} + \tfrac{c_1'}{2\gamma\eta}\big(\|f\|^2_{L^2_{\gamma,t}\mathcal{H}^3} + \|v\|^2_{L^2_{\gamma,t}\mathcal{H}^3}\big) \tag{2.14}$$

with $c_0' = \frac{\lambda^2}{2}\|A_0(0)\|_\infty$. The map w also belongs to $C^1(\overline{J}, \mathcal{H}^{-1})$ because of (2.3) and $f \in \mathcal{Z}^3(J)$. Set $\Phi v = \Lambda^{-3} w \in \mathcal{G}^3(\overline{J})$. Let $\overline{w}$ satisfy $L\overline{w} = g(f,\overline{v})$ and $\overline{w}(0) = w_0$ for some $\overline{v} \in C(\overline{J}, \mathcal{H}^3)$. For $w - \overline{w}$ estimate (2.14) applies with $u_0 = 0$ and $f = 0$ so that

$$\|\Phi(v - \overline{v})\|_{L^\infty_{\gamma,J}\mathcal{H}^3} = \|\Lambda^{-3}(w - \overline{w})\|_{L^\infty_{\gamma,J}\mathcal{H}^3} \le \frac{\lambda\sqrt{c_1' T}}{\sqrt{\gamma}\eta}\, \|v - \overline{v}\|_{L^\infty_{\gamma,J}\mathcal{H}^3}.$$

Fixing a large $\gamma = \gamma(r,\eta,T)$, we obtain a fixed point u of Φ in $L^\infty_{\gamma,J}\mathcal{H}^3$. It actually belongs to $\mathcal{G}^3(\overline{J})$ and satisfies $u(0) = u_0$. Equation (2.12) implies that $Lu = f$. Uniqueness of solutions was already shown in Proposition 2.3.

2) It remains to establish (2.13). We first insert $u = v$ and $w = \Lambda^3 u$ in (2.14) and take $\gamma \ge \max\Big\{\gamma_0(r,\eta), \frac{2\lambda\sqrt{c_1'}}{\eta}\Big\}$. Note that $\|u\|_{3,2} \le \lambda\|w\|_2$. Absorbing $\|u\|^2_{L^2_{\gamma,t}\mathcal{H}^3}$ by the left-hand side, we infer

$$\tfrac{\gamma\eta}{8}\|u\|^2_{L^2_{\gamma,t}\mathcal{H}^3} + \tfrac{\eta}{2}\|u\|^2_{L^\infty_{\gamma,t}\mathcal{H}^3} \le c_0'\lambda^2\|u_0\|^2_{\mathcal{H}^3} + \tfrac{c_1'\lambda^2}{2\gamma\eta}\, \|f\|^2_{L^2_{\gamma,t}\mathcal{H}^3}. \tag{2.15}$$

If we estimate $\partial_t u$ in $\mathcal{H}^2$ via (2.3) and (2.15), we obtain a constant depending on r in front of the norm of u_0. Instead we use that $\partial_t u \in C(\overline{J}, \mathcal{H}^2)$ satisfies

$$\begin{aligned} L\partial_t u &= \partial_t f - \partial_t D u - \sum\nolimits_{j=0}^3 \partial_t A_j \partial_j u =: h, \\ \partial_t u(0) &= A_0(0)^{-1} f(0) - A_0(0)^{-1} D(0) u_0 - \sum\nolimits_{j=1}^3 A_0(0)^{-1} A_j(0)\partial_j u_0 =: v_0. \end{aligned}$$

Lemma 2.8 yields

$$\begin{aligned} \|h(s)\|_{\mathcal{H}^2} &\le \|\partial_t f(s)\|_{\mathcal{H}^2} + \overline{c}(r)(\|u(s)\|_{\mathcal{H}^3} + \|\partial_t u(s)\|_{\mathcal{H}^2}), \qquad s \in J, \\ \|v_0\|_{\mathcal{H}^2} &\le c(r_0,\eta)(\|f(0)\|_{\mathcal{H}^2} + \|u_0\|_{\mathcal{H}^3}). \end{aligned}$$

The maps $[M_{\hat{A}_j}, \Lambda^2]\colon \mathcal{H}^1 \to L^2$ are bounded by Proposition 2.7 since $\partial_x^2 \hat{A}_j \in \mathcal{H}^1 \hookrightarrow L^3$. Starting from $L\partial_t u = h$, as in (2.12) and (2.14) we thus deduce

$$\begin{aligned}
&\tfrac{\gamma\eta}{4}\|\partial_t u\|^2_{L^2_{\gamma,t}\mathcal{H}^2} + \tfrac{\eta}{2}\|\partial_t u\|^2_{L^\infty_{\gamma,t}\mathcal{H}^2}\\
&\quad \le \hat{c}_0\lambda^2\big(\|u_0\|^2_{\mathcal{H}^3} + \|f(0)\|^2_{\mathcal{H}^2}\big) + \tfrac{\hat{c}_1\lambda^2}{2\gamma\eta}\big(\|\partial_t f\|^2_{L^2_{\gamma,t}\mathcal{H}^2} + \|u\|^2_{L^2_{\gamma,t}\mathcal{H}^3} + \|\partial_t u\|^2_{L^2_{\gamma,t}\mathcal{H}^2}\big)
\end{aligned}$$

for constants $\hat{c}_0 = \hat{c}_0(r_0,\eta)$ and $\hat{c}_1 = \hat{c}_1(r,\eta)$. Set $c_0 = 16\lambda^2\eta^{-1}(c_0' + \hat{c}_0)$ and $c_1 = \frac{8\lambda^2}{\eta^2}\max\{c_1', \hat{c}_1\}$. We add the above inequality to (2.15) and take $\gamma \ge \gamma_1(r,\eta) := \max\big\{\gamma_0(r,\eta), \sqrt{c_1}\big\}$. Estimate (2.13) follows after some calculations. □

In the above result we control more space than time derivatives. Under stronger assumptions on A_j, D and f, one can obtain analogous estimates on $\partial_t^2 u$ in $\mathcal{H}^1$ and $\partial_t^3 u$ in L^2 by differentiating (2.2) in time, cf. (2.27) in [**44**] or [**58**]. We discuss variants of the above theorem partly needed below.

Proposition 2.10. *Let A_j and D be as in Theorem 2.9, as well as $u_0 \in \mathcal{H}^2$ and $f \in L^2(J,\mathcal{H}^2)$. Then there is a unique solution $u \in C(\overline{J},\mathcal{H}^2) \cap C^1(\overline{J},\mathcal{H}^1)$ of (2.2). For $t \in \overline{J}$ and $\gamma \ge \tilde{\gamma}_1(r,\eta) := \max\big\{\gamma_0(r,\eta), \sqrt{\tilde{c}_1}\big\}$, it satisfies*

$$\gamma\|u\|^2_{L^2_{\gamma,t}\mathcal{H}^2} + \mathrm{e}^{-2\gamma t}\|u(t)\|^2_{\mathcal{H}^2} \le \tilde{c}_0\|u_0\|^2_{\mathcal{H}^2} + \tfrac{\tilde{c}_1}{\gamma}\,\|f\|^2_{L^2_{\gamma,t}\mathcal{H}^2}$$

for constants $\tilde{c}_0 = \tilde{c}_0(r_0,\eta)$ and $\tilde{c}_1 = \tilde{c}_1(r,\eta)$. If $\partial_t f \in L^2(J,\mathcal{H}^1)$, we also obtain

$$\gamma\|\partial_t u\|^2_{L^2_{\gamma,t}\mathcal{H}^1} + \mathrm{e}^{-2\gamma t}\|\partial_t u(t)\|^2_{\mathcal{H}^1} \le \tilde{c}_0(\|u_0\|^2_{\mathcal{H}^2} + \|f(0)\|^2_{\mathcal{H}^1}) + \tfrac{\tilde{c}_1}{\gamma}\,\|f\|^2_{\mathcal{Z}^2_\gamma(0,t)}.$$

The result is shown as Theorem 2.9, replacing Λ^3 by Λ^2 in its proof up to (2.15) and Λ^2 by Λ afterwards.

Remark 2.11. In Theorem 2.9 we have focused on the space $\mathcal{H}^3$ needed for the quasilinear problem. Actually, one obtains a unique solution $u \in \mathcal{G}^k(\overline{J})$ of (2.2) satisfying the analogue of (2.13) if $k \in \mathbb{N}$, $u_0 \in \mathcal{H}^k$, $f \in \mathcal{Z}^k(J)$, $A_j, D \in \mathcal{F}^k(J)$, $A_j = A_j^\top$, $A_0 \ge \eta I$, and $\partial_x^2 A_j \in L^\infty_J L^3$ if $k = 2$. This can be shown as for $k = 3$ still using Proposition 2.7. One only has to take care of estimates for products, inverse matrices and commutators, noting that the extra condition for $k = 2$ is preserved by products and inverses.

Moreover, there is no problem to change the range space $\mathbb{R}^6$ to $\mathbb{R}^n$. Also other spatial domains $\mathbb{R}^m$ can be treated analogously, though one has to modify the assumptions on the coefficients in this case. Invoking a bit harmonic analysis one can also work in fractional Sobolev spaces $\mathcal{H}^s$ instead of $\mathcal{H}^k$, see [**32**]. ◇

Remark 2.12. In (2.11) we have required that the derivatives of the coefficients belong to $\mathcal{H}^2$. So local singularities are allowed to some extent, but one enforces a certain decay at infinity which is an unnecessary restriction. Actually, Theorem 2.9 remains valid if we replace the space $\mathcal{F}^3(J)$ by $\mathcal{F}^3_\infty(J) = \mathcal{F}^3(J) + W^{3,\infty}(J\times\mathbb{R}^3)$, and $\hat{\mathcal{H}}^2$ by $\hat{\mathcal{H}}^2_\infty = \hat{\mathcal{H}}^2 + W^{2,\infty}$. (They have the norm $\|z\|_{X+Y} = \inf_{z=x+y}\|x\|_X + \|y\|_Y$ of sums $X+Y$. Observe $X, Y \hookrightarrow X+Y$.) To derive this fact, we note that $[M_A, \Lambda^2]\colon \mathcal{H}^2 \to \mathcal{H}^1$ is bounded uniformly in t if $A \in \mathcal{F}^3_\infty(J)$, and so the same is true for

$$[M_A, \Lambda^3] = [M_A, \Lambda]\Lambda^2 + \Lambda[M_A, \Lambda^2]\colon \mathcal{H}^2 \to L^2.$$

(Recall the boundedness of $[M_A, \Lambda]$ on L^2.) One can further check the appropriate bounds for products and inversions involving $\mathcal{F}^3_\infty(J)$ and $\hat{\mathcal{H}}^2_\infty$, as well as $\mathcal{G}^3(\overline{J})$. The analogue of Theorem 2.9 can now be proven as before. $\Diamond$

As a preparation for Theorem 2.19 on the wellposedness of the nonlinear problem we show an approximation result for the coefficients.

Lemma 2.13. *Let $u_0 \in L^2$, $f \in L^2_J L^2$, $n \in \mathbb{N} \cup \{\infty\}$, $j \in \{0,1,2,3\}$, $A^n_j \in \mathcal{F}^3_\infty(J)$ be symmetric with $A^n_0 \geq \eta I$, and $D^n \in \mathcal{F}^3_\infty(J)$. Assume that $\|A^n_j\|_{\mathcal{W}^{1,\infty}_J} \leq r$ and $\|D^n\|_{\mathcal{L}^\infty_J} \leq r$, as well as $A^n_j \to A^\infty_j$ and $D^n \to D^\infty$ in $\mathcal{L}^\infty_J$ as $n \to \infty$. Set $L_n = \sum_j A^n_j \partial_j + D^n$. There are maps $u_n \in C(\overline{J}, L^2)$ with $L_n u_n = f$ and $u_n(0) = u_0$ by Theorem 2.4. Then (u_n) tends to u_∞ in $C(\overline{J}, L^2)$.*

Proof. There are functions $u_{0,m}$ in $\mathcal{H}^3$ and f_m in $\mathcal{Z}^3(J)$ converging to u_0 and f in L^2 and $L^2_J L^2$, respectively, as $m \to \infty$. For these data Theorem 2.9 provides solutions $u_{n,m} \in \mathcal{G}^3(\overline{J})$ of $L_n u_{n,m} = f_m$ and $u_{n,m}(0) = u_{0,m}$. Fixing $\gamma = \gamma_0(r, \eta)$ from Lemma 2.1 and (2.4), Proposition 2.3 now shows

$$\|u_n - u_{n,m}\|_{L^\infty_J L^2} \leq c\|u_n - u_{n,m}\|_{L^\infty_{\gamma,J} L^2} \leq c\big(\|u_0 - u_{0,m}\|_{L^2} + \|f - f_m\|_{L^2_J L^2}\big).$$

with $c = c(r, \eta, T)$. The right-hand side tends to 0 as $m \to \infty$ uniformly for $n \in \mathbb{N} \cup \{\infty\}$. It is thus enough to take $u_0 \in \mathcal{H}^3$, $f \in \mathcal{Z}^3(J)$, and $u_n \in \mathcal{G}^3(\overline{J})$. We then compute

$$L_n(u_n - u_\infty) = L_\infty u_\infty - L_n u_\infty = \sum_{j=0}^{3} (A^\infty_j - A^n_j)\partial_j u_\infty + (D^\infty - D^n)u_\infty =: g_n.$$

Since $u_\infty \in \mathcal{G}^3(\overline{J})$, as above Lemma 2.1 yields

$$\|u_n - u_\infty\|_{L^\infty_J L^2} \leq c(\gamma, T)\, \|g_n\|_{L^\infty_{\gamma,J} L^2} \longrightarrow 0, \qquad n \to \infty. \qquad \Box$$

2.3. The quasilinear problem in $\mathcal{H}^3$

In this section we study the nonlinear system

$$L(u)u := \sum_{j=0}^{3} a_j(u)\partial_j u + d(u)u = f, \quad t \geq 0,\ x \in \mathbb{R}^3, \qquad u(0) = u_0, \tag{2.16}$$

under the assumptions

$$\begin{aligned} & a_j, d \in C^3(\mathbb{R}^3 \times \mathbb{R}^6, \mathbb{R}^{6\times 6}), \quad a_j = a_j^\top, \quad a_0 \geq \eta I, \quad \eta \in (0,1], \\ & \forall\, r > 0\colon \sup_{|\xi| \leq r} \max_{0 \leq |\alpha| \leq 3} \|\partial^\alpha_x a_j(\cdot, \xi)\|_{L^\infty}, \|\partial^\alpha_x d(\cdot, \xi)\|_{L^\infty} < \infty, \quad j \in \{0,1,2,3\}, \\ & u_0 \in \mathcal{H}^3, \quad \forall\, T > 0\colon f \in \mathcal{Z}^3(T) = \mathcal{Z}^3(J) = L^2(J, \mathcal{H}^3) \cap \mathcal{H}^1(J, \mathcal{H}^2), \quad J = (0, T). \end{aligned} \tag{2.17}$$

One can also treat coefficients only defined for $(x, \xi) \in \mathbb{R}^3 \times \mathcal{O}$ and an open subset $\mathcal{O} \subseteq \mathbb{R}^6$, see Remark 2.20. This is already needed in the Kerr Example 1.1 if κ is not non-negative. To simplify a bit, we focus on the case $\mathcal{O} = \mathbb{R}^6$ in (2.17).

We look for solutions u of (2.16) in $C([0, T_+), \mathcal{H}^3) \cap C^1([0, T_+), \mathcal{H}^2)$ for a maximally chosen final time $T_+ \in (0, \infty]$. As indicated in the next section, solutions may blow up and so T_+ could be finite. The solutions will be constructed in a fixed-point argument on the space

$$\mathcal{G}^{k-}(J) = L^\infty(J, \mathcal{H}^k) \cap W^{1,\infty}(J, \mathcal{H}^{k-1})$$

endowed with its natural norm, where $k = 3$. The strategy of this section and many techniques are typical for quasilinear evolution equations, though there are different (but related) approaches, see e.g. [**5**], [**6**], or [**28**]. The easier 'semilinear' case is discussed in [**46**].

We first state basic properties of substitution operators, which remain valid for Lipschitz domains instead of $\mathbb{R}^3$ with the same proof. (Recall Remark 2.12 concerning $\mathcal{F}^3_\infty(J)$ and $\hat{\mathcal{H}}^2_\infty$.) We set $E_\gamma = L^\infty_\gamma(J, \mathcal{H}^2)$ for a moment.

Lemma 2.14. *Let $a \in C^3(\mathbb{R}^3 \times \mathbb{R}^n, \mathbb{R}^{n\times n})$ fulfill the second line of* (2.17).

a) Let $v \in \mathcal{G}^{3-}(J)$ with $\|v\|_\infty \le r$. Then $\|a(v)\|_{\mathcal{F}^3_\infty(J)} \le \kappa(r)(1+\|v\|^3_{\mathcal{G}^{3-}(J)})$.

b) Let $v, w \in L^\infty_J \mathcal{H}^2$ with norm $\le r$. Then $\|a(v) - a(w)\|_{E_\gamma} \le \kappa(r)\|v - w\|_{E_\gamma}$ for all $\gamma \ge 0$. We can replace $L^\infty_J \mathcal{H}^2$ and E_γ by $\mathcal{G}^2(\overline{J})$ and $\mathcal{G}^2_\gamma(\overline{J})$, respectively.

c) Let $v_0 \in \mathcal{H}^2$ with $\|v_0\|_\infty \le r_0$. Then $\|a(v_0)\|_{\hat{\mathcal{H}}^2_\infty} \le \kappa_0(r_0)(1 + \|v_0\|^2_{\mathcal{H}^2})$.

d) Let $v_0, w_0 \in \mathcal{H}^2$ with norm $\le r_0$. Then $\|a(v_0) - a(w_0)\|_{\mathcal{H}^2} \le \kappa_0(r_0)\|v_0 - w_0\|^2_{\mathcal{H}^2}$.

Proof. We sketch the proof. (See §7.1 in [**56**] or §2 in [**57**] for more details.)

a) Take $\alpha \in \mathbb{N}^4_0$ with $1 \le |\alpha| \le 3$ and $\alpha_0 \in \{0, 1\}$. The latter refers to the time derivative. It is clear that the function $|(\partial^\beta a)(\cdot, v)|$ is bounded by $c(r)$ for all $0 \le |\beta| \le 3$ where $\beta = (\beta_x, \beta_\xi) \in \mathbb{N}^3_0 \times \mathbb{N}^6_0$. Note that $\partial^\alpha a(v)$ is a linear combination of products of $(\partial^\beta a)(\cdot, v)$ and $j \in \{0, 1, 2, 3\}$ factors $\partial^{\gamma_i} v$ with $\beta_x + \gamma_1 + \cdots + \gamma_j = \alpha$. Since $v \in \mathcal{W}^{1,\infty}_J$ by Sobolev's embedding, as in the proof of Lemma 2.8 one can estimate $\partial^\alpha a(v)$ in $L^\infty_J L^2$ if $j \ge 1$ and in $\mathcal{L}^\infty_J$ if $j = 0$, both by $c(r)(1 + \|v\|^3_{\mathcal{G}^3(\overline{J})})$. (Use $0 \le a, a^2 \lesssim 1 + a^3$.)

b) We start from the formula

$$a(w) - a(v) = \int_0^1 (\partial_\xi a)(\cdot, v + s(w - v))(w - v)\,\mathrm{d}s.$$

Let $\varphi_s = v + s(w - v)$. We then compute

$$\begin{aligned}\partial_x^2(a(w) - a(v)) = \int_0^1 (\partial_\xi a)(\cdot, \varphi_s)\partial_x^2(w - v)\,\mathrm{d}s + \int_0^1 \partial_x^2(\partial_\xi a)(\cdot, \varphi_s)(w - v)\,\mathrm{d}s \\ + 2\int_0^1 \partial_x(\partial_\xi a)(\cdot, \varphi_s)\,\partial_x(w - v)\,\mathrm{d}s \qquad (2.18)\end{aligned}$$

The factor $\mathrm{e}^{-\gamma t}$ is put in front of $\partial_x^j(w - v)$ on the right. We further have

$$\begin{aligned}\partial_x^2(\partial_\xi a)(\cdot, \varphi_s) = (\partial_x^2\partial_\xi a)(\cdot, \varphi_s) + 2(\partial_x\partial_\xi^2 a)(\cdot, \varphi_s)\partial_x\varphi_s + (\partial_\xi^2 a)(\cdot, \varphi_s)\partial_x^2\varphi_s \\ + (\partial_\xi^3 a)(\cdot, \varphi_s)[\partial_x\varphi_s, \partial_x\varphi_s].\end{aligned}$$

Using Sobolev's embedding, one can then bound the second term on the right-hand side of (2.18) in $L^\infty_\gamma(J, L^2)$ by $c(r)\|v - w\|_{E_\gamma}$. The other terms are handled more easily. Parts c) and d) are treated similarly. $\square$

As the space for the fixed-point argument we will use

$$\mathcal{E}(R,T) := \big\{v \in \mathcal{G}^{3-}(J) \,\big|\, \|v\|_{\mathcal{G}^{3-}(J)} \le R,\ v(0) = u_0\big\}.$$

for suitable $R > \|u_0\|_{\mathcal{H}^3}$ and $T > 0$. This set is non-empty as it contains the constant function $t \mapsto v(t) = u_0$. In view of Lemma 2.14 b) it is crucial that $\mathcal{E}(R,T)$ is complete for a metric involving only two derivatives, which can be shown by a standard application of the Banach–Alaoglu theorem. For this we recall that $L^\infty_J L^2$ is the dual space of $L^1_J L^2$, see Corollary 1.3.22 in [**26**]. (This is the reason to take L^∞ in time instead of C.)

Lemma 2.15. *The space $\mathcal{E}(R,T)$ is complete with the metric $\|u - v\|_{L^\infty_J \mathcal{H}^2}$.*

Proof. Let (u_n) be Cauchy in $\mathcal{E}(R,T)$ with this metric. Then (u_n) has a limit u in $C(\overline{J}, \mathcal{H}^2)$. Pick $\alpha \in \mathbb{N}_0^4$ with $\alpha_0 \le 1$ and $0 \le |\alpha| \le 3$. Applying Banach–Alaoglu iteratively, we obtain a subsequence (also denoted by (u_n)) such that $\partial^\alpha u_n$ tends to a function v_α weak* in $L^\infty_J L^2$ which also satisfies $\sum_{|\alpha|\le 3} \|v_\alpha\|^2_{L^\infty_J L^2} \le R^2$. It remains to check that $v_\alpha = \partial^\alpha u$. To this end, take $\varphi \in \mathcal{H}^3_0(J \times \mathbb{R}^3)$. We compute

$$\langle \partial^\alpha \varphi, u\rangle = \lim_{n\to\infty} \langle \partial^\alpha \varphi, u_n\rangle = \lim_{n\to\infty} (-1)^{|\alpha|}\langle \varphi, \partial^\alpha u_n\rangle = (-1)^{|\alpha|}\langle \varphi, v_\alpha\rangle$$

in the duality pairing $L^1_J L^2 \times L^\infty_J L^2$. There thus exists $\partial^\alpha u = v_\alpha$. □

In the next lemma we perform the core fixed-point argument.

Lemma 2.16. *Let (2.17) hold and $\rho^2 \ge \|u_0\|^2_{\mathcal{H}^3} + \|f(0)\|^2_{\mathcal{H}^2} + \|f\|^2_{\mathcal{Z}^3(1)}$. Then there is a radius $R = R(\rho) > \rho$ given by (2.19), a time $T_0 = T_0(\rho) \in (0,1]$ given by (2.20), and a unique solution $u \in \mathcal{E}(R, T_0)$ of (2.16).*

Proof. 1) Lemma 2.14 shows that $a_j(u_0)$ and $d(u_0)$ are bounded in $\hat{\mathcal{H}}^2_\infty$ by some $\kappa_0(\rho)$. This yields a constant $c_0 = c_0(\rho) \ge 1$ in (2.13), in the setting of Remark 2.12. We define

$$R = R(\rho) = \sqrt{\mathrm{e}c_0(\rho)\rho^2 + 1} > \rho. \tag{2.19}$$

Take $v, w \in \mathcal{E}(R,T)$ for some $T > 0$. We can use c_0 for v since $v(0) = u_0$. Let $a \in \{a_0, a_1, a_2, a_3, d\}$ and $\gamma \ge 0$. By Lemma 2.14 and $\mathcal{H}^2 \hookrightarrow L^\infty$ there is a constant $\kappa = \kappa(R)$ with

$$\|a(v)\|_{\mathcal{F}^3_\infty(J)} \le \kappa \qquad \text{and} \qquad \|a(v) - a(w)\|_{L^\infty_{\gamma,J}\mathcal{H}^2} \le \kappa \|v - w\|_{L^\infty_{\gamma,J}\mathcal{H}^2}\,.$$

Let $c_1 = c_1(\kappa,\eta)$, $\tilde{c}_1 = \tilde{c}_1(\kappa,\eta)$, and $\gamma_1 = \max\{\gamma_1(\kappa,\eta), \tilde{\gamma}_1(\kappa,\eta)\}$ be given by Theorem 2.9 and Proposition 2.10. We fix

$$\gamma = \gamma(\rho) = \max\big\{\gamma_1, \mathrm{e}c_1\rho^2, 2\mathrm{e}\tilde{c}_1(\overline{c}\kappa R)^2\big\}, \quad T_0 = T_0(\rho) = \min\{1, (2\gamma)^{-1}\}, \tag{2.20}$$

where the constant $\overline{c} > 0$ is introduced below.

2) Theorem 2.9 gives a solution $u \in \mathcal{G}^3(T_0)$ of $L(v)u = f$ and $u(0) = u_0$ with

$$\|u(t)\|^2_{\mathcal{H}^3} + \|\partial_t u(t)\|^2_{\mathcal{H}^2} \le \mathrm{e}^{2\gamma T_0}\big(c_0(\|u_0\|^2_{\mathcal{H}^3} + \|f(0)\|^2_{\mathcal{H}^2}) + c_1\gamma^{-1}\|f\|^2_{\mathcal{Z}^3(1)}\big) \le R^2$$

for $t \in [0, T_0]$. So the map $\Phi\colon v \mapsto u =: \hat{v}$ leaves invariant $\mathcal{E}(R, T_0)$. We note

$$L(v)(\hat{v} - \hat{w}) = (L(w) - L(v))\hat{w} = \sum_{j=0}^{3}(a_j(w) - a_j(v))\partial_j \hat{w} + (d(w) - d(v))\hat{w} =: g$$

and that $\|g(t)\|_{2,2} \le \overline{c}\kappa R\|v(t)-w(t)\|_{2,2}$ by Lemma 2.8 and the above estimate involving κ. Since $\hat{v}(0)=u_0=\hat{w}(0)$ and $T_0\le 1$, Proposition 2.10 then implies

$$\begin{aligned}\|\Phi(v)-\Phi(w)\|^2_{L^\infty_{T_0}\mathcal{H}^2} &\le \mathrm{e}^{2\gamma T_0}\|\Phi(v)-\Phi(w)\|^2_{L^\infty_{\gamma,T_0}\mathcal{H}^2} \qquad (2.21)\\ &\le \mathrm{e}\tilde{c}_1\gamma^{-1}(\overline{c}\kappa R)^2T_0\|v-w\|^2_{L^\infty_{\gamma,T_0}\mathcal{H}^2} \le \tfrac{1}{2}\|v-w\|^2_{L^\infty_{\gamma,T_0}\mathcal{H}^2}.\end{aligned}$$

The assertion now follows from the contraction mapping principle. □

The above result yields uniqueness only in the ball $\mathcal{E}(R,T_0)$, but the contraction estimate (2.21) itself will lead to a much more flexible uniqueness statement. Before showing it, we note that restrictions or translations of a solution $u\in\mathcal{G}^3(\overline{J})$ to (2.16) satisfy (obvious) variants of (2.16). Let $u\in\mathcal{G}^3(\overline{J})$ solve (2.16) and $v\in\mathcal{G}^3(\overline{J^*})$ with $v(T)=u(T)$ solve it on $\overline{J^*}=[T,T']$. Then the concatenation w of u and v belongs to $\mathcal{G}^3([0,T'])$ and fulfills (2.16). (Use (2.3) to check $\partial_t w\in C([0,T'],\mathcal{H}^2)$.)

Lemma 2.17. *Let (2.17) hold, $J_k=(0,T_k)$, $T_k\in(0,\infty]$, and $u^k\in\mathcal{G}^3(\overline{J_k})$ solve (2.16) on $\overline{J_k}$ for $k\in\{1,2\}$. We then have $u^1=u^2$ on $J_1\cap J_2=:J$.*

Proof. Let τ be the supremum of all $t\in[0,\sup J)$ for which $u^1=u^2$ on $[0,t]$. Note that $u^1(0)=u_0=u^2(0)$. We suppose that $\tau<\sup J$. Then $u^1=u^2$ on $[0,\tau]$ by continuity, and there exists a number $\overline{\delta}>0$ with $J_{\overline{\delta}}:=[\tau,\tau+\overline{\delta}]\subseteq J$. Let $\overline{R}$ be the maximum of the norms of u^1 and u^2 in $\mathcal{G}^3(J_{\overline{\delta}})$. Fix γ as in (2.20) (with $\overline{\kappa}=\kappa(\overline{R})$ and $\rho=0$) and take $\delta\in(0,\overline{\delta}]$. As in (2.21), Proposition 2.10 and a time shift yield a constant $\overline{c}_1=\tilde{c}_1(\overline{R})>0$ with

$$\|u^1-u^2\|^2_{L^\infty_\gamma(J_\delta,\mathcal{H}^2)}\le \mathrm{e}\overline{c}_1\gamma^{-1}(\overline{c\kappa}\overline{R})^2\delta\|u^1-u^2\|_{L^\infty_\gamma(J_\delta,\mathcal{H}^2)}.$$

Choosing a sufficently small $\delta>0$, we infer $u^1=u^2$ on $J_\delta=[\tau,\tau+\delta]$. This fact contradicts the definition of τ, so that $\tau=\sup J$ as asserted. □

We now use the above results to define a *maximal solution* u to (2.16) assuming (2.17). The *maximal existence time* is given by

$$T_+=T_+(u_0,f):=\sup\big\{T\ge 0\,\big|\,\exists\, u_T\in\mathcal{G}^3(T)\text{ solving (2.16) on }[0,T]\big\}\in(0,\infty].$$

Lemma 2.16 shows $T_+(u_0,f)>T_0(\rho)$ as we can restart the problem at time $T=T_0(\rho)$ with the initial value $u_T(T)$. Moreover, by Lemma 2.17 the solutions u_S and u_T coincide on $[0,S]$ for $0<S<T<T_+$. Setting $u(t)=u_T(t)$ for such times, we thus define a unique solution u of (2.16) in $\mathcal{G}^3([0,T_+))$.

In the proof of the blow-up criterion below, we need the following Moser-type estimates, which are still true if one replaces $\mathbb{R}^m$ by a Lipschitz domain in $\mathbb{R}^m$.

Lemma 2.18. *Let $k\in\mathbb{N}$ and $\alpha,\beta\in\mathbb{N}_0^m$.*

a) For $v,w\in L^\infty(\mathbb{R}^m)\cap\mathcal{H}^k(\mathbb{R}^m)$ and $|\alpha|+|\beta|=k$, we have

$$\|\partial^\alpha v\partial^\beta w\|_2\le c(\|v\|_\infty\|w\|_{k,2}+\|v\|_{k,2}\|w\|_\infty).$$

b) For $v,w\in W^{1,\infty}(\mathbb{R}^m)$ with $\partial^\alpha v,\partial^\beta w\in L^2(\mathbb{R}^m)$ for $1\le|\alpha|\le k$ and $|\alpha|+|\beta|=k+1$, we have

$$\|\partial^\alpha v\partial^\beta w\|_2\le c\|\nabla v\|_\infty\sum_{j=1}^m\|\partial_j w\|_{k-1,2}+c\|\nabla w\|_\infty\sum_{j=1}^m\|\partial_j v\|_{k-1,2}.$$

PROOF. We first recall the Gagliardo–Nirenberg inequality

$$\|\partial^\alpha\varphi\|_{2k/|\alpha|} \le c\|\varphi\|_\infty^{1-\frac{|\alpha|}{k}} \sum_{|\gamma|=k} \|\partial^\gamma\varphi\|_2^{\frac{|\alpha|}{k}}$$

where $|\alpha| \le k$ and $\varphi \in L^\infty(\mathbb{R}^m)$ with $\partial^\gamma\varphi \in L^2(\mathbb{R}^m)$ for all $|\gamma| = k$, see Lecture II in [41].

Assertion a) is clear if $|\alpha|$ is 0 or k. So let $k \ge 2$ and $1 \le |\alpha| \le k-1$. Note that $\frac{|\beta|}{k} = 1 - \frac{|\alpha|}{k}$. The inequalities of Hölder (with $\frac{1}{2} = \frac{|\alpha|}{2k} + \frac{|\beta|}{2k}$), Gagliardo–Nirenberg and Young yield

$$\begin{aligned}\|\partial^\alpha v\partial^\beta w\|_2 &\le \|\partial^\alpha v\|_{2k/|\alpha|}\|\partial^\beta w\|_{2k/|\beta|} \le c\|v\|_\infty^{1-\frac{|\alpha|}{k}}\|v\|_{k,2}^{\frac{|\alpha|}{k}}\|w\|_\infty^{1-\frac{|\beta|}{k}}\|w\|_{k,2}^{\frac{|\beta|}{k}} \\ &= (\|v\|_\infty\|w\|_{k,2})^{1-\frac{|\alpha|}{k}}(\|w\|_\infty\|v\|_{k,2})^{\frac{|\alpha|}{k}} \lesssim \|v\|_\infty\|w\|_{k,2} + \|v\|_{k,2}\|w\|_\infty.\end{aligned}$$

In part b) we can assume that $k \ge 3$ and $2 \le |\alpha| \le k-1$. There are $i,j \in \{1,\dots,m\}$ with $\alpha = \alpha' + e_i$ and $\beta = \beta' + e_j$, where $|\alpha'| + |\beta'| = k-1$. From a) we deduce

$$\|\partial^\alpha v\partial^\beta w\|_2 = \|\partial^{\alpha'}\partial_i v\,\partial^{\beta'}\partial_j w\|_2 \lesssim \|\partial_i v\|_\infty\|\partial_j w\|_{k-1,2} + \|\partial_i v\|_{k-1,2}\|\partial_j w\|_\infty$$

and thus statement b). □

We state the core local wellposedness result for $\mathcal{H}^3$-solutions of (2.16). It provides an improved blow-up condition in $W^{1,\infty}$ (and not only in $\mathcal{H}^3 \hookrightarrow W^{1,\infty}$). In quasilinear hyperbolic problems one can only expect continuity of the solution map, not even uniform continuity, see [25] or [32]. Let $\mathcal{B}_T((u_0,f),r)$ be the closed ball in $\mathcal{H}^3 \times \mathcal{Z}^3(T)$ with center (u_0,f) and radius $r > 0$.

THEOREM 2.19. *Let (2.17) hold and* $\rho^2 \ge \|u_0\|_{\mathcal{H}^3}^2 + \|f(0)\|_{\mathcal{H}^2}^2 + \|f\|_{\mathcal{Z}^3(1)}^2$. *Then the following assertions are true.*

a) There is a unique solution $u = \Psi(u_0,f)$ *of (2.16) on* $[0,T_+)$, *where* $T_+ = T_+(u_0,f) \in (T_0(\rho),\infty]$ *with* $T_0(\rho) > 0$ *from (2.20) and* $u \in \mathcal{G}^3(T)$ *for all* $T \in (0,T_+)$.

b) Let $T_+ < \infty$. *Then* $\lim_{t\to T_+}\|u(t)\|_{\mathcal{H}^3} = \infty$ *and* $\overline{\lim}_{t\to T_+}\|u(t)\|_{W^{1,\infty}} = \infty$.

c) Take $T \in [0,T_+)$. *Then there is a radius* $\delta > 0$ *such that for all* $(v_0,g) \in \mathcal{B}_T((u_0,f),\delta)$ *we have* $T_+(v_0,g) > T$ *and* $\Psi\colon \mathcal{B}_T((u_0,f),\delta) \to \mathcal{G}^3(T)$ *is continuous. Moreover,* $\Psi\colon (\mathcal{B}_T((u_0,f),\delta), \|\cdot\|_{\mathcal{H}^2\times\mathcal{Z}^2(T)}) \to \mathcal{G}^2(T)$ *is Lipschitz.*

PROOF. a)/b) Above we have shown part a). Let $T_+ < \infty$ and $u = \Psi(u_0,f)$.

1) Suppose there are $t_n \to T_+$ with $r := \sup_n \|u(t_n)\|_{3,2} < \infty$. Set $T = T_+ + 1$ and $\overline{\rho}^2 = r^2 + \|f\|_{\mathcal{Z}^3(T)}^2 + \sup_n \|f(t_n)\|_{2,2}^2 < \infty$. Let $\tau = T_0(\overline{\rho}) > 0$ be given by (2.20). Fix an index N such that $t_N + \tau > T_+$. Lemma 2.16 and a time shift yield a solution $v \in \mathcal{G}^3([t_N, t_N+\tau])$ of (2.16) with $v(t_N) = u(t_N)$. We thus obtain a solution on $[0, t_N+\tau]$. This fact contradicts the definition of T_+, and hence $\|u(t)\|_{3,2} \to \infty$ as $t \to T_+$.

2) Whereas the arguments for step 1) are fairly standard, the following steps are more sophisticated. Set $\omega = \sup_{0\le t<T_+}\|u(t)\|_{1,\infty}$ and suppose that $\omega < \infty$.

Let $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \le 3$. Using (2.3), we compute

$$\begin{aligned} L(u)\partial_x^\alpha u = \partial_x^\alpha f - \sum_{0<\beta\le\alpha} \binom{\alpha}{\beta} \Big[\sum_{j=1}^3 \partial_x^\beta a_j(u) \partial_x^{\alpha-\beta} \partial_j u + \partial_x^\beta d(u) \partial_x^{\alpha-\beta} u \\ + \partial_x^\beta a_0(u) \partial_x^{\alpha-\beta} \Big(a_0(u)^{-1} \Big(f - \sum_{j=1}^3 a_j(u)\partial_j u - d(u)u \Big)\Big)\Big] \\ =: f_\alpha = \partial_x^\alpha f - g_\alpha. \end{aligned} \tag{2.22}$$

In view of (the proofs of) Lemmas 2.8 and 2.14, the summands of f_α in the second line can be treated as the others (using Young's inequality for products of norms of f and u). Employing also Lemma 2.18 and $\mathcal{H}^3 \hookrightarrow W^{1,\infty}$, we estimate

$$\begin{aligned} \|f_\alpha(t)\|_2 &\le c(\omega)\Big[\|f(t)\|_{\mathcal{H}^3} + \|u(t)\|_{\mathcal{H}^3} + \sum_{k=1}^4 \sum_{|\gamma_i|\le 3, \sum_i |\gamma_i| \le 4} \|\partial_x^{\gamma_1} u(t) \cdots \partial_x^{\gamma_k} u(t)\|_2 \Big] \\ &\le c(\omega)\big(\|f(t)\|_{\mathcal{H}^3} + (1+\omega^3)\|u(t)\|_{\mathcal{H}^3}\big). \end{aligned}$$

Take $\gamma \ge \gamma_0(\omega)$ in Proposition 2.3. For $t \in [0, T_+)$, this proposition and the above inequality yield

$$\|\partial_x^\alpha u\|_{L^2_{\gamma,t}L^2}^2 + \tfrac{2\mathrm{e}^{-2\gamma t}}{\gamma} \|\partial_x^\alpha u(t)\|_{L^2}^2 \le \tfrac{c(\omega)}{\eta\gamma} \|u_0\|_{\mathcal{H}^3}^2 + \tfrac{c(\omega)}{\eta^2\gamma^2} \big[\|f\|_{L^2_{\gamma,t}\mathcal{H}^3}^2 + \|u\|_{L^2_{\gamma,t}\mathcal{H}^3}^2 \big].$$

We sum over $|\alpha| \le 3$ and fix a large γ to absorb the last summand. Hence, $\|u(t)\|_{3,2}$ is bounded for $t < T_+$ contradicting step 1); i.e., part b) is shown.

c) The proof of assertion c) is quite demanding. We first fix some constants, and then show continuity of Ψ at (u_0, f) on an interval $[0, b]$ assuming that we have solutions with uniform bounds on $[0, b]$. Using this fact and Lemma 2.16, we then prove inductively that solutions on $[0, T]$ exist and satisfy such bounds if we start in a certain ball around (u_0, f). Finally, we replace (u_0, f) by different data in this ball to obtain the asserted continuity statements.

1) Fix $T \in (0, T_+)$, write $J = (0, T)$, and let $c_S \ge 1$ be the norm of the embedding $\mathcal{Z}^3(T) \hookrightarrow C([0,T], \mathcal{H}^2)$. Choose $\tilde\rho^2 \ge \|u_0\|_{3,2}^2 + \|f\|_{\mathcal{Z}^3(T)}^2 + \|f\|_{L^\infty_J \mathcal{H}^2}^2$, $\delta_0 := \tilde\rho$, and $\tilde r \ge \max\{c_S\tilde\rho, \|u\|_{\mathcal{G}^3(T)}\}$. Below we take $R \ge \tilde r$, $b \le T$, and $v \in \mathcal{G}^3(b)$ with $\|v\|_{\mathcal{G}^3(b)} \le R$. Lemma 2.14 yields a constant $\overline\kappa = \overline\kappa(R)$ dominating the norms of $a_j(v)$ and $d(v)$ in $\mathcal{F}^3_\infty(b)$ and of $a_j(v)(0)$ and $d(v)(0)$ in $\hat{\mathcal{H}}^2_\infty$.

2) Assume there are $b \in (0, T]$, $v_0 \in \mathcal{H}^3$ and $g \in \mathcal{Z}^3(T)$ such that $T_+(v_0, g) > b$. We write $v = \Psi(v_0, g) \in \mathcal{G}^3(b)$. Let $R \ge \|v\|_{\mathcal{G}^3(b)}$ with $R \ge \tilde r$. Observe that

$$L(u)(v-u) = g - f + (L(u) - L(v))v = g - f + \sum_{j=0}^3 (a_j(u) - a_j(v))\partial_j v + (d(u) - d(v))v.$$

By Lemma 2.14, the function $(L(u) - L(v))v$ belongs to $\mathcal{G}^2_\gamma(b)$ with norm less than $c(\overline\kappa)R\|v - u\|_{\mathcal{G}^2_\gamma(b)}$ for $\gamma \ge 0$. Proposition 2.10 then yields

$$\|v - u\|_{\mathcal{G}^2_\gamma(b)}^2 \le \tilde c(\overline\kappa, \eta)\big(\|u_0 - v_0\|_{\mathcal{H}^2}^2 + \|f - g\|_{\mathcal{Z}^2_\gamma(b)}^2 + \gamma^{-1} R b \|v - u\|_{\mathcal{G}^2_\gamma(b)}^2 \big)$$

for $\gamma \geq \tilde{\gamma}_1(\overline{\kappa}, \eta) \geq 1$. Fixing a sufficiently large $\gamma = \overline{\gamma}_1(\overline{\kappa}, R, T, \eta) \geq \tilde{\gamma}_1(\overline{\kappa}, \eta)$, we can absorb the term by the right-hand side and deduce

$$\|v - u\|^2_{\mathcal{G}^2(b)} \leq \tilde{c}(\overline{\kappa}, R, T, \eta)\big(\|u_0 - v_0\|^2_{\mathcal{H}^2} + \|f - g\|^2_{\mathcal{Z}^2(b)}\big). \tag{2.23}$$

3) Estimate (2.23) leads to Lipschitz continuity of Ψ in $\mathcal{G}^2(T)$. The hard and core part of the proof is to check continuity of Ψ in $\mathcal{G}^3(T)$ at (u_0, f), assuming apriori bounds. So let $(u_{0,n}, f_n) \in \mathcal{B}_T((u_0, f), \tilde{\delta})$ tend to (u_0, f) on $\mathcal{H}^3 \times \mathcal{Z}^3(T)$ as $n \to \infty$, where $\tilde{\delta} > 0$. Hence, $f_n(0) \to f(0)$ in $\mathcal{H}^2$ and $f_n \to f$ in $\mathcal{Z}^3(T)$. Assume that $T_+(u_{0,n}, f_n) > b$ with $b \in (0, T]$ and that $u_n = \Psi(u_{0,n}, f_n)$ is bounded by some $R \geq \tilde{r}$ in $\mathcal{G}^3(b)$ for all $n \in \mathbb{N}$. Then u_n tends to u in $\mathcal{G}^2(b) \hookrightarrow \mathcal{L}^\infty_b$ as $n \to \infty$ by (2.23), and the coeffcients $a_j(u_n)$ and $d(u_n)$ satisfy the estimates of step 1) with a uniform $\overline{\kappa} = \overline{\kappa}(R)$.

The main idea is to split the n-convergence of the coefficients and the data. Let $\alpha \in \mathbb{N}_0^3$ with $|\alpha| = 3$. As in (2.22) we write $L(u_n)\partial_x^\alpha u_n = \partial_x^\alpha f_n - g_{n,\alpha}$ and $L(u)\partial_x^\alpha u = \partial_x^\alpha f - g_\alpha$. Theorem 2.4 yields solutions $w_n, z_n \in C([0, b], L^2)$ of

$$\begin{aligned} L(u_n)w_n &= \partial_x^\alpha f - g_\alpha, \quad w_n(0) = \partial_x^\alpha u_0,\\ L(u_n)z_n &= \partial_x^\alpha f_n - \partial_x^\alpha f + g_\alpha - g_{n,\alpha}, \quad z_n(0) = \partial_x^\alpha u_{0,n} - \partial_x^\alpha u_0. \end{aligned}$$

By uniqueness, we have $w_n + z_n = \partial_x^\alpha u_n$ and hence

$$\partial_x^\alpha u_n - \partial_x^\alpha u = w_n - \partial_x^\alpha u + z_n.$$

Since $a_j(u_n) \to a_j(u)$ and $d(u_n) \to d(u)$ in $\mathcal{L}^\infty_J$ as $n \to \infty$, Lemma 2.13 shows that $q_n := \|w_n - \partial_x^\alpha u\|_{L^\infty_J L^2}$ tends to 0. We thus have to prove $z_n \to 0$ in $L^\infty_J L^2$.

Choose $\gamma = \overline{\gamma}_1(R, T)$ as in step 2). For $t \in [0, b]$, Proposition 2.3 then implies

$$\begin{aligned} &\|\partial_x^\alpha(u_n(t) - u(t))\|^2_{L^2} \leq 2q_n^2 + 2\|z_n(t)\|^2_{L^2}\\ &\quad \leq 2q_n^2 + c(R, T')\big(\|\partial_x^\alpha(u_{0,n} - u_0)\|^2_{L^2} + \|\partial_x^\alpha(f_n - f)\|^2_{L^2_J L^2} + \|g_{n,\alpha} - g_\alpha\|^2_{L^2_J L^2}\big). \end{aligned}$$

To estimate $\|g_{n,\alpha} - g_\alpha\|_2$, let $a \in \{a_j, a_0^{-1}, d\}$, $v \in \{u, u_n\}$, and $w \in \{u, u_n, f\}$.

(i) First, we look at summands of the type

$$\partial_x^\beta a(v(t))\partial_x^{\gamma^2} a(v^2(t))\partial_x^{\gamma^3} a(v^3(t))\partial_x^{\gamma^1}(u_n(t) - u(t))$$

where the terms with the multiindices γ^2 or γ^3 may disappear, $|\beta| + |\gamma^1| + |\gamma^2| + |\gamma^3| \leq 4$, and $|\beta|, |\gamma^i| \leq 3$. By Lemma 2.8 and the bounds on the coefficients these terms are controlled in L^2 by $c(R)\|u_n(t) - u(t)\|_{3,2}$. Here the sup-norms of first-order factors are less than $c(R)$, and second-order factors are handled by Hölder and Sobolev. Summands with $f_n(t) - f(t)$ are treated analogously.

(ii) We next analyze the remaining terms, which look like

$$W = \partial_x^\beta[a(u_n(t)) - a(u(t))]\partial_x^{\gamma^2} a(v^2(t))\partial_x^{\gamma^3} a(v^3(t))\partial_x^{\gamma^1} w(t)$$

for multiindices as above. At first, we consider situations where we can estimate the first factor by $u - u_n$ in $L^\infty_J \mathcal{H}^2$ using Lemma 2.14. This works for $\beta = 0$ in L^∞ and all admitted γ^i, for $|\beta| = 1$ in L^6 if $|\gamma^i| \leq 2$ for some i and $|\gamma^j| \leq 1$ otherwise; and for $|\beta| = 2$ in L^2 if $|\gamma^i| \leq 1$ for all i. In this situation one obtains an estimate as in case (i).

This does not work if (and only if) $|\beta| = 3$ and $|\gamma^i| \leq 1$ for all i, or $|\beta| = 2$ and $|\gamma^1| = 2$ (then $w \in \{u, u_n\}$), or $|\beta| = 1$ and $|\gamma^1| = 3$. The factors with

∂_x^β are bounded in L^2, L^6 and L^∞, respectively, so that we have to estimate $a(u_n(t)) - a(u(t))$ in $\mathcal{H}^3$ by Sobolev. Similarly, the terms $\partial_x^{\gamma^1} w(t)$ are controlled by $\|w(t)\|_{3,2}$, and the other factors are bounded by $c(R)$ in L^∞. For the highest-order contributions we compute $\partial_x^\beta(a(u_n) - a(u))$ with $|\beta| = 3$ using the chain rule. For these terms we define

$$h_n(t) = \sum_a \sum_{k=1}^{3} \sum_{l_i=1}^{9} \|(\partial_{l_k} \cdots \partial_{l_1} a)(u_n(t)) - (\partial_{l_k} \cdots \partial_{l_1} a)(u(t))\|_{L^\infty}.$$

The L^2-norm of such W is then bounded by linear combinations of $c(R)$ times

$$h_n(t)\|\partial_x^{\gamma_1} v(t) \cdots \partial_x^{\gamma_{m-1}} v(t) \partial_x^{\gamma_m} w(t)\|_{L^2} + \|\partial_x^{\gamma_1} v(t) \cdots \partial_x^{\gamma_{m-1}} \varphi_n(t) \partial_x^{\gamma_m} w(t)\|_{L^2},$$

where $\varphi_n = u_n - u$, $m \in \{1,2,3,4\}$, $|\gamma_i| \le 3$, and $|\gamma_1| + \cdots + |\gamma_m| \le 4$. This sum can be estimated by $c(R)(h_n(t) + \|u_n(t) - u(t)\|_{3,2})$ due to Lemma 2.8 and the bounds on u and u_n. We have shown that

$$\|g_{n,\alpha} - g_\alpha\|^2_{L^2_t L^2} \le c(R,T)\Big(\|f_n - f\|^2_{L^2_J \mathcal{H}^2} + \|u_n - u\|^2_{L^\infty_J \mathcal{H}^2} + \int_0^T h_n(s)^2 \, \mathrm{d}s \\ + \int_0^t \sum_{|\gamma|=3} \|\partial_x^\gamma (u_n(s) - u(s))\|^2_{L^2} \, \mathrm{d}s\Big).$$

We write the last integrand as $\|\partial_x^3(u_n(s) - u(s))\|_2^2$. Note that $h_n(s)$ tends to 0 as $n \to \infty$ since $u_n \to u$ in $\mathcal{L}^\infty_J$ and that it is bounded uniformly in s and n. By dominated convergence $\int_0^T h_n^2 \, \mathrm{d}s$ tends to 0. Summing up, we conclude that

$$\|\partial_x^3(u_n(t) - u(t))\|^2_{L^2} \le c(R,T)\varepsilon_n + c(R,T) \int_0^t \|\partial_x^3(u_n(s) - u(s))\|^2_{L^2} \, \mathrm{d}s$$

for a null sequence (ε_n). By Gronwall, $\partial_x^3(u_n - u)$ tends to 0 in $C([0,b], L^2)$ as $n \to \infty$, and so $u_n \to u$ in $C([0,b], \mathcal{H}^3)$. Using (2.3) and Lemma 2.14, we infer $u_n \to u$ in $\mathcal{G}^3(b)$.

4) We now look for data to which we can apply steps 2) and 3). Let $(v_0, g) \in \mathcal{B}_T((u_0, f), \delta_0)$. We then obtain

$$\begin{aligned} \|v_0\|_{\mathcal{H}^3} &\le \|v_0 - u_0\|_{\mathcal{H}^3} + \|u_0\|_{\mathcal{H}^3} \le \delta_0 + \tilde{\rho} = 2\tilde{\rho} \le 2\tilde{r}, \\ \|g\|_{\mathcal{Z}^3(T)} &\le \|g - f\|_{\mathcal{Z}^3(T)} + \|f\|_{\mathcal{Z}^3(T)} \le 2\tilde{\rho} \le 2\tilde{r}, \\ \|g\|_{L^\infty_J \mathcal{H}^2} &\le c_S \|g\|_{\mathcal{Z}^3(T)} \le 2c_S\tilde{\rho} \le 2\tilde{r}. \end{aligned}$$

Lemma 2.16 thus yields a time $\tau = \tau(\tilde{r})$ and a solution $v \in \mathcal{G}^3(\tau)$ of (2.16) with data v_0 and g, where $\|v\|_{\mathcal{G}^3(\tau)} \le \tilde{R} = \tilde{R}(\tilde{r})$ and $\tilde{R} > 2\tilde{r}$. By part a), we have $v = \Psi(v_0, g)$ and $T_+(v_0, g) > \tau$. Fix $N \in \mathbb{N}$ with $(N-1)\tau < T \le N\tau$, set $t_k = k\tau$ for $k \in \{0, 1, \ldots, N-1\}$ and $t_N = T$.

Steps 2) and 3) show that (2.23) is true on $[0, \tau]$ for such v with a constant $\tilde{c} = \tilde{c}(\tilde{r})$ and that $\Psi \colon \mathcal{B}_T((u_0, f), \delta_0) \to \mathcal{G}^3(\tau)$ is continuous at (u_0, f). We can thus find a radius $\delta_1 \in (0, \delta_0]$ such that $\|v - u\|_{\mathcal{G}^3(\tau)} \le \tilde{r}$, and hence $\|v\|_{\mathcal{G}^3(\tau)} \le 2\tilde{r}$, for all $(v_0, g) \in \mathcal{B}_T((u_0, f), \delta_1)$.

5) We iterate the above argument. Assume that for some $k \in \{1, \ldots, N-1\}$ and $\delta_k \in (0, \delta_0]$, we have $T_+(v_0, g) > t_k$ and $\|v - u\|_{\mathcal{G}^3(t_k)} \le \tilde{r}$ for all $(v_0, g) \in \mathcal{B}_T((u_0, f), \delta_k)$ and the map $\Psi \colon \mathcal{B}_T((u_0, f), \delta_k) \to \mathcal{G}^3(t_k)$ is continuous at (u_0, f).

It follows $\|v\|_{\mathcal{G}^3(t_k)} \le 2\tilde{r}$. Since $\|v(t_k)\|_{3,2} \le 2\tilde{r}$, step 4) and a time shift provide a solution $\tilde{v} \in \mathcal{G}^3([t_k, t_{k+1}])$ of (2.16) with $\tilde{v}(t_k) = v(t_k)$ and norm less or equal $\tilde{R}$. We can thus extend v to a solution in $\mathcal{G}^3([0, t_{k+1}])$ bounded by $\tilde{R}$ and so $T_+(v_0, g) > t_{k+1}$. Because of this bound, steps 2) and 3) imply (2.23) on $[0, t_{k+1}]$ with $\tilde{c} = \tilde{c}(\tilde{r})$ for all $(v_0, g) \in \mathcal{B}_T((u_0, f), \delta_k)$ and the continuity of $\Psi\colon \mathcal{B}_T((u_0, f), \delta_k) \to \mathcal{G}^3(t_{k+1})$ at (u_0, f). Using the latter property, we find a radius $\delta_{k+1} \in (0, \delta_k]$ such that $\|v - u\|_{\mathcal{G}^3(t_{k+1})} \le \tilde{r}$ for $v = \Psi(v_0, g)$ and all $(v_0, g) \in \mathcal{B}_T((u_0, f), \delta_{k+1})$, and hence $\|v\|_{\mathcal{G}^3(t_{k+1})} \le 2\tilde{r}$.

Induction yields a radius $\delta = \delta_N$ such that for all $(v_0, g) \in \mathcal{B}_T((u_0, f), \delta)$ we have $T_+(v_0, g) > T$, the continuity of $\Psi\colon \mathcal{B}_T((u_0, f), \delta) \to \mathcal{G}^3(T)$ at (u_0, f), and $\|\Psi(v_0, g)\|_{\mathcal{G}^3(T)} \le 2\tilde{r}$. Moreover, (2.23) holds on $[0, T]$ for u and $v = \Psi(v_0, g)$.

6) Finally, we take data $(v_0, g), (w_0, h) \in \mathcal{B}_T((u_0, f), \delta)$ with solutions v and w. Replacing u by w and $\tilde{r}$ by $2\tilde{r}$ in step 2), we then obtain the last assertion in c). Also step 3) can be repeated on $[0, T]$ for data converging to (w_0, h) in $\mathcal{B}_T((u_0, f), \delta)$, since the corresponding solutions are bounded by $4\tilde{r}$ in $\mathcal{G}^3(T)$. □

Theorem 2.6 yields finite speed of progation for a solution $u \in \mathcal{G}^3(T)$ of (2.16), setting $A_j = a_j(u)$ and $D = d(u)$. We comment on variants of Theorem 2.19.

Remark 2.20. One can easily extend Theorem 2.19 to negative times (e.g., by time reversion). Moreover, in (2.17) one can replace the domain $\mathbb{R}^3 \times \mathbb{R}^6$ of a_j and d by $\mathbb{R}^3 \times \mathcal{O}$ for an open $\mathcal{O} \subseteq \mathbb{R}^6$, restricting ξ in the supremum not to each closed ball $\overline{B}(0, r) \subseteq \mathbb{R}^6$ but to each compact subset of $\mathcal{O}$. One further has to require that the closure K_0 of $u_0(\mathbb{R}^3)$ is contained in $\mathcal{O}$, and the solution u has to take values in $\mathcal{O}$. Theorem 2.19 is then valid with one modification. In part b) now $T_+ < \infty$ implies that $\limsup_{t<T_+} \|u(t)\|_{1,\infty} = \infty$ or that $u(t)$ leaves any compact subset of $\mathcal{O}$ as $t \to T_+$.

Indeed, the proofs are very similar in this more general case. In the fixed-point argument one chooses a bounded open set V with $K_0 \subseteq V \subseteq \overline{V} \subseteq \mathcal{O}$. Let $d > 0$ be the distance between V and ∂U. In $\mathcal{E}(R, T)$ one then also includes the condition that $\|v(t) - u_0\|_\infty \le d/2$ for all $t \in [0, T]$ which is preserved by limits in $L^\infty_J \mathcal{H}^2$. Other steps in the reasoning are modified accordingly. Compare Theorem 3.3 of **[57]**. ♢

As explained in Chapter 1, one can easily apply Theorem 2.19 to the Maxwell system (1.1) with material laws (1.9) and (1.11). We state the needed assumptions in a situation motivated by nonlinear optics.

Example 2.21. Let $\theta(x, E, H) = (\varepsilon_{\mathrm{lin}}(x)E + \varepsilon_{\mathrm{nl}}(x, E)E, \mu_{\mathrm{lin}}(x)H)$ be given as in Example 1.2 and (1.9) with $U = \mathbb{R}^3$. Assuming also $\varepsilon_{\mathrm{lin}}, \mu_{\mathrm{lin}} \in C_b^3(\mathbb{R}^3, \mathbb{R}^{3\times 3}_{\ge 2\eta})$ and $\kappa_i^{jkl} \in C_b^3(\mathbb{R}^3, \mathbb{R})$. Moreover, take $J_e = \sigma(x, E)E + J_0$ in (1.11) with $\sigma \in C^3(\mathbb{R}^6, \mathbb{R}^{3\times 3})$ satisfying $\sup_{|\xi| \le r} \|\partial_x^\alpha \sigma(\cdot, \xi)\|_{L^\infty} < \infty$ for all $r \ge 0$ and $0 \le |\alpha| \le 3$, respectively. Recall that for a suitable $\delta \in (0, \infty]$ and all $x, E, H \in \mathbb{R}^3$ with $|E| < \delta$ we obtain $\partial_{(E,H)}\theta(x, E, H) \ge \eta I$. Rewriting the system as in (1.15), we see that hypothesis (2.17) (modified as in Remark 2.20 if $\delta < \infty$) is fulfilled. For initial fields in $\mathcal{H}^3$ with $|E_0| < \delta/2$ and a current density $J_0 \in \mathcal{Z}^3(T)$ for all $T > 0$, Theorem 2.19 and Remark 2.20 thus provide wellposedness in $\mathcal{H}^3$ of the Maxwell system (1.1) with the above material laws. ♢

2.4. Energy and blowup

In the preceeding sections we have worked with the linear energy estimate which contains error terms caused by the time derivative of coefficients. (The space derivatives in C of (2.5) disappear in the Maxwell case.) These error terms have led to the inconvenient $\mathcal{H}^3$-setting. The time dependence arises since we freeze a function in the nonlinearities of (2.16). One may wonder whether this is really necessary and whether it is not better to solve (2.16) based on a nonlinear energy identity. Actually, this can be done in the semilinear case where $D = \varepsilon(x)E$, $B = \mu(x)H$, and $J_e = \sigma(x, E)E$ under appropriate conditions on σ, cf. [**19**]. Below we see that this does not seem to work in the quasilinear case.

In this section we first establish an energy equality for $J_e = 0$ and isotropic nonlinearities

$$D = \varepsilon_{\text{lin}} E + \beta_e(\cdot, |E|^2)E, \qquad B = \mu_{\text{lin}} H + \beta_m(\cdot, |H|^2)H. \tag{2.24}$$

Here ε_{lin} and μ_{lin} belong to $L^\infty(\mathbb{R}^3, \mathbb{R}^{3\times3}_{\geq\eta})$ for some $\eta > 0$ and the maps $\beta_e, \beta_m \colon \mathbb{R}^3 \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ are C^1, bounded in $x \in \mathbb{R}^3$ and non-decreasing in $s \in \mathbb{R}_{\geq 0}$. We set $u = (E, H)$ and

$$a_{\text{lin}} = \begin{pmatrix} \varepsilon_{\text{lin}} & 0 \\ 0 & \mu_{\text{lin}} \end{pmatrix}, \qquad \beta(|u|^2) = \begin{pmatrix} \beta_e(\cdot, |E|^2)I_{3\times3} & 0 \\ 0 & \beta_m(\cdot, |H|^2)I_{3\times3} \end{pmatrix},$$

$$M = \begin{pmatrix} 0 & \operatorname{curl} \\ -\operatorname{curl} & 0 \end{pmatrix} = -\sum_{j=1}^{3} A_j^{\text{co}} \partial_j, \qquad \mathcal{D}(M) = \mathcal{H}(\operatorname{curl}) \times \mathcal{H}(\operatorname{curl}),$$

where $\mathcal{H}(\operatorname{curl}) = \{v \in L^2(\mathbb{R}^3, \mathbb{R}^3) \mid \operatorname{curl} v \in L^2(\mathbb{R}^3, \mathbb{R}^3)\}$. The operator M is skew-adjoint in $L^2(\mathbb{R}^3, \mathbb{R}^6)$. Maxwell equations (1.1) then become

$$\partial_t[a_{\text{lin}} u(t) + \beta(|u(t)|^2)u(t)] = Mu(t), \quad t \geq 0, \qquad u(0) = u_0 = (E_0, H_0). \tag{2.25}$$

Omitting the argument x in the notation, we further define

$$b_j(s) = \int_0^s \beta_j(r)\,\mathrm{d}r, \qquad h_j(s) = s\beta_j(s) - \tfrac{1}{2} b_j(s).$$

We have $h_j(s) \geq \frac{s}{2}\beta_j(s)$ since β_j does not decrease and that $h_j'(s) = \frac{1}{2}\beta_j(s) + s\beta_j'(s)$, where $\beta_j' = \partial_2 \beta_j$. We now introduce the 'energy' for $u = (u_1, u_2)$ by

$$\mathcal{E}(u) = \int_{\mathbb{R}^3} \left[\tfrac{1}{2} a_{\text{lin}} u \cdot u + h_1(|u_1|^2) + h_2(|u_2|^2)\right] \mathrm{d}x$$

Note that $\mathcal{E}(u) \geq \frac{\eta}{2}\|u\|_2^2$ if $\beta_j \geq 0$. In the Kerr case $\varepsilon_{\text{lin}} = \mu_{\text{lin}} = 1$, $\beta_e(x, s) = \kappa(x)s$ and $\beta_m = 0$, we obtain

$$\mathcal{E}_{\mathrm{K}}(E, H) = \int_{\mathbb{R}^3} \left[\tfrac{1}{2}|E|^2 + \tfrac{3}{4}\kappa|E|^4 + \tfrac{1}{2}|H|^2\right] \mathrm{d}x.$$

Let $u \in \mathcal{G}^1(T)$ solve (2.25). The energy equality $\mathcal{E}(u(t)) = \mathcal{E}(u_0)$ for $t \in [0, T]$ follows from

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(u) &= \int_{\mathbb{R}^3} \left[u \cdot \partial_t(a_{\text{lin}} u) + \beta(|u|^2)u \cdot \partial_t u + 2|u|^2 \beta'(|u|^2)u \cdot \partial_t u\right] \mathrm{d}x \\ &= \int_{\mathbb{R}^3} \partial_t\left[a_{\text{lin}} u + \beta(|u|^2)u\right] \cdot u\,\mathrm{d}x = \int_{\mathbb{R}^3} Mu \cdot u\,\mathrm{d}x = 0. \end{aligned}$$

If $\beta_j \ge 0$ we can thus bound squares of 2-norms of solutions (and $\|E\|_4^4$ in the Kerr case if $\inf \kappa > 0$). This is not enough control to pass to a weak limit in the nonlinearity when performing an approximation argument (which would typically produce a global solution). One would need an estimate involving derivatives. Such estimates are not known, and the next result on blowup indicates that they do not hold.

We first stress that it is well known that the gradient of a solution to (2.25) may blow up in sup-norm in finite time, see [**38**]. However in the semilinear case one relies on estimates in $\mathcal{H}(\mathrm{curl})$, so we are interested in blowup in this space (or at least in $\mathcal{H}^1$). Below we give such an example on a domain with periodic boundary conditions, taken from [**17**]. Such conditions arise if one truncates a fullspace problem with periodic coefficients to a periodicity cell. (See [**17**] for a weaker result on $\mathbb{R}^3$.) We work in the following setting with $D = (1+\alpha(|E|))E$ and $B = H$. We set $a(s) = (1+\alpha(|s|))s$ for $s \in \mathbb{R}$ and assume

$$\begin{aligned} &a \in C^2(\mathbb{R},\mathbb{R}), \quad \exists\, s_- < 0 < s_0 < s_+ \colon\ a' > 0 \ \text{ on } \ S := (s_-, s_+), \\ &q\colon S \to \mathbb{R};\ q(s) = \frac{a''(s)}{2a'(s)^{3/2}}, \ \text{ has a global maximum at } \ s = s_0, \\ &q \text{ is } C^1 \text{ near } s_0, \quad q(s) > 0 \ \text{ for } \ 0 < s \le s_0. \end{aligned} \tag{2.26}$$

Let $\gamma > 2$ and $\alpha_0 > 0$. A simple example for (2.26) is furnished by any C^2-extension of $a\colon [0,s_+] \to \mathbb{R}$; $a(s) = s + \alpha_0 s^\gamma$, which is strictly growing on (s_-, s_+) for some $s_- < 0 < s_0 < s_+$ with

$$s_0 = \Big(\frac{2(\gamma-2)}{\alpha_0\gamma(\gamma+1)}\Big)^{\frac{1}{\gamma-1}}$$

in this case. We stress that the behavior of a for large s is arbitrary here.

Theorem 2.22. *Assume that* (2.26) *is true. Then there are numbers $M, T > 0$ and a map $(E,B) \in C^1([0,T) \times [-M,M]^3)$ which solves* (1.1) *on $(-M,M)^3$ with* $\operatorname{div} D = 0 = \operatorname{div} B$, *periodic boundary conditions and the above material laws, and which satisfies*

$$\|\mathrm{curl}\, E(t)\|_{L^2} \to \infty \qquad \text{as} \ \ t \to T^-.$$

We look for a solution of the form

$$(E(t,x), B(t,x)) = (u(t,x_2), 0, 0, 0, 0, v(t,x_2)).$$

for $x \in (-M,M)^3$ and $t \in [0,T)$. Observe that such E, B and $D = (1 + \alpha(|E|))E$ are divergence-free. If u and v have support in $[0,T) \times (-M,M)$, then E and B fulfill periodic boundary conditions. Moreover, (E,B) belong to $C^1([0,T) \times [-M,M]^3)$ satisfy (1.1) on $(-M,M)^3$ with the above material laws if and only if $(u,v) \in C^1([0,T) \times (-M,M))$ solve

$$\partial_t a(u) = \partial_x v, \qquad \partial_t v = \partial_x u, \qquad (u(0), v(0)) = (u_0, v_0),$$

for $t \in [0,T)$ and $x \in \mathbb{R}$. This system can be rewritten as

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} + A(u,v)\partial_x \begin{pmatrix} u \\ v \end{pmatrix} = 0 \qquad \text{with} \ \ A(u,v) = \begin{pmatrix} 0 & -a'(u)^{-1} \\ -1 & 0 \end{pmatrix} \tag{2.27}$$

on $\mathbb{R}$. Here we assume that u takes values in S from (2.26). Since also $\partial_x u = \operatorname{curl} E$, the theorem thus follows from the next one-dimensional result.

The following proof uses a standard construction from Section 1.4 of [**38**]. However, it requires a rather detailed analysis to find a class of initial values for which we get the blowup of $\partial_x u$ in L^2 instead of L^∞.

Proposition 2.23. *Assume that* (2.26) *is true. Then there exist initial data* $(u_0, v_0) \in C_c^1(\mathbb{R}, \mathbb{R}^2)$ *and a* C^1*-solution* (u, v) *to* (2.27) *on* $[0, T) \times \mathbb{R}$ *for some* $T \in (0, \infty)$ *which is compactly supported and which satisfies* $\|\partial_x u(t, \cdot)\|_{L^2(\mathbb{R})} \to \infty$ *as* $t \to T^-$.

Proof. 1) We first contruct the desired function (u, v). For $(s, z) \in S \times \mathbb{R}$, the matrix $A(s, z)$ has the eigenvalues and eigenvectors

$$\lambda_{1,2}(s, z) = \pm a'(s)^{-\frac{1}{2}}, \qquad w_{1,2}(s, z) = (\mp 1, a'(s)^{\frac{1}{2}}).$$

(Recall $S = (s_-, s_+)$, s_0 and q from (2.26).) In the following we take $\lambda = \lambda_1$ and $w = w_1$ and drop the index 1. Fix $(\xi, \zeta) \in (s_0, s_+) \times \mathbb{R}$ such that

$$q(s) > 0 \qquad \text{for} \quad 0 < s \le \xi.$$

Observe that the interval $\xi - S = (\xi - s_+, \xi - s_-)$ contains $[0, \xi]$. The C^2-function

$$\phi\colon \xi - S \to S \times \mathbb{R}; \quad \phi_1(s) = \xi - s, \quad \phi_2(s) = \zeta + \int_0^s a'(\xi - \tau)^{1/2}\,\mathrm{d}\tau,$$

solves the ordinary differential equation

$$\phi'(s) = w(\phi(s)), \quad s \in \xi - S, \qquad \phi(0) = (\xi, \zeta).$$

For later use, we note the identities

$$\nabla\lambda(\phi(s)) \cdot \phi'(s) = \nabla\lambda(\phi(s)) \cdot w(\phi(s)) = q(\xi - s), \qquad s \in \xi - S. \tag{2.28}$$

Let $\sigma_0\colon \mathbb{R} \to [0, \xi]$ be C^2 and equal to ξ outside a compact set. There is a unique C^1-solution σ of the scalar partial differential equation

$$\begin{aligned} \partial_t \sigma(t, x) + \lambda(\phi(\sigma(t, x)))\partial_x \sigma(t, x) = 0, \qquad & t \ge 0,\ x \in \mathbb{R}, \\ \sigma(0, x) = \sigma_0(x), \qquad & x \in \mathbb{R}, \end{aligned} \tag{2.29}$$

on a bounded time interval $[0, \bar{t})$, where σ takes values in $\xi - S$. See e.g. Theorems 2.1 and 2.2 Annex of [**38**] (a variant of Theorem 2.19). We now define

$$\begin{pmatrix} u(t, x) \\ v(t, x) \end{pmatrix} = \phi(\sigma(t, x)).$$

It is easy to check that (u, v) is a C^1-solution of (2.27) on $[0, \bar{t}) \times \mathbb{R}$. We observe

$$\partial_x u = \phi_1'(\sigma)\partial_x \sigma = -\partial_x \sigma. \tag{2.30}$$

2) By uniqueness, the solution of (2.29) fulfills the implicit formula

$$\begin{aligned} \sigma(t, x) &= \sigma_0\big(x - t\lambda(\phi(\sigma(t, x)))\big) = \sigma_0(y(t, x)), \\ y(t, x) &:= x - t\lambda\big(\phi(\sigma(t, x))\big) = x - ta'(\xi - \sigma(t, x))^{-1/2}. \end{aligned} \tag{2.31}$$

(Note that $\sigma_0(y)$ satisfies (2.29) as this is true for σ.) Hence, σ is bounded. We will need the inquality

$$1 + t\nabla\lambda\big(\phi(\sigma(t, x))\big) \cdot w\big(\phi(\sigma(t, x))\big)\sigma_0'\big(x - t\lambda(\phi(\sigma(t, x)))\big)$$

$$= 1 + t\sigma_0'\big(x - t\lambda(\phi(\sigma(t,x)))\big)q(\xi - \sigma(t,x)) \quad > 0, \tag{2.32}$$

where we use (2.28). We now set

$$\gamma(t) := \inf_{x\in\mathbb{R}} \sigma_0'(y(t,x))q(\xi - \sigma(t,x)) \quad \text{for} \quad t \in [0,\bar{t}).$$

Let $t_0 \geq 0$ be the supremum of $t \in [0,\bar{t})$ such that $\tau\gamma(\tau) > -1$ for all $\tau \in [0,t]$. In the following, we take $t \in [0,t_0)$ so that the inequality (2.32) is valid for all $x \in \mathbb{R}$. Equations (2.31) then imply

$$\partial_x\sigma(t,x) = \sigma_0'\big(x - t\lambda(\phi(\sigma(t,x)))\big)\Big(1 - tq(\xi - \sigma(t,x))\partial_x\sigma(t,x)\Big),$$

$$\partial_x\sigma(t,x) = \frac{\sigma_0'(y(t,x))}{1 + tq(\xi - \sigma(t,x))\sigma_0'(y(t,x))}.$$

In particular, $\partial_x\sigma$ is bounded on $[0,t_0-\delta]\times\mathbb{R}$ for each $\delta \in (0,t_0]$. We show below that the maps $\partial_x\sigma(t)$ tend to ∞ in L^2 and thus in L^∞ as $t \to t_0$. The blow-up condition in Theorem 2.2 Annex of [**38**] thus yields $\bar{t} = t_0$. From formula (2.31) we further deduce $\partial_x\sigma(t,x) = \sigma_0'(y(t,x))\partial_x y(t,x)$ and therefore

$$\partial_x y(t,x) = \frac{1}{1 + tq(\xi - \sigma(t,x))\sigma_0'(y(t,x))} > 0. \tag{2.33}$$

(In the case $\sigma_0'(y(t,x)) = 0$ the identity $\partial_x y(t,x) = 1 > 0$ follows from (2.31).) Using also (2.31), we see that the map $x \mapsto y(t,x)$ is a bijection from $\mathbb{R}$ to $\mathbb{R}$. This fact and (2.31) lead to the equation

$$\gamma(t) = \inf_{z\in\mathbb{R}} \sigma_0'(z)q(\xi - \sigma_0(z)) =: \gamma_0.$$

3) We now fix a C^1-function $\sigma_0\colon \mathbb{R} \to [0,\xi]$ which is equal to ξ outside some compact set and satisfies

$$\sigma_0(0) = \xi - s_0, \qquad \sigma_0'(0) = \min_{z\in\mathbb{R}} \sigma_0'(z) < 0.$$

In view of (2.26), we can determine

$$\gamma_0 = \sigma_0'(0)q(s_0) \quad \text{and} \quad t_0 = -\tfrac{1}{\gamma_0}. \tag{2.34}$$

Substituting $z = y(t,x)$ and using (2.33), we infer from (2.31) the identities

$$\|\partial_x\sigma(t,\cdot)\|_2^2 = \int_{\mathbb{R}} |\partial_x\sigma(t,x)|^2\,\mathrm{d}x = \int_{\mathbb{R}} |\sigma_0'(y(t,x))\partial_x y(t,x)|^2\,\mathrm{d}x$$
$$= \int_{\mathbb{R}} \frac{|\sigma_0'(z)|^2}{1 + tq(\xi - \sigma_0(z))\,\sigma_0'(z)}\,\mathrm{d}z.$$

Since q has a global maximum at s_0 while σ_0' has a global minimum at 0, we obtain the expansions

$$q(s) = q(s_0) - o_+(s - s_0), \qquad \sigma_0'(z) = \sigma_0'(0) + o_+(z), \qquad \sigma_0(z) = \xi - s_0 + O(z),$$

where $o_+(z)$ denotes any nonnegative function with the property $o_+(z)/z \to 0$ as $z \to 0$. Hence, (2.34) yields

$$1 + tq(\xi - \sigma_0(z))\,\sigma_0'(z) = 1 + t\gamma_0 + t\big[q(s_0)o_+(z) + o_+(z)\,|\sigma_0'(0)| - o_+(z)^2\big]$$
$$= 1 + t\gamma_0 + to_+(z)$$

for small $|z|$. Fix a number $\delta_0 > 0$ such that the above identity is true and $|\sigma_0'(z)|^2 \geq \frac{1}{2}|\sigma_0'(0)|^2 =: c_0$ if $|z| \leq \delta_0$. For each $\epsilon > 0$ there exists a radius $\delta \in (0, \delta_0]$ with $0 \leq o_+(z) \leq \epsilon\delta$ for $z \in [-\delta, \delta]$. We can then estimate

$$\|\partial_x \sigma(t, \cdot)\|_2^2 \geq \int_{-\delta}^{\delta} \frac{|\sigma_0'(z)|^2}{1 + t\gamma_0 + to_+(z)} \,\mathrm{d}z \geq \int_{-\delta}^{\delta} \frac{c_0}{1 + t\gamma_0 + t\epsilon\delta} \,\mathrm{d}z = \frac{2c_0\delta}{1 + t\gamma_0 + t\epsilon\delta}.$$

Because of $t_0 = -1/\gamma_0 =: T$ in (2.34), it follows

$$\liminf_{t \to T^-} \|\partial_x \sigma(t, \cdot)\|_2^2 \geq \frac{2c_0}{T\epsilon}.$$

Since $\epsilon > 0$ is arbitrary, equation (2.30) finally implies that

$$\liminf_{t \to T^-} \|\partial_x u(t, \cdot)\|_2^2 = \liminf_{t \to T^-} \|\partial_x \sigma(t, \cdot)\|_2^2 = +\infty.$$

4) Note that $\sigma(t, x) = \sigma_0(y(t, x)) = \xi$ if $|y|$ is large enough. This fact holds for some $x_0 > 0$ and all $t \in [0, T)$ and $|x| \geq x_0$ because of (2.31) and the strict positivity of a' on $[0, \xi]$. So $u = \xi - \sigma$ has compact support. Fixing

$$\zeta = -\int_0^{\xi} a'(\xi - \tau)^{1/2} \,\mathrm{d}\tau,$$

also the function

$$v = \zeta + \int_0^{\sigma} a'(\xi - \tau)^{1/2} \,\mathrm{d}\tau$$

has compact support. $\square$

CHAPTER 3

Background for Strichartz estimates

In this chapter we collect several results from functional and harmonic analysis needed to establish Strichartz estimates for the Maxwell system. In particular, we treat the Fourier transform of tempered distributions, Fourier multipliers, fractional derivatives and Sobolev spaces, and the Littlewood–Paley decomposition. The latter will lead to more flexible and general product and commutator estimates which are crucial for the analysis of partial differential equations. In the last section we discuss Strichartz estimates for wave equations which serve as background for our investigations of the Maxwell system. Much of this material is covered by other lectures. In these cases we partly indicate a derivation to explain main ideas in the area, but often we just refer to the literature or lecture notes for the proofs. From now we use $\mathbb{C}$ as scalar field.

3.1. Fourier transform and multipliers

It is convenient to extend the Fourier transform to the rather large space of 'tempered distributions.' To this aim, we first recall the *Schwartz space*

$$\mathcal{S} = \mathcal{S}_m = \big\{ v \in C^\infty(\mathbb{R}^m) \,\big|\, \forall\, k \in \mathbb{N}_0,\, \alpha \in \mathbb{N}_0^m \colon p_{k,\alpha}(v) := \| |x|^k \partial^\alpha v\|_\infty < \infty \big\}.$$

($|x|$ stands for the map $x \mapsto |x|$ etc.) A sequence (v_n) *converges* to v in $\mathcal{S}$ if $p_{k,\alpha}(v_n - v) \to 0$ as $n \to \infty$ for all $k \in \mathbb{N}_0$ and $\alpha \in \mathbb{N}_0^m$. This limit concept can be expressed by a complete metric. Recall that $\mathcal{S}$ is dense in all Sobolev spaces $W^{k,p}$ with $p < \infty$. (See Sections 3.1 and 3.6 in [**47**] and Section 5.1 of [**46**] for proofs omitted here and more information.) For further definitions, let $a > 0$, $x, y, \xi \in \mathbb{R}^m$, $e_{\mathrm{i}y}(x) = \mathrm{e}^{\mathrm{i}y\cdot x}$, and $v, w \in \mathcal{S}$. We define translations, dilations, reflection, Fourier transform, and convolution on $\mathcal{S}$ by

$$\tau_y v(x) = v(x+y), \quad \sigma_a v(x) = v(ax), \quad Rv(x) = v(-x),$$
$$\mathcal{F}v(\xi) = \hat{v}(\xi) = (2\pi)^{-\frac{m}{2}} \int_{\mathbb{R}^m} \mathrm{e}^{-\mathrm{i}\xi\cdot x} v(x)\,\mathrm{d}x, \quad (v * w)(x) = \int_{\mathbb{R}^m} v(x-y)w(y)\,\mathrm{d}y.$$

Extensions of these operators are denoted by the same symbols. We do not list properties of these objects on the level of $\mathcal{S}$, but state them immediately in greater generality. We only recall that $\mathcal{F}$ is a homeomorphism of $\mathcal{S}$ and that it can be extended to a unitary operator on $L^2 = L^2(\mathbb{R}^m)$ by Plancherel's theorem.

The space of *tempered distributions* $\mathcal{S}^\star = \mathcal{S}_m^\star$ consists of the continuous linear maps $\varphi \colon \mathcal{S} \to \mathbb{C}$. The 'weak' convergence $\varphi_n \to \varphi$ in $\mathcal{S}^\star$ means that $\varphi_n(v) = \langle v, \varphi_n\rangle_{\mathcal{S}} \to \varphi(v)$ for all $v \in \mathcal{S}$, as $n \to \infty$. Measurable functions f and Borel measures μ with at most polynomial growth as $|x| \to \infty$ belong to $\mathcal{S}^\star$. More precisely, assume that

$$\int_{A_k} |f| \,\mathrm{d}x \le ck^{\alpha}, \qquad \text{respectively,} \qquad \mu(A_k) \le ck^{\alpha} \tag{3.1}$$

for $A_k = \overline{B}(0, k+1) \setminus B(0,k)$, $k \in \mathbb{N}$ and some $c, \alpha \ge 0$. These objects induce elements φ_j of $\mathcal{S}^\star$ via

$$\varphi_f(v) = \langle v, f\rangle_{\mathcal{S}} := \int_{\mathbb{R}^m} vf \,\mathrm{d}x, \quad \text{respectively,} \quad \varphi_\mu(v) = \langle v, \mu\rangle_{\mathcal{S}} := \int_{\mathbb{R}^m} v \,\mathrm{d}\mu$$

for $v \in \mathcal{S}$. We also need the vector space and algebra

$$\mathcal{E} = \mathcal{E}_m = \big\{ f \in C^\infty(\mathbb{R}^m) \,\big|\, \forall\, \alpha \in \mathbb{N}_0^m \;\; \exists\, n_\alpha \in \mathbb{N}_0 \colon \; \sup\nolimits_{|x|\ge 1} |x|^{-n_\alpha} |\partial^\alpha f(x)| < \infty \big\}$$

of polynomially growing smooth functions. Note that both $f \in \mathcal{E}$ and $f \in L^p$ for $p \in [1,\infty]$ satisfy (3.1). Let $\varphi \in \mathcal{S}^\star$, $g \in \mathcal{E}$, and $\alpha \in \mathbb{N}_0^m$. For $v \in \mathcal{S}$ we define

a) $(M_g\varphi)(v) = (g\varphi)(v) = \langle v, g\varphi\rangle_{\mathcal{S}} := \langle gv, \varphi\rangle_{\mathcal{S}} = \varphi(gv)$,

b) $(\partial^\alpha \varphi)(v) = \langle v, \partial^\alpha\varphi\rangle_{\mathcal{S}} := (-1)^{|\alpha|}\langle \partial^\alpha v, \varphi\rangle_{\mathcal{S}} = (-1)^{|\alpha|}\varphi(\partial^\alpha v)$,

c) $\widehat{\varphi}(v) = (\mathcal{F}\varphi)(v) = \langle v, \mathcal{F}\varphi\rangle_{\mathcal{S}} := \langle \mathcal{F}v, \varphi\rangle_{\mathcal{S}} = \varphi(\mathcal{F}v)$,

d) $(R\varphi)(v) = \langle v, R\varphi\rangle_{\mathcal{S}} := \langle Rv, \varphi\rangle_{\mathcal{S}} = \varphi(Rv)$,

e) $(v * \varphi)(x) := \langle \tau_{-x}Rv, \varphi\rangle_{\mathcal{S}} = \varphi(\tau_{-x}Rv)$ for every $x \in \mathbb{R}^m$.

One can check that these maps belong to $\mathcal{S}^\star$. Observe that we multiply and convolve tempered distributions only with the (very regular) functions in $\mathcal{E}$ and $\mathcal{S}$, respectively. In view of the following examples and the proposition below, the above definitions extend the concepts on $\mathcal{S}$ in a natural way and allow to generalize several main properties of the Fourier transform to the space $\mathcal{S}^\star$.

Example 3.1. Let $v \in \mathcal{S}$, $g \in \mathcal{E}$, $\alpha \in \mathbb{N}_0^m$, and $x, y \in \mathbb{R}^m$.

a) Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^m)$ be as in (3.1). Then M_g acts as $g\varphi_f = \varphi_{gf}$ because of

$$(g\varphi_f)(v) = \int_{\mathbb{R}^m} vgf \,\mathrm{d}x = \varphi_{gf}(v).$$

b) Let $f \in W^{k,p}(\mathbb{R}^m)$ for some $p \in [1,\infty]$ and $|\alpha| \le k \in \mathbb{N}$. We then obtain $\partial^\alpha\varphi_f = \varphi_{\partial^\alpha f}$ since the definitions and the divergence theorem yield

$$\langle v, \partial^\alpha \varphi_f\rangle_{\mathcal{S}} = (-1)^{|\alpha|}\langle \partial^\alpha v, \varphi_f\rangle_{\mathcal{S}} = (-1)^{|\alpha|}\int_{\mathbb{R}^m} \partial^\alpha v f \,\mathrm{d}x = \int_{\mathbb{R}^m} v\partial^\alpha f \,\mathrm{d}x = \langle v, \varphi_{\partial^\alpha f}\rangle_{\mathcal{S}}.$$

c) Let $f \in L^2(\mathbb{R}^m)$. Then $\mathcal{F}\varphi_f = \varphi_{\mathcal{F}f}$ as Plancherel implies

$$\langle v, \mathcal{F}\varphi_f\rangle_{\mathcal{S}} = \langle \mathcal{F}v, \varphi_f\rangle_{\mathcal{S}} = \int_{\mathbb{R}^m} \widehat{v} f \,\mathrm{d}x = \int_{\mathbb{R}^m} v\widehat{f} \,\mathrm{d}x = \langle v, \varphi_{\mathcal{F}f}\rangle_{\mathcal{S}}\,.$$

d) The derivative of the point evaluation $\delta_y \colon v \mapsto v(y)$ is given by

$$\langle v, \partial^\alpha \delta_y\rangle_{\mathcal{S}} = (-1)^{|\alpha|}\langle \partial^\alpha v, \delta_y\rangle_{\mathcal{S}} = (-1)^{|\alpha|}\partial^\alpha v(y) =: (-1)^{|\alpha|}\delta_y^\alpha(v).$$

e) We have $\mathcal{F}\delta_y = (2\pi)^{-\frac{m}{2}} e_{-\mathrm{i}y}$ because of

$$\langle v, \mathcal{F}\delta_y\rangle_{\mathcal{S}} = \langle \mathcal{F}v, \delta_y\rangle_{\mathcal{S}} = (2\pi)^{-\frac{m}{2}}\int_{\mathbb{R}^m} \mathrm{e}^{-\mathrm{i}y\cdot x} v(x)\,\mathrm{d}x = \big\langle v, (2\pi)^{-\frac{m}{2}} e_{-\mathrm{i}y}\big\rangle_{\mathcal{S}}.$$

f) Conversely, $\mathcal{F}e_{\mathrm{i}y} = (2\pi)^{\frac{m}{2}}\delta_y$ follows from the inversion formula via

$$\langle v, \mathcal{F}e_{\mathrm{i}y}\rangle_{\mathcal{S}} = \langle \mathcal{F}v, e_{\mathrm{i}y}\rangle_{\mathcal{S}} = \int_{\mathbb{R}^m} \widehat{v}(\xi)\mathrm{e}^{\mathrm{i}y\cdot\xi}\,\mathrm{d}\xi = (2\pi)^{\frac{m}{2}}(\mathcal{F}^{-1}\widehat{v})(y) = (2\pi)^{\frac{m}{2}}v(y).$$

Assertion f) can also be deduced from e) since $\mathcal{F}^2$ is equal to R in $\mathcal{S}^\star$, too, as shown in the next proposition (with a similar proof as above).

g) Let $f \in L^1(\mathbb{R}^m)$. The convolutions then satisfy $v * \varphi_f = v * f$ because of

$$v * \varphi_f(x) = \langle \tau_{-x}Rv, \varphi_f\rangle_{\mathcal{S}} = \int_{\mathbb{R}^m} v(-(z-x))f(z)\,\mathrm{d}z = v * f(x). \qquad \Diamond$$

We now collect the main properties of the above objects on $\mathcal{S}^\star$. Observe that the second part of assertion c) does not work on $W^{k,2}$.

Proposition 3.2. *Let $\varphi \in \mathcal{S}^\star$, $u, v \in \mathcal{S}$, $a > 0$, and $\alpha \in \mathbb{N}_0^m$. The following assertions hold.*

a) $\mathcal{F}\colon \mathcal{S}^\star \to \mathcal{S}^\star$ is a homeomorphism with $\mathcal{F}^4 = I$ and $\mathcal{F}^{-1} = \mathcal{F}^3 = R\mathcal{F}$.

b) $\mathcal{F}(\sigma_a\varphi) = a^{-m}\sigma_{1/a}\hat{\varphi}$.

c) $\mathcal{F}(\partial^\alpha\varphi) = \mathrm{i}^{|\alpha|}\xi^\alpha\mathcal{F}\varphi$ and $\partial^\alpha(\mathcal{F}\varphi) = (-\mathrm{i})^{|\alpha|}\mathcal{F}(x^\alpha\varphi)$.

d) $v * \varphi \in \mathcal{E}$, and hence $v * \varphi$ induces a tempered distribution.

e) $\partial^\alpha(v * \varphi) = (\partial^\alpha v) * \varphi = v * \partial^\alpha\varphi$.

f) $\mathcal{F}(v * \varphi) = (2\pi)^{\frac{m}{2}}\widehat{v}\widehat{\varphi}$ and $\mathcal{F}(v\varphi) = (2\pi)^{-\frac{m}{2}}\widehat{v} * \widehat{\varphi}$.

g) $(u * v) * \varphi = u * (v * \varphi)$.

Fourier multipliers $a(D)$ with symbol a are an important tool in analysis and we often use them below. Let $a\colon \mathbb{R}^m \to \mathbb{C}$ be measurable and polynomially bounded. Then $a(D)v := \mathcal{F}^{-1}(a\hat{v})$ defines a linear map from L^2 into $\mathcal{S}^\star$. If a belongs to $\mathcal{E}$, then we can extend and restrict $a(D)$ to an operator from $\mathcal{S}^\star$ to $\mathcal{S}^\star$ and from $\mathcal{S}$ to $\mathcal{S}$. For two multipliers $a, b \in \mathcal{E}$, we obtain the algebra property

$$a(D)b(D)\varphi = \mathcal{F}^{-1}(ab\hat{\varphi}) = (ab)(D)\varphi = b(D)a(D)\varphi \tag{3.2}$$

for all $\varphi \in \mathcal{S}^\star$. These identities typically hold also for less regular symbols in adapted settings. For instance, by Plancherel $a(D)\colon L^2 \to L^2$ is bounded if and only if a is bounded, and then $\|a(D)\|_{\mathcal{B}(L^2)} = \|a\|_\infty$.

The L^p-boundedness of $a(D)$ is far more difficult for $p \neq 2$. Here the basic result is *Mikhlin's theorem.* Let $k = \lfloor \frac{n}{2} \rfloor + 1$, $a \in C^k(\mathbb{R}^m \setminus \{0\})$, $p \in (1, \infty)$, and

$$\|a\|_M := \sup\nolimits_{0\le|\alpha|\le k, \xi\neq 0} |\xi|^{|\alpha|}|\partial^\alpha a(\xi)| < \infty.$$

Then $a(D)\colon L^p \to L^p$ has norm less than $c_{m,p}\|a\|_M$ by Theorem 6.2.7 in [23], where also related theorems are discussed. The Mikhlin condition is satisfied for 0-homogeneous a in C^k. Indeed, the chain rule yields

$$\partial_j a(\xi) = \partial_j a(|\xi|^{-1}\xi) = (\partial_j a)(|\xi|^{-1}\xi)|\xi|^{-1} - (\nabla a)(|\xi|^{-1}\xi)\cdot\xi\,\xi_j|\xi|^{-3},$$

and thus the claim for $|\alpha| = 1$. This can be iterated. A typical example is $a(\xi) = \xi^\beta|\xi|^{-k}$ for $|\beta| = k$. Mikhlin also applies to $a(\xi) = \xi^\beta\langle\xi\rangle^{-k}$ with $\langle\xi\rangle = \sqrt{1 + |\xi|^2}$ because of $\partial_j\langle\xi\rangle^s = s\xi_j\langle\xi\rangle^{s-2}$ for $s \in \mathbb{R}$. In this context we write $\langle D\rangle^s = \mathcal{F}^{-1}\langle\xi\rangle^s\mathcal{F}$, noting that $\langle\xi\rangle^s \in \mathcal{E}$.

This fractional power of $I-\Delta$ plays a crucial role in the definition and treatment of the *Bessel-potential spaces* $\mathcal{H}^{s,p}$ given by

$$\mathcal{H}^{s,p} = \mathcal{H}^{s,p}(\mathbb{R}^m) = \{\varphi \in \mathcal{S}^\star \mid \langle D\rangle^s \varphi \in L^p(\mathbb{R}^m)\}$$

with norm $\|\varphi\|_{s,p} = \|\langle D\rangle^s\varphi\|_p$ for $s \in \mathbb{R}$ and $p \in [1,\infty]$. We often write $\mathcal{H}^{s,2} = \mathcal{H}^s$. Note that $\mathcal{H}^{0,p} = L^p$ and $\mathcal{S} \subseteq \mathcal{H}^{s,p}$. In the definition of $\mathcal{H}^s$ one can replace $\langle D\rangle^s$ by $\langle\xi\rangle^s\mathcal{F}$ due to Plancherel. Because of (3.2) the map $\langle D\rangle^{-s}\colon \mathcal{H}^{t,p} \to \mathcal{H}^{t+s,p}$ is an isometric isomorphism for $t \in \mathbb{R}$. Taking $t=0$, we see that $\mathcal{H}^{s,p}$ is a Banach space, which is reflexive for $p \in (1,\infty)$ and separable for $p \in [1,\infty)$, and $\mathcal{S}$ is dense in $\mathcal{H}^{s,p}$ in the latter case.

The dual of $\mathcal{H}^{s,p}$ coincides with $\mathcal{H}^{-s,p'}$ if $p \in [1,\infty)$. Indeed, the isomorphism $\langle D\rangle^s\colon \mathcal{H}^{s,p} \to L^p$ has the adjoint $\langle D\rangle^s\colon L^{p'} \to (\mathcal{H}^{s,p})^\star$ since

$$\langle\langle D\rangle^s u, v\rangle_{L^p} = \langle \mathcal{F}^{-1}\langle\xi\rangle^s\mathcal{F}u, v\rangle_{\mathcal{S}} = \langle u, \mathcal{F}\langle\xi\rangle^s\mathcal{F}^{-1}v\rangle_{\mathcal{S}} = \langle u, \mathcal{F}^{-1}\langle\xi\rangle^s\mathcal{F}v\rangle_{\mathcal{H}^{s,p}}$$

for $u \in \mathcal{S}$ and $v \in L^{p'}$. Here we identify $(L^p)^\star$ and $L^{p'}$ in the usual way and use $\mathcal{F}^{-1} = R\mathcal{F} = \mathcal{F}R$ in the last identity. Because of the density of $\mathcal{S}$ in $\mathcal{H}^{s,p}$, the claim about $\langle D\rangle^s$ is shown. Since also $\langle D\rangle^s\colon L^{p'} \to \mathcal{H}^{-s,p'}$ is an isomorphism, we can identify $(\mathcal{H}^{s,p})^\star$ and $\mathcal{H}^{-s,p'}$ by extending the usual L^p-$L^{p'}$ duality.

To use Mikhlin, we restrict to $p \in (1,\infty)$ in the treatment of $\mathcal{H}^{s,p}$. Then $\langle D\rangle^{-s}$ is bounded on L^p for $s \geq 0$ and thus $\mathcal{H}^{s,p} \hookrightarrow L^p$ in this case. Applying $\langle D\rangle^{t-s}$, the embedding $\mathcal{H}^{t,p} \hookrightarrow \mathcal{H}^{\tau,p}$ follows for $t \geq \tau$. Actually this remains true for $p \in [1,\infty]$ due to Corollary 1.2.6 in [**24**] and the isomorphisms.

By means of the L^p-boundedness of the Fourier multiplier for $\mathrm{i}^{|\alpha|}\xi^\alpha\langle\xi\rangle^{-k}$ with $|\alpha| \leq k$ and Proposition 3.2, we deduce the embedding $\mathcal{H}^{k,p} \hookrightarrow W^{k,p}$ for $k \in \mathbb{N}$ and $p \in (1,\infty)$. For the converse inclusion, note that $\langle\xi\rangle^{2k}$ can be written as a sum $\sum_{|\alpha|\leq k} c_\alpha \xi^{2\alpha}$ and hence Proposition 3.2 yields

$$\langle D\rangle^k v = \mathcal{F}^{-1}\sum_{|\alpha|\leq k} c_\alpha\xi^\alpha\langle\xi\rangle^{-k}\xi^\alpha\hat{v} = \sum_{|\alpha|\leq k}\mathrm{i}^{-|\alpha|}c_\alpha\mathcal{F}^{-1}\big(\xi^\alpha\langle\xi\rangle^{-k}\mathcal{F}(\partial^\alpha v)\big). \tag{3.3}$$

Since the last Fourier multiplier is bounded on L^p, we obtain the equality $\mathcal{H}^{k,p} = W^{k,p}$ with equivalent norms.

We thus have the Sobolev embedding $\mathcal{H}^{k,p} \hookrightarrow L^q$ for $q \in [p,\infty)$ if $k - \frac{m}{p} \geq -\frac{m}{q}$ and for $q = \infty$ if $k - \frac{m}{p} > 0$, still assuming $p \in (1,\infty)$. We use interpolation theory to extend the embeding to noninteger s, and also for other purposes. The Bessel-potential spaces behave well with respect to the *complex interpolation method*. We do not define it, but state its main property.

Let X_j and Y_j be Banach spaces which are subspaces of vector spaces Z_j with continuous inclusion, where Z_j has a metric for which its addition and scalar multiplication are continuous, where $j \in \{1,2\}$. Then, for $\theta \in (0,1)$ there exist Banach spaces $[X_j, Y_j]_\theta \hookrightarrow X_j + Y_j$ with the interpolation property: Let $T_X \in \mathcal{B}(X_1, X_2)$ and $T_Y \in \mathcal{B}(Y_1, Y_2)$ satisfy $T_X v = T_Y v =: Tv$ for $v \in X_1 \cap Y_1$. This map can then be extended to a bounded linear operator $T\colon X_1 + Y_1 \to X_2 + Y_2$. It has a restriction $T\colon [X_1, Y_2]_\theta \to [X_2, Y_2]_\theta$ with norm bounded by $\|T_X\|^{1-\theta}\|T_Y\|^\theta$. This theory is discussed in Chapter 2 in [**37**], for instance.

One can further show that

$$[\mathcal{H}^{s,p}(\mathbb{R}^m), \mathcal{H}^{t,q}(\mathbb{R}^m)]_\theta = \mathcal{H}^{\tau,r}(\mathbb{R}^m) \quad \text{with } \tau = (1-\theta)s + \theta t,\ \tfrac{1}{r} = \tfrac{1-\theta}{p} + \tfrac{\theta}{q}, \tag{3.4}$$

for $s, t \in \mathbb{R}$, $p, q \in (1, \infty)$ and $\theta \in (0,1)$. See Theorems 2.4.7 and 2.5.6 in [**68**]. In the case $s = 0$ one just interpolates L^p-spaces, where the result is true for $p, q \in [1, \infty]$ by the Riesz–Thorin theorem, see [**37**].

Let $k \in \mathbb{N}$, $k - \frac{m}{p} \geq -\frac{m}{q}$ and $p \leq q < \infty$, or $k - \frac{m}{p} > 0$ and $q = \infty$. We interpolate the embeddings $L^p \hookrightarrow L^p$ and $\mathcal{H}^{k,p} \hookrightarrow L^q$ with $\theta = \frac{\tau}{k} \in (0,1)$ and obtain $\mathcal{H}^{\tau,p} \hookrightarrow L^r$ with $\frac{1}{r} = \frac{1}{p} - \frac{\tau}{kp} + \frac{\tau}{kq}$. This yields the conditions $\tau - \frac{m}{p} \geq -\frac{m}{r}$ and $p \leq r < \infty$. The above isomorphisms then imply the Sobolev embedding

$$\mathcal{H}^{t,p}(\mathbb{R}^m) \hookrightarrow \mathcal{H}^{s,q}(\mathbb{R}^m) \qquad \text{if} \quad 1 < p \leq q < \infty, \quad t - \tfrac{m}{p} \geq s - \tfrac{m}{q}. \tag{3.5}$$

The same is true with $q = \infty$ if we have $t - \frac{m}{p} > s$.

The 'inhomogenous fractional derivative' $\langle D \rangle^s$ does not fit well to the scaling $x \mapsto \lambda x$, in contrast to the 'homogeneous' symbol $|\xi|^s$. However, this function is singular at 0 if $s < 0$, i.e., when we would expect a smoothing behavior. There are several ways to deal with this problem, where we take the most frequently used alternative. (See e.g. [**5**] for two other approaches.) For test functions, instead of $\mathcal{S}$ we use the space

$$\mathcal{S}_0 = \big\{ v \in \mathcal{S} \,\big|\, \forall\, \alpha \in \mathbb{N}_0^m \colon \ \partial^\alpha \hat{v}(0) = 0 \big\}$$

of Schwartz maps whose Fourier transform vanish at 0 together with their derivatives. It is a closed subspace of $\mathcal{S}$. For $v \in \mathcal{S}_0$ Taylor's theorem yields $|\partial^\alpha \hat{v}(\xi)| \leq c(N, \alpha, v)|\xi|^N$ for all $N \in \mathbb{N}$, $\alpha \in \mathbb{N}_0^m$, and $|\xi| \leq 1$. Hence, $|\xi|^s \hat{v}$ belongs to $\mathcal{S}$ and vanishes at 0 with all derivatives. We can thus define the *homogenous fractional derivative* $|D|^s = \mathcal{F}^{-1}|\xi|^s \mathcal{F} \colon \mathcal{S}_0 \to \mathcal{S}_0$ for $s \in \mathbb{R}$.

In a next step we look at the dual $\mathcal{S}_0^\star$ containing all continuous linear $\varphi \colon \mathcal{S}_0 \to \mathbb{C}$. The restriction to $\mathcal{S}_0$ of a distribution $\psi \in \mathcal{S}^\star$ belongs to this space. Moreover, for $v \in \mathcal{S}_0$ and a polynomial P we compute $\langle v, P \rangle_{\mathcal{S}} = \langle \mathcal{F}^{-1} v, \mathcal{F} P \rangle_{\mathcal{S}} = 0$ since $\mathcal{F} P$ is a linear combinations of derivatives of δ_0 by Example 3.1; i.e., $\varphi + P\restriction_{\mathcal{S}_0} = \varphi$ for $\varphi \in \mathcal{S}_0^\star$. Conversely, Hahn–Banach allows to extend $\varphi \in \mathcal{S}_0^\star$ to a map $\varphi_1 \in \mathcal{S}^\star$. Each extension $\varphi_2 \in \mathcal{S}^\star$ satisfies

$$0 = \langle v, \varphi_1 - \varphi_2 \rangle_{\mathcal{S}} = \langle \mathcal{F}^{-1} v, \mathcal{F}(\varphi_1 - \varphi_2) \rangle_{\mathcal{S}}$$

for all $v \in \mathcal{S}_0$, so that $\mathcal{F}(\varphi_1 - \varphi_2)$ vanishes on all $\chi \in C_c^\infty(\mathbb{R}^m \setminus \{0\})$. Then $\varphi_1 - \varphi_2$ is a polynomial by Proposition 2.4.1 in [**23**] combined with Example 3.1. As a result, $\mathcal{S}_0^\star$ can be identified with the quotient space $\mathcal{S}^\star$ over the set $\mathcal{P}$ of polynomials. See also §5.1.2 in [**68**].

We thus have the extensions $\mathcal{F} \colon \mathcal{S}_0^\star \to \mathcal{S}^\star$ and $|D|^s \colon \mathcal{S}_0^\star \to \mathcal{S}_0^\star$ observing that $|D|^s P(v) = \langle \mathcal{F} P, |\xi|^s \mathcal{F}^{-1} v \rangle_{\mathcal{S}} = 0$ for $v \in \mathcal{S}_0$. As in (3.2) we obtain that $|D|^{s+t} = |D|^s |D|^t = |D|^t |D|^s$ for $s, t \in \mathbb{R}$. For $\lambda > 0$ and $\varphi \in \mathcal{S}_0^\star$ Proposition 3.2 leads to the crucial scaling property

$$|D|^s(\sigma_\lambda \varphi) = \lambda^{-m} \mathcal{F}^{-1}(|\xi|^s \sigma_{\frac{1}{\lambda}} \hat{\varphi}) = \lambda^{-m} \mathcal{F}^{-1}\big(\sigma_{\frac{1}{\lambda}}(\lambda^s |\xi|^s \hat{\varphi})\big) = \lambda^s \sigma_\lambda(|D|^s \varphi). \tag{3.6}$$

Also, for $a \in \mathcal{E}$ the Fourier multiplier $a(D)$ leaves invariant $\mathcal{S}_0$ and thus $\mathcal{S}_0^\star$.

The *homogeneous (fractional) Sobolev spaces* are defined by

$$\dot{\mathcal{H}}^{s,p} = \dot{\mathcal{H}}^{s,p}(\mathbb{R}^m) = \big\{ \varphi \in \mathcal{S}_0^\star \,\big|\, |D|^s \varphi \in L^p(\mathbb{R}^m) \big\}$$

with norm $\|\varphi\|_{s,p} = \||D|^s \varphi\|_p$ for $s \in \mathbb{R}$ and $p \in (1, \infty)$. Again, $\dot{\mathcal{H}}^s := \dot{\mathcal{H}}^{s,2}$, $\dot{\mathcal{H}}^{0,p} = L^p$, and $|D|^t \colon \dot{\mathcal{H}}^{s,p} \to \dot{\mathcal{H}}^{s-t,p}$ is an isometric isomorphism. Hence, $\dot{\mathcal{H}}^{s,p}$

is a reflexive and separable Banach space with dual $\dot{\mathcal{H}}^{-s,p'}$ in which $\mathcal{S}_0$ is dense. (Use that $\mathcal{S}_0$ is dense in L^p by Theorems 5.1.5 and 5.2.3.1 in [68].)

Let $s > -m/p'$, $p \geq 2$, and $v \in \mathcal{S}$. Then $|\xi|^s$ belongs to $L^{p'}(B(0,1))$ and hence $|\xi|^s \hat{v}$ to $L^{p'} + \mathcal{S}$. The mapping properties of $\mathcal{F}^{-1}$ thus imply that $\mathcal{S}$ is contained in $\dot{\mathcal{H}}^{s,p}$ and thus dense.

Arguing as for $\mathcal{H}^{s,p}$ but now with $\xi^\alpha |\xi|^{-k}$, from Mikhlin we deduce that $\dot{\mathcal{H}}^{k,p}$ is isomorphic to the space $\dot{W}^{k,p}$ of $v \in \mathcal{S}_0^\star$ with $\partial^\alpha v \in L^p$ for all $|\alpha| = k$. In sharp contrast to the (inhomogeneous) Bessel-potential spaces there is no inclusion between $\dot{\mathcal{H}}^{s,p}$ and $\dot{\mathcal{H}}^{t,p}$ for $t \neq s$, cf. p.96 of [46]. For $s > 0$ one has $L^p \cap \dot{\mathcal{H}}^{s,p} = \mathcal{H}^{s,p}$ since $a(\xi) = (1 + |\xi|^s)(1 + |\xi|^2)^{-s/2}$ and $1/a$ satisfy Mikhlin's conditions. By duality, we infer that $\dot{\mathcal{H}}^{s,p}$ is embedded into $\mathcal{H}^{s,p}$ for $s < 0$.

The homogeneous spaces interpolate as in (3.4) via

$$[\dot{\mathcal{H}}^{s,p}(\mathbb{R}^m), \dot{\mathcal{H}}^{t,q}(\mathbb{R}^m)]_\theta = \dot{\mathcal{H}}^{\tau,r}(\mathbb{R}^m) \quad \text{with } \tau = (1-\theta)s + \theta t,\ \tfrac{1}{r} = \tfrac{1-\theta}{p} + \tfrac{\theta}{q}, \tag{3.7}$$

for $s, t \in \mathbb{R}$, $p, q \in (1, \infty)$ and $\theta \in (0,1)$, cf. §5.2.5 in [68]. Using the above characterization, one can show that $\dot{\mathcal{H}}^{k,p} \hookrightarrow L^q$ for $q \in [p, \infty)$ if $k - \frac{m}{p} = -\frac{m}{q}$. Compare the proof of Theorem 3.52 in [47], taking $f \in \mathcal{S}_0$ there. Interpolation and isomorphism then again imply the Sobolev embedding

$$\dot{\mathcal{H}}^{t,p}(\mathbb{R}^m) \hookrightarrow \mathcal{H}^{s,q}(\mathbb{R}^m) \qquad \text{if} \quad 1 < p \leq q < \infty, \ \ t - \tfrac{m}{p} = s - \tfrac{m}{q}. \tag{3.8}$$

Such an embedding can only be true if the 'Sobolev regularity exponents' are equal, see Remark 3.30 in [47]. Since $\dot{\mathcal{H}}^{s,p}$ is contained in L^r for some $r < \infty$ if $0 \leq s < \frac{m}{p}$, it cannot contain nonzero polynomials and so it is a subspace of $\mathcal{S}^\star$. Applying ∂^α with $|\alpha| \leq k$, we see that if $s - \frac{m}{p} < k$ only polynomials up to degree k have to be factored out in $\dot{\mathcal{H}}^{s,p}$.

3.2. Littlewood–Paley decomposition

The Fourier transform turns derivatives into multiplication operators via $\partial^\alpha v = \mathrm{i}^{|\alpha|} \mathcal{F}^{-1}(\xi^\alpha \hat{v})$. By Plancherel, ∂^α is thus bounded on L^2 by $\lambda^{|\alpha|}$ when acting on functions with $\operatorname{supp} \hat{v} \subseteq \overline{B}(0,\lambda)$. (The Fourier variable is often addressed as 'frequency'.) Actually such results are true in L^p as shown by the *Bernstein estimates* in the next lemma, which show in particular that functions with bounded Fourier support are smooth. By its part b), for functions with Fourier support in an annulus around λ the norm of k-th derivatives is proportional to λ^k. So the first part corresponds to Sobolev's embedding. Observe that $\|\sigma_a v\|_p = a^{-\frac{m}{p}} \|v\|_p$ for $v \in L^p$, $p \in [1, \infty]$, and $a > 0$ because of the transformation $y = ax$. We use the closed annulus $A(r,R) = \overline{B}(0,R) \setminus B(0,r)$.

Lemma 3.3. *Let $\lambda, r, r_1 > 0$, $r_2 > r_1$, $1 \leq p \leq q \leq \infty$, $k \in \mathbb{N}_0$, $v \in L^p(\mathbb{R}^m)$, and $\alpha \in \mathbb{N}_0^m$ with $|\alpha| = k$. For a constant $C = C(r, r_j, k, m)$ the following claims hold.*

a) The inclusion $\operatorname{supp} \hat{v} \subseteq \overline{B}(0, \lambda r)$ *yields* $v \in \mathcal{E}$ *and* $\|\partial^\alpha v\|_q \leq C \lambda^{k + \frac{m}{p} - \frac{m}{q}} \|v\|_p$.

b) $\operatorname{supp} \hat{v} \subseteq A(\lambda r_1, \lambda r_2)$ *yields* $C^{-1} \lambda^k \|v\|_p \leq \max_{|\alpha|=k} \|\partial^\alpha v\|_p \leq C \lambda^k \|v\|_p$.

c) One can replace the max-term in part b) by $\| |D|^k v\|_p$ if $k \in \mathbb{R}$ and the left term in part a) by $\| |D|^k v\|_q$ if $k \in \mathbb{R}_+$. If $\lambda \geq 1$, the same is true for $\langle D \rangle^k$.[1]

[1] Statement c) has been corrected and improved compared to the lecture.

PROOF. 1) Let $\lambda = 1$. For a), take $\phi \in C_c^\infty(\mathbb{R}^m)$ with $\phi = 1$ on $B := \overline{B}(0,r)$. Since $\hat{v} = \phi\hat{v}$, Proposition 3.2 yields that $v = \psi * v \in \mathcal{E}$ and $\partial^\alpha v = \partial^\alpha \psi * v$ with $\psi = (2\pi)^{-\frac{m}{2}}\mathcal{F}^{-1}\phi$. Estimate a) for $\lambda = 1$ then follows from Young's inequality with $\frac{1}{\rho} := 1 + \frac{1}{q} - \frac{1}{p} \in [0,1]$ and $\|\partial^\alpha\psi\|_r \le \max_{|\alpha|=k}\big(\|\partial^\alpha\psi\|_1 + \|\partial^\alpha\psi\|_\infty\big) =: C$.

To show part b), pick $\phi_0 \in C_c^\infty(\mathbb{R}^m \setminus \{0\})$ with $\phi_0 = 1$ on $A := A(r_1, r_2)$. For the lower bound note that there are constants $c_\alpha > 0$ such that $1 = \sum_{|\alpha|=k} c_\alpha(-\mathrm{i})^k\xi^\alpha|\xi|^{-2k}\mathrm{i}^k\xi^\alpha$. From $\hat{v} = \phi_0\hat{v}$, we then deduce

$$v = \sum\nolimits_{|\alpha|=k} \mathcal{F}^{-1}\big(c_\alpha(-\mathrm{i})^k\xi^\alpha|\xi|^{-2k}\phi_0\mathcal{F}(\partial^\alpha v)\big) = \sum\nolimits_{|\alpha|=k} \psi_\alpha * \partial^\alpha v$$

for suitable $\psi_\alpha \in \mathcal{S} \hookrightarrow L^1$. Another application of Young's inequality yields the lower bound if $\lambda = 1$, possibly after increasing C.

Let $k \in \mathbb{R}$. The upper estimate in claim b) for $|D|^k$ follows from the equation $|D|^k v = (2\pi)^{-\frac{m}{2}}\mathcal{F}^{-1}(|\xi|^k\phi_0) * v$ and Young since $|\xi|^k\phi_0 \in \mathcal{S}$. Similarly, the lower bound is derived from $v = \mathcal{F}^{-1}(|\xi|^{-k}\phi_0\,|\xi|^k\hat{v}) = (2\pi)^{-\frac{m}{2}}\mathcal{F}^{-1}(|\xi|^{-k}\phi_0) * |D|^k v$.

To derive the estimate in a) for $|D|^k$ and $k \in \mathbb{R}_+$, we split $\phi = \sum_{j\ge 0}\sigma_{2^j}\phi_0$ where ϕ_0 has support in $A(r/2, 2r)$ and is equal to 1 around $|\xi| = r$, cf. (3.9). Let v have Fourier support in B and set $v_j = \mathcal{F}^{-1}(\sigma_{2^j}\phi_0\hat{v}) = 2^{-mj}\sigma_{2^{-j}}v_0$ for $j \in \mathbb{N}_0$, so that $v = \sum_j v_j$. We also set $\tilde{v}_0 = \mathcal{F}^{-1}(\tilde{\phi}_0\hat{v})$ for a map in $C_c^\infty A(r/3, 3r)$ being 1 on $A(r/2, 2r)$. Equation (3.6) then yields $|D|^k v_j = 2^{-kj}2^{-mj}\sigma_{2^{-j}}|D|^k\phi_0(D)\tilde{v}_0$. The kernel $\mathcal{F}^{-1}(|\xi|^k\phi_0)$ also belongs to L^ρ with $\frac{1}{\rho} = 1 + \frac{1}{q} - \frac{1}{p} \in [0,1]$. Young and the transform $z = 2^{-j}y$ thus imply $\||D|^k v_j\|_q \le c2^{-kj}2^{-\frac{m}{\rho'}j}\|\tilde{v}_0\|_p \le c2^{-kj}\|v\|_p$ since $\tilde{\phi}_0(D)$ is L^p-bounded by step 1). As $k > 0$, summation yields assertion a) for $|D|^k$ and $\lambda = 1$.

2) The case $\lambda > 0$ is now shown for ∂^α and $|D|^k$ by a scaling argument. We apply the above estimates to $u = \sigma_{1/\lambda}v$ having Fourier support in B, respectively A, by Proposition 3.2. The result is a consequence of the identities $\|u\|_p = \lambda^{\frac{m}{p}}\|v\|_p$, $\|\partial^\alpha v\|_q = \lambda^{k-\frac{m}{q}}\|\partial^\alpha u\|_q$, and $\||D|^k v\|_q = \lambda^{k-\frac{m}{q}}\||D|^k u\|_q$, see (3.6).

3) To treat $\langle D\rangle^k$ in b), we use the cut-off $\phi_{0,\lambda}(\xi) = \phi_0(\lambda^{-1}\xi)$ and compute

$$(2\pi)^{\frac{m}{2}}\langle D\rangle^k v = \mathcal{F}^{-1}(\langle\xi\rangle^k\sigma_{1/\lambda}\phi_0) * v = \lambda^m\sigma_\lambda\mathcal{F}^{-1}(\langle\lambda\xi\rangle^k\phi_0) * v =: f_\lambda * v.$$

Transforming $z = \lambda y$, the 1-norm of f_λ is equal to that of $f := \mathcal{F}^{-1}(\langle\lambda\xi\rangle^k\phi_0)$. Because of the support of ϕ_0 and $\lambda \ge 1$ we obtain $\|\partial^\alpha(\langle\lambda\xi\rangle^k\phi)\|_1 \le c(N)\lambda^k$ for $|\alpha| \le N$. Thus one can show $\|f\|_1 \le c\lambda^k$ as in (3.10).

For part a) we proceed as for $|D|$ decomposing $\phi = \sum_{j\ge 0}\sigma_{2^j}\phi_0$ and setting $\phi_{0,\lambda}^j = \sigma_{2^j/\lambda}\phi_0$. We now estimate the analogue f_λ^j of f_λ in L^ρ. In this norm the transformation to f yields the needed factor $\lambda^m\lambda^{-\frac{m}{\rho}} = \lambda^{\frac{m}{p}-\frac{m}{q}}$. Similar as above, the ρ-norm of the resulting kernel f^j is bounded by $c(r,k)2^{-kj}\lambda^k$. Summing over $j \ge 0$, we obtain the estimate in s) for $\langle D\rangle^k$, $\lambda \ge 1$ and $k > 0$. □

In many areas of analysis it is an important technique to decompose functions into pieces with Fourier support in dyadic annuli, e.g., to control regularity in a very precise way. To introduce the resulting *Littlewood–Paley decompositions* of $\mathcal{H}^{s,p}$ and $\dot{\mathcal{H}}^{s,p}$ for $p \in (1,\infty)$ we fix a radial function $\chi \in C^\infty(\mathbb{R}^m)$ such that

$\chi \geq 0$, $\operatorname{supp}\chi \subseteq \{\frac{6}{7} \leq |\xi| \leq 2\}$, $\chi = 1$ on $\{1 \leq |\xi| \leq \frac{12}{7}\}$, and $\chi(\xi) + \chi(\frac{1}{2}\xi) = 1$ for $1 \leq |\xi| \leq \frac{24}{7}$, see §1.3.2 in [24]. Set $\chi_j = \sigma_{2^{-j}}\chi$ for $j \in \mathbb{Z}$. It follows

$$\sum\nolimits_{j\in\mathbb{Z}} \chi_j(\xi) = 1, \qquad \xi \neq 0. \tag{3.9}$$

The *Littlewood–Paley operators* are defined by $P_j = \mathcal{F}^{-1}\chi_j\mathcal{F}$. Clearly, $\chi_j\hat{v}$ has support in the annulus with radii 2^{j-1} and 2^{j+1}. We further set $P_{\leq k} = \sum_{j\leq k} P_j$ for $k \in \mathbb{Z}$ etc. These operators have the symbol $\chi^k = \sum_{j\leq k}\chi_j$, where $\chi^k(0) := 1$. All P_j and $P_{\leq j}$ leave invariant $\mathcal{S}$ and $\mathcal{S}_0$ so that they are defined on $\mathcal{S}^\star$ and on $\mathcal{S}_0^\star$, where $P_jp = 0$ for a polynomial p. They also behave well on L^p-spaces because of their representation as a convolution operator.

Remark 3.4. Proposition 3.2 implies that $P_j\varphi = (2\pi)^{-\frac{m}{2}}2^{jm}\sigma_{2^j}\psi * \varphi$ with $\psi = \mathcal{F}^{-1}\chi$ for $\varphi \in \mathcal{S}^\star$ and that P_j maps $\mathcal{S}^\star$ into $\mathcal{E}$. Since $\psi_j := 2^{jm}\sigma_{2^j}\psi$ has 1-norm $\|\psi\|_1$, the operators P_j are uniformly bounded on all L^p for $1 \leq p \leq \infty$, by Young. Replacing χ by χ^0, we obtain the same result for $P_{\leq j}$.

The kernel of P_j is 'essentially' supported near $\overline{B}(0, 2^{-j})$ because of

$$|\psi_j(y)| \leq c_N 2^{jm}(1 + 2^j|y|)^{-N} \qquad \text{for} \quad N \in \mathbb{N}_0, \quad y \in \mathbb{R}^m. \tag{3.10}$$

Indeed, this is clear if $2^j|y| \leq 1$. Otherwise, take an index k with $\sqrt{m}|y_k| \geq |y| > 2^{-j}$. Integrating by parts, we then compute

$$(2\pi)^{\frac{m}{2}}\sigma_{2^j}\psi(y) = \int_{\mathbb{R}^m} \frac{1}{\mathrm{i}2^jy_k}\partial_{\xi_k}\mathrm{e}^{\mathrm{i}2^jy\cdot\xi}\chi(\xi)\,\mathrm{d}\xi = \int_{\mathbb{R}^m} \frac{\mathrm{i}}{2^jy_k}\mathrm{e}^{\mathrm{i}2^jy\cdot\xi}\partial_k\chi(\xi)\,\mathrm{d}\xi,$$

obtaining $|\sigma_{2^j}\psi(y)| \leq c_1'/(2^j|y|) \leq c_1/(1+2^j|y|)$. Then (3.10) follows inductively. Since $0 \leq \chi^0 \in C^\infty(\mathbb{R}^m)$ satisfies $\chi^0 = 1$ on $\overline{B}(0,1)$ and $\operatorname{supp}\chi^0 \subseteq \overline{B}(0,2)$, we get the estimate (3.10) also for the kernel $(2\pi)^{-\frac{m}{2}}2^{jm}\sigma_{2^j}\mathcal{F}^{-1}\chi^0$ of $P_{\leq j}$.

The Littlewood–Paley operators are almost projections in the sense that

$$P_j = (P_{j-1} + P_j + P_{j+1})P_j =: \tilde{P}_jP_j \quad \text{and} \quad P_jP_k = 0 \ \text{ if } \ |j-k| \geq 2 \tag{3.11}$$

for $j,k \in \mathbb{Z}$, since $\chi_j\chi_k = 0$ and the 'enlarged' Littlewood–Paley operator $\tilde{P}_j$ has the symbol $\tilde{\chi}_j := \chi_{j-1} + \chi_j + \chi_{j+1}$ satisfying $\tilde{\chi}_j\chi_j = \chi_j$. There are also restrictions on the support of the product of frequency-localized functions, e.g.,

$$\text{for} \quad k \geq j+3\colon \quad \operatorname{supp}\mathcal{F}(P_ju\,P_kv) \subseteq A(2^{k-2}, 2^{k+2}) \tag{3.12}$$

for $u, v \in \mathcal{S}^\star$ with $\hat{u}, \hat{v} \in L^1_{\mathrm{loc}}$, say. Indeed the above Fourier transform is proportional to

$$c(\xi) := (\chi_j\hat{u} * \chi_k\hat{v})(\xi) = \int_{2^{k-1}\leq|\eta|\leq 2^{k+1}} (\chi_j\hat{u})(\xi-\eta)(\chi_k\hat{v})(\eta)\,\mathrm{d}\eta$$

To obtain $c(\xi) \neq 0$, we need $2^{j-1} \leq |\xi - \eta| \leq 2^{j+1}$. The triangle inequality then yields the claim via $|\xi| \geq 2^{k-1} - 2^{j+1} \geq 2^{k-2}$ and $|\xi| \leq 2^{k+1} + 2^{j+1} \leq 2^{k+2}$.

These operators allow us to reduce our analysis to frequency-localized functions. They yield an 'almost orthogonal' decomposition of $\dot{\mathcal{H}}^{s,p}$ as expressed by the following *Littlewood–Paley theorem.*

THEOREM 3.5. *Let $p \in (1,\infty)$ and $s \in \mathbb{R}$. If $v \in \dot{\mathcal{H}}^{s,p}$, we have*

$$\|v\|^*_{\dot{\mathcal{H}}^{s,p}} := \Big\| \Big(\sum\nolimits_{j\in\mathbb{Z}} 2^{2sj} |P_j v|^2 \Big)^{\frac{1}{2}} \Big\|_p \le c\|v\|_{\dot{\mathcal{H}}^{s,p}}.$$

If $v \in \mathcal{S}_0^*$ satisfies $\|v\|^*_{\dot{\mathcal{H}}^{s,p}} < \infty$, then v belongs to $\dot{\mathcal{H}}^{s,p}$ and $\|v\|_{\dot{\mathcal{H}}^{s,p}} \le c\|v\|^*_{\dot{\mathcal{H}}^{s,p}}$.

These results are also true with a different constant if one replaces the above χ by any $0 \le \chi \in C_c^\infty(\mathbb{R}^m \setminus \{0\})$ satisfying (3.9).

See Theorem 1.3.8 in [**24**] for a proof. Note that $\|v\|^*_{\dot{\mathcal{H}}^{s,p}}$ is the supremum over the p-norms of partial sums instead of $\sum_j$. We show the norm equivalence in the theorem for $p = 2$. Let $v \in \mathcal{S}_0$. Plancherel then yields

$$\|P_j v\|_2 = \|\chi_j \hat{v}\|_2 \le c2^{jm/2} \sup_{A(2^{j-1},2^{j+1})} |\hat{v}| \le c(s,v)2^{-2|sj|}$$

for all $j \in \mathbb{Z}$ since $\hat{v}$ tends to 0 as $|\xi| \to 0,\infty$ faster than any polynomial. Using Plancherel, (3.9) and (3.11), we compute

$$\begin{aligned}
\|v\|^2_{\dot{\mathcal{H}}^s} &= \int_{\mathbb{R}^m} \Big| \sum_{j\in\mathbb{Z}} \chi_j |\xi|^s \hat{v} \Big|^2 \mathrm{d}\xi = \sum_{j,k\in\mathbb{Z}} \int_{\mathbb{R}^m} |\xi|^s \chi_j |\xi|^s \chi_k |\hat{v}|^2 \,\mathrm{d}\xi \\
&\eqsim_s \sum_{j,k\in\mathbb{Z}} 2^{s(j+k)} \int_{\mathbb{R}^m} \chi_j \chi_k |\hat{v}|^2 \mathrm{d}\xi \lesssim_s \sum_{j\in\mathbb{Z}} 2^{2sj} \int_{\mathbb{R}^m} \chi_j \tilde{\chi}_j |\hat{v}|^2 \,\mathrm{d}\xi \\
&\le \sum_{j\in\mathbb{Z}} 2^{2sj} \int_{\mathbb{R}^m} |\tilde{\chi}_j \hat{v}|^2 \,\mathrm{d}\xi = \sum_{j\in\mathbb{Z}} 2^{2sj} \int_{\mathbb{R}^m} |\tilde{P}_j v|^2 \,\mathrm{d}x \lesssim_s \int_{\mathbb{R}^m} \sum_{k\in\mathbb{Z}} 2^{2sk} |P_k v|^2 \,\mathrm{d}x.
\end{aligned}$$

Here we inserted $\tilde{P}_j = P_{j-1} + P_j + P_{j+1}$, and the final $L^2\ell^2$-norm is finite due the observations above. The converse inequality is shown similarly.

To treat also $\mathcal{H}^{s,p}$, we replace the summands P_j for $j \le 0$ by $P_{\le 0}$ with symbol χ^0. In other words, we use multipliers with the property

$$\chi^0(\xi) + \sum\nolimits_{j\in\mathbb{N}} \chi_j(\xi) = 1, \qquad \xi \in \mathbb{R}^m. \tag{3.13}$$

We can then state the inhomogeneous version of the above result (see Theorem 1.3.6 in [**24**]).

THEOREM 3.6. *Let $p \in (1,\infty)$ and $s \in \mathbb{R}$. If $v \in \mathcal{H}^{s,p}$, we have*

$$\|v\|^*_{s,p} := \|P_{\le 0} v\|_p + \Big\| \Big(\sum\nolimits_{j\in\mathbb{N}} 2^{2sj} |P_j v|^2 \Big)^{\frac{1}{2}} \Big\|_p \le c\|v\|_{s,p}.$$

If $v \in \mathcal{S}^*$ satisfies $\|v\|^*_{s,p} < \infty$, then v belongs to $\mathcal{H}^{s,p}$ and $\|v\|_{s,p} \le c\|v\|^*_{s,p}$.

These results are also true with a different constant if one replaces the above χ and χ^0 by any $0 \le \chi \in C_c^\infty(\mathbb{R}^m \setminus \{0\})$ and $0 \le \chi^0 \in C_c^\infty(\mathbb{R}^m)$ satisfying (3.13).

Actually, all tempered distributions can be written as a 'Littlewood-Paley series.'

PROPOSITION 3.7. *a) Let $\varphi \in \mathcal{S}^\star$ and $\varphi_0 \in \mathcal{S}_0^\star$. Then the series $P_{\le 0}\varphi + \sum_{j\ge 1} P_j \varphi$ converges to φ in $\mathcal{S}^\star$ and $\sum_{j\in\mathbb{Z}} P_j \varphi_0$ to φ_0 in $\mathcal{S}_0^\star$.*

b) Let $p \in (1,\infty)$, $s \in \mathbb{R}$, $v \in \mathcal{H}^{s,p}$, and $v_0 \in \dot{\mathcal{H}}^{s,p}$. Then the series $P_{\le 0} v + \sum_{j\ge 1} P_j v$ converges to v in $\mathcal{H}^{s,p}$ and $\sum_{j\in\mathbb{Z}} P_j v_0$ to v_0 in $\dot{\mathcal{H}}^{s,p}$. In particular, $P_{\le n}$ strongly tends to I in these spaces, as $n \to \infty$.

Proof. We focus on the inhomogeneous case. Let $\varphi \in \mathcal{S}^\star$, $v \in \mathcal{S}$, and $n \in \mathbb{N}$. We compute

$$\langle v, P_{\le n}\varphi\rangle_{\mathcal{S}} = \Big\langle \Big(\chi^0 + \sum\nolimits_{j=1}^{n} \chi_j\Big)\mathcal{F}^{-1}v, \hat{\varphi}\Big\rangle_{\mathcal{S}} \longrightarrow \langle v, \varphi\rangle_{\mathcal{S}}$$

as $n \to \infty$, since the right-hand side tends to $\mathcal{F}^{-1}v$ in $\mathcal{S}$.

For b), we note that $P_{\le n}$ is uniformly bounded on $\mathcal{H}^{s,p}$ by Remark 3.4 and $\langle D\rangle^s P_{\le n} = P_{\le n}\langle D\rangle^s$. So it remains to show the strong convergence to 0 in $\mathcal{H}^{s,p}$ of $P_{\ge n+1} = I - P_{\le n}$ on the dense subset $\mathcal{S} \ni v$. Here we write $\langle D\rangle^s P_{\ge n+1}v = \langle D\rangle^{-1}P_{\ge n+1}\langle D\rangle^{s+1}v$. Let $\phi = \mathbb{1} - \chi^1$. The operator $\langle D\rangle^{-1}P_{\ge n+1}$ has the symbol $a_n = \langle \xi\rangle^{-1}\sigma_{2^{-n}}\phi$. It is bounded by $c2^{-n}$ and we have e.g.

$$\partial_j a_n(\xi) = -\xi_j\langle\xi\rangle^{-3}\phi(2^{-n}\xi) + \langle\xi\rangle^{-1}2^{-n}\phi'(2^{-n}|\xi|)\xi_j/|\xi|,$$

so that $|\xi|\,|\nabla a_n| \le c2^{-n}$ by the support of ϕ. This can be iterated and so Mikhlin shows that $\langle D\rangle^{-1}P_{\ge n+1}$ tends to 0 in $\mathcal{B}(L^p)$ as required. □

In the next proof we use the (Hardy–Littlewood) *maximal operator*

$$M(f)(x) = \sup_{r>0} \frac{1}{\operatorname{vol}(B(0,r))}\int_{B(0,r)} |f(x-y)|\,\mathrm{d}y, \qquad x \in \mathbb{R}^m,$$

for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^m)$. It satisfies $\|M(f)\|_p \le c(p,m)\|f\|_p$ for $p \in (1,\infty]$ by Theorem 2.1.6 in [**23**]. Fefferman and Stein showed the vector-valued variant

$$\Big\|\Big(\sum\nolimits_{j\in\mathbb{Z}} M(f_j)^2\Big)^{\frac12}\Big\|_p \le c(p,m)\Big\|\Big(\sum\nolimits_{j\in\mathbb{Z}} |f_j|^2\Big)^{\frac12}\Big\|_p \tag{3.14}$$

for $f = (f_j) \in L^p(\mathbb{R}^m, \ell^2)$ and $p \in (1,\infty)$, see Theorem 5.6.6 in [**23**]. Corollary 2.1.12 of [**23**] combined with (3.10) and the comments after it imply the pointwise bounds

$$|P_k f| \le cM(f) \qquad \text{and} \qquad |P_{\le k} f| \le cM(f) \tag{3.15}$$

for all $k \in \mathbb{Z}$, $f \in L^1_{\mathrm{loc}}$, and $c = c(m,\chi)$.

We will use the Littlewood–Paley series in the following product and commutator estimate which are crucial for later investigations. They considerably improve Lemma 2.8 and Proposition 2.7. The proof of the first result follows that of Proposition 3.3 in [**14**].

Proposition 3.8. *Let $s > 0$, $r, p_2, q_1 \in (1,\infty)$, and $p_1, q_2 \in (r,\infty]$ with $\frac{1}{r} = \frac{1}{p_i} + \frac{1}{q_i}$ for $i \in \{1,2\}$. Assume that $u \in L^{p_1}$, $|D|^s u \in L^{p_2}$, $v \in L^{q_2}$, and $|D|^s v \in L^{q_1}$. We then obtain $uv \in L^{r_0}$ for some $r_0 \in (1,\infty]$, $|D|^s(uv) \in L^r$, and*

$$\||D|^s(uv)\|_r \lesssim \|u\|_{p_1}\||D|^s v\|_{q_1} + \||D|^s u\|_{p_2}\|v\|_{q_2}. \tag{3.16}$$

Here one can replace $|D|^s$ by $\langle D\rangle^s$.

Proof. Let $p_1, q_2 < \infty$ so that we can use the Littlewood–Paley decomposition freely. See Theorem 1.1 in [**42**] and Proposition 2.1.1 in [**66**] for the remaining and also other cases.

1) We first show that $u \in L^{p_0}$ and $v \in L^{q_0}$ for some $p_0, q_0 \in (1,\infty)$ with $\frac{1}{p_0} + \frac{1}{q_0} =: \frac{1}{r_0} \in (0,1)$, and hence we can apply Hölder to products like $uv \in L^{r_0}$

below. We are done if $\frac{1}{p_1}+\frac{1}{q_2}<1$. If e.g. $s-\frac{m}{p_2}=:-\frac{m}{p_0}<0$, then the Sobolev embedding (3.8) shows that $u\in L^{p_0}$ with $p_0\in(1,\infty)$. In this case we obtain

$$\frac{1}{p_0}+\frac{1}{q_2}=\frac{1}{p_2}+\frac{1}{q_2}-\frac{s}{m}<\frac{1}{r}<1$$

by assumption. Otherwise, we have $\sigma:=s-\frac{m}{p_2}\geq 0$ and analoguously for q_1. One can now choose $\theta\in(0,1)$ with $\theta\sigma-(1-\theta)\frac{m}{p_1}=-\frac{m}{p_0}$ for some $p_0\in(p_1,\infty)$ satisfying $\frac{1}{p_0}+\frac{1}{q_2}<1$. The interpolation result (3.7) then yields $u\in\dot{\mathcal{H}}^{\tau,\rho}$ with $\frac{1}{\rho}=\frac{1-\theta}{p_1}+\frac{\theta}{p_2}$ and $\tau=\theta s$. Since $\tau-\frac{m}{\rho}=\theta\sigma-(1-\theta)\frac{m}{p_1}=-\frac{m}{p_0}$, by (3.8) the function u belongs to L^{p_0}.

It suffices to establish the homogeneous case as the inhomogeneous one follows by adding the estimate for $s=0$ which is just Hölder. To use (3.17) under our assumptions, we first restrict to u and v having a finite Littlewood–Paley series.

2) We abbreviate $Q_k=P_{\leq k-3}$ and set $\hat{P}_k=P_{k-2}+\cdots+P_{k+2}$. Formula (3.12) leads to $Q_kuP_kv=\hat{P}_k(Q_kuP_kv)$. Using Proposition 3.7, we can write

$$\begin{aligned} uv &= \sum_{k\in\mathbb{Z}}P_kv\,Q_ku+\sum_{k\in\mathbb{Z}}P_kv\sum_{j\geq k+3}P_ju+\sum_{|j-k|\leq 2}P_ju\,P_kv\\ &=\sum_{k\in\mathbb{Z}}P_kv\,Q_ku+\sum_{j\in\mathbb{Z}}P_ju\,Q_jv+\sum_{|j-k|\leq 2}P_ju\,P_kv\\ &=\sum_{k\in\mathbb{Z}}\hat{P}_k(P_kv\,Q_ku)+\sum_{j\in\mathbb{Z}}\hat{P}_j(P_ju\,Q_jv)+\sum_{|j-k|\leq 2}P_ju\,P_kv. \end{aligned}\tag{3.17}$$

To compute $\||D|^s(uv)\|_r$ be means of Theorem 3.5, we have to apply $2^{sl}P_l$ to the terms in (3.17) and form a square sum. As in (3.11) we obtain $P_l\hat{P}_k=0$ if $|k-l|\geq 4$ and, with $f_k=P_kv\,Q_ku$,

$$\begin{aligned} \sum_{k=l-3}^{l+3}P_l\hat{P}_kf_k &= P_lP_{l-1}f_{l-3}+P_l(P_{l-1}+P_l)f_{l-2}+P_l\tilde{P}_l(f_{l-1}+f_l+f_{l+1})\\ &\qquad+P_l(P_l+P_{l+1})f_{l+2}+P_lP_{l+1}f_{l+3}\\ &=\tilde{P}_lP_l\sum_{k=l-2}^{l+2}f_k=P_l\sum_{k=l-2}^{l+2}f_k. \end{aligned}$$

3) By (3.15), (3.14) and Theorem 3.5, the first term in (3.17) then leads to

$$\begin{aligned} &\Big\|\Big[\sum_{l\in\mathbb{Z}}2^{2sl}\Big|\sum_{k\in\mathbb{Z}}P_l\hat{P}_k(P_kv\,Q_ku)\Big|^2\Big]^{\frac{1}{2}}\Big\|_r\leq\Big\|\Big[\sum_{l\in\mathbb{Z}}2^{2sl}\Big(\sum_{|i|\leq 2}\big|P_l(P_{l+i}v\,Q_{l+i}u)\big|\Big)^2\Big]^{\frac{1}{2}}\Big\|_r\\ &\lesssim\Big\|\Big(\sum\nolimits_{l\in\mathbb{Z}}\sum\nolimits_{|i|\leq 2}\big|M(2^{sl}P_{l+i}v\,Q_{l+i}u)\big|^2\Big)^{\frac{1}{2}}\Big\|_r\lesssim\Big\|\Big(\sum\nolimits_{k\in\mathbb{Z}}\big|2^{sk}P_kv\,Q_ku\big|^2\Big)^{\frac{1}{2}}\Big\|_r\\ &\lesssim\Big\|\Big(\sum\nolimits_{k\in\mathbb{Z}}|2^{sk}P_kv|^2M(u)^2\Big)^{\frac{1}{2}}\Big\|_r=\Big\|M(u)\Big(\sum\nolimits_{k\in\mathbb{Z}}|2^{sk}P_kv|^2\Big)^{\frac{1}{2}}\Big\|_r\\ &\leq\|M(u)\|_{p_1}\Big\|\Big(\sum\nolimits_{k\in\mathbb{Z}}|2^{sk}P_kv|^2\Big)^{\frac{1}{2}}\Big\|_{q_1}\lesssim\|u\|_{p_1}\||D|^sv\|_{q_1}. \end{aligned}$$

The second summand is estimated in the same way. For the last term, observe that $P_l(P_ju\,P_kv)=0$ if $|j-k|\leq 2$ and $l\geq 4+j\vee k$ with $j\vee k=\max\{j,k\}$

since $\mathcal{F}(P_j u\, P_k v) \subseteq \overline{B}(0, 2^{j\vee k+2})$ in this case, cf. (3.12). Substituting $j = k - i$, $k = l + m$ and $l = n - m$, we deduce as above

$$\Big\| \Big[\sum_{l\in\mathbb{Z}} 2^{2sl} \Big| \sum_{|j-k|\le 2} P_l(P_j u\, P_k v) \Big|^2 \Big]^{\frac12} \Big\|_r \le \Big\| \Big[\sum_{l\in\mathbb{Z}} 2^{2sl} \Big(\sum_{\substack{|j-k|\le 2\\ j\vee k\ge l-3}} \big|P_l(P_j u\, P_k v)\big| \Big)^2 \Big]^{\frac12} \Big\|_r$$

$$\le \Big\| \Big[\sum\nolimits_{l\in\mathbb{Z}} \Big(\sum\nolimits_{|i|\le 2} \sum\nolimits_{m\ge -5} 2^{sl} \big|P_l(P_{l+m-i} u\, P_{l+m} v)\big| \Big)^2 \Big]^{\frac12} \Big\|_r$$

$$\lesssim \Big\| \sum\nolimits_{m\ge -5} \sum\nolimits_{|i|\le 2} \Big[\sum\nolimits_{l\in\mathbb{Z}} \big|P_l(P_{l+m-i} u\, 2^{sl} P_{l+m} v)\big|^2 \Big]^{\frac12} \Big\|_r$$

$$\lesssim \Big\| \sum\nolimits_{m\ge -5} \sum\nolimits_{|i|\le 2} 2^{-sm} \Big(\sum\nolimits_{n\in\mathbb{Z}} \big(M(P_{n-i} u\, 2^{sn} P_n v)\big)^2 \Big)^{\frac12} \Big\|_r$$

$$\lesssim \Big\| \sum\nolimits_{m\ge -5} \sum\nolimits_{|i|\le 2} 2^{-sm} \Big(\sum\nolimits_{n\in\mathbb{Z}} |P_{n-i} u|^2\, 2^{2sn} |P_n v|^2 \Big)^{\frac12} \Big\|_r$$

$$\lesssim \Big\| \sum\nolimits_{m\ge -5} \sum\nolimits_{|i|\le 2} 2^{-sm} \Big(\sum\nolimits_{n\in\mathbb{Z}} M(u)^2\, 2^{2sn} |P_n v|^2 \Big)^{\frac12} \Big\|_r$$

$$\lesssim \Big\| M(u) \Big(\sum\nolimits_{n\in\mathbb{Z}} 2^{2sn} |P_n v|^2 \Big)^{\frac12} \Big\|_r \lesssim \|u\|_{p_1} \| |D|^s v\|_{q_1},$$

also using the generalized Minkowski inequality Proposition 1.2.22 in [**26**].

4) So far we have shown (3.16) for $u_N = R_N u$ and $v_N = R_N v$ for $N \in \mathbb{N}$ with $R_N \coloneqq P_{\le N} P_{\ge -N}$ and constants independent of N. Proposition 3.7 shows the limits $u_N \to u$ in $L^{p_0} \cap L^{p_1}$, $|D|^s u_N \to |D|^s u$ in L^{p_2}, $v_N \to v$ in $L^{q_0} \cap L^{q_2}$, and $|D|^s v_N \to |D|^s v$ in L^{q_1}. So the terms on the right-hand side of (3.16) tend to those without N as $N \to \infty$. For the left-hand side we note that $\psi_l * (u_N v_N)$ converges to $\psi_l * (uv)$ in L^{r_0} as $N \to \infty$ since $\psi_l \in L^1$, cf. Remark 3.4. Passing to a subsequence, all summands $P_l(u_N v_N)$ tend to $P_l(uv)$ pointwise a.e., and analogously for their finite square sums. Fatou's lemma and (3.16) then imply

$$\begin{aligned}
\| |D|^s (uv)\|_r &\eqsim \sup_{L\in\mathbb{N}} \Big\| \Big[\sum\nolimits_{|l|\le L} 2^{2sl} \big|P_l(uv)\big|^2 \Big]^{\frac12} \Big\|_r \\
&\le \sup_{L\in\mathbb{N}} \liminf_{N\to\infty} \Big\| \Big[\sum\nolimits_{|l|\le L} 2^{2sl} \big|P_l(u_N v_N)\big|^2 \Big]^{\frac12} \Big\|_r \\
&\le \liminf_{N\to\infty} \Big\| \Big[\sum\nolimits_{l\in\mathbb{Z}} 2^{2sl} \big|P_l(u_N v_N)\big|^2 \Big]^{\frac12} \Big\|_r \\
&\lesssim \|u\|_{p_1} \| |D|^s v\|_{q_1} + \| |D|^s u\|_{p_2} \|v\|_{q_2}. \qquad \square
\end{aligned}$$

The proof of the commutator estimate is taken from the more general Theorem 1.4 in [**42**], where also a modified result is shown for $s > 1$. This type of inequalities goes back to Kato and Ponce. The argument in [**42**] relies on the following observation. Let $0 \le a_k \le \min\{2^{\alpha k} A, 2^{-\beta k} B\}$ for all $k \in \mathbb{Z}$ and some $\alpha, \beta, A, B > 0$. For $p \in [1, \infty]$ and $a = (a_k)$, we then obtain

$$\|a\|_p \lesssim_{\alpha,\beta} A^{\frac{\beta}{\alpha+\beta}} B^{\frac{\alpha}{\alpha+\beta}}. \tag{3.18}$$

Indeed, we have $2^{\alpha k}A \le 2^{-\beta k}B$ if and only if $k \le \ln(B/A)((\alpha+\beta)\ln 2)^{-1}$. Splitting $\sum_k a_k^p$ at this value, we get the bound $cA^p(B/A)^{\frac{p\alpha}{\alpha+\beta}} + cB^p(B/A)^{\frac{-p\beta}{\alpha+\beta}}$ if $p<\infty$, which is the desired one. The case $p=\infty$ is easier.

Proposition 3.9. *Let $s\in(0,1)$, $r\in(1,\infty)$, $p,q\in[r,\infty]$ with $\frac{1}{r}=\frac{1}{p}+\frac{1}{q}$. Assume that $a\in W^{1,p}$ and $v\in L^q$. We then obtain*

$$\|\langle D\rangle^s(av) - a\langle D\rangle^s v\|_r \lesssim \|\langle D\rangle^s a\|_p\|v\|_q + \|\nabla a\|_p\|\langle D\rangle^{s-1}v\|_q.$$

Proof. Again we restrict ourselves to $p,q<\infty$, see Theorem 1.4 in **[42]** for the other case. In our setting we can derive the inhomogenous version of (3.17) from Proposition 3.7 since Hölder, Lemma 3.3, and Remark 3.4 yield

$$\begin{aligned}\sum_{k\in\mathbb{N}}\sum_{j\ge k-3}\|P_k v P_j a\|_r &\lesssim \sum_{k\in\mathbb{N}}\sum_{j\ge k-3}\|v\|_q 2^{-j}\|\nabla P_j a\|_p \lesssim \|a\|_{1,p}\|v\|_q\sum_{k\in\mathbb{N}}\sum_{j\ge k-3}2^{-j}\\ &\lesssim \|a\|_{1,p}\|v\|_q\sum_{k\in\mathbb{N}}2^{-k}\lesssim \|a\|_{1,p}\|v\|_q.\end{aligned}$$

We thus have (3.17) for av and $a\langle D\rangle^s v$, apply $\langle D\rangle^s$ to the first equation, and subtract the second one, resulting in

$$\begin{aligned}[\langle D\rangle^s,a]v &= \sum_{k\in\mathbb{N}}[\langle D\rangle^s,Q_k a]P_k v + \sum_{k\in\mathbb{N}}[\langle D\rangle^s,P_k a]Q_k v + \sum_{k\in\mathbb{N}}[\langle D\rangle^s,P_k a]\hat P_k v\\ &\quad + [\langle D\rangle^s,P_{\le 1}a]P_{\le 1}v - [\langle D\rangle^s,P_1 a]P_1 v =: S_1+S_2+S_3+S_4. \qquad (3.19)\end{aligned}$$

Here we have redefined $Q_k=\sum_{j=1}^{k-3}P_j$ and analogously $\hat P_k$. As in (3.12) the Fourier support of $P_{\le 1}a\,P_{\le 1}v$ is contained in $\overline{B}(0,8)$. Hence Lemma 3.3 and Hölder show that $\|S_4\|_r\lesssim\|a\|_p\|v\|_q$. In a similar way we compute

$$\begin{aligned}\|[\langle D\rangle^s,&P_k a]\hat P_k v\|_r \le \|\langle D\rangle^s(P_k a\hat P_k v)\|_r + \|P_k a\,\langle D\rangle^s\hat P_k v\|_r\\ &\lesssim 2^{s(k+4)}\|P_k a\hat P_k v\|_r + \|P_k a\|_p\,2^{s(k+2)}\|\hat P_k v\|_q \lesssim 2^{sk}\|P_k a\|_p\|\hat P_k v\|_q.\end{aligned}$$

Using Bernstein again and Remark 3.4, we redistribute the weights to obtain

$$\|[\langle D\rangle^s,P_k a]\hat P_k v\|_r \lesssim \min\{2^{-(1-s)k}\|\nabla a\|_p\|v\|_q,\,2^{(1-s)k}\|\langle D\rangle^s a\|_p\|\langle D\rangle^{s-1}v\|_q\}.$$

Estimate (3.18) with $p=1$ and $\alpha=\beta=1-s$ then yields

$$\begin{aligned}\|S_3\|_r &\le \sum_{k\in\mathbb{N}}\|[\langle D\rangle^s,P_k a]\hat P_k v\|_r \lesssim \big(\|\nabla a\|_p\|v\|_q\big)^{\frac12}\big(\|\langle D\rangle^s a\|_p\|\langle D\rangle^{s-1}v\|_q\big)^{\frac12}\\ &\lesssim \|\nabla a\|_p\|\langle D\rangle^{s-1}v\|_q + \|\langle D\rangle^s a\|_p\|v\|_q.\end{aligned}$$

For S_1, Proposition 3.9 from **[42]**, Lemma 3.3 and Mikhlin show

$$\begin{aligned}a_k &:= \|[\langle D\rangle^s,Q_k a]P_k v\|_r \lesssim 2^{(s-1)k}\|\nabla Q_k a\|_p\|P_k v\|_q\\ &\lesssim \|\langle D\rangle^{1-s}Q_k\langle D\rangle^s a\|_p\|\langle D\rangle^{s-1}P_k v\|_q \lesssim 2^{(1-s)k}\|\langle D\rangle^s a\|_p\|\langle D\rangle^{s-1}v\|_q.\end{aligned}$$

(Note that we can estimate $\langle D\rangle^{1-s}Q_k$ by means of Lemma 3.3 since $1-s>0$.) On the other hand, we obtain

$$2^{(s-1)k}\|\nabla Q_k a\|_p \lesssim 2^{\frac12(s-1)k}\|\nabla a\|_p^{\frac12}\|\langle D\rangle^s a\|_p^{\frac12}$$

similarly, and hence

$$a_k \lesssim \min\big\{2^{(1-s)k}\|\langle D\rangle^s a\|_p\|\langle D\rangle^{s-1}v\|_q, 2^{-\frac{1}{2}(1-s)k}\|\nabla a\|_p^{\frac{1}{2}}\|\langle D\rangle^s a\|_p^{\frac{1}{2}}\|v\|_q\big\}.$$

From (3.18) with $\alpha = 1-s$ and $\beta = (1-s)/2$ it then follows

$$\begin{aligned}\|S_1\|_r &\lesssim \big(\|\langle D\rangle^s a\|_p\|\langle D\rangle^{s-1}v\|_q\big)^{\frac{1}{3}}\big(\|\nabla a\|_p^{\frac{1}{2}}\|\langle D\rangle^s a\|_p^{\frac{1}{2}}\|v\|_q\big)^{\frac{2}{3}}.\\ &= \big(\|\langle D\rangle^s a\|_p\|v\|_q\big)^{\frac{2}{3}}\big(\|\nabla a\|_p\|\langle D\rangle^{s-1}v\|_q\big)^{\frac{1}{3}}\\ &\lesssim \|\nabla a\|_p\|\langle D\rangle^{s-1}v\|_q + \|\langle D\rangle^s a\|_p\|v\|_q.\end{aligned}$$

The remaining term is expressed by

$$S_2 = \sum_{k\in\mathbb{N}}\langle D\rangle^s(P_k a\,Q_k v) - \sum_{k\in\mathbb{N}} P_k a\,\langle D\rangle^s Q_k v =: S_{21} + S_{22}.$$

Proposition 3.8 and Lemma 3.3 imply $\|S_{21}\|_r \lesssim 2^{sk}\|P_k a\|_p\|Q_k v\|_q$. As above Bernstein yields

$$\begin{aligned}\|P_k a\|_p &\lesssim \min\big\{2^{-sk}\|\langle D\rangle^s a\|_p, 2^{-k}\|\nabla a\|_p\big\}\\ \|Q_k v\|_q &\lesssim \min\big\{2^{(1-s)k}\|\langle D\rangle^{s-1}v\|_q, 2^{\frac{1}{2}(1-s)k}\|v\|_q^{\frac{1}{2}}\|\langle D\rangle^{s-1}v\|_q^{\frac{1}{2}}\big\},\end{aligned}$$

leading to

$$\|S_{21}\|_r \lesssim \min\big\{2^{(1-s)k}\|\langle D\rangle^s a\|_p\|\langle D\rangle^{s-1}v\|_q, 2^{-\frac{1}{2}(1-s)k}\|\nabla a\|_p\|\langle D\rangle^{s-1}v\|_p^{\frac{1}{2}}\|v\|_q^{\frac{1}{2}}\big\}.$$

Using (3.18), we conclude

$$\begin{aligned}\|S_{21}\|_r &\lesssim \big(\|\langle D\rangle^s a\|_p\|\langle D\rangle^{s-1}v\|_q\big)^{\frac{1}{3}}\big(\|\nabla a\|_p\|\langle D\rangle^{s-1}v\|_p^{\frac{1}{2}}\|v\|_q^{\frac{1}{2}}\big)^{\frac{2}{3}}\\ &\lesssim \|\nabla a\|_p\|\langle D\rangle^{s-1}v\|_q + \|\langle D\rangle^s a\|_p\|v\|_q.\end{aligned}$$

Applying Hölder, we obtain similarly

$$\|P_k a\,\langle D\rangle^s Q_k v\|_r \lesssim \min\big\{2^{(1-s)k}\|\langle D\rangle^s a\|_p\|\langle D\rangle^{s-1}v\|_q, 2^{-(1-s)k}\|\nabla a\|_p\|v\|_q\big\}.$$

Hence, S_{22} can be treated as S_3. □

Remark 3.10. To deal with the case $p=\infty$ in Proposition 3.9, let $v \in L^\infty$ and $s \in (0,1)$. Then it is known that $v \in C_b^s(\mathbb{R}^m)$ if and only if $\|P_j v\|_\infty \le C2^{-sj}$ for $j \in \mathbb{N}$, and then $C = c\|v\|_{C_b^s}$. Moreover, this estimate holds for $s = 1$ if $v \in W^{1,\infty}(\mathbb{R}^m)$. See §A.1 in **[65]**. The geometric series then implies $\|P_{\ge k}v\|_\infty \le c2^{-sk}\|v\|_{C_b^s}$. By Lemma 3.3 we have $\|\langle D\rangle^s P_j v\|_\infty \le c2^{sj}\|P_j v\|_\infty$.

Let $a \in C_b^s(\mathbb{R}^m)$. Proposition 3.7 yields $\langle D\rangle^s a = \langle D\rangle^s P_{\le 0}a + \sum_{j\ge 1}\langle D\rangle^s P_j a$ in $\mathcal{S}^\star$. Estimating as above, we deduce $\|\langle D\rangle^s a\|_\infty \lesssim_\delta \|a\|_{C_b^{s+\delta}}$ for $\delta > 0$. ♢

From Theorems A.8 and A.12 in **[34]** (which contains more general and precise results) we deduce the following homogeneous version which is more flexibel than Proposition 3.9, though $\sigma = 1$ is not admitted. The proofs in **[34]** use a similar approach as in Proposition 3.8. Since they are quite lenghty, we omit them.

Proposition 3.11. *Let* $s\in(0,1)$, $\sigma\in[0,s]$, $r, p_2, q_2 \in (1,\infty)$, *and* $p_1, q_1 \in [r,\infty]$ *with* $\frac{1}{r} = \frac{1}{p_i} + \frac{1}{q_i}$ *for* $i \in \{1,2\}$. *If* $\sigma = s$, *we can admit* $q_2 = \infty$. *We then obtain*

$$\||D|^s(av) - a|D|^s v\|_r \lesssim \||D|^s a\|_{p_1}\|v\|_{q_1} + \||D|^\sigma a\|_{p_2}\||D|^{s-\sigma}v\|_{q_2}.$$

3.3. Strichartz estimates for the wave equation

Before we study Strichartz estimates for the Maxwell system, it is important to recall corresponding results for the standard wave equation on $\mathbb{R}^m$ and discuss basic methods in this simpler case. Our treatment largely follows parts of Chapter 5 of [46]. We investigate the wave equation

$$\partial_t^2 u = \Delta u + f, \qquad u(0) = u_0, \quad \partial_t u(0) = u_1, \qquad t \in J, \quad x \in \mathbb{R}^m, \tag{3.20}$$

for $m \geq 2$, an interval J of positive length containing 0, and given initial maps $u_0, u_1 \colon \mathbb{R}^m \to \mathbb{C}$ and forcing $f \colon J \times \mathbb{R}^m \to \mathbb{C}$. Here u may represent the displacement of a vibrating object, the pressure, or a component of electromagnetic fields in vacuum.

One derives a solution formula to this equation taking the Fourier transform in $x \in \mathbb{R}^m$ (at first formally), which yields the ordinary differential equation

$$\partial_t^2 \hat{u}(t,\xi) + |\xi|^2 \hat{u}(t,\xi) = \hat{f}(t,\xi), \qquad \hat{u}(0,\xi) = \widehat{u_0}(\xi), \quad \partial_t \hat{u}(0,\xi) = \widehat{u_1}(\xi),$$

for fixed $\xi \in \mathbb{R}^m$. It is solved by

$$\hat{u}(t,\xi) = \cos(t|\xi|)\widehat{u_0}(\xi) + \sin(t|\xi|)\tfrac{1}{|\xi|}\widehat{u_1}(\xi) + \int_0^t \sin((t-\tau)|\xi|)\tfrac{1}{|\xi|}\hat{f}(\tau,\xi)\,\mathrm{d}\tau.$$

If we apply the inverse Fourier transform, we are led to the Fourier multipliers

$$C(t) = \mathcal{F}^{-1}\cos(t|\xi|)\mathcal{F}, \quad S(t) = \mathcal{F}^{-1}\sin(t|\xi|)\mathcal{F}, \quad Sc(t) = \mathcal{F}^{-1}\operatorname{sinc}(t|\xi|)\mathcal{F}$$

for $t \in \mathbb{R}$. They map L^2 into $\mathcal{S}^*$ for all $s \in \mathbb{R}$ since the symbols are bounded, they leave invariant $\mathcal{S}_0$ and thus $\mathcal{S}_0^*$, and they are uniformly bounded on $\mathcal{H}^s$ and $\dot{\mathcal{H}}^s$ by Plancherel. We obtain the solution formula

$$u(t) = C(t)u_0 + tSc(t)u_1 + \int_0^t (t-\tau)Sc(t-\tau)f(\tau)\,\mathrm{d}\tau, \qquad t \in J, \tag{3.21}$$

which implies the expressions

$$\begin{aligned} |D|u(t) &= C(t)|D|u_0 + S(t)u_1 + \int_0^t S(t-\tau)f(\tau)\,\mathrm{d}\tau, \qquad t \in J, \\ \partial_t u(t) &= -S(t)|D|u_0 + C(t)u_1 + \int_0^t C(t-\tau)f(\tau)\,\mathrm{d}\tau, \qquad t \in J. \end{aligned} \tag{3.22}$$

Proposition 5.6 of [46] shows that for $u_0 \in \dot{\mathcal{H}}^1$, $u_1 \in L^2$, and $f \in L^1_J L^2$ formula (3.21) provides the unique solution u of (3.20) in $C(J, \dot{\mathcal{H}}^1)$ with $\partial_t u \in C(J, L^2)$ and $\partial_t^2 u \in C(J, L^2 + \dot{\mathcal{H}}^{-1})$. In [46] the case $m = 2$ is excluded for the homogeneous spaces because of the definition of $\dot{\mathcal{H}}^s$ used there. However, the arguments there also work for our case. In $\mathcal{H}^1(\mathbb{R}^2)$ one only factors out constant functions. By means of the isomorphism $|D|^{1-s}$ we obtain a unique $\dot{\mathcal{H}}^s$-*solution* in $u \in C(J, \dot{\mathcal{H}}^s)$ with $\partial_t u \in C(J, \dot{\mathcal{H}}^{s-1})$ and $\partial_t^2 u \in C(J, \dot{\mathcal{H}}^{s-1} + \dot{\mathcal{H}}^{s-2})$ for data $u_0 \in \dot{\mathcal{H}}^s$, $u_1 \in \dot{\mathcal{H}}^{s-1}$, and $f \in L^1_J \dot{\mathcal{H}}^{s-1}$. Here one can omit the dots.

We note that u is real for real data u_0, u_1, and f in the above spaces, since then $\operatorname{Im} u$ solves (3.20) with 0 data and thus $\operatorname{Im} u = 0$ by uniqueness. Moreover, (3.22) imply the 'energy' estimate

$$\|(\partial_t, |D|)u(t)\|_{L^2} \leq c\big(\|(|D|u_0, u_1)\|_{L^2} + \|f\|_{L^1_t L^2}\big), \tag{3.23}$$

for $t \in J$, say with $t > 0$. We stress that the map $(u_0, u_1) \mapsto u(t)$ is just $\dot{\mathcal{H}}^1 \times L^2$-bounded, whereas $f \mapsto u$ gains one derivative and uniformity in time. Since $|\xi_k| \le |\xi|$, the left-hand side dominates $\|\nabla u(t)\|_2$ by Plancherel. In (3.23) one can also multiply all functions by $|D|^s$ or $\langle D\rangle^s$ with $s \in \mathbb{R}$. For the L^2-norm, (3.21) only yields the non-uniform estimate

$$\|u(t)\|_{L^2} \le \|u_0\|_{L^2} + |t|\,\|u_1\|_{L^2} + |t|\,\|f\|_{L^1_t L^2}.$$

For nonlinear problems one often needs to control the L^p-norms of solutions. Here this can be obtained to some extent using dispersive behavior.

REMARK 3.12. Let $\xi \in \mathbb{R}^m \setminus \{0\}$ and $\phi \in C^2(\mathbb{R}, \mathbb{R})$. Then the 'plane wave' $w_\xi(t,x) = \phi(t)\mathrm{e}^{-\mathrm{i}\xi\cdot x}$ solves the differential equation (3.20) with $f = 0$ if and only if $\phi'' + |\xi|^2\phi = 0$. So one obtains e.g. the solution

$$w_\xi(t,x) = a\exp(\mathrm{i}|\xi|(t - x\cdot\xi/|\xi|)), \qquad a > 0.$$

For $t \neq 0$, we have $w_\xi = a$ if and only if $x = t\xi/|\xi|$, and hence the plane waves travel in different directions $\xi/|\xi|$. As a result, superpositions of the functions w_ξ ('wave packets') disperse. This behavior is not present in space dimension $m = 1$ since, e.g., the solution $u(t,x) = \frac{1}{2}(u_0(x+t) + u_0(x-t))$ of (3.20) with $u_1 = 0$ and $f = 0$ just exhibits transport. $\Diamond$

There are several ways to quantify the effect of this phenomenon, where we focus on 'Strichartz inequalities.' To state them, we need time and space exponents $p, q \in [2, \infty]$ and a regularity loss parameter $\gamma \in \mathbb{R}$. Such numbers form an *admissible triple* (for $m \ge 2$ and the wave equation) if

$$\frac{2}{p} + \frac{m-1}{q} \le \frac{m-1}{2}, \quad \frac{1}{p} + \frac{m}{q} = \frac{m}{2} - \gamma, \quad \text{for } m = 3\colon\ (p,q,\gamma) \neq (2,\infty,1). \tag{3.24}$$

We call a triple *strict* if the first relation in (3.24) is an equality. In this case we have $\gamma = \frac{m+1}{p(m-1)}$. Otherwise the regularity loss $\frac{m+1}{p(m-1)} \le \gamma \le \frac{m}{2}$ is larger. So except for $p = \infty = q$ and thus $\gamma = \frac{m}{2}$ the space $\dot{\mathcal{H}}^\gamma$ is contained in $\mathcal{S}^\star$, and it always contains $\mathcal{S}$ as a dense subspace. By duality, $\dot{\mathcal{H}}^{-\gamma} \hookrightarrow \mathcal{S}^\star$ in all cases and $\mathcal{S}$ is dense in $\dot{\mathcal{H}}^{-\gamma}$ for $\gamma < \frac{m}{2}$. The loss γ is positive except for the energy case $(\infty, 2, 0)$, which is the 'trivial endpoint'. This is the only strict case with $p = \infty$, whereas the only strict triple with $q = \infty$ is $(4, \infty, \frac{3}{4})$ if $m = 2$, as $p \ge 2$. For $m > 3$, there exists the (strict) 'critical endpoint' $p = 2$, $q = 2\frac{m-1}{m-3}$ and $\gamma = \frac{m+1}{2(m-1)} < 1$. We now state the *Strichartz estimates* for the wave equation.

THEOREM 3.13. *Let (p,q,γ) and (r,s,θ) be admissible, $u_0 \in \dot{\mathcal{H}}^1$, $u_1 \in L^2$, and $f \in L^{r'}_J \dot{\mathcal{H}}^{\theta,s'}$. Then the solution u of (3.20) satisfies*

$$\||D|^{-\gamma}(|D|, \partial_t)u\|_{L^p_J L^q} \le C\big(\|(|D|u_0, u_1)\|_{L^2} + \||D|^\theta f\|_{L^{r'}_J L^{s'}}\big) \tag{3.25}$$

for a constant $C \ge 1$. If $q = 2$, then $(|D|, \partial_t)u$ belongs to $C_{\mathbb{R}}L^2$. For $q = \infty$ or $s = \infty$, the estimate has to be modified as in Remark 3.14 a).

We refer to e.g. Strichartz [**60**], Ginibre–Velo [**22**], Lindblad–Sogge [**36**], Keel–Tao [**33**], and also [**54**] and [**61**]. Keel and Tao developed a general approach to Strichartz estimates and settled the non-trivial endpoint cases.

In (3.25) the parameter γ measures a loss in regularity for the map $(u_0, u_1) \mapsto u$ and a reduced gain of $1-\gamma$ derivatives in the map $f \mapsto u$, compared to the energy estimate from (3.23); i.e., the case $(p, q, \gamma) = (\infty, 2, 0) = (r, s, \theta)$. Thus the Strichartz estimates trade regularity and boundedness in time to improve spatial integrability (and to obtain some decay as $|t| \to \infty$). More precisely, for $u_0 \in \dot{\mathcal{H}}^1$, $u_1 \in L^2$ and $f \in L^1_J L^2$, Sobolev's embedding, (3.23) and admissibility imply that the solution u of (3.20) belongs to $L^\infty_J \dot{\mathcal{H}}^{1-\gamma-\frac{1}{p},q}$, whereas (3.25) yields $L^p_J \dot{\mathcal{H}}^{1-\gamma,q}$. Later we focus on the energy case $(r', s', \theta) = (1, 2, 0)$ on the right-hand side which often suffices for applications, cf. Theorem 5.17 in [**46**].

We provide most of the proof below in various steps except for the harder critical endpoint. We first discuss Theorem 3.13 and versions of it.

Remark 3.14. a) In (3.25), one has to replace the spaces $L^\infty(\mathbb{R}^m)$ and $L^1(\mathbb{R}^m)$ for $q = \infty$ and $s = \infty$, respectively, by the *homogeneous Besov spaces*

$$\dot{B}^0_{\infty,2} = \{v \in \mathcal{S}_0^\star \,|\, (P_j v) \in \ell^2(\mathbb{Z}, L^\infty)\}, \quad \dot{B}^0_{1,2} = \{v \in \mathcal{S}_0^\star \,|\, (P_j v) \in \ell^2(\mathbb{Z}, L^1)\}$$

with their canonical norms, see Chapter 5 in [**68**]. (One defines $\dot{B}^\alpha_{p,q}$ for $p, q \in [1, \infty]$ and $\alpha \in \mathbb{R}$ analogously, replacing $P_j v$ by $2^{\alpha j} P_j v$ and using L^p and ℓ^q.) However, in the strict case the space exponent ∞ only occurs for the triple $(4, \infty, \frac{3}{4})$ and $m = 2$. Since the proof of Theorem 3.13 is reduced to strict triples by means of Sobolev's embedding, these Besov spaces rarely occur below.

b) It suffices to prove (3.25) for $J = \mathbb{R}$. Indeed, on the left the norm in $L^p_J L^q$ is dominated by that in $L^p_{\mathbb{R}} L^q$, and we can use the 0 extension of f from J to $\mathbb{R}$ which has the same norm and produces the same solution on J.

c) In (3.25) contains energy norms on the right (if $(r', s', \theta) = (1, 2, 0)$). Since (3.20) has constant coefficients, one can easily ransform (3.25) to versions on every regularity level. Let $\kappa \in \mathbb{R}$, $u_0 \in \dot{\mathcal{H}}^{1+\kappa}$, $u_1 \in \dot{\mathcal{H}}^\kappa$, and $f \in L^{r'}_J \dot{\mathcal{H}}^{\theta+\kappa,s'}$. Set $v_0 = |D|^\kappa u_0$, $v_1 = |D|^\kappa u_1$, and $g = |D|^\kappa f$. By (3.2), for these data (3.20) has the solution $v = |D|^\kappa u$. Applying (3.25) to it and using (3.2), we derive

$$\| |D|^{\kappa-\gamma}(|D|, \partial_t) u\|_{L^p_J L^q} \le C\big(\| |D|^\kappa (|D| u_0, u_1)\|_{L^2} + \| |D|^{\theta+\kappa} f\|_{L^{r'}_J L^{s'}}\big) \tag{3.26}$$

with the same modifications as a). By $|D|^{-\kappa}$, one sees that (3.26) implies (3.25).

d) The multiplier $\xi_k |\xi|^{-1}$ satisfies the Mikhlin condition, so that $\partial_k |D|^{-1}$ is L^q–bounded and in (3.25) and (3.26) one can replace $(|D|, \partial_t)$ by $(\nabla, \partial_t) =: \nabla_{t,x}$. This also works for $q = \infty$ because of part a) and Theorem 5.2.2 in [**68**]. $\Diamond$

Most of admissibility asssumptions in Theorem 3.13 are necessary.

Remark 3.15. a) The equality in (3.24) is needed for the Strichartz estimate with $f = 0$, which can be seen by a scaling argument. Let u solve (3.20) with $f = 0$ and $(u_0, u_1) \neq 0$. Then also $u_\lambda(t, x) = u(\lambda t, \lambda x)$ is a solution with initial values $\sigma_\lambda u_0$ and $\lambda \sigma_\lambda u_1$, for $\lambda > 0$. Let (3.25) hold for $(p, q, \gamma) \in [1, \infty]^2 \times \mathbb{R}$. By the transformation rule and (3.6), we conclude

$$\begin{aligned}
\lambda^{1-\gamma-\frac{1}{p}-\frac{m}{q}} \| |D|^{-\gamma}(|D|, \partial_t) u\|_{L^p_{\mathbb{R}} L^q} &= \lambda^{1-\gamma} \| (|D|^{-\gamma}(|D|, \partial_t) u)(\lambda \cdot, \lambda \cdot)\|_{L^p_{\mathbb{R}} L^q} \\
&= \| |D|^{-\gamma}(|D|, \partial_t) u_\lambda\|_{L^p_{\mathbb{R}} L^q} \le C \| (|D| \sigma_\lambda u_0, \lambda \sigma_\lambda u_1)\|_{L^2}
\end{aligned}$$

$$= C\lambda\|\sigma_\lambda(|D|u_0, u_1)\|_{L^2} = C\lambda^{1-\frac{m}{2}}\|(|D|u_0, u_1)\|_{L^2}.$$

Letting $\lambda \to 0$ and $\lambda \to \infty$, we infer $\frac{1}{p} + \frac{m}{q} = \frac{m}{2} - \gamma$. If $u_0 = 0 = u_1$, the Strichartz estimate holds for a wider range of exponents, see e.g. [**21**].

b) The inequality in (3.24) is necesssary for Theorem 3.13 because of Knapp's example: Let $\varepsilon \in (0,1]$, $x' = (x_2, \dots, x_m)$, $R_\varepsilon = [1,2] \times [-\varepsilon, \varepsilon]^{m-1}$, and $\varphi = \mathcal{F}^{-1}\mathbb{1}_{R_\varepsilon}$ (which belongs to H^k for all $k \in \mathbb{N}$). The H^2-solution of (3.20) with $u_0 = \varphi$, $u_1 = -\mathrm{i}|D|\varphi$ and $f = 0$ is given by

$$u(t,x) = (2\pi)^{-\frac{m}{2}} \int_{R_\varepsilon} \mathrm{e}^{\mathrm{i}x\cdot\xi}\mathrm{e}^{-\mathrm{i}t|\xi|}\,\mathrm{d}\xi = \mathcal{F}^{-1}(\mathrm{e}^{-\mathrm{i}t|\xi|}\mathbb{1}_{R_\varepsilon}),$$

see (3.27). By means of Plancherel, we first estimate (with $|\xi| = |\xi|_2$)

$$\||D|\varphi\|_2^2 = \int_{R_\varepsilon} |\xi|^2\,\mathrm{d}\xi \le (m+3)\lambda(R_\varepsilon) = (m+3)2^{m-1}\varepsilon^{m-1}.$$

To obtain a lower bound for u, we fix $\kappa = \frac{1}{4}\arccos\frac{1}{2} > 0$ and define

$$S_\varepsilon = \big\{(t,x) \in \mathbb{R}^{1+m} \,\big|\, (m-1)|t| \le \kappa\varepsilon^{-2},\ |x_1 - t| \le \kappa,\ |x'|_1 \le \kappa\varepsilon^{-1}\big\}.$$

Let $(t,x) \in S_\varepsilon$ and $\xi \in R_\varepsilon$. We aim at the inequality

$$\tfrac{1}{2} \le \operatorname{Re}\mathrm{e}^{\mathrm{i}(x\cdot\xi - |\xi|t)} = \cos\big[(x_1 - t)\xi_1 + x'\cdot\xi' + t\xi_1(1 - |\xi|/\xi_1)\big].$$

This lower bound is true since the definitions of S_ε and R_ε imply

$$|[\dots]| \le 2\kappa + \frac{\kappa}{\varepsilon}\varepsilon + \frac{\kappa}{(m-1)\varepsilon^2}2\Big(\sqrt{1 + |\xi'|^2\xi_1^{-2}} - 1\Big) \le 4\kappa$$

by a standard estimate for the square root. Let $E = L^p(\mathbb{R}, L^q)$. We infer

$$\||D|^{1-\gamma}u\|_E = \|\mathcal{F}^{-1}(|\xi|^{1-\gamma}\mathrm{e}^{-\mathrm{i}|\xi|t}\mathbb{1}_{R_\varepsilon})\|_E = c\Big\|\Big|\int_{R_\varepsilon} |\xi|^{1-\gamma}\mathrm{e}^{\mathrm{i}(x\cdot\xi - |\xi|t)}\,\mathrm{d}\xi\Big|\Big\|_{L_t^pL_x^q}$$

$$\ge c\Big\|\int_{R_\varepsilon} \operatorname{Re}\mathrm{e}^{\mathrm{i}(x\cdot\xi - |\xi|t)}\,\mathrm{d}\xi\Big\|_{L_t^pL_x^q(S_\varepsilon)} \ge \tfrac{c}{2}\lambda(R_\varepsilon)\|\mathbb{1}_{S_\varepsilon}\|_{L_{\mathbb{R}}^pL^q} = c\varepsilon^{m-1}\varepsilon^{-\frac{m-1}{q}}\varepsilon^{-\frac{2}{p}}$$

for some constants $c > 0$. On the other hand, estimate (3.25) yields $\||D|^{1-\gamma}u\|_E \le c\varepsilon^{\frac{m-1}{2}}$ for all $\varepsilon \in (0,1]$ so that $\frac{m-1}{2} - \frac{2}{p} - \frac{m-1}{q} \ge 0$.

c) The last condition in (3.24) is needed due to an example by Stein, see Exercise 2.44 in [**61**]. The inequality in (3.24) already implies that $q \ge 2$ and $p \ge 2$ if $m \le 3$. For $m > 3$ the condition $p \ge 2$ can be justified by a more complicated argument, see [**33**]. $\Diamond$

The solution operators $C(t)$ and $S(t)$ in (3.22) are inconvenient since they do not form groups. But one can easily express them by the *half-wave group* $G(t) = \mathrm{e}^{\mathrm{i}t|D|} = \mathcal{F}^{-1}\mathrm{e}^{\mathrm{i}t|\xi|}\mathcal{F}$ for $t \in \mathbb{R}$. As $C(t)$ and $S(t)$, the operators $G(t)$ map L^2 into $\mathcal{S}^*$ for all $t \in \mathbb{R}$, leave invariant $\mathcal{S}_0$ and $\mathcal{S}_0^*$, and are uniformly bounded and strongly continuous on $\mathcal{H}^s$ and $\dot{\mathcal{H}}^s$. (Use Plancherel and dominated convergence for the last point.) We set $(G *_+ f)(t) = \int_0^t G(t-\tau)f(\tau)\,\mathrm{d}\tau$. Observing that

$$C(t) = \tfrac{1}{2}(G(t) + G(-t)),\quad S(t) = \tfrac{1}{2\mathrm{i}}(G(t) - G(-t)),\quad G(t) = C(t) + \mathrm{i}S(t), \tag{3.27}$$

we next reduce the wave problem to a first-order one.

Let $f \in L^{r'}(\mathbb{R}, \dot{\mathcal{H}}^{\theta,s'})$ for an admissible triple. Then the above convolution and the Duhamel terms in (3.22) are defined in $\dot{\mathcal{H}}^{-\frac{1}{r}}$ (in L^2 if $r = \infty$), where $G(\cdot)$ is strongly continuous. Indeed, admissibility yields $\theta - \frac{m}{s'} = -\frac{1}{r} - \frac{m}{2}$ and $s' \le 2$. Sobolev (3.8) then implies $\dot{\mathcal{H}}^{\theta,s'} \hookrightarrow \dot{\mathcal{H}}^{-\frac{1}{r}}$ if $s < \infty$, where $\dot{\mathcal{H}}^{\theta,s'} \hookrightarrow L^2$ if $r = \infty$. If $s = \infty$, one has the same embeddings for $\dot{B}^{\theta}_{1,2}$ by §5.2.5 in **[68]** and the equality $\dot{B}^{\alpha}_{2,2} = \dot{\mathcal{H}}^{\alpha}$ for $\alpha \in \mathbb{R}$.

Lemma 3.16. *In the setting of Theorem 3.13, estimate (3.25) is equivalent to*

$$\||D|^{-\gamma} G(\cdot)\varphi\|_{L^p_{\mathbb{R}} L^q} \le c\|\varphi\|_{L^2} \quad \text{and} \quad \||D|^{-\gamma} G *_+ f\|_{L^p_{\mathbb{R}} L^q} \le c\||D|^{\theta} f\|_{L^{r'}_{\mathbb{R}} L^{s'}} \tag{3.28}$$

for $\varphi \in L^2$ and $f \in L^{r'}(\mathbb{R}, \dot{\mathcal{H}}^{\theta,s'})$. These inequalities are equivalent to

$$\||D|^{\kappa-\gamma} G(\cdot)\varphi\|_{L^p_{\mathbb{R}} L^q} \le c\||D|^{\kappa}\varphi\|_{L^2}, \quad \||D|^{\kappa-\gamma} G *_+ f\|_{L^p_{\mathbb{R}} L^q} \le c\|f\|_{L^{r'}_{\mathbb{R}} \dot{\mathcal{H}}^{\theta,s'}} \tag{3.29}$$

for $\kappa \in \mathbb{R}$, $\varphi \in \dot{\mathcal{H}}^{\kappa}$, and $f \in L^{r'}(\mathbb{R}, \dot{\mathcal{H}}^{\theta+\kappa,s'})$. (For $q = \infty$ or $s = \infty$ we have modifications as in Remark 3.14 a).) Moreover, for $q = 2$ the second part of (3.28) implies the addendum in Theorem 3.13.

Proof. The first part follows from (3.27) and (3.22), as the estimates for $G(t)$ and $G(-t)$ are equivalent by the transformation $t \mapsto -t$. The second statement is shown as in Remark 3.14 c). For the addendum, we note that $G * f$ is continuous in L^2 if $f \in C_c(\mathbb{R}, \mathcal{S}_0) \subseteq C_c(\mathbb{R}, L^2)$. Since the former space is dense in $L^{r'}_{\mathbb{R}} \dot{\mathcal{H}}^{\theta,s'}$, cf. Lemma 4.8 in **[46]**, by approximation we obtain that $G * f \in C_{\mathbb{R}} L^2$ if $f \in L^{r'}_{\mathbb{R}} \dot{\mathcal{H}}^{\theta,s'}$. For $s = \infty$ one argues in the same way, using Theorem 5.1.5 in **[68]**. Equations (3.27) and (3.22) then yield the last claim. $\square$

The first inequality in (3.28) or in (3.29) is called 'homogeneous,' the second one 'inhomogeneous'. We note that parts a) and b) of Remark 3.15 can easily be transferred to the half-wave case. We next reduce (3.25) to the strict case.

Lemma 3.17. *Let (3.28) hold for all strict admissible triples $(\tilde{p}, \tilde{q}, \tilde{\gamma})$ and $(\tilde{r}, \tilde{s}, \tilde{\theta})$. Then it is true for every admissible triples (p, q, γ) and (r, s, θ).*

Proof. Let (p, q, γ) be non-strict admissible. The numbers

$$\frac{1}{\tilde{q}} := \frac{1}{2} - \frac{2}{p(m-1)} > \frac{1}{q}, \qquad \tilde{\gamma} := \frac{m}{2} - \frac{1}{p} - \frac{m}{\tilde{q}} < \gamma,$$

yield a strict admissible triple $(p, \tilde{q}, \tilde{\gamma})$. (Note $p > 2$ if $m = 3$.) By admissibility we have $\gamma - \tilde{\gamma} = \frac{m}{\tilde{q}} - \frac{m}{q} > 0$. If (p, q, γ) is strict, we set $(p, \tilde{q}, \tilde{\gamma}) = (p, q, \gamma)$. We define $(r, \tilde{s}, \tilde{\theta})$ analogously, with $\tilde{\theta} - \theta = \frac{m}{\tilde{s}'} - \frac{m}{s'} < 0$ if (r, s, θ) is non-strict. It is enough to show (3.29) for $\kappa = \gamma$.

First, let $q, s < \infty$. The Sobolev embedding (3.8) and estimate (3.29) for $(p, \tilde{q}, \tilde{\gamma})$ and $(r, \tilde{s}, \tilde{\theta})$ then imply

$$\begin{aligned} \|G *_+ f\|_{L^p_{\mathbb{R}} L^q} &\le c\||D|^{\gamma-\tilde{\gamma}} G *_+ f\|_{L^p_{\mathbb{R}} L^{\tilde{q}}} \le cC\||D|^{\gamma+\tilde{\theta}} f\|_{L^{r'}_{\mathbb{R}} L^{\tilde{s}'}} \\ &= cC\||D|^{\tilde{\theta}-\theta}|D|^{\gamma+\theta} f\|_{L^{r'}_{\mathbb{R}} L^{\tilde{s}'}} \le cC\||D|^{\gamma+\theta} f\|_{L^{r'}_{\mathbb{R}} L^{s'}}. \end{aligned}$$

If infinite space exponents occur, we look at the frequency-localized piece $u_j = P_j G *_+ f = G *_+ (P_j f)$ for $j \in \mathbb{Z}$. Bernstein's Lemma 3.3 and (3.29) yield

$$\begin{aligned}\|u_j\|_{L^p_{\mathbb{R}} L^q} &\le c 2^{j(\frac{m}{\tilde q} - \frac{m}{q})} \|u_j\|_{L^p_{\mathbb{R}} L^{\tilde q}} \le c \||D|^{\gamma - \tilde\gamma} u_j\|_{L^p_{\mathbb{R}} L^{\tilde q}} \le cC \||D|^{\gamma + \tilde\theta} P_j f\|_{L^{r'}_{\mathbb{R}} L^{\tilde s'}} \\ &\le cC 2^{j(\frac{m}{\tilde s'} - \frac{m}{s'})} \||D|^{\gamma+\theta} P_j f\|_{L^{r'}_{\mathbb{R}} L^{\tilde s'}} \le cC \|P_j |D|^{\gamma+\theta} f\|_{L^{r'}_{\mathbb{R}} L^{s'}}.\end{aligned}$$

We square this estimate and sum over $j \in \mathbb{Z}$. For $\rho \in (1,\infty)$, the generalized Minkowski inequality and Littlewood–Paley yield $\dot B^0_{\rho,2} \hookrightarrow L^\rho$ if $\rho \ge 2$ and $L^\rho \hookrightarrow \dot B^0_{\rho,2}$ if $\rho \le 2$, since e.g.

$$\|v\|_{L^\rho} = \Big\| \Big(\sum\nolimits_{j\in\mathbb{Z}} |P_j v|^2 \Big)^{\frac12} \Big\|_\rho \le \Big(\sum\nolimits_{j\in\mathbb{Z}} \|P_j v\|_\rho^2 \Big)^{\frac12}$$

if $\rho \ge 2$. Similarly, using $q, p \ge 2$ and $2 \ge r', s'$, we deduce

$$\begin{aligned}\|G *_+ f\|_{L^p_{\mathbb{R}} L^q} &\lesssim \|G *_+ f\|_{L^p_{\mathbb{R}} \dot B^0_{q,2}} \le \Big[\sum_{j\in\mathbb{Z}} \|u_j\|^2_{L^p_{\mathbb{R}} L^q} \Big]^{\frac12} \lesssim \Big[\sum_{j\in\mathbb{Z}} \|P_j |D|^{\gamma+\theta} f\|^2_{L^{r'}_{\mathbb{R}} L^{s'}} \Big]^{\frac12} \\ &\lesssim \||D|^{\gamma+\theta} f\|_{L^{r'}_{\mathbb{R}} \dot B^0_{s',2}} \lesssim \||D|^{\gamma+\theta} f\|_{L^{r'}_{\mathbb{R}} L^{s'}},\end{aligned}$$

where the first or final step is omitted if $q = \infty$ or $s' = 1$, respectively. The homogeneous estimate is treated in the same way. $\square$

In the next result we show that the two parts of (3.28) are equivalent by means of a 'TT^*-argument'. One could formulate the equivalence in greater generality, see [**33**], but we stick to our setting to simplify a bit. We first introduce some notation for admissible triples (p, q, γ) and (r, s, θ).

We write $Y = \dot{\mathcal{H}}^{-\gamma, q}$ and $Y_* = \dot{\mathcal{H}}^{\gamma, q'}$ if $q \in [2, \infty)$ as well as $Y = \dot B^{-\gamma}_{\infty,2}$ and $Y_* = \dot B^{\gamma}_{1,2}$ if $q = \infty$. These spaces satisfy $Y^\star = Y_*$ if $q < \infty$ and $Y_*^\star = Y$ in both cases by the previous section and §5.2.5 in [**68**]. Hence, Y_* is a closed norming subspace of $Y^\star$; i.e, $\|v\|_Y = \sup_{\|u\|_{Y_*} \le 1} |\langle v, u\rangle_Y|$. We further set $E = L^p_{\mathbb{R}} Y$ and $E_* = L^{p'}_{\mathbb{R}} Y_*$. For $q < \infty$ we have $E_*^\star = E$ and $E^\star = E_*$ if also $p < \infty$ because of Corollary 1.3.22 in [**26**]. Otherwise, E_* is a closed norming subspace of $E^\star$ by Proposition 1.3.1 in [**26**]. For the triple (r, s, θ) the same results hold with the notation Z and F instead of Y and E.

To simplify notation, in the following we equip the duality pairing $\dot{\mathcal{H}}^\alpha \times \dot{\mathcal{H}}^{-\alpha}$ for $\alpha \in \mathbb{R}$ with the extension of the complex L^2-scalar product, so that the adjoint of $G(t)$ is $G(-t)$. We further set

$$\mathcal{G}\varphi = G(\cdot)\varphi, \qquad \mathcal{S} f = \int_{\mathbb{R}} G(-t) f(t)\,\mathrm{d}t, \qquad \mathcal{C} f = G * f$$

for $\varphi \in \dot{\mathcal{H}}^\alpha$ or $f \in C_c \dot{\mathcal{H}}^\alpha$. A priori the ranges of these operators are $C_{\mathbb{R}} \dot{\mathcal{H}}^\alpha$, $\dot{\mathcal{H}}^\alpha$ and $C_{\mathbb{R}} \dot{\mathcal{H}}^\alpha$, respectively, by the strong continuity of $G(\cdot)$ on $\dot{\mathcal{H}}^\alpha$.

Lemma 3.18. *In Lemma 3.16 the two statements in* (3.28) *are equivalent for admissible triples with* $r' < p$.

Proof. We use the notation and properties discussed above.

1) Let $\varphi \in \mathcal{H}^{\frac{1}{p}} \hookrightarrow \dot{\mathcal{H}}^{\frac{1}{p}} \hookrightarrow Y$ and $g \in C_c(\mathbb{R}, Y_*)$. Since $Y_* \hookrightarrow \dot{\mathcal{H}}^{-\frac{1}{p}}$ as noted before Lemma 3.16, by means of the above observations we compute

$$\langle \varphi, \mathcal{S}g\rangle_{\frac{1}{p}} = \int_{\mathbb{R}} \langle \varphi, G(-t)g(t)\rangle_{\frac{1}{p}}\,\mathrm{d}t = \int_{\mathbb{R}} \langle G(t)\varphi, g(t)\rangle_{\frac{1}{p}}\,\mathrm{d}t = \langle \mathcal{G}\varphi, g\rangle_E \tag{3.30}$$

in the duality of $\dot{\mathcal{H}}^{\frac{1}{p}}$ and $\dot{\mathcal{H}}^{-\frac{1}{p}}$. By density (cf. Lemma 4.8 in [**46**]), the boundedness of $\mathcal{G}\colon L^2 \to E$ and $\mathcal{S}\colon E_* \to L^2$ are equivalent, and $\mathcal{S}^\star = \mathcal{G}$ if $q < \infty$. (Here we can replace E by F.) E.g., let $\mathcal{S}$ be bounded. Then $|\langle \mathcal{G}\varphi, g\rangle_E| \le c\|\varphi\|_2 \|g\|_{E_*}$. Taking g with support in a compact J, we obtain $\|\mathcal{G}\varphi\|_{L^p(J,Y)} \le c\|\varphi\|_2$. The boundedness of $\mathcal{G}$ follows by Fatou and density.

2) Next, let $g \in E_*^0 = \{g \in C_c(\mathbb{R}, Y_*) \,|\, g(\mathbb{R}) \subseteq \mathcal{S}_0\}$ and $f \in F_*^0$. Also E_*^0 is dense in E_*. Similarly as in step 1), we derive

$$\begin{aligned} \langle \mathcal{C}f, g\rangle_E &= \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} G(t)G(-\tau)f(\tau)\,\mathrm{d}\tau \big| \overline{g}(t)\Big)_{l^2}\,\mathrm{d}t \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \big(G(-\tau)f(\tau)\big|G(-t)\overline{g}(t)\big)_{L^2}\,\mathrm{d}\tau\,\mathrm{d}t = (\mathcal{S}f|\mathcal{S}g)_{L^2}. \end{aligned} \tag{3.31}$$

First, let $\mathcal{S}\colon E_* \to L^2$ and $\mathcal{S}\colon F_* \to L^2$ have norm less than S. Using density and Hahn–Banach or the above comments, we conclude

$$\|\mathcal{C}f\|_E = \sup_{g\in E_*^0, \|g\|_{E_*}\le 1} |\langle \mathcal{C}f, g\rangle_E| \le S^2 \|f\|_{F_*}.$$

By density this is true for all $f \in F_*$. Conversely, let $\mathcal{C}\colon E_* \to E$ be bounded. With $f = g \in E_*^0$, identity (3.31) yields the continuity of $\mathcal{S}\colon E_* \to L^2$ via

$$\|\mathcal{S}f\|_{L^2}^2 = \langle \mathcal{C}f, f\rangle_E \le \|\mathcal{C}\|\,\|f\|_{E_*}^2.$$

We have now shown that the validity of the first part of (3.28) is equivalent to the boundedness of $\mathcal{C}\colon F_* \to E$, both for all admissible triple. By the following proposition, these facts imply the boundedness of the 'full' half-sided convolution $\mathcal{C}_+\colon F_* \to E$. If $t > 0$, it is equal to $G *_+ \tilde{f}(t)$ for the 0 extension $\tilde{f}$ of $f{\restriction}_{\mathbb{R}_+}$. The case $t < 0$ is then treated via the transform $t \mapsto -t$.

Conversely, let the second part of (3.28) be true. Take $f \in F_*$ with compact support, say in $[a, b]$. In this case we have $(\mathcal{C}_+ f)(t + a) = (G *_+ f(\cdot + a))(t)$, and so $\mathcal{C}_+\colon F_* \to E$ is continuous by density. As above it also follows that $\langle \mathcal{C}_+ f, g\rangle_E = \langle \mathcal{C}_- g, f\rangle_F$ for $g \in E_*^0$, $f \in F_*^0$, and $(\mathcal{C}_- g)(t) = \int_t^\infty G(t-\tau)g(\tau)\,\mathrm{d}\tau$. Hence, $\mathcal{C}\colon F_* \to E$ is bounded. □

For the *Christ–Kiselev lemma* stated below we refer to Lemma IV.2.1 in [**54**]. For $r = \infty$ one can by-pass it, see Lemma 5.13 in [**46**].

Proposition 3.19. *Let J be an interval, $Y, Z \subseteq X$ be Banach spaces, $1 \le p < q \le \infty$, and $K\colon J \times J \to \mathcal{B}(X)$ be strongly continuous and bounded. Set*

$$(\mathcal{K}f)(t) = \int_J K(t,s)f(s)\,\mathrm{d}s \quad \text{and} \quad (\mathcal{K}_+ f)(t) = \int_{\inf J}^t K(t,s)f(s)\,\mathrm{d}s$$

for $t \in J$ and $f \in C_c(J, Y)$. If $\mathcal{K}$ has a bounded extension from $L^p(J, Y)$ to $L^q(J, Z)$, then the same is true for $\mathcal{K}_+$.

It thus remains to show the homogeneous estimate in (3.28). A core step in this proof is the following reduction to a frequency-localized piece, which relies on the Littlewood–Paley decomposition.

LEMMA 3.20. *Let (p,q,γ) be admissible. Assume that*

$$\|P_0 G(\cdot)\varphi\|_{L^p_{\mathbb{R}} L^q} \le C\|\tilde{P}_0\varphi\|_{L^2} \tag{3.32}$$

for all $\varphi \in L^2$ and some $C > 0$. Then the first part of (3.29) with $\kappa = \gamma$ is true.

PROOF. By a scaling argument, from (3.32) we deduce

$$\|P_j G(\cdot)\varphi\|_{L^p_{\mathbb{R}} L^q} \le C 2^{\gamma j}\, \|\tilde{P}_j\varphi\|_{L^2} \tag{3.33}$$

for all $j \in \mathbb{Z}$ and $\varphi \in \dot{H}^\gamma$, see Lemma 5.15 in [**46**]. If $q = \infty$, we are done since

$$\|G(\cdot)\varphi\|_{L^p_{\mathbb{R}} \dot{B}^0_{\infty,2}} = \Big\| \Big(\sum\nolimits_{j\in\mathbb{Z}} \|P_j G(\cdot)\varphi\|_{L^\infty}^2 \Big)^{\frac{1}{2}} \Big\|_{L^p_{\mathbb{R}}} \le \Big(\sum\nolimits_{j\in\mathbb{Z}} \|P_j G(\cdot)\varphi\|_{L^p_{\mathbb{R}} L^\infty}^2 \Big)^{\frac{1}{2}}$$
$$\le C\Big(\sum\nolimits_{j\in\mathbb{Z}} 2^{2\gamma j}\, \|\tilde{P}_j\varphi\|_{L^2}^2 \Big)^{\frac{1}{2}} \lesssim \||D|^\gamma\varphi\|_{L^2},$$

using Minkowski's inequality if $p < \infty$ and Theorem 3.5 at the end.

For $q < \infty$ we employ the Littlewood–Paley decomposition. We let $p < \infty$, as the case $p = \infty$ just requires a minor modification. Take $\varphi \in \mathcal{H}^k$ for some $k \ge \frac{m}{2}$ and J be a compact interval. Then $G(t)\varphi$ belongs to $\mathcal{H}^k \hookrightarrow L^q$ by Sobolev's embedding. Theorem 3.5 yields

$$\|G(\cdot)\varphi\|_{L^p_J L^q}^2 \lesssim \Big\| \Big\| \Big(\sum\nolimits_{j\in\mathbb{Z}} |P_j G(\cdot)\varphi|^2 \Big)^{\frac{1}{2}} \Big\|_{L^q} \Big\|_{L^p_J}^2 = \Big\| \Big\| \|(|P_j G(\cdot)\varphi|^2)_j\|_{\ell^1} \Big\|_{L^{\frac{q}{2}}} \Big\|_{L^{\frac{p}{2}}_J}.$$

For fixed t, we interpret the inner terms as the norm in $L^{\frac{q}{2}}(\mathbb{R}^m)$ of the $L^{\frac{q}{2}}$-valued sum $\sum_j |P_j G(t)\varphi|^2$. We can take this norm in the sum since $q \ge 2$, obtaining

$$\|G(\cdot)\varphi\|_{L^p_J L^q}^2 \lesssim \Big\| \sum\nolimits_{j\in\mathbb{Z}} \|P_j G(\cdot)\varphi\|_{L^q}^2 \Big\|_{L^{\frac{p}{2}}_J}.$$

This procedure also works for the t-integral so that

$$\|G(\cdot)\varphi\|_{L^p_J L^q}^2 \lesssim \sum\nolimits_{j\in\mathbb{Z}} \|P_j G(\cdot)\varphi\|_{L^p_J L^q}^2$$

Estimates (3.33) and Theorem 3.5 now yield

$$\|G(\cdot)\varphi\|_{L^p_J L^q}^2 \le cC^2 \sum\nolimits_{j\in\mathbb{Z}} 2^{2\gamma j} \|\tilde{P}_j\varphi\|_{L^2}^2 = cC^2 \Big\| \sum\nolimits_{j\in\mathbb{Z}} 2^{2\gamma j} |\tilde{P}_j\varphi|^2 \Big\|_2^2 \lesssim \|\varphi\|_{\dot{\mathcal{H}}^\gamma}^2.$$

Fatou's lemma allows us to replace J by $\mathbb{R}$. The claim then follows from the density of $\mathcal{H}^k$ in $\dot{\mathcal{H}}^\gamma$. □

For Theorem 3.13 it remains to show (3.32) for strict triples by the above results. We restrict ourselves to the case $p \in (2,\infty)$. As seen above, in the strict case we have $p > 2$ if $m \le 3$, and $p = \infty$ only occurs in the energy case $(\infty, 2, 0)$ which we have settled in (3.23).

By a stationary phase argument one can show the following core frequency-localized dispersive estimate. See p.128 in [**54**], and also Lemma 5.16 in [**46**] for the easier case $m = 3$.

LEMMA 3.21. *We have* $\|\mathcal{F}^{-1}(\mathrm{e}^{\mathrm{i}t|\xi|}\chi)\|_{L^\infty} \le c(1+|t|)^{-\frac{m-1}{2}}$ *for all $t \neq 0$.*

We also need the Hardy-Littlewood-Sobolev inequality, Theorem 1.2.13 in [**23**].

LEMMA 3.22. *Let $1 < r < s < \infty$ and $0 < \lambda < n$ satisfy $1 + \frac{1}{s} = \frac{\lambda}{n} + \frac{1}{r}$. Then there is a constant $c > 0$ such that*

$$\Big(\int_{\mathbb{R}^n} \Big[\int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^\lambda}\,\mathrm{d}y\Big]^s \mathrm{d}x\Big)^{\frac{1}{s}} \le c\|f\|_r \qquad \textit{for all} \;\; f \in L^r(\mathbb{R}^n).$$

This result extends Young's convolution inequality to a borderline case since $\||t|^{-\lambda}\|_{n/\lambda}^{n/\lambda} = c\int_0^\infty \frac{1}{r}\,\mathrm{d}r = \infty$.

PROOF OF THEOREM 3.13 IN THE NON-ENDPOINT CASE. By Remark 3.14, Lemmas 3.16, 3.17, 3.18 and 3.20, as well as $P_0G(t) = P_0G(t)\tilde{P}_0$, it remains to show the boundedness of $P_0G(\cdot)\colon L^2 \to E := L^p(\mathbb{R}, L^q)$ for strict non-endpoint triples. Let $\varphi \in L^1 \cap L^2$ and $E' = L^{p'}_{\mathbb{R}} L^{q'}$. Lemma 3.21 and the formula

$$P_0G(t)\varphi = (2\pi)^{-\frac{m}{2}}\mathcal{F}^{-1}(\mathrm{e}^{\mathrm{i}t|\xi|}\chi) * \varphi$$

yield the basic (frequency-localized) dispersive estimate

$$\|P_0G(t)\varphi\|_{L^\infty} \le c|t|^{-\frac{m-1}{2}}\|\varphi\|_{L^1}.$$

Interpolating with $\|P_0G(t)\|_{\mathcal{B}(L^2)} \le 1$, see (3.7), we derive

$$\|P_0G(t)\varphi\|_{L^q} \le c|t|^{-(m-1)(\frac{1}{2}-\frac{1}{q})}\|\varphi\|_{L^{q'}} \tag{3.34}$$

if $q \in (2,\infty)$. (Recall that the triple $(4,\infty,\frac{3}{4})$ can occur if $m = 2$.) Strict admissibility and our setting yield $1 < p' < p < \infty$, $1 + \frac{1}{p} = (m-1)(\frac{1}{2} - \frac{1}{q}) + \frac{1}{p'}$, and $(m-1)(\frac{1}{2} - \frac{1}{q}) \in (0,1)$. Lemma 3.22 then implies

$$\|P_0G * f\|_E \le c\big\||t|^{-(m-1)(\frac{1}{2}-\frac{1}{q})} * \|f(\cdot)\|_{q'}\big\|_{L^p_{\mathbb{R}}} \le c\|f\|_{E'}, \quad f \in E'. \tag{3.35}$$

We now show (3.32) by a duality argument as in Lemma 3.18. Set $\mathcal{S}^0 f = \int_{\mathbb{R}} P_0G(-t)f(t)\mathrm{d}t$ for $f \in C_c(\mathbb{R}, L^2 \cap L^{q'})$. Using (3.35) and Remark 3.4, we compute

$$\begin{aligned} \|\mathcal{S}^0 f\|_{L^2}^2 &= \int_{\mathbb{R}}\int_{\mathbb{R}} \big(G(-\tau)P_0f(\tau)\big|G(-t)P_0f(t)\big)_{L^2}\,\mathrm{d}\tau\,\mathrm{d}t \\ &= \int_{\mathbb{R}}\Big(\int_{\mathbb{R}} P_0G(t-\tau)f(\tau)\,\mathrm{d}\tau\Big|P_0f(t)\Big)_{L^2}\mathrm{d}t = \langle P_0G * f, P_0\overline{f}\rangle_E \le c\|f\|_{E'}^2 \end{aligned}$$

By density, $\mathcal{S}^0\colon E' \to L^2$ is bounded. As in the proof of Lemma 3.18, we then deduce that $P_0G(\cdot)\colon L^2 \to E$ is bounded. ☐

We add some comments on the non-autonomous wave equation

$$\partial_t^2 u = \operatorname{div}(a\nabla u) + f, \quad u(0) = u_0, \quad \partial_t u(0) = u_1, \qquad t \in \mathbb{R}, \;\; x \in \mathbb{R}^m, \tag{3.36}$$

with bounded coefficients $a\colon \mathbb{R}\times\mathbb{R}^m \to \mathbb{R}^{m\times m}_{\ge\eta}$ and $\eta > 0$. First-order terms $b\cdot\nabla u + cu$ with bounded coefficients can be estimated by $\|u\|_{L^\infty\mathcal{H}^1}$ and thus by the energy inequality, at least locally in time (if a is regular enough).

Non-constants coefficients may prohibit global-in-time estimates as in (3.25). Locally in time, it is not difficult to control low frequencies using Bernstein estimates, as we see in Lemma 4.12. So one can pass to standard inhomogeneous

Sobolev spaces. In the proofs for smooth coefficients on $\mathbb{R}^m$ one may follow the strategy of the previous section, but one has to replace the arguments based on the Fourier transform by the sophisticated theory of Fourier integral operators. Theorem 7.5 and Remark 7.7 in [**30**] indeed imply (3.25) for solutions of (3.36) on bounded time intervals J, also assuming that $a \in C^\infty$ with bounded derivatives. The constant in (3.25) then depends on the length of J.

Global-in-time Stichartz estimates for varying coefficents need a 'nontrapping condition' and some decay of derivatives of a. We refer to [**55**], for instance, and to [**10**] for the description of a typical approach to this subject.

To treat quasilinear problems one needs rough coefficients. The methods used above do not work for non-smooth coefficients. Actually, if a possesses less than two derivatives, Strichartz estimates suffer an additional regularity loss.

Theorem 3.23. *Let $a \in C_b^\beta(\mathbb{R}\times\mathbb{R}^m, \mathbb{R}^{m\times m}_{\mathrm{sym}})$ with $a \geq \eta > 0$ and $\beta \in [0,2]$, (p,q,γ) and (r,s,θ) be admissible, and $\sigma := \frac{2-\beta}{2+\beta}$. Then the solution u of* (3.36) *satisfies*

$$\| |\bar{D}|^{1-\gamma-\frac{\sigma}{2}} u\|_{L^p_{\mathbb{R}} L^q} \leq C\big(\|\nabla_{t,x} u\|_{L^2_{\mathbb{R}} L^2} + \| |\bar{D}|^{-\sigma} f\|_{L^2_{\mathbb{R}} L^2}\big). \tag{3.37}$$

If $\beta \geq 1$ and $m \geq 3$, for $T > 0$ and $\kappa > 1$ ($\kappa = 1$ if $m \geq 4$) one obtains

$$\|\nabla_{t,x} u\|_{L^p_T \mathcal{H}^{-\gamma-\frac{\kappa\sigma}{p},q}} \leq C(T,\kappa)\big(\|u_0\|_{\mathcal{H}^1} + \|u_1\|_{L^2} + \|f\|_{L^{r'}_T \mathcal{H}^{\theta+\frac{\kappa\sigma}{r},s'}}\big). \tag{3.38}$$

See Theorem 2 and Corollary 6 in [**63**], and also Corollary 1.6 in [**64**]. The operator $|\bar{D}|^\alpha$ is defined via the Fourier transform on $\mathbb{R}^{1+m}$. With somewhat different methods the case $a \in C^{1,1}$ was treated earlier in [**52**], see also [**4**]. The regularity loss in Theorem 3.23 is sharp in general by an example in [**53**]. In the paper [**64**] variants with, e.g., $\nabla_{t,x} a \in L^1_T L^\infty$ are treated, which are needed for the study of the quasilinear problem, where $a = a(t,x,u)$ or $a = a(t,x,u,\nabla u)$.

Estimate (3.37) is global in time, provided one knows a priori that $|D_{t,x}|u$ belongs to $L^2_{\mathbb{R}} L^2$. Otherwise one has to replace u by ϕu with a cut-off in time. If $\beta \geq 1$, one can invoke the energy estimate (which involves $\|\partial_t a\|_{L^1_T L^\infty}$ if a depends on time) to bound $|\bar{D}|u$ by the data locally in time as in (3.38).

To explain the proof of Theorem 3.23 a bit, we first indicate Strichartz' approach to the homogeneous estimate for the half-wave equation in (3.28), cf. Section III.1 in [**54**]. For $f \in C_c(\mathbb{R}, L^2)$ we can write

$$\mathcal{S}f(y) = (2\pi)^{-\frac{m}{2}} \int_{\mathbb{R}^m}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}y\cdot\xi}\mathrm{e}^{-\mathrm{i}t|\xi|}\hat{f}(t,\xi)\,\mathrm{d}t\,\mathrm{d}\xi = (2\pi)^{-\frac{m-1}{2}} \int_{\mathbb{R}^m} \mathrm{e}^{\mathrm{i}y\cdot\xi}\tilde{f}(|\xi|,\xi)\,\mathrm{d}\xi$$

for the space-time Fourier transform $\tilde{f}$. Plancherel now yields

$$\|\mathcal{S}f\|^2_{L^2} = c\int_{\mathbb{R}^m} |\tilde{f}(|\xi|,\xi)|^2\,\mathrm{d}\xi = c\int_{\tau=|\xi|} |\tilde{f}|^2\,\mathrm{d}(\tau,\xi) \tag{3.39}$$

The last integral is taken over (one half of) the light cone $\{\tau = \pm|\xi|\}$ in $\mathbb{R}^{1+m}$, where omit the surface measure. One thus wants to estimate the L^2-norm of the restriction of the Fourier transform to a surface having non-zero curvature by a L^pL^q-norm of f. This topic is treated in, e.g., Section VIII.8 in [**59**].

In Theorem 3.23, one first reduces to pieces of u which are localized in (t, x) and (τ, ξ) up to error terms and to (at first C^2-) coefficients with a frequency cut-off. Instead of the space-time Fourier transform, one applies the 'FBI-transform', which maps into functions of (t, x, τ, ξ). The transformed problem is then split into a part away from the light cone $\mathcal{C}$ and the more difficult one close to it. With severe efforts, the latter is reduced to a Fourier restriction problem on $\mathcal{C}$ to which theory from [**59**] can be applied. Rougher coefficients are treated by another cut-off argument.

CHAPTER 4

Strichartz estimates for the Maxwell system

In this chapter we discuss very recent local-in-time Strichartz estimates for nonautonomous linear Maxwell equations and indicate two applications.

4.1. Introduction and the basic result

We start with an existence result and energy estimate for the Maxwell system under a bit weaker hypotheses than in Section 2.1. After a glimpse on dispersive properties, we treat properties of $L^p_{\mathbb{R}}L^q$ and related spaces. Then Strichartz estimates for the Maxwell equations with isotropic C^s-coefficients are presented. We discuss this result and some variants, and show first steps of the proof.

We study the (slightly generalized) Maxwell system

$$\begin{aligned} &\partial_t(\varepsilon E) = \operatorname{curl} H - \sigma_e E - J_e, \quad E(0) = E_0,\\ &\partial_t(\mu H) = -\operatorname{curl} E - \sigma_m H - J_m, \quad H(0) = H_0, \end{aligned} \qquad t \in J,\ x \in \mathbb{R}^3, \tag{4.1}$$

using somewhat modified notation. The time interval J of positive length $|J|$ contains 0. The unphysical magnetic 'conductivity' and 'current' will appear in our analysis later on. We further set

$$a = \operatorname{diag}(\varepsilon, \mu), \quad d = -\operatorname{diag}(\sigma_e, \sigma_m), \quad f = -\binom{J_e}{J_m}, \ u = \binom{E}{H}, \quad \rho = \binom{\rho_e}{\rho_m} = \operatorname{Div}(au)$$

with $\operatorname{Div} = \operatorname{diag}(\operatorname{div}, \operatorname{div})$. As in (1.4) one checks

$$\rho(t) = \rho(0) + \int_0^t \operatorname{Div}\big(f(\tau) + d(\tau)u(\tau)\big)\,\mathrm{d}\tau, \tag{4.2}$$

assuming (4.4) below, for instance. In our main results we actually focus on the fields $v = (D, B) = au$ which solve

$$\begin{aligned} &\partial_t D = \operatorname{curl}(\mu^{-1}B) - \sigma_e\varepsilon^{-1}D - J_e, \quad D(0) = D_0,\\ &\partial_t B = -\operatorname{curl}(\varepsilon^{-1}D) - \sigma_m\mu^{-1}B - J_m, \quad B(0) = B_0, \end{aligned} \qquad t \in J,\ x \in \mathbb{R}^3. \tag{4.3}$$

The above coefficients and data are required to satisfy

$$\begin{aligned} &\varepsilon, \mu \in L^\infty\big(J \times \mathbb{R}^3, \mathbb{R}^{3\times3}_{\geq\eta}\big), \quad \eta > 0, \quad \partial_t\varepsilon, \partial_t\mu, \sigma_i \in L^1\big(J, L^\infty(\mathbb{R}^3, \mathbb{R}^{3\times3})\big),\\ &E_0, H_0, D_0, B_0 \in L^2(\mathbb{R}^3, \mathbb{R}^3) = L^2, \quad J_i \in L^1(J, L^2), \quad i \in \{e, m\}, \end{aligned} \tag{4.4}$$

if J is bounded; otherwise one replaces $L^1(J, X)$ by $L^1_{\mathrm{loc}}(\overline{J}, X)$.

REMARK 4.1. Assume that (4.4) holds. Then there is a unique solution $u = (E, H) \in C(\overline{J}, L^2)$ of (4.1) and thus also $v = (D, B) \in C(\overline{J}, L^2)$ of (4.3). It fulfills the energy equality and estimate

$$\big(a(t)u(t)\,\big|\,u(t)\big)_{L^2} = \big(a(0)u_0\,\big|\,u_0\big)_{L^2} + \int_{J_t} \big((2d - \partial_t a)\cdot u + 2f\,\big|\,u\big)_{L^2}\,\mathrm{d}\tau, \tag{4.5}$$

$$\|u(t)\|_{L^2} \le c_0\big(\|u_0\|_{L^2} + \|f\|_{L^1(J_t,L^2)}\big) \exp \int_{J_t} c\|(d(\tau), \partial_t a(\tau))\|_{L^\infty}\, \mathrm{d}\tau \tag{4.6}$$

for $t \in \overline{J}$ with $u_0 = (E_0, H_0)$, $c_0 = c_0(\eta, \|a(0)\|_\infty)$, $J_t = (0,t)$ if $t \ge 0$, and $J_t = (t, 0)$ if $t < 0$. Here one can also allow for $\mathbb{C}$-valued data.

One can show these facts as in Theorem 2.4 and Example 2.5, using also Gronwall's inequality for (4.6). There are minor modifications, e.g., one first treats bounded J, takes $\gamma = 0$, and the operator L° in the theorem acts from $\big\{v \in W^{1,1}_T L^2 \cap L^1_T \mathcal{H}^1 \,\big|\, v(T) = 0\big\}$ to $L^1_T L^2$. $\Diamond$

In (1.13) we have seen that the isotropic autonomous Maxwell system without charges reduces to a wave system for E (and similarly for the other fields) whose components are coupled only in lower order. One obtains the basic wave equation (3.20) if $\varepsilon = 1 = \mu$, $\sigma_i = 0$, and the fields are divergence-free. So one should have the wave case in mind when treating the Maxwell system. However for the analysis of quasilinear systems one needs nonautonomous anisotropic linear systems, cf. Section 2.3, and already the presence of conductivity produces charges so that the wave case can just be a starting point. Indeed, we see below that charges and anisotropic coefficients change the behavior a lot.

For the wave equation plane waves exhibit dispersive behavior by Remark 3.12. We first discuss similar, but more complicated phenomena for simple Maxwell problems.

Remark 4.2. Let $\varepsilon, \mu \in \mathbb{R}^{3\times 3}_{>0}$ be constant and commute. Fix eigenvectors E^0 of ε and H^0 of μ with eigenvalues α_i and a wave vector $\xi \in \mathbb{R}^3$ such that $\{\xi, E^0, H^0\}$ is orthogonal with positive orientation. We then set

$$E(t,x) = \mathrm{e}^{\mathrm{i}(\omega_e t - \xi\cdot x)} E^0, \qquad H(t,x) = \mathrm{e}^{\mathrm{i}(\omega_m t - \xi\cdot x)} H^0$$

for numbers $\omega_i > 0$ and $(t,x) \in \mathbb{R}^{1+3}$. Observe that $\partial_k E_j = -\mathrm{i}\xi_k E_j$ and hence $\operatorname{div}(\varepsilon E) = -\mathrm{i}\xi \cdot \varepsilon E = -\mathrm{i}\alpha_e \mathrm{e}^{\mathrm{i}(\omega_e t - \xi\cdot x)}\, \xi \cdot E^0$ and analogously for H. As a result, the charges ρ vanish. Similarly, the Maxwell equations are equivalent to

$$\omega_e \alpha_e E^0 = \omega_e \varepsilon E^0 = -\xi \times H^0, \qquad \omega_m \alpha_m H^0 = \omega_m \mu H^0 = \xi \times E^0.$$

It remains to choose ω_i appropriately. Multiplying by E^0 and H^0, we obtain the *dispersion relations*

$$\omega_e = -\frac{(\xi \times H^0)\cdot E^0}{\varepsilon E^0 \cdot E^0} = \frac{(E^0 \times H^0)\cdot \xi}{\varepsilon E^0 \cdot E^0}, \qquad \omega_m = \frac{(\xi \times E^0)\cdot H^0}{\mu H^0 \cdot H^0} = \frac{(E^0 \times H^0)\cdot \xi}{\mu H^0 \cdot H^0}.$$

Hence, these plane waves move into the direction $E^0 \times H^0$ proportional to ξ. $\Diamond$

For scalar ε and μ, one can choose every $\xi \neq 0$ in the above construction, as for the wave equation (3.20). In the anisotropic case, the matrices ε and μ impose restrictions on the direction of ξ which hints at reduced dispersion. This effect becomes clearer in the discussion of the characteristic surface taken from Proposition 1.2 of **[39]**. Observe that the Fourier transform of curl results in

$$\mathcal{F}(\operatorname{curl} v) = \mathrm{i}\xi \times \hat{v} = \mathrm{i}\begin{pmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{pmatrix} \hat{v} =: \mathrm{i}c(\xi)\hat{v}. \tag{4.7}$$

REMARK 4.3. We consider diagonal and constant (homogeneous) coefficients $\varepsilon = \operatorname{diag}(\varepsilon_j)$ and $\mu = \operatorname{diag}(\mu_j)$ with $\varepsilon_j, \mu_j \in \mathbb{R}_+$. Inserting standing waves $E(t,x) = e^{i\tau t}E_0(x)$ and $H(t,x) = e^{i\tau t}H_0(x)$ into (4.1) with $\sigma_i = 0$ and $J_i = e^{i\tau t}J_{i,0}(x)$ with divergence free $J_{i,0}$, we obtain the *time-harmonic* system

$$i\tau\varepsilon E_0 - \operatorname{curl} H_0 = -J_{e,0}, \qquad i\tau\mu H_0 + \operatorname{curl} E_0 = -J_{m,0}, \tag{4.8}$$

on $\mathbb{R}^3$. The spatial Fourier transform then yields

$$i\tau\varepsilon\varphi - ic\psi = -h_e, \qquad ic\varphi + i\tau\mu\psi = -h_m, \qquad \xi \in \mathbb{R}^3,$$

where we set $\varphi = \mathcal{F}E_0$, $\psi = \mathcal{F}H_0$, $h_e = \mathcal{F}J_{e,0}$ and $h_m = \mathcal{F}J_{m,0}$. Applying

$$-i\begin{pmatrix} \tau\varepsilon^{-1} & \varepsilon^{-1}c\mu^{-1} \\ -\mu^{-1}c\varepsilon^{-1} & \tau\mu^{-1} \end{pmatrix}$$

to this system from the left, we arrive at two 3×3 systems

$$\left.\begin{aligned} (\tau^2 I + A_e)\varphi &:= \tau^2\varphi + \varepsilon^{-1}c\mu^{-1}c\varphi = i\tau\varepsilon^{-1}h_e + i\varepsilon^{-1}c\mu^{-1}h_m, \\ (\tau^2 I + A_m)\psi &:= \tau^2\psi + \mu^{-1}c\varepsilon^{-1}c\psi = i\tau\mu^{-1}h_m - i\mu^{-1}c\varepsilon^{-1}h_e, \end{aligned}\right. \quad \xi \in \mathbb{R}^3. \tag{4.9}$$

Observe that $A_e(\xi) =: a_e a_m$ and $A_m(\xi) = a_m a_e$ have the same characteristic polynomial $p(\tau,\xi)$. In (12) of [**39**] it is determined as

$$p(\tau,\xi) = \tau^2\big(\tau^4 - \tau^2 q_0(\xi) + q_1(\xi)\big), \qquad q_1(\xi) = \frac{\varepsilon\xi\cdot\xi}{\det\varepsilon}\,\frac{\mu\xi\cdot\xi}{\det\mu},$$

$$q_0(\xi) = \xi_1^2\Big(\frac{1}{\varepsilon_2\mu_3} + \frac{1}{\varepsilon_3\mu_2}\Big) + \xi_2^2\Big(\frac{1}{\varepsilon_1\mu_3} + \frac{1}{\varepsilon_3\mu_1}\Big) + \xi_3^2\Big(\frac{1}{\varepsilon_1\mu_2} + \frac{1}{\varepsilon_2\mu_1}\Big).$$

One can solve (4.9), and thus (4.8) via $\mathcal{F}^{-1}$, if ξ does not belong to the characteristic set $\mathcal{C}_\tau = \{\xi \mid p(\tau,\xi) = 0\}$ for $\tau \neq 0$. In the 'fully anisotropic' case where all $\frac{\varepsilon_j}{\mu_j}$ differ, by §3 of [**39**] the *Fresnel surface* $\mathcal{C}_\tau$ contains four singularities and four curves with one non-zero principal curvature. In the 'partially anisotropic' case $\mu_1 = \mu_2 = \mu_3$ and $\varepsilon_1 = \varepsilon_2 \neq \varepsilon_3$, the set $\mathcal{C}_\tau$ consists of two ellipsoids touching at two points, see §2.3 of [**43**]. In crystal optics these cases are called 'biaxial', resp. 'uniaxial'. If also ε is scalar, $\mathcal{C}_\tau$ is a (doubly sheeted') sphere. $\Diamond$

Hence the anisotropy of the coefficients drastically changes the properties of the characteristic surface (or light cone). In view of the comments at the end of the previous section, one can expect weaker dispersive porperties in this case. This is in fact true, as reported next.

REMARK 4.4. Let $\varepsilon = \operatorname{diag}(\varepsilon_j)$ and $\mu = \operatorname{diag}(\mu_j)$ be fully anisotropic as in the previous remark, $u_0 \in W^{k,1}$ for a sufficiently large $k \in \mathbb{N}$, $f = 0 = d$ and $\operatorname{Div}(au) = 0$. The solution to (4.1) then satisfies $\|u(t)\|_\infty \le ct^{-\frac{1}{2}}\|u_0\|_{k,1}$ for $t > 0$ due to Theorem 1.3 in [**35**]. This corresponds to (3.34) for the wave equation on $\mathbb{R}^2$ only! In the partially anisotropic case one recovers the 3D decay t^{-1}, see item (4) in [**35**]. $\Diamond$

The previous two remarks strongly indicate that the dispersive behavior of the anisotropic Maxwell system is considerably harder to study. Indeed, here the available Strichartz estimates are restricted to special cases, exhibit additional regularity loss, and in the fully anisotropic case the triples are wave admissible only for $m = 2$. See [**43**] for the partially anisotropic and [**49**] for the fully anisotropic case. One has 3×3 Maxwell systems on $\mathbb{R}^2$ for polarized fields, see (4.69). In [48] the results from the wave case are recovered here. In these papers local-in-time estimates for (t,x)-depending coefficients are treated. Below we focus on the isotropic case on $\mathbb{R}^3$, where we also find results as for (3.36).

In our arguments we use the harmonic analysis tools on space-time $\mathbb{R}^{1+3}$ and not only on $\mathbb{R}^3$. To distinguish from the $\mathbb{R}^3$-case, we overline symbols; i.e.,

$$\bar{x}=(\bar{x}_j)_{j=0}^3=(t,x),\quad \bar{\xi}=(\tau,\xi),\quad \bar{\mathcal{F}}=\mathcal{F}_{\bar{x}},\quad |\bar{D}|=\bar{\mathcal{F}}^{-1}|\bar{\xi}|\bar{\mathcal{F}},\quad \bar{P}_j=\bar{\mathcal{F}}^{-1}\bar{\chi}_j\bar{\mathcal{F}},$$

$\bar{\nabla}=\nabla_{\bar{x}}$, and so on. When using nonhomogeneous Littlewood–Paley decompositions $(\bar{P}_j)_{j\in\mathbb{N}_0}$ and $(P_j)_{j\in\mathbb{N}_0}$, we redefine $\bar{P}_{\le 0}$ and $P_{\le 0}$ as $\bar{P}_0$ and P_0, respectively. We further write $\bar{P}_j'$ and $\bar{P}_j''$ instead of $\tilde{\bar{P}}_j$ and $\hat{\bar{P}}_j$ for the enlarged operators. Fortunately, the relevant results from L^q remain valid in $L^p_{\mathbb{R}}L^q$. We start with Sobolev embeddings and Bernstein estimates, before collecting deeper results.

LEMMA 4.5. *Let* $\alpha-\frac{m}{q}-\frac{1}{p}=-\frac{m}{s}-\frac{1}{r}$, $\alpha\in(0,m+1)$, $1<p<r<\infty$, $1\le q\le s<\infty$, *and* $\beta\in\mathbb{R}$. *We then obtain* $\||\bar{D}|^{\beta-\alpha}g\|_{L^r_{\mathbb{R}}L^s}\lesssim\||\bar{D}|^{\beta}g\|_{L^p_{\mathbb{R}}L^q}$ *if* $|\bar{D}|^{\beta}g\in L^p_{\mathbb{R}}L^q$. *and analogously for* $\langle\bar{D}\rangle$.

PROOF. It suffices to treat $\beta=0$ by isomorphisms. We have $|\bar{D}|^{-\alpha}g=c_\alpha|\bar{x}|^{\alpha-m-1}*g$ by p.10 in [24], where we may assume that $g\in\mathcal{S}_0$ by density. The assumptions imply $\frac{1}{\sigma}:=1+\frac{1}{s}-\frac{1}{q}\in(0,1]$, $\frac{1}{\rho}:=1+\frac{1}{r}-\frac{1}{p}\in(0,1)$, and

$$m+1-\alpha=m\big(1+\tfrac{1}{s}-\tfrac{1}{q}\big)+1+\tfrac{1}{r}-\tfrac{1}{p}=\tfrac{m}{\sigma}+\tfrac{1}{\rho}>\tfrac{m}{\sigma}.$$

Using Minkowski, Young and a transformation, we deduce

$$\begin{aligned}
\|(|\bar{D}|^{-\alpha}g)(t,\cdot)\|_{L^s}&\lesssim\int_{\mathbb{R}}\Big(\int_{\mathbb{R}^m}\Big(\int_{\mathbb{R}^m}|\bar{x}-\bar{y}|^{\alpha-m-1}|g(\tau,y)|\,\mathrm{d}y\Big)^s\,\mathrm{d}x\Big)^{\frac{1}{s}}\,\mathrm{d}\tau\\
&\le\int_{\mathbb{R}}\|g(\tau,\cdot)\|_{L^q}\Big(\int_{\mathbb{R}^m}|(t-\tau,z)|^{\sigma(\alpha-m-1)}\,\mathrm{d}z\Big)^{\frac{1}{\sigma}}\,\mathrm{d}\tau\\
&=\int_{\mathbb{R}}|t-\tau|^{\alpha-m-1}\|g(\tau,\cdot)\|_{L^q}\Big(\int_{\mathbb{R}^m}\big|\big(1,\tfrac{z}{t-\tau}\big)\big|^{\sigma(\alpha-m-1)}\,\mathrm{d}z\Big)^{\frac{1}{\sigma}}\,\mathrm{d}\tau\\
&=\int_{\mathbb{R}}|t-\tau|^{-\frac{1}{\rho}-\frac{m}{\sigma}+\frac{m}{\sigma}}\|g(\tau,\cdot)\|_{L^q}\Big(\int_{\mathbb{R}^m}|(1,\zeta)|^{\sigma(\alpha-m-1)}\,\mathrm{d}\zeta\Big)^{\frac{1}{\sigma}}\,\mathrm{d}\tau.
\end{aligned}$$

Since the last integral is finite, Lemma 3.22 for exponents $1+\frac{1}{r}=\frac{1}{\rho}+\frac{1}{p}$ then yields the first assertion.

The second claim can be treated analogously since $\langle\bar{D}\rangle^{-\alpha}$ is a convolution with kernel bounded by $c_\alpha'\big(|\bar{x}|^{\alpha-m-1}+\mathrm{e}^{-|\bar{x}|/2}\big)$ due to Proposition 1.2.5 in [24]. $\square$

REMARK 4.6. Let $1\le\bar{p}\le p\le\infty$, $1\le\bar{q}\le q\le\infty$, $\lambda,r,r_1>0$, $r_2>r_1$, $v\in L^p_{\mathbb{R}}L^q$, and $s\in\mathbb{R}$. Applying Young twice, we see that the convolution with $\psi\in L^\rho_{\mathbb{R}}L^\sigma$ maps $L^{\bar{p}}_{\mathbb{R}}L^{\bar{q}}$ into $L^p_{\mathbb{R}}L^q$ continuously if $1+\frac{1}{p}=\frac{1}{\rho}+\frac{1}{\bar{p}}$ and $1+\frac{1}{q}=\frac{1}{\sigma}+\frac{1}{\bar{q}}$. So the following facts can be shown as in Lemma 3.3 (with implicit constants independent of λ and v).

a) $\operatorname{supp}\bar{\mathcal{F}}v\subseteq A(\lambda r_1,\lambda r_2)$ yields $\lambda^s\|v\|_{L^p_{\mathbb{R}}L^q}\lesssim\||\bar{D}|^s v\|_{L^p_{\mathbb{R}}L^q}\lesssim\lambda^s\|v\|_{L^p_{\mathbb{R}}L^q}$.

b) $\operatorname{supp}\bar{\mathcal{F}}v\subseteq\overline{B}(0,\lambda r)$ yields $\||\bar{D}|^s v\|_{L^p_{\mathbb{R}}L^q}\lesssim\lambda^{s+\frac{m}{\bar{q}}+\frac{1}{\bar{p}}-\frac{m}{q}-\frac{1}{p}}\|v\|_{L^{\bar{p}}_{\mathbb{R}}L^{\bar{q}}}$ if $s\le 0$. $\lozenge$

Remark 4.7. Let $1 < p, q < \infty$. The Littlewood–Paley Theorems 3.5 and 3.6 remain true for $L^p_{\mathbb{R}} L^q$ because of Theorem 5.5.22 in [**26**] and Theorems 6.2.4 and 7.2.13 in [**27**]. One then shows Proposition 3.8 and 3.9 also within these spaces. Moreover, as in Theorem 2.4.2.1 of [**69**] one can derive interpolation properties for the spaces $|\bar{D}|^\alpha L^p_{\mathbb{R}} L^q$ (with norm $\||\bar{D}|^\alpha v\|_{L^p_{\mathbb{R}} L^q}$) and $\langle \bar{D}\rangle^\alpha L^p_{\mathbb{R}} L^q$ as those for $\dot{\mathcal{H}}^{\alpha,q}$, resp. $\mathcal{H}^{\alpha,q}$, where $\alpha \in \mathbb{R}$. As in Section 3.1 one sees that $\langle \bar{D}\rangle^{-\alpha} L^{p'} L^{q'}$ is the dual space of $\langle \bar{D}\rangle^\alpha L^p L^q$, and analogously for $|\bar{D}|$. Corollary 8.3.22 and Example 8.1.9 in [**27**] yield Mikhlin's theorem with the condition $\xi^\beta \partial^\beta a \in L^\infty$ for all $\beta \in \{0,1\}^m$. This result implies $\|\langle \bar{D}\rangle^{-\alpha} v\|_{L^p_{\mathbb{R}} L^q} \lesssim \||\bar{D}|^{-\alpha} v\|_{L^p_{\mathbb{R}} L^q}$ for $\alpha > 0$. The converse is true if $\operatorname{supp} \bar{\mathcal{F}} v \subseteq \mathbb{C} \setminus B(0,\delta)$ for some $\delta > 0$. $\Diamond$

The next theorem invokes the seminorm $\|v\|_{\dot{C}^s}$ which is the highest-order part of the norm in $C^s_b(\mathbb{R}^{1+m})$ for $s \geq 0$. For $p, q, r \in [1,\infty]$ and $s \in \mathbb{R}$ we use the space-time Besov spaces $\dot{B}^s_{p,q,r}$ and $B^s_{p,q,r}$ of distributions $\varphi_0 \in \mathcal{S}^\star_0(\mathbb{R}^{1+m})$ and $\varphi \in \mathcal{S}^\star(\mathbb{R}^{1+m})$ with finite norms given by

$$\|\varphi_0\|^r_{\dot{B}^s_{p,q,r}} = \sum\nolimits_{j\in\mathbb{Z}} 2^{rsj} \|\bar{P}_j \varphi_0\|^r_{L^p_{\mathbb{R}} L^q}, \qquad \|\varphi\|^r_{B^s_{p,q,r}} = \sum\nolimits_{j\in\mathbb{N}_0} 2^{rsj} \|\bar{P}_j \varphi\|^r_{L^p_{\mathbb{R}} L^q},$$

respectively, for $r < \infty$, and similar for $r = \infty$. One can interpolate between these spaces as for standard Besov spaces in Theorem 2.4.2 of [**68**]. Recall that wave admissible triples for $m = 3$ satisfy

$$p, q \in [2,\infty], \quad \tfrac{1}{p} + \tfrac{1}{q} \leq \tfrac{1}{2}, \quad \tfrac{3}{2} - \gamma = \tfrac{1}{p} + \tfrac{3}{q}, \quad (p,q,\gamma) \neq (2,\infty,1), \tag{4.10}$$

and that the triple is strict if the inequality is an identity.

We now state Theorems 1.1 and 1.2 of [**43**], writing $Lv = (\partial_t + M a^{-1})v = f$ for (4.3). These results are proved in this and the next two sections. The general strategy of the proof originates from [**63**]. We only aware of one earlier (local-in-time) Strichartz estimate for the Maxwell system from [**18**], in the charge-free case and for smooth scalar coefficients being constant outside a compact set.

Theorem 4.8. *Let $\varepsilon, \mu \in C^s_b(\mathbb{R} \times \mathbb{R}^3, \mathbb{R})$ with $\varepsilon, \mu \geq \eta$ for some $\eta \in (0,1]$ and $s \in (0,2]$, $\sigma := \frac{2-s}{2+s}$, $\sigma_e = 0 = \sigma_m$, (p,q,γ) be admissible, but $(p,q,\gamma) \neq (\infty, 2, 0)$, and $v \in L^2_{\mathbb{R}} L^2$. Set $Lv = f$ and $\rho = \operatorname{Div} v$. Then v satisfies*

$$\||\bar{D}|^{-\gamma-\frac{\sigma}{2}} v\|_{L^p_{\mathbb{R}} L^q} \lesssim \kappa \|v\|_{L^2_{\mathbb{R}} L^2} + \tfrac{1}{\kappa} \||\bar{D}|^{-\sigma} f\|_{L^2_{\mathbb{R}} L^2} + \||\bar{D}|^{-\frac{1}{2}-\frac{\sigma}{2}} \rho\|_{L^2_{\mathbb{R}} L^2}, \tag{4.11}$$

$$\|\langle \bar{D}\rangle^{-\gamma-\frac{\sigma}{2}} v\|_{L^p_{\mathbb{R}} L^q} \lesssim \kappa \|v\|_{L^2_{\mathbb{R}} L^2} + \tfrac{1}{\kappa} \|\langle \bar{D}\rangle^{-\sigma} f\|_{L^2_{\mathbb{R}} L^2} + \|\langle \bar{D}\rangle^{-\frac{1}{2}-\frac{\sigma}{2}} \rho\|_{L^2_{\mathbb{R}} L^2}, \tag{4.12}$$

if the terms on the right are finite and $\|(\varepsilon,\mu)\|_{\dot{C}^s} \leq \kappa^{2+s}$ with $\kappa \gtrsim 1$ for (4.12). If $p = \infty$ and $q > 2$ or if $q = \infty$, one has to replace L^q by $\dot{B}^0_{p,q,2}$, resp., $B^0_{p,q,2}$.

By the assumptions Lv and ρ belong to $\dot{\mathcal{H}}^{-1}(\mathbb{R}^4)$. Note that ρ is given by the data via (4.2) and that $\|a^{-1}\|_{C^s_b} \lesssim \|a\|_{C^s_b}$. By Lemma 4.12, we can mainly restrict the reasoning to the inhomogeneous norms in (4.12). One deduces the case $s < 2$ from $s = 2$ using the control on $\|(\varepsilon,\mu)\|_{\dot{C}^s}$ in the estimates, see Lemma 4.19. The above Strichartz estimates are the versions closest to the core of the proof, where we work on $\mathbb{R}^{1+3}$ and the L^2-setting on the right is very convenient. The occuring error terms will end up in $\|v\|_{L^2 L^2}$. In Corollary 4.25 we treat a local-in-time variant which avoids (non-causal!) fractional time-derivatives. We discuss the results in several remarks and lemmas.

REMARK 4.9. a) There is a unique solution $v \in C_{\mathbb{R}}L^2$ of (4.3) if (4.4) is true, e.g. if $s \geq 1$, see Remark 4.1. In this case, by density it is enough to consider $f \in \mathcal{S}_0(\mathbb{R}^4)$ and $v_0 \in \mathcal{S}_0(\mathbb{R}^3)$, respectively $f \in \mathcal{S}(\mathbb{R}^4)$ and $v_0 \in \mathcal{S}(\mathbb{R}^3)$.

b) If $f = 0$ and ε and μ do not depend on time, the solution v to (4.3) does not belong to $L^2_{\mathbb{R}}L^2$ since $\|v(t)\|_2 \approx \|v_0\|$ by (4.5). Actually, one should consider Theorem 4.8 as a local-in-time result. Assume that (4.4) holds and that (after a cut-off in time) the solution has compact support I in time containing 0. Then the energy estimate (4.6) allows to bound $\|v\|_{L^2_I L^2}$ by $c(|I|)\big(\|v_0\|_{L^2} + \|f\|_{L^1_I, L^2}\big)$.

c) The cases $p = \infty$ and $q = \infty$ can only occur for non-strict triples, except for the energy triple $(\infty, 2, 0)$ which has already been treated in (4.6). In Lemma 4.11 we reduce to the strict case, so that infinite exponents rarely occur later on. In the rest of this remark we let $p, q < \infty$ to simplify.

d) The regularity loss $\frac{\sigma}{2}$ on the left is sharp for the corresponding result on $\mathbb{R}^2$ by [**48**]. It varies between $\sigma = 0$ if $s = 2$ and $\sigma = 1$ if $s = 0$ (where (4.11) and (4.12) directly follow from Lemma 4.5, cf. the next item). We have $\sigma = \frac{1}{3}$ for the case $s = 1$ which is used to treat quasilinear problems, see [**43**], [**48**], [**49**], and also [**64**] for the wave case.

e) Let $s = 2$. The Sobolev embedding in Lemma 4.5 shows that the left-hand side of (4.11) is bounded by $\||\bar{D}|^{1/2}v\|_{L^2_{\mathbb{R}}L^2}$, and analogously for (4.12). The Strichartz estimates in the theorem thus improve on Sobolev by half a derivative – compared to a gain of $\frac{1}{p}$ derivatives in Theorem 3.13.

f) Note that $\|\operatorname{Div} v\|_{\dot{\mathcal{H}}^{-1}(\mathbb{R}^4)}$ is bounded by $\|v\|_{L^2_{\mathbb{R}}L^2}$ a priori. In the Strichartz estimates with $s = 2$ we require that the charge is a half derivative better, in accordance with statement e). In contrast to the wave case, we need a condition on the charges since initial data $v_0 = a(0)(\nabla\varphi_e, \nabla\varphi_m) \in L^2$ yield equilibria of the Maxwell system (with σ_i, J_i, $\partial_t a$ being 0). Since $-3/2$ is smaller than $-\gamma - 3/q$ by admissibility, there will be functions φ_i such that v_0 does not belong to $\dot{\mathcal{H}}^{-\gamma,q}$ (which is left invariant by $a(0)^{-1}$ for $s = 2$). In other words, the charge condition is needed to control the huge kernel of curl.

g) One can easily extend (4.12) to non-zero $\sigma_i \in C^s_b(\mathbb{R}^4)$ if $\sigma < s$. To this aim, redefine f as $f + da^{-1}v$, which does not change ρ by (4.2). We can then estimate $\|da^{-1}v\|_{\mathcal{H}^{-\sigma}}$ by $\|v\|_{L^2_{\mathbb{R}}L^2}$. Here one has to use the product estimate Proposition 3.8, Remark 3.10, and the duality of $\mathcal{H}^{\sigma}(\mathbb{R}^4)$ and $\mathcal{H}^{-\sigma}(\mathbb{R}^4)$. $\Diamond$

The Strichartz estimate (4.11) is scaling invariant which will be important for the proof. Using this fact, we first restrict to the case $\kappa = 1$.

LEMMA 4.10. *In Theorem 4.8 for* (4.11) *it is enough to take* (ε, μ) *with* $\kappa = 1$.

PROOF. Let (4.11) be true for $\kappa = 1$. We assume that $p, q < \infty$. The other case is shown similarly. We set $v_\lambda(t, x) = v(\lambda t, \lambda x)$ for $\lambda > 0$ and analogously for the other maps. Let L_λ be the operator for $a_\lambda = (\varepsilon_\lambda, \mu_\lambda)$. Note $\operatorname{Div} v_\lambda = \lambda\rho_\lambda$, $L_\lambda v_\lambda = \lambda f_\lambda$, and $\|a_\lambda\|_{\dot{C}^s} = \lambda^s\|a\|_{\dot{C}^s} \leq \lambda^s\kappa^{2+s}$. We choose $\lambda = \kappa^{-\frac{2+s}{s}}$ to obtain $\|a_\lambda\|_{\dot{C}^s} \leq 1$. Equation (3.6), estimate (4.11) for v_λ, and (4.10) imply

$$\begin{aligned}\||\bar{D}|^{-\gamma-\frac{\sigma}{2}}v\|_{L^p_{\mathbb{R}}L^q} &= \lambda^{\frac{1}{p}+\frac{3}{q}}\|(|\bar{D}|^{-\gamma-\frac{\sigma}{2}}v)_\lambda\|_{L^p_{\mathbb{R}}L^q} = \lambda^{\frac{1}{p}+\frac{3}{q}+\gamma+\frac{\sigma}{2}}\||\bar{D}|^{-\gamma-\frac{\sigma}{2}}v_\lambda\|_{L^p_{\mathbb{R}}L^q} \\ &\lesssim \lambda^{\frac{3}{2}+\frac{\sigma}{2}}\big(\|v_\lambda\|_{L^2_{\mathbb{R}}L^2} + \||\bar{D}|^{-\sigma}\lambda f_\lambda\|_{L^2_{\mathbb{R}}L^2} + \||\bar{D}|^{-\frac{1}{2}-\frac{\sigma}{2}}\lambda\rho_\lambda\|_{L^2_{\mathbb{R}}L^2}\big)\end{aligned}$$

$$= \lambda^{\frac{\sigma}{2}-\frac{1}{2}} \|v\|_{L^2_{\mathbb{R}} L^2} + \lambda^{-\frac{\sigma}{2}+\frac{1}{2}} \| |\bar{D}|^{-\sigma} f\|_{L^2_{\mathbb{R}} L^2} + \| |\bar{D}|^{-\frac{1}{2}-\frac{\sigma}{2}} \rho\|_{L^2_{\mathbb{R}} L^2}.$$

Since $1-\sigma = \frac{2s}{2+s}$, the definition of λ leads to (4.11) for v. □

We next reduce the reasoning to strict triples by Sobolev's embedding.

LEMMA 4.11. *In Theorem 4.8 it suffices to show the estimates for strict triples.*

PROOF. We focus on (4.11), as (4.12) is treated similarly. By Lemma 4.10 we may assume $\kappa = 1$, where $\kappa \geq 1$ would be sufficient. Let (p, q, γ) be non-strict admissible. Choose $\bar{p} \in (2, p)$ and $\bar{q} \in [2, q]$ with $\bar{q} < \infty$ and

$$\tfrac{1}{\bar{p}} + \tfrac{1}{\bar{q}} = \tfrac{1}{2} \qquad \text{and set} \quad \bar{\gamma} = \tfrac{3}{2} - \tfrac{1}{\bar{p}} - \tfrac{3}{\bar{q}} < \gamma,$$

so that $(\bar{p}, \bar{q}, \bar{\gamma})$ is strict admissible. Let (4.11) be true for $(\bar{p}, \bar{q}, \bar{\gamma})$.

First, let $p, q < \infty$. Since $\gamma - \bar{\gamma} - \frac{3}{\bar{q}} - \frac{1}{\bar{p}} = -\frac{3}{q} - \frac{1}{p}$ by (4.10), Lemma 4.5 yields

$$\| |\bar{D}|^{-\gamma-\frac{\sigma}{2}} v\|_{L^p_{\mathbb{R}} L^q} \lesssim \| |\bar{D}|^{-\bar{\gamma}-\frac{\sigma}{2}} v\|_{L^{\bar{p}}_{\mathbb{R}} L^{\bar{q}}}$$

so that (4.11) for $(\bar{p}, \bar{q}, \bar{\gamma})$ implies the statement.

Next, let $p = \infty$ or $q = \infty$. Remark 4.6 and the above relations yield

$$2^{-(\gamma+\frac{\sigma}{2})j} \|\bar{P}_j v\|_{L^p_{\mathbb{R}} L^q} \lesssim 2^{-(\bar{\gamma}+\frac{\sigma}{2})j} \|\bar{P}_j v\|_{L^{\bar{p}}_{\mathbb{R}} L^{\bar{q}}} \lesssim 2^{(\frac{1}{2}-\frac{\sigma}{2})j} \|\bar{P}_j v\|_{L^2_{\mathbb{R}} L^2}$$

for $j \in \mathbb{Z}$. Using Remark 4.6, $\sigma < 1$ and the L^2-boundedness of $\bar{P}_j$, we deduce

$$\begin{aligned} \| |\bar{D}|^{-\gamma-\frac{\sigma}{2}} v\|^2_{\dot{B}^0_{p,q,2}} &\eqsim \|v\|^2_{\dot{B}^{-\gamma-\frac{\sigma}{2}}_{p,q,2}} \lesssim \sum_{j<j_0} 2^{(1-\sigma)j} \|\bar{P}_j v\|^2_{L^2_{\mathbb{R}} L^2} + \sum_{j \geq j_0} 2^{-2(\bar{\gamma}+\frac{\sigma}{2})j} \|\bar{P}_j v\|^2_{L^{\bar{p}}_{\mathbb{R}} L^{\bar{q}}} \\ &\lesssim_{j_0} \|v\|^2_{L^2_{\mathbb{R}} L^2} + \sum\nolimits_{j \geq j_0} 2^{-2(\bar{\gamma}+\frac{\sigma}{2})j} \|\bar{P}_j v\|^2_{L^{\bar{p}}_{\mathbb{R}} L^{\bar{q}}} \end{aligned}$$

for any $j_0 \in \mathbb{Z}$. For $s = 2$ we will see later that

$$2^{-2(\bar{\gamma}+\frac{\sigma}{2})j} \|\bar{P}_j v\|^2_{L^{\bar{p}}_{\mathbb{R}} L^{\bar{q}}} \lesssim \|\bar{P}_j v\|^2_{L^2_{\mathbb{R}} L^2} + 2^{-2\sigma j} \|\bar{P}_j f\|^2_{L^2_{\mathbb{R}} L^2} + 2^{-2(\sigma+\frac{1}{2})j} \|\bar{P}_j \rho\|^2_{L^2_{\mathbb{R}} L^2}$$

for all large $j \geq 0$, cf. (4.16). So the Littlewood–Paley decomposition in $L^2_{\mathbb{R}} L^2$ yields the assertion. For $s < 2$ one obtains this estimate plus a term whose square sum is bounded by $\|v\|^2_{L^2_{\mathbb{R}} L^2}$, cf. (4.20) and (4.21). □

In our proof we will work with frequency-localized pieces of v, see (4.16), and the next result will allow us to restrict to large $\lambda = 2^j$.

LEMMA 4.12. *In Theorem 4.8 estimate (4.12) implies (4.11), and conversely if $s = 2$. In (4.11) and (4.12) it suffices to take $\bar{P}_{\geq k} v$ on the left for a $k \in \mathbb{N}_0$.*

PROOF. Let $k \in \mathbb{N}_0$. Thanks to Lemmas 4.10 and 4.11, we may assume that $\kappa = 1$ for (4.11) and that (p, q, γ) is strict and hence $p, q < \infty$. Lemma 4.5, admissibility, and Remark 4.6 yield

$$\| |\bar{D}|^{-\gamma-\frac{\sigma}{2}} \bar{P}_{<k} v\|_{L^p_{\mathbb{R}} L^q} \lesssim \| |\bar{D}|^{\frac{1}{2}-\frac{\sigma}{2}} \bar{P}_{<k} v\|_{L^2_{\mathbb{R}} L^2} \lesssim_k \|\bar{P}_{<k} v\|_{L^2_{\mathbb{R}} L^2} \lesssim \|v\|_{L^2_{\mathbb{R}} L^2}$$

since $\sigma < 1$. The same works with $\langle \bar{D} \rangle$ and $\kappa \|v\|_{L^2_{\mathbb{R}} L^2}$. So the last claim holds.

Observe that $\| |\bar{D}|^{-\gamma-\frac{\sigma}{2}} \bar{P}_{\geq k} v\|_{L^p_{\mathbb{R}} L^q} \eqsim_k \| \langle \bar{D} \rangle^{-\gamma-\frac{\sigma}{2}} \bar{P}_{\geq k} v\|_{L^p_{\mathbb{R}} L^q}$ due to Mikhlin, cf. Remark 4.7. Assuming (4.12), we then deduce

$$\| |\bar{D}|^{-\gamma-\frac{\sigma}{2}} \bar{P}_{\geq k} v\|_{L^p_{\mathbb{R}} L^q} \lesssim \|\bar{P}_{\geq k} v\|_{L^2_{\mathbb{R}} L^2} + \|f\|_{\mathcal{H}^{-\sigma}} + \|L \bar{P}_{<k} v\|_{\mathcal{H}^{-\sigma}} + \|\bar{P}_{\geq k} \rho\|_{\mathcal{H}^{-\frac{1}{2}-\frac{\sigma}{2}}}.$$

On the right we can use that the operators $\bar{P}_{\geq k}$ are uniformly bounded and $\dot{\mathcal{H}}^{-\alpha}(\mathbb{R}^4) \hookrightarrow \mathcal{H}^{-\alpha}(\mathbb{R}^4)$ for $\alpha > 0$. The remaining term is estimated via

$$\begin{aligned} \||\bar{D}|^{1-\sigma}(a^{-1}\bar{P}_{<k}v)\|_{L^2_{\mathbb{R}}L^2} &\lesssim \||\bar{D}|^{1-\sigma}a^{-1}\|_\infty \|\bar{P}_{<k}v\|_2 + \|a^{-1}\|_\infty \||\bar{D}|^{1-\sigma}\bar{P}_{<k}v\|_2 \\ &\lesssim_k \|a\|_{C^s_b} \|\bar{P}_{<k}v\|_{L^2_{\mathbb{R}}L^2} \lesssim \|v\|_{L^2_{\mathbb{R}}L^2}, \end{aligned}$$

by means of Proposition 3.8, Remark 3.10 and $1-\sigma < s$. (Recall that $\|a^{-1}\|_{C^s_b} \lesssim \|a\|_{C^s_b}$.) Altogether (4.11) follows.

The converse is proven in the same way for $s = 2$, using that $\||D|^{-\sigma}f\|_{L^2_{\mathbb{R}}L^2} = \|f\|_{L^2_{\mathbb{R}}L^2}$ in this case. □

We can now show two variants of Theorem 4.8. In the first one, regularity in the Strichartz estimates is shifted up to a level given by s.

Remark 4.13. Let $\alpha \in (0, s+\sigma-1)$ or $\alpha \in (0,1]$ if $s=2$. Then (4.12) implies

$$\|\langle\bar{D}\rangle^{\alpha-\gamma-\frac{\sigma}{2}}v\|_{L^p_{\mathbb{R}}L^q} \lesssim \kappa\|v\|_{\mathcal{H}^\alpha(\mathbb{R}^4)} + \tfrac{1}{\kappa}\|f\|_{\mathcal{H}^{\alpha-\sigma}(\mathbb{R}^4)} + \|\rho\|_{\mathcal{H}^{\alpha-\frac{1}{2}-\frac{\sigma}{2}}(\mathbb{R}^4)}. \tag{4.13}$$

The converse implication is true for $\alpha < s+\sigma-1$.

Proof. If $s = 2$ and $\alpha = 1$, we differentiate (4.3) in $\bar{x} = (t,x)$ obtaining the equation for $\bar{\nabla}v$ with right-hand side $\bar{\nabla}f + M(\bar{\nabla}a^{-1}v)$ and charge $\bar{\nabla}\rho$. In $L^2_{\mathbb{R}}L^2$ the extra term is bounded by $\|v\|_{L^2_{\mathbb{R}}\mathcal{H}^1}$ as $a \in C^2_b$. Thus, (4.12) and $\kappa \geq 1$ lead to

$$\|\langle\bar{D}\rangle^{-\gamma}\bar{\nabla}v\|_{L^p_{\mathbb{R}}L^q} \lesssim \kappa\|v\|_{\mathcal{H}^1(\mathbb{R}^4)} + \tfrac{1}{\kappa}\|f\|_{\mathcal{H}^1(\mathbb{R}^4)} + \|\rho\|_{\mathcal{H}^{\frac{1}{2}}(\mathbb{R}^4)}.$$

Using Mikhlin and (3.3), we can replace $\langle\bar{D}\rangle^{-\gamma}\bar{\nabla}$ by $\langle\bar{D}\rangle^{1-\gamma}$ and deduce (4.13) in this case. Let $\alpha < s+\sigma-1$. Estimate (4.12) for $\tilde{v} = \langle\bar{D}\rangle^\alpha v$ yields

$$\begin{aligned} \|\langle\bar{D}\rangle^{\alpha-\gamma-\frac{\sigma}{2}}v\|_{L^p_{\mathbb{R}}L^q} &\lesssim \kappa\|v\|_{\mathcal{H}^\alpha(\mathbb{R}^4)} + \tfrac{1}{\kappa}\|f\|_{\mathcal{H}^{\alpha-\sigma}(\mathbb{R}^4)} + \|\rho\|_{\mathcal{H}^{\alpha-\frac{1}{2}-\frac{\sigma}{2}}(\mathbb{R}^4)} \\ &\quad + \|\langle\bar{D}\rangle^{1-\sigma}[a^{-1}, \langle\bar{D}\rangle^\alpha]v\|_{L^2_{\mathbb{R}}L^2}. \end{aligned}$$

The last term can be rewritten as $[\langle\bar{D}\rangle^{1-\sigma}, a^{-1}]\langle\bar{D}\rangle^\alpha v + [a^{-1}, \langle\bar{D}\rangle^{1-\sigma+\alpha}]v$. If $1-\sigma+\alpha = 1$ we replace σ by some $\sigma' \in (1+\alpha-s, \sigma)$. The first commutator is L^2-bounded by Proposition 1.2 in **[67]**, and the same is true for the second if $1-\sigma+\alpha < 1$. In the case $1-\sigma+\alpha > 1$, we instead use Theorem 1.4 of **[42]** to obtain boundednes from $\mathcal{H}^{\alpha-\sigma}$ to L^2. The restriction on α is needed for the second commutator.

The converse is shown similarly, starting from $\langle\bar{D}\rangle^{-\gamma-\frac{\sigma}{2}}v = \langle\bar{D}\rangle^{\alpha-\gamma-\frac{\sigma}{2}}\langle\bar{D}\rangle^{-\alpha}v$ and estimating the commutator term $\|[\langle\bar{D}\rangle^\alpha, a^{-1}]\langle\bar{D}\rangle^{-\alpha}v\|_{\mathcal{H}^{1-\sigma}(\mathbb{R}^4)}$. □

We can also pass from the fields (D,B) back to (E,H).

Remark 4.14. Let $\gamma < s$, $1 < s + \frac{\sigma}{2}$ and $p,q < \infty$ (which hold for $s \geq 1$ and strict triples) for (4.12), and $\gamma + \frac{\sigma}{2} < s$ and $p,q < \infty$ for (4.11). In Theorem 4.8 we can then replace $v = (D,B)$ by $u = (E,H)$ which solves (4.1).

Proof. Remark 4.13 also works for u, so that it is enough to show (4.13) for u and $\alpha = \frac{\sigma}{2}$ noting that $\frac{\sigma}{2} < s+\sigma-1$ by the assumption. Using Remarks 3.10 and 4.7, $a \in C^s_b$, and $\frac{\sigma}{2}, \gamma < s$, we see that the multiplication with a or a^{-1} leaves invariant $\langle\bar{D}\rangle^{\frac{\sigma}{2}}L^2_{\mathbb{R}}L^2$, $\langle\bar{D}\rangle^\gamma L^{p'}L^{q'}$, and thus $\langle\bar{D}\rangle^{-\gamma}L^pL^q$ by duality. These facts show the result for (4.12) since f and ρ are the same in (4.1) and (4.3).

It is not clear that muliplication leaves invariant $|\bar{D}|^{\gamma}L^{p'}L^{q'}$. But we can argue as in Lemma 4.12. First, again one obtains $\| |\bar{D}|^{-\gamma-\frac{\sigma}{2}}P_{<0}u\|_{L^p_{\mathbb{R}}L^q} \lesssim \|u\|_{L^2_{\mathbb{R}}L^2}$. Next, Remarks 4.7 and 4.6 and a variant of the first step yield

$$\begin{aligned}\| |\bar{D}|^{-\gamma-\frac{\sigma}{2}}P_{\geq 0}u\|_{L^p_{\mathbb{R}}L^q} &\lesssim \|\langle\bar{D}\rangle^{-\gamma-\frac{\sigma}{2}}P_{\geq 0}u\|_{L^p_{\mathbb{R}}L^q} \lesssim \|\langle\bar{D}\rangle^{-\gamma-\frac{\sigma}{2}}u\|_{L^p_{\mathbb{R}}L^q} \\ &\lesssim \|\langle\bar{D}\rangle^{-\gamma-\frac{\sigma}{2}}v\|_{L^p_{\mathbb{R}}L^q} \lesssim \| |\bar{D}|^{-\gamma-\frac{\sigma}{2}}v\|_{L^p_{\mathbb{R}}L^q}.\end{aligned}$$

Estimate (4.11) for u now follows from that for v. □

In four more lemmas we reduce Theorem 4.8 to compactly supported solutions, frequency-localized pieces, coeffcients with Fourier cut-off, and to $s = 2$. By the above results, in these steps it is enough to show (4.12) for v with Fourier support off 0 assuming that (p, q, γ) is strict with $p < \infty$ and $\kappa = 1$ if $s = 2$.

Lemma 4.15. *It suffices to show Theorem 4.8 for compactly supported v.*

Proof. We[1] fix a function $0 \leq \varphi \in C^\infty(\mathbb{R}^4)$ with support in $B(0, 2)$ satisfying $\sum_{k\in\mathbb{Z}^4}\varphi_k = 1$ for $\varphi_k = \varphi(\cdot - k)$. (For instance, let $0 \leq \psi \in C_c^\infty(B(0, 2))$ be 1 on $B(0, 1)$. Since $\Psi = \sum_k \psi_k \geq 2$ is uniformly bounded, one can choose $\varphi_k = \psi_k/\Psi$.) Moreover, take smooth maps $0 \leq \tilde{\varphi}_k$ with support in $B(k, 3)$ being 1 on $B(k, 2)$. Let $v \in L^2_{\mathbb{R}}L^2$. We can then write

$$\langle\bar{D}\rangle^{-\gamma-\frac{\sigma}{2}}v = \sum_{k,l\in\mathbb{Z}^4}\varphi_k\langle\bar{D}\rangle^{-\gamma-\frac{\sigma}{2}}(\varphi_l v) = \sum_{|k-l|<8} G_{kl}v + \sum_{|k-l|\geq 8} G_{kl} =: S_< + S_\geq$$

abbreviating $G_{kl} = \varphi_k\langle\bar{D}\rangle^{-\gamma-\frac{\sigma}{2}}\varphi_l$. Set $\hat{\varphi}_k := \sum_{l:|l-k|<8}\varphi_l = \hat{\varphi}_0(\cdot - k)$. The near-diagonal part then becomes $S_< = \sum_k \varphi_k\langle\bar{D}\rangle^{-\gamma-\frac{\sigma}{2}}\hat{\varphi}_k v$.

To obtain square sums in k, we note that for some $K \in \mathbb{N}$ and each $\bar{x}$ the series $\sum_k \varphi_k(\bar{x})$ has at most K nonzero summands. From Hölder we thus deduce

$$\big|S_<+S_\geq\big|^2 \lesssim \sum_{k\in\mathbb{Z}^4}\big|\langle\bar{D}\rangle^{-\gamma-\frac{\sigma}{2}}\hat{\varphi}_k v\big|^2 + \sum_{k\in\mathbb{Z}^4}\Big(\sum_{l:|l-k|\geq 8}|G_{kl}(\tilde{\varphi}_l v)|\Big)^2 =: \sum_{k\in\mathbb{Z}^4}\big(\hat{a}_k^2+\tilde{a}_k^2\big),$$

also inserting $\varphi_l v = \varphi_l\tilde{\varphi}_l v$. Minkowski's inequality then implies

$$\begin{aligned}\big\|\langle\bar{D}\rangle^{-\gamma-\frac{\sigma}{2}}\big\|^2_{L^p_{\mathbb{R}}L^q} &\lesssim \Big\|\sum\nolimits_{k\in\mathbb{Z}^4}\big(\hat{a}_k^2+\tilde{a}_k^2\big)\Big\|_{L^{\frac{p}{2}}_{\mathbb{R}}L^{\frac{q}{2}}} \qquad (4.14)\\ &\leq \sum_{k\in\mathbb{Z}^4}\big\|\langle\bar{D}\rangle^{-\gamma-\frac{\sigma}{2}}\hat{\varphi}_k v\big\|^2_{L^p_{\mathbb{R}}L^q} + \sum_{k\in\mathbb{Z}^4}\Big(\sum_{l:|l-k|\geq 8}\|G_{kl}(\tilde{\varphi}_l v)\|_{L^p_{\mathbb{R}}L^q}\Big)^2\end{aligned}$$

since $p, q \geq 2$. The operator G_{kl} has the form

$$G_{kl}w(\bar{x}) = \lim_{\delta\to 0}\int_{\mathbb{R}^4}\int_{\mathbb{R}^4}\phi(\delta\bar{\xi})e^{i(\bar{x}-\bar{y})\cdot\bar{\xi}}\varphi_k(\bar{x})\varphi_l(\bar{y})\langle\bar{\xi}\rangle^{-\gamma-\frac{\sigma}{2}}\,d\bar{\xi}\,w(\bar{y})\,d\bar{y}, \quad \bar{x}\in\mathbb{R}^4,$$

with $\delta \in (0, 1]$ and a fixed cutoff $0 \leq \phi \in C^\infty(\mathbb{R}^4)$ being 1 on $B(0, 1)$ and supported in $B(0, 2)$, if $w \in \mathcal{S}(\mathbb{R}^4)$ say. The limit exists in $\mathcal{H}^3(\mathbb{R}^4) \hookrightarrow L^p_{\mathbb{R}}L^q$, for instance.

[1] This proof was omitted in the lectures.

For $|k-l| \geq 8$, the approximate kernels $g_{kl}^{\delta}(\bar{x},\bar{y})$ can be integrated by parts in $\bar{\xi}$, cf. (3.10), yielding

$$\begin{aligned}|g_{kl}^{\delta}(\bar{x},\bar{y})| &\lesssim_N \langle \bar{x}-\bar{y}\rangle^{-2N} \sum_{k=0}^{N} c_k \delta^k \int_{\mathbb{R}^4} \big|(\bar{\nabla}^k \phi)(\delta\bar{\xi})\big| \langle \bar{\xi}\rangle^{-\gamma-\frac{\sigma}{2}-N+k} \,\mathrm{d}\bar{\xi} \\ &\lesssim_N \langle \bar{x}-\bar{y}\rangle^{-2N} \sum_{k=0}^{N} c_k \delta^k \int_0^{\frac{2}{\delta}} \langle r\rangle^{-\gamma-\frac{\sigma}{2}-N+k+3} \,\mathrm{d}r \lesssim_N \langle \bar{x}-\bar{y}\rangle^{-N} \langle k-l\rangle^{-N}\end{aligned}$$

for $N \in \mathbb{N}$ with $N > 4-\gamma$. At the end we use that $|\bar{x}-\bar{y}| \geq |k-l|-4$ by the supports of φ_k and φ_l. Thanks to Young's inequality, see Remark 4.6, the double sum in (4.14) can then be bounded via

$$c \sum_{k\in\mathbb{Z}^4} \Big(\sum_{l:|l-k|\geq 8} \langle k-l\rangle^{-N} \|\tilde{\varphi}_l v\|_{L^2_{\mathbb{R}} L^2} \Big)^2 \lesssim \sum_{l\in\mathbb{Z}^4} \|\tilde{\varphi}_l v\|^2_{L^2_{\mathbb{R}} L^2} \lesssim \|v\|^2_{L^2_{\mathbb{R}} L^2}$$

for a fixed sufficiently large N, because of $p,q \geq 2$ and $\sum_l \tilde{\varphi}_l \leq \tilde{K}$. By density, this estimate is true for all $v \in L^2_{\mathbb{R}} L^2$. Equation (4.14) now leads to

$$\big\|\langle \bar{D}\rangle^{-\gamma-\frac{\sigma}{2}} v\big\|^2_{L^p_{\mathbb{R}} L^q} \lesssim \|v\|^2_{L^2_{\mathbb{R}} L^2} + \sum\nolimits_{k\in\mathbb{Z}^4} \big\|\langle \bar{D}\rangle^{-\gamma-\frac{\sigma}{2}} \hat{\varphi}_k v\big\|^2_{L^p_{\mathbb{R}} L^q}.$$

The assumption allows us to apply (4.12) to $\hat{\varphi}_k v$. We arrive at

$$\begin{aligned}\big\|\langle \bar{D}\rangle^{-\gamma-\frac{\sigma}{2}} v\big\|^2_{L^p_{\mathbb{R}} L^q} &\lesssim \kappa \|v\|^2_{L^2_{\mathbb{R}} L^2} + \sum_{k\in\mathbb{Z}^4} \Big[\tfrac{1}{\kappa} \|L(\hat{\varphi}_k v)\|^2_{\mathcal{H}^{-\sigma}(\mathbb{R}^4)} + \|\operatorname{Div}(\hat{\varphi}_k v)\|^2_{\mathcal{H}^{-\frac{1}{2}-\frac{\sigma}{2}}} \Big] \\ &\lesssim \kappa \|v\|^2_{L^2_{\mathbb{R}} L^2} + \sum_{k\in\mathbb{Z}^4} \Big[\tfrac{1}{\kappa} \|\hat{\varphi}_k f\|^2_{\mathcal{H}^{-\sigma}(\mathbb{R}^4)} + \|\hat{\varphi}_k \rho\|^2_{\mathcal{H}^{-\frac{1}{2}-\frac{\sigma}{2}}(\mathbb{R}^4)} \Big] \end{aligned} \tag{4.15}$$

since $\sum_k \|\psi_k w\|_2^2 = \big\|\sum_k \psi_k w\big\|_2^2 \lesssim \|w\|_2^2$ for $\psi_k = \hat{\varphi}_k$ or $\psi_k = \|\bar{\nabla}\hat{\varphi}_0\|_\infty \mathbb{1}_{\operatorname{supp}\hat{\varphi}_k}$.

It remains to show the boundedness of $g \mapsto (\hat{\varphi}_k g)$ from $\mathcal{H}^{-\alpha}$ to $\ell^2(\mathcal{H}^{-\alpha})$, or by duality of $S\colon (h_k) \mapsto \sum_k \hat{\varphi}_k h_k$ from $\ell^2(\mathcal{H}^{\alpha})$ to $\mathcal{H}^{\alpha}$, for $\alpha \in [0,1]$. As above, the support property of $\hat{\varphi}_k$ yields

$$\Big|\sum\nolimits_{k\in\mathbb{Z}^4} \hat{\varphi}_k h_k\Big|^2 \lesssim \sum\nolimits_{k\in\mathbb{Z}^4} \big|\hat{\varphi}_k h_k\big|^2 \lesssim \sum\nolimits_{k\in\mathbb{Z}^4} \big|h_k\big|^2$$

and analogously for $\bar{\nabla}(\hat{\varphi}_k h_k)$. By integration, S is bounded for $\alpha \in \{0,1\}$ and thus for $\alpha \in (0,1)$ by interpolation. Hence, (4.15) implies (4.12) for v. $\square$

As in Lemma 3.20 we now reduce to Littlewood–Paley pieces. We first restrict $s=2$ in order to handle commutators of $\bar{P}_j$ with coefficients.

Lemma 4.16. *Let $s=2$. Then Theorem 4.8 follows from the estimate*

$$2^{-\gamma j} \|\bar{P}_j v\|_{L^p_{\mathbb{R}} L^q} \leq C\big(\|\bar{P}_j v\|_{L^2_{\mathbb{R}} L^2} + \|L\bar{P}_j v\|_{L^2_{\mathbb{R}} L^2} + 2^{-\frac{1}{2}j} \|\bar{P}_j \rho\|_{L^2_{\mathbb{R}} L^2}\big) \tag{4.16}$$

for strict triples, compactly supported $v \in L^2_{\mathbb{R}} L^2$, $j \geq j_0$, and some fixed $j_0 \in \mathbb{N}_0$.

Proof. By the previous lemmas it suffices to show (4.12) with $\bar{P}_{>j_0} v$ on the left and $\kappa = 1$. As in Lemma 3.20, Remark 4.7 and estimate (4.16) yield

$$\|\langle \bar{D}\rangle^{-\gamma} \bar{P}_{>j_0} v\|^2_{L^p_{\mathbb{R}} L^q} \lesssim \sum\nolimits_{j\geq j_0} 2^{-2\gamma j} \|\bar{P}_j v\|^2_{L^p_{\mathbb{R}} L^q} \tag{4.17}$$

$$\lesssim \sum\nolimits_{j\ge j_0} \big(\|\bar P_j v\|^2_{L^2_{\mathbb R}L^2} + \|L\bar P_j v\|^2_{L^2_{\mathbb R}L^2} + 2^{-j}\|\bar P_j\rho\|^2_{L^2_{\mathbb R}L^2}\big)$$
$$\lesssim \|v\|^2_{L^2_{\mathbb R}L^2} + \|f\|^2_{L^2_{\mathbb R}L^2} + \|\rho\|^2_{\mathcal H^{-\frac12}(\mathbb R^4)} + \sum\nolimits_{j\ge j_0}\|[L,\bar P_j]v\|^2_{L^2_{\mathbb R}L^2}.$$

The commutator term is rewritten as

$$[L,\bar P_j] = [L,\bar P_j]\bar P_j'' - \bar P_j L(I-\bar P_j'') = M[a^{-1},\bar P_j]\bar P_j'' - \bar P_j L(I-\bar P_j'')$$

using the enlarged Littlewood–Paley projector $\bar P_j'' = \bar P_{j-2}+\cdots+\bar P_{j+2}$. In view of Remark 3.4, the last commutator is given by

$$\begin{aligned}[a^{-1},\bar P_j]w(\bar x) &= c\int_{\mathbb R^4}\big(a^{-1}(\bar x)-a^{-1}(\bar y)\big)2^{4j}\psi\big(2^j(\bar x-\bar y)\big)w(\bar y)\,\mathrm d\bar y\\ &= c\int_{\mathbb R^4}\int_0^1(\bar\nabla a^{-1})\big(\bar y+r(\bar x-\bar y)\big)\cdot(\bar x-\bar y)2^{4j}\psi\big(2^j(\bar x-\bar y)\big)w(\bar y)\,\mathrm dr\mathrm d\bar y\\ &= c\int_{\mathbb R^4}\int_0^1(\bar\nabla a^{-1})\big(\bar y+r(\bar x-\bar y)\big)\,\mathrm dr\cdot 2^{3j}\tilde\psi\big(2^j(\bar x-\bar y)\big)w(\bar y)\,\mathrm d\bar y\end{aligned}$$

with the Schwartz function $\tilde\psi(\bar x)=\bar x\psi(\bar x)$. We can now differentiate in $\bar x$ and use that $2^{4j}\sigma_{2^j}\tilde\psi$ has a fixed 1-norm. Young's convolution inequality then yields

$$\big\|M[a^{-1},\bar P_j]\bar P_j''v\big\|^2_{L^2_{\mathbb R}L^2}\lesssim 2^{-2j}\|\bar P_j''v\|^2_{L^2_{\mathbb R}L^2} + \|\bar P_j''v\|^2_{L^2_{\mathbb R}L^2}$$

since $a\in C_b^2$. The sum of the right-hand side is bounded by $\|v\|^2_{L^2_{\mathbb R}L^2}$.

As in (3.12) we see that $\bar P_j M\big(\bar P_{<j-2}a^{-1}(I-\bar P_j'')v\big)=0$. The remaining piece is bounded by

$$\big\|M\bar P_j\big(\bar P_{\ge j-2}a^{-1}(I-\bar P_j'')v\big)\big\|_{L^2_{\mathbb R}L^2}\lesssim 2^j\|\bar P_{\ge j-2}a^{-1}\|_\infty\|v\|_{L^2_{\mathbb R}L^2}\lesssim 2^{-j}\|a\|_{2,\infty}\|v\|_2$$

due to Remarks 4.6 and 3.10. So the last term in (4.17) is less than $c\|v\|^2_{L^2_{\mathbb R}L^2}$. □

Remark 4.17. In the above and the next lemma one can replace P_j by, e.g., $\bar P_j'$ on the right-hand side. Note that P_jv is not compactly supported anymore, but decays faster than any polynomial. Indeed, Remark 3.4 and (3.10) lead to

$$|P_jv(\bar x)|\le c_N 2^{4j}\int_K\frac{|v(\bar y)|}{\langle 2^j(\bar x-\bar y)\rangle^N}\,\mathrm d\bar y\le c_N' 2^{(4-N)j}\|v\|_1\langle\bar x\rangle^{-N}$$

for $j,N\in\mathbb N$, $|\bar x|\ge 2\max_K|\bar y|$, and the compact set $K=\operatorname{supp} v$. ◊

Multiplication with a^{-1} destroys the frequency localization of v, in general. As a remedy, one applies a Fourier cut-off to the coefficients. We set $P_{\le\alpha}=\sum_{j\le\alpha}P_j$ etc. for $\alpha\in\mathbb R$ and define $a^k=P_{\le k/2}a^{-1}$ for $k\in\mathbb Z$. Remark 3.10 implies

$$\|a^{-1}-a^k\|_\infty = \|P_{>k/2}a^{-1}\|_\infty\lesssim 2^{-\frac{sk}{2}}\|a^{-1}\|_{C_b^s}\lesssim 2^{-\frac{sk}{2}}\|a\|_{C_b^s},\qquad k\in\mathbb N. \tag{4.18}$$

So there is index $k_0\in\mathbb N$ such that $\frac{2}{\eta}\ge a^k\ge(2\|a\|_\infty)^{-1}I$ for $k\ge k_0$. Using this fact, one also obtains $\|a-(a^k)^{-1}\|_\infty\lesssim\|a^{-1}-a^k\|_\infty\lesssim 2^{-sk/2}\|a\|_{C_b^s}$ and thus $(a^k)^{-1}\ge\eta/2$, possibly after increasing k_0. Finally, we have $\|(a^k)^{-1}\|_{C_b^s}\lesssim\|a^k\|_{C_b^s}\lesssim\|a^{-1}\|_{C_b^s}\lesssim\|a\|_{C_b^s}$. For $k\ge k_0$ we define $L^k=\partial_t+Ma^k$.

LEMMA 4.18. *Let $s=2$. To establish Theorem 4.8, we only have to show the frequency-localized and -truncated estimate*

$$2^{-\gamma j}\|\bar P_j v\|_{L^p_{\mathbb R}L^q} \le C\big(\|\bar P_j v\|_{L^2_{\mathbb R}L^2} + \|L^j \bar P_j v\|_{L^2_{\mathbb R}L^2} + 2^{-\frac12 j}\|\bar P_j\rho\|_{L^2_{\mathbb R}L^2}\big) \tag{4.19}$$

for strict triples, compactly supported $v\in L^2_{\mathbb R}L^2$, and $j\ge k_0$.

PROOF. We pass from (4.19) to (4.16) via

$$\begin{aligned}\|L^j\bar P_j v\|_{L^2_{\mathbb R}L^2} &\le \|L\bar P_j v\|_{L^2_{\mathbb R}L^2} + \big\|M\big(\bar P_{>j/2}a^{-1}\bar P_j v\big)\big\|_{L^2_{\mathbb R}L^2}\\ &\lesssim \|L\bar P_j v\|_{L^2_{\mathbb R}L^2} + \big(\|a\|_{C^1_b} + 2^j\|\bar P_{>j/2}a^{-1}\|_\infty\big)\|\bar P_j v\|_{L^2_{\mathbb R}L^2}\\ &\lesssim \|L\bar P_j v\|_{L^2_{\mathbb R}L^2} + \|a\|_{C^2_b}\|\bar P_j v\|_{L^2_{\mathbb R}L^2}\end{aligned}$$

again using Remarks 3.10 and 4.6. □

The above reasoning fails if $s<2$. However, this case can be reduced to $s=2$ by another frequency-cutoff which causes the regularity loss in Theorem 4.8.

LEMMA 4.19. *It suffices to show Theorem 4.8 for $s=2$.*

PROOF. Let $s\in(0,2)$ and $\kappa\gtrsim 1$. Take Fourier-truncated coefficients $\hat a^l = \bar P_{\le l}a^{-1}$ for $l=\frac{2}{2+s}j =: \nu j$ and $j\ge j_0 := \frac{k_0}{\nu}$. Lemma 3.3 and Remark 3.10 yield

$$\|\hat a^l\|_{\dot C^2} \lesssim 2^{2\nu j}\|\bar P_{\le l}a^{-1}\|_\infty \lesssim 2^{2\sigma j}\|a^{-1}\|_{C^s_b} \lesssim 2^{2\sigma j}\|a\|_{C^s_b} \lesssim 2^{2\sigma j}\kappa^4$$

since $2\nu j = 2\sigma j + sl$ and $\|a\|_{C^s_b} \lesssim \|a\|_{\dot C^s} + \|a\|_\infty \lesssim \kappa^4$. Hence, $\hat a^l$ satisfies the assumptions of Theorem 4.8 for $s=2$ with $\kappa' = c\kappa 2^{\frac{\sigma}{2}j}$. Together with Bernstein, estimate (4.12) for $s=2$, κ' and $\bar P_j v$ thus leads to

$$2^{-(\gamma+\frac{\sigma}{2})j}\|\bar P_j v\|_{L^p_{\mathbb R}L^q} \lesssim \kappa\|\bar P_j v\|_{L^2_{\tilde x}} + 2^{-\sigma j}\tfrac{1}{\kappa}\|\hat L^l\bar P_j v\|_{L^2_{\tilde x}} + 2^{-(\frac12+\frac{\sigma}{2})j}\|\bar P_j\rho\|_{L^2_{\tilde x}} \tag{4.20}$$

for all $v\in L^2_{\mathbb R}L^2 = L^2_{\tilde x}$ and the operator $\hat L^l = \partial_t + M\hat a^l$. If we can show

$$\sum_{j\ge j_0} 2^{-2\sigma j}\|\hat L^l\bar P_j v - \bar P_j L v\|^2_{L^2_{\mathbb R}L^2} \lesssim \|v\|^2_{L^2_{\mathbb R}L^2}, \tag{4.21}$$

then (4.12) for $s<2$ follows from (4.20) as in (4.17). Let $\hat a^{>l} = \bar P_{>l}a^{-1}$. After factoring out the derivatives and decomposing a^{-1}, it suffices to establish

$$\sum_{j\ge j_0} 2^{2(1-\sigma)j}\Big[\big\|[\hat a^l,\bar P_j]v\big\|^2_{L^2_{\tilde x}} + \big\|\bar P_j(\hat a^{>l}\bar P''_j v)\big\|^2_{L^2_{\tilde x}} + \big\|\bar P_j(\hat a^{>l}(I-\bar P''_j)v)\big\|^2_{L^2_{\tilde x}}\Big] \lesssim \|v\|^2_{L^2_{\tilde x}}. \tag{4.22}$$

For the first summand, if $s\ge 1$ the map $\hat a^l$ is Lipschitz with norm bounded by $\|a^{-1}\|_{1,\infty} \lesssim \|a\|_{C^s_b}$. Otherwise we have

$$\|\hat a^l\|_{1,\infty} \lesssim \|a^{-1}\|_\infty + 2^{\nu(1-s)j}2^{s\nu j}\|\bar P_{\le \nu j}a^{-1}\|_\infty \lesssim 2^{\nu(1-s)j}\|a^{-1}\|_{C^s_b} \lesssim 2^{\nu(1-s)j}\|a\|_{C^s_b}$$

by Bernstein and Remark 3.10. Hence the first summand in (4.22) can be handled by means of the commutator estimate

$$\big\|[\hat a^l, 2^j\bar P_j]v\big\|_{L^2_{\tilde x}} \lesssim \|\hat a^l\|_{1,\infty}\|v\|_{L^2_{\tilde x}} \lesssim 2^{\nu(1-s)+j}\|a\|_{C^s_b}\|v\|_{L^2_{\tilde x}},$$

see (3.6.2) in **[65]**, since $\nu(1-s) = \sigma - s/(2+s) < \sigma$ if $s<1$.

Remark 3.10 and $s\nu = 1-\sigma$ also yield $2^{(1-\sigma)j}\|\hat a^{>l}\|_\infty \lesssim \|a^{-1}\|_{C^s_b} \lesssim \|a\|_{C^s_b}$. The second sum in (4.22) is thus dominated by $\|v\|^2_{L^2_{\mathbb R}L^2}$ in view of Theorem 3.6.

In the third term enter only the frequencies $|\bar{\xi}| \in [2^{j-1}, 2^{j+1}]$ of the product $\hat{a}^{>l}(I - \bar{P}_j'')v =: \tilde{g}\tilde{h}$, where those of second factor satisfy $|\bar{\zeta}| \ge 2^{j+2}$ or $|\bar{\zeta}| \le 2^{j-2}$. Hence, in $g * h(\bar{\xi}) = \int g(\bar{\xi} - \bar{\zeta})h(\bar{\zeta})\,\mathrm{d}\bar{\zeta}$ we obtain $|\bar{\zeta} - \bar{\xi}| \ge 2^{j+2} - 2^{j+1} \ge 2^{j+1}$ or $|\bar{\xi} - \bar{\zeta}| \ge 2^{j-1} - 2^{j-2} \ge 2^{j-2}$. As a result, in the above product we can replace $\bar{P}_{>\nu j}$ by $\bar{P}_{\ge j-3}$, possibly after increasing j_0 to $j_0 \ge 3/(1-\nu)$. Then the third summand can be estimated by

$$\big\|\bar{P}_j\big(\bar{P}_{\ge j-3}a^{-1}(I - \bar{P}_j'')v\big)\big\|_{L^2_{\bar{x}}} \le \|\bar{P}_{\ge j-3}a^{-1}\|_{L^\infty_{\bar{x}}}\big\|(I - \bar{P}_j'')v\big\|_{L^2_{\bar{x}}} \lesssim 2^{-js}\|a\|_{C^s_b}\|v\|_{L^2_{\bar{x}}}.$$

Since $1 - \sigma - s < 0$, the desired inequality (4.22) follows. □

4.2. Reduction to a half-wave problem

We have seen in the above chain of lemmas that Theorem 4.8 follows from (4.19) for $j \ge k_0$, strict triples and compactly supported $v \in L^2_{\mathbb{R}}L^2$. In this section we reduce this inequality to a Strichartz estimate for a half-wave equation with coefficients, which is essentially shown in [**63**] and discussed in the next section. This reduction is based on a diagonalization of the 'principal symbol' of the operator $L^j = \partial_t + Ma^j$, as explained next. We will pass from the symbols to the estimate (4.19) at the end of this section. For this step and also later on, we need the so-called FBI transform and tools from the theory of pseudo-differential operators, which are treated in the section's middle part.

A) Diagonalization of the principal symbol. Recall that we use the scalar, truncated coefficients $\varepsilon^j := \bar{P}_{\le j/2}\varepsilon^{-1}$ and $\mu^j := \bar{P}_{\le j/2}\mu^{-1}$ for $j \ge k_0$, where we set $a^j = \operatorname{diag}(\varepsilon^j, \mu^j)$. The isotropy is crucially used in the sequel. The 1-homogeneous *principal symbol* of L^j is given by

$$\ell^j(\bar{x}, \bar{\xi}) = \mathrm{i}\begin{pmatrix} \tau I & -\mu^j(\bar{x})c(\xi) \\ \varepsilon^j(\bar{x})c(\xi) & \tau I \end{pmatrix}$$

with $I = I_{3\times 3}$. The precise relation of ℓ^j and L^j is discussed in the next subsection. We compute the eigenvalues $\lambda(\bar{x}, \bar{\xi}) \in \mathbb{R}$ and eigenvectors $w(\bar{x}, \bar{\xi}) = (w^1, w^2) \in \mathbb{R}^{3+3}$ of the symmetric matrix $\frac{1}{\mathrm{i}}\ell^j$. Take $\xi \ne 0$, as $\ell^j(\bar{x}, \tau, 0) = \mathrm{i}\tau I_{6\times 6}$.

We set $\omega = \lambda - \tau$ and normalize $\xi^* = |\xi|^{-1}\xi$. We have to solve the system

$$-\mu^j c w^2 = \omega w^1, \qquad \varepsilon^j c w^1 = \omega w^2. \tag{4.23}$$

Corresponding to N(curl), we have the eigenvectors $(\xi^*, 0)$ and $(0, \xi^*)$ for $\lambda = \tau$, i.e, $\omega = 0$. Let $\lambda \ne \tau$. Set $\nu^j = (\varepsilon^j\mu^j)^{\frac{1}{2}}$. The system (4.23) yields $-(\nu^j)^2 c^2 w^l = \omega^2 w^l$. The matrix $c^2 = \xi\xi^\top - |\xi|^2 I$ has a kernel spanned by ξ and the double eigenvalue $-|\xi|^2$ with eigenvectors orthogonal to ξ. (Below we use multiples of $(\xi_2, -\xi_1, 0)$ and $(\xi_3, 0, -\xi_1)$.) Hence, ω is equal to one of the numbers $\pm\nu^j|\xi| =: \omega_\pm$ and we obtain the remaining eigenvalues $\lambda_\pm = \tau \pm \nu^j|\xi|$ having multiplicity 2. The eigenvalues of ℓ^j are collected in the diagonal matrix

$$d^j(\bar{x}, \bar{\xi}) = \mathrm{i}\operatorname{diag}\big(\tau, \tau, \tau + \nu^j(\bar{x})|\xi|, \tau - \nu^j(\bar{x})|\xi|, \tau + \nu^j(\bar{x})|\xi|, \tau - \nu^j(\bar{x})|\xi|\big). \tag{4.24}$$

It remains to find suitably normalized eigenvectors $w = (w^1, w^2)$ which yield a transformation matrix with good properties. There is an index i with $|\xi_i^*| \ge \frac{1}{3}$. We take $i = 1$, as the other cases are handled analogously. See §3.1 in [**43**], where the coefficients ε^{-1} and μ^{-1} are treated, however. Set $\xi_{kl} = (\xi_k^2 + \xi_l^2)^{\frac{1}{2}}$

and $\hat\nu^j = \nu^j/\mu^j = (\varepsilon^j/\mu^j)^{\frac{1}{2}}$. Note that ξ_{kl} is positive if $k=1$ or $l=1$. We first choose $w^1 = (\xi_2\xi_{12}^{-1}, -\xi_1\xi_{12}^{-1}, 0)$ which satisfies

$$\begin{aligned}
\varepsilon^j c w^1 &= \varepsilon^j \begin{pmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{pmatrix} \begin{pmatrix} \xi_2\xi_{12}^{-1} \\ -\xi_1\xi_{12}^{-1} \\ 0 \end{pmatrix} = \varepsilon^j \begin{pmatrix} -\xi_1\xi_3\xi_{12}^{-1} \\ \xi_2\xi_3\xi_{12}^{-1} \\ -\xi_{12} \end{pmatrix} = \omega_\pm w_\pm^2, \\
w_\pm^2 &:= \pm\frac{\hat\nu^j}{|\xi|} \begin{pmatrix} \xi_1\xi_3\xi_{12}^{-1} \\ \xi_2\xi_3\xi_{12}^{-1} \\ -\xi_{12} \end{pmatrix}.
\end{aligned}$$

We then also obtain $\mu^j c w_\pm^2 = -\omega_\pm w_1$, and have found first eigenvectors for $\lambda_\pm$.

Next we take $\tilde w^1 = (\xi_3\xi_{13}^{-1}, 0, -\xi_1\xi_{13}^{-1})$ and compute

$$\begin{aligned}
\varepsilon^j c \tilde w^1 &= \varepsilon^j \begin{pmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{pmatrix} \begin{pmatrix} \xi_3\xi_{13}^{-1} \\ 0 \\ -\xi_1\xi_{13}^{-1} \end{pmatrix} = \varepsilon^j \begin{pmatrix} -\xi_1\xi_2\xi_{13}^{-1} \\ \xi_{13} \\ -\xi_2\xi_3\xi_{13}^{-1} \end{pmatrix} = \omega_\pm \tilde w_\pm^2, \\
\tilde w_\pm^2 &:= \pm\frac{\hat\nu^j}{|\xi|} \begin{pmatrix} -\xi_1\xi_2\xi_{13}^{-1} \\ \xi_{13} \\ -\xi_2\xi_3\xi_{13}^{-1} \end{pmatrix}.
\end{aligned}$$

Note that $\mu^j c \tilde w_\pm^2 = -\omega_\pm \tilde w^1$ and that $\{w^1, \tilde w^1\}$ are linearly independent. We have thus computed a basis of six eigenvectors which are 0-homogeneous in ξ. It is important that $w_1 = (\xi^*, 0)$ and $w_2 = (0, \xi^*)$ are orthogonal to the other four ones and to each other. The eigenvectors form the transformation matrix

$$m_1^j(\bar x, \bar\xi) = \begin{pmatrix}
\xi_1^* & 0 & \frac{\xi_2}{\xi_{12}} & \frac{\xi_2}{\xi_{12}} & \frac{\xi_3}{\xi_{13}} & \frac{\xi_3}{\xi_{13}} \\
\xi_2^* & 0 & -\frac{\xi_1}{\xi_{12}} & -\frac{\xi_1}{\xi_{12}} & 0 & 0 \\
\xi_3^* & 0 & 0 & 0 & -\frac{\xi_1}{\xi_{13}} & -\frac{\xi_1}{\xi_{13}} \\
0 & \xi_1^* & \hat\nu^j\frac{\xi_1\xi_3}{\xi_{12}|\xi|} & -\hat\nu^j\frac{\xi_1\xi_3}{\xi_{12}|\xi|} & -\hat\nu^j\frac{\xi_1\xi_2}{\xi_{13}|\xi|} & \hat\nu^j\frac{\xi_1\xi_2}{\xi_{13}|\xi|} \\
0 & \xi_2^* & \hat\nu^j\frac{\xi_2\xi_3}{\xi_{12}|\xi|} & -\hat\nu^j\frac{\xi_2\xi_3}{\xi_{12}|\xi|} & \hat\nu^j\frac{\xi_{13}}{|\xi|} & -\hat\nu^j\frac{\xi_{13}}{|\xi|} \\
0 & \xi_3^* & -\hat\nu^j\frac{\xi_{12}}{|\xi|} & \hat\nu^j\frac{\xi_{12}}{|\xi|} & -\hat\nu^j\frac{\xi_2\xi_3}{\xi_{13}|\xi|} & \hat\nu^j\frac{\xi_2\xi_3}{\xi_{13}|\xi|}
\end{pmatrix}. \tag{4.25}$$

The final step is to invert m_1^j. This is not done explicitely, we rather want to show the properties of the inverse n_1^j needed below. First, their orthogonality properties imply that w_1 and w_2 are the first two rows of n_1^j. Next we compute the determinant of m_1^j. Adding the third to the fourth row and the fifth to the sixth, we eliminate the lower parts of the fourth and sixth row, but double their upper parts. These factors can be taken out and then the inverse operations annulate the upper parts of the third and fifth rows. Permuting the rows, we obtain a block structure and hence

$$\det m_1^j = -4 \begin{vmatrix} \xi_1^* & \frac{\xi_2}{\xi_{12}} & \frac{\xi_3}{\xi_{13}} \\ \xi_2^* & -\frac{\xi_1}{\xi_{12}} & 0 \\ \xi_3^* & 0 & -\frac{\xi_1}{\xi_{13}} \end{vmatrix} \begin{vmatrix} \xi_1^* & \hat\nu^j\frac{\xi_1\xi_3}{\xi_{12}|\xi|} & -\hat\nu^j\frac{\xi_1\xi_2}{\xi_{13}|\xi|} \\ \xi_2^* & \hat\nu^j\frac{\xi_2\xi_3}{\xi_{12}|\xi|} & \hat\nu^j\frac{\xi_{13}}{|\xi|} \\ \xi_3^* & -\hat\nu^j\frac{\xi_{12}}{|\xi|} & -\hat\nu^j\frac{\xi_2\xi_3}{\xi_{13}|\xi|} \end{vmatrix} =: -4\delta'\delta''$$

One easily computes $\delta' = \xi_1|\xi|\xi_{12}^{-1}\xi_{13}^{-1}$. In δ_2 one can take out factors and add multiples of the first row to conclude

$$\delta'' = \frac{(\hat{\nu}^j)^2}{|\xi|^3\xi_{12}\xi_{13}} \begin{vmatrix} \xi_1 & \xi_1\xi_3 & -\xi_1\xi_2 \\ \xi_2 & \xi_2\xi_3 & \xi_1^2+\xi_3^2 \\ \xi_3 & -\xi_1^2-\xi_2^2 & -\xi_2\xi_3 \end{vmatrix} = \frac{(\hat{\nu}^j)^2}{|\xi|^3\xi_{12}\xi_{13}} \begin{vmatrix} \xi_1 & 0 & 0 \\ \xi_2 & 0 & |\xi|^2 \\ \xi_3 & -|\xi|^2 & 0 \end{vmatrix} = \frac{(\hat{\nu}^j)^2\xi_1|\xi|}{\xi_{12}\xi_{13}}$$

and hence

$$\delta_1^j(\bar{x},\bar{\xi}) := \det m_1^j(\bar{x},\bar{\xi}) = -\frac{4\varepsilon^j(\bar{x})\xi_1^2|\xi|^2}{\mu^j(\bar{x})\xi_{12}^2\xi_{13}^2}. \tag{4.26}$$

The determinant has a positive distance to 0 since $|\xi_1| \geq \frac{1}{3}|\xi|$. Moreover, it is 0-homogeneous in $\bar{\xi}$ and it inherits the regularity properties of ε^j and μ^j.

To use Cramer's rule, we write $(m_i^j)^{(kl)}$ for the determinant of m_i^j with deleted kth row and lth column, which is a homogeneous polynomial in the components of m_i^j. We summarize

$$\ell^j = m_i^j d^j n_i^j,\ (n_i^j)_{kl} = \frac{(-1)^{k+l}(m_i^j)^{(lk)}}{\delta_i^j},\ (n_i^j)_{1\bullet} = (\xi^*,0),\ (n_i^j)_{2\bullet} = (0,\xi^*) \tag{4.27}$$

if $3\xi_i \geq |\xi|$, omitting the arguments $(\bar{x},\bar{\xi})$. The components c of m_i^j and n_i^j are 0-homogeneous in $\bar{\xi}$ and those of d^j are 1-homogeneous; all are smooth in $\bar{x}$. For later use, we admit coefficients $\varepsilon,\mu \in C_b^1$. In this case we infer the core bounds

$$\left|\partial_{\bar{x}}^\alpha\partial_{\bar{\xi}}^\beta c\right| \lesssim_{|\alpha|,|\beta|} 2^{\frac{1}{2}(|\alpha|-1)j}|\xi|^{-|\beta|}, \qquad \left|\partial_{\bar{x}}^\alpha\partial_{\bar{\xi}}^\beta d_k^j\right| \lesssim_{|\alpha|,|\beta|} 2^{\frac{1}{2}(|\alpha|-1)j}|\bar{\xi}|^{1-|\beta|} \tag{4.28}$$

for $\alpha,\beta \in \mathbb{N}_0^4$, $j \geq k_0$ and $k \in \{1,\cdots,6\}$. With significant more work and partly losing regularity, one can show similar results in the partially anisotropic case, see [**43**]. The approach fails in the fully anisotropic case, cf. [**49**].

B) Pseudo-differential operators and FBI transform. We want to turn the factorization in (4.27) into an operator equation. Since our symbols also depend on $\bar{x}$, this requires pseudo-differential operators which we discuss first. A smooth function a on $\mathbb{R}^m \times (\mathbb{R}^m \setminus \{0\})$ belongs to the *symbol class* $S_{\omega,\kappa}^\nu$ for some $\nu \in \mathbb{R}$ and $\omega,\kappa \in [0,1]$ if

$$\sup_{\xi\neq 0,x}\left|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)\right| \leq c_{\alpha,\beta}\langle\xi\rangle^{\nu-\omega|\beta|+\kappa|\alpha|}, \qquad \alpha,\beta \in \mathbb{N}_0^m. \tag{4.29}$$

We then define the *pseudo-differential operator* $\mathrm{Op}(a) = a(x,D) : \mathcal{S}_m \to \mathcal{S}_m$ by

$$\mathrm{Op}(a)\varphi(x) = (2\pi)^{-\frac{m}{2}}\int_{\mathbb{R}^m} \mathrm{e}^{\mathrm{i}x\cdot\xi}a(x,\xi)\hat{\varphi}(\xi)\,\mathrm{d}\xi. \tag{4.30}$$

Observe that one has $\mathrm{Op}(a) = bc(D)$ if $a(x,\xi) = b(x)c(\xi)$ where $c(D)$ is just a Fourier multiplier. Hence our principal symbol yields the operator $l^j(x,D) = \partial_t + a^jM$ which differs from L^j by the $L^2_{\mathbb{R}}L^2$-bounded perturbation $Ev = (-\nabla\mu^j \times v_2, \nabla\varepsilon^j \times v_1)$. The mapping properties of $a(x,D)$ have been studied in detail. For instance, let $\kappa < 1$. Then one has $a(x,D) : \mathcal{S}_m^\star \to \mathcal{S}_m^\star$. If also $a \in S_{1,\kappa}^\nu$, the operator $a(x,D) : \mathcal{H}^{s+\nu,q} \to \mathcal{H}^{s,q}$ is bounded for $s \in \mathbb{R}$ and $q \in (1,\infty)$, by Lemma 0.1.A and Proposition 0.5.E in [**65**].

We work with symbols being frequency-localized at $\lambda = 2^j$ and thus need to know the λ-dependence of the constants. Typically our symbols are of the form $a_j(\bar{x},\bar{\xi}) = a(\bar{x},2^{-j}\bar{\xi})$ for a smooth function a that vanishes if $|\bar{\xi}| \notin (\frac{1}{2},2)$, say, and satisfies $|\partial_{\bar\xi}^\alpha a| \lesssim 2^{(\nu-|\alpha|)j}$, cf. (4.28). Such operators can be controlled via the next elementary lemma. In the matrix case it is applied componentwise.

Lemma 4.20. *Let $p,q \in [1,\infty]$ and $a : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{C}$ be smooth with $a(\bar x, \bar\xi) = 0$ for $\bar\xi \notin B(0,2)$ and*

$$C := \sup_{\bar x \in \mathbb{R}^4} \sum_{0\le|\alpha|\le 5} \|\partial_{\bar\xi}^\alpha a(\bar x, \cdot)\|_{L^1} < \infty.$$

Then the operator $\operatorname{Op}(a)$ is bounded on $L^p_{\mathbb{R}}L^q$ with norm less than cC.

Let $a_j(\bar x, \bar\xi) = a(\bar x, 2^{-j}\bar\xi)$ for a as above with $|\partial_{\bar\xi}^\alpha a| \le c_\alpha 2^{\nu j}$ for $0 \le |\alpha| \le 5$ and some $\nu \in \mathbb{R}$ and $j \in \mathbb{Z}$. We then obtain $\|\operatorname{Op}(a_j)\| \le cC2^{\nu j}$ in $L^p_{\mathbb{R}}L^q$.

Proof. 1) Observe that $a(\bar x, \bar\xi) = \beta(\bar\xi) a(\bar x, \bar\xi)$ for a function $\beta \in C^\infty(\mathbb{R}^4)$ with support in $Q = [-\pi, \pi]^4$ and $\beta = 1$ on $B(0,2)$. To separate variables, we expand βa into a Fourier series in $\bar\xi$, namely

$$\beta(\bar\xi)a(\bar x, \bar\xi) = \beta(\bar\xi) \sum\nolimits_{k\in\mathbb{Z}^4} a_k(\bar x) e^{ik\cdot\bar\xi} \quad \text{with} \quad a_k(\bar x) = \frac{1}{16\pi^4}\int_Q e^{-ik\cdot\bar\xi} a(\bar x, \bar\xi)\, d\bar\xi$$

for every fixed $\bar x \in \mathbb{R}^4$, obtaining

$$\operatorname{Op}(a)g(\bar x) = \sum_{k\in\mathbb{Z}^4} \frac{1}{4\pi^2}\int_{\mathbb{R}^4} e^{i(\bar x+k)\cdot\bar\xi} a_k(\bar x)\beta(\bar\xi)\bar{\mathcal{F}}g(\bar\xi)\, d\bar\xi = \sum_{k\in\mathbb{Z}^4} a_k(\bar x)(\beta(\bar D)g)(\bar x + k).$$

Translation invariance and Young's inequality (via Remarks 3.4 and 4.6) yield $\|\beta(\bar D)g(\cdot + k)\|_{L^p_{\mathbb{R}}L^q} \lesssim \|g\|_{L^p_{\mathbb{R}}L^q}$. As in (3.10), integrating by parts we infer

$$|a_k(\bar x)| \lesssim \langle k\rangle^{-5} \sum\nolimits_{0\le|\alpha|\le5} \|\partial_{\bar\xi}^\alpha a(\bar x, \cdot)\|_{L^1} \le C\langle k\rangle^{-5}$$

for $\bar x \in \mathbb{R}^4$. The above formula for $\operatorname{Op}(a)g$ can thus be estimated as asserted.

2) For the second part we compute $a_j(\bar x, D)g = \sigma_{2^j}\hat a^j(\bar x, D)\sigma_{2^{-j}}g$ with symbol $a(2^{-j}\bar x, \bar\xi)$ by substituting $\bar\eta = 2^{-j}\bar\xi$ in (4.30). The claim follows from the transformation rule and step 1). □

We want to factorize L^j as its symbol ℓ^j in (4.27). For this we use the Kohn–Nirenberg formula for the product of pseudo-differential operators, see Proposition 0.3.C in [65]. We write $D_\xi = -i\partial_\xi$. Let a_k be contained in $S^{\nu_k}_{\omega_k,\kappa_k}$ with $\nu_k \in \mathbb{R}$, $\omega_k, \kappa_k \in [0,1]$, and $\kappa_2 < \omega := \min\{\omega_1, \omega_2\}$. Set $\kappa = \max\{\kappa_1, \kappa_2\}$ and $\nu = \nu_1 + \nu_2$. Then $a_1(x,D)a_2(x,D)$ is a pseudo-differential operator with symbol $a_1 \circ a_2 \in S^\nu_{\omega,\kappa}$ having the asymptotic expansion

$$a_1 \circ a_2 = \sum\nolimits_{0\le|\alpha|\le N} \tfrac{1}{\alpha!}(D_\xi^\alpha a_1\, \partial_x^\alpha a_2) + r_N, \tag{4.31}$$

where $r_N \in S^{\nu-N}_{1,0}$ for $N \in \mathbb{N}$ and the symbol is given by

$$a_1 \circ a_2(x,\xi) = \frac{1}{(2\pi)^m} \lim_{\delta\to0} \int_{\mathbb{R}^{2m}} e^{i(x-z)\cdot(\zeta-\xi)} \phi(\delta(\zeta,z)) a_1(x,\zeta) a_2(z,\xi)\, d(z,\zeta) \tag{4.32}$$

for any cut-off $0 \le \phi \in C_c^\infty(\mathbb{R}^{2m})$ being 1 on $B(0,1)$.

We apply this result to symbols of the form $a_j(\bar{x},\bar{\xi}) = a(\bar{x},2^{-j}\bar{\xi})$ and $b_j(\bar{x},\bar{\xi}) = b(\bar{x},2^{-j}\bar{\xi})$ for some $j\in\mathbb{Z}$ and a,b as in Lemma 4.20. We assume

$$|\partial_{\bar{x}}^{\alpha}\partial_{\bar{\xi}}^{\beta}a| \le c_{\alpha,\beta}2^{\nu j}2^{\frac{1}{2}(|\alpha|-1)+j}, \quad |\partial_{\bar{x}}^{\alpha}\partial_{\bar{\xi}}^{\beta}b| \le c_{\alpha,\beta}2^{\nu' j}2^{\frac{1}{2}(|\alpha|-1)+j} \tag{4.33}$$

for $\alpha,\beta\in\mathbb{N}_0^4$ and some $\nu,\nu'\in\mathbb{R}$, cf. (4.28). Here, the transformation in the proof of Lemma 4.20 turns the products π_α in the sum of (4.31) into the symbols $\hat{\pi}_\alpha^j(\bar{x},\bar{\xi}) = 2^{-|\alpha|j}(D_{\bar{\xi}}^{\alpha}a)(2^{-j}\bar{x},\bar{\xi})\,(\partial_{\bar{x}}^{\alpha}b)(2^{-j}\bar{x},\bar{\xi})$. Lemma 4.20 then implies

$$\|\mathrm{Op}(D_\xi^\alpha a_j\,\partial_x^\alpha b_j)\|_{\mathcal{B}(L_{\mathbb{R}}^pL^q)} \le c_\alpha 2^{(\nu+\nu')j}2^{-\frac{1}{2}(|\alpha|+1)j} \tag{4.34}$$

for $1\le|\alpha|\le N$. The case $|\alpha|=1$ dominates and yields the bound $2^{(\nu+\nu'-1)j}$ for these operators. We still have to control the remainder.

Lemma 4.21. *In the above setting we obtain* $\mathrm{Op}(a_j)\mathrm{Op}(b_j) = \mathrm{Op}(a_jb_j)+E$, *where the error term E is bounded by $2^{(\nu+\nu'-1)j}$ on $L_{\mathbb{R}}^pL^q$ for $p,q\in[1,\infty]$.*

Proof. By (4.31) and (4.34), it remains to check $\|\mathrm{Op}(r_N)\|\le c_N2^{(\nu+\nu'-1)j}$ for a fixed $N\in\mathbb{N}$. Let $\tilde{\xi}=2^{-j}\bar{\xi}$. Formula (4.32) for $c_j := a_j\circ b_j$ reads as

$$\begin{aligned}
c_j(\bar{x},\xi) &= \frac{1}{(2\pi)^4}\lim_{\delta\to0}\int_{\mathbb{R}^8}\mathrm{e}^{\mathrm{i}(\bar{x}-\bar{z})\cdot(\bar{\zeta}-2^j\tilde{\xi})}\phi(\delta(\bar{z},\bar{\zeta}))a(\bar{x},2^{-j}\bar{\zeta})b(\bar{z},\tilde{\xi})\,\mathrm{d}(\bar{z},\bar{\zeta})\\
&= \frac{2^{4j}}{(2\pi)^4}\lim_{\delta\to0}\int_{\mathbb{R}^8}\mathrm{e}^{\mathrm{i}2^j(\bar{x}-\bar{z})\cdot(\tilde{\zeta}-\tilde{\xi})}\phi(\delta(\bar{z},2^j\tilde{\zeta}))a(\bar{x},\tilde{\zeta})b(\bar{z},\tilde{\xi})\,\mathrm{d}(\bar{z},\tilde{\zeta})\\
&= \frac{2^{4j}}{(2\pi)^4}\lim_{\delta\to0}\int_{\mathbb{R}^8}\mathrm{e}^{\mathrm{i}2^j\bar{y}\cdot\bar{\eta}}\phi\big(\delta(\bar{x}-\bar{y},2^j(\tilde{\xi}+\bar{\eta}))\big)a(\bar{x},\tilde{\xi}+\bar{\eta})b(\bar{x}-\bar{y},\tilde{\xi})\mathrm{d}(\bar{y},\bar{\eta}),
\end{aligned}$$

transforming $\tilde{\zeta}=2^{-j}\bar{\zeta}$, $\bar{y}=\bar{x}-\bar{z}$, and $\bar{\eta}=\tilde{\zeta}-\tilde{\xi}$, where we can take $\delta\le2^{-j}$. We can assume that $\tilde{\xi}$ belongs to $\overline{B}(0,2)$ and then $\bar{\eta}$ to $\overline{B}(0,4)$ as b and a vanish off $B(0,2)$. We further take a cut-off $\phi_0\in C_c^\infty(\mathbb{R}^4)$ being 1 on $B(0,1)$, and split the integral into summands with $\phi_0(\bar{y})$ and $1-\phi_0(\bar{y})$ in the integrand. Observe that the δ-cut-off disappears in the first term (for fixed $\bar{x}$). In the second term, we can integrate by parts as in (3.10) starting from $\mathrm{e}^{\mathrm{i}2^j\bar{y}\cdot\bar{\eta}} = -\mathrm{i}2^{-j}\bar{y}_k^{-1}\partial_{\bar{\eta}_k}\mathrm{e}^{\mathrm{i}2^j\bar{y}\cdot\bar{\eta}}$ with $|\bar{y}_k|\ge|\bar{y}|/4$. Doing this n times, we obtain a prefactor of the form $c_n2^{(4-n)j}\langle\bar{y}\rangle^{-n}$ and harmless $\bar{\eta}$-derivatives in the integrand (as $\delta2^j\le1$). After replacing $\tilde{\xi}=2^{-j}\bar{\xi}$ one can apply the second part of Lemma 4.20 and obtains the norm bound $C_n2^{(4+\nu+\nu'-n)j}$ for the resulting pseudo-differential operator.

Taking $n=5$, we are left with the 'diagonal part'

$$c_j^d(\bar{x},\xi) = \frac{2^{4j}}{(2\pi)^4}\int_{\mathbb{R}^8}\mathrm{e}^{\mathrm{i}2^j\bar{y}\cdot\bar{\eta}}\phi_0(\bar{y})a(\bar{x},\tilde{\xi}+\bar{\eta})b(\bar{x}-\bar{y},\tilde{\xi})\,\mathrm{d}(\bar{y},\bar{\eta})$$

whose integrand vanishes outside a fixed compact set in $(\bar{y},\bar{\eta})$. We now insert the Taylor polynomial of $h(\bar{y},\bar{\eta})=\phi_0(\bar{y})a(\bar{x},\tilde{\xi}+\bar{\eta})b(\bar{x}-\bar{y},\tilde{\xi})$ of order N at 0. In this way one actually derives the expansion (4.31), cf. the text after equation (24) in [**48**]. Here we need the remainder term given by

$$r_N^d(\bar{y},\bar{\eta}) = \sum_{|\beta|=N+1}\tfrac{N+1}{\beta!}\int_0^1(1-\theta)^N\big(\partial_{\bar{y},\bar{\eta}}^{\beta}h\big)(\theta(\bar{y},\bar{\eta}))\,\mathrm{d}\theta\,(\bar{y},\bar{\eta})^\beta.$$

Writing $\beta = \beta_{\bar y} + \beta_{\bar\eta}$, $\beta_{\bar y} = \beta'_{\bar y} + \beta''_{\bar y}$, and omitting the θ-integral and factors, we obtain terms of the form

$$\hat r^d_\beta(\bar x, \tilde\xi) = 2^{4j}\int_{\mathbb{R}^8} \mathrm{e}^{\mathrm{i}2^j\bar y\cdot\bar\eta}\, \bar y^{\beta_{\bar y}}\bar\eta^{\beta_{\bar\eta}}\, \big(\partial^{\beta'_{\bar y}}\phi_0\big)(\theta\bar y)\, \big(\partial_2^{\beta_{\bar\eta}}a\big)(\theta(\bar x, \tilde\xi+\bar\eta))\, \big(\partial_1^{\beta''_{\bar y}}b\big)(\theta(\bar x-\bar y, \tilde\xi)).$$

We insert $\bar\eta^{\beta_{\bar\eta}}\mathrm{e}^{\mathrm{i}2^j\bar y\cdot\bar\eta} = (\mathrm{i}2)^{-|\beta_{\bar\eta}|j}\partial_{\bar y}^{\beta_{\bar\eta}}\mathrm{e}^{\mathrm{i}2^j\bar y\cdot\bar\eta}$ and integrate by parts. This gives at most $\beta - \beta'_{\bar\eta} =: \beta^*_{\bar y}$ spatial derivatives of b, where $0 \le \beta'_{\bar\eta} \le \min\{\beta_{\bar y}, \beta_{\bar\eta}\}$ is the number of derivatives hitting $\bar y^{\beta_{\bar y}}$. The factor $\bar y^{\beta_{\bar y}-\beta'_{\bar\eta}}$ is treated in the same way, giving the total prefactor $2^{(4-|\beta|+|\beta'_{\bar\eta}|)j}$ and harmless frequency derivatives.

We now replace again $\tilde\xi = 2^{-j}\bar\xi$. The resulting integral is a symbol in $(\bar x, \bar\xi)$ of the form treated in the second part of Lemma 4.20. The estimates (4.33) give a bound of the corresponding operator $\mathrm{Op}(\hat r^d_\beta)$ by

$$c_\beta 2^{(4-|\beta^*_{\bar y}|)j}2^{\nu j}2^{\nu' j}2^{\frac12(|\beta^*_{\bar y}|-1)j} = c_\beta 2^{(\nu+\nu'+\frac72)j}2^{-\frac12|\beta^*_{\bar y}|j} \le c_\beta 2^{(\nu+\nu'+\frac72)j}2^{-\frac14|\beta|j}.$$

For $N = |\beta| - 1 = 17$ the above is less than $c2^{(\nu+\nu'-1)j}$. The main part of $r_N(\bar x, \bar\xi)$ is the θ-integral and linear combination of such terms and thus satisfies the same estimate. ☐

We further need the *Fourier–Bros–Iagolnitzer (FBI) transform*, see [**62**].[2] For $g \in L^1_{\mathrm{loc}}(\mathbb{R}^m, \mathbb{C})$ as in (3.1) and a frequency $\lambda > 0$ we define

$$\begin{aligned} T_\lambda g(z) &\coloneqq C_m\lambda^{\frac{3m}{4}}\int_{\mathbb{R}^m}\mathrm{e}^{-\frac{\lambda}{2}(z-y)^2}g(y)\,\mathrm{d}y, \qquad z = x - \mathrm{i}\xi \in \mathbb{C}^m,\\ &= C_m\lambda^{\frac{3m}{4}}\mathrm{e}^{\frac{\lambda}{2}|\xi|^2}\mathrm{e}^{\mathrm{i}\lambda\xi\cdot x}\int_{\mathbb{R}^m}\mathrm{e}^{-\frac{\lambda}{2}|x-y|^2}\mathrm{e}^{-\mathrm{i}\lambda\xi\cdot y}g(y)\,\mathrm{d}y,\\ &= C_m\lambda^{\frac{5m}{4}}\mathrm{e}^{\frac{\lambda}{2}|\xi|^2}\int_{\mathbb{R}^m}\mathrm{e}^{-\frac{\lambda}{2}|\xi-\zeta|^2}\mathrm{e}^{\mathrm{i}\lambda x\cdot\zeta}\hat g(\lambda\zeta)\,\mathrm{d}\zeta, \end{aligned} \tag{4.35}$$

using also basic properties of the Fourier transform, see (15) in [**63**]. Here we let $C_m = 2^{-\frac{m}{2}}\pi^{-\frac{3m}{4}}$, $z^2 = \sum_k z_k^2$, $x, \xi \in \mathbb{R}^m$, and identify $\mathbb{C}^m$ and $\mathbb{R}^{2m}$ via $z = x - \mathrm{i}\xi$. It can be seen that $T_\lambda\colon L^2(\mathbb{R}^m) \to L^2_\Phi(\mathbb{R}^{2m}) = L^2_\Phi$ is isometric onto the closed subspace of anti-holomorphic functions in L^2_Φ, where $\Phi(\xi) = \mathrm{e}^{-\lambda|\xi|^2}$ and L^2_Φ has the measure $\Phi\,\mathrm{d}(x,\xi)$. The FBI transform posseses the left inverse

$$T^\star_\lambda G(y) = C_m\lambda^{\frac{3m}{4}}\int_{\mathbb{R}^{2m}}\mathrm{e}^{-\frac{\lambda}{2}(\bar z-y)^2}\Phi(\xi)G(z)\,\mathrm{d}(x,\xi).$$

The Fourier transform is a superposition of plane waves. In T_λ one replaces them by (L^2-normalized) ‘coherent states’

$$\varphi_{x_0,\xi_0}(y) = \lambda^{\frac{m}{4}}\pi^{-\frac{m}{4}}e^{-\frac{\lambda}{2}|y-x_0|^2}\mathrm{e}^{\mathrm{i}\lambda(y-x_0)\cdot\xi_0},$$

which are localized in space in a $\lambda^{-\frac12}$ neighborhood of x_0 and in frequency in a $\lambda^{\frac12}$ neighborhood of $\lambda\xi_0$. One can compute

$$T_\lambda\varphi_{x_0,\xi_0}(z) = \lambda^{-\frac{m}{4}}\pi^{\frac{m}{4}}\mathrm{e}^{-\frac{\lambda}{4}(|x-x_0|^2+|\xi-\xi_0|^2)}\mathrm{e}^{\frac{\lambda}{2}|\xi|^2}\mathrm{e}^{\mathrm{i}\frac{\lambda}{2}(x-x_0)\cdot(\xi+\xi_0)}$$

which is localized in the same way near $z_0 = x_0 - \mathrm{i}\xi_0$. One directly checks that $T_\lambda(yg) = \big(x + \frac{\mathrm{i}}{\lambda}\big(\partial_\xi - \lambda\xi\big)\big)T_\lambda g$, $\quad T_\lambda\big(\frac1\lambda D_y g\big) = \big(\xi - \frac1\lambda(\mathrm{i}\partial_x + \lambda\xi)\big)T_\lambda g = \frac1\lambda D_x T_\lambda g$.

[2] This transform goes back to work by Bros and Iagolnitzer in the 70’s.

This observation leads one to the following commutator properties of T_λ.

Let $a(x,\xi)$ be a symbol vanishing for $|\xi|\geq 2$ and which is C_b^s in x for some $s\in(0,2]$. Then $a_\lambda(x,\xi):=a(x,\lambda^{-1}\xi)$ is supported up to frequencies 2λ. Set $A_\lambda=\mathrm{Op}(a_\lambda)$, $\tilde a_\lambda^s=a$ for $s\in(0,1]$,

$$\tilde a_\lambda^s=a+\tfrac{\mathrm{i}}{\lambda}\partial_x a\,(\partial_\xi-\lambda\xi)-\tfrac{1}{\lambda}\partial_\xi a\,(\mathrm{i}\partial_x+\lambda\xi)=a+\tfrac{2}{\lambda}(\overline{\partial}a)(\partial-\mathrm{i}\lambda\xi),\quad s\in(1,2],\tag{4.36}$$

and $R_{\lambda,a}^s=T_\lambda A_\lambda-\tilde a_\lambda^s T_\lambda$, with $2\partial=\partial_x+\mathrm{i}\partial_\xi$ and $2\overline{\partial}=\partial_x-\mathrm{i}\partial_\xi$. The second equality in (4.36) is true on anti-holomorphic maps where $\overline{\partial}=0$. The following core remainder estimates are taken from Theorem 1 and Remark 2.2. of [**62**].

Theorem 4.22. *Let $a\in C_b^sC_c^\infty$ be as above, $s\in(0,2]$, and $\lambda>0$. We obtain*

$$\|R_{\lambda,a}^s\|_{\mathcal{B}(L^2,L^2_\Phi)}\lesssim\lambda^{-\frac{s}{2}}\qquad\text{and}\qquad\|(\partial_\xi-\lambda\xi)R_{\lambda,a}^s\|_{\mathcal{B}(L^2,L^2_\Phi)}\lesssim\lambda^{\frac12-\frac{s}{2}}\quad\text{if } s>1.$$

We will apply T_λ with $\lambda=2^j\geq1$ to $\bar P_j v$. Here one can restrict to $(\bar x,\bar\xi)\in\overline{B}(0,2)\times A(\frac14,4)=:K$. Indeed formula (4.35) leads to

$$\Phi(\bar\xi)\big|T_\lambda\bar P_j v(\bar z)\big|^2=C_4^2\lambda^{10}\Big|\int_{\mathbb{R}^4}\mathrm{e}^{-\frac{\lambda}{2}|\bar\xi-\bar\eta|^2}\mathrm{e}^{\mathrm{i}\lambda\bar x\cdot\bar\eta}(\bar{\mathcal{F}}\bar P_j v)(\lambda\bar\eta)\,\mathrm{d}\bar\eta\Big|^2.$$

Let $(\bar x,\bar\xi)\notin K$. Note that $\sigma_\lambda(\bar{\mathcal{F}}\bar P_j v)$ is supported in $A(\frac12,2)$ since $\lambda=2^j$, implying $|\bar\xi-\bar\eta|\geq\frac14$. Starting from $\mathrm{e}^{\mathrm{i}\lambda\bar x\cdot\bar\eta}=-\mathrm{i}(\lambda\bar x_k)^{-1}\partial_{\bar\eta_k}\mathrm{e}^{\mathrm{i}\lambda\bar x\cdot\bar\eta}$ we can integrate by parts gaining a factor $\lambda^{-N}|\bar x|^{-N}$, but losing $\lambda^N|\bar\xi-\bar\eta|^n$ for some $n\leq N$. Moreover, the Gaussian times $\lambda^{-\frac{n}{2}}\lambda^{\frac{n}{2}}|\bar\xi-\bar\eta|^n$ can be estimated by $\mathrm{e}^{-c'\lambda}\mathrm{e}^{-\frac{\lambda}{4}|\bar\xi-\bar\eta|^2}$ for $c'=(8\cdot16)^{-1}$. Using $N=3$, Young and Remark 4.17, we obtain

$$\|T_\lambda\bar P_j v\|_{L^2_\Phi(K^c)}\lesssim\lambda^5\lambda^{-4}\mathrm{e}^{-c'\lambda}\|\bar{\mathcal{F}}\bar P_j v\|_{\mathcal{H}^3(\mathbb{R}^4)}\lesssim\mathrm{e}^{-c\lambda}\|v\|_{L^2_{\mathbb{R}}L^2}.\tag{4.37}$$

As $\lambda=2^j$ this term is square summable, and it suffices to estimate $T_\lambda\bar P_j v$ on K.

C) Diagonalization of L^j. We now transform (4.27) into a factorization of L^j. For this we need a refined frequency decomposition taking care of the cases in (4.27) and of the case $|\tau|\gg|\xi|$ off the characteristic surface $\mathcal{C}=\{\ell^j=0\}$.

For the latter point, as noted after (4.18) we have $|\varepsilon^j|,|\mu^j|\leq\frac{2}{\eta}$ for $j\geq k_0$. Let $|\bar\xi|\in[\frac12,2]$ and $|\tau|\geq\frac{4}{\eta}|\xi|=:c_0|\xi|$. Using also $\eta\leq1$, we derive $|\tau|\geq|\bar\xi|-|\xi|\geq\frac12-\frac14|\tau|$, and so $|\tau|\geq\frac25$. For $w=(w_1,w_2)$ this inequality yields the lower bound

$$|\ell^j w|\geq|\tau||w_1|-\tfrac{2}{\eta}|\xi||w_2|\geq|\tau||w_1|-\tfrac12|\tau||w_1|\geq\tfrac{1}{10}|w|,\tag{4.38}$$

if $|w_1|\geq|w_2|$ and hence $2|w_1|\geq|w|$, for instance.

Next, for the angle cases in (4.27) we fix maps $0\leq\omega_i\in C_c^\infty(\mathbb{S}^2)$ with $\omega_i(\theta)=0$ if $|\theta_i|\leq\frac13$ for $i\in\{1,2,3\}$ and $\omega_1+\omega_2+\omega_3=1$. To incorporate the condition $|\tau|\geq c_0|\xi|$, we fix $\check n\in\mathbb{N}$ with $2^{\check n}\geq2(c_0+1)$ and use the cut-off $\check\chi=\chi_{-\check n}+\dots+\chi_1$ acting in ξ only. Let $\chi^\tau(|\bar\xi|):=\chi(|\bar\xi|)(1-\check\chi(|\xi|))$ be non-zero. We then obtain $|\bar\xi|\in[\frac12,2]$ and $|\xi|\leq2^{-\check n}$, cf. the lines before (3.9). It follows $|\tau|\geq\frac12-|\xi|\geq(\frac12 2^{\check n}-1)|\xi|\geq c_0|\xi|$. Therefore we can apply (4.38) if $\chi^\tau\neq0$. We now define

$$\begin{aligned}\chi_{ij}(\bar\xi)=\chi(2^{-j}|\bar\xi|)\check\chi(2^{-j}|\xi|)\omega_i(|\xi|^{-1}\xi),\quad\check\chi_j=\sigma_{2^{-j}}\check\chi,\quad\chi_j^\tau=\sigma_{2^{-j}}\chi^\tau,\\ \bar P_{ij}=\chi_{ij}(\bar D),\quad\check P_j=\check\chi_j(D),\quad\bar P_j^\tau=\chi_j^\tau(\bar D),\quad\Omega_i=\omega_i(D),\end{aligned}\tag{4.39}$$

for $j \in \mathbb{Z}$ and $i \in \{1,2,3\}$. Here we consider χ as a map $\mathbb{R}_{>0} \to \mathbb{R}_{>0}$. Observe

$$\check{P}_j \bar{P}_j = \bar{P}_{1j} + \bar{P}_{2j} + \bar{P}_{3j}, \qquad (I - \check{P}_j)\bar{P}_j = \bar{P}_j^{\tau}. \tag{4.40}$$

We can now define the operators corresponding to the symbols in (4.24), (4.25) and (4.27), namely

$$\begin{aligned} \mathcal{M}_{ij} &= \mathrm{Op}(m_i^j \chi_{ij}), \qquad \mathcal{N}_{ij} = \mathrm{Op}(n_i^j \chi_{ij}), \\ \mathcal{D}_j &= \mathrm{Op}(d^j) = \mathrm{diag}\big(\partial_t, \partial_t, \partial_t + \mathrm{i}\nu^j |D|, \partial_t - \mathrm{i}\nu^j|D|, \partial_t + \mathrm{i}\nu^j|D|, \partial_t - \mathrm{i}\nu^j|D|\big) \end{aligned} \tag{4.41}$$

for $j \geq k_0$ and $i \in \{1,2,3\}$. We write E_ν for any operator on $L^p_{\mathbb{R}}L^q$ with norm less than $c2^{\nu j}$ and $\bar{P}'_{ij}$ for 'enlarged' versions of $\bar{P}_{ij}$. We collect main properties.

Proposition 4.23. *Let $\varepsilon, \mu \in C_b^1(\mathbb{R}^4)$ satisfy $\varepsilon, \mu \geq \eta > 0$, and $p, q \in [1,\infty]$. Then the operators in (4.39) and $\mathcal{O}_{ij}$ in (4.41) fulfill*

$$\mathcal{O}_{ij}\bar{P}_{ij} = \bar{P}'_{ij}\mathcal{O}_{ij}\bar{P}_{ij} + E_{-1}, \quad \mathcal{M}_{ij}\mathcal{N}_{ij}\bar{P}_{ij} = \bar{P}_{ij} + E_{-1}, \quad \mathcal{N}_{ij}\mathcal{M}_{ij}\bar{P}_{ij} = \bar{P}_{ij} + E_{-1},$$
$$\mathcal{O}_{ij}\bar{P}_j = \bar{P}'_j \mathcal{O}_{ij}\bar{P}_j + E_{-1}, \quad L^j \bar{P}_{ij} = \mathcal{M}_{ij}\mathcal{D}_j\mathcal{N}_{ij}\bar{P}_{ij} + E_0 \quad \textit{for } j \geq k_0,\ i \in \{1,2,3\}.$$

Proof. The result follow from the corresponding identities of the symbols as in (4.27), estimate (4.28), the localization $|\xi| \geq 2^{k_0 - \check{n} - 1}$ as well as Lemmas 4.20 and 4.21. □

After these preparations we can reduce (4.19) to a scalar half-wave problem treated in the next section. We write T_j for T_{2^j} and $L^2_{\bar{x}}$ for $L^2_{\mathbb{R}}L^2$. Recall from Remark 4.17 that we can accept enlarged operators $\bar{P}'_j$ on the right of (4.19).

Proposition 4.24. *Let $\varepsilon, \mu \in C_b^1(\mathbb{R}^4)$ and $\varepsilon, \mu \geq \eta > 0$. Then Theorem 4.26 implies (4.19) for $j \geq k_0$ with additional term $\|L^j \bar{P}_{ij} v\|_{L^2_{\bar{x}}}$ on the right. In particular, Theorem 4.8 follows if $\varepsilon, \mu \in C_b^2(\mathbb{R}^4)$.*

Proof. We split $\bar{P}_j v = \bar{P}_j^\tau v + \sum_i \bar{P}_{ij} v$, see (4.40), and estimate them separately. We first use Bernstein's inequalities, the isometry of T_j, Theorem 4.22 and (4.38) to obtain

$$\begin{aligned} 2^{-\gamma j}\|\bar{P}_j^\tau v\|_{L^p_{\mathbb{R}}L^q} &\lesssim 2^{\frac12 j}\|\bar{P}_j^\tau v\|_{L^2_{\bar{x}}} = 2^{\frac12 j}\|T_j \bar{P}_j^\tau \bar{P}'_j v\|_{L^2_\Phi} \\ &\lesssim 2^{\frac12 j}\|\chi^\tau T_j \bar{P}'_j v\|_{L^2_\Phi} + \|\bar{P}'_j v\|_{L^2_{\bar{x}}} \lesssim 2^{\frac12 j}\|\ell^j T_j \bar{P}'_j v\|_{L^2_\Phi} + \|\bar{P}'_j v\|_{L^2_{\bar{x}}} \\ &\lesssim 2^{\frac12 j}2^{-j}\|T_j L^j \bar{P}'_j v\|_{L^2_\Phi} + \|\bar{P}'_j v\|_{L^2_{\bar{x}}} = 2^{-\frac12 j}\|L^j \bar{P}'_j v\|_{L^2_{\bar{x}}} + \|\bar{P}'_j v\|_{L^2_{\bar{x}}}, \end{aligned} \tag{4.42}$$

which fits to (4.19).

To prepare the part including the light cone $\mathcal{C}$, we first compute

$$\|\bar{P}_{ij} g\|_{L^2_{\bar{x}}} \lesssim \|\mathcal{N}_{ij}\bar{P}'_{ij}\mathcal{M}_{ij}\bar{P}_{ij} g\|_{L^2_{\bar{x}}} + 2^{-j}\|\bar{P}_{ij}g\|_{L^2_{\bar{x}}} \lesssim \|\mathcal{M}_{ij}\bar{P}_{ij}g\|_{L^2_{\bar{x}}} + 2^{-j}\|\bar{P}_{ij}g\|_{L^2_{\bar{x}}}$$

using Proposition 4.23 and Lemma 4.20. Possibly increasing k_0, we conclude

$$\|\bar{P}_{ij} v\|_{L^2_{\bar{x}}} \lesssim \|\mathcal{M}_{ij}\bar{P}_{ij} v\|_{L^2_{\bar{x}}}. \tag{4.43}$$

These results and Bernstein imply

$$\begin{aligned} 2^{-\gamma j}\|\bar{P}_{ij} v\|_{L^p_{\mathbb{R}}L^q} &\lesssim 2^{-\gamma j}\|\mathcal{M}_{ij}\bar{P}'_{ij}\mathcal{N}_{ij}\bar{P}_{ij} v\|_{L^p_{\mathbb{R}}L^q} + 2^{-(\gamma+1)j}\|\bar{P}_{ij} v\|_{L^p_{\mathbb{R}}L^q} \\ &\lesssim 2^{-\gamma j}\|\bar{P}'_{ij} w\|_{L^p_{\mathbb{R}}L^q} + 2^{-\frac12 j}\|\bar{P}_{ij} v\|_{L^2_{\bar{x}}} \lesssim 2^{-\gamma j}\|\bar{P}'_j w\|_{L^p_{\mathbb{R}}L^q} + \|\bar{P}_j v\|_{L^2_{\bar{x}}} \end{aligned} \tag{4.44}$$

with $w := \mathcal{N}_{ij}\bar{P}_{ij}v$. The first two components of w correspond to the degenerate part of L^j. Here the charges come into play via

$$2^{-\gamma j}\|\bar{P}_j' w_{1/2}\|_{L^p_{\mathbb{R}}L^q} \lesssim 2^{\frac{j}{2}}\||\xi|^{-1}\chi_{ij}'\bar{\mathcal{F}}\operatorname{Div} v\|_{L^2_{\bar{x}}} \lesssim 2^{-\frac{1}{2}j}\|\bar{P}_j'\rho\|_{L^2_{\bar{x}}} \tag{4.45}$$

invoking Bernstein, Plancherel and the form of n_i^j described in (4.27).

The other components w_k are treated by means of Theorem 4.26. Using also Proposition 4.23, Lemma 4.20 and (4.43), we arrive at

$$\begin{aligned}2^{-\gamma j}\|\bar{P}_j' w_k\|_{L^p_{\mathbb{R}}L^q} &\lesssim \|\bar{P}_j' w_k\|_{L^2_{\bar{x}}} + \|\mathcal{D}_{j,k}\bar{P}_j' w_k\|_{L^2_{\bar{x}}} \lesssim \|\bar{P}_{ij}' v\|_{L^2_{\bar{x}}} + \|\mathcal{M}_{ij}\mathcal{D}_j\mathcal{N}_{ij}\bar{P}_{ij}v\|_{L^2_{\bar{x}}}\\ &\lesssim \|\bar{P}_j' v\|_{L^2_{\bar{x}}} + \|L^j\bar{P}_{ij}v\|_{L^2_{\bar{x}}}.\end{aligned} \tag{4.46}$$

Estimates (4.42), (4.45) and (4.46) imply the first claim. The second follows from the commutator

$$[L^j, \check{P}_i\Omega_i] = M[a^j(t), \check{P}_j\Omega_i]$$

which is bounded on L^2 by $c\|a^j(t)\|_{C_b^2} \le c'\|a(t)\|_{C_b^2}$. This can be checked as in the proof of Lemma 4.16. □

We add a variant of Theorem 4.8 only involving fractional space derivatives, which is better suited to the study of evolution equations. (See also §4.4 A).) The basic idea is that on the range of $\check{P}_j$ space and space-time derivatives are equivalent, whereas L behaves much better off the light cone, as seen in the above proof. For simplicity we take $q < \infty$ and inhomogeneous norms. The energy estimate (4.6) yields $\|v\|_{L^\infty_{J_T}L^2} \lesssim_T \|v_0\|_{L^2} + \|f\|_{L^1_T L^2}$ in (4.47) if $s \ge 1$.

Corollary 4.25. *In the setting of Theorem 4.8, let $q < \infty$, $T \ge 1$, and $J_T = (-\frac{1}{2}, \frac{T}{2})$. We then obtain*

$$\|\langle D\rangle^{-\gamma-\frac{\sigma}{2}}v\|_{L^p_T L^q} \lesssim_T \|v\|_{L^\infty_{J_T}L^2} + \|\langle D\rangle^{-\sigma}f\|_{L^2_T L^2} + \|\langle D\rangle^{-\frac{1}{2}-\frac{\sigma}{2}}\rho\|_{L^2_T L^2}, \tag{4.47}$$

if the terms on the right are finite. The analogous result is true on $[-T, 0]$.

Proof. The last claim is shown by reflection. Since we allow for $s < 2$, we have to replace the coefficients a^{-1} by $\hat{a}^l = \bar{P}_{\le l}$ for $l = \frac{2}{2+s}j$ and $j \ge j_0$ with $j_0 \in \mathbb{N}_0$ from the proof of Lemma 4.19. Note that $\bar{P}_j = \bar{P}_j P_{\le j_0}$ for $j < j_0$, where $\bar{P}_0$ means $\bar{P}_{\le 0}$. Thus the lower frequencies can be treated as in Lemma 4.12, but now using Sobolev's embedding on $\mathbb{R}^3$ with $-\gamma - \frac{3}{q} = \frac{1}{p} - \frac{3}{2}$. It follows

$$\|\langle D\rangle^{-\gamma-\frac{\sigma}{2}}\bar{P}_j v\|_{L^p_T L^q} \lesssim \|P_{\le j_0}v\|_{L^p_T\mathcal{H}^{\frac{1}{p}-\frac{\sigma}{2}}} \lesssim T^{\frac{1}{p}}\|v\|_{L^\infty_T L^2} =: S_0. \tag{4.48}$$

As in Lemma 4.16, Theorem 3.6 and Minkowski's inequality allow us to pass to frequency-localized terms with $\bar{P}_j$ which we then split by means of $\check{P}_j$, cf. (4.39). However, on the range of $\bar{P}_j$ the fractional derivative $\langle D\rangle^\alpha$ is not comparable to $2^{j\alpha}$ in norm estimates, so that we have to keep it. Moreover, to pass to the time interval $\mathbb{R}$, we extend f by odd reflection and a time cut-off being 1 on $(-\frac{1}{2}, T+\frac{1}{2})$ to a map supported in $(-1, T+1)$. The solution of the corresponding problem (4.3) on $\mathbb{R}$ is still denoted by v. Note that f and ρ on $(-1, T+1)$ are bounded by the functions on $(0,T)$. We set $\tilde v = \phi v$ for a map $\phi \in C^\infty(\mathbb{R})$ with $\phi = 1$ on $[0,T]$ and support in $(-\frac12, T+\frac12)$. We start with

$$\begin{aligned}\|\langle D\rangle^{-\gamma-\frac{\sigma}{2}} v\|^2_{L^p_T L^q} &\le \|\langle D\rangle^{-\gamma-\frac{\sigma}{2}} \tilde v\|^2_{L^p_{\mathbb{R}} L^q} \lesssim S_0^2 + \sum\nolimits_{j\ge j_0} \|\langle D\rangle^{-\gamma-\frac{\sigma}{2}} \bar P_j \tilde v\|^2_{L^p_{\mathbb{R}} L^q}\\ &\lesssim S_0^2 + \sum\nolimits_{j\ge j_0} \|\langle D\rangle^{-\gamma-\frac{\sigma}{2}} \bar P_j^\tau \tilde v\|^2_{L^p_{\mathbb{R}} L^q} + \sum\nolimits_{j\ge j_0} \|\langle D\rangle^{-\gamma-\frac{\sigma}{2}} \check P_j \bar P_j \tilde v\|^2_{L^p_{\mathbb{R}} L^q}. \quad (4.49)\end{aligned}$$

On the range of $\check P_j$ we can replace D by $\bar D$ and vice versa. We use $\hat L^l$ with coefficients $\hat a^l$ which preserve the frequency localization. Theorem 4.8 yields

$$\begin{aligned}&\|\langle D\rangle^{-\gamma-\frac{\sigma}{2}} \check P_j \bar P_j \tilde v\|_{L^p_{\mathbb{R}} L^q} \qquad (4.50)\\ &\lesssim \|\check P_j \bar P_j \tilde v\|_{L^2_{\mathbb{R}} L^2} + \|\langle \bar D\rangle^{-\sigma} \hat L^l \check P_j \bar P_j \tilde v\|_{L^2_{\mathbb{R}} L^2} + \|\langle \bar D\rangle^{-\frac12-\frac{\sigma}{2}} \check P_j \bar P_j(\phi\rho)\|_{L^2_{\mathbb{R}} L^2}\\ &\lesssim \|\bar P_j \tilde v\|_{L^2_{\mathbb{R}} L^2} + 2^{-\sigma j} \|(\hat L^l \check P_j \bar P_j - \check P_j \bar P_j L)(\phi v)\|_{L^2_{\mathbb{R}} L^2} + \|\check P_j \bar P_j(\phi f)\|_{L^2_{\mathbb{R}} \mathcal{H}^{-\sigma}}\\ &\qquad + \|\check P_j \bar P_j(\phi' v)\|_{L^2_{\mathbb{R}} L^2} + \|\bar P_j(\phi\rho)\|_{L^2_{\mathbb{R}} \mathcal{H}^{-\frac12-\frac{\sigma}{2}}}\\ &\lesssim \|\bar P_j'(\phi v)\|_{L^2_{\mathbb{R}} L^2} + 2^{-\delta j}\|\phi v\|_{L^2_{\mathbb{R}} L^2} + \|\bar P_j(\phi f)\|_{L^2_{\mathbb{R}} \mathcal{H}^{-\sigma}} + \|\bar P_j(\phi' v)\|_{L^2_{\mathbb{R}} L^2} + \|\bar P_j(\phi\rho)\|,\end{aligned}$$

for some $\delta > 0$, arguing as in Lemma 4.19 in the last step. (For $s = 2$ one can omit $\bar P_{\le l}$ and treat $[L, \check P_j \bar P_j]$ as in the proof of Lemma 4.16.) Using Littlewood–Paley, the last square sum in (4.49) is thus bounded by

$$\begin{aligned}&\|\phi v\|^2_{L^2_{\mathbb{R}} L^2} + \|\phi' v\|^2_{L^2_{\mathbb{R}} L^2} + \|\phi f\|^2_{L^2_{\mathbb{R}} \mathcal{H}^{-\sigma}} + \|\phi\rho\|^2_{L^2_{\mathbb{R}} \mathcal{H}^{-\frac12-\frac{\sigma}{2}}}\\ &\qquad \le_T \|v\|^2_{L^\infty(J_T, L^2)} + \|f\|^2_{L^2_T \mathcal{H}^{-\sigma}} + \|\rho\|^2_{L^2_T \mathcal{H}^{-\frac12-\frac{\sigma}{2}}}.\end{aligned}$$

By means of Sobolev in space and time, $|\tau| \le |\bar\xi|$, and $|\xi| \le |\bar\xi|$ if $\frac1p - \frac{\sigma}{2} \ge 0$, the penultimate square sum in (4.49) is dominated via

$$\|\langle D\rangle^{-\gamma-\frac{\sigma}{2}} \bar P_j^\tau \tilde v\|_{L^p_{\mathbb{R}} L^q} \lesssim \|\langle D_t\rangle^{\frac12-\frac1p} \langle D\rangle^{\frac1p-\frac{\sigma}{2}} \bar P_j^\tau \tilde v\|_{L^2_{\mathbb{R}} L^2} \lesssim 2^{\frac12 j} \|\bar P_j^\tau \tilde v\|_{L^2_{\mathbb{R}} L^2}.$$

We can now proceed as in (4.42) gaining a derivative and deduce

$$\|\langle D\rangle^{-\gamma-\frac{\sigma}{2}} \bar P_j^\tau \tilde v\|_{L^p_{\mathbb{R}} L^q} \lesssim 2^{-\frac12 j} \|\hat L^l \bar P_j' \tilde v\|_{L^2_{\bar x}} + \|\bar P_j' \tilde v\|_{L^2_{\mathbb{R}} L^2}$$

As $\sigma \le 1$, we estimate the square sum of the above terms as in (4.50) by

$$\|\phi v\|^2_{L^2_{\mathbb{R}} L^2} + \|\phi' v\|^2_{L^2_{\mathbb{R}} L^2} + \|\langle \bar D\rangle^{-\sigma}(\phi f)\|^2_{L^2_{\mathbb{R}} L^2} \lesssim_T \|v\|^2_{L^\infty(J_T, L^2)} + \|f\|^2_{L^2_T \mathcal{H}^{-\sigma}}$$

using $\|\langle \bar D\rangle^{-\sigma}(\phi f)\|^2_{L^2_{\mathbb{R}} L^2} \lesssim \|\langle D\rangle^{-\sigma}(\phi f)\|^2_{L^2_{\mathbb{R}} L^2}$. Combined with (4.48), (4.49) and (4.50), the above inequality yields (4.47). $\square$

4.3. A localized Strichartz estimate for a half-wave equation

For Theorem 4.8 it remains to show the following Strichartz estimate, taken from [**63**]. We state it for coefficients $a \in C_b^2(\mathbb{R}^4, \mathbb{R}^{3\times 3}_{\eta'})$ with $\eta' > 0$. We set $|\xi|_{a(\bar x)} = \sqrt{a(\bar x)\xi \cdot \xi}$, $q_\pm(\bar x, \bar\xi) := \tau \pm |\xi|_{a(\bar x)}$, and $Q_\pm = \mathrm{Op}(\mathrm{i} q_\pm)$. We need the case $a = \varepsilon^j \mu^j I$ with $Q_\pm = \partial_t \pm \mathrm{i}\sqrt{\varepsilon^j \mu^j}|D|$, satisfying the condition below.

THEOREM 4.26. *In the above setting assume* $|\partial_{\bar{x}}^{\alpha} a| \lesssim_{\alpha} 2^{\frac{1}{2}(|\alpha|-2)j}$ *for* $\alpha \in \mathbb{N}_0^4$ *and some sufficiently large* $j \in \mathbb{N}$. *Let* $w \in L^2_{\mathbb{R}} L^2$ *decay rapidly outside a compact set and* (p, q, γ) *be strict admissible, see* (4.10). *We then have*

$$2^{-\gamma j} \|\bar{P}_j w\|_{L^p_{\mathbb{R}} L^q} \leq C\big(\|\bar{P}_j w\|_{L^2_{\mathbb{R}} L^2} + \|Q_{\pm} \bar{P}_j w\|_{L^2_{\mathbb{R}} L^2}\big).$$

In the following we restrict to the case $q_- =: q$ and $Q_- =: Q$, as the other one is handled in the same way (or via time reversal). We sketch the proof given in pp. 397–415 in [**63**], following the treatment in Subsection 4.1 of [**48**].

A) Preparations. Let $\lambda = 2^j$ for $j \geq k_0$ from above. We write $L^2_{\bar{x}} = L^2_{\mathbb{R}} L^2$ and also $\bar{P}_\lambda$, T_λ etc. instead of $\bar{P}_j$, T_j, etc. since j is fixed and we do not sum over it. We further set $w_\lambda = \bar{P}_\lambda w$ for w with finite norms $c(w) := \|\bar{P}_\lambda w\|_{L^2_{\mathbb{R}} L^2} + \|Q \bar{P}_\lambda w\|_{L^2_{\mathbb{R}} L^2}$, $\tilde{w}_\lambda = T_\lambda w_\lambda$, and write E_ν for any operator with norm $c\lambda^\nu$ in L^2_Φ, $L^2_{\bar{x}}$ or $L^p_{\mathbb{R}} L^q$, depending on the context. Moreover $g \in L^2_\Phi$ means that $\|g\|_{L^2_\Phi} \lesssim c(w)$, and analogously for other spaces. As before we set $\phi_\lambda(\bar{x}, \bar{\xi}) = \phi(\bar{x}, \lambda^{-1}\bar{\xi})$. We start with a useful observation.

LEMMA 4.27. *In the setting of Theorem 4.8, let* $\phi \in C^\infty(\mathbb{R}^4)$ *have bounded derivatives and assume that* $\|\phi \tilde{w}_\lambda\|_{L^2_\Phi} \lesssim \lambda^{-\frac{1}{2}} \|w_\lambda\|_{L^2_{\bar{x}}}$. *Then it is enough to treat* $(1 - \phi_\lambda) w_\lambda$ *in Theorem 4.8, where* $T_\lambda\big((1 - \phi_\lambda) w_\lambda\big) = (1 - \phi)\tilde{w}_\lambda + E_{-\frac{1}{2}} w_\lambda$.

PROOF. Sobolev's embedding, the isometry of T_λ and Theorem 4.22 yield

$$\begin{aligned}
\lambda^{-\gamma} \|w_\lambda\|_{L^p_{\mathbb{R}} L^q} &\lesssim \lambda^{\frac{1}{2}} \|\phi_\lambda w_\lambda\|_{L^2_{\bar{x}}} + \lambda^{-\gamma} \|(1 - \phi_\lambda) w_\lambda\|_{L^p_{\mathbb{R}} L^q} \\
&= \lambda^{\frac{1}{2}} \|T_\lambda \phi_\lambda w_\lambda\|_{L^2_\Phi} + \lambda^{-\gamma} \|(1 - \phi_\lambda) w_\lambda\|_{L^p_{\mathbb{R}} L^q} \\
&\lesssim \lambda^{\frac{1}{2}} \|\phi T_\lambda w_\lambda\|_{L^2_\Phi} + \|w_\lambda\|_{L^2_{\bar{x}}} + \lambda^{-\gamma} \|(1 - \phi_\lambda) w_\lambda\|_{L^p_{\mathbb{R}} L^q} \\
&\lesssim \|w_\lambda\|_{L^2_{\bar{x}}} + \lambda^{-\gamma} \|(1 - \phi_\lambda) w_\lambda\|_{L^p_{\mathbb{R}} L^q}.
\end{aligned}$$

The second claim also follows from Theorem 4.22. $\square$

Note that $\phi_\lambda \chi_\lambda$ satisfies Mikhlin's conditions in $\bar{\xi}$ and that (4.37) also works if one replaces the compactly supported map v by w which decays rapidly outside a compact set. Using the above lemma, we thus have to estimate $\tilde{w}_\lambda$ only on $K = \overline{B}(0, 2) \times A(\frac{1}{4}, 4)$. Similary, take a neighborhood $U_\kappa = \{|q(\bar{x}, \bar{\xi})| < \kappa\}$ of the 'light cone' $\mathcal{C} = \{q = 0\}$ for some $\kappa > 0$. On the complement of U_κ we can invert q and thus argue as in (4.42) to estimate $\|\phi \tilde{w}_\lambda\|_{L^2_\Phi} \lesssim \lambda^{-1} \|w_\lambda\|_{L^2_{\bar{x}}}$ for ϕ having support in $U^c_{\kappa/2}$. By Lemma 4.27 it suffices to bound $\tilde{w}_\lambda$ on $K \cap U_\kappa =: K_\kappa$.

We start with estimates for $\tilde{w}_\lambda$. Theorem 4.22 with $s = 2$ and $a = q$ and Lemma 2.1 in [**62**] yield

$$\tilde{w}_\lambda \in L^2_\Phi, \qquad (\lambda q + 2(\bar{\partial} q)(\partial - \mathrm{i}\lambda\bar{\xi}))\tilde{w}_\lambda \in L^2_\Phi, \tag{4.51}$$

$$\lambda^{-\frac{1}{2}} (\partial_{\bar{\xi}} - \lambda\bar{\xi})\tilde{w}_\lambda \in L^2_\Phi, \qquad \lambda^{-\frac{1}{2}} (\partial_{\bar{\xi}} - \lambda\bar{\xi})(\lambda q + 2(\bar{\partial} q)(\partial - \mathrm{i}\lambda\bar{\xi}))\tilde{w}_\lambda \in L^2_\Phi. \tag{4.52}$$

Using $\partial_{\bar{x}} = \mathrm{i}\partial_{\bar{\xi}}$ on anti-holomorphic functions, the second and third property above and the calculation after formula (26) in [**63**] lead to

$$\lambda^{\frac{1}{2}} q \tilde{w}_\lambda \in L^2_\Phi \qquad \text{and} \qquad q(\partial - \mathrm{i}\lambda\bar{\xi}))\tilde{w}_\lambda \in L^2_\Phi. \tag{4.53}$$

Hence, the second part of (4.51) remains valid if we replace q by hq for a map h in $W^{1,\infty}(\mathbb{R}^8)$. The equality $\partial_{\bar{x}} = \mathrm{i}\partial_{\bar{\xi}}$ and (4.51) also imply

$$\big[\mathrm{i}(\partial_{\bar{x}}q\,\partial_{\bar{\xi}} - \partial_{\bar{\xi}}q\,\partial_{\bar{x}}) + \lambda(q - \mathrm{i}\bar{\xi}\partial_{\bar{x}}q - \bar{\xi}\partial_{\bar{\xi}}q)\big]\tilde{w}_\lambda \in L^2_\Phi,$$
$$\big[(\partial_{\bar{x}}q\,\partial_{\bar{x}} + \partial_{\bar{\xi}}q\,\partial_{\bar{\xi}}) + \lambda(q - \bar{\xi}\partial_{\bar{\xi}}q - \mathrm{i}\bar{\xi}\partial_{\bar{x}}q)\big]\tilde{w}_\lambda \in L^2_\Phi.$$

These expressions simplify a bit if we remove the weight Φ by setting $\hat{w}_\lambda = \Phi^{\frac{1}{2}}\tilde{w}_\lambda$. Since $\partial_{\bar{\xi}}\Phi^{\frac{1}{2}} = -\lambda\bar{\xi}\Phi^{\frac{1}{2}}$, we deduce

$$\big[(\partial_{\bar{x}}q\,\partial_{\bar{\xi}} - \partial_{\bar{\xi}}q\,\partial_{\bar{x}}) - \mathrm{i}\lambda(q - \bar{\xi}\partial_{\bar{\xi}}q)\big]\hat{w}_\lambda \in L^2_z, \tag{4.54}$$
$$\big[(\partial_{\bar{x}}q\,\partial_{\bar{x}} + \partial_{\bar{\xi}}q\,\partial_{\bar{\xi}}) + \lambda(q - \mathrm{i}\bar{\xi}\partial_{\bar{x}}q)\big]\hat{w}_\lambda \in L^2_z. \tag{4.55}$$

with $L^2_z = L^2(\mathbb{R}^8)$. The first property corresponds to an ODE along the 'Hamiltonian flow' for q and is used below to control regularity on $\mathcal{C}$, whereas the second relates to a gradient flow for q and allows to estimate off $\mathcal{C}$. Observe that $q - \bar{\xi}\partial_{\bar{x}}q$ vanishes by 1-homogeneity. We write $H_q = \partial_{\bar{x}}q\,\partial_{\bar{\xi}} - \partial_{\bar{\xi}}q\,\partial_{\bar{x}}$ for the Hamiltonian vector field for q.

The first estimates in (4.51) and (4.52) translate into $\hat{w}_\lambda, \lambda^{-\frac{1}{2}}\partial_{\bar{\xi}}\hat{w}_\lambda \in L^2_z$. The (proof of the) trace theorem and (4.51) imply

$$\lambda^{-\frac{1}{4}}\|\hat{w}_\lambda\|_{L^2(\mathcal{C}\cap K)} \lesssim \lambda^{-\frac{1}{4}}\|\hat{w}_\lambda\|_{L^2_z} + \lambda^{-\frac{1}{4}}\|\hat{w}_\lambda\|^{\frac{1}{2}}_{L^2_z}\|\partial_{\bar{\xi}}\hat{w}_\lambda\|^{\frac{1}{2}}_{L^2_z} \lesssim c(w). \tag{4.56}$$

Based on the second part of (4.51), as above one computes $\lambda^{-\frac{1}{2}}\partial_{\bar{\xi}}H_q\hat{w}_\lambda \in L^2_z$. Together with $H_q\hat{w}_\lambda \in L^2_z$ from (4.55), we similarly derive

$$\lambda^{-\frac{1}{4}}\|H_q\hat{w}_\lambda\|_{L^2(\mathcal{C}\cap K)} \lesssim c(w). \tag{4.57}$$

In the following we only need these two estimates.

B) Reduction to an estimate for an oscillatory integral. We want to estimate $\hat{w}_\lambda$ by its trace on $\mathcal{C}\cap K$ which is controlled via (4.56) and (4.57). To this aim we parametrize a neighborhood K'_κ of (a part of) $\mathcal{C}\cap K$ by (r,ζ) for the distance r to $\mathcal{C}\cap K$ and C^2-coordinates ζ on $\mathcal{C}\cap K$. Here r is given by

$$(\bar{x},\bar{\xi}) = (\bar{x}_0,\bar{\xi}_0) + r\frac{\nabla_{\bar{x},\bar{\xi}}q(\bar{x}_0,\bar{\xi}_0)}{|\nabla_{\bar{x},\bar{\xi}}q(\bar{x}_0,\bar{\xi}_0)|} = \zeta + r(\nabla_{\bar{x},\bar{\xi}}r)(\bar{x}_0,\bar{\xi}_0)$$

for the base point $(\bar{x}_0,\bar{\xi}_0) \in \mathcal{C}\cap K$. It can be seen that r/q is Lipschitz, see p. 400 in [**63**]. We can replace q by r in (4.51) and hence in (4.55). Moreover, the differential expression for r in (4.55) transforms into

$$\mathrm{d}_r\hat{w}_\lambda := \big[\partial_r + \lambda\big(r - \mathrm{i}\partial_{\bar{x}}r(\zeta)(\bar{\xi}(\zeta) + r\partial_{\bar{\xi}}r(\zeta))\big)\big]\hat{w}_\lambda =: g \in L^2_z. \tag{4.58}$$

Here we consider $\bar{\xi}$, $\bar{x}$, $\partial_{\bar{x}}r$ and $\partial_{\bar{\xi}}r$ as functions of $\zeta \in \mathcal{C}\cap K$, which is partly suppressed below.

Below we use the transformation $\mathrm{d}\bar{x}\,\mathrm{d}\bar{\xi} = h(r,\zeta)\,\mathrm{d}r\,\mathrm{d}\zeta$, where h is strictly positive, $h(0,\zeta) = 1$ and $\mathrm{d}\zeta$ denotes the surface measure on $\mathcal{C}\cap K$. We split $\hat{w}_\lambda$ by means of the ordinary differential equations

$$\mathrm{d}_r\hat{w}^1 = g \quad \text{on} \quad K'_\kappa, \qquad \hat{w}_1 = 0 \quad \text{on} \quad \mathcal{C}\cap K,$$
$$\mathrm{d}_r\hat{w}^2 = 0 \quad \text{on} \quad K'_\kappa, \qquad \hat{w}_2 = \hat{w}_\lambda =: \varphi \quad \text{on} \quad \mathcal{C}\cap K,$$

$$\mathrm{d}_r \hat{w}^3 = -\partial_r \tfrac{1}{h}\, \hat{w}^2 \quad \text{on} \quad K'_\kappa, \qquad \hat{w}_3 = 0 \quad \text{on} \quad \mathcal{C} \cap K.$$

We have $\hat{w}^3 = (1-\frac{1}{h})\hat{w}^2$ and $\hat{w}_\lambda = \hat{w}^1 + \hat{w}^3 + \frac{1}{h}\hat{w}^2$. We set $w^1 = T^*_\lambda \Phi^{-\frac{1}{2}}(\hat{w}^1 + \hat{w}^3)$ and $w^2 = T^*_\lambda \Phi^{-\frac{1}{2}} \frac{1}{h}\hat{w}^2$ so that $w^1 + w^2 = T^\star_\lambda T_\lambda w_\lambda = w_\lambda$. Energy-type estimates show that $\lambda^{\frac{1}{2}}\hat{w}_1 \in L^2_z$ and $\lambda^{\frac{1}{2}}\|\hat{w}_3\|_2 \lesssim \|\hat{w}_2\|_2 \lesssim \lambda^{-\frac{1}{4}}\|\varphi\|_2$, see (37) in [**63**]. Hence (4.56) implies $\lambda^{\frac{1}{2}}\|w^1\|_2 \lesssim c(w)$.

It remains to treat w^2. The ode for $\hat{w}^2$ can be solved explicitely. Applying $T^*_\lambda \Phi^{-\frac{1}{2}}$ and transforming to (r, ζ), we deduce

$$w^2(\bar{y}) = C_4 \lambda^3 \int \mathrm{e}^{-\frac{\lambda}{2}(\bar{y}-\bar{x}-r r_{\bar{x}} - \mathrm{i}(\bar{\xi}+r r_{\bar{\xi}}))^2} \mathrm{e}^{-\frac{\lambda}{2}(\bar{\xi}+r r_{\bar{\xi}})^2} \mathrm{e}^{-\frac{\lambda}{2}r^2} \mathrm{e}^{i\lambda(r r_{\bar{x}}\bar{\xi} + \frac{1}{2}r^2 r_{\bar{x}} r_{\bar{\xi}})} \varphi(\zeta)\, \mathrm{d}r\, \mathrm{d}\zeta$$

with the abbreviations $r_{\bar{x}} = \partial_{\bar{x}} r$ and $r_{\bar{\xi}} = \partial_{\bar{\xi}} r$. Astonishingly the r-integral can be calculated explicitely, leading to

$$w^2(\bar{y}) = C_4 \lambda^{-\frac{1}{2}} \lambda^3 \int_{\mathcal{C}} \mathrm{e}^{\mathrm{i}\lambda\bar{\xi}\cdot(\bar{y}-\bar{x})} \mathrm{e}^{-\frac{\lambda}{2}\omega_\zeta(\bar{y}-\bar{x})} \alpha(\zeta)\varphi(\zeta)\, \mathrm{d}\zeta =: C_4 \lambda^{-\frac{1}{2}} V_\lambda(\alpha^{-1}\varphi)(\bar{y}),$$

$$\omega_\zeta(\bar{y}-\bar{x}) = (\bar{y}-\bar{x})^2 - \frac{[(r_{\bar{x}} + \mathrm{i} r_{\bar{\xi}})\cdot(\bar{y}-\bar{x})]^2}{r^2_{\bar{\xi}} + 2r^2_{\bar{x}} + \mathrm{i} r_{\bar{x}} \cdot r_{\bar{\xi}}}, \qquad \alpha(\zeta) = (r^2_{\bar{\xi}} + 2r^2_{\bar{x}} + \mathrm{i} r_{\bar{x}} r_{\bar{\xi}})^{-\frac{1}{2}}.$$

The map α is well defined since $|\nabla r| = 1$. We can include it into φ as it is Lipschitz and so $[H_q, \alpha]$ is bounded. It can been seen that $\mathrm{Re}\,\omega > 0$ and the above functions are continuous in $\bar{x}$ and smooth in $\bar{\xi}$.

Because of (4.56) and (4.57), we have to show

$$\|V_\lambda \varphi\|_{L^p_{\mathbb{R}} L^q} \lesssim \lambda^{\gamma+\frac{1}{4}} \big(\|\varphi\|_{L^2(\mathcal{C})} + \|H_q \varphi\|_{L^2(\mathcal{C})}\big) \tag{4.59}$$

for φ with support in $\mathcal{C} \cap K$. To exploit the extra regularity on the right of (4.59), we use the Hamiltonian flow $(\bar{x}_t, \bar{\xi}_t)$ for q solving

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\bar{x}_t = q_{\bar{\xi}}(\bar{x}_t, \bar{\xi}_t), \qquad \tfrac{\mathrm{d}}{\mathrm{d}t}\bar{\xi}_t = -q_{\bar{x}}(\bar{x}_t, \bar{\xi}_t), \qquad (\bar{x}_0, \bar{\xi}_0) = (\bar{x}, \bar{\xi}). \tag{4.60}$$

Note that $(\bar{x}_t, \bar{\xi}_t)$ belongs to $\mathcal{C}$ if $(\bar{x}, \bar{\xi}) \in \mathcal{C} = \{q = 0\}$ since $\frac{\mathrm{d}}{\mathrm{d}t}q(\bar{x}_t, \bar{\xi}_t) = (\nabla_{\bar{x},\bar{\xi}} q) q(\bar{x}_t, \bar{\xi}_t) \cdot \frac{\mathrm{d}}{\mathrm{d}t}(\bar{x}_t, \bar{\xi}_t) = 0$ by (4.60). We set $R\varphi(\bar{x}, \bar{\xi}) = \int_0^\infty \mathrm{e}^{-t}\varphi(\bar{x}_t, \bar{\xi}_t)\,\mathrm{d}t$ noting that $(I - H_q)R\psi = \psi$. Setting $\varphi = R\psi$, estimate (4.59) and thus Theorem 4.26 follow from the next result, which is Theorem 6 of [**63**].

Proposition 4.28. *In the above setting, let $b \in C_c^\infty(\mathbb{R}^4 \times \mathbb{R}^4 \setminus \{0\})$ being one for $|\xi| \in [\frac{1}{4}, 4]$. We then obtain*

$$\|V_\lambda b R\psi\|_{L^p_{\mathbb{R}} L^q} \lesssim \lambda^{\gamma+\frac{1}{4}} \|\psi\|_{L^2(\mathcal{C})}.$$

Equivalent statements are that $(V_\lambda bR)^*\colon L^{p'}_{\mathbb{R}} L^{q'} \to L^2(\mathcal{C})$ is bounded by $\lambda^{\gamma+\frac{1}{4}}$ or

$$Z = V_\lambda b R R^* b V^*_\lambda : L^{p'}_{\mathbb{R}} L^{q'} \to L^p_{\mathbb{R}} L^q \text{ by } \lambda^{2\gamma+\frac{1}{2}}. \tag{4.61}$$

since $\|T^\star g\|_2^2 = \langle TT^\star g, g\rangle$.

C) On the proof of the core estimate. We give a brief overview of the demanding and lengthy derivation of Proposition 4.28. First, for the energy triple $(\infty, 2, 0)$ it suffices to show to the stronger estimate $\|\tilde{V}_{\lambda,t,\tau}\varphi\|_2 \lesssim \mathrm{e}^{-c\lambda(t-\tau)^2}\|\varphi\|_2$ for $t,\tau \in \mathbb{R}$ and the operator $V_{\lambda,t,\tau}\colon L^2(\mathbb{R}^6) \to L^2(\mathbb{R}^3)$ given by

$$\tilde{V}_{\lambda,t,\tau}\varphi(y) = \lambda^3 \int_{\mathbb{R}^6} \mathrm{e}^{\mathrm{i}\lambda\bar{\xi}\cdot(\bar{x}-\bar{y})}\mathrm{e}^{-\lambda\omega_{\bar{x},\xi}(\bar{x}-\bar{y})} b(\bar{x},\bar{\xi})\varphi(x,\xi)\,\mathrm{d}(x,\xi).$$

This estimate follows from the inequality

$$|\tilde{K}_{t,\tau}(y,y')| \lesssim_N \lambda^3 \mathrm{e}^{-c\lambda(t-\tau)^2}\big(1+\lambda|y-y'|\big)^{-N}$$

for the kernel of $\tilde{V}_{\lambda,t,\tau}\tilde{V}^\star_{\lambda,t,\tau}$. See p. 404 of [**63**].

The main step in the proof of Proposition 4.28 is the case $2 < p < q$. (We have $p > 2$ for strict triples and $m = 3$.) To treat any strict triple $(\bar{p},\bar{q},\bar{\gamma}) \neq (\infty,2,0)$, we choose $p < \bar{p}$ and $q > \max\{\bar{q},p\}$ and interpolate using $\theta := p/\bar{p} \in (0,1)$ and

$$\frac{1}{\bar{p}} = \frac{\theta}{p} + \frac{1-\theta}{\infty}, \qquad \frac{1}{\bar{q}} = \frac{\theta}{q} + \frac{1-\theta}{2}, \qquad \bar{\gamma} = \frac{3}{2} - \frac{3}{\bar{q}} - \frac{1}{\bar{p}} = \theta\gamma + (1-\theta)0.$$

So let $2 < p \le q$. We note that $R^\star$ is given like R with $(\bar{x}_{-t},\bar{\xi}_{-t})$ and hence $RR^\star\varphi(\bar{x},\bar{\xi}) = \int_{\mathbb{R}} \mathrm{e}^{-|t|}\varphi(\bar{x}_t,\bar{\xi}_t)\,\mathrm{d}t =: \mathcal{R}\varphi(\bar{x},\bar{\xi})$. We can thus write

$$Z = V_\lambda b\mathcal{R}\delta_{q=0}bV_\lambda^*.$$

Let $q^{-\theta}(\bar{x},\bar{\xi}) = (\tau - a(\bar{x})\xi\cdot\xi)_+^{-\theta}$ for $\operatorname{Re}\theta < 1$. As a distribution-valued map, $q^{-\theta}$ can be analytically extended to $\operatorname{Re}\theta \in [1,2)$. Moreover, for $\theta = 1$ it coincides with the restriction operator $\delta_{q=0}$ to $\mathcal{C}$. (Compare §IX.1.2 in [**59**] or the proof of Proposition 3.7 in [**49**].)

One estimates Z by an interpolation argument. For this we fix $\phi \in C_c^\infty(\mathbb{R}^4)$ being 1 near $\bar{x} = 0$ and set $\phi^\delta(\bar{x}) = \phi(\delta\bar{x})$. For an integral operator J with kernel k we define J^δ with kernel $\phi^\delta(\bar{x}-\bar{y})k(\bar{x},\bar{y})$. Also, for $\operatorname{Re}\theta \ge 0$ let $\mathcal{R}_\theta$ be given by inserting $|t|^{1-\theta}$ in the integrand of $\mathcal{R}$. We then introduce the operators

$$W_\theta = V_\lambda bq^{-\theta}\mathcal{R}_\theta \mathrm{e}^{\mathrm{i}\lambda q}bV_\lambda^*, \qquad X_\theta = V_\lambda bq^{-\theta}\mathcal{R}_\theta bV_\lambda^*, \qquad Z_\theta = W_\theta^{\sqrt{\lambda}} + X_\theta - X_\theta^{\sqrt{\lambda}}.$$

Here $W_\theta^{\sqrt{\lambda}}$ possesses a short-range kernel and $X_\theta - X_\theta^{\sqrt{\lambda}}$ a long-range kernel, and we have $Z_1 = Z$.

The maps Z_θ for $\operatorname{Re}\theta = 0$ are estimated in $\mathcal{B}(L^2_{\mathbb{R}}L^2)$ by c, and for $\operatorname{Re}\theta = \vartheta := \frac{q}{2}$ in $\mathcal{B}(L^{r'}_{\mathbb{R}}L^1, L^r_{\mathbb{R}}L^\infty)$ by $c\lambda^{\vartheta(2\gamma+\frac{1}{2})}$. Here $r > p$ is chosen such that $\frac{1}{p} = \frac{\vartheta}{2} + \frac{1-\vartheta}{r}$. Hence the interpolation result in § IX.1.2.5 of [**59**] will yield Proposition 4.28.

The estimate for $\operatorname{Re}\theta = 0$ in $\mathcal{B}(L^2_{\mathbb{R}}L^2)$ is again based on analysis of the kernel of $V_\lambda V_\lambda^*$, see p. 405f. of [**63**]. The treatment of the case $\operatorname{Re}\theta = \vartheta$ is reduced to the operators $Z^t = V_\lambda bF^t\delta_{q=0}bV_\lambda^*$ for $t \in \mathbb{R}$, where F^t is the substitution with $(\bar{x}_t,\bar{\xi}_t)$. This is achieved by handling the t- and τ-integrals separately, see p. 406f. of [**63**]. The operator Z^t has the kernel

$$K_t(\bar{y},\bar{y}') = \lambda^6 \int_{\mathbb{R}^6} \mathrm{e}^{\lambda\omega(\bar{x}-\bar{y})}\mathrm{e}^{\lambda\omega(\bar{x}_t-\bar{y}')} b(\bar{x},\bar{\xi})b(\bar{x}_t,\bar{\xi}_t)\mathrm{e}^{\mathrm{i}\lambda\bar{\xi}\cdot(\bar{x}-\bar{y})}\mathrm{e}^{-\mathrm{i}\lambda\bar{\xi}_t\cdot(\bar{x}_t-\bar{y}')}\,\mathrm{d}(\bar{x},\xi).$$

The desired norm inequality for Z_θ with $\operatorname{Re}\theta = \vartheta$ follows from the kernel bound

$$|K_t(\bar{y},\bar{y}')| \lesssim \lambda^4 \mathrm{e}^{-c\lambda(s-s'-t)^2}(1+\lambda|\bar{y}-\bar{y}'|)^{-1} \tag{4.62}$$

for $\bar{y} = (s, y)$ and $\bar{y}' = (s', y')$ in $\mathbb{R}^4$ and $t \in \mathbb{R}$, established in Theorem 7 of [**63**]. This result is based on a detailed regularity analysis of the Hamiltonian flow $(\bar{x}_t, \bar{\xi}_t)$ in Lemmas 9-12 of [**63**], which relies on the assumption on $\partial_{\bar{x}}^{\alpha} a$.

Using these lemmas, the inequality (4.62) is proven for the short range $t < \sqrt{\lambda}$ by reducing it to standard oscillatory integral estimates from Sections VIII.3 and IX.1 of [**59**]. These results depend on the number 2 of non-vanishing principal curvatures of $\mathcal{C}$. The long range $t \geq \sqrt{\lambda}$ is split into a 'non-oscillatory' part $c|\bar{y} - \bar{y}'| \leq t^2$ and the 'oscillatory' part $\lambda \leq t^2 \leq c|\bar{y} - \bar{y}'|$. The former can be handled by more direct estimates, whereas the latter is more complicated and involves the Fourier transform.

4.4. Variants and applications

Theorem 4.8 is not suited for applications to quasilinear problems. In this section we discuss appropriate variants and an application to the local wellposedness theory. However, for such results one has to differentiate the nonlinear terms, which leads to matrix-valued coefficients in the linearized equation, cf. Example 1.1. In the anisotropic case this can be done only for rather special situations so far, see [**43**] and [**49**]. On the other hand, in [**48**] we have developed theory for the anisotropic Maxwell system on $\mathbb{R}^2$, using the same arguments as presented here. These Strichartz estimates lead to an improved local wellposedness theory on $\mathbb{R}^2$. Theorem 4.8 can still be used to establish local wellposedness of semilinear Maxwell systems arising from retarded problems. This application is treated at the end of the section. Below we can only sketch the arguments in a more informal way, and we cannot discuss variants as we did in Section 4.1. We further note that for scalar coefficients and certain additional conditions, there are global-in-time Strichartz estimates, see [**16**], and (local-in-time) on domains in the presence of boundary conditions, see [**11**].

A) Strichartz estimates in an $L^1_{\mathbb{R}}$-setting. For the treatment of evolution equations, one prefers a true local-in-time estimate without (non-local and non-causal) time regularity, as we have achieved it in Corollary 4.25. There we have passed to an error term for $v = (D, B)$ in $L^\infty_T L^2$, as $L^2_T L^2$ does not fit to some of the arguments anymore, cf. (4.48). However, the data are still measured in L^2_T on the right-hand side of the Strichartz inequalities. In view of the results for the wave equation in Theorem 3.13 or the energy estimate (4.6), we rather expect and prefer L^1_T here. Even more importantly, the applications to quasilinear problems lead to linearized systems with $\bar{\nabla}(\varepsilon, \mu) \in L^2_T L^\infty$, for instance. As a first step towards such Strichartz estimates we look at Lipschitz coefficients with $\partial_{\bar{x}}^2(\varepsilon, \mu) \in L^1_{\mathbb{R}} L^\infty$. We state a slightly weaker version of Theorem 1.3 of [**43**], recalling that $L = \partial_t + Ma^{-1}$ represents the Maxwell system (4.3) for $v = (D, B)$.

THEOREM 4.29. *Let $\varepsilon, \mu \in C^1_b(\mathbb{R}^4, \mathbb{R})$ with $\partial_{\bar{x}}^2(\varepsilon, \mu) \in L^1_{\mathbb{R}} L^\infty$ and $\varepsilon, \mu \geq \eta$ for some $\eta > 0$, $\sigma_e = 0 = \sigma_m$, (p, q, γ) be admissible with $p < \infty$, $T > 0$, and $v \in L^\infty_T L^2$. Set $Lv = f$ and $\rho = \operatorname{Div} v$. Then v satisfies*

$$\| |D|^{-\gamma} v \|_{L^p_T L^q} \lesssim \kappa^{\frac{1}{p}} \|v\|_{L^\infty_T L^2} + \kappa^{-\frac{1}{p'}} \|f\|_{L^1_T L^2} + T^{\frac{1}{2}} \big(\| |D|^{-\frac{1}{2}} \rho(0) \|_{L^2}$$

$$+ \| |D|^{-\frac{1}{2}} \partial_t \rho \|_{L^1_T L^2} \big), \tag{4.63}$$

if the terms on the right are finite and $T\|\partial^2_{\bar{x}}(\varepsilon,\mu)\|_{L^1_T L^\infty} \le \kappa^2$ *with* $\kappa \gtrsim 1$*. If* $q = \infty$*, one has to replace* L^q *by* $\dot{B}^0_{\infty,2}$*.*

Note that inequality (4.63) holds trivially for the energy triple $(\infty, 2, 0)$. Other (non-strict) triples (∞, q, γ) are excluded to avoid technical problems in steps 2) and 3) below. In step 2) we see that we can replace $|D|$ by $\langle D\rangle$ in (4.63). The different form of the charge term is caused by the arguments in step 4) below. Note that $\partial_t \rho = \operatorname{Div} f$ by (4.2).

As for Theorem 4.8, the proof of (4.63) involves a lenghty reduction process to a frequency-localized and -truncated inequality, namely

$$2^{-\gamma j} \|v_j\|_{L^p_{\mathbb{R}} L^q} \lesssim \|\bar{P}'_j v\|_{L^\infty_{\mathbb{R}} L^2} + \|L^j v_j\|_{L^1_{\mathbb{R}} L^2} + 2^{-\frac{1}{2}j} \|\check{P}_j \bar{P}_j \rho\|_{L^2_{\mathbb{R}} L^2} \tag{4.64}$$

for strict triples, $v_j = \check{P}_j \bar{P}_j v$, $v \in L^\infty_{\mathbb{R}} L^2$ with support in a ball of fixed size, $\kappa = 1$ and $j \ge k_0$. As before we set $L^j = \partial_t + M a^j$ and $a^j = \bar{P}_{\le j/2} \operatorname{diag}(\varepsilon^{-1}, \mu^{-1})$. In contrast to (4.19) we have L^∞_T and L^1_T instead of $L^2_{\mathbb{R}}$. Moreover, we have already reduced to frequencies $|\xi| \gtrsim |\tau|$ in order to deal with the purely spatial regularity in (4.63). We first explain why (4.64) implies (4.63) in several steps, before we discuss its proof.

1) Using the scaled function $v_\lambda(t,x) = v(\lambda t, \lambda x)$ for $\lambda = T\kappa^{-2}$ we can reduce Theorem 4.29 to $T = \kappa^2$. As in p. 426f. of **[64]** one then sees that it is enough to consider the case $T\|\partial^2_{\bar{x}}(\varepsilon,\mu)\|_{L^1_T L^2} \le 1$. Finally, by scaling with $\lambda = T$, we can restrict to $T = 1 = \kappa$.

2) The small frequency part $w := \bar{P}_{\le j_0} v = \bar{P}_{\le j_0} P_{\le j_0+1} v$ for $j \le j_0$ can be treated as in (4.48). If $q < \infty$, we now involve the Sobolev embedding $\mathcal{H}^{\frac{1}{p}} \hookrightarrow \dot{\mathcal{H}}^{\frac{1}{p}} \hookrightarrow \dot{\mathcal{H}}^{-\gamma,q}$. For $q = \infty$, we first use $\mathcal{H}^{\beta,r} \hookrightarrow L^\infty$ for some $\delta := \beta - \frac{3}{r} > 0$ and $r \in (3p, \infty)$, and then $\mathcal{H}^{\frac{1}{p} - \frac{3}{r}} \hookrightarrow \dot{\mathcal{H}}^{-\gamma,r}$. It follows

$$\| |D|^{-\gamma} P_k w(t)\|_{L^\infty} \lesssim \|P_k w(t)\|_{\mathcal{H}^{\frac{1}{p}+\delta}} \lesssim_{j_0} \|P_k w(t)\|_{L^2}.$$

Littlewood–Paley in L^2 then yields $\| |D|^{-\gamma} w\|_{L^p_1 \dot{B}^0_{\infty,2}} \lesssim_{j_0} \|w\|_{L^p_1 L^2} \le \|v\|_{L^\infty_1 L^2}$.

Hence, we only have to treat $\tilde{v} := P_{>j_0} v$ on the left-hand side. As in Lemma 4.12, we see that (4.63) is true for $\tilde{v}$ on the left and with $\langle D\rangle$ instead of $|D|$. Here we use $\varepsilon, \mu \in C^1_b$.

3) If (p,q,γ) is not strict, we pass to the strict triple $(p, \bar{q}, \bar{\gamma})$ with $\frac{1}{\bar{q}} = \frac{1}{2} - \frac{1}{p} > 0$ as in Lemma 3.17. If $q < \infty$, the Sobolev embedding $\dot{\mathcal{H}}^{-\bar{\gamma},\bar{q}} \hookrightarrow \dot{\mathcal{H}}^{-\gamma,q}$ and (4.63) for $(p, \bar{q}, \bar{\gamma})$ imply the estimate for (p,q,γ). For $q = \infty$, we use Bernstein, Minkowski and (4.63) for $\tilde{v}$ to compute

$$\begin{aligned} \| |D|^{-\gamma} \tilde{v}\|^2_{L^p_1 \dot{B}^0_{\infty,2}} &\lesssim \Big\| \sum\nolimits_{k \ge j_0} 2^{-2k\bar{\gamma}} \|P_k v\|_{L^{\bar{q}}} \Big\|^2_{L^p_1} \le \sum\nolimits_{k \ge j_0} \| |D|^{-\bar{\gamma}} P_k v\|^2_{L^p_1 L^{\bar{q}}} \\ &\lesssim \sum_{k \ge j_0} \big[\|P_k v\|^2_{L^\infty_1 L^2} + \|L P_k v\|^2_{L^1_1 L^2} + \| |D|^{-\frac{1}{2}} P_k \rho(0)\|^2_{L^2} + \| |D|^{-\frac{1}{2}} \partial_t P_k \rho\|^2_{L^1_1 L^2} \big]. \end{aligned}$$

The square sum can be taken into the L^1-norms by Minkowski. A bit sloppy, LP_kv is written as $P_kf + Da^{-1}P_kv + [a^{-1}, MP_k]v$. The second term is bounded in L^2 by $\|P_kv(t)\|_2$, whereas the third is split into

$$[a^{-1}, MP_k]P_k''v + P_kM\big(P_{\geq k-2}a^{-1}(I - P_k'')v\big)$$

as in the proof of Lemma 4.16. Both summands are estimated as in this proof by $\|a^{-1}(t)\|_{2,2}\|P_k''v(t)\|_2$. All together the square sum of $\|LP_kv\|_{L^1_1L^2}$ is less than $c\big(\|v\|^2_{L^\infty_1L^2} + \|f\|^2_{L^1_1L^2}\big)$. In the summand with L^∞_1 we use the energy estimate (4.6) to obtain $\|P_kv(0)\|^2_{L^2}+\|f\|^2_{L^1_1L^2}$ and then $\|v_0\|^2_{L^2} \leq \|v\|^2_{L^\infty L^2}$ by Littlewood–Paley, arriving at (4.63). From now on we can assume $p, q < \infty$.

4) In this core step we deduce (4.63) for $\kappa = 1 = T$, strict triples and $\tilde v$ on the left from the estimate

$$\||D|^{-\gamma}\tilde v\|_{L^p_1L^q} \leq \||D|^{-\gamma}\tilde v\|_{L^p_2L^q} \lesssim \|v\|_{L^2_{\mathbb{R}}L^2} + \|f\|_{L^2_{\mathbb{R}}L^2} + \|\langle D\rangle^{-\frac{1}{2}}\rho\|_{L^2_{\mathbb{R}}L^2} \quad (4.65)$$

involving again only $L^2_{\mathbb{R}}L^2$ on the right. (Observe that $|D|$ and $\langle D\rangle$ are equivalent on the range of $P_{>j_0}$ by Mikhlin.) Indeed, let (4.65) be true. The solution v of (4.3) is given by Duhamel's formula for solution operators $U(t,r)$ with $t, r \in \mathbb{R}$. The part $P_{>j_0}U(\cdot,0)v_0$ has the constant charge $P_{>j_0}\rho(0)$. We apply (4.65) to $w = \phi P_{>j_0}U(\cdot,0)v_0$ for $\phi \in C_c^\infty(\mathbb{R})$ being 1 on $[0,1]$. Note that the resulting inhomogeneity $Lw = \phi' P_{>j_0}U(\cdot,0)v_0$ can be bounded by $c\|v_0\| \leq c\|v\|_{L^\infty_1L^2}$ by the energy estimate (4.6), so that

$$\|\langle D\rangle^{-\gamma}P_{>j_0}U(\cdot,0)v_0\|_{L^p_2L^q} \lesssim \|v(0)\|_{L^2} + \|\langle D\rangle^{-\frac{1}{2}}\operatorname{Div} v_0\|_{L^2}.$$

It remains to treat the part $\tilde v_1(t) = P_{>j_0}\int_0^t U(t,r)f(r)\,\mathrm{d}r$. The above estimate for $f(r)$ instead of v_0 then yields

$$\|\langle D\rangle^{-\gamma}\tilde v_1\|_{L^p_1L^q} \leq \int_0^2 \big\|\langle D\rangle^{-\gamma}U(\cdot,r)f(r)\big\|_{L^p_1L^q}\,\mathrm{d}r \lesssim \|f\|_{L^1_2L^2} + \|\langle D\rangle^{-\frac{1}{2}}\partial_t\rho\|_{L^1_2L^2}$$

by Minkowski's inequality and $\operatorname{Div} f = \partial_t\rho$.

5) As in Lemma 4.15, we now restrict to v supported in balls of a radius $R>0$.

6) We strenghten (4.65) to the time interval $\mathbb{R}$ on the left, and treat the frequency region $|\xi| \leq \frac{1}{2}\eta|\tau|$. For the principal symbol ℓ of L and such $\bar\xi$, one has $|\ell(\bar\xi)| \leq |\tau|/2$ and $|\tau| \leq c_0|\bar\xi|$ with $c_0 = 2/(2+\eta)$, cf. (4.38). We choose a smooth function χ_τ with bounded derivatives and support in this region, being 1 on $\{|\xi| \leq \frac{1}{4}\eta|\tau|\}$, and $P_\tau = \chi_\tau(\bar D)$. Using Sobolev and Plancherel, we infer

$$\begin{aligned}\|\langle D\rangle^{-\gamma}P_\tau\tilde v\|_{L^p_{\mathbb{R}}L^q} &\lesssim \|\langle D\rangle^{-\gamma}P_\tau\tilde v\|_{L^p_{\mathbb{R}}\mathcal{H}^{\frac{1}{p}}} \lesssim \|\langle D\rangle^{-\gamma}P_\tau\tilde v\|_{\mathcal{H}^{\frac{1}{p}-\frac{1}{2}}_{\mathbb{R}}\mathcal{H}^{\frac{1}{p}}}\\ &\lesssim \|\langle\bar\xi\rangle^{\frac{1}{2}}\ell^{-1}\chi_\tau\ell\bar{\mathcal{F}}\tilde v\|_{L^2_{\mathbb{R}}L^2} \lesssim \|(\partial_t - a^{-1}M)\tilde v\|_{L^2_{\mathbb{R}}L^2}\\ &\lesssim \|v\|_{L^2_{\mathbb{R}}L^2} + \|f\|_{L^2_{\mathbb{R}}L^2}.\end{aligned}$$

As a result, we can replace in (4.65) $\tilde v$ by $\hat v = (I - P_\tau)\tilde v$.

7) We next infer (4.65) for $\hat v$ and thus Theorem 4.29 from the frequency localized version

$$\||D|^{-\gamma}v_j\|_{L^p_{\mathbb{R}}L^q} \lesssim \|v_j\|_{L^2_{\mathbb{R}}L^2} + \|Lv_j\|_{L^r_{\mathbb{R}}L^2} + 2^{-\frac{1}{2}j}\|\bar P_j\check P_j\rho\|_{L^2_{\mathbb{R}}L^2} \quad (4.66)$$

for $v_j = \bar{P}_j \check{P}_j v$, $j \geq j_0$ and some $r \in (1,2)$. (Here we slighty adjust the definition of $\check{\chi}$.) Because of Littlewood–Paley and the fixed size of $\operatorname{supp} f \subseteq \operatorname{supp} v$, as in step 3) it remains to show

$$\sum\nolimits_{j\geq j_0} \big\|[a^{-1}, M\bar{P}_j\check{P}_j]\bar{P}_j' v\big\|^2_{L^2_{\mathbb{R}}L^2} + \big\|\check{P}_j M\bar{P}_j\big(\bar{P}_{\geq j-2}a^{-1}(I-\bar{P}_j')v\big)\big\|^2_{L^r_{\mathbb{R}}L^2} \lesssim \|v\|^2_{L^2_{\mathbb{R}}L^2}.$$

The commutator can be treated as in Lemma 4.16 only using $\varepsilon, \mu \in C^1_b$. Further, Bernstein yields

$$\big\|2^j \bar{P}_{\geq j-2} g\big\| \lesssim \sum\nolimits_{k\geq j-2} 2^{j-k}\big\|\bar{P}_k \bar{\nabla} \bar{P}_{\geq j-3} g\big\| \lesssim \big\|\bar{\nabla}\bar{P}_{\geq j-3} g\big\|$$

in any $L^p_{\mathbb{R}}L^q$-norm. Let $r^* = 2r/(2-r)$. Using also Hölder, the second summand above is thus is controlled via

$$\big\|\bar{P}_{\geq j-3}\bar{\nabla}a^{-1}\big\|_{L^{r^*}_{\mathbb{R}}L^\infty}\|v\|_{L^2_{\mathbb{R}}L^2} \lesssim \big\|\bar{\nabla}a^{-1}\big\|^{1-\frac{1}{r^*}}_{L^\infty_{\mathbb{R}}L^\infty}\big\|2^{-j}\bar{P}_{j-3}D^2_{\bar{x}}a^{-1}\big\|^{\frac{1}{r^*}}_{L^1_{\mathbb{R}}L^\infty}\|v\|_{L^2_{\mathbb{R}}L^2}$$

and thus by $2^{-\frac{1}{r^*}j}\|v\|_{L^2_{\mathbb{R}}L^2}$ which is square summable.

As in Remark 4.17 we see that $\partial_{\bar{x}} v_j$ decays rapidly outside a compact set so that (4.66) follows from the analogous estimate with $L^1_{\mathbb{R}}$ instead of $L^r_{\mathbb{R}}$ and with $2^{-j}\|v\|_{L^\infty_{\mathbb{R}}L^2}$ on the right. Moreover, by the energy estimate on compact time intervals $\|v_j(0)\|$ behaves like $\|v_j(t)\|$ up to summands $\|Lv_j\|_{L^1_{\mathbb{R}}L^2}$. Hence it suffices to prove (4.66) with $\|\bar{P}_j v\|_{L^\infty_{\mathbb{R}}L^2}$ on the right.

8) We deduce (4.66) with the above modifications from estimate (4.64) as in Lemma 4.18 but applying Hölder in time differently, similar to step 7).

Next we indicate the proof of (4.64). In the argument we split $\bar{P}_j\check{P}_j = \bar{P}_{1j} + \bar{P}_{2j} + \bar{P}_{3j}$, see (4.40). The terms $\bar{P}_{ij}v$ are then estimated as in (4.44) observing that the symbol compositions in Proposition 4.23 work for Lipschitz coefficients. For the non-degenerate components one uses the estimate

$$2^{-\gamma j}\|\bar{P}_j w\|_{L^p_{\mathbb{R}}L^q} \lesssim \|\bar{P}_j w\|_{L^\infty_{\mathbb{R}}L^2} + \|Q\bar{P}_j w\|_{L^2_{\mathbb{R}}L^2} \tag{4.67}$$

from Proposition 1.8 in [**43**], which is a variant of Theorem 4.26. As in step 4) above one can replace $L^2_{\mathbb{R}}$ by $L^1_{\mathbb{R}}$ on right. In this way one reduces (4.66) to (4.67). The latter result is shown in pp. 429–436 of [**64**], cf. § 4.2 of [**48**]. This proof is similar to Theorem 4.26, but adapted to $\partial^2_{\bar{x}}a \in L^1_{\mathbb{R}}L^\infty$ and $w \in L^\infty_{\mathbb{R}}L^2$. In particular, Theorem 4.22 is replaced by its variant Theorem 2.3 in [**64**], which yields estimates of the transformed solution which are adapted to the new setting. Also the core part involving the oscillatory integrals is modified according to the lower regularity of the coefficients.

We finally deduce a corollary for coefficients possessing only one derivative. To this aim, for $s \in \mathbb{R}$ we introduce the space $\mathcal{X}_s$ of $\varphi \in \mathcal{S}_0^\star(\mathbb{R}^4)$ with finite norm

$$\|\varphi\|_{\mathcal{X}_s} = \sup\nolimits_{j\in\mathbb{Z}} 2^{sj}\|\bar{P}_j\varphi\|_{L^1_{\mathbb{R}}L^\infty}.$$

Note that $\bar{\nabla}L^1_{\mathbb{R}}L^\infty \hookrightarrow \mathcal{X}_1$ by Bernstein and Young, cf. Remarks 4.6 and 3.4. For strictly positive $\varepsilon, \mu \in \mathcal{X}_s$ one can show a Strichartz estimate with the norm $\sup_j 2^{-\gamma-\frac{\sigma}{p}}\|\bar{P}_j v\|_{L^p_{\mathbb{R}}L^q}$ on the left. This is deduced from Theorem 4.29 by an argument as in Lemma 4.19, see Theorem 1.5 in [**43**]. As a consequence, we obtain an estimate involving space regularity only.

Corollary 4.30. *Assume that* $\varepsilon, \mu \in \mathcal{X}_s$ *satisfies* $\varepsilon, \mu \geq \eta > 0$, $\|(\varepsilon, \mu)\|_{\mathcal{X}_s} \lesssim 1$ *and* $\|\partial_{\bar{x}}(\varepsilon, \mu)\|_{L^2_T L^\infty} \lesssim 1$ *for some* $s \in [1, 2)$ *and* $T > 0$. *Let* $\sigma = \frac{2-s}{2+s}$, (p, q, γ) *be admissible,* $\alpha > \gamma + \frac{\sigma}{p}$, $Lv = f$, *and* $\operatorname{Div} v = \rho$. *We then obtain*

$$\|\langle D\rangle^{-\alpha} v\|_{L^p_T L^q} \lesssim_T \|v_0\|_{L^2} + \|f\|_{L^1_T L^2} + \|\rho(0)\|_{\mathcal{H}^{-\frac{1}{2}-\frac{\sigma}{p}}} + \|\partial_t \rho\|_{L^1_T \mathcal{H}^{-\frac{1}{2}-\frac{\sigma}{p}}}.$$

This result can be shown as Corollary 1.7 in [**48**], passing to Littlewood–Paley blocks. On a frequency range $|\tau| \lesssim |\xi|$ one can use the result mentioned above and the assumption involving $\mathcal{X}_s$. For $|\tau| \gg |\xi|$, we argue as in (4.44) employing $\partial_{\bar{x}}(\varepsilon, \mu) \in L^2_{\mathbb{R}} L^\infty$. The extra regularity loss $\alpha - \gamma - \frac{\sigma}{p}$ is needed to bound certain square sums.

To treat a quasilinear problem in next paragraph, we pass to the *two-dimensional* Maxwell system

$$\begin{aligned} \partial_t D &= \nabla_\perp(\mu^{-1} B) - J_e, \quad D(0) = D_0, \\ \partial_t B &= -\operatorname{curl}(\varepsilon^{-1} D) - J_m, \quad B(0) = B_0, \end{aligned} \qquad t \in J,\ x \in \mathbb{R}^2. \tag{4.68}$$

with $D, J_e \colon \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}^2$, $B, J_m, \mu \colon \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}$, $\varepsilon \colon \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}^{2\times 2}_{\geq\eta}$, $\rho_e = \operatorname{div} D$, $\operatorname{curl}(\varphi_1, \varphi_2) = \partial_1\varphi_2 - \partial_2\varphi_1$, and $\nabla_\perp = (\partial_2, -\partial_1)^\top$.

In contrast to (4.3) for $(\mathbf{D}, \mathbf{B})$, in (4.68) the fields and coefficients only depend on the variables (t, x_1, x_2). Morever, compared to (4.3) the components $\mathbf{D}_3$, $\mathbf{B}_1$, and $\mathbf{B}_2$ vanish, so that $\mathbf{B}$ is orthogonal to the (x_1, x_2)-plane. This behavior is called TM polarization ('transversal magnetic'). If $\varepsilon_{j3} = \varepsilon_{3j} = 0$ for $j \in \{1, 2\}$ and μ is scalar, solutions of (4.3) preserve this structure if it satisfied by $(\mathbf{D}_0, \mathbf{B}_0)$ and $\mathbf{J}$. The theory of Chapter 2 can be transfered to the above setting (adapting the Sobolev embeddings a bit). One can also infer the wellposedness from the three-dimensional case, see Appendix A in [**11**].

Remark 4.31. In the above setting, let $p, q \in [2, \infty]$, $\frac{2}{p} + \frac{1}{q} \leq \frac{1}{2}$, and $\gamma = 1 - \frac{2}{q} - \frac{1}{p}$. Then the analogues of Theorem 4.8, Theorem 4.29, and Corollary 4.30 are true. See Theorems 1.1–1.3 and Corollary 1.7. in [**48**]. One can use the same arguments with obvious modifications in the context of Sobolev and Bernstein inequalities and the admissibility relations. The core estimates of oscillatory integrals are modfied since $\mathcal{C}$ now has only one non-zero principal curvature, cf. part C) of Section 4.3. The diagonalization of the symbol is easier in this case and thus also works in the matrix case. Here one can compute both the transformation matrix m and its inverse m^{-1}, see (30) and (31) in [**48**] for $\mu = 1$. They mainly involve $\xi_j/|\xi|_{\tilde{\varepsilon}}$ for $|\xi|^2_{\tilde{\varepsilon}} = \tilde{\varepsilon}\xi\cdot\xi$ and $\tilde{\varepsilon} = \det(\varepsilon)\,\varepsilon^{-1}$. In [**48**] one finds more Strichartz estimates with different exponents on the right-hand side. $\Diamond$.

B) A quasilinear problem in lower regularity. We apply Corollary 4.30 on $\mathbb{R}^2$ to the quasilinear equation

$$\begin{aligned} &\partial_t D = \nabla_\perp(B), \quad D(0) = D_0, \\ &\partial_t B = \partial_2(\varepsilon^{-1}(D) D_1) - \partial_1(\varepsilon^{-1}(D) D_1), \quad B(0) = B_0, \end{aligned} \qquad t \in J,\ x \in \mathbb{R}^2. \tag{4.69}$$

with $\varepsilon^{-1}(D) = \psi(|D_1|^2 + |D_2|^2)$ and $\operatorname{div} D_0 = 0$. Here $\psi : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is smooth and increasing with $\psi(0) = 1$. This covers the Kerr case $\varepsilon(E) = 1 + |E|^2$. We set $v = (D, B) = (\tilde{v}, v_3)$ and state Theorem 1.9 of [**48**].

THEOREM 4.32. *Under the above assumptions, let $s > 11/6$ and $v_0 \in \mathcal{H}^s(\mathbb{R}^2)$. Then there is a unique local solution $v \in C\big([0,T], \mathcal{H}^s(\mathbb{R}^2)\big)$ with $\bar\nabla v \in L^1_T L^\infty$ of (4.69) for some $T = T(\|v_0\|_s) > 0$. It depends continuously on v_0 in $\mathcal{H}^s$.*

We note that energy methods alone yield the result for $s > 2$, see [**32**]. For the wave equation the same improvement for the wave equation was established in Theorem 5.1 in [**64**]. This seems to be the borderline for an approach only using linearization and Strichartz estimates. Here we use an approximation method presented in [**28**], which starts from the existence of regular solutions for regular data as provided by [**32**] or Chapter 2.[3] This has the advantage that one use (4.69), which fits to the Strichartz, and the version where one applies product and chain rule to $\operatorname{curl}(\varepsilon^{-1}(D)D)$, which fits to energy estimates. We write the resulting equation as

$$\partial_t v = \mathcal{A}^1(v)\partial_1 + \mathcal{A}^2(v)\partial_2 \tag{4.70}$$

with the coefficient matrices

$$\mathcal{A}^1(v) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ -2\psi'(|\tilde v|^2)v_1v_2 & -2\psi'(|\tilde v|^2)v_2^2 - \psi(|\tilde v|^2) & 0 \end{pmatrix},$$

$$\mathcal{A}^2(v) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 2\psi'(|\tilde v|^2)v_1^2 + \psi(|\tilde v|^2) & 2\psi'(|\tilde v|^2)v_1v_2 & 0 \end{pmatrix}.$$

Let v be a smooth solution, $\|v_0\|_{\mathcal{H}^s} \le r_0$, set $r(t) = \|\nabla_x \tilde v(t)\|_{L^\infty}$ and write c for constants only depending on $\|v\|_{L^\infty_T L^\infty}$.

1) Using commutator and Moser-type estimates, one can show the energy inequality

$$\|v(t)\|_{\mathcal{H}^s} \le c e^{c\int_0^t r\,\mathrm{d}\tau} \|v_0\|_{\mathcal{H}^s}, \tag{4.71}$$

see Proposition 6.1 in [**48**]. Moreover, there is a time $T = T(\|v_0\|_s)$ such that $\|v(t)\|_{\mathcal{H}^s} \le c\|v_0\|_{\mathcal{H}^s}$.

2) To show this fact we fix $R = c_0 r_0$ for a suitable $c_0 > 0$. Let $T_1 > 0$ be the supremum of $T > 0$ such v exists on $[0,T]$ and $\|\nabla_x w\|_{L^4_T L^\infty} \le R$. Let $T < T_1$ and set $R_1 = \|r\|_{L^1_T} \le T^{\frac34} R$. We can control the norms

$$\|\nabla_x \varepsilon(\tilde v)\|_{L^2_T L^\infty} \le cT^{\frac14} R \le 1, \qquad \|\nabla_x \varepsilon(\tilde v)\|_{L^1_T L^\infty} \le cT^{\frac34} R \le 1$$

for small $T = T(r_0) \in (0, T_1)$. Hence we can apply Corollary 4.30 in the version of Remark 4.31 with a unform constant. We note that

$$\|P_{\le j_0} \nabla_x v\|_{L^4_T L^\infty} \lesssim T^{\frac14} \|v\|_{L^\infty_T L^2} \le c e^{cR_1} T^{\frac14} r_0 \le c T^{\frac14} r_0$$

by Hölder, Bernstein and the energy estimate. For high frequencies we use $\tilde v = \langle D\rangle^s v$. Since $L(v)v = 0$, as above one can bound

$$\|L(v)\tilde v(t)\|_{L^2} = \|[L(v), \langle D\rangle^s] v(t)\|_{L^2} \le c r(t) \|v(t)\|_{\mathcal{H}^s}.$$

[3] In the lecture an erroneous argument was presented.

We take the admissible triple $(4, \infty, \frac{3}{4})$ and note that $1 - s < -\frac{5}{6} = \gamma + \frac{\sigma}{p}$ since $\sigma = \frac{1}{3}$ for $s = 1$. Hence the Strichartz inequality and step 1) yield

$$\begin{aligned}\|P_{\geq j_0}\nabla_x v\|_{L^4_T L^\infty} &\lesssim \|\langle D\rangle^{1-s}\tilde{v}(t)\|_{L^4_T L^\infty} \lesssim \|\tilde{v}(0)\|_{L^2} + \|L(v)\tilde{v}\|_{L^1_T L^2}\\ &\lesssim c\big(\|v_0\|_{\mathcal{H}^s} + R_1 \sup_{t\leq T}\|v(t)\|_{\mathcal{H}^s}\big) \lesssim c\big(r_0 + T^{\frac{3}{4}} \mathrm{e}^{cRT^{\frac{3}{4}}} r_0\big) < R.\end{aligned}$$

for sufficiently small $T = T(r_0) > 0$, as claimed. (We avoid the Besov norm on the left by a slight regularity loss.)

3) As in step 1) one can show the contraction $\|v(t) - \hat{v}(t)\|_{L^\infty_T \mathcal{H}^s} \leq c\|\hat{v}_0 - \hat{v}_0\|_{\mathcal{H}^s}$. for the solution $\hat{v}$ of (4.69) with initial value $\hat{v}_0$ and $\|\hat{v}_0\|_{\mathcal{H}^s} \leq r_0$, see Proposition 6.2 in [**48**]. These ingredients are enough to show Theorem 4.32 using [**28**], as explained in §6 of [**48**].

C) A retarded nonlinear problem in low regularity. This is very recent joint work with C. Bresch, see [**8**]. In nonlinear optics the typical nonlinearities exhibit retardations in time, see [**7**], [**12**] or [**20**] and the short discussion around (1.10). We only look at the typical example

$$\begin{aligned}&\partial_t(\varepsilon E) = \operatorname{curl} H - \partial_t P(E), \quad \partial_t(\mu H) = -\operatorname{curl} E, \qquad t \geq 0,\ x \in \mathbb{R}^3,\\ &E(t) = E^0(t), \quad H(t) = H^0(t), \qquad t \in [-b, 0],\ x \in \mathbb{R}^3,\end{aligned} \tag{4.72}$$

with $\varepsilon, \mu \in C_b^2(\mathbb{R}^3, \mathbb{R})$, $\varepsilon, \mu \geq \eta > 0$, $\kappa \in W^{1,\infty}(\mathbb{R}_{\geq 0}, L_3(\mathbb{R}^3, \mathbb{R}^3))$, $a > 0$, and

$$P(E)(t) = \int_{[-a,t]^3} \kappa(t - r_1, t - r_2, t - r_3)[E(r_1), E(r_2), E(r_3)]\,\mathrm{d}(r_1, r_2, r_3)$$

for $t \geq 0$. Here one has to impose conditions for the 'prehistory' E^0 for $t \in [-a, 0]$ and not just at time $t = 0$. In [**8**] we treat finite sums of analogous n-linear terms also for the magntization with kernels depending on x and r-integrals over $(-\infty, t]^n$ assuming also that κ is $W^{1,1}$ in time. Moreover, we allow for conductivity. In typical examples, κ is given by trigonometric polynomials times decaying exponentials.

In the above setting, one can differentiate $P(E)$ in time obtaining

$$\begin{aligned}\partial_t P(E)(t) = &\int_{[-a,t]^3} \partial_1\kappa(t - r_1, t - r_2, t - r_3)[E(r_1), E(r_2), E(r_3)]\,\mathrm{d}(r_1, r_2, r_3)\\ &+ \int_{[-a,t]^2} \kappa(0, t - r_2, t - r_3)[E(t), E(r_2), E(r_3)]\,\mathrm{d}(r_2, r_3) + \cdots\end{aligned}$$

We stress that no derivative hits E, resulting in a semilinear non-local problem. Existence and unqiqueness of such problems was shown in [**3**] in $\mathcal{H}^s(\mathbb{R}^3)$ for $s > \frac{3}{2}$, using that this space embedds into $L^\infty(\mathbb{R}^3)$.

We now treat the case $s \in (1, \frac{3}{2}]$, strict admissible (p, q, γ), $s > 1 + \frac{1}{q}$, $\alpha = s - \gamma > \frac{3}{q}$ and use that $\mathcal{H}^{\alpha,q}(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$. We set

$$Z(b) = C_b\big([-a, b], \mathcal{H}^s(\mathbb{R}^3)^6\big) \cap L^p\big([-a, b], \mathcal{H}^{\alpha,q}(\mathbb{R}^3)^6\big)$$

for $b \geq 0$, endowed with the canonical norm. Moreover let $E^t(\tau) = E(t+\tau)$ for $\tau \in [-a, 0]$ andf $t \geq 0$, $u = (E, H)$, $L = a\partial_t + M$, and $F(E^t) = \frac{1}{\varepsilon}\partial_t P(E)(t)$. One can the check that $F : Z(b) \to L^1\big([0, b], \mathcal{H}^s \cap \mathcal{H}^{\alpha,q}\big)$ is Lipschitz on balls, using

$\mathcal{H}^{\alpha,q}(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$. Moreover we can shift Theorem 4.29 to the regularity level $s > 1$.

Corollary 4.33. *In the above setting, let $T \in (0,1]$, $u_0 \in \mathcal{H}^s(\mathbb{R}^3)$, and $f \in L^1(0,T), \mathcal{H}^s(\mathbb{R}^3))$. We then obtain*

$$\|\langle D\rangle^\alpha u\|_{L^p_T \mathcal{H}^{\alpha,q}} \lesssim \|u_0\|_{\mathcal{H}^s} + \|f\|_{L^1_T \mathcal{H}^s} + \|\rho(0)\|_{\mathcal{H}^{s-\frac{1}{2}}} + \|\partial_t \rho\|_{L^1_T \mathcal{H}^{s-\frac{1}{2}}}.$$

if the terms with $\rho = \mathrm{Div}(au)$ are finite.

Here we pass to u as in Lemma 4.14, to inhomogeneous derivatives as after Theorem 4.29, and to u_0 using the mapping properties of the C_0–semigroup generated by $a^{-1}M$. The regularity lift to $s > 1$ is more complicated as in Remark 4.13. For the charge terms one has to exploit the negative regularity in the charge terms in Theorem 4.29. See Theorem 3.4 in [**8**].

Unfortunately the charge terms in Corollary 4.33 would spoil the local wellposedness result. To deal with them, we use the projection Q_θ on N(curl) with kernel given by $\mathrm{div}(\theta\varphi) = 0$ and set $Q = \mathrm{diag}(Q_\varepsilon, Q_\mu)$ as well as $\tilde{Q} = I - Q$. One can check that these operators behave well in $\mathcal{H}^s$ and $\mathcal{H}^{\alpha,q}$, see Lemma 4.6 in [**8**]. We can split (4.72) with ‘frozen' nonlinearity into

$$\partial_t \tilde{Q} u(t) = a^{-1} M \tilde{Q} u(t) + \tilde{Q} F(v_t), \qquad \partial_t Q u(t) = Q F(v_t),$$

where v is taken from a suitable ball in $Z(b)$. Here the inhomogeneity $f(t) = \tilde{Q} F(u_t)$ is charge-free (for the operator L) and the second equation can be simply integrated.

Assuming that $u^0 \in Z(0)$ and $Qu^0(0) \in \mathcal{H}^{\alpha,q}$, we can then solve the fixed point problem $v \mapsto u$ and establish a local wellposedness theory in $Z(b)$, see Section 5 in [**8**].

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